# Talk 2: Equivalence Relations and Quotienting Vector Spaces

Lecture Notes

September 11, 2025

## 1 Equivalence Relations

**Definition 1.1** (Equivalence Relation)**.** Let $X$ be a set. An **equivalence relation** on $X$ is a binary relation $\sim$ on $X$ that satisfies:

(i) **Reflexivity:** $x \sim x$ for all $x \in X$

(ii) **Symmetry:** If $x \sim y$, then $y \sim x$

(iii) **Transitivity:** If $x \sim y$ and $y \sim z$, then $x \sim z$

**Definition 1.2** (Equivalence Class)**.** Given an equivalence relation $\sim$ on $X$ and an element $x \in X$, the **equivalence class** of $x$ is:

$$[x] = \{y \in X : y \sim x\}$$

**Definition 1.3** (Quotient Set)**.** The **quotient set** (or **quotient space**) of $X$ by $\sim$ is:

$$X/\sim = \{[x] : x \in X\}$$

the set of all equivalence classes.

### Examples

**Example 1.4.** Let $X = \mathbb{Z}$ and define $a \sim b$ if $a - b$ is even. This is an equivalence relation:

- Reflexive: $a - a = 0$ is even
- Symmetric: If $a - b$ is even, then $b - a = -(a - b)$ is even
- Transitive: If $a - b$ and $b - c$ are even, then $a - c = (a - b) + (b - c)$ is even

The equivalence classes are $[0] = \{\dots, -4, -2, 0, 2, 4, \dots\}$ (even integers) and $[1] = \{\dots, -3, -1, 1, 3, 5, \dots\}$ (odd integers). Thus $\mathbb{Z}/\sim = \{[0], [1]\} \cong \mathbb{Z}_2$.

**Example 1.5** (Modular Arithmetic)**.** For $n \in \mathbb{N}$, define $a \sim b$ if $n \mid (a - b)$ (i.e., $a \equiv b \pmod{n}$). This gives $\mathbb{Z}/n\mathbb{Z} = \{[0], [1], \dots, [n-1]\}$.

**Example 1.6.** Let $X = \mathbb{R}^2 \setminus \{(0,0)\}$ and define $(x_1, y_1) \sim (x_2, y_2)$ if there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $(x_2, y_2) = (\lambda x_1, \lambda y_1)$. The equivalence classes are lines through the origin (minus the origin itself). The quotient space $\mathbb{R}^2 \setminus \{(0,0)\}/\sim$ is the **real projective line** $\mathbb{RP}^1$.

## 2 Quotient Vector Spaces

### Setup

Let $V$ be a vector space over a field $\mathbb{F}$ and let $W \subseteq V$ be a subspace.

**Definition 2.1** (Coset)**.** For $v \in V$, the **coset** of $W$ containing $v$ is:

$$v + W = \{v + w : w \in W\}$$

**Key observation:** Define $v_1 \sim v_2$ if $v_1 - v_2 \in W$. This is an equivalence relation, and the equivalence class of $v$ is exactly $v + W$.

**Definition 2.2** (Quotient Vector Space)**.** The **quotient vector space** $V/W$ is the set of all cosets:

$$V/W = \{v + W : v \in V\}$$

with operations:

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$$
$$c(v + W) = (cv) + W \quad \text{for } c \in \mathbb{F}$$

**Well-definedness:** These operations are well-defined because if $v_1 + W = v_1' + W$ and $v_2 + W = v_2' + W$, then $v_1 - v_1' \in W$ and $v_2 - v_2' \in W$, so $(v_1 + v_2) - (v_1' + v_2') = (v_1 - v_1') + (v_2 - v_2') \in W$.

**Theorem 2.3** (Dimension Formula)**.** *If $V$ is finite-dimensional, then:*

$$\dim(V/W) = \dim(V) - \dim(W)$$

*Proof sketch.* Choose a basis $\{w_1, \ldots, w_k\}$ of $W$ and extend it to a basis $\{w_1, \ldots, w_k, v_1, \ldots, v_m\}$ of $V$. Then $\{v_1 + W, \ldots, v_m + W\}$ is a basis for $V/W$. □

### Examples

**Example 2.4.** Let $V = \mathbb{R}^3$ and $W = \{(x, y, 0) : x, y \in \mathbb{R}\}$ (the $xy$-plane). Then:

- Each coset has the form $(0, 0, z) + W$ for $z \in \mathbb{R}$
- These are planes parallel to the $xy$-plane
- $\dim(V/W) = 3 - 2 = 1$
- $V/W \cong \mathbb{R}$ (isomorphic as vector spaces)

**Example 2.5.** Let $V = \mathbb{R}[x]$ (polynomials) and $W = \{p \in \mathbb{R}[x] : p(0) = 0\}$. Then $W = \langle x \rangle$ (polynomials divisible by $x$), and:

$$V/W \cong \mathbb{R}$$

via the isomorphism $[p] \mapsto p(0)$ (evaluation at 0).

**Example 2.6.** Let $V = C([0, 1])$ (continuous functions on $[0, 1]$) and $W = \{f : f(1/2) = 0\}$. Then:

$$V/W \cong \mathbb{R}$$

via the map $[f] \mapsto f(1/2)$.

## 3 The First Isomorphism Theorem

**Theorem 3.1** (First Isomorphism Theorem for Vector Spaces)**.** *Let $T : V \to U$ be a linear map. Then:*

$$V/\ker(T) \cong \operatorname{im}(T)$$

*The isomorphism is given by $\bar{T} : V/\ker(T) \to \operatorname{im}(T)$ where $\bar{T}(v + \ker(T)) = T(v)$.*

*Proof.* (i) **Well-defined:** If $v + \ker(T) = v' + \ker(T)$, then $v - v' \in \ker(T)$, so $T(v) = T(v')$.

(ii) **Linear:** $\bar{T}((v_1+\ker(T))+(v_2+\ker(T))) = \bar{T}((v_1+v_2)+\ker(T)) = T(v_1+v_2) = T(v_1)+T(v_2)$.

(iii) **Injective:** If $\bar{T}(v + \ker(T)) = 0$, then $T(v) = 0$, so $v \in \ker(T)$, thus $v + \ker(T)$ is the zero element of $V/\ker(T)$.

(iv) **Surjective:** For any $u \in \operatorname{im}(T)$, there exists $v \in V$ with $T(v) = u$, so $\bar{T}(v + \ker(T)) = u$.

□

**Corollary 3.2** (Rank-Nullity Theorem)**.** *For a linear map $T : V \to U$ where $V$ is finite-dimensional:*

$$\dim(V) = \dim(\ker(T)) + \dim(\operatorname{im}(T))$$

*Proof.* By the First Isomorphism Theorem, $\dim(\operatorname{im}(T)) = \dim(V/\ker(T)) = \dim(V) - \dim(\ker(T))$.

□

## 4 Canonical Projection

**Definition 4.1.** The **canonical projection** (or **quotient map**) is:

$$\pi : V \to V/W, \quad \pi(v) = v + W$$

**Properties:**

- $\pi$ is linear
- $\pi$ is surjective
- $\ker(\pi) = W$
- By the First Isomorphism Theorem: $V/\ker(\pi) = V/W \cong \operatorname{im}(\pi) = V/W$

## 5 Universal Property of Quotients

**Theorem 5.1** (Universal Property)**.** *Let $V, U$ be vector spaces, $W \subseteq V$ a subspace, and $\pi : V \to V/W$ the canonical projection. For any linear map $T : V \to U$ with $W \subseteq \ker(T)$, there exists a unique linear map $\bar{T} : V/W \to U$ such that $T = \bar{T} \circ \pi$.*

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\pi"'] & U \\ V/W \arrow[ur, dashed, "\bar{T}"'] & \end{tikzcd}$$

*The map $\bar{T}$ is defined by $\bar{T}(v + W) = T(v)$.*

**Intuition:** The quotient $V/W$ is the "best" way to collapse $W$ to zero while preserving the vector space structure.

## Key Takeaways

- Equivalence relations partition sets into disjoint equivalence classes
- Quotient vector spaces "collapse" a subspace to zero
- The dimension formula: $\dim(V/W) = \dim(V) - \dim(W)$
- First Isomorphism Theorem connects kernels and images
- The universal property characterizes quotients up to isomorphism