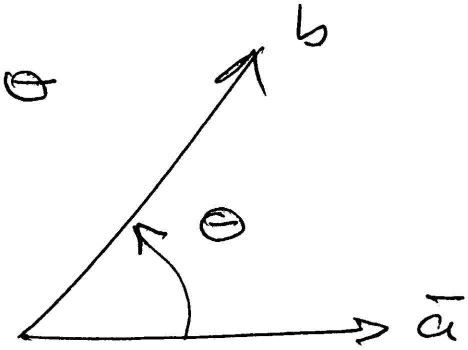
Dot product between vectors.

Def: $\bar{a}\cdot\bar{b} = a\, b \cos\theta$



$$\begin{cases} 0 \le \theta \le \pi \\ 0 \le \cos\theta \le 1 \end{cases}$$

$a = |\bar{a}|$; $b = |\bar{b}|$

Properties:

1) Commutativeness (symmetry)

$\bar{a}\cdot\bar{b} = a b \cos\theta = b a \cos\theta = \bar{b}\cdot\bar{a}$

2) Positivity: (positive definiteness)

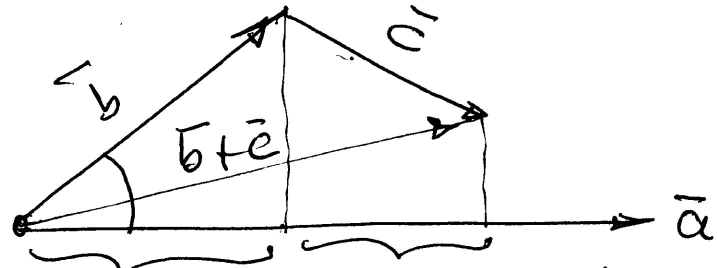
$\bar{a}\cdot\bar{a} = a\cdot a = a^2 \geq 0$

$\bar{a}\cdot\bar{a} = 0$ iff $a = 0$

3) Distributivity: (linearity).

$\bar{a}\cdot(\bar{b}+\bar{c}) = \bar{a}\cdot\bar{b} + \bar{a}\cdot\bar{c}$ (second factor)

$(\bar{a}+\bar{b})\cdot\bar{c} = \bar{a}\cdot\bar{c} + \bar{b}\cdot\bar{c}$ (first factor).



$Ba = b\cos(\bar a,\bar b)$ $\quad$ $Ca = c\cos(a,c)$

$(B+C)_a = |b+c|\cos(b+c,a)$

$$\bar a.\bar b = a.b\underbrace{\cos(\bar a,\bar b)}_{Ba} = a\,Ba.$$

$$\bar a.\bar c = a.\underbrace{c\cos(\bar a,\bar b)}_{Ca} = a.Ca$$

$$\bar a.(\bar b+\bar c) = a\underbrace{|\bar b+\bar c|\cos(\bar a,\bar b+\bar c)}_{(B+C)a} =$$

$$= a(B+C)_a$$

From figure above:

$(B+C)a = Ba + Ca$; $\times |\bar a|$ both sides

$a(B+C)a = a\,Ba + Ca \Rightarrow$

$$\boxed{\bar a.(\bar b+\bar c) = \bar a.\bar b + \bar a.\bar c}$$

Polar expression of dot product

- $|\bar{a}+\bar{b}|^2 = (\bar{a}+\bar{b})\cdot(\bar{a}+\bar{b}) =$

$= (\bar{a}+\bar{b})\cdot\bar{a} + (\bar{a}+\bar{b})\cdot\bar{b}$

$= (\bar{a}\cdot\bar{a}) + (\bar{b}\cdot\bar{a}) + (\bar{a}\cdot\bar{b}) + (\bar{b}\cdot\bar{b})$

$= |\bar{a}|^2 + 2\,\bar{a}\cdot\bar{b} + |\bar{b}|^2$ (a)

- $|\bar{a}-\bar{b}|^2 = (\bar{a}-\bar{b})\cdot(\bar{a}-\bar{b}) =$

$= (\bar{a}-\bar{b})\cdot\bar{a} - (\bar{a}-\bar{b})\,\bar{b}$

$= (\bar{a}\cdot\bar{a}) - (\bar{b}\cdot\bar{a}) - (\bar{a}\cdot\bar{b}) + (\bar{b}\cdot\bar{b})$

$= |a|^2 - 2\,\bar{a}\cdot\bar{b} + |b|^2$ (b)

Sustracting (b) from (a):

$|\bar{a}+\bar{b}|^2 - |\bar{a}-\bar{b}|^2 = 4\,\bar{a}\cdot\bar{b} \rightarrow$

$$\boxed{\bar{a}\cdot\bar{b} = \frac{1}{4}\left[(\bar{a}+\bar{b})^2 - (\bar{a}-\bar{b})^2\right]}$$