MEASUREMENT IN QM:

Start with the postulates of QM (like axioms in math):

$\boxed{P1}$: STATES:
the state of an isolated quantum system is completely described by (normalized) unit vector in a Hilbert space, called a STATE SPACE, H.

- we can take it to be a finite dim^n complex vector space with standard inner product.
- compare to classical mechanics where a system of n particles $\bar{c}$ positions $\{\underline{r}_i\}_{i=1}^{n}$ is completely described by a state vector, $[\underline{r}_1, \dot{\underline{r}}_1, \underline{r}_2, \dot{\underline{r}}_2, \dots, \underline{r}_n, \dot{\underline{r}}_n]^T \in \mathbb{R}^{3n}$.

$\Rightarrow$ QM just abstracts this.

$\Rightarrow$ all future states can be calculated (predicted) from the current state (see P .).

$\boxed{P2}$: EVOLUTION:

A state can only change in time under the application of a unitary operator / transform. For $|\psi\rangle_i \in H$:

$$|\psi\rangle_i \longrightarrow U|\psi\rangle = |\psi\rangle_f .$$

- it is a separate issue on where this U comes from or how it is calculated.

$\boxed{P3}$: OBSERVABLE:

Any physical observable A (energy, position, spin, ...) is associated with a Hermitian operator $\hat{A} \in L(H)$ that acts on the Hilbert state space of the quantum system.

- e.g. energy $\longleftrightarrow \hat{H}$ (Hamitonian operator)

  pos^n $\longleftrightarrow \hat{x}$

  momentum $\longleftrightarrow \hat{p}$.

P4: MEASUREMENT OUTCOMES:

The only possible values of a measurement of an observable $\hat{A}$, are the real ($\because$ Hermitian) eigenvalues of the operator, $\hat{A}$.

- e.g., suppose the energy operator (Hamiltonian) of a system is somehow calculated to be, (as a matrix in a basis),

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

→ the only values of energy, E, that are possible for the system are the eigenvalues:

$$\lambda_1 = 1, \quad \lambda_2 = 0, \quad \lambda_3 = -1.$$

P5: POST-MEASUREMENT STATE:

Immediately after a measurement, regardless of the initial state of the system, if the value λ_i is observed the post-meas't state will be $|i\rangle$, the eigenvector of λ_i.

- I'm assuming non-degeneracy for simplicity.
- e.g., for the matrix A above, the eigenvector in the given basis are:

$$\underset{|+\rangle}{|1\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}}, \quad \underset{|0\rangle}{|2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}, \quad \underset{|-\rangle}{|3\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}}$$

→ if the energy was measured to be $0 = \lambda_2$, the post-measurement state of the system is,

$$|\psi\rangle_f = |2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

P6: PROBY OF MEASUREMENT OUTCOME:

The measurement values and possible post-meas't states DO NOT depend on the initial state of the quantum system.

BUT, the probability of a particular measurement outcome does:

The prob. of observing outcome λ_i is equal to:
$\boxed{P(\lambda_i) = |\langle i|\psi\rangle_i|^2}$ for an initial state $|\psi\rangle_i$.

- this is just the magnitude squared of the projection of the system state onto the i'th eigenvector.
- e.g., suppose the initial state is $|\psi\rangle_i = \frac{1}{2}\begin{pmatrix} 1-i \\ 0 \\ 1+i \end{pmatrix}$

$= \frac{1}{2}|+\rangle + \frac{i}{\sqrt{2}}|0\rangle + \frac{1}{2}|-\rangle$

$\Rightarrow |\langle +|\psi_i\rangle|^2 = \frac{1}{4}$ = prob. ($\lambda = +1$)

$|\langle 0|\psi_i\rangle|^2 = \frac{1}{2}$ = prob ($\lambda = 0$)

$|\langle -|\psi_i\rangle|^2 = \frac{1}{4}$ = prob ($\lambda = -1$).

$\Sigma = 1$

A1 Degenerate Eigenvalues Case

- if an eigenvalue a is degenerate $\bar{c}$ eigenvectors $\{|a\rangle_1, |a\rangle_2, \cdots, |a\rangle_k\}$, form the projector onto the eigenspace,

$$\Pi_a = |a\rangle_1\langle a|_1 + |a\rangle_2\langle a|_2 + \cdots + |a\rangle_k\langle a|_k.$$

Post-Measurement State:

- given a measured a the final state is

$$|\psi\rangle_f = \frac{P_a|\psi\rangle}{\sqrt{\langle\psi|P_a|\psi\rangle}}$$

Measurement Probability:

- given measurement outcome a its probability is:

$$p(a) = \langle\psi|P_a|\psi\rangle$$

Average of Observable Over Repeat Measurements:

- set up an experiment with a steady supply of some given fixed state $|\psi\rangle$.

- have an observable $A = \sum_m m\,|m\rangle\langle m|$ (eigenvalues/vectors)

$$= \sum_m m\,P_m \quad (\;\;\text{''}\;\;/\text{projections})$$

$$\boxed{\psi} \xrightarrow{|\psi\rangle,\,|\psi\rangle,\,|\psi\rangle} \boxed{\nearrow} \longrightarrow m.$$

- over many measurements, always starting $\bar{c}$ state $|\psi\rangle$, we get a dist$^{\underline{n}}$:

$$\left.\begin{array}{l} m_1 : p(m_1) \\ m_2 : p(m_2) \\ \quad\vdots \\ m_n : p(m_n). \end{array}\right\} \sim p(m).$$

- the average of the measurements is then:

$$\bar{m} = \sum_m m \cdot p(m)$$

$$= \sum_m m \cdot \langle\psi|P_m|\psi\rangle.$$

$$= \langle\psi|\sum_m m P_m|\psi\rangle$$

$$\boxed{\bar{m} = \langle\psi|A|\psi\rangle. \equiv \langle A\rangle}$$

- **notice:** the randomness here is purely quatum mechanical in nature.
  There is no uncertainty in the state, $|\psi\rangle$, being measured.
  Only the observable.

Mixed States and Density Matrices

- now extend this to where we have a distribution of possible states to measure.
- recall in Shannon's information theory we had an alphabet of symbols, $\mathcal{X} = \{X_1, X_2, \cdots, X_n\}$ $\bar{c}$ a probability dist^{n} over, $p_X(X = X_i) = p_i$.
- now treat a set of quantum states as an alphabet $\bar{c}$ a prob. dist^{n} over:

$$\{ |\psi_i\rangle, p_i \}.$$

$\ast \rightarrow$ this reflects a classical uncertainty in the state, distinct from uncertainty in a measured observable, A

Previous: Source $\xrightarrow{|\psi\rangle}$ $\boxed{A}$ $\longrightarrow$ measurements m

New: Source $\xrightarrow{\{|\psi_i\rangle, p_i\}}$ $\boxed{A}$ $\longrightarrow$ measurements m

$\rightarrow$ we now have a classical pdf, too, $p(|\psi_i\rangle) = p_i$.

What is the average observed value now? $\bar{m}$.

As before:

$$\bar{m} = \sum_m m \cdot p(m).$$

$\leftarrow$ note $p(m) = \sum_k p(m|\psi_k) \cdot p(\psi_k)$.

$$= \sum_m m \cdot \sum_i p(m|\psi_i) \cdot p(\psi_i).$$

$$= \sum_{m,i} m \cdot \langle \psi_i | m \rangle \langle m | \psi_i \rangle p(\psi_i).$$

$$= \sum_{m,i} m \langle m | \psi_i \rangle \langle \psi_i | m \rangle \, p(\psi_i).$$

$$= \sum_m m \langle m | \left(\sum_i p(\psi_i) |\psi_i\rangle\langle\psi_i| \right) | m \rangle$$

$$\rightarrow = \sum_m m \langle m | \rho | m \rangle$$

where:

$$\boxed{\rho \equiv \sum_i p(\psi_i) |\psi_i\rangle\langle\psi_i|}$$

$\uparrow$ Density Operator/Matrix.

Continuing, ...

$$\overline{m} = \sum_m m \langle m|\rho|m\rangle$$

$$= \sum_m m\cdot \langle m|\rho \underbrace{\sum_i |i\rangle\langle i|}_{I} |m\rangle$$

$$= \sum_{m,i} m\cdot \langle i|m\rangle\langle m|\rho|i\rangle$$

$$= \sum_i \langle i|\underbrace{\sum_m m\cdot|m\rangle\langle m|}_{A}\, \rho|i\rangle$$

$$= \sum_i \langle i|A\rho|i\rangle$$

$\boxed{\overline{m} = \operatorname{tr} A\rho}$ ⟵ very compact in terms of d.m., ρ. where $\rho = \sum_i p(\psi_i)|\psi_i\rangle\langle\psi_i|$

• ρ compactly describes both quantum and classical behaviour of measurements of the system.

Properties of Density Operator:

• matrix elts $\rho_{ij} = \langle i|\rho|j\rangle$.

• $\rho^2 = \rho$ for $\rho = |\psi\rangle\langle\psi|$ a pure state. (a projector).

$\rho^2 \neq \rho$ for rank $\rho > 1$ a mixed state.

• $\rho = \rho^\dagger$ is Hermitian.

• $\operatorname{tr}\rho = 1$ → Why?

$$\Rightarrow \operatorname{tr}\rho = \sum_i \langle i|\rho|i\rangle$$

$$= \sum_i \langle i| \sum_k p_k |\psi_k\rangle\langle\psi_k|\, |i\rangle$$

$$= \sum_k p_k \sum_i \langle i|\psi_k\rangle\langle\psi_k|i\rangle$$

$$= \sum_k p_k \sum_i \langle\psi_k|i\rangle\langle i|\psi_k\rangle$$

$$= \sum_k p_k \langle\psi_k|\cdot\sum_i |i\rangle\langle i|\cdot|\psi_k\rangle$$

$$= \sum_k p_k \langle\psi_k|\psi_k\rangle$$

$$= \sum_k p_k$$

$$\boxed{\operatorname{tr}\rho = 1}$$

- We have the overall average measurement value,

$$\bar{m} = \langle m \rangle = \mathrm{tr}\, \rho A .$$

- what about individual measurement outcomes? – i.e.,
  1. – prob. of measurement values (eigenvalues of A).
  2. – post-measurement states as mixtures.

① Probability of a Measurement

- recall for a given state $|\psi\rangle$, the prob. of measuring m is:

$$p(m) = \langle \psi | P_m | \psi \rangle = \mathrm{tr}(P_m |\psi\rangle\langle\psi|) .$$

- we now have a set of ψ's : $|\psi_k\rangle$, each c̄ prob. $p(\psi_k)$.

  ∴ $p(m) \longrightarrow p(m|\psi_k)$. instead

- So now, using $P(A) = \sum_B p(B)\, p(A|B)$.

$$p(m) = \sum_k p(\psi_k) \cdot p(m|\psi_k).$$

$$= \sum_k p(\psi_k) \cdot \mathrm{tr}(P_m |\psi_k\rangle\langle\psi_k|)$$

$$= \mathrm{tr}\, P_m \underbrace{\sum_k p(\psi_k) |\psi_k\rangle\langle\psi_k|}_{\rho}$$

$$\boxed{p(m) = \mathrm{tr}\, P_m \rho}$$

Compare: $\boxed{\bar{m} = \mathrm{tr}\, A\rho}$

- note for a pure state $\rho = |\psi\rangle\langle\psi|$ we recover:

$$p(m) = \mathrm{tr}\, P_m |\psi\rangle\langle\psi| = \langle\psi|P_m|\psi\rangle$$

$$\bar{m} = \mathrm{tr}\, A |\psi\rangle\langle\psi| = \langle\psi|A|\psi\rangle .$$

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② Post-Measurement State:

Note: if the eigenvalues are not degenerate, so $P_m = |m\rangle\langle m|$ for all m, then the post-measurement state given m was observed is just:

$$\rho_f = |m\rangle\langle m|$$

- this derivation handles the degenerate ($\text{rk}\, P_m > 1$ for some m) case.

⇒ this has a few pieces to put together...

① taking into account possible degeneracy, the state after observing measurement is the projection onto the m'th eigenspace:

$$|\psi\rangle_{f|m} = \frac{P_m|\psi\rangle}{\sqrt{\langle\psi|P_m|\psi\rangle}}$$

($= |m\rangle$ when $P_m = |m\rangle\langle m|$).

② where before we had the set of states & probabilities,

$$\{|\psi_k\rangle, p(\psi_k)\},$$

we now condition on the observed m:

$$\{|\psi_k\rangle_{f|m}, p(\psi_k|m)\}$$

← aren't just the $p(\psi_k)$'s given? No dependence on m? ⇒ we will calc. this from Bayes Rule. See below.

③ To get the $p(\psi_k|m)$ recall identity:

$$P(A,B) = P(A|B) \cdot P(B) = P(B|A) \cdot P(A).$$

→ so we can write:

$$\underbrace{P(\psi_k|m) \cdot p(m)}_{} = p(m|\psi_k) \cdot p(\psi_k)$$

$p(m)$: $\text{tr}\,\rho P_m$

$p(m|\psi_k)$: $\langle\psi_k|P_m|\psi_k\rangle$

$p(\psi_k)$: given from ensemble provided

- So we can rewrite:

$$\boxed{P(\psi_k|m) = \frac{\langle\psi_k|P_m|\psi_k\rangle \cdot p(\psi_k)}{\mathrm{tr}\rho P_m}}$$

④ So, the post-measurement state, conditioned on measurement outcome m, is:

$$\rho_f(m) = \sum_k p(\psi_k|m) \cdot |\psi_k\rangle_f \langle\psi_k|_f$$

$$= \sum_k \frac{\cancel{\langle\psi_k|P_m|\psi_k\rangle}\, p(\psi_k)}{\mathrm{tr}\rho P_m} \cdot \frac{P_m|\psi_k\rangle\langle\psi_k|P_m}{\cancel{\langle\psi_k|P_m|\psi_k\rangle}}$$

$$= \frac{1}{\mathrm{tr}\rho P_m} P_m\left(\sum_k p(\psi_k)|\psi_k\rangle\langle\psi_k|\right)P_m$$

$$\boxed{\rho_f(m) = \frac{P_m\rho P_m}{\mathrm{tr}\rho P_m}}$$

where: • ρ is the initial mixed state
• P_m is the projector onto the m'th eigen-space.