Foundations of Machine Learning — Homework Assignment 1

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C. Randomized Halving

1. Psuedo code

Algorithm 1 Randomized Halving

1: $H_1 \leftarrow H$
2: **for** t $\leftarrow$ 1 to T **do**
3:     $RECEIVE(x_t)$
4:     $r_t \leftarrow \frac{\sum_{i:y_{t,i}=1} 1}{|H_t|}$
5:     $p_t \leftarrow 1$
6:     **if** $r_t \leq \frac{3}{4}$ **then**
7:          $p_t \leftarrow [\frac{1}{2} \log_2 \frac{1}{1-r_t}]$
8:     $\hat{y}_t \leftarrow GetRandomNumberWithProbability([1,0],[p_t, 1-p_t])$
9:     $RECEIVE(y_t)$
10:    **if** $\hat{y}_t \neq y_t$ **then**
11:        $H_{t+1} \leftarrow \{C \in H_t : C(x_t) = y_t\}$
    return H_{T+1}

2. Prove $\forall t \geq 1, E[\mu] \leq \frac{\phi_t - \phi_{t+1}}{2}$

Given: Potential function: $\phi_t = \log_2 | H_t |$ and $\mu_t = 1_{y_t \neq \hat{y}_t}$

Proof:

We are only considering the case when the predicted value $\hat{y}_t$ is not equal to the received value y_t. The value of expectation can be written as

$$\begin{aligned} E[\mu_t] &= p_t * 1 + (1 - p_t) * 0 \\ &= p_t * 1 \end{aligned}$$

The probability of predicting 1 by the randomized algorithm is the probability of making a mistake since we are only considering the cases in which we make mistakes $(\mu_t = 1_{y_t \neq \hat{y}_t})$

Therefore,

$$\begin{aligned} E[\mu_t] &= p_t \\ &= [\frac{1}{2} \log_2 \frac{1}{1 - r_t}] 1_{r_t \leq \frac{3}{4}} + 1_{r_t > \frac{3}{4}} \\ &\leq [\frac{1}{2} \log_2 \frac{1}{1 - r_t}] \end{aligned}$$

Since, the Expectation will be 1 when $r_t > \frac{3}{4}$ which corresponds to the maximum expectation here, we can upper bound the expectation by using $r_t \leq \frac{3}{4}$ as $[\frac{1}{2} \log_2 \frac{1}{1-r_t}]$ equals 1 when $r_t = \frac{3}{4}$

$$E[\mu_t] \leq [\frac{1}{2} \log_2 \frac{1}{1-r_t}]$$

$$= [\frac{\log_2 \frac{1}{1-r_t}}{2}]$$

$$= [\frac{\log_2 \frac{1}{1-r_t}}{2}]$$

$$= [\frac{\log_2 \frac{1}{1-\frac{|H_1|}{|H_t|}}}{2}]$$

$$= [\frac{\log_2 \frac{|H_t|}{|H_t|-|H_1|}}{2}]$$

$$= [\frac{\log_2 \frac{|H_t|}{|H_0|}}{2}]$$

$$= [\frac{\log_2 |H_t| - \log_2 |H_0|}{2}]$$

$$= [\frac{\log_2 |H_t| - \log_2 |H_{t+1}|}{2}]$$

$$= [\frac{\phi_t - \phi_{t+1}}{2}]$$

$$\therefore E[\mu_t] \leq \frac{\phi_t - \phi_{t+1}}{2}$$

3. **Expected number of mistakes.**

4. **[Bonus Question]**