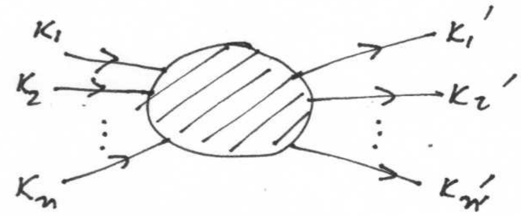
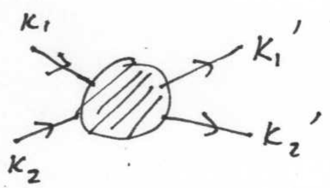
Process : n particles transforming to n' particles



Transition amplitude for this process is given by LSZ reduction formula (5.15):

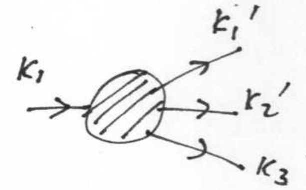
$$\langle f|i\rangle = \langle k_1' k_2' \dots k_{n'}' | k_1 k_2 \dots k_n\rangle$$
$$= i^{n+n'} \int d^4x_1\, e^{ik_1\cdot x_1}(-\partial_1^2+m^2)\dots$$
$$\int d^4x_1'\, e^{-ik_1' x_1'}(-\partial_{1'}^2+m^2)\dots$$
$$\langle 0|T\,\varphi(x_1)\dots\varphi(x_1')\dots|0\rangle$$

In next chapter (11), these transition amplitudes will be used to calculate



scattering cross sections

and



decay rates (e.g. $n\to pe^-\bar{\nu}_e$)

On the other hand,

$$\langle 0|T\varphi(x_1)\varphi(x_2)\dots|0\rangle = \frac{1}{i}\frac{\delta}{\delta J(x_1)}\frac{1}{i}\frac{\delta}{\delta J(x_2)}\dots Z(J)\Big|_{J=0}$$
$$= \delta_1\,\delta_2\dots Z(J)_{J=0}$$

where $Z(J)$ is the vacuum-vacuum transition amplitude in the presence of an external source J.

$$Z(J) = e^{iW(J)}$$

$W(J) \equiv$ sum of all connected diagrams with no ~~tad~~poles and at least 2 sources

We evaluated $W(J)$ in chapter 9. This chapter (10) is concerned with using $W(J)$ to simplify expressions for the transition amplitudes in $\langle f|i\rangle$, and in particular for developing Feynman rules in momentum space.

Renormalization condition

$$0 = \langle 0 | \varphi(x_1) | 0 \rangle = \underbrace{\frac{1}{i}\frac{\delta}{\delta J(x_1)}}_{\equiv \delta_1} Z(J)\Big|_{J=0} = \delta_1 e^{iW(J)}\Big|_{J=0} = \delta_1 \, iW(J)\Big|_{J=0}$$

since $W(0) = 0$

$$\Rightarrow \delta_1 \, iW(J)\Big|_{J=0} = 0$$

This is accomplished by adjusting the $Y\varphi(x)$ counterterm.

Exact propagator

$$\frac{1}{i}\boldsymbol{\Delta}(x_1 - x_2) \equiv \langle 0 | T\varphi(x_1)\varphi(x_2) | 0 \rangle = \delta_1 \delta_2 \, e^{iW(J)}\Big|_{J=0}$$

$$= \delta_1 \delta_2 \, iW(J)\Big|_{J=0} + \delta_1 \, iW(J)\Big|_{J=0} \cdot \delta_2 \, iW(J)\Big|_{J=0}$$

$$= \delta_1 \delta_2 \, iW(J)\Big|_{J=0}$$

Only 2-point (source) diagrams (with no tadpoles) contribute.
From Fig. 9.13 (ignoring counterterms)

$iW(J) = \frac{1}{2}$  $+ \frac{1}{4}$  $+ O(g^4)$

$+ (n > 2)$-point diagrams

$$= \frac{1}{2}\int d^4x \, iJ(x) \int d^4y \, iJ(y) \, \frac{1}{i}\Delta(x-y)$$

$$+ \frac{1}{4}(ig)^2 \int d^4x \, iJ(x) \int d^4y \, iJ(y) \int d^4z \int d^4u$$

$$\frac{1}{i}\Delta(x-z)\left(\frac{1}{i}\Delta(z-u)\right)^2 \frac{1}{i}\Delta(u-y)$$

$$+ \cdots$$

$$\Rightarrow \frac{1}{i}\boldsymbol{\Delta}(x_1 - x_2) = \delta_1 \delta_2 \, iW(J)\Big|_{J=0}$$

$$= \frac{1}{i}\Delta(x_1 - x_2) - \frac{g^2}{2}\int d^4z \, d^4u \, \Delta(x_1 - z)\Delta^2(z-u)\Delta(z - x_2) + \cdots$$

x_1 x_2 $=$ x_1 —— x_2 $+ \frac{1}{2}$ x_1 x_2 $+ \cdots$

4-point correlation function

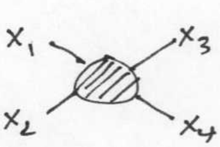
$$\langle 0 | T\, \varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4) | 0 \rangle$$

$$= \delta_1 \delta_2 \delta_3 \delta_4 \, Z(J)\big|_{J=0} = \delta_1 \delta_2 \delta_3 \delta_4 \, e^{iW(J)}\big|_{J=0}$$

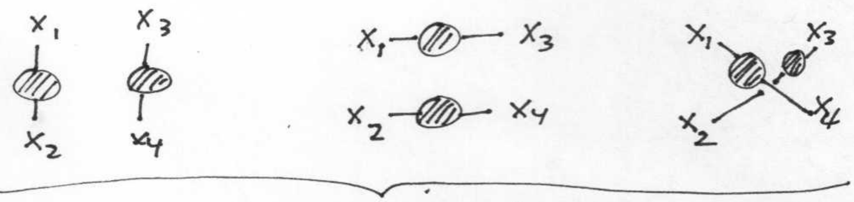
$$\begin{aligned}
= e^{iW(J)} \big(& \delta_1\delta_2\delta_3\delta_4 iW + \delta_1 iW\, \delta_2\delta_3\delta_4 iW + \delta_2 iW\, \delta_1\delta_3\delta_4 iW \\
& + \delta_3 iW\, \delta_1\delta_2\delta_4 iW + \delta_4 iW\, \delta_1\delta_2\delta_3 iW + \delta_1\delta_2 iW\, \delta_3\delta_4 iW \\
& + \delta_1\delta_3 iW\, \delta_2\delta_4 iW + \delta_1\delta_4 iW\, \delta_2\delta_3 iW + \delta_1\delta_2 iW\, \delta_3 iW\, \delta_4 iW \\
& + \delta_1\delta_3 iW\, \delta_2 iW\, \delta_4 iW + \delta_1\delta_4 iW\, \delta_2 iW\, \delta_3 iW + \delta_2\delta_3 iW\, \delta_1 iW\, \delta_4 iW \\
& + \delta_2\delta_4 iW\, \delta_1 iW\, \delta_3 iW + \delta_3\delta_4 iW\, \delta_1 iW\, \delta_2 iW + \delta_1 iW\, \delta_2 iW\, \delta_3 iW\, \delta_4 iW \big)\Big|_{J=0}
\end{aligned}$$

$$\begin{aligned}
= \big(& \delta_1\delta_2\delta_3\delta_4 iW \\
& + \delta_1\delta_2 iW\, \delta_3\delta_4 iW + \delta_1\delta_3 iW\, \delta_2\delta_4 iW + \delta_1\delta_4 iW\, \delta_2\delta_3 iW \big)\big|_{J=0}
\end{aligned}$$

$$= \delta_1\delta_2\delta_3\delta_4 iW\big|_{J=0} + \frac{\Delta(x_1-x_2)}{i}\frac{\Delta(x_3-x_4)}{i} + \frac{\Delta(x_1-x_3)}{i}\frac{\Delta(x_2-x_4)}{i} + \frac{\Delta(x_1-x_4)}{i}\frac{\Delta(x_2-x_3)}{i}$$

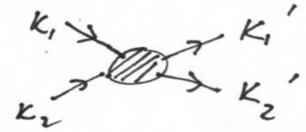


connected



disconnected

For 2-particle scattering, disconnected diagrams are uninteresting (or zero)



$$\langle f|i\rangle = \langle k_1' k_2' | k_1 k_2 \rangle$$

$$= i^{2+2} \int d^4x_1\, d^4x_2\, d^4x_1'\, d^4x_2'\; e^{i(k_1\cdot x_1 + k_2\cdot x_2 - k_1'\cdot x_1' - k_2'\cdot x_2')}$$

$$(\partial_1^2 + m^2)(-\partial_2^2+m^2)(-\partial_1'^2+m^2)(-\partial_2'^2+m^2)\langle 0|T\,\varphi(x_1)\varphi(x_2)\varphi(x_1')\varphi(x_2')|0\rangle$$

$$\langle 0|T\,\varphi(x_1)\varphi(x_2)\varphi(x_1')\varphi(x_2')|0\rangle = \delta_1\delta_2\delta_1'\delta_2'\, iW\big|_{J=0} \quad \text{(connected)}$$

$$+\, i^{-2}\left(\Delta(x_1-x_2)\Delta(x_1'-x_2') + \Delta(x_1-x_1')\Delta(x_2-x_2') + \Delta(x_1-x_2')\Delta(x_2-x_1')\right)$$

$$\langle f|i\rangle = i^2\left[\int d^4x_1\, d^4x_1'\, e^{i(k_1\cdot x_1 - k_1'\cdot x_1')}(-\partial^2+m^2)(-\partial_1'^2+m^2)\,\Delta(x_1-x_1')\right]\left[\,1\to 2\,\right] + \dots$$

change variables, for each μ,

$$x_+^\mu = x_1^\mu + x_1'^\mu \qquad x_1^\mu = \tfrac{1}{2}(x_+^\mu + x_-^\mu) \qquad \partial_1^\mu = \tfrac{1}{2}(\partial_+^\mu + \partial_-^\mu)$$

$$x_-^\mu = x_1^\mu - x_1'^\mu \qquad x_1'^\mu = \tfrac{1}{2}(x_+^\mu - x_-^\mu) \qquad \partial_1'^\mu = \tfrac{1}{2}(\partial_+^\mu - \partial_-^\mu)$$

$$dx_1^\mu \wedge dx_1'^\mu = \tfrac{1}{4}(dx_+^\mu + dx_-^\mu)\wedge(dx_+^\mu - dx_-^\mu) = \tfrac{1}{2}\, dx_-^\mu \wedge dx_+^\mu$$

$$k_1 x_1 - k_1' x_1' = k_1\cdot \tfrac{1}{2}(x_+ + x_-) - k_1'\cdot\tfrac{1}{2}(x_+ - x_-)$$

$$= \tfrac{1}{2}(k_1 - k_1')\cdot x_+ + \underbrace{\tfrac{1}{2}(k_1+k_1')}_{\equiv \bar{k}_{11'}}\cdot x_-$$

then

$$[\;\;] = \int \frac{1}{2^4}\, d^4x_+\, dx_-^4\; e^{i\frac{1}{2}(k_1-k_1')\cdot x_+ + i\bar{k}_{11'}\cdot x_-}\left(-\tfrac{1}{4}\partial_-^2 + m^2\right)^2 \Delta(x_-)$$

$$= \frac{1}{2^4}(2\pi)^4\,\delta^4\!\left(\tfrac{1}{2}(k_1-k_1')\right) F(\bar{k}_{11'}) = (2\pi)^4\,\delta^4(k_1-k_1')\,F(\bar{k}_{11'})$$

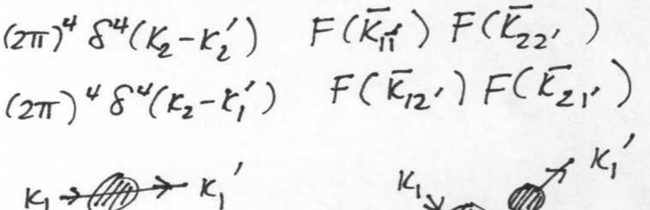
And so the contribution to the scattering amplitude from these disconnected diagram is

$$\langle k_1' k_2' | k_1 k_2\rangle = i^2\,(2\pi)^4\delta^4(k_1+k_2)\,(2\pi)^4\delta^4(k_1'+k_2')\;F(\tfrac{1}{2}(k_1-k_2))F(\tfrac{1}{2}(k_1'-k_2'))$$

$$+\, i^2\,(2\pi)^4\delta^4(k_1-k_1')\,(2\pi)^4\delta^4(k_2-k_2')\;F(\bar{k}_{11'})F(\bar{k}_{22'})$$

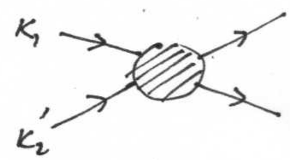
$$+\, i^2\,(2\pi)^4\delta(k_1-k_2')\,(2\pi)^4\delta^4(k_2-k_1')\;F(\bar{k}_{12'})F(\bar{k}_{21'})$$

vanishes since $k_1^0 + k_2^0 \geq 2m$



no scattering has occurred

2-particle scattering, to lowest order in perturbation theory



$$\langle f|i\rangle = \langle k_1' k_2' | k_1 k_2\rangle$$

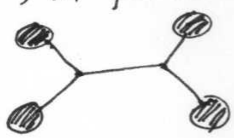
$$= i^{2+2}\int d^4x_1\, d^4x_2\, dx_1'\, dx_2'\; e^{i(k_1\cdot x_1 + k_2\cdot x_2 - k_1'\cdot x_1' - k_2'\cdot x_2')}$$

$$(-\partial_1^2+m^2)(-\partial_2^2+m^2)(-\partial_1'^2+m^2)(-\partial_2'^2+m^2)\,\langle 0|T\,\varphi(x_1)\varphi(x_2)\varphi(x_1')\varphi(x_2')|0\rangle_c$$

where the connected correlation function is

$$\langle 0|T\,\varphi(x_1)\varphi(x_2)\varphi(x_1')\varphi(x_2')|0\rangle_c \equiv \delta_1\delta_2\delta_1'\delta_2'\, iW(J)\Big|_{J=0}$$

we proceed to evaluate this to lowest order (g^2) in perturbation theory. The relevant diagram is Fig. 9.10:



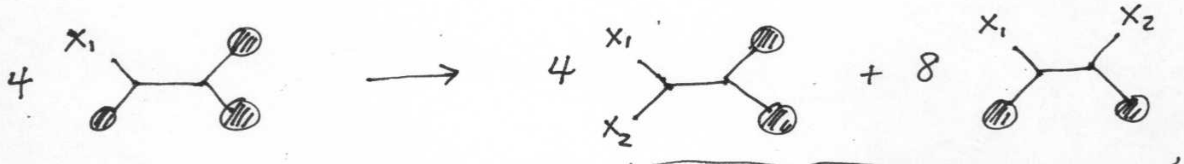
$S = 2^3$

$$iW(J) = \frac{1}{8}(ig)^2\int d^4x_1\, iJ(x_1)\int d^4x_2\, iJ(x_2)\int d^4x_3\, iJ(x_3)\int d^4x_4\, iJ(x_4)$$

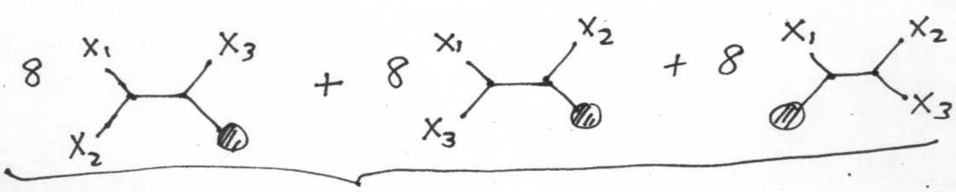
$$\int d^4y\, d^4z\, \frac{\Delta(y-z)}{i}$$

$$\times\frac{\Delta(x_1-y)}{i}\frac{\Delta(x_2-y)}{i}\cdot\frac{\Delta(z-x_3)}{i}\frac{\Delta(z-x_4)}{i}$$

The functional derivatives $\delta_i \equiv \frac{1}{i}\frac{\delta}{\delta J(x_i)}$ will remove the symmetry factor of 8:

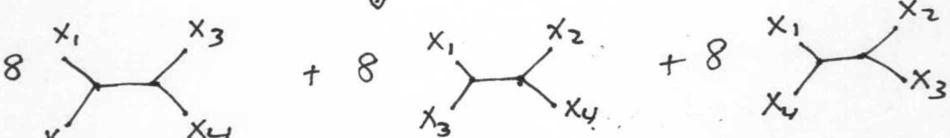


$\delta_1 iW$ $\qquad\qquad$ $\delta_1\delta_2 iW$



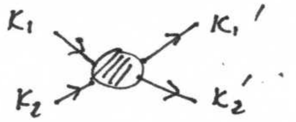
$\delta_1\delta_2\delta_3 iW$

$\downarrow$



General result: the symmetry factor of tree level diagrams contributing to the transition amplitude is 1.

2-particle scattering amplitude in momentum space



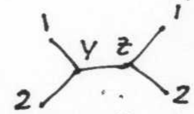
$$\langle f|i\rangle = \langle k_1' k_2' | k_1 k_2 \rangle$$

$$= i^{2+2} \int d^4x_1\, d^4x_2\, d^4x_1'\, d^4x_2'\; e^{i(k_1\cdot x_1 + k_2\cdot x_2 - k_1'\cdot x_1' - k_2'\cdot x_2')}$$
$$(-\partial_1^2+m^2)(-\partial_2^2+m^2)(-\partial_1'^2+m^2)(-\partial_2'^2+m^2)\langle 0|T\,\varphi(x_1)\varphi(x_2)\varphi(x_1')\varphi(x_2')|0\rangle_c$$

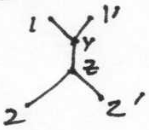
where

$$\langle 0|T\,\varphi(x_1)\varphi(x_2)\varphi(x_1')\varphi(x_2')|0\rangle_c = \delta_1\delta_2\delta_1'\delta_2'\, iW(J)\big|_{J=0}$$

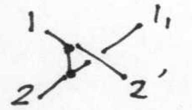
$$= (ig)^2 \frac{1}{i} \int d^4y\, d^4z\; \Delta(y-z)$$
$$\times \big[\Delta(x_1-y)\Delta(x_2-y)\Delta(x_1'-z)\Delta(x_2'-z)$$



$$+ \Delta(x_1-y)\Delta(x_1'-y)\Delta(x_2-z)\Delta(x_2'-z)$$



$$+ \Delta(x_1-y)\Delta(x_2'-y)\Delta(x_2-z)\Delta(x_1'-z)\big]$$



$$+ O(g^4)$$

Now apply the Klein-Gordon operators to remove all propogators associated with incoming and outgoing particles, e.g.

$$(-\partial_1^2+m^2)\Delta(x_1-y) = \delta^4(x_1-y), \quad \text{etc.}$$

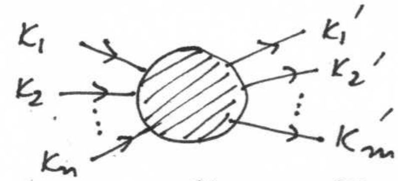
$$\langle k_1'k_2'|k_1k_2\rangle = (ig)^2\frac{1}{i}\int d^4y\, d^4z\;\Delta(y-z)\big[e^{i(k_1\cdot y + k_2\cdot y - k_1'\cdot z - k_2'\cdot z)}$$
$$+ e^{i(k_1\cdot y + k_2\cdot z - k_1'\cdot y - k_2'\cdot z)}$$
$$+ e^{i(k_1\cdot y + k_2\cdot z - k_1'\cdot z - k_2'\cdot y)}\big]$$
$$+ O(g^4)$$

Now using the integral representation, $\Delta(y-z) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot(y-z)}}{k^2+m^2-i\epsilon}$, we can evaluate the integrals overy y and z:

$$\langle k_1'k_2'|k_1k_2\rangle = ig^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2-i\epsilon}\big[(2\pi)^4\delta(k_1+k_2+k)(2\pi)^4\delta(k_1'+k_2'+k)$$
$$+(2\pi)^4\delta(k_1-k_1'+k)(2\pi)^4\delta(k_2'-k_2+k)$$
$$+(2\pi)^4\delta(k_1-k_2'+k)(2\pi)^4\delta(k_1'-k_2+k)\big] + O(g^4)$$

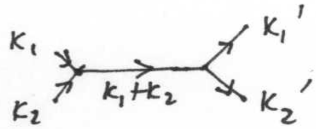
$$= ig^2(2\pi)^4\delta(k_1+k_2-k_1'-k_2')$$
$$\times\left[\frac{1}{(k_1+k_2)^2+m^2} + \frac{1}{(k_1-k_1')^2+m^2} + \frac{1}{(k_1-k_2')^2+m^2}\right] + O(g^4)$$

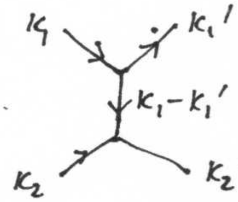
Scattering matrix element T

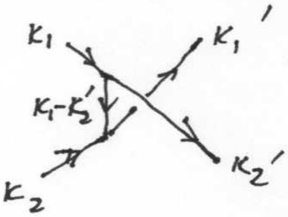


$$\langle k_1' k_2' \ldots k_m' | k_1 k_2 \ldots k_n \rangle \equiv (2\pi)^4 \delta^4\left(\sum_{i=1}^{m} k_i' - \sum_{i=1}^{m} k_i\right) iT$$

To lowest order in g in our φ^3 theory, the scattering matrix element is







$$iT = ig^2\left[\frac{1}{(k_1+k_2)^2+m^2} + \frac{1}{(k_1-k_1')^2+m^2} + \frac{1}{(k_1-k_2')^2+m^2}\right] + O(g^4)$$

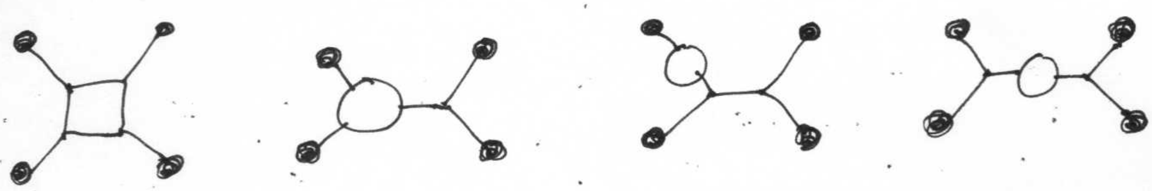
Introducing the Mandelstam variables,

$$s = -(k_1+k_2)^2 = -(k_1'+k_2')^2$$
$$t = -(k_1-k_1')^2 = -(k_2-k_2')^2$$
$$u = -(k_1-k_2')^2 = -(k_2-k_1')^2,$$

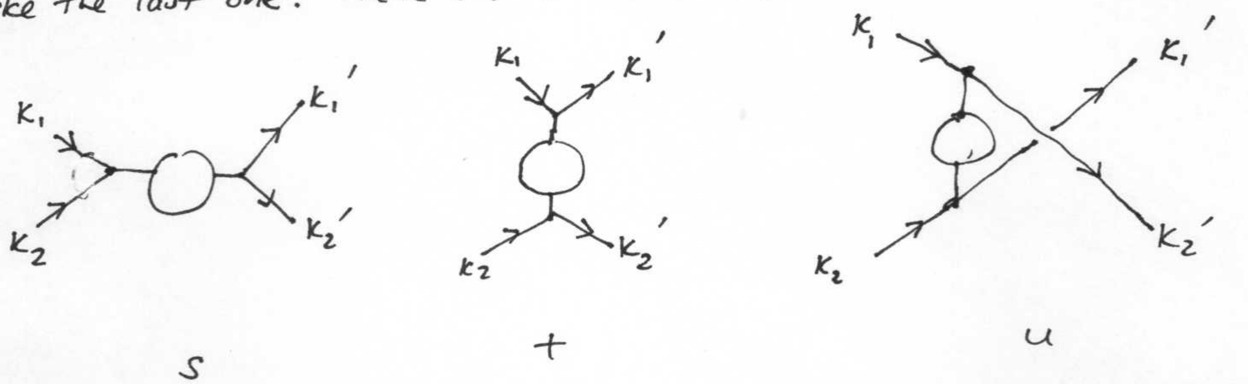
$$iT = ig^2\left[\frac{1}{-s+m^2} + \frac{1}{-t+m^2} + \frac{1}{-u+m^2}\right] + O(g^4)$$

Example : 2-particle scattering matrix T at order g^4 (ignore counterterms)

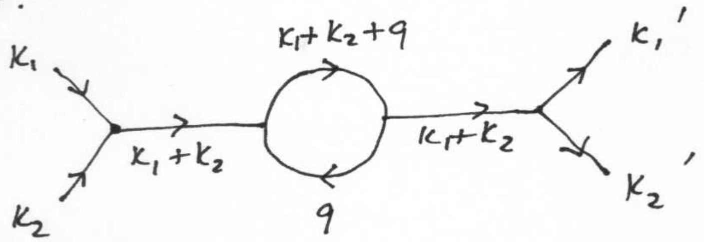
From Fig. 9.13, the connected diagrams with $E=4$ and $V=4$ are



Take the last one. There are 3 channels:



Take the s-channel and enforce momentum conservation at all vertices:



The amplitude for this process is

$$iT = \dots + \frac{(ig)^4}{2} \left(\frac{-i}{(k_1+k_2)^2+m^2} \right)^2 \int \frac{d^4q}{(2\pi)^4} \frac{-i}{(k_1+k_2+q)^2+m^2} \cdot \frac{-i}{q^2+m^2}$$