# Back to Quotienting Out a Subgroup:

- same as quotienting out a vector subspace.

Two ways to look at it:

① • have a big group $G$ & smaller (normal) subgroup $H$.
  • quotienting $H$ in $G$ forms a new group, $G/H$.
  • the cosets of $H$ in $G$ act as a group under composition of cosets.

  → $[x] = \{x + h \mid h \in H\}$. for $x \in G$.

  $\boxed{[x] \circ_c [y] \equiv [x \circ_G y]}$ → this works & is well defined (see proofs).

  → this effectively condense the big group down to a smaller one while preserving some of its properties.


e.g., taking $G = \mathbb{Z}$ & $H = 2\mathbb{Z} \subseteq G$.

We showed $G/H = \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}_2$

| $\mathbb{Z}_2$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 0 |

→ takes us to a finite group of 2 elements but carries the idea of eveness & oddness of integers from the full infinite $\mathbb{Z}$.

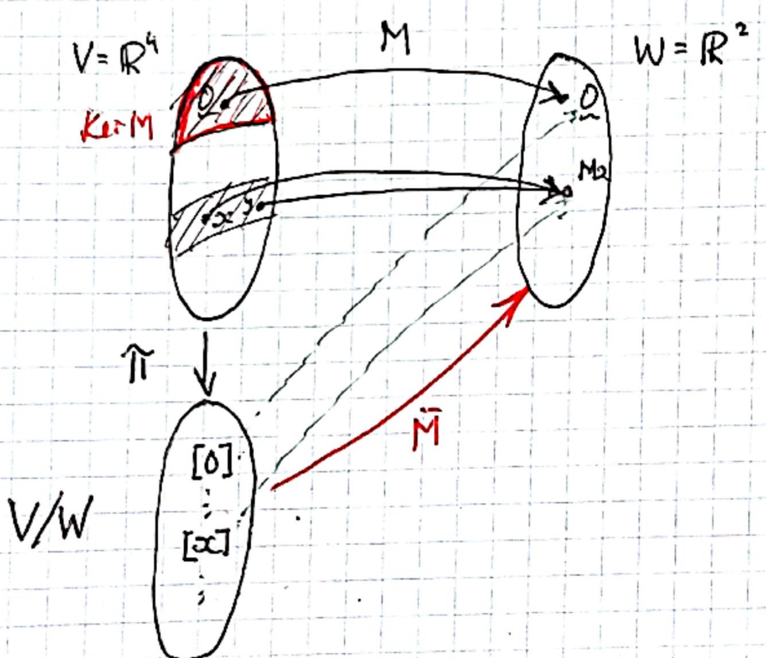
② • Start $\bar{c}$ a big group $G$ and a target group $K$ we want to capture some behaviour of $G$ in.
  • find somehow a map (homomorphism) $\phi : G \to K$
  • amazingly (see proof): (FIRST ISOMORPHISM THM) → ① $\ker\phi$ is a (normal) subgroup of $G$ ② $G/\ker\phi \simeq K$. (isomorphic)
  • the quotient group acts like the target group we wanted, $K$.
  • the kernel of $\phi$ provides constraints or relations on $G$ to force it to behave like $K$

## Example:

⇒ Say we have $V = \mathbb{R}^4$ and we want to make it "look like" $\mathbb{R}^2$ ($W=\mathbb{R}^2$)

- just find a map $M: \mathbb{R}^4 \to \mathbb{R}^2$, $V \to W$
  - ⇒ since we need a homomorphism for vector spaces $M$ must be linear.

e.g., Pick $M = \begin{bmatrix} -1 & -2 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix}_{2\times 4}$




- $\ker M = \{ \underline{x} \in V \mid M\underline{x} = \underline{0} \}$
  $=$ null space of $M$.

- have:

$$\begin{tikzcd} V \arrow[r, "M"] \arrow[d, "\pi"'] & W \\ V/\ker M \arrow[ur, "\bar{M}"', "\cong"] & \end{tikzcd}$$

- what are the constraints introduced by $M$ to make $V$ look like $W$?
- $\ker M$ = null space $M$ = span$\{ (1\ 0\ 1\ 0)^T, (-2\ 1\ 0\ 2)^T \} = N$
- note $\dim V = 4$, $\underset{\dim N}{\dim \ker M} = 2 \to \dim V/N = 2$.
- as a basis elt of nullspace: $(-2,1,0,2) \xrightarrow{M} (0,0)$

So in $V/N$ as a coset: $[(-2\ 1\ 0\ 2)] = [\underline{0}]$

in basis of $V$: $[-2\underline{e}_1 + \underline{e}_2 + 2\underline{e}_4] = [\underline{0}]$

$-2[\underline{e}_1] + [\underline{e}_2] + 2[\underline{e}_4] = [0]$

$\Rightarrow \boxed{[e_2] = 2[e_1] - 2[e_3]}$

Similarly for $(1\ 0\ 1\ 0) \in N$:

$\boxed{[e_1] = -[e_3]}$

- so if we have a vector space we can force it to behave like another vector space by quotienting out constraints or relations.

→ they can be: ① given explicitly
② defined implicitly as a kernel of a homomorphic map between the spaces.

* This provides a surprisingly rich way of creating new mathematical structures. *

## On to Free Vector Spaces OVER VECTOR SPACES

- recall we defined a free vector space over a set $S$.

$$F(S) = \left\{ \sum_{k=1}^{K} a_k \, \delta_{s_k} \right\} \quad \text{for } s_k \in S, \; a_k \in \mathbb{R}, \; K \text{ finite.}$$

basis indices

real coefficients.

- we looked at the case $S = \mathbb{R}$, the set of reals
  - $F(\mathbb{R})$ with elts $\sum_k^K a_k \, \delta_{s_k}$
  - we can consider $\mathbb{R} = \mathbb{R}^1$, a one-dim$^n$ vector space.
  - in general, take $S = V$, an arbitrary finite dim$^n$ vector space

$$\Rightarrow F(V) = \left\{ \sum_{k=1}^{K} a_k \, \delta_{v_k} \right\}$$

$v_k \in V$ as indices

real coeffs.

- note: this is not $\sum_k a_k v_k$ which is in $V$, not $F(V)$.

- previous we defined the moment map $M$, from $F(\mathbb{R}) \to \mathbb{R}^n$ and showed the quotient $F(\mathbb{R})/\ker M \cong \mathbb{R}^n$

→ the cosets formed an $n$-dim vector space as per the 1st isom$^n$ th$^m$.

Example: Set $S = V$, a finite dim vector space, $V \simeq \mathbb{R}^n$

- we have the free vector space $F(V)$ with basis elts, $\delta_v$ for $v \in V$
- we know this game involves: ① a map $\phi$ (v.s. homom$^m$) and ② its kernel we quotient $F(V)$ by: $F(V)/\ker\phi$.

①• define the homom$^m$ map:

$$\phi : F(V) \longrightarrow V$$
$$\delta_v \longmapsto v$$

and extend linearly, so that

$$\phi(\underbrace{x = \sum_h a_h \delta_{v_h}}_{\in F(V)}) = \sum_h a_h v_h \in V$$

② • now define a vector subspace of $F(V)$ as the span of:

$$R_{lin} = \text{span}\{\underbrace{\delta_{v+w} - \delta_v - \delta_w}_{\text{additivity}},\ \underbrace{\delta_{\alpha v} - \alpha\delta_v}_{\text{homogeneity}},\ \underbrace{\delta_0}_{\text{zero}} \mid v, w \in V, \alpha \in \mathbb{R}\}$$

– note: in the interpretation that moduloing out a subspace effectively sets its elts to zero:

$\Rightarrow$ this $R_{lin}$ forces the bases to act like vectors in $V$

$$\left.\begin{array}{lll} \Rightarrow \delta_{v+w} = \delta_v + \delta_w & \sim (v+w) = v + w \\ \delta_{\alpha v} = \alpha\delta_v & \sim (\alpha v) = \alpha \cdot v \\ \delta_0 = 0 & \underline{0} = \underline{0} \end{array}\right\} \text{Hmm...}$$

③ Now look at the kernel of $\phi$ (it's $\mathcal{R}_{lin}$!).

Claim: $\ker\phi = \mathcal{R}_{lin}$ (use $A \subseteq B$ & $B \subseteq A \to A = B$ trick).

Why?

① If every generator of $\mathcal{R}_{lin} \xrightarrow{\phi} \underline{0}$ then all elts in $\mathcal{R}_{lin} \subseteq \ker\phi$.

But: i) $\phi(\delta_{v+w} - \delta_v - \delta_w) = (v+w) - v - w = \underline{0}$

ii) $\phi(\delta_{\alpha v} - \alpha\delta_v) = \alpha v - \alpha \cdot v = \underline{0}$

iii) $\phi(\delta_{\underline{0}}) = \underline{0}$

$\therefore$ all linear comb's of the generators $\xrightarrow{\phi} \underline{0}$

$\therefore$ $\boxed{\mathcal{R}_{lin} \subseteq \ker\phi}$

② Now want to show $\ker\phi \subseteq \mathcal{R}_{lin}$

That is, if $\phi(x) = 0 \longrightarrow x \in \mathcal{R}_{lin}$

i) let $x = \sum_i a_i \delta_{v_i}$ such that $\sum_i a_i v_i = \underline{0}$

$\rightarrow$ $\boxed{\text{i.e., } \phi(x) = 0}$

ii) now we want to show this $x \in F(V)$ is in $\mathcal{R}_{lin}$.

- we know for each $i$:

$$a_i\delta_{v_i} - \delta_{a_i v_i} \in \mathcal{R}_{lin} \text{ (by homog'ty).}$$

$\therefore \sum_i a_i \delta_{v_i} - \left(\sum_i \delta_{a_i v_i}\right) \in \mathcal{R}_{lin}$ (by closure of $\mathcal{R}_{lin}$).

- repeatedly use $\delta_u + \delta_v = \delta_{u+v}$ to put:

$$\sum_i \delta_{a_i v_i} = \delta_{\sum_i a_i v_i}$$

$\therefore \underbrace{\sum_i a_i \delta_{v_i}}_{x} - \delta_{\sum_i a_i v_i} \in \mathcal{R}_{lin}$

- but $\sum_i a_i v_i = \underline{0}$ by assumption

$\therefore x - \delta_0 \in \mathcal{R}_{lin}$

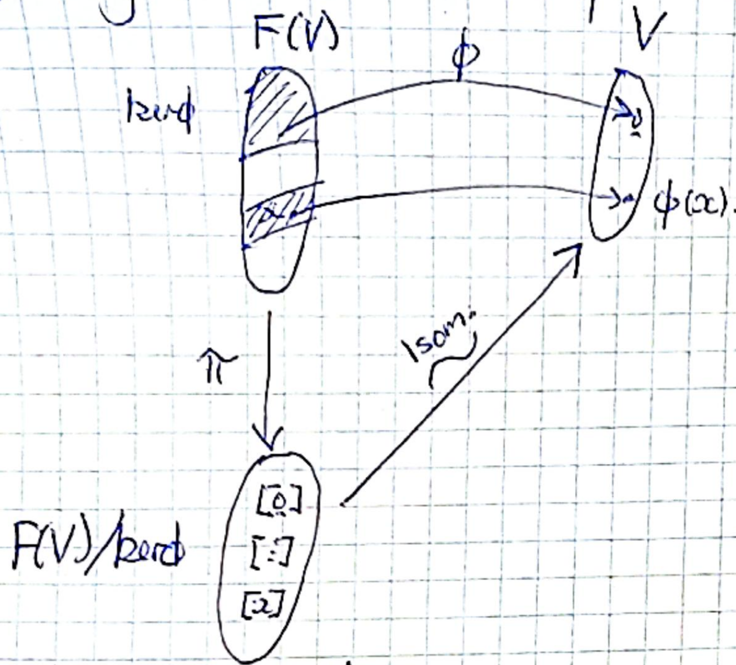
- but $\delta_0 \in \mathcal{R}_{lin}$ by defn

$\therefore x \in \mathcal{R}_{lin}$

$\therefore$ $\boxed{\ker\phi \subseteq \mathcal{R}_{lin}}$

$\boxed{③ \therefore \ker\phi = \mathcal{R}_{lin}}$

④ By the 1'st isomorph$^{=}$ th$^{m}$:




⑤ $\therefore F(V)/\mathcal{R}_{lin} \simeq V.$

$\Rightarrow$ We've reconstructed $V$ from $F(V)$ by imposing the correct (linear) relations.

---

Let's do the same but now involving:

① Two vector spaces, $V$ & $W$

② New relations, $\mathcal{R}_{bilinear}$.

---

Ⓐ Take two finite-dim vector spaces $V, W$ ($\simeq \mathbb{R}^m, \mathbb{R}^n$, e.g.).

Ⓑ Look at their Cartesian product $V \times W$ with elts

$$V \times W = \{ (v,w) \mid v \in V, w \in W \}.$$

Ⓒ Take a free vector space on the set $S = (V \times W)$ now

$\rightarrow$ has basis elts $\delta_{(v,w)}$ $\forall v,w \in V,W$.), span $F(V \times W)$

Ⓓ Now take the subspace of bilinear relations/constraints.

$$\mathcal{R}_{bilin} = \text{span} \begin{cases} \underbrace{\delta_{(v_1+v_2,w)} - \delta_{(v_1,w)} - \delta_{(v_2,w)}}_{\text{linear in } V \text{ alone}}, & \underbrace{\delta_{(\alpha v,w)} - \alpha \delta_{(v,w)}}_{\text{homogeneous in } V \text{ alone}}, \\ \underbrace{\delta_{(v,w_1+w_2)} - \delta_{(v,w_1)} - \delta_{(v,w_2)}}_{\text{linear in } W \text{ alone}}, & \underbrace{\delta_{(v,\alpha w)} - \alpha \delta_{(v,w)}}_{\text{homogeneous in } W \text{ alone}}. \end{cases}$$

(E) Now take the quotient

$$T \equiv F(V \times W) / \mathcal{R}_{bilin.}$$

(F) In the equivalence classes of $T$ define notationally $[\delta_{(u,v)}]$ as $u \otimes v$.

→ it's purely notational here but cleaner.

(G) As equivalence classes, since $[\mathcal{R}_{bilin.}] = [0]$ we have:

$$\delta_{(v_1+v_2,w)} - \delta_{(v,w)} - \delta_{(v_2,w)} \in \mathcal{R}_{bil} \Rightarrow (v_1+v_2)\otimes w - v_1 \otimes w - v_2 \otimes w = 0$$

$$\Rightarrow \boxed{(v_1+v_2)\otimes w = v_1 \otimes w + v_2 \otimes w}$$

Similarly from the bilin. rel^s:

$$\boxed{\begin{aligned} &(\alpha v)\otimes w = \alpha(v \otimes w). \\ &v \otimes (w_1+w_2) = v\otimes w_1 + v \otimes w_2 \\ &v \otimes (\alpha w) = \alpha(v \otimes w). \end{aligned}}$$

(H) We identify $T$ as the tensor product of vector spaces $V \otimes W$.

(I) Example:

Let $V \simeq \mathbb{R}^2$, basis $e_1, e_2$

$W \simeq \mathbb{R}^3$, basis $f_1\ f_2\ f_3$.

Let $v = 2e_1 - e_2$

$w = 3f_1 + 0f_2 + 4f_3$

$$\begin{aligned} \Rightarrow v \otimes w &= (2e_1 - e_2)\otimes w \\ &= (2e_1)\otimes w - e_2 \otimes w \\ &= 2e_1 \otimes (3f_1 + 4f_3) - e_2 \otimes (3f_1 + 4f_3) \\ &= 2e_1 \otimes (3f_1) + 2e_1 \otimes (4f_3) - e_2 \otimes (3f_1) - e_2 \otimes (4f_3) \\ &= 6\, e_1 \otimes f_1 + 8 e_1 \otimes f_3 - e_2 \otimes f_1 - 4 e_2 \otimes f_3 \\ &\in V \otimes W \end{aligned}$$

Note: this can be organized as.

$$v\otimes w = \begin{pmatrix}2\\-1\end{pmatrix} \otimes \begin{pmatrix}3\\0\\4\end{pmatrix} \rightarrow \begin{pmatrix}2\\-1\end{pmatrix}(3\ 0\ 4) = \begin{array}{c|ccc} & f_1 & f_2 & f_3 \\ \hline e_1 & 6 & 0 & 8 \\ e_2 & -3 & 0 & -4 \end{array}$$

- $(i,j)$'th entry is the coefficient of $e_i \otimes f_j$.

(J) In general:

Pick basis $\{e_i\}_{i=1}^{m}$ for $V$

" " $\{f_j\}_{j=1}^{n}$ for $W$

- given $v = \sum_i v_i e_i$

$w = \sum_j w_j f_j$

→ expanding in the bilin. relations gives:

$$\boxed{v\otimes w = \sum_{i,j}^{m,n} v_i w_j \, (e_i \otimes f_j)} \in V\otimes W.$$

⇒ the $(e_i \otimes f_j)$ form a basis for $V\otimes W$.

Any $v\otimes w$ can be expanded in ~~at most~~ $m\cdot n$ basis vectors.

$$\therefore \boxed{\dim V\otimes W = m\cdot n}$$

# Tensor Products In Coordinates:

- again let $V \sim$ basis $\{e_1 \cdots e_m\}$. $\sim \mathbb{R}^m$

  $W \sim$ basis $\{f_1 \cdots f_n\} \sim \mathbb{R}^n$.

- Let $\underset{\sim}{v} = \sum_i v_i e_i$

  $\underset{\sim}{w} = \sum_j w_j f_j$

$$\Rightarrow \underset{\sim}{v} \otimes \underset{\sim}{w} = \sum_{ij}^{m,n} (v_i w_j)(e_i \otimes f_j)$$

- this can be written as a column vector in basis $e_i \otimes f_j$

$$\begin{bmatrix} v_1 w_1 \\ v_1 w_2 \\ \vdots \\ v_1 w_n \\ v_2 w_1 \\ v_2 w_2 \\ \vdots \\ v_2 w_n \\ \vdots \\ v_m w_n \end{bmatrix}_{mn \times 1} \begin{array}{l} \leftarrow e_1 \otimes f_1 \\ \leftarrow e_1 \otimes f_2 \\ \quad \vdots \\ \quad e_1 \otimes f_n \\ \quad e_2 \otimes f_1 \\ \quad e_2 \otimes f_2 \\ \quad \vdots \\ \quad e_2 \otimes f_n \\ \quad \vdots \\ \quad e_m \otimes f_n \end{array} = \begin{bmatrix} v_1 \underset{\sim}{w} \\ v_2 \underset{\sim}{w} \\ \vdots \\ v_m \underset{\sim}{w} \end{bmatrix} = \underset{\sim}{v} \otimes \underset{\sim}{w}$$

This is the Kronecker product of vectors.

---

## Linear Operators on Tensors

- Suppose we have a linear mappings: $A : V \longrightarrow V'$

  $B : W \longrightarrow W'$

- can define the operator on $V \times W$ via:

$$A \otimes B : V \otimes W \xrightarrow{\text{linear}} V' \otimes W'$$

$$v \otimes w \longmapsto (Av) \otimes (Bw).$$

and extend linearly. That is,

$$(A \otimes B)(v_1 \otimes w_1 + v_2 \otimes w_2) = Av_1 \otimes Bw_1 + Av_2 \otimes Bw_2.$$

**Summary:**

1. $(A \otimes B)(v \otimes w) = Av \otimes Bw$
2. $(A \otimes B)(C \otimes D) = AC \otimes BD.$

- in co-ord's & as matrices:

$$A \otimes B = \begin{bmatrix} a_{11} B & a_{12} B & & a_{1m} B \\ a_{21} B & a_{22} B & \ddots & \vdots \\ \vdots & \vdots & & \vdots \\ a_{m1} B & & \cdots & \end{bmatrix}$$

$\leftarrow$ the Kronecker product of matrices

kron( ) in Matlab, python...