Axiomatic definition of inner product.

V: vector space

The inner product, $\langle x, y\rangle$, of two vectors is a bilinear mapping.

$$\langle x, y\rangle : V \times V \to \mathbb{R}$$

satisfying the following properties:

1) Linearity:

$\forall u, v, w \in V$; and $\forall a, b \in \mathbb{R}$.

$$\langle au + b.v, w\rangle = a\langle u, w\rangle + b\langle v, w\rangle$$

$$\langle u, av + bw\rangle = a\langle u, v\rangle + b\langle u, w\rangle$$

2) Symmetry:

$\forall u, v \in V$, $\langle u, v\rangle = \langle v, u\rangle$

3) Positive definite:

$\forall u \in V$; $\langle u, u\rangle \geq 0$ & $\langle u, u\rangle = 0$ iff $u = 0$.

The inner product allows for defining the norm of a vector as:

$\|x\|^2 = \langle x, x \rangle$; always a positive number unless $x = 0$.

Using the inner product properties, we can derive the <u>polarization identity</u>:

Let's compute $\|x+y\|^2$ & $\|x-y\|^2$:

- $$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y \rangle \\ &= \langle x, x+y \rangle + \langle y, x+y \rangle \\ &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\ &= \|x\|^2 + \|y\|^2 + 2\langle x, y \rangle \quad (a) \end{aligned}$$

- $$\begin{aligned} \|x-y\|^2 &= \langle x-y, x-y \rangle \\ &= \langle x, x-y \rangle - \langle y, x-y \rangle \\ &= \langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle \\ &= \|x\|^2 + \|y\|^2 - 2\langle x, y \rangle \quad (b) \end{aligned}$$

Substracting (b) from (a):

$$\|x+y\|^2 - \|x-y\|^2 = 4\langle x, y\rangle$$

from which:

$$\boxed{\langle x, y\rangle = \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2\right]}$$

(Polarization identity).

This expression can be used as a "geometric" definition of the dot product between two geometric vectors $\bar{a}$ & $\bar{b}$.

$$\bar{a}\cdot\bar{b} = \frac{1}{4}\left[|\bar{a}+\bar{b}|^2 - |\bar{a}-\bar{b}|^2\right],$$

with $|\cdot|$, length of a vector.

If $\bar{a} = \bar{b}$; $\underline{\bar{a}\cdot\bar{a}} = \frac{1}{4}\left[|2\bar{a}|^2 - |\cancelto{0}{\bar{a}-\bar{a}}|^2\right] =$

$$= \frac{1}{4}\left[4|\bar{a}|^2\right] = \underline{|\bar{a}|^2}$$