Singular Value Decomposition - quick review

- called SVD
- a basic tool from linear algebra.
- generalizes eigen-decomposition to non-square matrices.

→ w/o proof:

Any matrix $A \in \mathbb{C}^{m \times n}$ can be written in product form:

$$A_{m\times n} = U_{m\times m}\, \Sigma_{m\times n}\, V^{\dagger}_{n\times n}$$

with: U, V unitary,

: $\Sigma = \begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \; 0 \end{bmatrix}_{m\times n}$ $\bar{c}\ \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0.$

→ positive & descending.

- example,

$$\begin{bmatrix} & A & \end{bmatrix}_{m\times n} = \begin{bmatrix} U \end{bmatrix}_{m\times m} \begin{array}{c} \quad r \quad \Sigma \\ r \begin{bmatrix} \cdot & & & & \\ & \cdot & & & \\ & & 0 & & \\ & & & \ddots & 0 \end{bmatrix}_{m\times n} \end{array} \begin{bmatrix} & V^{\dagger} & \\ & & \\ & & \end{bmatrix}_{n\times n}$$

rot^{n} in output space

scaling along r axis.

rotation in input space

$$\begin{bmatrix} & A & \end{bmatrix}_{m\times n} \begin{bmatrix} v \\ \\ \\ \end{bmatrix}_{n\times 1} = \begin{bmatrix} \; \end{bmatrix}_{m\times 1}$$ as matrix-vector mult^{n}.

- r is the rank of matrix A with $r \leq \min(m,n)$ here.
- special case:

① $r=1$ rank 1: $A = \sigma\, \underline{u}\, \underline{v}^{\dagger} = \sigma |u\rangle\langle v|$

② select $k < r$ of the largest k singular values, $\{\sigma_i\}_{i=1}^{r}$:

→ yields the closest in Euclidean norm rank k approximation of A → used in PCA, recommendor systems, photogrammetry, signal processing, ...

Schmidt Decomposition

- a useful canonical form in QIT.
- takes a (generally) messy superpositioned state into a simpler standard form by a co-ord transformation.
- it's the same state, just more compact.
- does this for a state in the tensor product of two Hilbert spaces:

Theorem: Schmidt Decomposition.

Let H_A & H_B be two Hilbert spaces and $|\psi\rangle \in H_A \otimes H_B$ a unit vector.

Then there exist ONB $\{|i_A\rangle\} \subset H_A$ and $\{|i_B\rangle\} \subset H_B$ and real numbers $\{\lambda_i\} > 0$ $\bar{c}$ $\sum_i \lambda_i^2 = 1$ such that:

$$\boxed{|\psi\rangle = \sum_{i=1}^{r} \lambda_i \, |i_A\rangle \otimes |i_B\rangle}.$$

- it says that even though in an arbitrary ONB of the two Hilbert spaces a state vector can be expressed of a sum of up to $d_A d_B$ product basis states, a unique ONB exists such that is the sum of only r product basis, with $r \leq \min(d_A, d_B)$.

Proof:

- in general you can always write some $|\psi\rangle \in H_A \otimes H_B$ as:

$$|\psi\rangle = \sum_{jk}^{d_A d_B} c_{jk} |j\rangle_A \otimes |k\rangle_B$$

for any ONB $\{|j\rangle_A\}$ in H_A and $\{|k\rangle_B\}$ in H_B.

- we are going to rotate them to a new basis that reduces the sum from $d_A d_B$ terms (in general) to $r \leq \min(d_A, d_B)$.
  $\Rightarrow$ use SVD.
- put the coeff's c_{ij} into a matrix, C, such that

$$[C]_{jk} = c_{jk}$$

- take its SVD,

$$C = UDV^\dagger \quad \& \quad D = \mathrm{diag}(\lambda_1, \lambda_2, \cdots \lambda_r, 0, \cdots, 0)$$

- then,

$$|\psi\rangle = \sum_{jk} [UDV^\dagger]_{jk} |j\rangle_A \otimes |k\rangle_B$$

$$= \sum_{jk} \sum_i U_{ji} D_{ii} [V^\dagger]_{ik} |j\rangle_A \otimes |k\rangle_B$$

$$= \sum_{jk} \sum_i U_{ji} \underbrace{D_{ii}}_{\lambda_i} V^*_{ki} |j\rangle_A \otimes |k\rangle_B$$

$$= \sum_i \lambda_i \left(\sum_j U_{ji} |j\rangle_A\right) \otimes \left(\sum_k V^*_{ki} |k\rangle_B\right)$$

$$\boxed{|\psi\rangle = \sum_i \lambda_i |i_A\rangle \otimes |i_B\rangle} \quad \text{where:} \quad |i_A\rangle \equiv \sum_j U_{ji} |j\rangle_A$$

$$|i_B\rangle \equiv \sum_k V^*_{ki} |k\rangle_B$$

Example:

- Suppose we want to build rank 1 decomposition for a qubit pair $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$.
- since there are only two singular values,
$$\lambda_1 = 1 \quad , \quad \lambda_2 = 0 .$$
- $\therefore \; D = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}$
- take U & V to be real orthogonal:
$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad V = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

- the coefficient matrix is:
$$C = U D V^{\dagger}$$
$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
where $a = \cos\theta\cos\phi$, $b = \cos\theta \sin\phi$, $c = \sin\theta\cos\phi$, $d = \sin\theta\sin\phi$

- we have the new basis:
$$|i'_A\rangle = \sum_j U_{ji} |j\rangle_A \Rightarrow |0'_A\rangle = \cos\theta|0\rangle_A + \sin\theta|1\rangle_A$$
$$|0'_B\rangle = \cos\phi|0\rangle_B + \sin\phi|1\rangle_B .$$

- $|\psi\rangle$ in the new basis is:
$$\boxed{|\psi\rangle = 1 \cdot |0'_A\rangle \otimes |0'_B\rangle}$$
← a pure state as expected with $r=1$

- note the $|1'_A\rangle \otimes |1'_B\rangle$ is multiplied by $\lambda_2 = 0$.
- Check: expand $|0'_A\rangle \otimes |0'_B\rangle = (c_\theta|0\rangle_A + s_\theta|1_A\rangle) \otimes (c_\phi|0\rangle_B + s_\phi|1_B\rangle)$
$$= \underbrace{c_\theta c_\phi}_{a}|00\rangle + \underbrace{c_\theta s_\phi}_{b}|01\rangle + \underbrace{s_\theta c_\phi}_{c}|10\rangle + \underbrace{s_\theta s_\phi}_{d}|11\rangle$$

→ see $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ entries above. ✓

Question?

Where did the degrees of freedom go?

- a general ket $|\psi\rangle$ in $H_A \otimes H_B$ has the form

$$|\psi\rangle = \sum_{j,k=1}^{d_A, d_B} c_{jk} \, |j\rangle_A \otimes |k\rangle_B$$

for an arbitrary ONB $\{|j\rangle_A\}$ and $\{|k\rangle_B\}$ on H_A & H_B.

- this can have up to $d_A \cdot d_B$ non-zero coefficients (dof)
- but under a new ONB, $\{|i_A\rangle\}, \{|j_B\rangle\}$, we can write it:

$$|\psi\rangle = \sum_{i=1}^{r} \lambda_i \, |i_A\rangle \otimes |i_B\rangle$$

$\Rightarrow$ where $r \leq \min(d_A, d_B)$.

- how did we go from $d_A d_B$ dof to $\min(d_A, d_B)$ without losing something?

$\Rightarrow$ they're in the special basis vectors that make it work!

Quick Explanation:

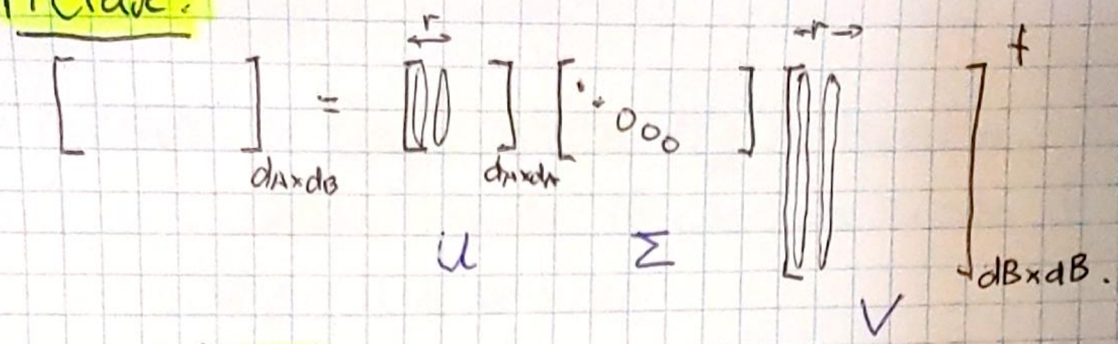
- original c_{jk} coefficients are in the $C_{d_A \times d_B}$ matrix.
  $\Rightarrow 2 d_A \cdot d_B - 1$ dof. (subtract 1 for $\sum_{jk} |c_{jk}|^2 = 1$ norm. constraint)
  2 ∵ complex!
- we have the SVD:

$$\Big[\quad \Big]_{d_A \times d_B} = \Big[\overbrace{\,|||\,}^{r} \; U \Big]_{d_A \times d_A} \begin{bmatrix} \cdot & & & \\ & \cdot & & \emptyset \\ & & 0_{\,0_{\,0}} & \\ \emptyset & & & \end{bmatrix}_{d_A \times d_B} \Big[\overbrace{\,||\,}^{r} \; V \Big]^{\dagger}_{d_B \times d_B}$$

- note: Only the first r columns of U & V actually count in the matrix mult:

$\Rightarrow$ - the first r singular values keep them.
- the rest of the 0-diagonal elts nullify the rest of the columns.

Picture:



Count of dof:

- singular values: r (real)
- $\mathcal{U}$ matrix: $2rd_A$ (complex)
- $-r^2$ (orthonormality/inner product constraint)
- V matrix: $2rd_B$
  $-r^2$
- phases btwn A-B basis vectors $-r$
- global phase -1

TOTAL: $\boxed{2(d_A+d_B)r - 2r^2 - 1}$

- say d_A is the smallest dim^n & $r = d_A$ to maximize dof:

$$\Rightarrow 2(d_A+d_B)d_A - 2d_A^2 - 1$$

$$= 2d_A^2 + 2d_Ad_B - 2d_A^2 - 1$$

$$\boxed{= 2d_Ad_B - 1}$$ $\Rightarrow$ equals dof of C matrix!

ENTANGLEMENT (finally!)

- consider a single state $|\psi\rangle_{AB}$ in $H_A \otimes H_B$.
- it is a product state if we can write it in the form
$$|\psi\rangle_{AB} = |u\rangle_A \otimes |v\rangle_B .$$
  $\Rightarrow$ it is determined by a single vector in H_A and a single vector in H_B, separatly!
- this is a special (trivial) case of a Schmidt decomposition,
$$|\psi\rangle = \sum_{i=1}^{r} \sigma_i |i_A\rangle \otimes |i_B\rangle$$
  where: $r=1$
  $\sigma_1 = 1$
  $|i_A\rangle = |u\rangle_A, \ |i_B\rangle = |v\rangle_B$.
- suppose we have some other $|\phi\rangle \in H_A \otimes H_B$ and do a S.D.
$$|\phi\rangle = \sum_{i=1}^{r} \sigma_i |i_A\rangle \otimes |i_B\rangle \quad \text{where } r>1.$$
  $\rightarrow$ this decomp^n is unique and r is the smallest possible
  $\Rightarrow$ if $r>1$ we say the state is entangled.

That is, if we can't write a state vector as a single tensor product, $|\psi\rangle_{AB} = |u\rangle_A \otimes |v\rangle_B$, and must include more as a sum, e.g. $|\psi\rangle \propto |u_1\rangle_A \otimes |v_1\rangle_B + |u_2\rangle_A \otimes |v_2\rangle_B + \cdots$, then it is entangled.

- so by def^n: "Entangled" $\Leftrightarrow$ Schmidt rank >1.

The intuition is that if we can't describe the joint state in $H_A \otimes H_B$ by two separate independent substate vectors, alone in H_A & H_B, they are entangled in their behaviour.

"Not entangled" $\Leftrightarrow$ "Separable/Factored" $\Leftrightarrow$ Schmidt $r=1$

Here is where it gets confusing:

You are dealing $\bar{c}$ 3 different binary conditions.

1. Compositeness: i) 1 qubit, ii) 2 qubit
2. Purity: i) single state vector, $|\psi\rangle$ (pure)
   ii) ensemble of state vectors, $\{|\psi_i\rangle, p_i\}$ (mixed)
3. Entanglement: i) Separable (can be factored).
   ii) entangled.

Start with:

1. Single Qubit/Quantum System:
   - so a single Hilbert state space, H_A.
     NOT: $H_A \otimes H_B$.

   ∴ entanglement (i.e., being separable or entangled) does not enter the picture.

   Only two cases to consider then for Purity: i) pure, or ii) mixed

   i) Purity = pure:
   - a state is a single vector, $|\psi\rangle_A \in H_A$
   - density matrix $\rho = |\psi\rangle_A\langle\psi|_A$

   Test: i) rank 1
   ii) single non-zero eigenvalue
   iii) $\mathrm{tr}\,\rho^2 = 1$

   } equivalent.

   ii) Purity = mixed:
   - $|\psi\rangle$ is drawn from an ensemble or distribution, $\{|\psi_i\rangle, p_i\}$
   - a density matrix $\rho = \sum_{i=1}^{n>1} p_i |\psi_i\rangle\langle\psi_i|$

   Test: i) rank > 1
   ii) more than one non-zero eigenvalue
   iii) $\mathrm{tr}\,\rho^2 < 1$.

Now consider:

2 Bipartite (composite) 2 Qubit System:

- a Hilbert space given by $H_A \otimes H_B$.

→ entanglement is possible.

A Purity = pure:

- system described by a single state vector: $|\psi\rangle_{AB}$
- density matrix $\rho_{AB} = |\psi\rangle_{AB} \langle\psi|_{AB}$
- again test for purity is:

$$\boxed{\begin{array}{l} \rho \text{ is: } \cdot \text{ rank } 1 \\ \quad\quad \cdot \text{ one eigenvalue} \\ \quad\quad \cdot \ tr\rho^2 = 1 \end{array}}$$

Now test for entanglement: Two cases:

i) Separable

→ we can factorize the state into a single tensor product: $|\psi\rangle_{AB} = |\phi_A\rangle \otimes |\phi_B\rangle$ for some $|\phi_A\rangle \in H_A$, $|\phi_B\rangle \in H_B$.

Test: $\boxed{\text{Schmidt rank } r = 1}$.

- quick test for two level bipartite system:
- general $|\psi\rangle_{AB} = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$.
- coeff. matrix $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0$.

→ iff a singular value / eigenvalue is 0 then C is singular and $\det C = 0$.

⇒ then $r = 1$ & vector is separable.

∴ $ab - bc = 0 \iff$ not entangled

$\neq 0 \iff$ entangled.

ii) Entangled: Schmidt rank $r > 1$ (cannot cleanly factor state)

Example:

① $|\psi\rangle_{AB} = |00\rangle + |01\rangle$

$$\therefore C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \neq 0$$

$\det C = 1\cdot 0 - 1\cdot 0 = 0$.

$\therefore$ C is rank 1 & separable (not entangled).

Why?: Write:

$$|\psi\rangle_{AB} = |00\rangle + |01\rangle$$
$$= |0\rangle_A \otimes |0\rangle_B + |0\rangle_A \otimes |1\rangle_B$$
$$= \underset{\in H_A}{|0\rangle_A} \otimes \underbrace{\left(|0\rangle_B + |1\rangle_B \right)}_{\in H_B}$$

② $|\psi\rangle_{AB} = (|00\rangle + |11\rangle) / \sqrt{2}$.

$$\Rightarrow C \propto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}.$$

$$\det C = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} - 0\cdot 0$$
$$= \frac{1}{2} \neq 0.$$

$\therefore$ C is Schmidt rank 2 and is entangled.

What about mixed bipartite systems?

$\Rightarrow$ much more complicated situation

- need to define what "factoring" a state/density matrix even means

$\Rightarrow$ generally, an NP-hard problem!

Categories of Compositeness, Purity, and Entanglement

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Table 1: Quantum system conditions — Compositeness / Purity / Entanglement

Condition	Type	
Compositeness	**Single Qubit: A** $\|\psi\rangle_A = \|0\rangle_A$ $\|\psi\rangle_A = \|0\rangle_A + \|1\rangle_A$	**Two Qubits: AB (Composite, Bipartite)** $\|\psi\rangle_{AB} = \|01\rangle_{AB}$ $\|\psi\rangle_{AB} = \|00\rangle_{AB} + \|01\rangle_{AB}$
Purity	**Single State Vector (Pure)** $\|\psi\rangle_A, \ \|\psi\rangle_{AB}$ $\rho_A = \|\psi\rangle_A \langle\psi\|_A$ $\rho_{AB} = \|\psi\rangle_{AB} \langle\psi\|_{AB}$	**Ensemble of SV's (Mixed, Mixture)** $\{\|\psi_i\rangle, p_i\}$ $\rho = \sum_i p_i \|\psi_i\rangle \langle\psi_i\|$
Entanglement	**Separable (Factored)** $\|\psi\rangle_{AB} = \|1\rangle_A \otimes \|0\rangle_B$ $\|\psi\rangle_{AB} = \|0\rangle_A \otimes (\|0\rangle_B + \|1\rangle_B)$	**Entangled** $\|\psi\rangle_{AB} = \|0\rangle_A \otimes \|0\rangle_B + \|1\rangle_A \otimes \|1\rangle_B$

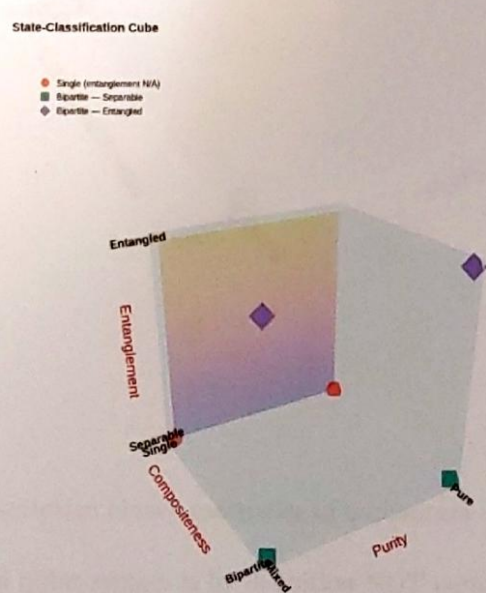

Figure 1: The three descriptive binary categories of a quantum system: Compositness, Purity and Engtanglement. **Note**: A single isolated qubit system is by definition NOT entangled with another. So two of the top corners of the cube, (Single, Entangled, Pure/Mixed), are not valid and left empty.

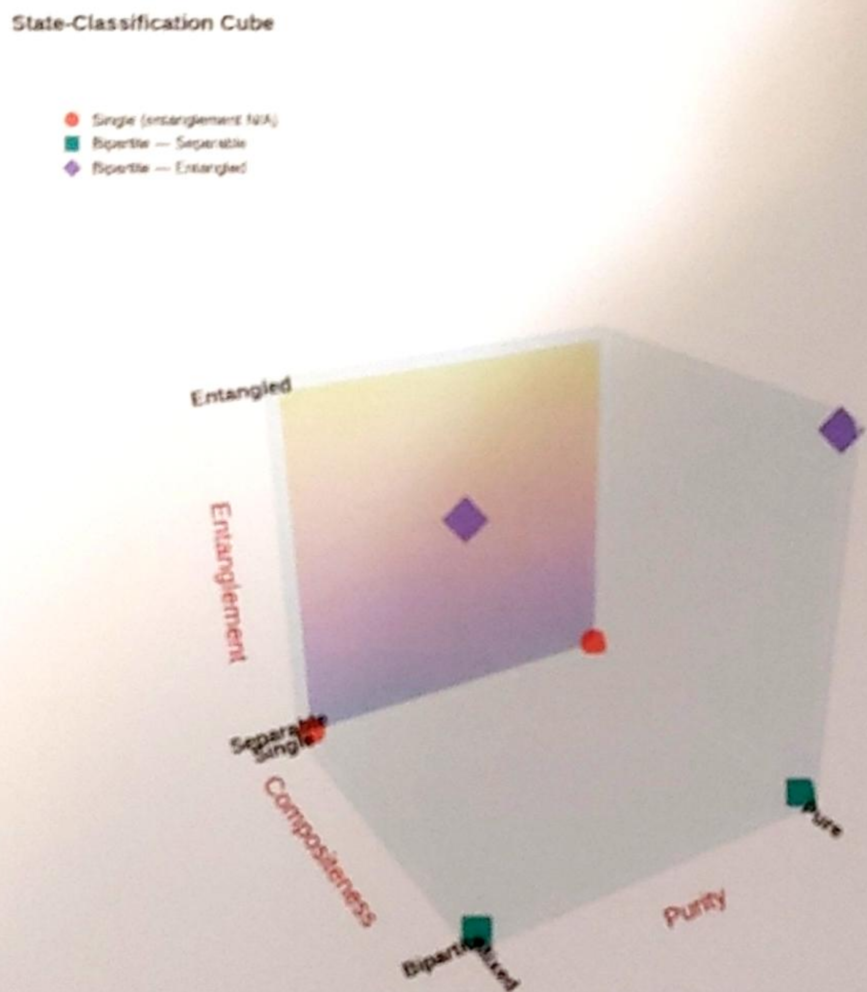

Figure 2: The three descriptive binary categories of a quantum system: Compositness, Purity and Engtanglement.
Note: A single isolated qubit system is by definition NOT entangled with another. So two of the top corners of the cube, (Single, Entangled, Pure/Mixed), are not valid and left empty.