Physics with Friends – QFT

The Quantum Dirac Field

Chapter 38, QFT for Gifted Amateurs

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Canonical quantization – Dirac equation

Euler-Lagrange equation for classical fields

$$\frac{\partial \mathcal{L}}{\partial \psi^\alpha} - \partial^\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\alpha)} \right) = 0$$

The Dirac Lagrangian density

$$\mathcal{L}_{\text{Dirac}} := \overline{\psi} \cdot (\, \mathbf{i}\, \gamma^\mu \,\partial_\mu - m\,)\, \psi = \mathbf{i}\, \overline{\psi}\, \gamma^\mu \,\partial_\mu \psi - m \cdot \overline{\psi} \cdot \psi$$

Dirac equation as Euler-Lagrange equation of $\mathcal{L}_{\text{Dirac}}$ with respect to $\overline{\psi}$

$$\frac{\partial \mathcal{L}_{\text{Dirac}}}{\partial(\partial \overline{\psi})} = 0 \qquad \text{and} \qquad \frac{\partial \mathcal{L}_{\text{Dirac}}}{\partial \overline{\psi}} = \mathbf{i}\, \gamma^\mu \,\partial_\mu \psi - m \cdot \psi$$

Hence:

$$0 = \frac{\partial \mathcal{L}_{\text{Dirac}}}{\partial \overline{\psi}} - \frac{\partial \mathcal{L}_{\text{Dirac}}}{\partial(\partial \overline{\psi})} = \left(\mathbf{i}\, \gamma^\mu \,\partial_\mu \psi - m \cdot \psi \right) - \left(0 \right) = \mathbf{i}\, \gamma^\mu \,\partial_\mu \psi - m \cdot \psi$$

Canonical quantization – conjugate momentum

The Dirac Lagrangian density

$$\mathcal{L}_{\text{Dirac}} \;:=\; \overline{\psi} \cdot (\,\mathbf{i}\,\gamma^{\mu}\,\partial_{\mu} \,-\, m\,)\,\psi \;=\; \mathbf{i}\,\overline{\psi}\,\gamma^{\mu}\,\partial_{\mu}\psi \,-\, m \cdot \overline{\psi} \cdot \psi$$

Conjugate momentum

$$\Pi^{0}_{\psi} \;:=\; \frac{\partial \mathcal{L}_{\text{Dirac}}}{\partial(\partial_0 \psi)} \;=\; \mathbf{i}\,\overline{\psi}\,\gamma^{0} \qquad \text{and} \qquad \Pi^{0}_{\overline{\psi}} \;:=\; \frac{\partial \mathcal{L}_{\text{Dirac}}}{\partial(\partial_0 \overline{\psi})} \;=\; 0$$

$$\begin{aligned} \mathcal{H}_{\text{Dirac}} \;&=\; \Pi^{0}_{\psi} \cdot \partial_0 \psi \,+\, \Pi^{0}_{\overline{\psi}} \cdot \partial_0 \overline{\psi} \,-\, \mathcal{L}_{\text{Dirac}} \;=\; \mathbf{i}\,\overline{\psi}\,\gamma^{0}\,\partial_0\psi \,-\, \mathcal{L}_{\text{Dirac}} \;=\; \cdots \\ &=\; \psi^{\dagger}\,\mathbf{i}\,\partial_0\psi \end{aligned}$$

Canonical quantization – mode expansions

Second quantization of Eqn's (36.45) and (36.47), p.331, [2]

$$\begin{aligned}\widehat{\psi}(x) &= \int \left(\frac{1}{(2E_p)^{1/2}} \sum_{s=1}^{2} \Big(u^s(p)\widehat{a}_{s,\mathbf{p}}\, e^{-\mathbf{i}\,p\cdot x} + v^s(p)\widehat{b}^\dagger_{s,\mathbf{p}}\, e^{+\mathbf{i}\,p\cdot x} \Big) \right) \frac{\mathrm{d}^3 p}{(2\pi)^{3/2}} \\ \widehat{\overline{\psi}}(x) &= \int \left(\frac{1}{(2E_p)^{1/2}} \sum_{s=1}^{2} \Big(\overline{u}^s(p)\widehat{a}^\dagger_{s,\mathbf{p}}\, e^{+\mathbf{i}\,p\cdot x} + \overline{v}^s(p)\widehat{b}_{s,\mathbf{p}}\, e^{-\mathbf{i}\,p\cdot x} \Big) \right) \frac{\mathrm{d}^3 p}{(2\pi)^{3/2}}\end{aligned}$$

Anti-commutation relations

$$\left.\begin{aligned} \Big\{ \widehat{\psi}_a(t,\mathbf{x}) \,,\, \widehat{\psi}^\dagger_b(t,\mathbf{y}) \Big\} &= \delta_{ab}\cdot\delta^{(3)}(\mathbf{x}-\mathbf{y}) \\ \Big\{ \widehat{\psi}_a(t,\mathbf{x}) \,,\, \widehat{\psi}_b(t,\mathbf{y}) \Big\} &= 0 \\ \Big\{ \widehat{\psi}^\dagger_a(t,\mathbf{x}) \,,\, \widehat{\psi}^\dagger_b(t,\mathbf{y}) \Big\} &= 0 \end{aligned}\right\} \Longrightarrow \left\{\begin{aligned} \Big\{ \widehat{a}_{s,\mathbf{p}} \,,\, \widehat{a}^\dagger_{r,\mathbf{q}} \Big\} &= \delta_{sr}\cdot\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\ \Big\{ \widehat{b}_{s,\mathbf{p}} \,,\, \widehat{b}^\dagger_{r,\mathbf{q}} \Big\} &= \delta_{sr}\cdot\delta^{(3)}(\mathbf{p}-\mathbf{q}) \end{aligned}\right.$$

Feynman propagator for fermions

$$\langle\, 0 \,|\, T\,\psi(x)\,\overline{\psi}(y)\,|\, 0\,\rangle \;=\; G_0(x,y) \;=\; \mathbf{i}\cdot S(x-y) \;=\; \cdots$$

$$=\; \int \left(\frac{\mathbf{i}\cdot e^{-\,\mathbf{i}\,p\cdot(x-y)}}{\not{p} - m + \mathbf{i}\,\varepsilon} \right) \frac{\mathrm{d}^4 p}{(2\pi)^4}$$

For a derivation of last equality that does NOT use path integrals, see §11.6.1, p.423, [1].

Feynman rules and scattering

?????

Lagrangian of QED

- Lorentz invariant
- gauge invariant (w.r.t. local $U(1)$ symmetry)
- renormalizability

$$
\begin{aligned}
\mathcal{L}_{\text{QED}} &= \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Yang-Mills}} + \mathcal{L}_{\text{Interaction}} \\
&= \left(\mathbf{i}\,\overline{\psi}\,\gamma^\mu\,\partial_\mu\psi - m\cdot\overline{\psi}\cdot\psi \right) - \frac{1}{4}\,(F^{(A)})_{\mu\nu}\,(F^{(A)})^{\mu\nu} - q\cdot\overline{\psi}\,\gamma^\mu\,A_\mu\,\psi \\
&= \left(\mathbf{i}\,\overline{\psi}\,\gamma^\mu\,(\partial_\mu + \mathbf{i}\,q\,A_\mu)\,\psi - m\cdot\overline{\psi}\cdot\psi \right) - \frac{1}{4}\,(F^{(A)})_{\mu\nu}\,(F^{(A)})^{\mu\nu} \\
&= \left(\mathbf{i}\,\overline{\psi}\,\gamma^\mu\,D^{(A)}_\mu\psi - m\cdot\overline{\psi}\cdot\psi \right) - \frac{1}{4}\,(F^{(A)})_{\mu\nu}\,(F^{(A)})^{\mu\nu}
\end{aligned}
$$

Thank You!

[1] D'Auria, R., and Trigiante, M.
From Special Relativity to Feynman Diagrams: A Course in Theoretical Particle Physics for Beginners.
Springer, 2016.

[2] Lancaster, T., and Blundell, S. J.
Quantum Field Theory for the Gifted Amateur.
Oxford University Press, 2014.