Problems

A. Finite-Dimensional Hilbert Spaces (Sec. 2.1)

A1. Direct-sum $\leftrightarrow$ tensor-product isomorphism.

Ref.: §2.1; Eqs. (2.1.18)–(2.1.19); PDF p. 8.

Let H be a finite-dimensional Hilbert space and $H^{\oplus m}$ its m-fold direct sum. Show that the linear map $U : H^{\oplus m} \to \mathbb{C}^m \otimes H$ defined by

$$U\left(\bigoplus_{j=0}^{m-1} |x_j\rangle\right) = \sum_{j=0}^{m-1} |j\rangle \otimes |x_j\rangle$$

is unitary. Compute $U^\dagger$ explicitly and verify that orthogonal projectors onto the j-th summand are mapped to $|j\rangle\langle j| \otimes \mathbb{I}_H$.

A2. Block embeddings as isometries.

Ref.: §2.1; Eq. (2.1.13); PDF p. 6.

Using the same U, prove that for any operators $A_j \in \mathcal{L}(H)$ we have

$$U\left(\bigoplus_{j=0}^{m-1} A_j\right) U^\dagger = \sum_{j=0}^{m-1} |j\rangle\langle j| \otimes A_j.$$

Deduce that $\operatorname{rank}\left(\bigoplus_j A_j\right) = \sum_j \operatorname{rank}(A_j)$ and that $\operatorname{supp}\left(\bigoplus_j A_j\right) = \bigoplus_j \operatorname{supp}(A_j)$.

C. Trace and Cyclicity (Sec. 2.2.3)

C1. Cyclicity with rectangular factors.

Ref.: §2.2.3; Eqs. (2.2.25)–(2.2.26); PDF p. 14.

For $X \in \mathcal{L}(H_A, H_B)$, $Y \in \mathcal{L}(H_B, H_C)$, $Z \in \mathcal{L}(H_C, H_A)$, prove $\operatorname{Tr}[XYZ] = \operatorname{Tr}[YZX] = \operatorname{Tr}[ZXY]$. Then show $\operatorname{Tr}[(X \otimes Y)(Z \otimes W)] = \operatorname{Tr}[XZ]\operatorname{Tr}[YW]$.

D. Transpose, Adjoint, and Basis Changes (Sec. 2.2.4)

D1. Basis-dependence of transpose, basis-independence of adjoint.

Ref.: §2.2.4; Eqs. (2.2.27)–(2.2.30) and (2.2.28); PDF pp. 14–15.

Let $A \in \mathcal{L}(H_1, H_2)$ and let U, V be unitaries on H_1, H_2. Define $A' = VAU^\dagger$. Show that $(A')^T = (U^T)^{-1} A^T V^T$ (transpose depends on the chosen bases) but $(A')^\dagger = UA^\dagger V^\dagger$ (adjoint is basis independent).

D2. **Transpose superoperator in an operator basis.**

Ref.: §2.2.4; Eq. (2.2.30) and Exercise 2.7; PDF p. 15.

Let $\{E_{ij}\}$ be the standard operator basis. Show that the linear map $T(\cdot) = (\cdot)^T$ has representation $T(X) = \sum_{ij} E_{ji} X E_{ji}$.

E. Hilbert–Schmidt, Vectorization, and the Transpose Trick (Sec. 2.2.5)

E1. Vectorization identities.

Ref.: §2.2.5; Eqs. (2.2.31)–(2.2.33); PDF pp. 15–16.

Prove $\mathrm{vec}(AXB^T) = (B \otimes A)\mathrm{vec}(X)$ and $\mathrm{Tr}\big[A^\dagger B\big] = \langle \mathrm{vec}(A), \mathrm{vec}(B)\rangle$. Show that $\|A\|_2 = \|\mathrm{vec}(A)\|_2$.

Just state results for E1.

E2. **Maximally entangled vector calculus.**

Ref.: §2.2.5; Eqs. (2.2.34)–(2.2.36) and (2.2.45); PDF pp. 16–18.

Let $|\Gamma\rangle = \sum_i |i\rangle \otimes |i\rangle$. Show the "transpose trick" $(\mathbb{I} \otimes A)\,|\Gamma\rangle = (A^T \otimes \mathbb{I})\,|\Gamma\rangle$ and use it to derive $\mathrm{Tr}[X] = \langle\Gamma|\,(X \otimes \mathbb{I})\,|\Gamma\rangle$.

E3. **State-operator isomorphisms.**

Ref.: §2.2.5; Eqs. (2.2.43)–(2.2.45); PDF p. 18.

Show that every $|\psi\rangle \in H_A \otimes H_B$ can be uniquely written as $|\psi\rangle = (\mathbb{I} \otimes A)\,|\Gamma\rangle$ for some A, and also as $|\psi\rangle = (B \otimes \mathbb{I})\,|\Gamma\rangle$ for some B, with $A = T(B)$.

F. SVD, Schmidt, and Polar Decompositions (Sec. 2.2.7)

F1. **Best rank-r approximation (unitarily invariant).**

Ref.: §2.2.7; Theorem 2.1 (SVD) and Eqs. (2.2.56)–(2.2.59); PDF pp. 22–23.

Let $A = U\Sigma V^\dagger$ be an SVD. Show that among all rank-r operators X, the maximizer of $\mathrm{Re}\,\mathrm{Tr}\big[X^\dagger A\big]$ is $X = U\Sigma_r V^\dagger$, where Σ_r keeps the top r singular values and zeros out the rest.

F2. **Schmidt spectra of marginals.**

Ref.: §2.2.7; Theorem 2.2 (Schmidt) Eqs. (2.2.60)–(2.2.62) and vec Eqs. (2.2.34)–(2.2.36); PDF pp. 22–24 16–17.

If $|\psi\rangle = \sum_{i=1}^{r} \sqrt{\lambda_i}\,|a_i\rangle \otimes |b_i\rangle$ is a Schmidt decomposition, prove that the reduced operators $\rho_A = \mathrm{Tr}_B\,|\psi\rangle\!\langle\psi|$ and $\rho_B = \mathrm{Tr}_A\,|\psi\rangle\!\langle\psi|$ have the same nonzero spectra $\{\lambda_i\}_{i=1}^{r}$.