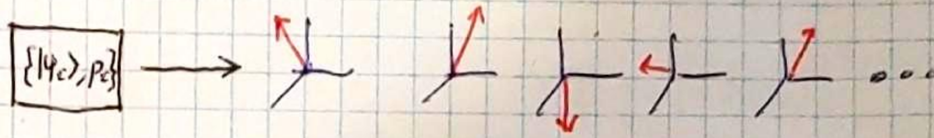
Summary - Recap.

- a **mixed state** is a enemble of quantum states with a prob. dist^{n} over them:

$$\{|\psi_i\rangle, p_i\} \quad \rightarrow |\psi_i\rangle\text{'s are arbitrary and not an ONB in general.}$$

- like

$$\left\{\begin{matrix} |\psi_1\rangle, p_1 \\ |\psi_2\rangle, p_2 \\ |\psi_3\rangle, p_3 \\ \vdots \\ |\psi_n\rangle, p_n \end{matrix}\right\}$$



- **Def^n:** Density matrix of ensemble

$$\rho = \sum_{i=1}^{n} p_i |\psi_i\rangle\langle\psi_i|$$

- **Measurements:** Given an observable's Hermitian operator, A

$$\rightarrow A = \sum_m m |m\rangle\langle m|$$

, • m = eigenvalue/measurement outcome.

• $\{|m\rangle\}$ an ONB

- define projectors $P_m = |m\rangle\langle m|$

1/ Prob. of measurement outcome, m:

$$p(m) = \mathrm{tr}\, P_m \rho$$

2/ Post-measurement state conditioned on/given measurement m:

$$\rho'(m) = \frac{\mathrm{tr}\, P_m \rho P_m}{\mathrm{tr}\, P_m \rho} = \frac{\mathrm{tr}\, P_m \rho P_m}{p(m)}.$$

3/ Average of measurements over the ensemble, $\overline{m}$:

$$\overline{m} = \mathrm{tr}\, A\rho$$

Properties of Density Matrix / Operator:

① Hermitian, $\rho^\dagger = \rho$:

$$\rho^\dagger = \left(\sum_i p_i |\psi_i\rangle\langle\psi_i|\right)^\dagger$$

$$= \sum_i p_i \left(|\psi_i\rangle\langle\psi_i|\right)^\dagger$$

$$= \sum_i p_i |\psi_i\rangle\langle\psi_i|^\dagger$$

$$= \rho \checkmark$$

② Positive semi-definite, $\rho \geq 0$:

For any $|\phi\rangle$,

$$\langle\phi|\rho|\phi\rangle = \langle\phi|\left(\sum_i p_i |\psi_i\rangle\langle\psi_i|\right)\cdot|\phi\rangle$$

$$= \sum_i p_i \langle\phi|\psi_i\rangle\langle\psi_i|\phi\rangle$$

$$= \sum_i p_i \left|\langle\phi|\psi_i\rangle\right|^2$$

- but $p_i \geq 0$ & $|\langle\phi|\psi_i\rangle|^2 \geq 0$.
- $\therefore$ the sum ≥ 0

$\rightarrow \langle\phi|\rho|\phi\rangle \geq 0 \;\; \forall|\phi\rangle$ & ρ psd $\checkmark$

③ $\operatorname{tr}\rho = 1$:

- use ensemble average trick.

$\rightarrow$ have $\bar{m} = \operatorname{tr} A\rho$ for eigenvalues m of A.

- take $A = I \rightarrow$ all eigenvalues $= 1$

$\therefore m = \;\; \forall m \rightarrow \bar{m} = 1$

$\therefore 1 = \operatorname{tr} I\rho.$

$\Rightarrow \operatorname{tr}\rho = 1 \checkmark$

④ $\rho^2 = \rho$ iff ρ is pure, i.e., $\rho = |\psi\rangle\langle\psi|$:

iff $\Rightarrow$: $\rho^2 = \rho \rightarrow \rho = |\psi\rangle\langle\psi|$:

If $\rho^2 = \rho$ then $\sum_i \lambda_i^2 |i\rangle\langle i| = \sum_i \lambda_i |i\rangle\langle i|$

$\langle m|\rho^2|m\rangle = \langle m|\rho|m\rangle$:

$$\sum_i \lambda_i^2 \langle m|i\rangle\langle i|m\rangle = \sum_i \lambda_i \langle m|i\rangle\langle i|m\rangle$$

$$\sum_i \lambda_i^2 \delta_{im} = \sum_i \lambda_i \delta_{im}$$

$\Rightarrow \lambda_m^2 = \lambda_m \;\; \forall m.$

But: $\operatorname{tr}\rho = 1 \rightarrow \sum_i \lambda_i = 1$

$\rho \geq 0 \rightarrow \lambda_i \geq 0$

$\Rightarrow \therefore 0 \leq \lambda_i \leq 1$

So, $\lambda_m^2 = \lambda_m$ means $\lambda_m = 0, 1.$

$\because \sum_m \lambda_m = 1$; then only one $\lambda_m = 1$ & the rest equal 0.

$\therefore \rho = |m\rangle\langle m|$ for that $\lambda_m = 1.$

④ $\rho^2 = \rho$ iff $\rho = |\psi\rangle\langle\psi|$, cont'd.

iff: $\Leftarrow$: $\rho = |\psi\rangle\langle\psi| \rightarrow \rho^2 = \rho$.

- just sub. into lhs:

$$
\begin{aligned}
\rho^2 &= \rho \cdot \rho \\
&= |\psi\rangle\langle\psi|\psi\rangle\langle\psi| \\
&= |\psi\rangle \cdot 1 \cdot \langle\psi| \\
&= |\psi\rangle\langle\psi| \\
&= \rho \checkmark .
\end{aligned}
$$

Postulates of QM in Density Operators

Easy to show:

① State of a system is described by a density operator, ρ.

② A state evolves by a unitary transformation U, s.t.

$$\rho \xrightarrow{U} U\rho U^\dagger$$

③ Under the projections P_m of an observable, given a measured observation, m:

i.) $p(m) = \mathrm{tr}\, P_m \rho$ (prob. of observation).

ii.) $\rho'(m) = \dfrac{P_m \rho P_m^\dagger}{\mathrm{tr}\, P_m \rho}$ (conditional post measurement state).

④ A composite system of quantum systems is the tensor product of their state spaces.

If n systems are individually prepared in states ρ_i, $i = 1 \dots n$, the joint state of the composite system is:

$$\rho = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_n .$$

Example:

Suppose we have source generating a mixture of the eigenvectors of an observable.

$$\rho = \sum_i p_i |i\rangle\langle i|$$

If we do a measurement and conditional outcome m, we would expect to have a post-measurement state $|m\rangle$ occuring with prob. p_m.

Check:

$$\rho = \sum_i p_i |i\rangle\langle i| \quad \text{and} \quad P_m = |m\rangle\langle m|$$

- prob. of measurement:

$$\begin{aligned} p(m) &= \operatorname{tr} \rho P_m \\ &= \operatorname{tr} \sum_i p_i |i\rangle \underbrace{\langle i| \cdot |m\rangle}_{\delta_{im}} \langle m| \\ &= \operatorname{tr}(p_m |m\rangle\langle m|) \\ &= p_m \langle m|m\rangle \\ &= p_m \quad \text{as expected.} \end{aligned}$$

- post-meas't state:

$$\rho'(m) = \frac{P_m \rho P_m}{p(m)} = \frac{P_m \rho P_m}{p_m}$$

- but $P_m \rho P_m = |m\rangle\langle m| \cdot \sum_i p_i |i\rangle\langle i| \cdot |m\rangle\langle m|$

$$\begin{aligned} &= |m\rangle \cdot \sum_i p_i \underbrace{\langle m|i\rangle\langle i|m\rangle}_{\delta_{im}} \cdot \langle m| \\ &= |m\rangle p_m \langle m| \\ &= p_m |m\rangle\langle m| \end{aligned}$$

$\therefore \rho'(m) = \dfrac{p_m |m\rangle\langle m|}{p_m} = |m\rangle\langle m|$ as expected.

Non-Uniqueness of Decompositions

- many different ensembles of $\{|\psi_i\rangle, p_i\}$ can give rise to the same density matrix, $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$.
- example, regardless of the actual mixture of p_i's & $|\psi_i\rangle$'s used to generate a given ρ, you can always do an eigendecomposition,

$$\rho = \sum_i \lambda_i |i\rangle\langle i|,$$

So the eigenvalues are the prob's & the ONB $\{|i\rangle\}$ a new ensemble.

Example:

Let $\rho = \frac{I}{2}$ on $\mathbb{C}^2$.

① $\rho = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) = \frac{1}{2}\begin{pmatrix} 1 & \\ & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & \\ & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ ✓.

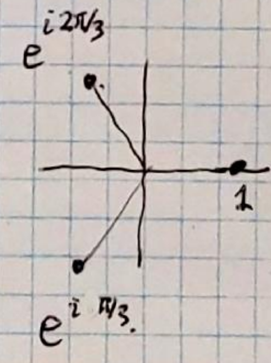
② Let $|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$, $|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$.

Also have: $\frac{1}{2}\left(|+\rangle\langle +| + |-\rangle\langle -|\right) = \frac{I}{2}$.

③ Let $|\psi_0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$

$$|\psi_1\rangle = \frac{|0\rangle + e^{i2\pi/3}|1\rangle}{\sqrt{2}}$$

$$|\psi_2\rangle = \frac{|0\rangle + e^{i4\pi/3}|1\rangle}{\sqrt{2}}$$



The $\rho = \frac{1}{3}\sum_{k=0}^{2} |\psi_k\rangle\langle\psi_k| = \frac{I}{2}$.

$\because$ off diagonal entries cancel: $1 + e^{i2\pi/3} + e^{i4\pi/3} = 0$.

Claim: Given a density matrix ρ, in general there are more than one ensemble of $\{|\psi_i\rangle, p_i\}$ that may generate it.

Proof:

- Given a d.m. ρ form its unique eigendecomposition:

$$\rho = \sum_{i=1}^{r} \lambda_i |i\rangle\langle i| \qquad \bar{c}\ \lambda_i > 0 \text{ and } \sum_i \lambda_i = 1.$$

- define the renormalized kets,

$$|x_i\rangle = \sqrt{\lambda_i}\,|i\rangle \qquad \text{(still mutually orthogonal, not normalized)}$$

- we can write,

$$\rho = \sum_i |x_i\rangle\langle x_i| \qquad (*)$$

- now take any unitary $U \in \mathcal{U}(r) \sim r \times r$. and write.

$$|x_j'\rangle = \sum_{i=1}^{r} U_{ji} |x_i\rangle, \quad j = 1, \ldots, r$$

- now look at:

$$\sum_j |x_j'\rangle\langle x_j'| = \sum_j \sum_{lm} U_{jl} |x_l\rangle\langle x_m| U_{jm}^*$$

$$= \sum_{lm} |x_l\rangle\langle x_m| \underbrace{\sum_j U_{jl} U_{jm}^*}_{\delta_{ml}} \longrightarrow \sum_j U_{jl} U_{jm}^* = \sum_j [U^\dagger]_{mj} [U]_{jl} = [U^\dagger U]_{ml} = I_{ml} = \delta_{ml}.$$

$$= \sum_l |x_l\rangle\langle x_l|$$

$$= \rho \quad \text{(same } \rho \text{ in } (*)\text{!)}.$$

- now let:

$$p_j = \| |x_j'\rangle \|^2 = \langle x_j' | x_j' \rangle$$

$$\phi_j = \frac{|x_j'\rangle}{\sqrt{p_j}}$$

$$\Rightarrow \boxed{\rho = \sum_j p_j |\phi_j\rangle\langle \phi_j|}$$

← p_j & $|\phi_j\rangle$ depend on U continuously to get an infinite number of decompositions!

Non-orthogonal mixtures:

- note the generated $|x_i'\rangle$'s from the $|x_i'\rangle = \sqrt{\lambda_i}\,|i\rangle$ are not generally orthogonal.

Example:

- Let $\rho = \lambda_0 |0\rangle\langle 0| + \lambda_1 |1\rangle\langle 1| = \begin{bmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{bmatrix}$ and $\lambda_0 \neq \lambda_1$
- take $U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
- $\Rightarrow$ $|x_0'\rangle = \frac{1}{\sqrt{2}}\Big(\underbrace{\sqrt{\lambda_0}\,|0\rangle}_{|x_0\rangle} + \underbrace{\sqrt{\lambda_1}\,|1\rangle}_{|x_1\rangle}\Big)$

$$|x_1'\rangle = \frac{1}{\sqrt{2}}\Big(\sqrt{\lambda_0}\,|0\rangle - \sqrt{\lambda_1}\,|1\rangle\Big).$$

$$\langle x_0'|x_1'\rangle = \frac{1}{2}\Big(\lambda_0\langle 0|0\rangle + \underbrace{\sim}_{\text{c.t.'s}} \lambda_1\langle 1|1\rangle\Big)$$

$$= \frac{1}{2}(\lambda_0 - \lambda_1)$$

$\neq 0$ in general.

- so $|\phi_0\rangle$ & $|\phi_1\rangle$ are not orthogonal in general.

Why?

From the eigen-decomp^n $\rho = \sum_i \lambda_i |i\rangle\langle i|$, we defined the rescaled,

$|x_i\rangle = \sqrt{\lambda_i}\,|i\rangle$, then

$|x_j'\rangle = \sum_{i=1}^{r} u_{ji}\,|x_i\rangle$

Substituting and expanding, using $\langle x_i|x_j\rangle = \lambda_i\delta_{ij}$, we have

$\langle x_i'|x_j'\rangle = [U\Lambda U^\dagger]_{ij}$ as matrix elements where $\Lambda = \mathrm{diag}(\lambda_i)$.

$\rightarrow$ Note: If all eigenvalues equal, $\Lambda \propto I$ & $U\Lambda U^\dagger \propto I \Rightarrow \langle x_i'|x_j'\rangle \propto \delta_{ij}$.

Mixed State Arising From Forgetfull Measurement

- idea of a mixed state coming from an ensemble of different pure states $\bar{c}$ a prob. distⁿ seems forced.
- can one arise naturally from a pure state?
  $\Rightarrow$ YES - by performing the measurement but ignoring the outcome — non-selective measurement (details to follow...)

Recall: Measurements as Projections:

① $p(m) = \mathrm{tr}\, P_m \rho$ (prob.)

② $\rho'(m) = \dfrac{P_m \rho P_m}{p(m)}$ (post-meas. state)

$\Big\}$ Conditioned on measurement outcome 'm'.

Setup: Two measurements in series.

$$\rho \longrightarrow \boxed{\mathcal{P}} \xrightarrow{\rho'} \boxed{Q} \longrightarrow \rho_{out}.$$

$\mathcal{P}$: has projectors $\{P_m\}$ for some observable $\bar{c}$ outcomes m.
Q: has projectors $\{Q_x\}$ " " " " " x.

Ⓐ ① Initial input state, ρ.

② First measurement $\mathcal{P}$:

- has meas. outcome m $\bar{c}$ prob.: $\boxed{p(m) = \mathrm{tr}\, P_m \rho}$
- state conditioned on outcome m: $\boxed{\rho'(m) = \dfrac{P_m \rho P_m}{p(m)}}$

Ⓑ ③ Take a second measurement Q:

- conditioned on the first outcome, m, the prob. of measuring x is:

$$p(x|m) = \mathrm{tr}\, Q_x \rho'(m) = \mathrm{tr}\, \frac{Q_x P_m \rho P_m}{p(m)}.$$

4 The joint prob. of measuring (m,x) is:

$$p(m,x) = p(m)\cdot p(x|m)$$
$$= p(m)\cdot \frac{\operatorname{tr} Q_x P_m \rho P_m}{p(m)}.$$

$$\boxed{p(m,x) = \operatorname{tr} Q_x P_m \rho P_m}$$

5 If the first outcome of $\mathcal{P}$ is not revealed the prob. of the later outcome x is just the marginal of $p(m,x)$ over m:

$$p(x) = \sum_m p(m,x)$$

$$= \sum_m \operatorname{tr} Q_x P_m \rho P_m$$

$$\boxed{p(x) = \operatorname{tr} Q_x \sum_m P_m \rho P_m}$$

6 But we know from the measurement postulate that there must exist a ρ' for the input of $\mathcal{Q}$ such that:

$$p(x) = \operatorname{tr} Q \rho'.$$

$$\rho \to \boxed{\mathcal{P}} \xrightarrow{\rho'} \boxed{\mathcal{Q}} \to \rho_{out}$$

7 So we require:

$$\operatorname{tr} Q\rho' = \operatorname{tr} Q_x \sum_m P_m \rho P_m \qquad (*)$$

and must solve for ρ'

Note: this (*) must hold for all possible projectors P_m, Q_x as they were arbitrary to begin with.

8 Claim: The unique well defined sol^{n} to (*) is:

$$\boxed{\rho' = \sum_m P_m \rho P_m}$$

- Seems true by comparison (it is), but how to show it's the only sol^{n}?

(9) Restate:

$\boxed{\text{If } \operatorname{tr} QX = \operatorname{tr} QY \text{ for all projectors } Q\text{, then } X = Y}$

- so for a given Y, say, the only sol^{n} for X is precisely $X = Y$.

Proof:

Restate as: $\operatorname{tr} Q(X-Y) = 0$

Let $A = X - Y$, so we are saying,

- if $\operatorname{tr} QA = 0$ $\forall$ projectors Q, then $A = 0$.

- take the case $Q = |q\rangle\langle q|$ for some state $|q\rangle$
  ($\rightarrow$ it's general, though, for rank > 1 & non-orthogonal)

- so, $\operatorname{tr} QA = 0$
  $\rightarrow \operatorname{tr} |q\rangle\langle q| A = 0$
  $\langle q|A|q\rangle = 0 \quad \forall\ |q\rangle$.

- use the polarization identity,
  Given a quadratic form function,
  $$f(\omega) = \langle \omega | A | \omega \rangle$$
  known for all $|\omega\rangle$ for some Hermitian A, then this fixes all off-diagonal elements by,
  $$\langle u|A|v\rangle = \frac{1}{4}\left[f(u+v) - f(u-v) - i f(u+iv) + i f(u-iv)\right]$$
  $$(\text{OR}) = \frac{1}{4}\left[f(u+v) - f(u-v)\right]$$ for real symmetric A, u, v.

- seems surprising $\because$ it implies the diagonal matrix elts of A determine & fix the off diagonal elts.

  $\Rightarrow$ but knowing the diagonal elts of an operator's matrix rep^{n} holds only for the vectors $|e_i\rangle_{i=1}^{n}$. $\Leftarrow$ n constraints
  Not for all $|\omega\rangle$ in $f(\omega) = \langle \omega | A | \omega\rangle$. $\Leftarrow$ ∞ number of constraints

- so we have $\mathrm{tr}\, QA = 0$, $Q = |q\rangle\langle q|$

$$\rightarrow \langle q|A|q\rangle \equiv f(q) = 0 \quad \text{for \underline{all} } |q\rangle.$$

$$\therefore \langle u|A|v\rangle = \frac{1}{4}\Big[\underbrace{f(u+v)}_{=0} - \underbrace{f(u-v)}_{=0} + \cdots\Big]$$

$$= 0 \quad \forall\, u, v.$$

- and in any particular basis,

$$\langle e_i|A|e_j\rangle = A_{ij} = 0$$

- so, $\boxed{A=0}$ identically.

- $\therefore \mathrm{tr}\,(Q(X-Y)) = 0 \quad \forall$ proj. Q:

$$\Rightarrow \boxed{X=Y} \text{ identically.}$$

(10) Back to,

$$\mathrm{tr}\, Q_x \rho' = \mathrm{tr}\, Q_x \sum_m P_m \rho P_m \qquad \forall \text{ proj. } Q.$$

$\Rightarrow \boxed{\rho' = \sum_m P_m \rho P_m}$ is the single unique solution.

(11) This is the same result as averaging over possible outcomes:

$$p(m) = \mathrm{tr}\, P_m \rho$$

$$\rho'(m) = \frac{P_m \rho P_m}{p(m)}$$

$$\Rightarrow \rho' = \sum_m p(m)\, \rho'(m)$$

$$= \sum_m p(m) \frac{P_m \rho P_m}{\cancel{p(m)}}$$

$$\rho' = \sum_m P_m \rho P_m$$

Think of it as:

- we know a measurement was made
- the proj. measurement carves the state into branches
- we only know one of the branches happened, not which one.
- best description is the classical average of possible states, weighted by their prob. p(m)