# Foundations of Machine Learning — Homework Assignment 1

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# A. PAC Learning

### 1

**Algorithm $\mathcal{A}$:** Given a sample $\mathcal{S}$, the algorithm returns the tightest interval I' = $I_s$ containing all points labeled with 1 in sample S.

**Error intervals:** We define error regions as the intervals that lie between [a,b] but outside I'.

**Proof of PAC learnability** :

Let I be the target concept — [a,b].
Let $\epsilon > 0$ be the error.
Let m be the number of samples in S.
Let I$_s$ be the tightest interval formed by the the points labelled 1 from sample S.
Let Pr[$I_s$] denote the probability mass of the interval defined by I$_s$.

$$\Pr[I_s] > \epsilon$$

Lets assume that the interval I$_s$ (denoted by [a',b']) has error $\epsilon$. The error can be found in intervals [a,a') and (b',b] denoted by $I_1 and I_2$ respectively. If we assume that the error is equally distributed across the two regions, we can denote the error to be $\epsilon/2$ for each of $I_1$ and $I_2$. Each point in the error region has a probability of (1-$\frac{\epsilon}{2}$). So Probability of the error in sample being greater than $\epsilon$ can be written as

$$\begin{aligned}
\Pr_{S\sim\mathcal{D}^m}[R(I_S) > \epsilon] \leq& \sum_{i=1}^{2} \Pr_{S\sim\mathcal{D}^m}[I \bigcap I_i = \emptyset] \\
\leq& 2(1-\epsilon/2)^m \\
\leq& 2\mathrm{e}^{-m\epsilon/2}
\end{aligned}$$

Equating the RHS to $\delta$ gives us the sample complexity.

$$\begin{aligned}
2\mathrm{e}^{-m\epsilon/2} =& \delta \\
\frac{2}{\delta} =& \mathrm{e}^{m\epsilon/2} \\
\log\frac{2}{\delta} =& \frac{m\epsilon}{2} \\
m =& \frac{2}{\epsilon}\log\frac{2}{\delta}
\end{aligned}$$

## 2

**Algorithm $\mathcal{A}$:** Given a sample $\mathcal{S}$ for target concept C containing p closed intervals,

1. If there are p separate sequence of positively labeled points in the training data, then return the union of the p tightest intervals containing the positive points.
2. Otherwise, return (p - i) tightest intervsals, each containing a sequence of positie labels separfated by i negative labelsi can take the values 0 to (p -1).

**Error intervals:** Let $[a_1, b_1] \cup [a_2, b_2] \cup ...[a_p, b_p]$ be the target concept C. Let $\epsilon > 0$. We can assume that $\Pr[a_i, b_i] > \epsilon/(p+1)$. The actual error on the training set is gap between the target concept $[a_i, b_i]$ and the learned concept $[a'_i, b'_i]$. Each of the error regions $r_i$ can occur with a probability $\epsilon/2(p+1)$. (Factor of 2 for $[a_i, a'_i] and [b'_i, bi]$.

**Proof of PAC learnability** :

If error($\mathrm{h_S}$) $> \epsilon$, then either the union of the intervals misses one of the regions $r_i$ or $\Pr[b_i, a_{i+1}] > \epsilon/(p+1)$. Thus by union bound, we have

$$\begin{aligned}\Pr[\mathrm{error(h_S)}] &\leq \Pr[\exists i \in [1,p] : h_S \text{ misses } r_i] + e^{-m\epsilon/(p+1)} \\ &\leq 2p(1 - \frac{\epsilon}{2(p+1)})^m + e^{-m\epsilon/(p+1)} \\ &\leq 2pe^{-m\epsilon/2(p+1)} + e^{-m\epsilon/(p+1)} \\ &\leq (2p+1)e^{-m\epsilon/2(p+1)}\end{aligned}$$

Setting $\delta = 0$ and solving the RHS will give us a bound for m.

$$(2p+1)e^{-m\epsilon/2(p+1)} = \delta \tag{1}$$

$$\frac{2p+1}{\delta} = e^{m\epsilon/2(p+1)} \tag{2}$$

$$\log \frac{2p+1}{\delta} = \frac{m\epsilon}{2(p+1)} \tag{3}$$

$$m = \frac{2(p+1)}{\epsilon} \log \frac{2p+1}{\delta} \tag{4}$$

Therefore, for probability of error to be less than $\epsilon$ m should be greater than the value obtained in (4) (Sample complexity).

**When p is 2** :

Substituting p = 2 in the equation (4) above gives us $m \geq \frac{6}{\epsilon} \log \frac{5}{\delta}$ as the sample complexity.

**Time complexity** :

The time complexity is O(p).

# B. Rademacher complexity, growth function

## 1

The upper bound on the growth function can be found by finding out the number of dichotomies possible using the given hypothesis set for a given m. For a given value of $\theta$, the functions in H can classify either the points to the left of $\theta$ as '+' ($x \mapsto 1_{x \leq \theta} : \theta \in \mathbb{R}$) or as '$-$' ($x \mapsto 1_{x \geq \theta} : \theta \in \mathbb{R}$). Similarly H can classify points to the right of $\theta$ as '+' or '$-$'.

If we have a sample of size $m$, we have a total of $(m + 1)$ ways of classifying the sample. Each classification can be done in two ways (as seen in the previous paragraph). Hence we have a total of 2(m+1) ways of classification. Of these classifications, two classification will be repeated (the one at the extremeties). So we have to subtract 2 from the total ways of classification. Hence total ways of classification is $2(m + 1) - 2 = 2m$.

$$\prod_H (m) = 2m$$

### Upper bound on $\mathfrak{R}_m(H)$

The upper bound is given by

$$\begin{aligned}\mathfrak{R}_m(H) &\leq \sqrt{\frac{2log \prod_H (m)}{m}} \\ &\leq \sqrt{\frac{2log(2m)}{m}}\end{aligned}$$

## 2

$$\begin{aligned}
\hat{\mathfrak{R}}_S(H) =& \frac{1}{m} \underset{\sigma}{E}[\sup_{h \in H} \sum_{i=1}^{m} \sigma_i h_1(x_i) h_2(x_i)] && (5)\\
=& \frac{1}{m} \underset{\sigma}{E}[\sup_{h \in H} \sum_{i=1}^{m} \sigma_i (\max(0, h_1(x_i) + h_2(x_i) - 1)] && (6)\\
\leq& \frac{1}{m} \underset{\sigma}{E}[\sup_{h \in H} \sum_{i=1}^{m} \sigma_i (h_1(x_i) + h_2(x_i))] && (7)\\
\leq& \frac{1}{m} \underset{\sigma}{E}[\sup_{h \in H} \sum_{i=1}^{m} \sigma_i h_1(x_i) + \sigma_i h_2(x_i)] && (8)\\
\leq& \frac{1}{m} \underset{\sigma}{E}[\sup_{h \in H} \sum_{i=1}^{m} \sigma_i h_1(x_i) + \sum_{i=1}^{m} \sigma_i h_2(x_i)] && (9)\\
\leq& \frac{1}{m} \underset{\sigma}{E}[\sup_{h \in H_1} \sum_{i=1}^{m} \sigma_i h_1(x_i) + \sup_{h \in H_2} \sum_{i=1}^{m} \sigma_i h_2(x_i)] && (10)\\
\leq& \frac{1}{m} \underset{\sigma}{E}[\sup_{h \in H_1} \sum_{i=1}^{m} \sigma_i h_1(x_i)] + \frac{1}{m} \underset{\sigma}{E}[\sup_{h \in H_2} \sum_{i=1}^{m} \sigma_i h_2(x_i)] && (11)\\
\leq& \hat{\mathfrak{R}}_S(H_1) + \hat{\mathfrak{R}}_S(H_2) && (12)
\end{aligned}$$

The equations and its explanations are as follows:

6 — by rewriting $h_1(x_i)h_2(x_i)$ in 1-lipshitz function form. The lipshitz form is valid as $h_1(x_i$ and $h_2(x_i$ take values 0,1.

7 — due to Talagrand's lemma

10 — Since $sup(f + g) \leq sup(f) + sup(g)$

8, 9, 11, 12 — Expanding the equation and replacing with Rademacher complexity terms.

The Rademacher complexity is bounded by

$$\begin{aligned}
\mathfrak{R}_m(U) \leq& \hat{\mathfrak{R}}_S(U) + \sqrt{\frac{log\frac{2}{\delta}}{2m}}\\
\leq& \hat{\mathfrak{R}}_S(C_1) + \hat{\mathfrak{R}}_S(C_2) + \sqrt{\frac{log\frac{2}{\delta}}{2m}}
\end{aligned}$$

This gives the bound for Rademacher complexity of the family U formed by the intersection of two concepts $C_1$ and $C_2$