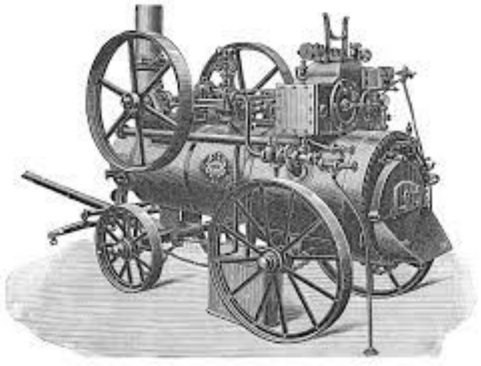
Spin-Statistics Theorem

Bosons have integer spin
Fermions half ½ integer spin

Early 1800s - Steam Engines



1803 Atomic Model (Dalton)

1824 Carnot cycle

1834 Ideal gas law (Clapeyron) $PV=nRT$

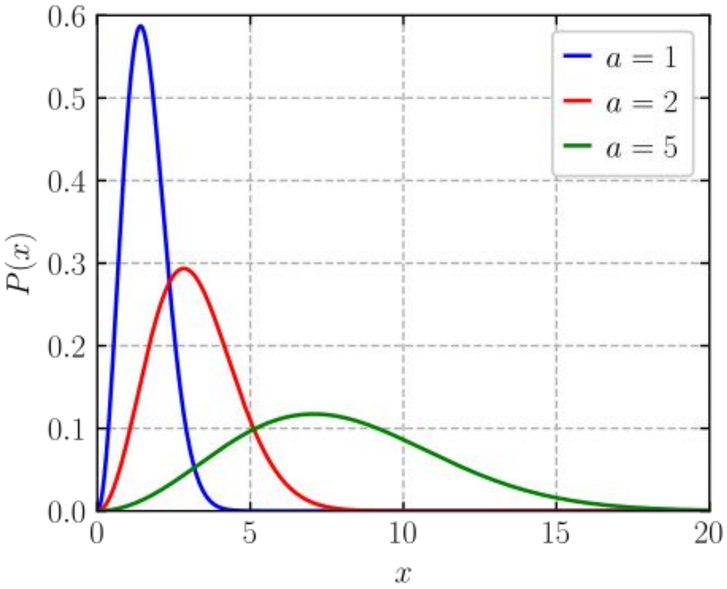
1840s Energy-Heat relation (Joule et al) $E \sim= T$

1845 Equipartition Theorem (Waterston)

1850 Maxwell Distribution

Velocity distribution for an ideal gas.

$$f(\mathbf{v})\, d^3\mathbf{v} = \left[\frac{m}{2\pi k_B T}\right]^{3/2} \exp\left(-\frac{mv^2}{2k_B T}\right) d^3\mathbf{v},$$



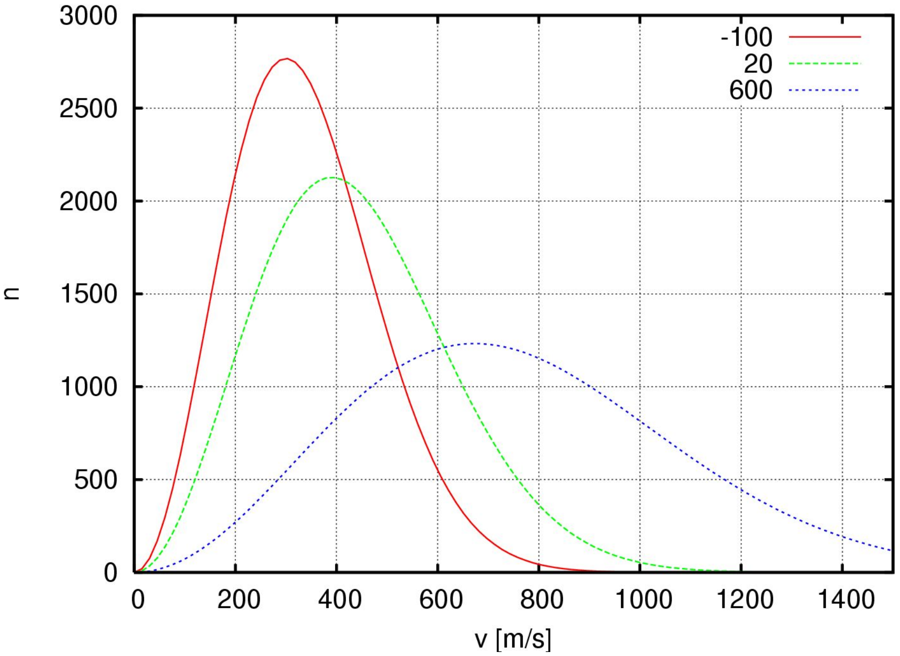
1870 Boltzmann Statistics

Generalizes Maxwell distribution to 'states of **distinguishable** particles'

$$\langle N_i \rangle = \frac{g_i}{e^{(\varepsilon_i - \mu)/k_B T}}$$

N_i = Number of particles N in state i
e_i = Energy of state i

Describes properties of systems in equilibrium including specific heat capacity of many materials (but not metals...)



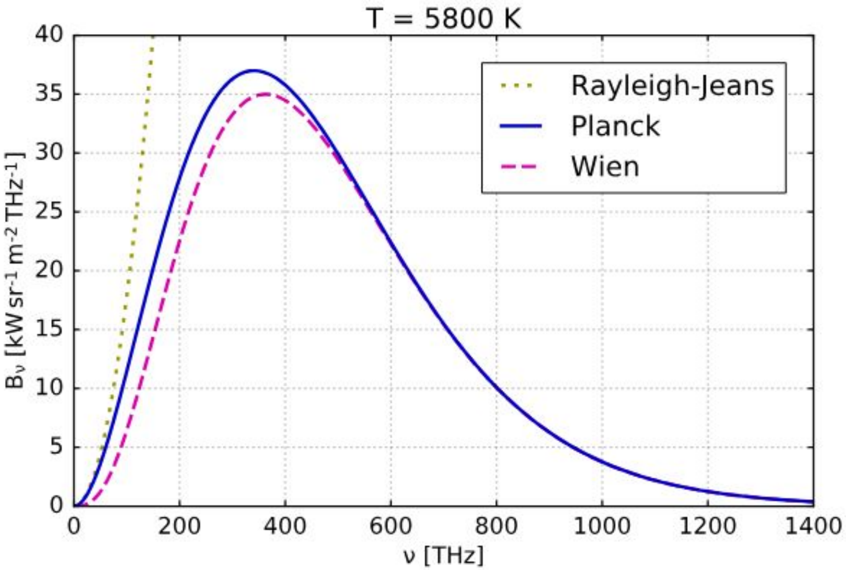
These are now called the Maxwell-Boltzmann Distribution and Maxwell-Boltzmann Statistics

1890s Blackbody Radiation

Rayleigh-Jeans Law uses equipartition technique but fails at high frequencies, yielding the 'Ultraviolet Catastrophe'

Wien's Law uses thermodynamic arguments but fails at low frequencies (later this would be shown to be equivalent to Boltzmann statistics)

Enter Planck...



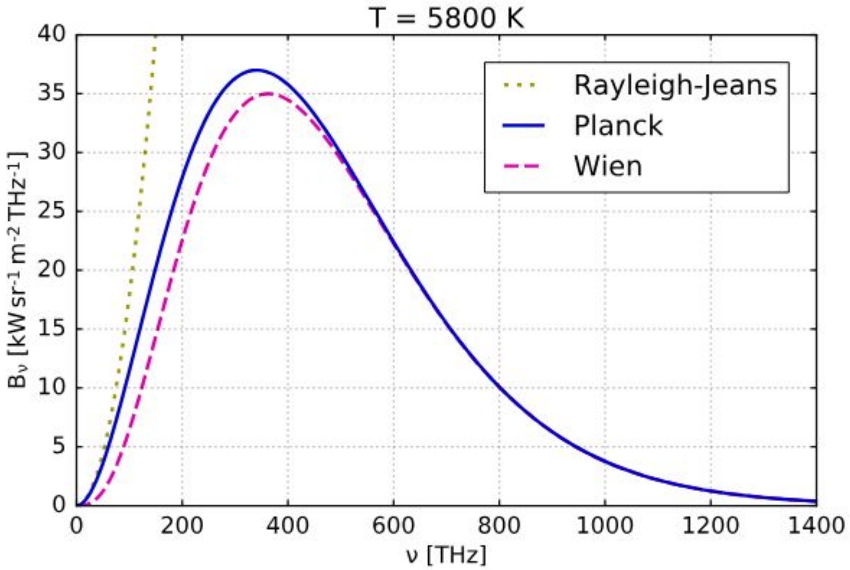
1900 Planck's Law

Figures out 1/exp-1 curve matches the data

Postulates **indistinguishable** quanta of energy emitted from distinguishable 'radiators'

$$B_\nu(\nu, T) = \frac{2h\nu^3}{c^2} \cdot \frac{1}{e^{\frac{h\nu}{k_B T}} - 1}$$

Both 'quanta' and 'indistinguishable' are required, otherwise you get 1/exp Boltzmann curve (like Wien)



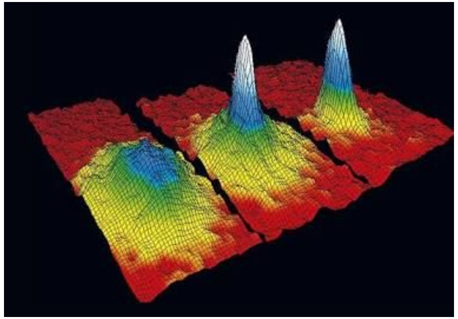
1924 Bose-Einstein

Bose reformulates Planck's law as 'indistinguishable photon gas'

So indistinguishable particles (photons) that can occupy any of a set of distinguishable energy levels.

Yields identical math and equation of Planck's law.

$$\bar{n}_i = \frac{g_i}{e^{(\varepsilon_i - \mu)/k_{\mathrm{B}}T} - 1}$$



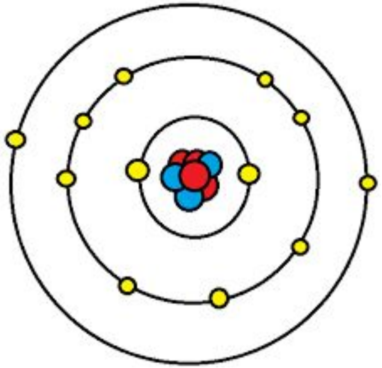
Einstein extends to massive particles (postulating Bose-Einstein condensate verified in 1995)

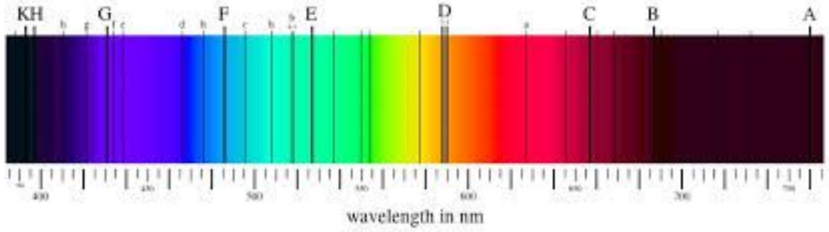
1925 Pauli Exclusion Principle

To justify spectral doublets and the periodic table, Pauli postulates two bold claims

1) Electrons can't be in the same state
2) Electrons two-valued 4th quantum number (ylm numbers already known)

Months later electron spin is discovered.



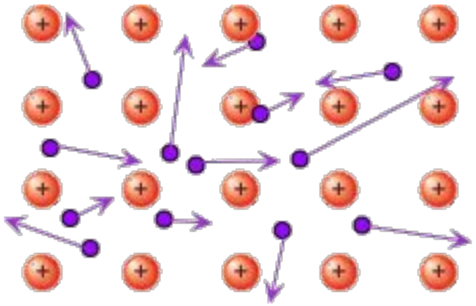


1925 Fermi-Dirac Statistics

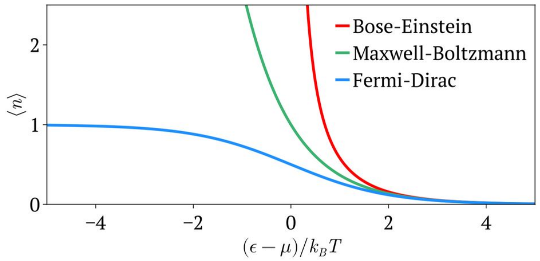
Independently, Fermi and Dirac math-out what indistinguishable particles that can't be in the same state means. The result is Fermi-Dirac statistics and fermions.

$$\bar{n}_i = \frac{1}{e^{(\varepsilon_i - \mu)/k_B T} + 1},$$

In a matter of months experiments modeling 'an electron gas' obeying Fermi-Dirac statistics verify the the specific heat capacities of metals.



Statistics Summary



		Same state occupation	Distinguishability
Fermi-Dirac	$\bar{n}_i = \frac{1}{e^{(\varepsilon_i-\mu)/k_\mathrm{B}T}+1},$	Disallowed	Indistinguishable
Bose-Einstein	$\bar{n}_i = \frac{g_i}{e^{(\varepsilon_i-\mu)/k_\mathrm{B}T}-1}$	Allowed	Indistinguishable
Maxwell-Boltzmann	$\langle N_i \rangle = \frac{g_i}{e^{(\varepsilon_i-\mu)/k_\mathrm{B}T}}$	Allowed	Distinguishable

Indistinguishability

In modern notation, wave functions indistinguishably means either of the following

$$|\Psi_1\Psi_2\rangle = |\Psi_2\Psi_1\rangle \qquad \text{symmetric}$$

$$|\Psi_1\Psi_2\rangle = -|\Psi_2\Psi_1\rangle \qquad \text{antisymmetric}$$

The minus sign is allowed because we observer the square of the wave function, and the -1 squares away.

Note if these equations are applied to the same state, symmetric wave functions work just fine, while the antisymmetric results in X=-X, which is only true for 0. So we can interpret bosons as having symmetric wave functions, and fermions as having antisymmetric.

In 'regular' QM these properties have to be manually added.

1928 Dirac Equation

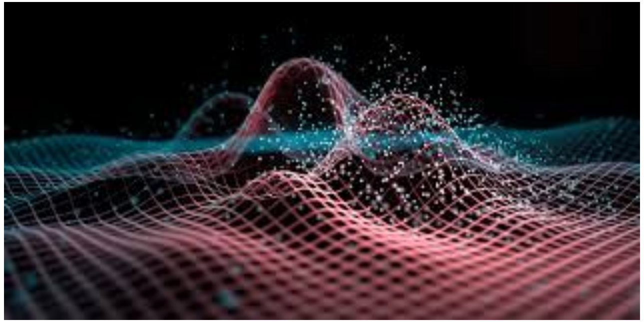
To extend QM to relativity, Dirac formulates his equation.

This equation implicitly describes spin ½ particles, that implicitly require anti-commutators, have the anti-symmetric property and correspondingly obey fermi-dirac statistics.

$$(i\partial\!\!\!/ - m)\psi = 0$$

1930s QFT

As QFT is invented and developed, the correlation between spin and commutators continues: systems that describe integer spin particles require commutators, and systems that describe half-integer spin require anti-commutators.



1940 Pauli Spin-Statistics Theorem

Pauli formulates the Spin-Statistics Theorem, relating

Integer Spin $\Longleftrightarrow$ Bosons (commutators,particles can occupy the same state)

½ Integer Spin $\Longleftrightarrow$ Fermions (anticommutators,particles cannot occupy the same state)

Postulates:

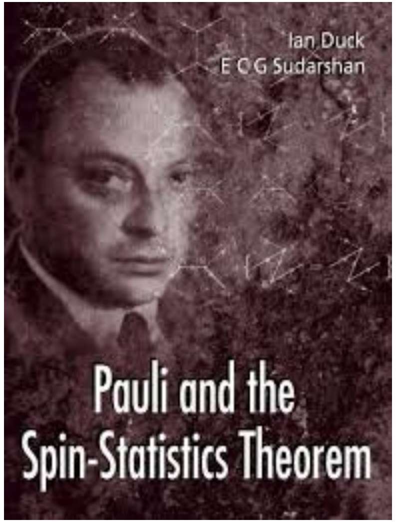
Relativistic QFT
Positive probabilities
Positive energy
Spatial commutativity
(and implicitly restricted analysis to commutators and anticommutators)

Modern Spin-Statistics Theorem

Later these postulates were relaxed to:

QM
Lorentz invariance
Locality

To this day, a very complicated theorem.



Spin-Statistics Derivation

Srednicki: Chap 4 (for spin 0, limited to commutators/anticommutators)

Mark Weitzman: https://www.youtube.com/watch?v=my2ZuQWdI14

Weinberg: 233-238