Jarrow-Rudd Risk Neutral

$$u = \exp(r-\frac{\sigma^2}{2})\Delta t + \sigma\sqrt{\Delta t} \qquad P_u = \frac{e^{r\Delta t}-d}{u-d}$$

$$d = \exp(r-\frac{\sigma^2}{2})\Delta t - \sigma\sqrt{\Delta t} \qquad P_d = \frac{u-e^{r\Delta t}}{u-d}$$

first moment

$$E[S_x] = S_0\left[u\left(\frac{e^{r\Delta t}-d}{u-d}\right) + d\left(\frac{u-e^{r\Delta t}}{u-d}\right)\right]$$

$$= S_0\left[\frac{e^{r\Delta t}u - du + du - e^{r\Delta t}d}{u-d}\right]$$

$$= S_0\left[\frac{e^{r\Delta t}(u-d)}{u-d}\right] = S_0 e^{r\Delta t}$$

second moment

$$E[S_x^2] = S_0^2\left[u^2\left(\frac{e^{r\Delta t}-d}{u-d}\right) + d^2\left(\frac{u-e^{r\Delta t}}{u-d}\right)\right]$$

$$= S_0^2\left[\frac{e^{r\Delta t}(u^2-d^2) - du^2 + ud^2}{u-d}\right]$$

$$= S_0^2\left[\frac{e^{r\Delta t}(u+d)(u-d)}{u-d} - \frac{du(u-d)}{u-d}\right]$$

$$= S_0^2\left[e^{r\Delta t}(u+d) - du\right]$$

$$= S_0^2\left[e^{(2r-\frac{\sigma^2}{2})\Delta t + \sigma\sqrt{\Delta t}} + e^{(2r-\frac{\sigma^2}{2})\Delta t - \sigma\sqrt{\Delta t}} - e^{2(r-\frac{\sigma^2}{2})\Delta t}\right.$$

note:

$e^x \approx 1$ as x approahes 0

proof: $1+x \leq 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = e^x$

$$\therefore\ E[S_x^2] \approx S_0^2\left[(1+2r-\frac{\sigma^2}{2})\Delta t + (2r-\frac{\sigma^2}{2})\Delta t + \sigma\sqrt{\Delta t} - 2(r-\frac{\sigma^2}{2})\Delta t\right]$$

$$\approx S_0^2[1+2r\Delta t]$$

approaches zero as $\Delta t \to 0$

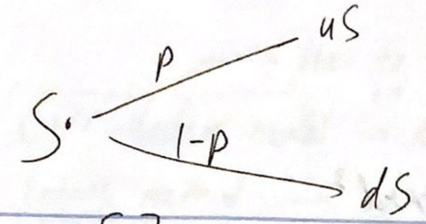
$$\approx S_0^2 e^{2r\Delta t}$$

Given a small enough time interval Δt, the variance σ^2 approaches zero and the second moment of Jarrow-Rudd $\approx$ second moment of GBM.

Tree Model

$$p = \frac{e^{r\Delta t} - d}{u-d} \qquad u = e^{\sigma\sqrt{\Delta t}}$$

$$1-p = \frac{u - e^{r\Delta t}}{u-d} \qquad d = e^{-\sigma\sqrt{\Delta t}}$$



$$E[S_t] = S_0 \cdot p \cdot u + S_0 \cdot (1-p) \cdot d$$

$$= S_0\left(\frac{e^{r\Delta t}-d}{u-d}\right)e^{\sigma\sqrt{\Delta t}} + S_0\left(\frac{u-e^{r\Delta t}}{u-d}\right)e^{-\sigma\sqrt{\Delta t}}$$

$$= \frac{S_0}{u-d}\left[e^{r\Delta t+\sigma\sqrt{\Delta t}} - 1 + 1 - e^{r\Delta t - \sigma\sqrt{\Delta t}}\right]$$

$$= S_0 e^{r\Delta t}\left(\frac{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}{u-d}\right)$$

$$E[S_0] = S_0 e^{r\Delta t} \Leftarrow \text{first moment of GBM}$$

$$E[S_0^2] = S_0^2 \cdot p \cdot u^2 + S_0^2 (1-p) \cdot d^2$$

$$= S_0^2\left(\frac{e^{r\Delta t}-d}{u-d}\right)\cdot u^2 + S_0^2\left(\frac{u-e^{r\Delta t}}{u-d}\right)d^2$$

$$= S_0^2\left[\frac{e^{r\Delta t}(u^2-d^2) - du^2 + ud^2}{u-d}\right]$$

$$= S_0^2\left[\frac{e^{r\Delta t}(u+d)(u-d)}{u-d} - \frac{du(u-d)}{u-d}\right]$$

$$= S_0^2\left(e^{r\Delta t}(u+d) - du\right)$$

$$= S_0^2\left(e^{r\Delta t+\sigma\sqrt{\Delta t}} + e^{r\Delta t-\sigma\sqrt{\Delta t}} - e^0\right)$$

$$S_0^2\left[r\Delta t + \sigma\sqrt{\Delta t} + 1 + r\Delta t - \sigma\sqrt{\Delta t}\right]$$

$$E[S_0^2] = S_0^2[1 + 2r\Delta t] = S_0^2 e^{2r\Delta t}$$

$e^x \approx 1+x$ when x is sufficiently small

as σ becomes small, due to small time interval, the second moment converges to GBM

$$E[S_t^3] = S_0^3 \cdot p(u^3) + S_0^3(1-p)\cdot d^3$$

$$= S_0^3 \left(\frac{e^{r\Delta t}-d}{u-d}\right)u^3 + S_0^3\left(\frac{u-e^{r\Delta t}}{u-d}\right)\cdot d^3$$

$$= S_0^3\left[\frac{e^{r\Delta t}\cdot u^3 - du^3 + u\cdot d^3 - e^{r\Delta t}\cdot d^3}{u-d}\right]$$

$$= S_0^3\left[\frac{e^{r\Delta t}(u^3-d^3)}{u-d} - \frac{ud(u^2-d^2)}{u-d}\right]$$

$$= S_0^3\left[\frac{e^{r\Delta t}(u^2+ud+d^2)\cancel{(u-d)}}{\cancel{u-d}} - \frac{u\cdot d\,(u+d)\cancel{(u-d)}}{\cancel{u-d}}\right]$$

$$= S_0^3\left[e^{r\Delta t}(u^2+ud+d^2) - ud(u+d)\right]$$

$$= S_0^3\left[e^{r\Delta t + 2\sigma\sqrt{\Delta t}} + e^{r\Delta t + \sigma\sqrt{\Delta t} - \sigma\sqrt{\Delta t}} \dots + e^{r\Delta t - 2\sigma\sqrt{\Delta t}} - e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}\right]$$

when x is sufficiently small

$$= S_0^3\left[e^{r\Delta t + 2\sigma\sqrt{\Delta t}} + e^{r\Delta t - 2\sigma\sqrt{\Delta t}} + e^{r\Delta t} - e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}\right]$$

$e^x \approx 1+x$

$$= S_0^3\left[1 + \underline{r\Delta t} + 2\sigma\sqrt{\Delta t} + 1 + \underline{r\Delta t} - 2\sigma\sqrt{\Delta t} + 1 + \underline{r\Delta t} - 1 - \sigma\sqrt{\Delta t} - 1 + \sigma\sqrt{\Delta t}\right]$$

$$= S_0^3\left[1 + 3r\Delta t\right] = S_0^3 e^{3r\Delta t}$$

as $\Delta t \to 0$, $\sigma^2 \to 0$ $\therefore$

the 3rd moment converges to GBM 3rd moment.