Advanced Math for QM

Chapter 3

Problems 1-5

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Exercise 1:

Answer:

We use the addition and subtraction of the same term trick as deriving the derivative of a product. Writing $\phi(t)$ as ϕ_t, etc., for neatness, we have,

$$\begin{aligned}
\frac{d}{dt}\langle\phi,\psi\rangle &= \lim_{h\to 0}\frac{\langle\phi_{t+h},\psi_{t+h}\rangle-\langle\phi_t,\psi_t\rangle}{h} && (1)\\
&= \lim_{h\to 0}\frac{\langle\phi_{t+h},\psi_{t+h}\rangle-\langle\phi_{t+h},\psi_t\rangle+\langle\phi_{t+h},\psi_t\rangle-\langle\phi_t,\psi_t\rangle}{h} && (2)\\
&= \lim_{h\to 0}\frac{\langle\phi_{t+h},\psi_{t+h}-\psi_t\rangle}{h}+\frac{\langle\phi_{t+h}-\phi_t,\psi_t\rangle}{h} && (3)\\
&= \lim_{h\to 0}\langle\phi_{t+h},\frac{\psi_{t+h}-\psi_t}{h}\rangle+\langle\frac{\phi_{t+h}-\phi_t}{h*},\psi_t\rangle && (4)\\
&= \langle\phi_t,\frac{d}{dt}\psi_t\rangle+\langle\frac{d}{dt}\phi_t,\psi_t\rangle && (5)
\end{aligned}$$

Exercise 2:

Answer:

We are given that $AB - BA$ has the form cI for some constant c in finite non-zero dimension n. This does *not* mean that the commutator cannot take another form, just that we are restricting to the case it equals cI for some c.

Generally, when dealing with a problem involving dimensionality and the identity matrix the trace operation is usually around.

So, given $AB - BA = cI$, take the trace of both sides. Then,

$$\begin{aligned}
\mathrm{tr}\ (AB-BA) &- \mathrm{tr}\ cI && (6)\\
\mathrm{tr}\ AB-\mathrm{tr}\ BA &= cn && (7)\\
\mathrm{tr}\ AB-\mathrm{tr}\ AB &= cn && (8)\\
0 &= cn. && (9)
\end{aligned}$$

Since $n > 0$ this forces $c = 0$.

Doing some reading, this is also true in the infinite dimensional case for *bounded* operators. It does not hold in general for unbounded operators.

Exercise 3:

Answer:

(a) First, we have

$$\langle \phi, cA\psi \rangle = \langle (cA)^* \phi, \psi \rangle \tag{10}$$

by definition.

We can then expand the left hand side as,

$$\begin{aligned} \langle \phi, cA\psi \rangle &= c\langle \phi, A\psi \rangle & (11) \\ &= \langle c^* \phi, A\psi \rangle & (12) \\ &= \langle A^* c^* \phi, \psi \rangle & (13) \\ &= \langle c^* A^* \phi, \psi \rangle. & (14) \end{aligned}$$

But this means,

$$\langle (cA)^* \phi, \psi \rangle = \langle c^* A^* \phi, \psi \rangle. \tag{15}$$

And so, $(cA)^* = c^* A^*$.

(b) We have,

$$\begin{aligned} \left[\frac{1}{i\hbar}(AB - BA) \right]^* &= \frac{-1}{i\hbar}(B^* A^* - A^* B^*) & (16) \\ &= \frac{1}{i\hbar}(A^* B^* - B^* A^*) & (17) \\ &= \frac{1}{i\hbar}(AB - BA). & (18) \end{aligned}$$

Therefore it's self adjoint.

Exercise 4:

Answer:

By definition, $\langle A \rangle_\psi = \langle \psi, A\psi \rangle$.

And we have from proposition 3.14,

$$\frac{d}{dt}\langle A \rangle_\psi = \langle \frac{1}{i\hbar}[A, H] \rangle_\psi. \tag{19}$$

(a) From the above we have,

$$\begin{aligned} \frac{d}{dt}\langle X \rangle_\psi &= \langle \frac{1}{i\hbar}(XH - HX) \rangle_\psi & (20) \\ &= \frac{1}{i\hbar}\left(\langle \psi, XH\psi \rangle - \langle \psi, HX\psi \rangle \right). & (21) \end{aligned}$$

Now substitute the Hamilton, $H = (-\hbar/2m \cdot \partial_{xx} + V)\psi$, and X operator, $X = x$.

$$\frac{d}{dt}\langle X\rangle_\psi = \frac{1}{i\hbar}\left[\langle\psi, x \cdot \frac{-\hbar^2}{2m}\psi_{xx} + xV\psi\rangle - \langle\psi, \left(\frac{-\hbar^2}{2m}\partial_{xx} + V\right)(x\psi)\rangle\right] \tag{22}$$

Noting $\partial_{xx}(x\psi) = x\psi_{xx} + 2\psi_x$ we have,

$$\begin{aligned}
\frac{d}{dt}\langle X\rangle_\psi &= \frac{1}{i\hbar}\left[\langle\psi, \frac{-\hbar^2}{2m}x\psi_{xx} + xV\psi\rangle - \langle\psi, \frac{-\hbar^2}{2m}(x\psi_{xx} + 2\psi_x) + xV\psi\rangle\right] && (23)\\
&= \frac{1}{i\hbar}\langle\psi, \frac{-\hbar^2}{2m}x\psi_{xx} + xV\Psi + \frac{\hbar^2}{2m}x\psi_{xx} + \frac{\hbar^2}{2m}\psi_x - xV\psi\rangle && (24)\\
&= \frac{1}{i\hbar}\langle\psi, \frac{\hbar^2}{m}\psi_x\rangle && (25)\\
&= \frac{\hbar}{im}\langle\psi, \psi_x\rangle. && (26)
\end{aligned}$$

But $P\psi = -i\hbar\psi_x$. Substituting $\psi_x = \frac{-1}{i\hbar}P\psi$ yields,

$$\begin{aligned}
\frac{d}{dt}\langle X\rangle_\psi &= \frac{\hbar}{im}\langle\psi, \frac{-1}{i\hbar}P\psi\rangle && (27)\\
&= \frac{1}{m}\langle\psi, P\psi\rangle, && (28)
\end{aligned}$$

giving the desired result,

$$\frac{d}{dt}\langle X\rangle_\psi = \frac{1}{m}\langle P\rangle_\psi. \tag{29}$$

(b) Similar calculation.

Exercise 5:

Answer:

In terms of notation I'm writing,

$$\langle\psi, f(x)\psi\rangle = \int_{-\infty}^{\infty} f(x)|\psi(x)|^2\ dx = \langle F\rangle_\psi. \tag{30}$$

We have,

$$\begin{aligned}
\int_{-\infty}^{\infty}(x-a)^2|\psi(x)|^2\ dx &= \int_{-\infty}^{\infty}(x^2 - 2ax + a^2)|\psi(x)|^2\ dx && (31)\\
&= \int_{-\infty}^{\infty} x^2|\psi(x)|^2\ dx - 2a\int_{-\infty}^{\infty} x|\psi(x)|^2\ dx + a^2\int_{-\infty}^{\infty}|\psi(x)|^2\ dx && (32)
\end{aligned}$$

Note that $|\psi(x)|^2$ is a unit normed probability distribution function. Thus,

$$\int_{-\infty}^{\infty}(x-a)^2|\psi(x)|^2\ dx = \langle X^2\rangle_\psi - 2a\langle X\rangle_\psi + a^2. \tag{33}$$

The left hand side is clearly positive. Thus,

$$\langle X^2 \rangle_\psi > 2a\langle X \rangle_\psi - a^2. \tag{34}$$

Setting $a = \langle X \rangle_\psi$ we have the desired result,

$$\langle X^2 \rangle_\psi > \langle X \rangle_\psi^2. \tag{35}$$