Quantum Theory for Mathematicians
Chapter 3
Problems 7,9

Wayne Dam

January 31, 2025

Exercise 7:

Prove proposition 3.23:

$$\frac{dX}{dt} = \frac{1}{m}P(t) \tag{1}$$
$$\frac{dP}{dt} = -V'\left(X(t)\right). \tag{2}$$

Hint. Show:

$$[X(t), H] = ([X, H])(t) \tag{3}$$
$$[P(t), H] = ([P, H])(t) \tag{4}$$

Answer:

First show H commutes with $e^{itH/\hbar}$. We will use this.

We know that H must satisfy,

$$H(t) = e^{itH/\hbar} H e^{-itH/\hbar}. \tag{5}$$

Since H does not evolve in time then,

$$H = e^{itH/\hbar} H e^{-itH/\hbar}. \tag{6}$$

So,

$$e^{\pm itH/\hbar} H = H e^{\pm itH/\hbar}. \tag{7}$$

And we then have for the momentum,

$$\begin{aligned}
[P(t), H] &= \left[e^{itH/\hbar} P e^{-itH/\hbar}, H\right] && (8)\\
&= e^{itH/\hbar} P e^{-itH/\hbar} H - H e^{itH/\hbar} P e^{-itH/\hbar} && (9)\\
&= e^{itH/\hbar} P H e^{-itH/\hbar} - e^{itH/\hbar} H P e^{-itH/\hbar} && (10)\\
&= e^{itH/\hbar} [P, H] e^{-itH/\hbar} && (11)\\
[P(t), H] &= ([P, H])(t). && (12)
\end{aligned}$$

And similarly for position,

$$[X(t), H] = ([X, H])(t). \tag{13}$$

For position we have,

$$\begin{aligned} \frac{dX}{dt} &= \frac{1}{i\hbar}[X(t), H] && (14) \\ &= \frac{1}{i\hbar}([X, H])(t). && (15) \end{aligned}$$

For $H = \frac{P^2}{2m} + V(x)$, we have $[X, H] = \left[X, P^2/2m + V\right] = 1/2m[X, P^2]$. And from the commutator identities, $[X, P^2] = [X, P \cdot P] = [X, P]P + P[X, P]$. Thus,

$$[X, P^2] = 2i\hbar P, \tag{16}$$

and

$$[X, H] = \frac{i\hbar}{m}P. \tag{17}$$

So,

$$\frac{dX}{dt} = \frac{1}{2m}P(t). \tag{18}$$

The momentum is derived similarly.

Exercise 9:

Dropping the constants we can just let $H = \frac{d^2}{dx^2}$. We want to show $\langle\phi|H\psi\rangle = \langle H\phi|\psi\rangle$ provided $\phi(0) = \phi(L) = 0$ and the same for ψ on $[0, L]$.

Answer:

(a)

$$\begin{aligned} lhs &= \langle\phi|H\psi\rangle && (19) \\ &= \int_0^L \overline{\phi}(x)\psi_{xx}(x)\ dx && (20) \\ &= -\int_0^L \overline{\phi}_x(x)\psi_x(x)\ dx + \overline{\phi}(x)\psi_x(x)\Big|_0^L . && (21) \end{aligned}$$

$$\begin{aligned} rhs &= \langle H\phi|\psi\rangle && (22) \\ &= \int_0^L \overline{\phi}_{xx}(x)\psi(x)\ dx && (23) \\ &= -\int_0^L \overline{\phi}_x(x)\psi_x(x)\ dx + \overline{\phi}_x(x)\psi(x)\Big|_0^L . && (24) \end{aligned}$$

Since both $\phi(x)$ and $\psi(x)$ equal zero at $x = 0$ and L the boundary terms are equal to zero and the two inner products are equal.

(b) For general $\phi(x)$ and $\psi(x)$ the boundary terms are not zero and not necessarily equal and so the identity does not hold in general.