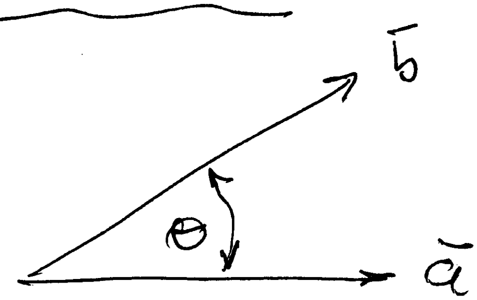
Dot product between vectors.



$$\bar{a} \cdot \bar{b} \overset{\text{def}}{=} a\, b \cos\theta$$

$$0 \leq \theta \leq \pi$$

$$-1 \leq \cos\theta \leq 1$$

$a = |\bar{a}|$; $b = |\bar{b}|$ (length of vectors).

Properties of dot product:

1) Commutative (symmetry):

$$\bar{a} \cdot \bar{b} = a\, b \cos\theta = b\, a \cos\theta = \bar{b} \cdot \bar{a}$$

2) Positivity (positive definite):

$$\bar{a} \cdot \bar{a} = a \cdot a = a^2 \geq 0$$

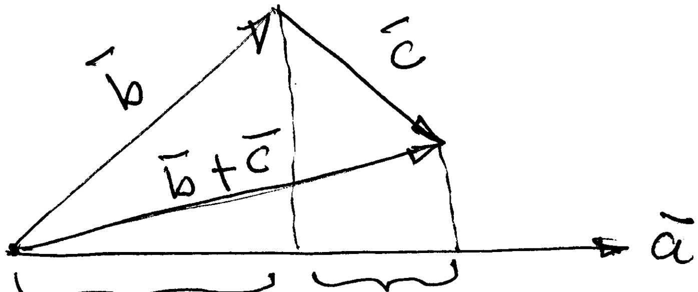
$\bar{a} \cdot \bar{a} = 0$ iff $\bar{a} = \bar{0}$.

3) Distributivity (linearity):

$\bar{a} \cdot (\bar{b} + \bar{c}) = \bar{a} \cdot \bar{b} + \bar{a} \cdot \bar{c}$ (2nd factor)

$(\bar{a} + \bar{b}) \cdot \bar{c} = \bar{a} \cdot \bar{c} + \bar{b} \cdot \bar{c}$ (1st factor).

Let's consider the following picture:



$B_a = b\cos(\bar{a},\bar{b})$ $\quad C_a = c\cos(\bar{a},\bar{c})$

$$(B+C)_a = |(\bar{b}+\bar{c})|\cos(\bar{b}+\bar{c},\bar{a}).$$

It shows that the projections of $\bar{b}$ & $\bar{c}$ onto $\bar{a}$, added, equals the projection of $\bar{b}+\bar{c}$ onto $\bar{a}$.

$$\begin{cases} \bar{a}\cdot\bar{b} = a\,\underbrace{b\cos(\bar{a},\bar{b})}_{B_a} = a\cdot B_a \\[2ex] \bar{a}\cdot\bar{c} = a\,\underbrace{c\cos(\bar{a},\bar{c})}_{C_a} = a\cdot C_a \\[2ex] \bar{a}\cdot(\bar{b}+\bar{c}) = a\,\underbrace{|\bar{b}+\bar{c}|\cos(\bar{a},\bar{b}+\bar{c})}_{(B+C)_a} = a\cdot(B+C)_a \end{cases}$$

From figure above:

$$(B+C)_a = B_a + C_a$$

Multiplying both members by a:

$$a(B+C)a = a(Ba+Ca) = aBa + aCa \therefore$$

$$\boxed{\bar{a}\cdot(\bar{b}+\bar{c}) = \bar{a}\cdot\bar{b} + \bar{a}\cdot\bar{c}}$$

Polar form of the dot product:

- $|\bar{a}+\bar{b}|^2 = (\bar{a}+\bar{b})\cdot(\bar{a}+\bar{b})$

$$= (\bar{a}+\bar{b})\cdot\bar{a} + (\bar{a}+\bar{b})\cdot\bar{b}$$

$$= \bar{a}\cdot\bar{a} + \bar{b}\cdot\bar{a} + \bar{a}\cdot\bar{b} + \bar{b}\cdot\bar{b}$$

$$= a^2 + 2\bar{a}\cdot\bar{b} + b^2 \qquad (a)$$

- $|\bar{a}-\bar{b}|^2 = (\bar{a}-\bar{b})\cdot(\bar{a}-\bar{b})$

$$= (\bar{a}-\bar{b})\cdot a - (\bar{a}-\bar{b})\bar{b}$$

$$= \bar{a}\cdot\bar{a} - \bar{b}\cdot\bar{a} - \bar{a}\cdot\bar{b} + \bar{b}\cdot\bar{b}$$

$$= a^2 - 2\bar{a}\cdot\bar{b} + b^2 \qquad (b)$$

Substracting (b) from (a):

$$\bar{a}\cdot\bar{b} = \frac{1}{4}\left[(\bar{a}+\bar{b})^2 - (\bar{a}-\bar{b})\right] \quad \left(\begin{matrix}\text{polar form} \\ \text{of } \bar{a}\cdot\bar{b}\end{matrix}\right)$$