Road to Reality Problems and Notes

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Chapter 4

4.1

[4.1], [4.2], [4.3] is just simple complex algebra.

4.4

Show that

$$1 + x^2 + x^4 + x^6 + ... = (1 - x^2)^{-1}$$

Solution Assume that the series converges to a function $f(x)$ Then

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} x^{2n} && \Longrightarrow && (1)\\
f(x) &= 1 + x^2 \sum_{n=0}^{\infty} x^{2n} && \Longrightarrow && (2)\\
f(x) &= 1 + x^2 f(x) && \Longrightarrow && (3)\\
f(x) &= \frac{1}{1-x^2} && && (4)
\end{aligned}$$

4.5

$(1 + x^2)^{-1}$ can be obtained from $(1 - x^2)^{-1}$ by the transformation $x \to ix$

Chapter 5

5.1, 5.2

Geometrically, addition of complex numbers is vector addition in R^2 and multiplication is a combination of scaling and rotation.

In addition, parallelograms degenerate to lines when the two complex numbers are co-linear. In other words if for $a + ib$ and $c + id$ we have $a/b = c/d$. Exact inverses degenerate to a point.

Similarly, in multiplication triangles degenerate to lines where the numbers are co-linear.

5.3

Show that multiplication in the complex plane preserves shapes and angles with direct computation and without trigonometry.

TODO

5.4

This becomes obvious once exponential notation is introduced. Otherwise, it is a boring trigonometry exercise.

5.5

Show $e^{(a+b)} = e^a e^b$ using the Taylor series expansion.

Solution

$$e^{(a+b)} = \textstyle\sum_{n=0}^{\infty} \frac{(a+b)^n}{n!} \tag{5}$$

$$= \textstyle\sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{a^k b^{n-k}}{k!(n-k)!} \tag{6}$$

On the other hand

$$e^a e^b = \sum_{n=0}^{\infty} \frac{a^n}{n!} \sum_{m=0}^{\infty} \frac{b^m}{m!} \tag{7}$$

Introduce a change of variable by setting $m = n - k$. Then the new variable k ranges from n (when $m = 0$) to 0 (when $m = \infty$).

$$e^a e^b = \textstyle\sum_{n=0}^{\infty} \frac{a^n}{n!} \sum_{k=0}^{n} \frac{b^{n-k}}{(n-k)!} \tag{8}$$

$$= \textstyle\sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{a^k b^{n-k}}{k!(n-k)!} \tag{9}$$

Which produces the same result as in (**??**).

5.6

Show that $z + i\pi$ is a logarithm of $-w$.

Solution

$$w = e^z \implies log(-w) = log(-e^z) = log(-1) + log(e^z) \tag{10}$$

$$= log(e^{i\pi}) + z = z + i\pi \tag{11}$$

5.8

Show

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$
$$\sin 3\theta = 3\sin\theta\cos^2\theta - \sin^3\theta$$

Solution Expand $e^{3i\theta} = (e^{i\theta})^3$ in real and imaginary parts

5.9

I do not understand what the plot is plotting.

5.10

Resolve the paradox:

$$e = e^{1+2\pi i} = (e^{1+2\pi i})^{1+2\pi i} = e^{1+4\pi i - 4\pi^2} = e^{1-4\pi^2}$$

Solution This is explained by 5.15. Because of the multi-valuedness of the complex power, we cannot willi-nilly conclude that $w^{ab} = e^{ab \ln w}$.

5.11

$$z = lob_b z \implies w = b^z \implies lnw = zlnb \implies z = \frac{lnw}{lnb}$$

And since we can add $2ki\pi$ to lnw, we can add $2ki\pi/lnb$ to z.

5.12

Why is it allowable to specify $logi = \frac{1}{2}\pi i$?

Solution because $i = e^{i\pi/2}$

5.13

$$e^{2\pi n} = e^i \cdot e^{-i2\pi n} = e^i$$

5.14

Multivaluedness of $w^{1/n}$

Idea

Starting from $z^n = w$ and taking the log of both sides we have

$nlogz = logw + 2ik\pi$ where the second term results from the multi-valuedness of log. $logz = (1/n)logw + \frac{2ik\pi}{n} = logw^{1/n} + \frac{2ik\pi}{n}$

From which $ze^{\frac{2ik\pi}{n}} = w^{1/n}$

5.15

Describe the conditions under which $(w^a)^b = w^{ab}$.

Solution

Fix a branch for $logw$. By the definition of complex power, we have for the right hand side: $w^{ab} = e^{ab \ln w}$.

For the left hand side we have $(w^a)^b = e^{b \ln w^a}$

For the two sides to much we must specify that $\ln w^a = a \ln w$

Chapter 6

6.1

Show that the Heaviside function is given by

$$\theta(x) = \frac{|x| + x}{2x}$$

If $x > 0$, then $\theta(x) = \frac{x+x}{2x} = 1$.
If $x < 0$ then $\theta(x) = \frac{-x+x}{2x} = 0$.

6.2

Show that the following function is C^∞

$$h(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-1/x} & \text{if x ¿ 0} \end{cases} \tag{12}$$

Sketch On $x > 0$, $h(x)$ is a composition of smooth functions, therefore smooth (easy to show from chain rule). So if suffices to show that h is smooth at 0.

Show that all derivatives of h are sums of the form $e^{-1/x} \cdot (1/x)^n$

Do the substitution $u = 1/x$. Then show that each of $e^{-u} \cdot u^n$ goes to 0 as $u \to \infty$.

6.3

Just differentiate the expansion and evaluate at 0

6.4

Show that e^{-1/x^2} is smooth but not analytic at 0.

Sketch As in 6.2 show that all derivatives are of the form $e^{-1/x^2} \cdot (1/x)^n$ and show that they are all 0 at 0. Then, use an argument as in page 113 to show that if it had a Taylor expansion all coefficients have to vanish so that the function is 0 at 0.

[6.5] to [6.10] are calculus review