Physics with Friends – QFT

The Dirac Equation

Chapter 36, QFT for Gifted Amateurs

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The Schrödinger equation

- Newtonian Energy-Momentum Relation for a free particle (no force acting on it; zero potential):

$$E = \frac{\| \mathbf{p} \|^2}{2m} \qquad \Longleftrightarrow \qquad E - \frac{1}{2m}\left(p_1^2 + p_2^2 + p_3^2\right) = 0 \tag{1}$$

Proof: $$E := \left(\begin{array}{c}\text{total energy}\\ \text{of the}\\ \text{free particle}\end{array}\right) = \left(\begin{array}{c}\text{kinetic energy}\\ \text{of the}\\ \text{particle}\end{array}\right) = \frac{1}{2}\, m \| \mathbf{v} \|^2 = \frac{(\| m \cdot \mathbf{v} \|)^2}{2m} = \frac{\| \mathbf{p} \|^2}{2m}$$

- **First quantization:** Replacing E, p_1, p_2, p_3 respectively with differential operators:

$$E \longmapsto \mathbf{i}\,\frac{\partial}{\partial t}\,, \qquad p_j \longmapsto -\,\mathbf{i}\,\frac{\partial}{\partial x_j}\,, \qquad \text{and} \tag{2}$$

letting the resulting differential operator act on the wave function $\psi(t, x_1, x_2, x_3)$ yields the Schrödinger equation for a free particle:

$$\left(\mathbf{i}\,\frac{\partial}{\partial t} + \frac{1}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)\right)\psi = 0 \tag{3}$$

- Schrödinger equation is NOT Lorentz-invariant (e.g., asymmetry between time and space) – incompatible with Special Relativity.

The Klein-Gordon equation

- Special-relativistic Energy-Momentum Relation for a free particle (see §3.4, [1]):

$$E^2 = p^2 + m^2 \tag{4}$$

- First quantization on (4):

$$E \longmapsto \mathbf{i}\,\frac{\partial}{\partial t}\,, \qquad p_j \longmapsto -\mathbf{i}\,\frac{\partial}{\partial x_j}$$

  now yields the Klein-Gordon equation for a free particle:

$$\left(\Box + m^2\right)\psi = \left(\partial_\mu \partial^\mu + m^2\right)\psi = \left(\frac{\partial^2}{\partial t^2} - \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) + m^2\right)\psi = 0 \tag{5}$$

- The Klein-Gordon equation is indeed Lorentz-invariant but:
  - it permits negative-energy eigenstates; see Example 6.1, [4]
  - probabilistic interpretation of (squared modulus of) Schrödinger's wave function is lost; see §6.2, [4]
- The Dirac equation addresses the second problem (i.e., restores dynamical/probabilistic interpretation).

The Dirac equation

- Dirac sought a new equation of motion by seeking a first-order linear differential operator

$$D = \mathbf{i}\,\gamma^\mu \partial_\mu = \mathbf{i}\left(\gamma^0 \frac{\partial}{\partial x_0} + \gamma^1 \frac{\partial}{\partial x_1} + \gamma^2 \frac{\partial}{\partial x_2} + \gamma^3 \frac{\partial}{\partial x_3}\right)$$

  whose square is minus the Minkowskian d'Alembertian $\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}$.

- Dirac arrived at the famous Dirac equation:

$$(\mathbf{i} \cdot \gamma^\mu \partial_\mu - m)\,\psi = 0 \tag{6}$$

  The requirement $-g^{\mu\nu}\partial_\mu\partial_\nu = -\Box = (\mathbf{i} \cdot \gamma^\mu \partial_\mu)^2 = -\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu = -\left(\frac{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu}{2}\right)\partial_\mu\partial_\nu$

  implies that γ^μ must satisfy:

$$\{\gamma^\mu, \gamma^\nu\} = 2 \cdot g^{\mu\nu}, \tag{7}$$

  where $\{\gamma^\mu, \gamma^\nu\} := \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu$ and $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

- Dirac equation $\iff$ $\mathbf{i}\,\gamma^\mu\partial_\mu\psi = m\,\psi \implies \underbrace{\mathbf{i}\,\gamma^\mu\partial_\mu\left(\mathbf{i}\,\gamma^\nu\partial_\nu\psi\right)}_{-\Box\psi} = m \cdot \underbrace{\mathbf{i}\,\gamma^\mu\partial_\mu\psi}_{m\,\psi} \implies$ Klein-Gordon equation

The Dirac equation (cont'd)

- Condition (7) cannot be satisfied by complex numbers:

$$\left.\begin{array}{r} \{\,\gamma^\mu\,,\,\gamma^\nu\,\} \;=\; 2\cdot g^{\mu\nu} \\ \gamma_\mu \;\in\; \mathbb{C},\;\; \forall\,\mu = 0,1,2,3 \end{array}\right\} \;\Longrightarrow\; \text{contradiction}$$

But, Dirac observed that Condition (7) could indeed be satisfied by a set of 4×4 complex matrices:

$$\gamma^0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -\mathbf{i} \\ 0 & 0 & \mathbf{i} & 0 \\ 0 & \mathbf{i} & 0 & 0 \\ -\mathbf{i} & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}$$

- So, Dirac sought solutions $\psi \in C^\infty(\mathcal{M}, \mathbb{C}^4)$ for the Dirac equation: $\left(\mathbf{i}\,\gamma^\mu \partial_\mu - m\right)\psi = 0.$

What is a *spinor*?

Clifford algebra, Lorentz group & Spin group

- Condition (7) ($\{ \gamma^\mu, \gamma^\nu \} = 2\, g^{\mu\nu}$, with $g^{\mu\nu} = \mathrm{diag}(1, -1 - 1 - 1)$) is the defining relations of what is known as the Clifford algebra $\mathrm{Cl}(1,3)$; see §6.3, [3].
- $\mathrm{Cl}(1,3)$ contains the orthochronous spin group $\mathrm{Spin}^{\uparrow}(1,3)$; see Proposition 6.5.4, [3].
- $\mathrm{Spin}^{\uparrow}(1,3)$ is the universal covering of the proper orthochronous Lorentz group $\mathrm{SO}^{\uparrow}(1,3)$; see Definition 6.1.16 and Corollary 6.5.16, [3]. $\ker\left(\mathrm{Spin}^{\uparrow}(1,3) \xrightarrow{\pi} \mathrm{SO}^{\uparrow}(1,3)\right) = \{\pm 1\}$; see Theorem 6.5.13, [3].
- Condition (7) $\Longrightarrow$ $\mathrm{Spin}^{\uparrow}(1,3) \cong \mathrm{SL}(2,\mathbb{C})$ acts on the copy of $\mathbb{C}^4$ in question.

Relationship between $\mathrm{SU}(2)$ **and** $\widetilde{\mathrm{SO}^{\uparrow}(1,3)} = \mathrm{Spin}^{\uparrow}(1,3) \cong \mathrm{SL}(2,\mathbb{C})$

- $\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{su}(2) \oplus \mathbf{i}\,\mathfrak{su}(2)$ (the copies of $\mathfrak{su}(2)$ commute)

  (Skew self-adjoint matrices with trace zero plus self-adjoint matrices with trace zero gives all matrices with trace zero.)

Symmetry group of a quantum system & action of $\mathrm{SO}(3)$ **on the system**

- The state space of a quantum system is the projective space $\mathbb{P}(H)$ over a complex Hilbert space H.
- The **symmetry group** of a quantum system with state space $\mathbb{P}(H)$ is the projective unitary group of H, i.e.

$$\mathbb{P}U(H) := U(H) \Big/ \{ e^{\mathbf{i}\,\theta} \cdot \mathbf{1}_H \}_{\theta \in \mathbb{R}}$$

- The action of $\mathrm{SO}(3)$ on a quantum system with state space $\mathbb{P}(H)$ is thus a (Lie group) homomorphism $\mathrm{SO}(3) \longrightarrow \mathbb{P}U(H)$, i.e., a projective unitary representation of $\mathrm{SO}(3)$.

What is a *spinor*? (cont'd)

Relations between $\mathrm{SO}(3) \longrightarrow \mathbb{P}U(H)$, $\mathrm{SO}(3) \longrightarrow U(H)$, **and** $\mathrm{SU}(2) \longrightarrow U(H)$

- Every $\mathrm{SO}(3) \xrightarrow{\theta} U(H)$ gives rise to a projective unitary representation (symmetry group of a quantum system) simply via composition $\mathrm{SO}(3) \xrightarrow{\theta} U(H) \xrightarrow{Q} \mathbb{P}U(H) := U(H) \,/\, \{ e^{\mathbf{i}\,\theta} \cdot \mathbf{1}_H \}$.

  Converse is false.

$$\begin{tikzcd} & U(H) \arrow[d, "Q"] \\ \mathrm{SO}(3) \arrow[r, "\rho"'] \arrow[ur, dashed, "\exists\,\theta\,?"] & \mathbb{P}U(H) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{SU}(2) \arrow[r, "\widetilde{\rho}"] \arrow[d, "\pi"'] & U(H) \arrow[d, "Q"] \\ \mathrm{SO}(3) \arrow[r, "\rho"'] & \mathbb{P}U(H) \end{tikzcd}$$

- However, a **partial converse** is true: If $\dim_{\mathbb{C}}(H) < \infty$, then, for each projective unitary representation $\mathrm{SO}(3) \xrightarrow{\rho} \mathbb{P}U(H)$, there exists an (ordinary) unitary representation $\widetilde{\mathrm{SO}(3)} = \mathrm{SU}(2) \xrightarrow{\widetilde{\rho}} U(H)$ such that $\rho \circ \pi = Q \circ \widetilde{\rho}$; see Theorem 16.47, [2].

- So, there are two types of finite-dimensional projective unitary representations $\mathrm{SO}(3) \xrightarrow{\rho} \mathbb{P}U(H)$:
  - either ρ is induced by an (ordinary) unitary representation $\mathrm{SO}(3) \xrightarrow{\theta} U(H)$
  - or ρ cannot be so induced (still have $\mathrm{SU}(2) \xrightarrow{\widetilde{\rho}} U(H)$; elements of H are then called *spinors*)

- The notion of spinors "extends" to the scenario:

$$\mathrm{SO}(3) \rightsquigarrow \mathrm{SO}^{\uparrow}(1,3)\,, \qquad \mathrm{SU}(2) \rightsquigarrow \mathrm{Spin}^{\uparrow}(1,3) \cong \mathrm{SL}(2,\mathbb{C})\,, \qquad U(H) \rightsquigarrow \mathrm{Aut}(H)$$

What is a *spinor*? (cont'd)

- The notion of spinors "extends" to the scenario:

$$\mathrm{SO}(3) \rightsquigarrow \mathrm{SO}^{\uparrow}(1,3)\,, \quad \mathrm{SU}(2) \rightsquigarrow \mathrm{Spin}^{\uparrow}(1,3) \cong \mathrm{SL}(2,\mathbb{C})\,, \quad U(H) \rightsquigarrow \mathrm{Aut}(H)$$

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$$\begin{tikzcd} & \mathrm{Aut}(H) \arrow[d, "Q"] \\ \mathrm{SO}^{\uparrow}(1,3) \arrow[ur, dashed, "\exists\ \theta\ ?"] \arrow[r, "\rho"'] & \mathbb{P}\mathrm{Aut}(H) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Spin}^{\uparrow}(1,3) \arrow[r, "\widetilde{\rho}"] \arrow[d, "\pi"'] & \mathrm{Aut}(H) \arrow[d, "Q"] \\ \mathrm{SO}^{\uparrow}(1,3) \arrow[r, "\rho"'] & \mathbb{P}\mathrm{Aut}(H) \end{tikzcd}$$

- So, there are two types of finite-dimensional projective representations $\mathrm{SO}^{\uparrow}(1,3) \xrightarrow{\rho} \mathbb{P}\mathrm{Aut}(H)$:
  - either ρ is induced by an (ordinary) representation $\mathrm{SO}^{\uparrow}(1,3) \xrightarrow{\theta} \mathrm{Aut}(H)$
  - or ρ cannot be so induced (still have $\mathrm{Spin}^{\uparrow}(1,3) \xrightarrow{\widetilde{\rho}} \mathrm{Aut}(H)$; elements of H are also called *spinors*)

Representations of $\mathrm{Spin}^{\uparrow}(1,3) \cong \widetilde{\mathrm{SO}^{\uparrow}(1,3)}$

- $\widetilde{\mathrm{SO}^{\uparrow}(1,3)} = \mathrm{Spin}^{\uparrow}(1,3) \cong \mathrm{SL}(2,\mathbb{C})$.
- $\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{su}(2) \oplus \mathbf{i}\,\mathfrak{su}(2)$ (the copies of $\mathfrak{su}(2)$ commute)
  (Skew self-adjoint matrices with trace zero plus self-adjoint matrices with trace zero gives all matrices with trace zero.)
- Generators of $\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{spin}(1,3)$:

  $$\textbf{spatial rotations: } J_x, J_y, J_z\,, \quad \textbf{boosts: } K_x, K_y, K_z$$

  $$[\,J_i, J_j\,] = \mathbf{i}\,\epsilon_{ijk}\,J_k\,, \quad [\,J_i, K_j\,] = \mathbf{i}\,\epsilon_{ijk}\,K_k\,, \quad [\,K_i, K_j\,] = -\,\mathbf{i}\,\epsilon_{ijk}\,J_k$$

  $$N_i^{\pm} := \frac{1}{2}(\,J_i \pm \mathbf{i}\,K_i\,) \implies [\,N_i^+, N_j^+\,] = \mathbf{i}\,\epsilon_{ijk}\,N_k^+\,, \quad [\,N_i^-, N_j^-\,] = \mathbf{i}\,\epsilon_{ijk}\,N_k^-\,, \quad [\,N_i^-, N_j^+\,] = 0$$

  Hence, $\langle\, N_x^+, N_y^+, N_z^+ \,\rangle \cong \langle\, N_x^-, N_y^-, N_z^- \,\rangle \cong \mathfrak{su}(2)$.
- Aside: The $\frac{1}{2}$ above $\implies$ for spinors, a spatial (planar) rotation of 360° induces multiplication by -1.
- The finite-dimensional irreducible representations of $\mathrm{SU}(2)$ have been classified.

  For each $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots,$ there exists a unique representation $\rho_s : \mathrm{SU}(2) \longrightarrow \mathrm{GL}(\mathbb{R}, 2s+1)$
- The irreducible representations of $\mathrm{Spin}^{\uparrow}(1,3)$ are parametrized by the ordered pairs (s_+, s_-) of non-negative multiples of $\frac{1}{2}$, where s_+ refers to $\mathfrak{su}(2) \subset \mathfrak{sl}(2,\mathbb{C})$ and s_- refers to $\mathbf{i}\,\mathfrak{su}(2) \subset \mathfrak{sl}(2,\mathbb{C})$. See Theorem on page 517, [6].

Left- & Right-handed Weyl Spinors + Dirac spinors

Representations of $\mathrm{Spin}^{\uparrow}(1,3) \;\cong\; \widetilde{\mathrm{SO}^{\uparrow}(1,3)}$

- The irreducible representations of $\mathrm{Spin}^{\uparrow}(1,3)$ are parametrized by the ordered pairs (s_+, s_-) of non-negative multiples of $\frac{1}{2}$, where s_+ refers to $\mathfrak{su}(2) \subset \mathfrak{sl}(2,\mathbb{C})$ and s_- refers to $\mathbf{i}\,\mathfrak{su}(2) \subset \mathfrak{sl}(2,\mathbb{C})$. See Theorem on page 517, [6].

Weyl spinors

- Left-handed: $\left(\frac{1}{2},0\right)$ representation of $\mathrm{Spin}^{\uparrow}(1,3)$; right-handed: $\left(0,\frac{1}{2}\right)$
- Aside: The parity transformation transforms left-handed Weyl spinors to right-handed ones, and vice versa. See page 174, [5]. Full Lorentz invariance implies both types of Weyl spinors must "occur in nature."

Dirac spinors: The reducible $\left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right)$ representation of $\mathrm{Spin}^{\uparrow}(1,3)$

- $\left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right)$ corresponds to action of Dirac γ-matrices on $\mathbb{C}^4$. See §4.3.7, [5] or §41.2, [6].
- Aside: $\lambda \in \left(\frac{1}{2},0\right) \implies \mathbf{i}\,\sigma^2\left(\overline{\lambda}\right) \in \left(0,\frac{1}{2}\right)$. See §4.3.12, [5].

  Hence, can write general element of $\left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right)$ as $\begin{pmatrix} \lambda \\ \mathbf{i}\,\sigma^2(\overline{\rho}) \end{pmatrix}$, where $\lambda,\ \rho \in \left(\frac{1}{2},0\right)$. Thus,

$$\mathbf{i}\,\gamma^2(\,\overline{\cdot}\,)\ :\ \left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right) \longrightarrow \left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right)\ :\ \begin{pmatrix} \lambda \\ \mathbf{i}\,\sigma^2(\overline{\rho}) \end{pmatrix} \longmapsto \begin{pmatrix} \rho \\ \mathbf{i}\,\sigma^2\left(\overline{\lambda}\right) \end{pmatrix}$$

The Chirality Operator

- Recall the **chiral representation** of the Clifford algebra $\mathrm{Cl}(1,3)$:

$$\gamma^0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -\mathbf{i} \\ 0 & 0 & \mathbf{i} & 0 \\ 0 & \mathbf{i} & 0 & 0 \\ -\mathbf{i} & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}$$

- Define the **chirality operator**:

$$\gamma^5 \ := \ \mathbf{i}\,\gamma^0\gamma^1\gamma^2\gamma^3 \ = \ \cdots \ = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}$$

$$P_L \ := \ \frac{1}{2}\left(1-\gamma^5\right) \ = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_R \ := \ \frac{1}{2}\left(1+\gamma^5\right) \ = \begin{pmatrix} 0 & 0 \\ 0 & I_2 \end{pmatrix}$$

- Hence,

$\psi_L \ := \ P_L\,\psi \ := \ P_L\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \ = \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix}$ is a left-handed Weyl spinor, and

$\psi_R \ := \ P_R\,\psi \ := \ P_R\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \ = \begin{pmatrix} 0 \\ \psi_2 \end{pmatrix}$ a right-handed one.

Thank You!

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