Gröbner bases, commutative case

Consider an ideal $I \subseteq k[x_1, \dots, x_n] = k[\mathbb{X}]$, the aim is to devide modulo I

DEF Let $M = \{\prod_{i=1}^{d} x_i^{p_i} \mid p_i \in \mathbb{Z}_{\geq 0}\}$ the set of monic monomials of $k[\mathbb{X}]$. A monomial order on M is a total (well) order in which

$1 \leq m \quad \forall m \in M$

$m_1 \leq m_2$ implies $m_1 m_3 \leq m_2 m_3 \quad \forall m_1, m_2, m_3 \in M$

Our favourite examples are lexicographic and degree lexicographic
Our favourite non example is degree partition lexicographic.

We now fix a monomial order on M

DEF Let $f = \sum_{i=0}^{s} c_i m_i \in k[\mathbb{X}]$ with $m_i \in M_i$. Then we call $\operatorname{init}(f) := \bigwedge_{i=0}^{s} m_i$ the initial monomial of f.

Similarly, for an ideal I of $k[\mathbb{X}]$, we call $\operatorname{init}(I) := \langle \operatorname{init}(f) \mid f \in I \rangle$ the initial ideal of I.
Monomials $m \in \operatorname{init}(I)$ are called non standard, otherwis they are standard.

A finite subset $G \subseteq I$ is called a Gröbner basis of I provided that

$$\operatorname{init}(I) = \langle \operatorname{init}(g) \mid g \in G \rangle = \langle \operatorname{init}(G) \rangle$$

and if it additionally satisfies that for all distinct $g_1, g_2 \in G$, we have

$$\operatorname{init}(g_1) \nmid m \quad \text{whenever } m \text{ is a monomial belonging to } g_2$$

we call G reduced (for uniqueness one can demand the g_i to be monic).

LEMMA Let $I \subseteq k[\mathbb{X}]$ be a monomial ideal, then I is finitely generated by monomials.
PROOF Define $I_i := \langle m \in M \cap k[x_1, \dots, x_{n-1}] \mid m x_n^i \in I \rangle$ and observe $I_i \subseteq I_{i+1}$ for all i.
Inductively, we can assume $\bigcup_{i \geq 0} I_i$ to be finitely generated by monomials so that $(I)_{i\geq 0}$ stabilizes, say $\bigcup_{i\geq 0} I_i = \bigcup_{i=0}^{r} I_i$, then I is generated by $m_{ij} x_n^i$ where I_i is generated by monomials m_{ij} when j varies. □

COR Monomial orders are well orders.

LEM Every ideal $I \subseteq k[\mathbb{X}]$ has a Gröbner basis and it generates I

PROOF Existence follows from the previous lemma. Now let G a Gröbner basis for I and assume $I \setminus \langle G \rangle \neq \emptyset$, then $\bigvee \{\operatorname{init}(f) \mid f \in I \setminus \langle G \rangle\} =: \operatorname{init}(f_0)$ is contained $\operatorname{init}(I) = \langle \operatorname{init}(G) \rangle$ so there is a $g \in G$ such that $\operatorname{init}(f_0) = m \operatorname{init}(g)$. But then $f_0 - mg \in I \setminus \langle G \rangle$, yet $\operatorname{init}(f_0 - mg) < \operatorname{init}(f_0)$ □

COR Hilbert's basis theorem

THM The standard monomials form a basis for $k[\mathbb{X}]/I$
PROOF Let G be a Gröbner basis, then we compute the "normal form" like so,

Given a polynomial p, determine if p has nonstandard monomials.

if not, it is in standard form so return p

if so, let h be the largest one and find $g \in G$ such that $\mathrm{init}(g) \mid h$, say $h = \mathrm{init}(g)\, m$, now return the standard form of $p - mg$

The beforementioned well foundedness of monomially ordered sets guarantees this procedure terminates and the uniqueness follows from the fact that every polynomial in I has an initial monomial. $\square$

There are ways to compute and verify Gröbner bases, relying on the s-polynomial

$$f = c\,\mathrm{init}(f) + r, \quad g = d\,\mathrm{init}(g) + s, \quad m = \mathrm{lcm}(\mathrm{init}(f), \mathrm{init}(g))$$

$$S(f,g) = m\left(f\,(c\,\mathrm{init} f)^{-1} - g\,(d\,\mathrm{init}(g))^{-1}\right)$$

LEM G is a Gröbner basis if and only if $S(g_1, g_2) \in \langle G \rangle$ for all $g_1 \neq g_2$ in G

Buchberger algorithm

Let $I = (f_1, \dots, f_t)$ and set $G_0 = \{f_1, \dots, f_t\}$.
Define recursively $G_i = G_{i-1} \cup \{ S(g_1, g_2) \mid g_1, g_2 \in G_{i-1} \text{ distinct},\ S(g_1, g_2) \in \langle G_{i-1} \rangle \}$
Cleary G_ℓ will be a Gröbner basis for I at some point, albeit superfluous.

We will not concern ourselves with the construction of unique reduced Gröbner bases at this time.

Non commutative case

Since path algebras have a canononical vectorspace basis, consisting of all paths, so assume a basis B is given and well ordered. The set $\mathrm{Fin}(B)$ of finite subsets of B is again a well order with the following order

LEM $A_1, A_2 \in \mathrm{Fin}(B)$ $\quad A_1 \leq A_2 :\Leftrightarrow$

$$\begin{aligned} &A_1 = \emptyset \\ &A_1 \neq \emptyset \quad \text{and } \wedge A_1 < \wedge A_2 \\ &A_1 \neq \emptyset \quad \text{and } \wedge A_1 = \wedge A_2 \text{ and } A_1 \setminus \{\wedge A_1\} \leq A_2 \setminus \{\wedge A_2\} \end{aligned}$$

is a well order.

PROOF Clearly the order is a total one and a brief moment of contemplation will convince one that it inherits the wellness from the order on B $\square$

The order on $\mathrm{Fin}(B)$ respects the partial order given by inclusion and has the property that if the largest element of a set is replaced by strictly smaller elements (in the order on B), the the set will shrink strictly in $\mathrm{Fin}(B)$.

We can order the vectorspace V, spanned by B using the order on $\mathrm{Fin}(B)$ using the assignment

$$\mathrm{supp} : v = \sum_{\beta \in B} \lambda_\beta \beta \quad \mapsto \quad \{ \beta \mid \lambda_\beta \neq 0 \}.$$

An important observation is that

OB If $v, w \in V$ and $\wedge\, \mathrm{supp}(w) \in \mathrm{supp}(v)$, there is a unique $\lambda \in k$ such that $\wedge\, \mathrm{supp}(w) \notin \mathrm{supp}(v - \lambda w)$ so $v - \lambda w < v$

DEF Let $W \subseteq V$ be a subspace. We denote

$$\begin{aligned} B_W &:= B \setminus \{ \wedge\, \mathrm{supp}(w) \mid w \in W \} \\ &= \{ \beta \in B \mid \beta \neq \wedge\, \mathrm{supp}(w) \ \forall w \in W \} \end{aligned}$$

LEM $V = W \oplus \langle B_W \rangle$

PROOF By construction $W \cap \langle B_W \rangle = 0$, now suppose $W + \langle B_W \rangle \subsetneq V$ and take $u \in V \setminus (W + \langle B_W \rangle)$ minimal. Since $u \notin \langle B_W \rangle$, we can find $\beta \in \mathrm{supp}(v)$ such that $\beta = \wedge\, \mathrm{supp}(w)$ for some $w \in W$ and per the previous observation we get $v - \lambda w < v$ belongs to $W + \langle B_W \rangle$ □

We have $0 \longrightarrow W \longrightarrow V \underset{\pi}{\overset{b}{\rightleftarrows}} V/W \cong \langle B_W \rangle \longrightarrow 0$ with $\pi b = \mathrm{id}$ and we call $b\pi(v)$ the normal form of v mod W.

DEF A Gröber generating set for $W \subseteq V$ is a subset $G \subseteq W$ such that

$\forall \beta \in B \ (\exists w \in W \ \wedge\, \mathrm{supp}(w) = \beta \ \Rightarrow \exists g \in G \ \wedge\, \mathrm{supp}(g) = \beta)$, that is

$\forall w \in W \ \exists g \in G \ \wedge\, \mathrm{supp}(w) = \wedge\, \mathrm{supp}(g)$

LEM If G is a Gröbner generating set for W, every $v \in V$ reduces to its normal form over G

PROOF The vector v can only be reduced a finite number of times, after which the complete reduction clearly lies in $\langle B_W \rangle$ □

COR If G is a Gröbner generating set for W then it also spans it as vector space

DEF A vector $u \in W$ is called sharp when

$$\mathrm{supp}(u) \cap \{ \wedge\, \mathrm{supp}(w) \mid w \in W \} = \{ \wedge\, \mathrm{supp}(u) \}$$

and the coefficient of $\wedge\, \mathrm{supp}(u)$ in u is 1.
The collection of sharp vectors in W will be denoted $W^\#$

THM $W^\#$ is a Gröbner generating set for W such that no two distinct elements can be reduced over each other.

□

Now we move from vector spaces to associative algebras. In particular algebras kQ where is a quiver (Q_0, Q_1, s, t) and kQ has a canonical vector space basis P consisting of paths in P.

We order P degree lexicographically with the vertices in degree zero and the arrows in degree one.

We get tree structures on kQ that we can exploit.

- a well order on kQ coming from the one on P
- a partial order on kQ coming from divisibility.
- the algebraic structure on kQ

The paper mentions the following 5 properties as crucial

M_1 $P \cup \{0\}$ is closed under multiplication
M_2 divisibility is reflexive on P
M_3 Each $p \in P$ has finitely many factors
M_4 Multiplication respects the order on P
M_5 The the order on P refines the one coming from divisibility

Let $I \subseteq kQ$ be an admissible (two-sided) ideal

DEF A Gröbner basis for I is a subset $G \subseteq I$ such that

$$\forall r \in I \setminus \{0\} \quad \exists g \in G \qquad \wedge supp(g) \mid \wedge supp(r)$$

DEF A simple reduction ρ of an element $c \in kQ$ is a tuple (λ, p, d, q) where $\lambda \in k^*$, $p, q \in P$ and $d \in kQ \setminus \{0\}$ such that

$$p(\wedge supp(d))q \in supp(c)$$
$$p(\wedge supp(d))q \notin supp(c - \lambda pdq)$$

REM By the observation made early on, this indeed reduces c, that is $c - \lambda pdq < c$

A sequence of simple reduction will be called a reduction and if all the d's in a reduction belong to the same set S, then the reduction is said to be over S

THM If G is a Gröbner basis for I then every $c \in kQ$ reduces to its normal form mod I over G

PROOF This follows completely analogous to the case of vector spaces. $\square$

DEF The vectors in $I^{\#}$ that are minimal with respect to divisibility are called minimal sharp and the set of minimal sharp vectors we denote

$$\mathrm{atom}(I^{\#})$$

THM $\mathrm{atom}(I^{\#})$ is a Gröbner generating set such that no two members reduce over another.

PROOF Take $r_1 \in I$ and consider the minimum of $\{r_2 \in I \mid \wedge\mathrm{supp}(r_2) \mid \wedge\mathrm{supp}(r_1)\} =: r_m$ with respect to divisibility, then r_m can be rescaled to be minimal sharp. Pairwise irreducibility follows from minimality $\square$

LEM Let R be a set of relations in kQ and $c \in kQ$ a further relation, that reduces to 0 over R, then pcq reduces to 0 over R

PROOF Take a simple reduction of c, say $c - \lambda_1 p_1 d_1 q_1$, then we get a simple reduction $pcq - \lambda_1 pp_1 d q_1 q$ of pcd and the statement follows inductively $\square$

THM Let G be a set of generators of I such that
- The coefficient of each $\wedge\mathrm{supp}(g)$ is 1 $g \in G$
- For each pair of distinct g_1 and g_2 in G, g_1 and g_2 don't reduce over oneanother.
- For each $g_1, g_2 \in G$, all the overlap differences reduce to 0 over G

DEF An overlap for $p, q \in P$ is a pair of factorizations

$$\begin{aligned} p &= bo \qquad & b, a, o \in P\setminus\{0\} \\ q &= oa \qquad & p \neq b \quad q \neq a \end{aligned}$$

We say $c_1, c_2 \in kQ$ have overlap if $\wedge\mathrm{supp}(c_1)$ and $\wedge\mathrm{supp}(c_2)$ do and if λ_1 and λ_2 are the respective coefficients of $\wedge\mathrm{supp}(c_1)$ and $\mathrm{supp}(c_2)$ and setting

$$\begin{aligned} \wedge\mathrm{supp}(c_1) &= bo \\ \wedge\mathrm{supp}(c_2) &= oa \end{aligned}$$

we call $\lambda_2 c_1 a - \lambda_1 o c_1$ the overlap diffirence

Hochschild cohomology

Fix a field k and let Λ ($= kQ/I$) be a k algebra. Then Λ is a bimodule over itself, or a right $\Lambda^{op} \otimes \Lambda =: \Lambda^e$ module if you will. If we take a further right Λ^e module M we can form the semisimplicial k-module

$$0 \leftarrow M \leftarrow M \otimes \Lambda \leftleftarrows M \otimes \Lambda^{\otimes 2} \Lleftarrow M \otimes \Lambda^{\otimes 3} \cdots$$

$$\partial_i = id^{\otimes i} \otimes \mu \otimes id^{\otimes n-i-1} \qquad i = 0, \dots, n-1$$
$$\partial_n = \mu \otimes id^{n-1} \circ (1 \cdots n)$$

where $\sigma \in S_n$ permutes the constituents of n-fold tensor products. We thus get a chain map

$$d = \sum_{n \geq 0} \sum_{k=0}^{n} (-1)^k \partial_k$$

and the homology of this complex is the Hochschild homology of Λ with coefficients in M. To get the Hochschild cohomology we simply look at

$$0 \to M \to \mathrm{Hom}(\Lambda, M) \rightrightarrows \mathrm{Hom}(\Lambda^{\otimes 2}, M) \Rrightarrow \mathrm{Hom}(\Lambda^{\otimes 3}, M) \cdots$$

$$\partial_0 = (\mu \circ (id \otimes ev))^{\#}$$
$$\partial_i = (ev \circ (id^{\otimes i-1} \otimes \mu \otimes id^{\otimes n-i} \otimes id))^{\#} \qquad i = 1, \dots, n$$
$$\partial_{n+1} = (\mu \circ (ev \otimes id))^{\#}$$

Here the $\#$ symbol denotes the right adjoint

$$\mathrm{Hom}(\Lambda^{\otimes n+1} \otimes \mathrm{Hom}(\Lambda^{\otimes n}, M)) \underset{b}{\overset{\#}{\rightleftarrows}} \mathrm{Hom}(\mathrm{Hom}(\Lambda^{\otimes n}, M), \mathrm{Hom}(\Lambda^{\otimes n+1}, M))$$

And by liberal and shameless abuse of notation, μ can mean any of

$$\mu : \Lambda^{\otimes 2} \to \Lambda \qquad \mu : \Lambda \otimes M \to M \qquad \mu : M \otimes \Lambda \to M$$

We denote the Hochschild homology and cohomology $HH_n(\Lambda, M)$ and $HH^n(\Lambda, M)$. If $M = \Lambda$ we simply write $HH_n(\Lambda)$ and $HH^n(\Lambda)$, interesting to note is that

$HH^0(\Lambda) = Z(\Lambda)$ $\qquad$ $HH^1(\Lambda) =$ outer derivations

Lemma $\quad HH^\bullet(\Lambda, M) \cong \mathrm{Ext}^\bullet_{\Lambda^e}(\Lambda, M)$
$\quad HH_\bullet(\Lambda, M) \cong \mathrm{Tor}^{\Lambda^e}_\bullet(\Lambda, M)$

Proof $\quad$ Look at the bar resolution of Λ,

$$0 \leftarrow \Lambda \xleftarrow{\beta} \Lambda^{\otimes 2} \xleftarrow{\beta} \Lambda^{\otimes 3} \xleftarrow{\beta} \Lambda^{\otimes 4} \cdots$$

$$\beta = \sum_{i=0}^{n} (-1)^i id^{\otimes i} \otimes \mu \otimes id^{\otimes n-i-1}$$

To compute $\mathrm{Hom}(\Lambda^{\otimes n}, M) \xrightarrow{d} \mathrm{Hom}(\Lambda^{\otimes n+1}, M)$ we write

$$d = \Big(ev \circ \sum_{i=0}^{n+1} (-1)^i (id^{\otimes i} \otimes \mu \otimes id^{\otimes n-i-1})\Big)^{\#} = \sum_{i=0}^{n+1} (-1) \partial_i$$

Because the bar resolution is a resolution of projective right Λ^e modules $\square$

Recall $\Lambda = kQ/I$ in which $I = \langle r_i \mid i = 1, \dots, n \rangle$ is an admissible ideal generate by the relations r_i. The aim is to construct a projective resolution of Λ as right Λ^e module.
To this end we take

$$\bigoplus_{i=1}^{n} \Lambda e_{s(r_i)} \otimes e_{t(r_i)} \Lambda \xrightarrow{\pi_2} \bigoplus_{\alpha \in Q_1} \Lambda e_{s(\alpha)} \otimes e_{t(\alpha)} \Lambda \xrightarrow{\pi_1} \bigoplus_{i \in Q_0} \Lambda e_i \otimes e_i \Lambda \xrightarrow{m} \Lambda \longrightarrow 0 \qquad \mathrm{I}$$

m is the multiplication map an π_1 is defined by

$$\pi_1(e_{s(\alpha)} \otimes e_{t(\alpha)}) = \alpha \otimes e_{t(\alpha)} - e_{s(\alpha)} \otimes \alpha$$

To define π_2 we note that the r_i are chosen to be generators in $\operatorname{atom}(I^{\#})$ so that they form a Gröbner basis. We write $r_i = \sum_{p \in \operatorname{supp}(r_i)} \lambda_p p$ and for a path $p \in \operatorname{supp}(r_i)$ we write

$$p = e_{s(r_i)} \alpha_{p1} \alpha_{p2} \cdots \alpha_{p|p|} e_{t(r_i)} \quad \text{and}$$

$$\hat{p}_j = e_{s(r_i)} \alpha_{p1} \cdots \alpha_{pj-1} \otimes \alpha_{pj+1} \cdots \alpha_{p|p|} e_{t(r_i)} \quad j = 1, \dots, |p| \quad \text{now define}$$

$$\pi_2(e_{s(r_i)} \otimes e_{t(r_i)}) = \sum_{p \in \operatorname{supp}(r_i)} \lambda_p \sum_{j=1}^{|p|} \hat{p}_j$$

We shall now esthablish exactness of (I).
Obiously, m is surjective. To see that $\ker m = \operatorname{im} \pi_1$ we can argue similar to the case the standard projective resolution of right kQ modules.

$$m \circ \pi_1(e_{s(\alpha)} \otimes e_{t(\alpha)}) = m(\alpha \otimes e_{t(\alpha)} - e_{s(\alpha)} \otimes \alpha) = 0$$

Now take $x = \sum_{i \in Q_0} \lambda_{p_i q_i} p_i e_i \otimes e_i q_i \in \ker m$, and order the canonical basis for $P_0 = \bigoplus_{i \in Q_0} \Lambda e_i \otimes e_i \Lambda$ by $p_1 e_{i_1} \otimes e_{i_1} q_1 \leq p_2 e_{i_2} \otimes e_{i_2} q_2 \Leftrightarrow p_1 q_1 \leq p_2 q_2$ or $p_1 q_1 = p_2 q_2 \wedge e_{i_1} \leq e_{i_2}$. Denote

$$\wedge\operatorname{supp}\Big(\sum_{\substack{i \in Q_0 \\ p_i q_i \in \Lambda e_i \otimes e_i \Lambda}} \lambda_{p_i q_i} p_i e_i \otimes e_i q_i\Big) = p_\wedge \otimes q_\wedge \text{ (in tensor normal form), with } |p_\wedge q_\wedge| > 0 \text{ otherwise } x = 0$$

assume by virtue of symmetry, that $p_\wedge = r_\wedge \alpha_\wedge$ $\alpha_\wedge \in Q_1$, then

$$\wedge\operatorname{supp}(x - \pi_1(\lambda_{p_\wedge q_\wedge} r_\wedge \otimes q_\wedge)) < \wedge\operatorname{supp}(x) \quad \text{so inductively } x \in \operatorname{im} \pi_1 .$$

The inclusion $\operatorname{im} \pi_2 \subseteq \ker \pi_1$ is again a straight forward computation.
First we note that $\pi_1(\sum_{j=1}^{|p|} \hat{p}_j) = p \otimes e_{t(p)} - e_{s(p)} \otimes p$, namely

$$\begin{aligned}
\pi_1\Big(\sum_{j=1}^{|p|} \hat{p}_j\Big) &= \sum_{j=1}^{|p|} e_{s(p)} \cdots \alpha_{pj-1} \pi_1(e_{s(\alpha_{pj})} \otimes e_{t(\alpha_{pj})}) \alpha_{pj+1} \cdots e_{t(p)} \\
&= \sum_{j=1}^{|p|} e_{s(p)} \cdots \alpha_{pj-1} (\alpha_{pj} \otimes e_{t(p_j)} - e_{s(p_j)} \otimes \alpha_{pj}) \alpha_{pj+1} \cdots e_{t(p)} \\
&= p \otimes e_{t(p)} - e_{s(p)} \otimes p
\end{aligned}$$

Now we calculate

$$\begin{aligned}\pi_1\circ\pi_2(e_{s(r_i)}\otimes e_{t(r_i)}) &= \pi_1\Big(\sum_{p\in supp(r_i)} \lambda_p \sum_{j=1}^{|p|} \hat{p}_j\Big)\\ &= \sum_{p\in supp(r_i)} \lambda_p\, \pi_1\Big(\sum_{j=1}^{|p|} \hat{p}_j\Big)\\ &= \sum_{p\in supp(r_i)} \lambda_p\,(p\otimes e_{t(p)} - e_{s(p)}\otimes p)\\ &= \sum_{p\in supp(r_i)} \lambda_p\, p\otimes e_{t(p)} - e_{s(p)}\otimes \sum_{p\in supp(r_i)} \lambda_p\, p\\ &= 0\end{aligned}$$

To show $\ker\pi_1 \subseteq \operatorname{im}\pi_2$ we introduce yet another order.

Using a monomial (degree lexicographic) order on P, we define a well order on $\bigoplus_{c\in P}\Lambda e_{s(c)}\otimes e_{t(c)}\Lambda$ as follows: take $p_1\otimes q_1 \in \Lambda e_{s(c_1)}\otimes e_{t(c_1)}\Lambda$, $p_2\otimes q_2\in\Lambda e_{s(c_2)}\otimes e_{t(c_2)}\Lambda$

$$p_1\otimes q_1 < p_2\otimes q_2 \iff \begin{array}{lllll} |q_1|>|q_2| & & & & \text{or}\\ |q_1|=|q_2| & \wedge & |p_1|<|p_2| & & \text{or}\\ |q_1|=|q_2| & \wedge & |p_1|=|p_2| & \wedge\ q_1>q_2 & \text{or}\\ q_1=q_2 & \wedge & |p_1|=|p_2| & \wedge\ p_1<p_2 & \text{or}\\ q_1=q_2 & \wedge & p_1=p_2 & \wedge\ c_1<c_2 & \end{array}$$

This gives us an order on P_0, $P_1=\bigoplus_{\alpha\in Q_1}\Lambda e_{s(\alpha)}\otimes e_{t(\alpha)}\Lambda$ and $P_2=\bigoplus_{i=1}^{n}\Lambda e_{s(r_i)}\otimes e_{t(r_i)}\Lambda$

Alternatively, it would be nice if the argument of the paper could be modified to work with the order on $\bigoplus_{c\in P}\Lambda e_{s(c)}\otimes e_{t(c)}\Lambda$ given by

$$p_1 e_{s(c_1)}\otimes e_{t(c_1)} q_1 \leq p_2 e_{s(c_2)}\otimes e_{t(c_2)} q_2 \iff \begin{array}{ll} p_1c_1q_1 \leq p_2c_2q_2 & \text{or}\\ p_1c_1q_1 = p_2c_2q_2 \ \wedge\ c_1\leq c_2 & \end{array}$$

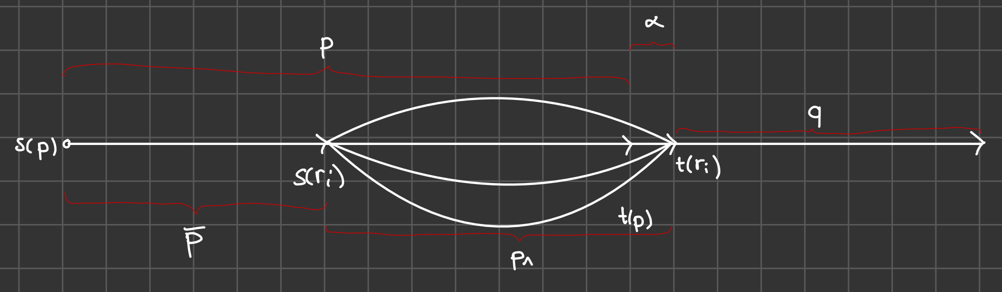
To make this work we will also assume that the vertexidempotents are ordered so that

$$\alpha: i\longrightarrow j \quad \text{implies} \quad e_i < e_j$$

and the arrows are ordered in such a way that

$$e_{t(\alpha)} < e_{t(\alpha)} \vee (e_{t(\alpha)} = e_{t(\beta)} \wedge e_{s(\alpha)} < e_{s(\beta)}) \text{ implies } \alpha<\beta$$

Now take $x\in\ker\pi_1$ written in tensor normal form and $\wedge supp(x) = pe_{s(\alpha)}\otimes e_{t(\alpha)}q$. Since $\pi_1(pe_{s(\alpha)}\otimes e_{t(\alpha)}q) = p\alpha\otimes q - p\otimes\alpha q$ has initial monomial $p\alpha\otimes q$ and any other monomial $p'\otimes q'$ satisfies $p\alpha'\otimes q' < p\alpha\otimes q$, $p\alpha$ must be part of a relation r_i for some r_i with $t(r_i)=t(\alpha)$. We illustrate the situation



Let $p_\wedge = \wedge\operatorname{supp}(r_i)$, then $p\alpha \mid p_\wedge$ and

$$\wedge\operatorname{supp}(\pi_2(e_{s(r_i)} \otimes e_{t(r_i)})) = e_{s(r_i)}\alpha_{p_\wedge 1} \cdots \alpha_{p_\wedge |p_\wedge|-1}\, e_{s(\alpha)} \otimes e_{t(r_i)}$$

$$s(r_i) \xrightarrow{\alpha_{p_\wedge 1}} \cdot \xrightarrow{\alpha_{p_\wedge 2}} \cdots \xrightarrow{\alpha_{p_\wedge |p_\wedge|-1}} \cdot \xrightarrow{\alpha = \alpha_{p_\wedge |p_\wedge|}} t(r_i)$$

so that

$\alpha_{p_\wedge 1} \cdots \alpha_{p_\wedge |p_\wedge|-1} \otimes e_{t(r_i)} (\bar{p} \otimes q) = p \otimes q$, meaning that

$\wedge\operatorname{supp}(\pi_2(e_{s(r_i)} \otimes e_{t(r_i)})) \mid \wedge\operatorname{supp}(x) = p \otimes q$, so we can for

$x - \lambda_{pq} \lambda_{p_\wedge}^{-1} (\pi_2(e_{s(r_i)} \otimes e_{t(r_i)})(\bar{p} \otimes q)) < x$ and the result follows $\square$

In the usual setting with $\Lambda = kQ/I$ with I an admissible ideal, we now impose the further condition that I is a monomial ideal, i.e. generated by monomials. Denote

$$\mathrm{atom}(I^\#) = \{p_1, \dots, p_m\}, \quad p_i \in \mathcal{P} \quad i = 1, \dots, m$$

DEF We call a sequence of paths $(q_1, \dots, q_n)$ $q_i \in \mathrm{atom}(I^\#)$ an admissible sequence of order n, provided that $(q_1, \dots, q_{n-1})$ is an admissible sequence of order $n-1$ and

$$O_n := \{c \in \mathrm{atom}(I^\#) \mid c \mid q_{n-1} \text{ and } c \mid q_n \text{ and } |c| \geq 1\} \neq \emptyset$$

$$\{d \in \mathrm{atom}(I^\#) \mid d \mid q_{n-2} \text{ and } d \mid q_n \text{ and } |d| \geq 1\} = \emptyset$$

and q_n maximizes $\max O_n$. Or $n = 1$ and $q_1 \in \mathrm{atom}(I^\#)$.

In the paper of interest, this is defined as follows:

DEF For a directed subgraph $\Gamma \subseteq Q$ that is a path, the left associate sequence of paths to $p_i \in \mathrm{atom}(I^\#)$ of order k is
- (p_i) if $k = 1$ and p_i lies along Γ
- $(q_1 = p_i, q_2)$ if $k = 2$ and $q_2 \in \mathrm{atom}(I^\#)$, if it exists, is the path of minimal source such that $s(p_i) < s(q_2) < t(p_i)$
- $(q_1 = p_i, q_2, \dots, q_k)$ if $k > 2$ and $(q_1, \dots, q_{k-1})$ is a left associated sequence of paths of order $k-1$ and $q_k \in \mathrm{atom}(I^\#)$, if it exists, is the path of minimal source lying along Γ for which $t(q_{k-2}) \leq s(q_k) < t(q_{k-1})$

DEF For a directed subgraph $\Gamma \subseteq Q$ that is a path, the right associate sequence of paths to $p_i \in \mathrm{atom}(I^\#)$ of order k is
- (p_i) if $k = 1$ and p_i lies along Γ
- $(q_1 = p_i, q_2)$ if $k = 2$ and $q_2 \in \mathrm{atom}(I^\#)$, if it exists, is the path of maximal source such that $s(q_2) < s(p_i) < t(q_2)$
- $(q_1 = p_i, q_2, \dots, q_k)$ if $k > 2$ and $(q_1, \dots, q_{k-1})$ is a right associated sequence of paths of order $k-1$ and $q_k \in \mathrm{atom}(I^\#)$, if it exists, is the path of maximal source lying along Γ for which $s(q_{k-1}) < t(q_k) \leq s(q_{k-2})$

DEF $AP(n) = \left\{ e_{s(q_1)}\, p\, e_{t(q_n)} \in \mathcal{P} \;\middle|\; \begin{array}{l} (q_1, \dots, q_{n-1}) \text{ is an admissible sequence of order } n-1 \text{ and every} \\ \text{idempotent dividing one of the } q_i \text{, devides } p \end{array} \right\}$

$$= \bigcup_{i=1}^{m} \left\{ e_{s(q_1)}\, p\, e_{t(q_n)} \in \mathcal{P} \;\middle|\; \begin{array}{l} (q_1, \dots, q_{n-1}) \text{ is an associated sequence of paths to } p_i \text{ of order } n-1 \\ \text{and every idempotent dividing one of the } q_i \text{, divides } p \end{array} \right\}$$

LEM $AP(n) = AP(n)^{op}$

PROOF Start with $p \in AP(n)$ coming from $(q_1, \dots, q_{n-1})$ and build a right associated sequence of paths of order $n-1$ along p, starting at q_{n-1}, which exists as is witnessed by $(q_1, \dots, q_{n-1})$. If we call the thus obtained sequence $(\vec{q}_{n-1}, \dots \vec{q}_1 = q_{n-1})$ we see that $t(\vec{q}_k) \geq t(q_{n-k})$ for each k, so in particular $s(q_1) \leq s(\vec{q}_{n-1})$. If we now turn around again and start forming left associated sequence of paths of order $n-1$ along p starting at $\vec{q}_{n-1}$, we realise that in fact $s(q_1) = s(\vec{q}_{n-1})$, so as not to contradict our minimality condition in the creation of $(q_1, \dots, q_{n-1})$ $\square$

LEM Suppose m even and $p, q \in AP(m)$ $p \neq q$ lie along the same path T, such that $\alpha q = p\beta$ and $\alpha, \beta \in P$ are not divisible by any $p_i \in \mathrm{atom}(I^{\#})$. Then there exists a $c \in AP(m+1)$ that divides the path from $s(p)$ to $t(q)$ along T.

PROOF Since $p \neq q$, we have that $\alpha \neq 0$ or $\beta \neq 0$ and in view of the previous lemma $\alpha \neq 0$ and $\beta \neq 0$. For $m = 2$ the statement is clear. Now suppose the statement holds for all multiples $2k$ of two for $k < \frac{1}{2}m$, then for m we can consider

considering the right construction of p, we can omit the first two arrows and considering the left construction of q, we can omit the last two arrows. Using the induction hypothesis, we get an associated sequence of paths of lenght $m-2$ dividing the path $\bar{c}$ from $s(q)$ to $t(p)$ along T. Now we can prepend a path d_{pre}, viewing $\bar{c}$ as a right construction, and append a path d_{app}, viewing $d_{pre}\bar{c}$ as a left construction, such that $d_{pre}\bar{c}d_{app}$ decides the path from $s(p)$ to $t(q)$ along T and $d_{pre}, d_{app} \in \mathrm{atom}(I^{\#})$ $\square$

DEF Let $p \in AP(n)$, then $\mathrm{sub}(p) := \{\tilde{p} \in AP(n-1) \mid \tilde{p} \mid p\}$

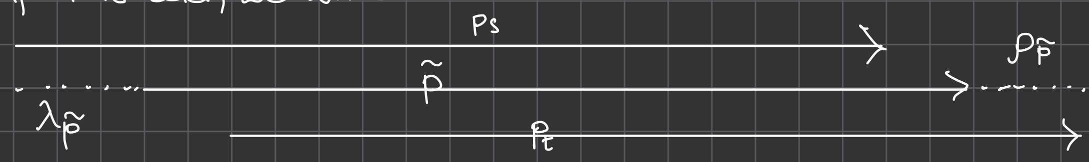
LEM $\mathrm{sub}(p)$ contains two paths: $e_{s(p)}p_s$ and $p_t e_{t(p)}$ and if n is odd, then these are all the paths in $\mathrm{sub}(p)$

PROOF Let n be odd, then we are in the following situation

Now $\alpha, \beta, \gamma, \delta$ are all nonzero, with $\tilde{p}$ be a hypothetical element of $\mathrm{sub}(p) \setminus \{p_s, p_t\}$. Then by the previous lemma, we can find a $q \in AP(n)$ dividing the path from $s(p_s)$ to $t(\tilde{p})$, meaning that it divides p properly $\lightning$ $\square$

DEF Consider $p, q \in AP(n)$, n even, lying along the same graph that is a path

If n is odd, we write



REM All these λ's and ρ's are not divisible by any of the paths in atom($I^{\#}$)

With the convention that $AP(0) = Q_0$ and $AP(1) = Q_1$ we now set

$$P_n = \bigoplus_{p \in AP(n)} \Lambda e_{s(p)} \otimes e_{t(p)} \Lambda$$ and with n odd, we define

$$\pi_n : e_{s(p)} \otimes e_{t(p)} \longmapsto \lambda_{p_s p_t} \otimes e_{t(p)} - e_{s(p)} \otimes \rho_{p_s p_t}$$

$$\pi_{n+1} : e_{s(p)} \otimes e_{t(p)} \longmapsto \sum_{\hat{p} \in sub(p)} \lambda_{\tilde{p}} \otimes \rho_{\tilde{p}}$$

Recall $\Lambda = kQ/I$, I a monomial ideal

$$P_0 = \bigoplus_{i\in Q_0} \Lambda e_i \otimes e_i \Lambda \,,\quad P_1 = \bigoplus_{\alpha \in Q_1} \Lambda e_{s(\alpha)} \otimes e_{t(\alpha)}\Lambda \,,\quad P_n = \bigoplus_{p\in AP(n)} \Lambda e_{s(p)} \otimes e_{t(p)}\Lambda \qquad n\geq 2$$

for n odd, $\pi_n : P_n \longrightarrow P_{n-1} \;:\; e_{s(p)}\otimes e_{t(p)} \longmapsto \lambda_{p_t} \otimes e_{t(p)} - e_{s(p)} \otimes \rho_{p_s}$

$\pi_{n+1} : P_{n+1} \longrightarrow P_n \;:\; e_{s(p)} \otimes e_{t(p)} \longmapsto \sum_{q\in sub(p)} \lambda_q \otimes \rho_q$

THM $\quad \cdots \longrightarrow P_2 \xrightarrow{\pi_2} P_1 \xrightarrow{\pi_1} P_0 \xrightarrow{m} \Lambda \quad$ is a minimal projective resolution

PROOF $\;rad(\Lambda^e) = (e_i\otimes\alpha, \alpha\otimes e_i \mid i\in Q_0, \alpha\in Q_1)$, it follows that $\pi_n(P_n) \subseteq rad(\Lambda^e)P_{n-1}$ which shows minimality.
Using the order defined earlier, we see that for $p_1\otimes q_1, p_2\otimes q_2$, $p_1\otimes q_1 < p_2\otimes q_2$ implies $\Lambda supp(\pi_n(p_1\otimes q_1)) < \Lambda supp(\pi_n(p_2\otimes q_2))$
this is very easy to see in our order and not so easy in Bardzells

$im\,\pi_3 = ker\,\pi_2$:
Take $p\in AP(3)$ with $sub(p) = \{p_i, p_j\} \subseteq atom(I^\#)$, we get

$$\pi_2\circ\pi_3(e_{s(p)}\otimes e_{t(p)}) = \pi_2(\lambda_p \otimes e_{t(p)} - e_{s(p)} \otimes \rho_p)$$

$$= \lambda_p \pi_2(e_{s(p_j)}\otimes e_{t(p_j)}) - \pi_2(e_{s(p_i)}\otimes e_{t(p_i)})\rho_p$$

$$= \lambda_p \sum_{k=1}^{|p_j|} \widehat{(p_j)}_k - \sum_{\ell=1}^{|p_i|} \widehat{(p_i)}_\ell \rho_p$$

$$= \lambda_p \sum_{k:\alpha_{p_j k} | p_i} \widehat{(p_j)}_k - \sum_{\ell:\alpha_{p_i \ell} | p_j} \widehat{(p_i)}_\ell \rho_p = 0$$

The situation can be illustrated as follows:

$\alpha_{p_i k}$ dividing p_j $=$ $\alpha_{p_j \ell}$ dividing p_i

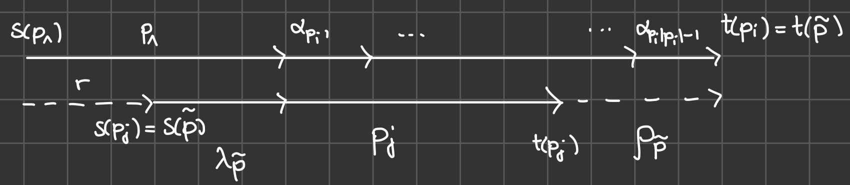
To show the reverse inclusion, we take $x\in ker\,\pi_2$ in tensor normal form modulo I. Write $\Lambda supp(x) = p_\lambda \otimes q_\lambda \in \Lambda e_{s(p_i)} \otimes e_{t(p_i)}\Lambda$ for some $p_i \in atom(I^\#)$, then

$$\Lambda supp(\pi_2(p_\lambda \otimes q_\lambda)) = p_\lambda \alpha_{p_i 1} \cdots \alpha_{p_i |p_i|-1} \otimes q_\lambda \,.$$

Since any other element in the support of x is mapped by π_2 to something which is supported only on elements strictly less than $\Lambda supp(\pi_2(p_\lambda\otimes q_\lambda))$, there must be a maximal $p_j \in atom(I^\#)$ such that

$$p_j \mid p_\lambda \alpha_{p_i 1} \cdots \alpha_{p_i|p_i|-1} \,,\quad p_k \nmid p_\lambda \,,\quad p_k \nmid \alpha_{p_i 1}\cdots \alpha_{p_i|p_i|-1}$$

Here r is the path from $s(p_\lambda)$ to $s(\tilde{p})$.



So $\tilde{p} \in AP(3)$ and

$$\Lambda supp(\pi_3(e_{s(\tilde{p})}\otimes e_{t(\tilde{p})})(r\otimes q_\lambda)) = r\lambda_{\tilde{p}} \otimes q_\lambda = p_\lambda \otimes q_\lambda ,\text{ we conclude as usual.}$$

$\mathrm{im}\,\pi_4 = \ker\pi_3$:

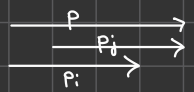
Take $p \in AP(4)$, then we can compute

$$\begin{aligned}
\pi_3\circ\pi_4(e_{s(p)}\otimes e_{t(p)}) &= \sum_{\tilde p\in \mathrm{sub}(p)} \pi_3(e_{s(\tilde p)}\otimes e_{t(\tilde p)})(\lambda_{\tilde p}\otimes\rho_{\tilde p}) \\
&= \sum_{\tilde p\in \mathrm{sub}(p)} (\lambda_{\tilde p_t}\otimes e_{t(\tilde p)} - e_{s(\tilde p)}\otimes\rho_{\tilde p_s})(\lambda_{\tilde p}\otimes\rho_{\tilde p}) \\
&= \sum_{\tilde p\in \mathrm{sub}(p)} \lambda_{\tilde p}\lambda_{\tilde p_t}\otimes\rho_{\tilde p} - \lambda_{\tilde p}\otimes\rho_{\tilde p_s}\rho_{\tilde p} \\
&= \lambda_{p_t}\lambda_{p_{tt}}\otimes e_{t(p)} - e_{s(p)}\otimes\rho_{p_{ss}}\rho_{p_s} \\
&= 0
\end{aligned}$$

where the last equality follows by considering p as a left construction to derive $\rho_{p_{ss}}\rho_{p_s} = 0$ and as a right construction to derive that $\lambda_{p_t}\lambda_{p_{tt}} = 0$

For the reverse inclusion, take $x\in\ker\pi_4$ in tensor normal form mod I with

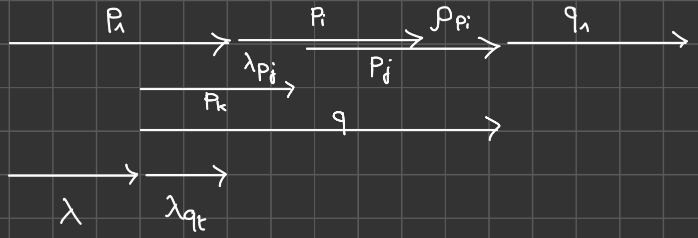
$$\wedge\mathrm{supp}(x) = p_\wedge\otimes q_\wedge \in \Lambda e_{s(p_j)}\otimes e_{s(p_i)}\Lambda,\quad p_i,p_j\in \mathrm{atom}(I^\#)$$



$p\in AP(3)$

$$\begin{aligned}
\wedge\mathrm{supp}(p_\wedge\otimes q_\wedge) &= \wedge\mathrm{supp}((\lambda_{p_j}\otimes e_{t(p_j)} - e_{s(p_i)}\otimes\rho_{p_i})(p_\wedge\otimes q_\wedge)) \\
&= p_\wedge\lambda_{p_j}\otimes q_\wedge \\
&= \wedge\mathrm{supp}(\pi_3(x)) \text{ in view of a previous remark}
\end{aligned}$$

Again we get $p_\wedge\otimes q_\wedge \neq y\in\mathrm{supp}(x) \Rightarrow p_\wedge\lambda_{p_j}\otimes q_\wedge \in \mathrm{supp}(\pi_3(y))$, so there must be a $p_k\in\mathrm{atom}(I^\#)$ such that $p_k \mid p_\wedge\lambda_{p_j}$, $p_k\nmid p_\wedge$, $p_k\nmid\lambda_{p_j}$ and we take it to be maximal.

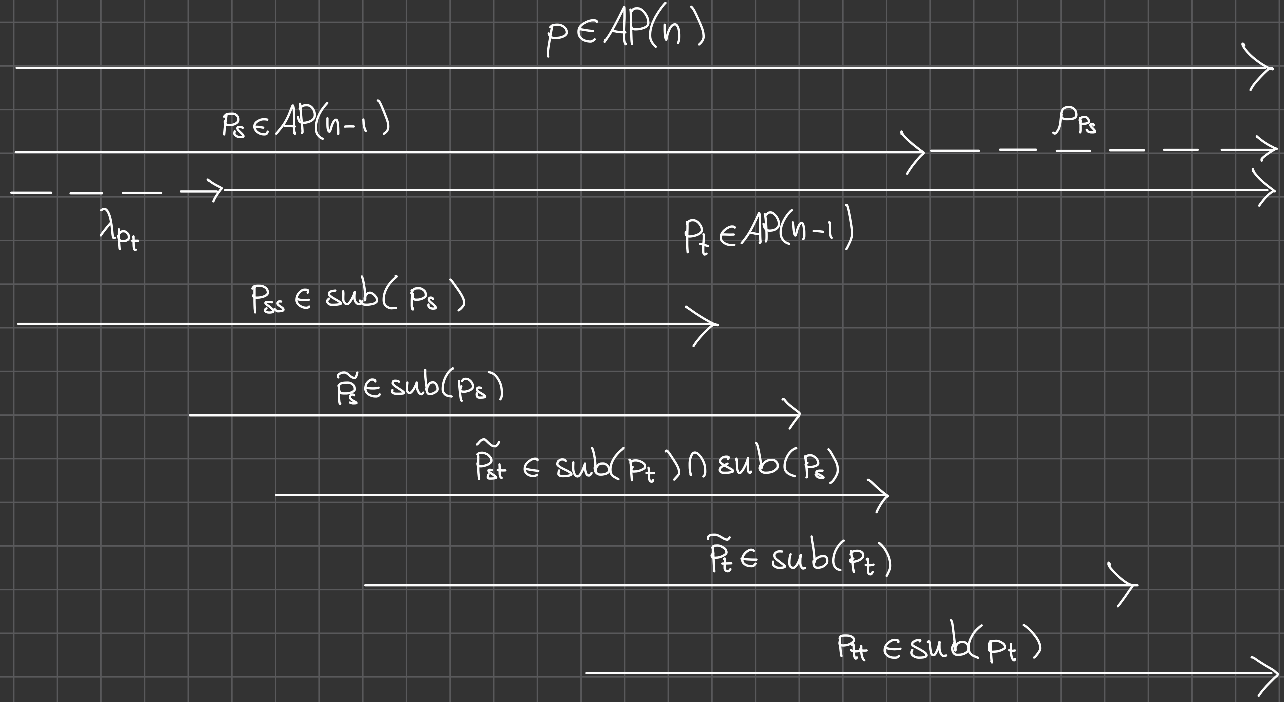


$\mathrm{sub}(q) = \{q_s, \dots, q_t = p\}$

$$\wedge\mathrm{supp}(\pi_4(e_{s(q)}\otimes e_{t(q)})) = \wedge\mathrm{supp}\Big(\sum_{\tilde q\in\mathrm{sub}(q)}\lambda_{\tilde q}\otimes\rho_{\tilde q}\Big) = \lambda_{q_t}\otimes e_{t(q)}$$

So we get $(\lambda_{q_t}\otimes e_{t(q)})(\lambda\otimes q_\wedge) = p_\wedge\otimes q_\wedge$, in other words $\wedge\mathrm{supp}(\pi_4(e_{s(q)}\otimes e_{t(q)})) \mid \wedge\mathrm{supp}(x)$ showing that x reduces over $\mathrm{im}\,\pi_4$, yielding the desired conclusion.

Now assume $P_{n-1}\to P_{n-2}\to\cdots\to P_0\to\Lambda\to 0$ to be exact for $n\geq 5$ odd then $P_{n+1}\to P_n\to P_{n-1}\to\cdots\to 0$ is exact at P_n for similar reasons as those showing exactness at P_3. For exactness at P_{n-1} we compute for $p\in AP(n)$ $\pi_{n-1}\circ\pi_n(e_{s(p)}\otimes e_{t(p)})$. To do this we draw a nice picture



Note that $\mathrm{sub}(p_s) \cap \mathrm{sub}(p_t) \neq \emptyset$ as can be seen by considering either p_t or p_s as a left and right construction in the appropriate order.

$$\begin{aligned}
\pi_{n-1} \circ \pi_n \left(e_{s(p)} \otimes e_{t(p)} \right) &= \pi_{n-1} \left(\lambda_{p_t} \otimes e_{t(p)} - e_{s(p)} \otimes \rho_{p_s} \right) \\
&= \lambda_{p_t} \sum_{\tilde{p}_t \in \mathrm{sub}(p_t)} \lambda_{\tilde{p}_t} \otimes \rho_{\tilde{p}_t} - \sum_{\tilde{p}_s \in \mathrm{sub}(p_s)} \lambda_{\tilde{p}_s} \otimes \rho_{\tilde{p}_s} \rho_{p_t} \\
&= \lambda_{p_t} \sum_{\tilde{p}_t \in \mathrm{sub}(p_t) \setminus \mathrm{sub}(p_s)} \lambda_{\tilde{p}_t} \otimes \rho_{\tilde{p}_t} - \sum_{\tilde{p}_s \in \mathrm{sub}(p_s) \setminus \mathrm{sub}(p_t)} \lambda_{\tilde{p}_s} \otimes \rho_{\tilde{p}_s} \rho_{p_t} \\
&= 0
\end{aligned}$$

The inclusion $\ker \pi_{n-1} \subseteq \mathrm{im}\, \pi_n$ is done using the familiar reduction techniques and will be omitted $\square$