THE INTERACTION PICTURE

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Motivated by Chapter 7 of Klauber's "Student Friendly Quantum Field Theory Vol. 1"

In the Schrödinger Picture (SP), operators remain constant in time, while the states evolve in time.

In the Heisenberg Picture (HP), operators evolve in time, while the states remain constant in time.

Regardless of which picture the states and operators are represented in, the expectation values of operators are the same. Mathematically, this means:

$$ {}_S\langle\varphi|\mathcal{O}^S|\varphi\rangle_S = {}_H\langle\varphi|\mathcal{O}^H|\varphi\rangle_H, $$

where the picture is denoted by the superscript on operators and the subscript on states. To see the connection between the two, we make use of the full-Hamiltonian unitary operators by first multiplying the operator on both sides by $1 = UU^\dagger$:

$$ {}_S\langle\varphi|\mathcal{O}^S|\varphi\rangle_S = {}_S\langle\varphi|\left(UU^\dagger\right)\mathcal{O}^S\left(UU^\dagger\right)|\varphi\rangle_S $$
$$ = \left({}_S\langle\varphi|U\right)\left(U^\dagger\mathcal{O}^S U\right)\left(U^\dagger|\varphi\rangle_S\right) $$

Looking at just the $U^\dagger|\varphi\rangle_S$ term, we call this the HP representation of the state, $|\varphi\rangle_H$. We can see how we go from a time evolving state of the SP to a constant-in-time state of the HP.

Some Preliminaries

$$ \underbrace{H}_{\text{Full Hamiltonian}} = \underbrace{H_0}_{\text{Free-Particle Part}} + \underbrace{H_I}_{\text{Interacting Part}} $$

Below are the time evolution operators that utilize the full Hamiltonian:

$$ UU^\dagger = 1 $$
$$ U = e^{-iHt} $$
$$ U^\dagger = e^{iHt} $$

We can also construct unitary operators from the free and interacting parts of the Hamiltonian:

$$ U_0U_0^\dagger = 1 \qquad U_IU_I^\dagger = 1 $$
$$ U_0 = e^{-iH_0t} \qquad U_I = e^{-iH_It} $$
$$ U_0^\dagger = e^{iH_0t} \qquad U_I^\dagger = e^{iH_It} $$

All of these operators are unitary and preserve the norm of the states on which they operate.

We'll use U_0, $U_0^\dagger$, U_I, and $U_I^\dagger$ later when we introduce the Interaction Picture (IP).

For example, if $|\varphi\rangle_S$ has the form of $Ae^{-i(Ht-kx)}$, then, using the definition of $U^\dagger$, we find:

$$ U^\dagger|\varphi\rangle_S = e^{iHt}\left(Ae^{-i(Ht-kx)}\right) = Ae^{ikx} = |\varphi\rangle_H, $$

resulting in a time-independent HP state. In summary, the transformations of states and operators from the SP to the HP are written as:

$$ U^\dagger\underbrace{|\varphi\rangle_S}_{\text{SP State}} = \underbrace{|\varphi\rangle_H}_{\text{HP State}} \qquad U^\dagger\underbrace{\mathcal{O}^S}_{\text{SP Operator}}U = \underbrace{\mathcal{O}^H}_{\text{HP Operator}} $$

Schrödinger Picture (and Heisenberg Picture) to Interaction Picture

We can get to the IP from the SP similarly to how we got to the HP, but this time we will operate on $|\varphi\rangle_S$ with $U_0^\dagger = e^{iH_0t}$, which only depends on the free part of the Hamiltonian. Doing so yields:

$$ U_0^\dagger|\varphi\rangle_S = e^{iH_0t}\left(Ae^{-i(Ht-kx)}\right) = e^{iH_0t}\left(Ae^{-i((H_0+H_I)t-kx)}\right) = Ae^{-i(H_It-kx)} = |\varphi\rangle_I, $$

resulting in an IP state that evolves based only on the interaction term of the Hamiltonian. The transformations of states and operators from the SP to the IP are thus written as:

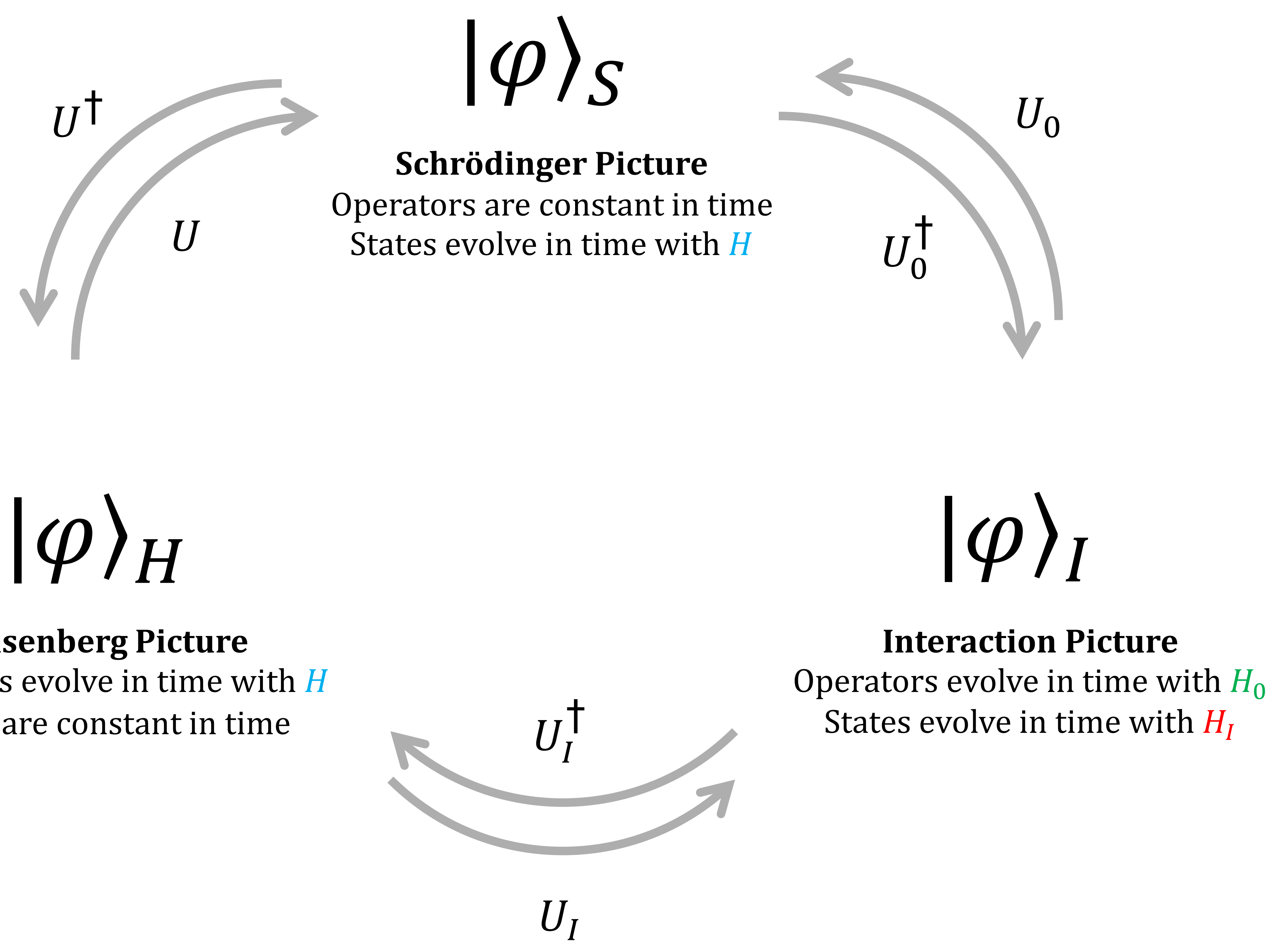
$$ U_0^\dagger\underbrace{|\varphi\rangle_S}_{\text{SP State}} = \underbrace{|\varphi\rangle_I}_{\text{IP State}} \qquad U_0^\dagger\underbrace{\mathcal{O}^S}_{\text{SP Operator}}U_0 = \underbrace{\mathcal{O}^I}_{\text{IP Operator}} $$

An important result of this is the operator $\mathcal{O}^I$ will be equivalent to the free-particle $\mathcal{O}^H$. This is because the transformation from an SP operator to an IP operator imparts time-dependence based on only the free-particle part of the Hamiltonian, H_0, which is equivalent to the SP-to-HP transformation for free particles (i.e., when $H = H_0$ and $H_I = 0$). Thus, in the IP, operators behave in accordance with the equations of motion for free fields.

A similar approach can be used to get from the HP to the IP via $U_I|\varphi\rangle_H = |\varphi\rangle_I$.

Summary

The diagram below shows how to alternate between the various pictures using the unitary operators defined above.



The unitary operators allow for you to "reallocate" the time dependence between the operators and the states.

The equation of motion for a state in the IP is $i\frac{d}{dt}|\varphi\rangle_I = H_I|\varphi\rangle_I$ because states in the IP only depend on H_I.

Thus, when we determine the form of the S operator in the IP, we use $i\frac{d}{dt}(S|\varphi\rangle_I) = H_I(S|\varphi\rangle_I)$. This is why the Dyson expansion of the S operator will only contain the interacting part of the Hamiltonian and not the free-particle part (which is reallocated to the operators).