

Lecture 18: October 19

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18.1 More Reductions

$$E_{TM} = \{\langle N \rangle : N \text{ is a TM s.t. } L(N) = \emptyset\}$$

Theorem 18.1 (Sipser 5.1): E_{TM} is undecidable.

Proof: We know A_{TM} is undecidable. We want to prove $A_{TM} \leq_T E_{TM}$. Then we can conclude that E_{TM} is undecidable. How to do reduction? Assume that R is a (hypothetical) TM solving E_{TM} . We must somehow construct a TM S that decides A_{TM} . $A_{TM} = \{\langle M, w \rangle : M \text{ accepts } w\}$.

Design of S :

On input X :

1. If x is not of form $\langle M, w \rangle$, reject.
2. Construct description of N like this:
 Let N be TM: On input y , (ignore input), simulate M on input w :
 - (a) If M accepts N accepts.
 - (b) If M rejects N rejects.

S is a decider since R is.

What is $L(N)$:

Case 1: M accepts w . Then $L(N) = \Sigma^*$.

Case 2: M does not accept w :

2a: M rejects w . Then N rejects all inputs, so $L(N) = \emptyset$.

2b: M runs forever on input w . Then N runs forever on all inputs, so $L(N) = \emptyset$.

Analysis:

In Case 1, then R rejects $\langle N \rangle$, so S accepts (M accepts w). Otherwise, $L(N) = \emptyset$, so R accepts $\langle N \rangle$, so S rejects. ■

$$REGULAR_{TM} = \{\langle N \rangle : N \text{ is a TM s.t. } L(N) \text{ is regular}\}$$

Proof: We know A_{TM} is undecidable. We want to prove $A_{TM} \leq_T E_{TM}$. Then we can conclude that $REGULAR_{TM}$ is undecidable.

What is $L(N)$:

Case 1: M accepts w . Then $L(N) = \{0^n 1^n : n \in \mathbb{N}\}$ (NOT REGULAR!).

Case 2: M does not accept w :

2a: M rejects w . $L(N) = \emptyset$ is regular.

2b: M runs forever on input w . $L(N) = \emptyset$ is regular.

Analysis:

If M accepts w , then $L(N)$ is not regular, so R rejects, so S accepts. Otherwise, $L(N) = \emptyset$, so R accepts $\langle N \rangle$, so S rejects. ■