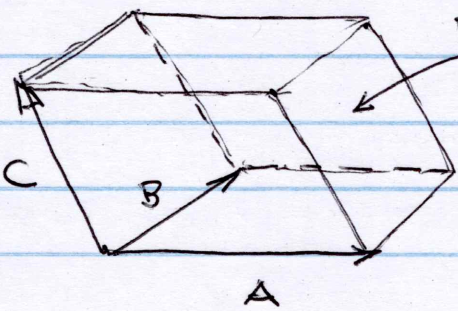
Modern Classical Physics
(Thorne & Blandford)

Exercise 1.7: Properties of the Levi-Civita Tensor

(i) The volume vanishes unless the legs are linearly independent.

Given 3 vectors, A, B, C, $\epsilon(A, B, C)$ = Volume of parallelepiped formed by them.



3- Volume = $\epsilon(A, B, C)$

3-Volume = $\epsilon(A, B, C) = \epsilon_{ijk} A_i B_j C_k$

$$= \det \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

The determinant is different from zero, iff the rows (and also the vectors) are linerly independent.

(ii) Once the volume has been specified for one parallelepiped of linearly independent legs, it is thereby for all parallelepipeds.

(iii) We require only one number plus antisymmetry to determine ϵ -

Antisymmetry: $\epsilon(A, B, C) = -\epsilon(B, A, C)$

(e_1, e_2, e_3): orthonormal basis

$$g(e_i, e_j) = e_i \cdot e_j = \delta_{ij} \begin{cases} 1; & i=j \\ 0; & i \neq j \end{cases}$$

Then: $\epsilon(e_i, e_j, e_k) = \epsilon_{ijk} =$

- $= 0$, if the parallelepipe contains one or more repeated vetors.
- $= 1$, if (e_i, e_j, e_k) forms a right-handed parallelepiped
- $= -1$; if (e_i, e_j, e_k) forms a left-handed parallelepipe.

So; $\epsilon(e_1, e_2, e_3) = \epsilon_{123} = 1$

from antisymmetry: $\epsilon(e_2, e_1, e_3) = \epsilon_{213} = -1$ (one permutation)

$\epsilon(e_2, e_3, e_1) = \epsilon_{231} = 1$ (two permutations)

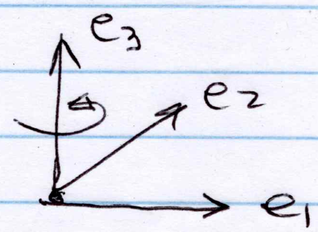
$\epsilon(e_3, e_2, e_1) = \epsilon_{321} = -1$ (three perm.).

In general,

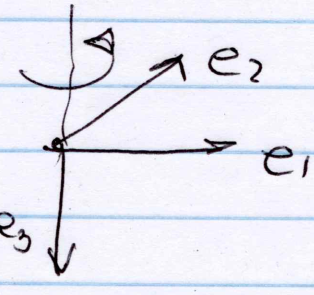
$$\varepsilon_{123} = +1$$

$$\varepsilon_{ijk} = \begin{cases} +1, & \text{if } i,j,k \text{ is an even permutation of } 1,2,3. \\ -1; & \text{if } i,j,k \text{ is an odd permutation of } 1,2,3 \\ 0, & \text{if } i,j,k \text{ are not all different} \end{cases}$$

Then, (IV), ε is fully determined by its antisymmetry, compatibility with the metric, and a single sign (positive or negative volume).



(+ volume)



(− volume)

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