Advanced Math for QM
Chapter 2

Wayne Dam

January 20, 2025

Exercise 1:

Answer:

We can write the given formula (2.6) as,

$$dx = \pm\sqrt{\frac{2(E_0 - V(x))}{m}}\, dt. \tag{1}$$

So,

$$dt = \pm\sqrt{\frac{m}{2(E_0 - V(x))}}\, dx \tag{2}$$

Integrating and choosing the plus sign for left to right motion we get,

$$t = \int_{x=x_0}^{x_1} \sqrt{\frac{m}{2(E_0 - V(x))}}\, dx \tag{3}$$

Exercise 3:

Answer:

The question asks for the leading term behaviour of $T(\delta)$, the time of travel of the pendulum from $\theta = \pi - \delta$, where $0 < \delta \ll 1$, to some θ_0 significantly enough away from the starting point.

The potential energy of the pendulum mass is determined by its vertical height, h, from some arbitrary level. Choose $h = 0$ as the position of the mass when it sits vertically down from the origin, i.e., $\theta = 0$.

From the geometry you can show that,

$$h = L(1 - \cos\theta). \tag{4}$$

At $\theta = 0$ this gives $h = 0$, as expected, and at $\theta = \pi$ (at the vertical top) $h = 2L$.

The potential energy as a function of θ is thus,

$$V(\theta) = mgL(1 - \cos\theta). \tag{5}$$

So the initial energy of the system is,

$$\begin{aligned} E_0 &= V(\theta = \pi - \delta) & (6)\\ &= mgL(1 - \cos\theta) & (7) \end{aligned}$$

Substituting into the time of travel equation in problem 1 we get,

$$\begin{aligned} T &= \int_{\theta=\pi-\delta}^{\theta_0} \sqrt{\frac{m}{2\left[mgL(1-\cos(\pi-\delta)) - mgL(1-\cos\theta)\right]}}\, d\theta & (8)\\ &\propto \int_{\theta=\pi-\delta}^{\theta_0} \frac{1}{\sqrt{\cos\theta - \cos(\pi-\delta)}}\, d\theta & (9)\\ &= \int_{\theta=\pi-\delta}^{\theta_0} \frac{1}{\sqrt{\cos\theta + \cos\delta}}\, d\theta & (10) \end{aligned}$$

This is a usual elliptic function as an exact solution.

Since we are interested in $\delta \ll 1$ we can expand $\cos\delta$ to second order as, $1 - \delta^2/2$. However keeping only the leading 1 and dropping the second order term, while retaining the first order δ in the integral's lower limit, seems to give the log behaviour the question asks for. So we'll drop the second order term entirely. Continuing,

$$T(\delta) \sim \int_{\theta=\pi-\delta}^{\theta_0} \frac{1}{\sqrt{\cos\theta + 1}}\, d\theta \tag{11}$$

Evaluating the integral in SageMath gives us,

$$\frac{1}{\sqrt{2}}\left[\log\left(2 + 2\sin\left(\frac{1}{2}\theta\right)\right) - \log\left(2 - 2\sin\left(\frac{1}{2}\theta\right)\right)\right]\Bigg|_{\pi-\delta}^{\theta_0} \tag{12}$$

This simplifies nicely. Since we are only interested the behaviour in terms of δ we can drop the θ_0 substitution. It will only act as a constant wrt δ. Also, $\sin\frac{\pi-\delta}{2} = \cos\delta/2 \approx 1 - \delta^2/8$.

So,

$$\begin{aligned} T(\delta) &\sim -\log\big(2(1 + 1 - \delta^2/8)\big) + \log\big(2(1 - 1 - \delta^2/8)\big) & (13)\\ &= -\log(4 - \delta^2/4) + \log(\delta^2/4) & (14) \end{aligned}$$

The first term for $\delta \ll 1$ is just a constant $-\log(4)$ and can be ignored. The last term gives us,

$$T(\delta) \sim 2\log\frac{\delta}{2}, \tag{15}$$

as required.

Exercise 8:

Answer:

Energy is conserved if,

$$\frac{d}{dt}E(x,v) = 0. \tag{16}$$

Just substitute and evaluate as in the text ...

$$\begin{aligned}
\frac{d}{dt}\frac{1}{2}m|v|^2 + V(x) &= m\sum_j \dot{x}_j(t)\ddot{x}_j(t) + \sum_j \frac{\partial V}{\partial x_j}\dot{x}_j(t) && (17)\\
&= \dot{x}(t)\cdot(m\ddot{x}(t) + \nabla V(x)) && (18)\\
&= \dot{x}\cdot(F(x,\dot{x}) + \nabla V(x)) && (19)\\
&= \dot{x}\cdot(-\nabla V(x) + F_2(x,\dot{x}) + \nabla V(x)) && (20)\\
&= \dot{x}\cdot F_2(x,\dot{x}) && (21)\\
&\equiv 0 \text{ by assumption.} && (22)
\end{aligned}$$

Exercise 18:

Answer:

We are give $f(x,p) = xp$. So,

$$\begin{aligned}
\frac{dx}{dt} &= \frac{\partial}{\partial p}xp = x && (23)\\
\frac{dp}{dt} &= -\frac{\partial}{\partial x}xp = -p. && (24)
\end{aligned}$$

This has the solution,

$$\begin{aligned}
x &= ae^t && (25)\\
p &= be^{-t} && (26)
\end{aligned}$$

As a physical system this looks concerning since the position x increases at an exponential rate, and so should the momentum you would think. But the momentum, p, here is *decreasing* at an exponential rate.

The discussion on page 38 and 39 of the text covers this, I think. The variable, t, is not really time. It's just a parameter. It would be good to discuss this further in class.