Problem 9.5: The Interaction Picture

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In this problem, we will derive a formula for

$$\langle 0|T\,\varphi(x_n)\cdots\varphi(x_1)|0\rangle$$

without using path integrals.

Suppose we have a Hamiltonian density

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1,$$

where

$$\mathcal{H}_0 = \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2,$$

and $\mathcal{H}_1$ is a function of $\Pi(\mathbf{x},0)$ and $\varphi(\mathbf{x},0)$ and their spatial derivatives. (It should be chosen to preserve Lorentz invariance, but we will not be concerned with this issue.)

We add a constant $-E_0$ to H so that

$$H|0\rangle = 0.$$

Let $|\emptyset\rangle$ be the ground state of H_0, with a constant $-E_\emptyset$ added so that

$$H_0\,|\emptyset\rangle = 0.$$

(H_1 is then defined as $H - H_0$.)

The Heisenberg-picture field is

$$\varphi(\mathbf{x},t) \equiv e^{iHt}\varphi(\mathbf{x},0)e^{-iHt}. \tag{9.33}$$

We now define the *interaction-picture* field

$$\varphi_I(\mathbf{x},t) \equiv e^{iH_0t}\varphi(\mathbf{x},0)e^{-iH_0t}. \tag{9.34}$$

(a) Show that $\varphi_I(x)$ obeys the Klein–Gordon equation, and hence is a free field.

$$\varphi_I(x) \equiv e^{iH_0t}\varphi(\vec{x},0)e^{-iH_0t}$$

$$\frac{\partial\varphi_I}{\partial t}(x) = e^{iH_0t}[iH_0, \varphi(\vec{x},0)]e^{-iH_0t}$$

$$= e^{iH_0t}\, i\int d^3x' \left[\tfrac{1}{2}\pi^2(\vec{x}',0)+\tfrac{1}{2}(\vec{\nabla}'\varphi(\vec{x}',0))^2+\tfrac{1}{2}m^2\varphi^2(\vec{x}',0) - E_\phi,\ \varphi(\vec{x},0)\right]e^{-iH_0t}$$

Using $[\pi(\vec{x}',0), \varphi(\vec{x},0)] = -i\,\delta^3(\vec{x}'-\vec{x})$,

$$\frac{\partial\varphi_I(x)}{\partial t} = e^{iH_0t}\int d^3\vec{x}'\ \pi(\vec{x}',0)\,\delta^3(\vec{x}'-\vec{x})\,e^{-iH_0t}$$

$$= e^{iH_0t}\pi(\vec{x},0)e^{-iH_0t}$$

Then

$$\frac{\partial^2}{\partial t^2}\varphi_I(t) = e^{iH_0t}[iH_0, \pi(\vec{x},0)]e^{-iH_0t}$$

$$= e^{iH_0t}\, i\int d^3\vec{x}'\left[\tfrac{1}{2}\pi^2(\vec{x}',0)+\tfrac{1}{2}(\vec{\nabla}'\varphi(\vec{x}',0))^2+\tfrac{1}{2}m^2\varphi^2(\vec{x}',0)-E_\phi,\ \pi(\vec{x},0)\right]e^{-iH_0t}$$

$$= -e^{iH_0t}\int d^3\vec{x}'\left[\vec{\nabla}'\varphi(\vec{x}',0)\cdot\vec{\nabla}'\delta^3(\vec{x}'-\vec{x})+m^2\varphi(\vec{x}',0)\delta^3(\vec{x}'-\vec{x})\right]e^{-iH_0t}$$

$$= e^{iH_0t}\left[\nabla^2\varphi(\vec{x},0)-m^2\varphi(\vec{x},0)\right]e^{-iH_0t}$$

$$= \nabla^2\varphi_I(\vec{x},t) - m^2\varphi_I(\vec{x},t)$$

$$\therefore\quad 0 = \frac{\partial^2}{\partial t^2}\varphi_I(t) - \nabla^2\varphi_I(\vec{x},t) + m^2\varphi_I(\vec{x},t)$$

$$= (-\partial_\mu\partial^\mu + m^2)\varphi_I(\vec{x},t)$$

(b) Show that

$$\varphi(x) = U^\dagger(t)\varphi_I(x)U(t),$$

where $U(t) = e^{iH_0t}e^{-iHt}$ is unitary.

$$\varphi(\vec{x},t) = e^{iHt}\varphi(\vec{x},0)e^{-iHt}$$

$$= \underbrace{e^{iHt}e^{-iH_0t}}_{U^\dagger(t)}\underbrace{e^{iH_0t}\varphi(\vec{x},0)e^{-iH_0t}}_{\varphi_I(x)}\underbrace{e^{iH_0t}e^{-iHt}}_{U(t)}$$

$$= U^\dagger(t)\varphi_I(x)U(t)$$

(c) Show that $U(t)$ obeys the differential equation

$$i\frac{d}{dt}U(t) = H_I(t)U(t),$$

where

$$H_I(t) = e^{iH_0t}H_1e^{-iH_0t}$$

is the interaction Hamiltonian in the interaction picture, and that $U(0) = 1$.

$$U(t) = e^{iH_0t}e^{-iHt} \Rightarrow U(0) = 1$$

$$i\frac{dU(t)}{dt} = ie^{iH_0t}(iH_0)e^{-iHt} + ie^{iH_0t}(-iH)e^{-iHt}$$

$$= e^{iH_0t}(H-H_0)e^{-iHt}$$

$$= e^{iH_0t}H_1e^{-iH_0t}e^{iH_0t}e^{-iHt}$$

$$= H_I(t)U(t)$$

(d) If $\mathcal{H}_1$ is specified by a particular function of the Schrodinger-picture fields $\Pi(\mathbf{x},0)$ and $\varphi(\mathbf{x},0)$, show that $\mathcal{H}_I(t)$ is given by the same function of the interaction-picture fields $\Pi_I(\mathbf{x},t)$ and $\varphi_I(\mathbf{x},t)$.

$$\mathcal{H}(\vec{x},0) = \sum_{m,n} C_{m,n}\,\pi^m(\vec{x},0)\,\varphi^n(\vec{x},0)$$

$$\mathcal{H}_I(\vec{x},t) = e^{iH_0t}\mathcal{H}_1(\vec{x},0)e^{-iH_0t}$$

$$= \sum_{m,n} C_{m,n}\left(e^{iH_0t}\pi(\vec{x},0)e^{-iH_0t}\right)^m\left(e^{iH_0t}\varphi(\vec{x},0)e^{-iH_0t}\right)^n$$

$$= \sum_{m,n} C_{m,n}\,\pi_I^m(\vec{x},t)\,\varphi_I^n(\vec{x},t)$$

(e) Show that, for $t > 0$,

$$U(t) = T\exp\left[-i\int_0^t dt'\, H_I(t')\right] \tag{9.35}$$

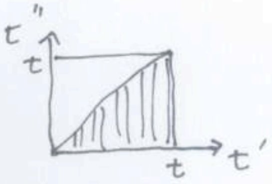
obeys the differential equation and boundary condition of part (c). What is the comparable expression for $t < 0$? (Hint: you may need to define a new ordering symbol.)

$$U(0) = 1 \quad , \quad \frac{i\,dU(t)}{dt} = H_I(t)\,U(t)$$

Re-express as an integral equation: $U(t) = 1 - i\int_0^t dt'\, H_I(t')\,U(t')$

Interate:

$$\begin{aligned} U(t) &= 1 - i\int_0^t dt'\, H_I(t')\,U(t') \\ &= 1 - i\int_0^t dt'\, H_I(t') + (-i)^2\int_0^t dt'\, H_I(t')\int_0^{t'} dt''\, H_I(t'')\,U(t'') \\ &= 1 - i\int_0^t dt'\, H_I(t') + (-i)^2\int_0^t dt'\, H_I(t')\int_0^{t'} dt''\, H_I(t'') + \dots \end{aligned}$$



Consider the second-order term:

$$\begin{aligned} &\int_0^t dt'\int_0^{t'} dt''\, H_I(t')\,H_I(t'') \\ &= \frac{1}{2}\left(\int_0^t dt'\int_0^{t'} dt''\, H_I(t')\,H_I(t'') + \int_0^t dt''\int_0^{t''} dt'\, H_I(t'')\,H_I(t')\right) \\ &= \frac{1}{2}\left(\int_0^t dt'\int_0^{t'} dt''\, H_I(t')\,H_I(t'') + \int_0^t dt'\int_{t'}^{t} dt''\, H_I(t'')\,H_I(t')\right) \\ &= \frac{1}{2}T\int_0^t dt'\int_0^t dt''\, H_I(t')\,H_I(t'') \end{aligned}$$

In general, $U(t) = \sum_{n=0}^{\infty} T\,\frac{(-i)^n}{n!}\left(\int_0^t dt'\, H_I(t')\right)^n$

$$= T\exp\left(-i\int_0^t dt'\, H_I(t')\right)$$

But this holds only for $t > 0$.

For $t < 0$, $U(t) = \overline{T}\exp\left(-i\int_0^t dt'\, H_I(t')\right)$

where $\overline{T}$ is the anti-time ordering operator.

Note that $U^\dagger(t) = \begin{cases} \overline{T}\exp\left(+i\int_0^t dt'\, H_I(t')\right), & t > 0 \\ T\exp\left(+i\int_0^t dt'\, H_I(t')\right), & t < 0 \end{cases}$

so that, for $t > 0$,

$$U(t)\,U^\dagger(t) = T\exp\left(-i\int_0^t dt'\, H_I(t')\right)\cdot\overline{T}\exp\left(+i\int_0^t dt'\, H_I(t')\right) = 1$$

(f) Define

$$U(t_2, t_1) = U(t_2)U^\dagger(t_1).$$

Show that, for $t_2 > t_1$,

$$U(t_2, t_1) = T\exp\left[-i\int_{t_1}^{t_2} dt'\, H_I(t')\right]. \tag{9.36}$$

What is the comparable expression for $t_1 > t_2$?

We rely on the identity, that if $A = A(t_2)$ and $B = B(t_1)$ are two Heisenberg picture operators, and $t_2 > t_1$, then $Te^{A+B} = Te^A \cdot Te^B$.

There are 3 case to consider:

$t_2 > t_1 > 0$:

$$\begin{aligned} U(t_2, t_1) &= U(t_2)U(t_1)^\dagger \\ &= T\exp\left[-i\int_0^{t_2} dt' H_I(t')\right] \cdot \overline{T}\exp\left[+i\int_0^{t_1} dt' H_I(t')\right] \\ &= T\exp\left[-i\int_{t_1}^{t_2} dt' H_I(t') - i\int_0^{t_1} dt' H_I(t')\right] \cdot \quad '' \\ &= T\exp\left[-i\int_{t_1}^{t_2} dt' H_I(t')\right] \cdot T\left[-i\int_0^{t_1} dt' H_I(t')\right] \cdot \overline{T}\left[+i\int_0^{t_1} dt' H_I(t')\right] \\ &= T\exp\left[-i\int_{t_1}^{t_2} dt' H_I(t')\right] \end{aligned}$$

$t_2 > 0 > t_1$

$$\begin{aligned} U(t_2, t_1) &= T\exp\left[-i\int_0^{t_2} dt' H_I(t')\right] T\exp\left[+i\int_0^{t_1} dt' H_I(t')\right] \\ &= T\exp\left[-i\int_{t_1}^{t_2} dt' H_I(t')\right] \end{aligned}$$

$0 > t_2 > t_1$

$$\begin{aligned} U(t_2, t_1) &= \overline{T}\exp\left[-i\int_0^{t_2} dt' H_I(t')\right] T\exp\left[i\int_0^{t_2} dt' H_I(t') + i\int_{t_2}^{t_1} dt' H_I(t')\right] \\ &= \overline{T}\exp\left[-i\int_0^{t_2} dt' H_I(t')\right] T\exp\left[i\int_0^{t_2} dt' H_I(t')\right] \cdot T\exp\left[-i\int_{t_1}^{t_2} dt' H_I(t')\right] \\ &= T\exp\left[-i\int_{t_1}^{t_2} dt' H_I(t')\right] \end{aligned}$$

For $t_1 > t_2$, note that

$$\begin{aligned} U^\dagger(t_2, t_1) &= U(t_1)U(t_2)^\dagger = U(t_1, t_2) \\ &= T\exp\left[-i\int_{t_2}^{t_1} dt' H_I(t)\right] \end{aligned}$$

$$\Rightarrow U(t_2, t_1) = \overline{T}\exp\left[-i\int_{t_1}^{t_2} dt' H_I(t')\right]$$

(g) For any time ordering, show that

$$U(t_3,t_1) = U(t_3,t_2)U(t_2,t_1) \quad \text{and} \quad U^\dagger(t_1,t_2) = U(t_2,t_1).$$

$$U(t_3,t_2)U(t_2,t_1) = U(t_3)U(t_2)^\dagger U(t_2)U(t_1)^\dagger = U(t_3)U(t_1)^\dagger = U(t_3,t_1)$$

$$U^\dagger(t_1,t_2) = (U(t_1)U(t_2)^\dagger)^\dagger = U(t_2)U(t_1)^\dagger = U(t_2,t_1)$$

(h) Show that

$$\varphi(x_n)\cdots\varphi(x_1) = U^\dagger(t_n,0)\varphi_I(x_n)U(t_n,t_{n-1})\varphi_I(x_{n-1})\cdots U(t_2,t_1)\varphi_I(x_1)U(t_1,0). \tag{9.37}$$

From part (b), $\varphi(\vec{x},t) = U^\dagger(t)\varphi_I(\vec{x},t)U(t)$,

$$\Rightarrow \varphi(x_n)\cdots\varphi(x_1) = \underbrace{U^\dagger(t_n)}_{\substack{U(0)U^\dagger(t_n) \\ = U^\dagger(t_n,0)}}\varphi_I(x_n)\underbrace{U(t_n)U^\dagger(t_{n-1})}_{U(t_n,t_{n-1})}\cdots\underbrace{U(t_2)U^\dagger(t_1)}_{U(t_2,t_1)}\varphi_I(x_1)\underbrace{U(t_1)}_{\substack{U(t_1)U(0)^\dagger \\ = U(t_1,0)}}$$

$$= U^\dagger(t_n,0)\varphi_I(x_n)U(t_n,t_{n-1})\cdots U(t_2,t_1)\varphi_I(x_1)U(t_1,0)$$

(i) Show that

$$U^\dagger(t_n,0) = U^\dagger(\infty,0)U(\infty,t_n), \quad \text{and} \quad U(t_1,0) = U(t_1,-\infty)U(-\infty,0).$$

For any $T > t_n$,

$$U^\dagger(T,0)U(T,t_n) = U^\dagger(T)U(T)U^\dagger(t_n) = U(0)U^\dagger(t_n) = U^\dagger(t_n,0)$$

For any $T < t_1$,

$$U(t_1,T)U(T,0) = U(t_1)U(T)^\dagger U(T)U^\dagger(0) = U(t_1)U(0)^\dagger = U(t_1,0)$$

(j) Replace H_0 with $(1-i\epsilon)H_0$, and show that

$$\langle 0|U^\dagger(\infty,0) = \langle 0|\emptyset\rangle\langle\emptyset|, \quad \text{and} \quad U(-\infty,0)|0\rangle = |\emptyset\rangle\langle\emptyset|0\rangle.$$

$$\lim_{T\to-\infty} U(T,0)|0\rangle = \lim_{T\to-\infty} U(T)|0\rangle = \lim_{T\to-\infty} e^{iH_0(1-i\epsilon)T} e^{-iHT}|0\rangle$$

$$= \lim_{T\to-\infty} e^{iH_0(1-i\epsilon)T} \sum_n |n\rangle\langle n|0\rangle = |\phi\rangle\langle\phi|0\rangle$$

similarly, $\lim_{T\to\infty} \langle 0|U^\dagger(T,0) = \langle 0|\phi\rangle\langle\phi|$

(k) Show that

$$\langle 0|\varphi(x_n)\cdots\varphi(x_1)|0\rangle = \langle\emptyset|U(\infty,t_n)\varphi_I(x_n)U(t_n,t_{n-1})\varphi_I(x_{n-1})\cdots U(t_2,t_1)\varphi_I(x_1)U(t_1,-\infty)|\emptyset\rangle \times |\langle\emptyset|0\rangle|^2. \tag{9.38}$$

$\langle 0|\varphi(x_n)\ldots\varphi(x_1)|0\rangle$

$$= \underbrace{\langle 0|U^\dagger(t_n,0)}_{\langle 0|U^\dagger(\infty,0)U(\infty,t_n)}\varphi_I(x_n)U(t_n,t_{n-1})\ldots U(t_2,t_1)\varphi_I(x_1)\underbrace{U(t_1,0)|0\rangle}_{U(t_1,-\infty)U(-\infty,0)|0\rangle}$$

$= \langle 0|\phi\rangle\langle\phi|U(\infty,t_n)$

$= U(t_1,-\infty)|\phi\rangle\langle\phi|0\rangle$

$$= |\langle\phi|0\rangle|^2\langle\phi|U(\infty,t_n)\varphi_I(x_n)U(t_n,t_{n-1})\ldots U(t_2,t_1)\varphi_I(x_1)U(t_1,-\infty)|\phi\rangle$$

(l) Show that

$$\langle 0|T\,\varphi(x_n)\cdots\varphi(x_1)|0\rangle = \langle\emptyset|T\,\varphi_I(x_n)\cdots\varphi_I(x_1)e^{-i\int d^4x\,\mathcal{H}_I(x)}|\emptyset\rangle \times |\langle\emptyset|0\rangle|^2. \tag{9.39}$$

From part (k),

$$\langle 0|\varphi(x_n)\cdots\varphi(x_1)|0\rangle = |\langle\phi|0\rangle|^2$$
$$\times \langle\phi|U(\infty,t_n)\varphi_I(x_n)U(t_n,t_{n-1})\cdots U(t_2,t_1)\varphi(x_1)U(t_1,-\infty)|\phi\rangle$$

$$\Rightarrow \quad \langle 0|T\,\varphi(x_n)\cdots\varphi(x_1)|0\rangle = |\langle\phi|0\rangle|^2$$
$$\times \langle\phi|T\,\varphi_I(x_n)\cdots\varphi_I(x_1)\cdot U(\infty,t_n)\underbrace{U(t_n,t_{n-1})}_{T\exp\left[-i\int_{t_{n-1}}^{t_n}dt'\,H_I(t')\right]}\cdots U(t_2,t_1)U(t_1,-\infty)|\phi\rangle$$
$$= |\langle\phi|0\rangle|^2\langle\phi|T\,\varphi_I(x_n)\cdots\varphi_I(x_1)\exp\left[-i\int d^4x\,\mathcal{H}_I(x)\right]|\phi\rangle$$

(m) Show that

$$|\langle\emptyset|0\rangle|^2 = \frac{1}{\langle\emptyset|Te^{-i\int d^4x\,H_I(x)}|\emptyset\rangle}. \tag{9.40}$$

Thus we have

$$\boxed{\langle 0|T\,\varphi(x_n)\cdots\varphi(x_1)|0\rangle = \frac{\langle\emptyset|T\,\varphi_I(x_n)\cdots\varphi_I(x_1)e^{-i\int d^4x\,H_I(x)}|\emptyset\rangle}{\langle\emptyset|Te^{-i\int d^4x\,H_I(x)}|\emptyset\rangle}.} \tag{9.41}$$

We can now Taylor expand the exponentials on the right-hand side of Eq. (9.41), and use free-field theory to compute the resulting correlation functions.

Evaluate part (l) for $\varphi(\vec{x},0)=1 \;\Rightarrow\; \varphi(\vec{x},t)=1$
$\varphi_I(\vec{x},t)=1$

Then $\langle 0|1|0\rangle = \langle 0|0\rangle = 1$
$= |\langle\phi|0\rangle|^2\langle\phi|T\exp\left[-i\int d^4x\,\mathcal{H}_I(x)\right]|\phi\rangle$