THINKING LIKE A MATHEMATICIAN

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1. Let $\vec{u}$ be a vector. Show that, if $\|\vec{u}\| = 0$, then $\vec{u} = \vec{0}$. In a similar vein, prove the converse – that is, show that, if $\vec{u} = \vec{0}$, then $\|\vec{u}\| = 0$.

2. Prove that, for any $n \geq 0$, it holds that $\displaystyle\int_{[0,1]} \left(x^2+1\right)^n \, dx \geq 1$. **HINT:** this can be done via induction on n or – an easier approach – by breaking down and reconstructing the inequality.

3. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous, differentiable function such that, for any $x < y$, we have $f(x) < f(y)$. Suppose that, for some $x_1 \in \mathbf{R}$, we have that $f(x_1) < 0$ and for some $x_2 \in \mathbf{R}$, we have that $f(x_2) > 0$. Show that f has one and *only* one zero.

4. Using only upward and rightward moves, how many ways can the point $(8,5)$ be reached? How many ways can $(8,5)$ be reached without going through the point $(3,3)$?

5. How many diagonals does an n-sided, regular polygon have? **HINT:** try some small cases – begin with $n = 3$ and try to identify a pattern.

6. For what positive value of k is the following equation true?
$$\int_e^{k^e} \frac{dx}{x \int_k^{kx} \frac{dy}{y}} = 1$$

7. Evaluate $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{\sqrt{n^2 - i^2}}$.

8. Let $S = \{x \in \mathbf{R} \ : \ m \leq x < n\}$. What is the smallest upper bound (the supremum) on S? Is the supremum a member of S? Why or why not?

9. Find upper and lower bounds for the double integral $\displaystyle\iint_R \frac{1}{\sin(x+y)^2 + 1} \, dA$, where $R = [0,a] \times [0,b]$ for positive a and b.

10. Let C be the upper semicircle of radius 3. *Without using calculus*, show that the slope of any line tangent to a point p on C is $m = -2p/\sqrt{9-p^2}$.

11. Is the upper half-plane a subspace of $\mathbf{R}^2$? Why or why not?

12. How many ways can the number 4 be written as the sum of five non-negative integers?

13. Let A be a set containing $n+1$ integers. Prove that it is always possible to choose $a \in A$ and $b \in A$ such that $a - b$ is divisible by n.

14. Consider the interval $I = [0,1]$. Show that
$$I \subset \bigcup_{a \in \mathbf{R}_+} (-a, a).$$
Further, show that
$$\mathbf{R} = \bigcup_{a \in \mathbf{R}_+} (-a, a).$$

15. Calculate the value of $\displaystyle\sum_{i=0}^{n} \binom{n}{i} 2^n$.

16. How many *injective* functions are there from a set with k elements to a set with n elements?