Equivalence Relations.

- given a set S it's a way of dividing it into subsets where elts of each subset are considered "the same".
- say elts $a, b \in S$ are equivalent as:

  $a \sim b$ if: $a \sim a$

  $a \sim b$ and $b \sim c \rightarrow a \sim c$

  $a \sim b \rightarrow b \sim a$
- it is equivalent to partitioning the set S.
- e.g., $S = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z} \right\}$
- define $\frac{a}{b} \sim \frac{c}{d}$ if one can be reduced to another by a factor.

  e.g. $\frac{8}{12} \sim \frac{4}{6} \sim \frac{2}{3}$.
- 

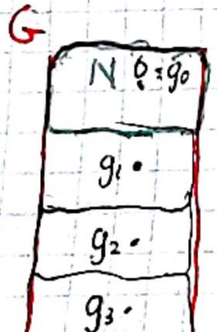
- pick a representative from each partition, x.
- denote the partition, or equivalence class as $[x] = \{ y \in S \mid y \sim x \}$

SEE PG 2 FOR EX.

CONT. HERE

- recall with groups:

  If $N \trianglelefteq G$ is a normal subgroup of of G, we can form the quotient group G/N. but taking cosets of N



$[0] = [g_0] = 0 + N$

$[g_1] = g_1 + N$

$[g_2] = g_2 + N$

$[g_3] = g_3 + N$

$[g] = \{ g + n \mid n \in N \}$

- we have $g \sim h$ if $g - h \in N$
- we write the equiv. classes at G/N.

An Important Equivalence Relation

Sept. 11/25 ②

Let $f: X \to Y$ be any function.

Define $a \sim_f b$ if $f(a) = f(b)$ is an equiv. relⁿ.

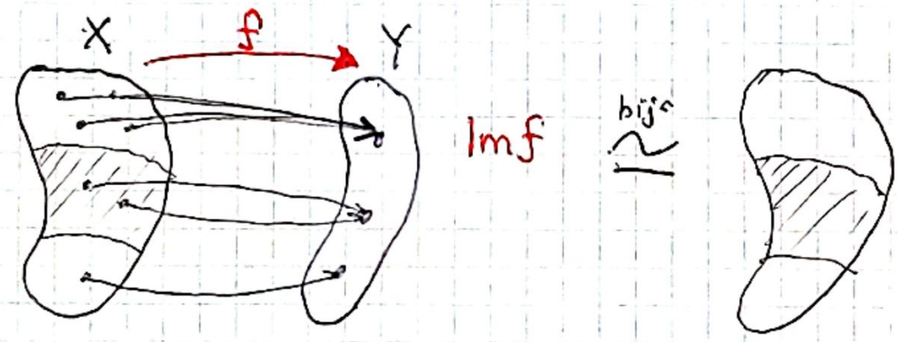
Why? – check e.r. conditions:

① reflexive: ∵ $f(a) = f(a)$ then $a \sim a$ ✓

② symmetric: if $f(a) = f(b)$ then $a \sim b$
but, $f(b) = f(a)$, too, so $b \sim a$. } ✓

③ transitive: clear ∵ $f(a) = f(b) = f(c)$ it follows if $\underset{\to f(a) = f(b)}{a \sim b}$ and $\underset{\to f(b) = f(c)}{b \sim c}$ then, $f(a) = f(c)$ and $a \sim c$.

E.g.,



• a set of three elts
• denoted $X/\sim_f$

⇒ CONT. HERE.

• note there's more structure here than a set
→ there is a group operation btwn elements!

⇒ this translates to the cosets. and quotient group ↪ is why it's called a group.

e.g. $\mathbb{Q}^* = \mathbb{Q} \setminus 0$ as a set & multⁿ as group operation.

• we say $\frac{a}{b} \sim \frac{c}{d}$ if $a = kc$, $b = kd$ for some $k \in \mathbb{Q}^*$.

e.g. $\frac{2}{3} \sim \frac{4}{6}$ ∵ $4 = 2 \cdot 2$, $6 = 2 \cdot 3$

- cosets respect ~~rational mult^n~~ group operation.
  → so given $g_1 \circ_G g_2 = g'$ as group elts.
  - we can define the operation on cosets as
  $$[g_1] \circ_c [g_2] = [g_1 \circ_G g_2] = [g']$$
- as cosets under $\circ_c$ this is a well defined group denoted G/N.
- note since N is a subgroup it contains the identity element 0 (or e).
- thus all elements of N are equivalent to the identity.

⇒ the elements of N can in a sense by considered to set to zero.

Relation to Vector Spaces:

- note that in the def^n of a vector space, V, all its elements, $\underline{v}$, behave as a group under vector add^n

① Closure: $v_1 + v_2 \in V \quad \forall\, v_1, v_2 \in V$

② Identity: $e = \underline{0}$ (so $\underline{v} + \underline{0} = \underline{0} + \underline{v} = \underline{v} \quad \forall\, \underline{v} \in V$)

③ Inverse: inverse of $\underline{v}$ is $-\underline{v}$, so $\underline{v} + (-\underline{v}) = \underline{0} \quad \forall\, \underline{v} \in V$

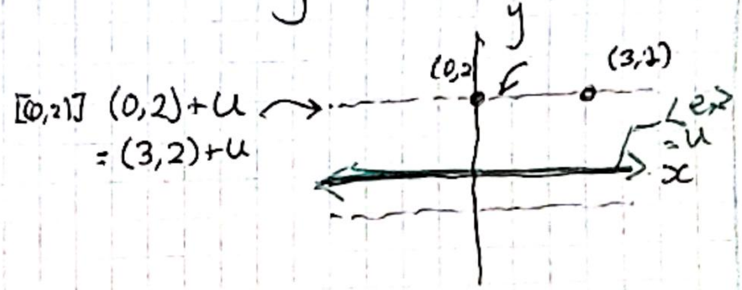
④ Assoc'ue: $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3 \quad \forall\, v_1, v_2, v_3 \in V$.

⇒ A vector subspace, U, can be considered a subgroup of a vector space V wrt vector addition.

∴ $\boxed{\text{WE CAN QUOTIENT OUT A VECTOR SUBSPACE.}}$

E.g., $\mathcal{U} = \langle e_x \rangle$ ~"span of" in $V = \mathbb{R}^2$

- like introduces a relation between x & y that sets $x = 0$.



- $\langle e_x \rangle = \text{Span}(e_x)$ Note: a vector space in its own right.

① equiv. relation:

$v_1 \sim v_2$ if $v_1 - v_2 \in \mathcal{U}$

or, $(x_1, y_1) - (x_2, y_2) = \alpha \cdot (1,0)$ for some $\alpha \in \mathbb{R}$

$\therefore$ $\left.\begin{array}{l} x_1 - x_2 = \alpha \\ y_1 - y_2 = 0 \end{array}\right\}$ • no restriction of x's. • but y_1 must equal y_2.

② coset are points $\{(x, y_0) \mid x \in \mathbb{R}$ & fixed $y_0\}$

③ natural coset representatives are $(0,y) \simeq \mathbb{R}$

④ cosets are $\{(0,y) + \mathcal{U}\} = \{(0,y) + \langle e_x \rangle\}$.

⑥ any point (x,y) can be projected onto its coset rep've by setting $x = 0$: $[(x,y)] \longrightarrow [(0,y)]$.

⑦ Coset vector add^{n} example:

$\Rightarrow$ from group theory we know $[v_1] +_c [v_2] = [v_1 +_v v_2]$

e.g. lhs $= [(3,5)] +_c [(2,1)]$ $\qquad$ rhs $= [(3,5) +_v (2,1)]$

$= [(0,5)] +_c [(0,1)]$ $\qquad$ $= [(5,6)]$

$= [(0,5) +_v (0,1)]$ $\qquad$ $= [(0,6)]$

$= [(0,6)]$ $\qquad$ $=$

8 Why is this like setting $x = 0$?

- take a general point (x, y).
- its coset is $[(x, y)]$.
- since cosets have their own add^n & mult^n by scalars, we can write

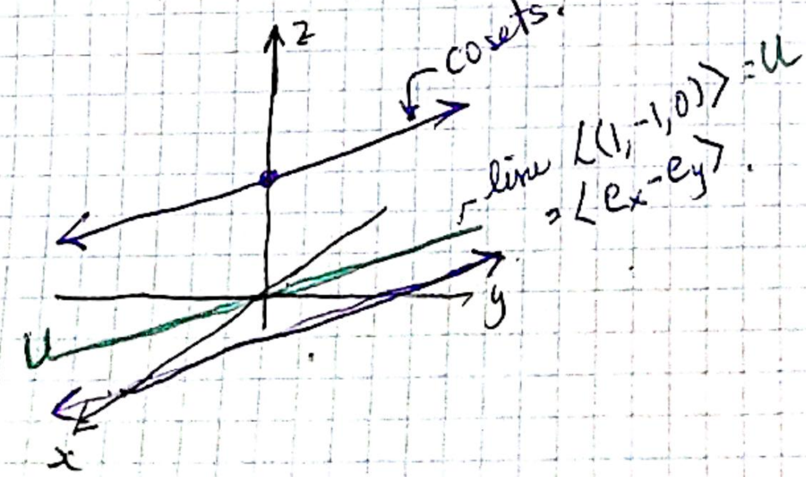
$$
\begin{aligned}
[(x,y)] &= [x e_x + y e_y] \\
&= x[e_x] +_c y[e_y] \\
&= x[\underset{\sim}{0}] +_c y[e_y] \\
&= [x \cdot \underset{\sim}{0}] +_c y[e_y] \\
&= [\underset{\sim}{0}] +_c y[e_y] \\
&= y[e_y]. \simeq \mathbb{R} \quad \text{is one dim}^n \text{ vector space.}
\end{aligned}
$$

E.g., let $V = \mathbb{R}^3$ and we want to impose the relation "$x = y$".

- this is the same as $x - y = 0$.

$\Rightarrow$ so quotient $\mathbb{R}^3$ by the subspace $\langle e_x - e_y \rangle$.

$$\mathbb{R}^3 / \langle e_x - e_y \rangle.$$



- to get the coset rep've of an arbitrary $v = (x,y,z)$, project onto a unit vector $\hat{u}$ in U and subtract

$\rightarrow$ so we'll have $[v] = [\hat{v}]$, where

$$\mathrm{proj}_{\hat{u}} v = (\underline{v} \cdot \hat{\underline{u}})\, \hat{\underline{u}}$$
$$= \left(v \cdot \frac{u}{|u|}\right) \frac{u}{|u|} \quad \text{for any } u \in U$$
$$= \frac{1}{|u|^2} (\underline{v} \cdot \underline{u})\, \underline{u}$$

- take $u = (1,-1,0)$.

$$\Rightarrow \mathrm{proj}_u v = \frac{1}{2}\left[(1,-1,0) \cdot (x,y,z)\right] (\ , \ , 0).$$
$$= \frac{x-y}{2} (1,-1,0).$$
$$= \left(\frac{x-y}{2}, \frac{y-x}{2}, 0\right).$$

$$\therefore \hat{v} = (x,y,z) - \frac{1}{2}(x-y, y-x, 0).$$
$$= \left(\frac{x+y}{2}, \frac{x+y}{2}, z\right) \simeq \mathbb{R}^2.$$

$$\therefore [(x,y,z)] = [(\underbrace{\tfrac{x+y}{2}, \tfrac{x+y}{2}}_{\text{"averaged out" } x \,\&\, y}, z)] \simeq \mathbb{R}$$

In what sense have we set "$x=y$"?

- Letting $\left.\begin{array}{l} e_x = (1,0,0) \\ e_y = (0,1,0) \end{array}\right\}$, we have.

$$\left.\begin{array}{l} [e_x] = [(1,0,0)] = [(\tfrac{1}{2},\tfrac{1}{2},0)] \\ [e_y] = [(0,1,0)] = [(\tfrac{1}{2},\tfrac{1}{2},0)] \end{array}\right\}$$ they are equal! $\Rightarrow$ call them $[a]$

- Note $[e_x]-[e_y] = [(\tfrac{1}{2},\tfrac{1}{2},0)-(\tfrac{1}{2},\tfrac{1}{2},0)] = [(0,0,0)] = [\underset{\sim}{0}]$

$\rightarrow$ as expected. $= [a]-[a] = [\underset{\sim}{0}]$.

- we can then reduce an arbitrary $[(x,y,z)]$ as:

$$\begin{aligned} [(x,y,z)] &= [x e_x + y e_y + z e_z] \\ &= x[e_x] + y[e_y] + z[e_z] \\ &= x[a] + y[a] + z[e_z] \\ &= (x+y)[a] + z[e_z]. \end{aligned}$$

$\Rightarrow$ We have set "$x=y$" in the sense the x & y directions are no longer distinguishable!

$\Rightarrow$ we can treat them as the same and are free to add components in a common basis. Here we called it $[a]$.

How does this get us to tensor products of vector spaces?

$\Rightarrow$ Basic idea is to start with a free vector space and the quotient out the relations we want.

$\rightarrow$ this gives us a lot of flexibility in defining vector spaces with a structure we want. e.g. direct sums,

FREE VECTOR SPACE - Def^n.

Given any set S, the free vector space is the set of all formal sums of elts of S with coefficients in $\mathbb{R}$. (can be $\mathbb{C}$). $= \sum_{v \in S} a_v \cdot v$

Not^n: $V(S)$.

Ex.

① Say $S = \{x\}$ is the singleton set

Then $v = ax$ for $a \in \mathbb{R}/\mathbb{C}$.

$\rightarrow$ is one dim^n'l

So, $V(\{x\}) \simeq \mathbb{R}$.

Note: Let $a = 0$. Then $v = 0x \equiv \underline{0}$

② Say $S = \{x_1, x_2, \dots, x_n\}$

• an arbitrary $v \in V(S)$ looks like,

$$v = a_1 x_1 + a_2 x_2 + \dots + a_n x_n \quad \bar{c} \; a_i \in \mathbb{R}$$

$\rightarrow$ this is n-dim'l and $V(S) \simeq \mathbb{R}^n$

Say $a_i = 0 \; \forall i$. Then $v = 0x_1 + \dots + 0x_n$ = "empty sum" $\equiv \underline{0}$

$\Rightarrow V(\emptyset) \equiv \underline{0}$

$\rightarrow$ note there are no relations btwn the x_i so this does not simplify down any further.

- we can mapp 1:1, $x_i \to e_i$: $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ← i'th pos^n.

③ More exotic : Let $S = \mathbb{R}$. The set of real numbers

- note we distinguish the real coefficients from the reals from $S = \mathbb{R}$.
- a $v \in V(\mathbb{R})$ is given by.

$$v = \sum_{i=1}^{n} a_i \cdot v_i \quad \text{where } a_i \in \mathbb{R}_1, \; v_i \in \mathbb{R}_2.$$

} different reals → they don't multiply!

- e.g. $v = 3(7) + 2(-4)$

different basis vectors of & don't combine.

$$v = \underbrace{2.1}_{\in \mathbb{R}}(\overbrace{\pi}^{\in S}) - \sqrt{5}(87).$$

- notationally write : $v = \sum_{i=1}^{n} a_i e_{x_i}$ where $a_i \in \mathbb{R}$, $x_i \in \mathbb{R}$.
  - distinguishes real coeffs & elts of S.

→ $v = 2.1 e_\pi - \sqrt{5} e_{87}$

NOTE: V(S) is uncountably infinite in dimension.

Claim: For any set S, V(S) is a vector space.

Why? (via def^n) ① define $\underset{\sim}{0}$ = "empty sum"

$$\to v + 0 = 0 + v = 0$$

② add^n : let $v = \sum_{i=1}^{n} a_i v_i$, $w = \sum_{j=1}^{m} b_j w_j$

→ define $v + w = \sum_i a_i v_i + b_j w_j$ and let:

$a_i v_i + b_j w_j = (a_i + b_j) v_i$ if $v_i = w_j$, leave alone if not equal

③ Let the negative of $v = \sum_i a_i v_i$ be,

$$-v \equiv \sum_i (-a_i) v_i$$

$\Rightarrow$ works as add've inverse.

④ etc. ...

④ E.g.. Let $S = \{(v,w) \mid v \in V, w \in W\}$.

$\rightarrow$ no relations!

Let $V * W = \mathcal{V}(S)$.

- a typical elt is: $\sum_{\substack{v \in V \\ w \in W}} a_{v,w} (v,w)$.
- V and W could be $\mathbb{R}^n$ and $\mathbb{R}^m$ for example.

$\Rightarrow$ this is a huge vector space c̄ uncountably infinite dimensions. in general.

e.g. i) Let $V = W = \underset{\sim}{0}$, zero vector space.

- $S = \{(0,0)\}$ $\rightarrow$ a single elt:

$\therefore \mathcal{V}(S) = \mathcal{V}(\{(0,0)\}) \Rightarrow V * W$ is one dim^n $\cong \mathbb{R}$.

ii) Let $V = \mathbb{Q}$ & $W = \mathbb{R}$.

$\Rightarrow S = \{(0,a) \mid a \in \mathbb{R}\}$

$\therefore \mathcal{V}(S)$ is uncoutntably infinite in dim^n.