GR Problems 3 (Guidry)

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1 Chapter 11 Problem 5

Show that $L = g_{\mu\nu}u^\mu u^\nu$ with $u^\mu = \frac{dx^\mu}{d\tau}$ for motion of a test particle having unit mass in a Schwarzschild spacetime can be written

$$L = -c^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 (u^t)^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} (u^r)^2 + r^2 (u^\phi)^2$$

and this is equivalent to $E^2 = p^2c^2 + m^2c^4$ in flat space.

Answer:
Taking the dot product of the particles four velocity in this spacetime we get.

$$-c^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 (u^t)^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} (u^r)^2 + r^2 sin(\theta)^2 (u^\phi)^2 + r^2 (u^\theta)^2$$

We can set the theta component equal to a constant due to the rotational symmetry of the spacetime. We can find a plane in which the particle moving on a geodesic would not move from. We will set this plane to be $\theta = \frac{\pi}{2}$. This will give us the equation above. We can think of this particle as having unit mass therefore $p = mu = u$ with energy being the related to the conserved quantity $-p^2 = -m^2u^2 = m^2c^2 = c^2$

For flat space we have

$$-c^2 = u * u = -c^2 \left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2 \tag{1}$$

$$p = m_0 u \tag{2}$$

$$p^2 = m_0^2 u^2 = -m_0^2 c^2 = m_0^2 (-c^2 \left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2) \tag{3}$$

$$m_0^2 c^2 = m_0^2 \gamma^2 (c^2 - v^2) \tag{4}$$

$$m_0^2 c^4 \gamma^2 = m_0^2 c^4 + (\vec{p})^2 c^2 \tag{5}$$

$$E^2 = m_0^2 c^4 + (\vec{p})^2 c^2 \tag{6}$$

2 Chapter 11 Problem 14

Find an expression for the acceleration of a stationary observer in Schwarzschild spacetime. Show that the length of the acceleration 4 vector tends to infinity at the Schwarzschild radius.

Answer:

By stationary we mean that $dr = 0$, $d\phi = 0$, and $d\theta = 0$ or stationary to a distant observer. So we end up with the simple line element given below. Before I do any math I wanted to give some intuition for what is happening here. We can immediately see that if $r < 2M$, $ds^2 > 0$ which is impossible for a timelike path so it is no surprise we are going to get issues as we approach the horizon. In fact at the horizon $r = 2M$, $ds^2 = 0$ meaning we have a lightlike surface "stationary" at the horizon. I would like to note that at these extremes its helpful to give up on the intuition that r is a distance from some center and just think of it as a coordinate. A local observer would still measure the light traveling at c relative to them as they plunged to their doom.

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2$$

We get the proper acceleration (locally felt force) by measuring how much the geodesic equation fails. Specifically ...

$$|A^c| = |u^b \nabla_b u^c|$$

Our only non zero coordinate acceleration is for t leaving us with the only component being.

$$A^r = \Gamma^r_{tt} \left(\frac{dt}{d\tau}\right) \left(\frac{dt}{d\tau}\right) \tag{7}$$

$$A^r = \frac{M(r-2M)}{r^3} \left(1 - \frac{2M}{r}\right)^{-1} \tag{8}$$

$$A^r = \frac{M}{r^2} \left(1 - \frac{2M}{r}\right) \left(1 - \frac{2M}{r}\right)^{-1} \tag{9}$$

$$A^r = \frac{M}{r^2} \tag{10}$$

This may look like the Newtonian answer of $\frac{GM}{r^2}$ but we want the length and the metric is not flat.
For the force/acceleration needed

$$F = m|A| = m\sqrt{(A^r)^2 g_{rr}} = \frac{GMm}{r^2} \left(1 - \frac{2GM}{rc^2}\right)^{-\frac{1}{2}} \tag{11}$$

Which goes to infinity as we approach the horizon. I have also restored the factors of G and c.

3 Effective Potential for Orbits

Starting with the dot product for a timelike path.

$$-1 = -\left(1 - \frac{2M}{r}\right)(u^t)^2 + \left(1 - \frac{2M}{r}\right)^{-1}(u^r)^2 + r^2(u^\phi)^2$$

We can use symmetries of the spacetime to find constants of motion along geodesic paths. Specifically I am using killing vectors.

$$\xi_t = (1, 0, 0, 0) \tag{12}$$
$$\xi_\phi = (0, 0, 0, 1) \tag{13}$$
$$e^2 = u * \xi_t = \left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right) \tag{14}$$
$$l^2 = u * \xi_\phi = r^2\left(\frac{d\phi}{d\tau}\right) \tag{15}$$

Where e is an energy per unit mass and l is an angular momentum per unit mass.

$$1 = e^2\left(1 - \frac{2M}{r}\right)^{-1} - \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 - \frac{l^2}{r^2} \tag{16}$$
$$\left(1 - \frac{2M}{r}\right) = e^2 - \left(\frac{dr}{d\tau}\right)^2 - \frac{l^2}{r^2}\left(1 - \frac{2M}{r}\right) \tag{17}$$
$$e^2 + \left(\frac{dr}{d\tau}\right)^2 = \left(1 + \frac{l^2}{r^2}\right)\left(1 - \frac{2M}{r}\right) \tag{18}$$
$$\frac{e^2 - 1}{2} + \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 = -\frac{M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3} \tag{19}$$
$$V(r) = -\frac{M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3} \tag{20}$$

The first two terms are the usual Newtonian terms for the effective orbital potential. The last term is a new correction term that shows up in GR only.