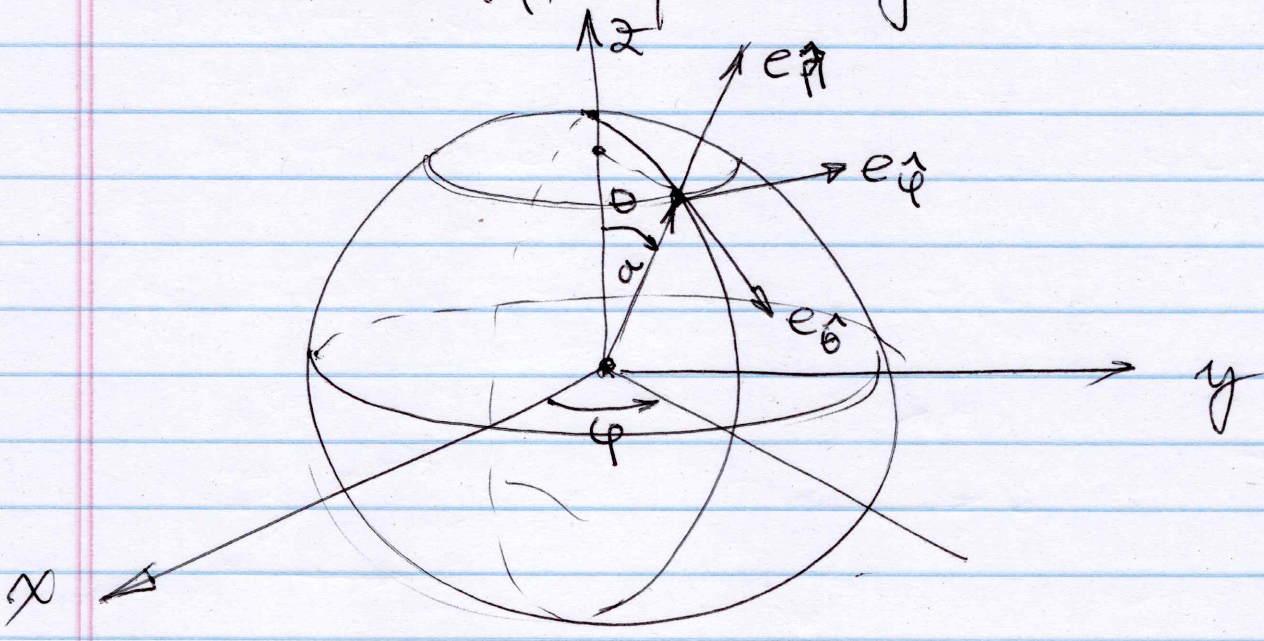
Modern Classical Physics
(Thorne & Blandford)

Exercise 1.11: Integral of vector field over sphere

Integrate the vector field $A = z e_z$ over a sphere with radius a, centered at the origin of the Cartesian coordinate system (i.e., compute $\int A \cdot d\Sigma$).

a) Introduce spherical coordinates on the sphere, and construct the vectorial element, $d\Sigma$, using the two legs $a\, d\theta\, e_{\hat\theta}$ and $a \sin\theta\, d\varphi\, e_{\hat\varphi}$.

$e_{\hat\theta}$ & $e_{\hat\varphi}$ are unit length vectors along the directions θ & φ, respectively.



$(e_{\hat r}, e_{\hat\theta}, e_{\hat\varphi})$ spherical basis vectors (orthonormal basis).

Explain the factors $a\,d\theta$ and $a\sin\theta$ in the definition of the legs.

$e_{\hat\theta}$ & $e_{\hat\varphi}$ are unit vectors tangent to the coordinate lines of θ & φ, respectively.

Legs to construt the vectorial element, $d\Sigma$.

$$\begin{cases} A = \dfrac{\partial \bar{R}}{\partial \theta} d\theta = \left|\dfrac{\partial \bar{R}}{\partial \theta}\right| e_{\hat\theta}\, d\theta & \& \\ \\ B = \dfrac{\partial \bar{R}}{\partial \varphi} d\varphi = \left|\dfrac{\partial \bar{R}}{\partial \varphi}\right| e_{\hat\varphi}\, d\varphi; & \text{but} \end{cases}$$

$$\bar{R} = x\, e_{\bar{x}} + y\, e_{\bar{y}} + z\, e_{\bar{z}}$$

$$= (a\sin\theta\cos\varphi) e_{\bar{x}} + (a\sin\theta\sin\varphi) e_{\bar{y}} + a\cos\theta\, e_{\bar{z}}$$

from which:

$$\begin{cases} \dfrac{\partial \bar{R}}{\partial \theta} = (a\cos\theta\cos\varphi) e_{\bar{x}} + (a\cos\theta\sin\varphi) e_{\bar{y}} - a\sin\theta\, e_{\bar{z}}. \\ \\ \dfrac{\partial \bar{R}}{\partial \varphi} = (-a\sin\theta\sin\varphi) e_{\bar{x}} + (a\sin\theta\cos\varphi) e_{\bar{y}} \end{cases}$$

We can now calculate the norms (lengths) of these two tangent vetors:

$$\left|\frac{\partial \bar{R}}{\partial \theta}\right|^2 = a^2\cos^2\theta\cos^2\varphi + a^2\cos^2\theta\sin^2\varphi + a^2\sin^2\theta$$

$$= a^2\cos^2\theta(\cos^2\varphi+\sin^2\varphi) + a^2\sin^2\theta$$

$$= a^2(\cos^2\theta+\sin^2\theta) = a^2 \;\therefore$$

$$\left|\frac{\partial R}{\partial \theta}\right| = a.$$

Also,

$$\left|\frac{\partial \bar{R}}{\partial \varphi}\right|^2 = a^2\sin^2\theta\sin^2\varphi + a^2\sin^2\theta\cos^2\varphi$$

$$= a^2\sin^2\theta(\sin^2\varphi+\cos^2\varphi) = a^2\sin^2\theta$$

$$\left|\frac{\partial R}{\partial \varphi}\right| = a\sin\theta$$

Consequently, the legs A & B are:

$$A = a\,d\theta\, e_{\hat{\theta}} \;\&\; B = a\sin\theta\, d\varphi\, e_{\hat{\varphi}}. \checkmark$$

We can then construct the vectorial element $d\Sigma$ as:

$$d\Sigma = A\times B = \varepsilon(-, A, B).$$

$$= \varepsilon(-, a\,d\theta\, e_{\hat{\theta}}, a\sin\theta\, d\varphi\, e_{\hat{\varphi}}).$$

Due to the linearity of the Levi-Civita tensor, $\mathcal{E}$, we can write:

$$d\Sigma = \mathcal{E}(\_, e_{\hat{\theta}}, e_{\hat{\varphi}})\, a^2 \sin\theta\, d\varphi\, d\theta \checkmark$$

b) Using $z = a\cos\theta$ and

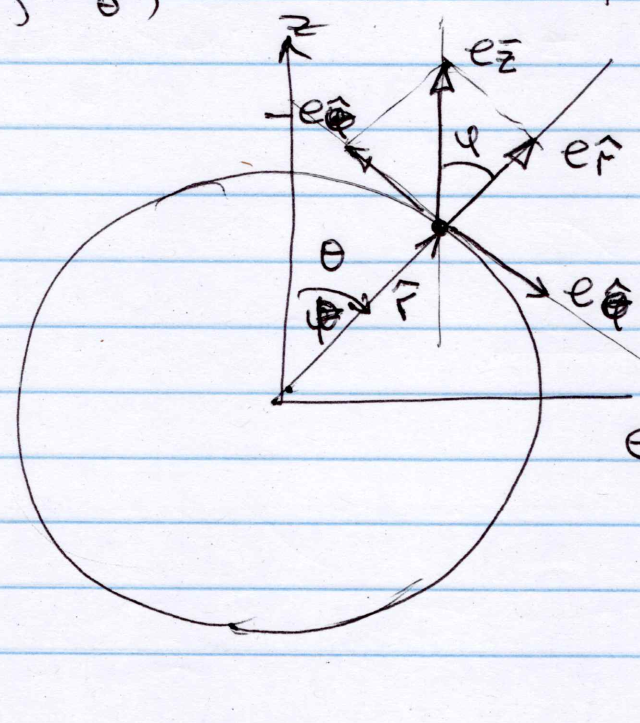
$$e_{\hat{z}} = \cos\theta\, e_{\hat{r}} - \sin\theta\, e_{\hat{\theta}},$$

Show that:

$$A \cdot d\Sigma = a\cos^2\theta\, \mathcal{E}(e_{\hat{r}}, e_{\hat{\theta}}, e_{\varphi})\, a^2 \sin\theta\, d\theta\, d\varphi$$

Note: $A = z e_{\hat{z}}$. is the vector field here, not one of the legs used above-

To derive the expression of $e_{\hat{z}}$ in terms of $e_{\hat{r}}$ & $e_{\hat{\theta}}$, we can use the following figure.



$e_{\hat{z}} = a e_{\hat{r}} + b e_{\hat{\theta}}$

$a = \cos\varphi;\ b = -\sin\varphi$

$e_{\hat{z}} = \cos\varphi\, e_{\hat{r}} - \sin\theta\, e_{\theta}$ $\checkmark$

So that:

$$A = z\, e_{\hat{z}} = a \cos\theta \left(\cos\theta\, e_{\hat{r}} - \sin\theta\, e_{\hat{\theta}} \right)$$

$$= a \cos^2\theta\, e_{\hat{r}} - a \cos\theta \sin\theta\, e_{\hat{\theta}}.$$

Consequently:

$$A \cdot d\Sigma = \left(a \cos^2\theta\, e_{\hat{r}} - a \cos\theta \sin\theta\, e_{\hat{\theta}} \right) \cdot$$

$$\varepsilon(\_, e_{\hat{\theta}}, e_{\hat{\varphi}})\, a^2 \sin\theta\, d\theta\, d\varphi$$

$$= a \cos^2\theta\, e_{\hat{r}} \cdot \varepsilon(\_, e_{\hat{\theta}}, e_{\hat{\varphi}})\, a^2 \sin\theta\, d\theta\, d\varphi$$

$$- a \cos\theta \sin\theta\, e_{\hat{\theta}} \cdot \varepsilon(\_, e_{\hat{\theta}}, e_{\hat{\varphi}})\, a^2 \sin\theta\, d\theta\, d\varphi$$

Now: $e_{\hat{r}} \cdot \varepsilon(\_, e_{\hat{\theta}}, e_{\hat{\varphi}}) = e_{\hat{r}} \cdot (e_{\hat{\theta}} \times e_{\hat{\varphi}})$

$$= \varepsilon(e_{\hat{r}}, e_{\hat{\theta}}, e_{\hat{\varphi}}) = 1$$

and:

$$e_{\hat{\theta}} \cdot \varepsilon(\_, e_{\hat{\theta}}, e_{\hat{\varphi}}) = e_{\hat{\theta}} \cdot (e_{\hat{\theta}} \times e_{\varphi})$$

$$= \varepsilon(e_{\hat{\theta}}, e_{\hat{\theta}}, e_{\hat{\varphi}}) = 0.$$

Then:

$$A \cdot d\Sigma = a \cos^2\theta\, \varepsilon(e_{\hat{r}}, e_{\hat{\theta}}, e_{\hat{\varphi}})\, a^2 \sin\theta\, d\theta\, d\varphi.$$

c) Explain $\mathcal{E}(e_{\hat{r}}, e_{\hat{\theta}}, e_{\hat{\varphi}}) = 1$.

Since $(e_{\hat{r}}, e_{\hat{\theta}}, e_{\hat{\varphi}})$ is an orthonormal, right sided basis; $\mathcal{E}(e_{\hat{r}}, e_{\hat{\theta}}, e_{\hat{\varphi}})$. represents a positive, unit volume.

$$\mathcal{E}(e_{\hat{r}}, e_{\hat{\theta}}, e_{\hat{\varphi}}) = +1 \checkmark$$

d) Perform the integral $\int A \cdot d\Sigma$ over the entire sphere -

$$\int_\Sigma A \cdot d\Sigma = \int_{\varphi=0}^{2\pi} d\varphi \int_{\theta=0}^{\pi} a^3 \cos^2\theta \sin\theta \, d\theta$$

$$= a^3 \int_{\varphi=0}^{2\pi} d\varphi \int_{\theta=0}^{\pi} \cos^2\theta \sin\theta \, d\theta .$$

It is easy to see that

$$\frac{d}{d\theta}\left(-\frac{1}{3}\cos^3\theta\right) = \cos^2\theta \sin\theta, \text{ so that,}$$

$$\int \cos^2\theta \sin\theta \, d\theta = \frac{(-1)}{3}\cos^3\theta + C .$$

Consequently,

$$\int_{\varphi=0}^{\pi} \cos^2\theta \sin\theta \, d\theta = -\frac{1}{3}\cos^3\theta \Big|_0^{\pi} = \left(-\frac{1}{3}\right)(-1-1)$$

$$= \frac{2}{3}$$

$$\therefore \int_\Sigma A \cdot d\Sigma = a^3 \int_{\varphi=0}^{2\pi} d\varphi \, (2/3) = \frac{2}{3} a^3 \varphi \Big|_0^{2\pi}$$

$$= \frac{4}{3}\pi a^3$$

$$\int_\Sigma A \cdot d\Sigma = \frac{4}{3}\pi a^3$$ ✓ Volume of sphere.