Quantum Theory for Mathematicians

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Problem 1. Consider a particle moving in the real line in the presence of a force coming from a potential function V. Given some value E_0 for the energy of the particle, suppose that $V(x) < E_0$ for all x in some closed interval $[x_0, x_1]$. Then a particle with initial position x_0 andparticle moving in the real line in the presence of a force coming from a potential function V. Given some value E_0 for the energy of the particle, suppose that $V(x) < E_0$ for all x in some closed interval $[x_0, x_1]$. Then a particle with initial position x_0 and positive initial velocity will continue to move to the right until it reaches x_1. Using (2.6), show that the time needed to travel from x_0 to x_1 is given by

$$t = \int_{x_0}^{x_1} \sqrt{\frac{m}{2\left(E_0 - V(y)\right)}} dy$$

Note: This shows that we can solve Newton's equation in $\mathbb{R}^1$ more or less explicitly for time as a function of position, which in principle determines the position as a function of time.

Proof. The total energy of the above particle is given by

$$E_0 = \frac{1}{2} m \dot{x}^2 + V(x)$$

Solving for velocity as a function of position gives

$$\dot{x} = \sqrt{\frac{2(E_0 - V(x))}{m}}$$

Since $V(x) < E_0$, this velocity is positive on the entire interval $[x_0, x_1]$. By the inverse function theorem, we may conclude that the time is a function of position and moreover that

$$\frac{dt}{dx} = \sqrt{\frac{m}{2(E_0 - V(x))}}$$

The total time elapsed between x_0 and x_1 is then given by

$$\int_{t_0}^{t_1} dt = \int_{x_0}^{x_1} \frac{dt}{dx} dx = \int_{x_0}^{x_1} \sqrt{\frac{m}{2\left(E_0 - V(y)\right)}} dy$$

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Problem 3. Consider the equation of motion of a pendulum of length L,

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

where g is the acceleration of gravity. Here θ is the angle between the pendulum and the negative y-axis in the plane. This system has a stable equilibrium at $\theta = 0$ and an unstable equilibrium at $\theta = \pi$. Consider initial conditions of the form $\theta(0) = \pi - \delta, \dot{\theta}(0) = 0$, for $0 < \delta < \pi/4$. Fix some angle θ_0 and let $T(\delta)$ denote the time it takes for the pendulum with the given initial conditions to reach the angle θ_0. (Here θ_0 represents an arbitrarily chosen cutoff point at which the pendulum is no longer "close" to $\theta = \pi$.) Show that $T(\delta)$ grows only logarithmically as δ tends to zero. Note: Logarithmic growth of T as a function of δ corresponds to exponential decay of δ as a function of T. Thus, if we want T to be large, we must choose δ to be very small.

Proof. The total energy for the pendulum is given by

$$E = \frac{1}{2}m\left(L\frac{d\theta}{dt}\right)^2 - mgL\cos\theta$$

This is the same as the Hamiltonian for a particle with mass mL^2 and potential energy $mgL\cos\theta$. Therefore, by the result of problem 1, the time to travel is given by

$$T(\delta) = -\int_{\pi-\delta}^{\theta_0}\sqrt{\frac{mL^2}{2(-mgL\cos(\pi-\delta) + mgL\cos\theta)}}d\theta = \sqrt{\frac{L}{2g}}\int_{\pi-\delta}^{\theta_0}\frac{-1}{\sqrt{\cos\theta - \cos(\pi-\delta)}}d\theta$$

Rewriting this we get

$$T(\delta) = \sqrt{\frac{L}{2g}}\int_{\delta}^{\pi-\theta_0}\frac{1}{\sqrt{cos(\delta) - \cos(\theta)}}d\theta$$

Since we are only interested in the case of small δ, we may instead examine the approximation

$$T(\delta) \approx \sqrt{\frac{L}{2g}}\int_{\delta}^{\pi-\theta_0}\frac{1}{\sqrt{1 - \cos(\theta)}}d\theta = \sqrt{\frac{L}{4g}}\int_{\delta}^{\pi-\theta_0}\frac{1}{\sin(\theta/2)}d\theta$$

Thus, we have the approximation

$$T(\delta) \approx \sqrt{\frac{L}{g}}\left(\log\left(\tan\left(\frac{\pi-\theta_0}{4}\right)\right) - \log\left(\tan\left(\frac{\delta}{4}\right)\right)\right)$$

By using a taylor expansion for $\tan(\delta/4)$, we can see that

$$T(\delta) \sim -\sqrt{\frac{L}{g}}\log(\delta)$$

This shows that $T(\delta)$ grows logarithmically in δ as $\delta \to 0$.

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Problem 8. Consider a particle moving in $\mathbb{R}^n$ with a velocity-dependent force law given by

$$\mathbf{F}(\mathrm{x}, \mathrm{v}) = -\nabla V(\mathrm{x}) + \mathrm{F}_2(\mathrm{x}, \mathrm{v})$$

where the velocity-dependent term $\mathbf{F}_2$ acts perpendicularly to the velocity of the particle. (That is, we assume that $\mathrm{v} \cdot \mathrm{F}_2(\mathrm{x}, \mathrm{v}) = 0$ for all x and v .) Let E denote the usual energy function $E(\mathrm{x}, \mathrm{v}) = \frac{1}{2}m|\mathrm{v}|^2 + V(\mathrm{x})$, unmodified by the presence of the velocity-dependent term in the force. Show that E is conserved.

Proof. Notice that

$$\begin{aligned}
\frac{dE}{dt} &= m\dot{x} \cdot \ddot{x} + \nabla V \cdot \dot{x} \\
&= \dot{x} \cdot (m\ddot{x} + \nabla V \cdot x) \\
&= \dot{x} \cdot (F(x, v) + \nabla V \cdot x) \\
&= \dot{x} \cdot F_2(x, v)
\end{aligned}$$

If $F_2(x, v)$ is always orthogonal to $\dot{x}$, then $\frac{dE}{dt} = 0$.

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Problem 18. Determine the Hamiltonian flow on $\mathbb{R}^2$ generated by the function $f(x, p) = xp$.

Proof. We calculate the gradient of the Hamiltonian to be

$$\nabla f(x, p) = (p, x)$$

By Hamilton's equations, we then have

$$\dot{x} = x, \quad \dot{p} = -p$$

These have the unique solutions

$$x(t) = x_0 e^t, \quad p(t) = p_0 e^{-t}$$

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