Back to Chernoff Bnd

- we use the MGF to capture the moments & apply Markov's inequality to the expectation.

- if X is a r.v., then for any $a \in \mathbb{R}$ we have,

$$P(X \geq a) = P(e^{sX} \geq e^{a}), \quad \text{for } s>0$$
$$P(X \leq a) = P(e^{sX} \geq e^{a}) \quad \text{for } s<0.$$

- note e^{sX} is a positive r.v. for all $s \in \mathbb{R}$.
- Markov's inequality $\rightarrow$ $P(X \geq a) = P(e^{sX} \geq e^{sa})$

$$\leq \frac{\mathbb{E}\, e^{sX}}{e^{sa}}, \quad s>0$$

$$\Rightarrow \boxed{\begin{array}{l} P(X \geq a) \leq e^{-sa} M_X(s), \quad s>0 \\ P(X \leq a) \leq e^{-sa} M_X(s), \quad s<0 \end{array}}.$$

- note: this holds for all s values $\rightarrow$ we can choose an optimal one freely.

$$\boxed{P(X \geq a) \leq \min_{s>0} e^{-sa} M_X(s)}$$

Example: Bernoulli (n,p) - n coin flips, 1 c prob p
0 c " $q = 1-p$.

$$M_X(s) = (pe^s + q)^n$$

- $P(X \geq a) \leq e^{-sa} (pe^s + q)^n$

$\rightarrow$ take deriv. of rhs & solve for optimal s:

$$e^{s_0} = \frac{aq}{p(n-a)}$$

- for $a = \frac{3}{4}n$, $p = 1/4$: $\boxed{P(X \geq \frac{3}{4}n) \leq 3^{-n/2}}$ - decays exponentially with n.