# Ch. 2 Khatri et al Problems.



**A1.** For $H$ finite dim$^n$ & $H^{\oplus m}$ its direct sum $m$-times, show the linear map, $U$, is unitary. "flagging"

$$U: H^{\oplus m} \longrightarrow \mathbb{C}^m \otimes H$$

$$\bigoplus_{j=1}^{m} |x_j\rangle \longmapsto \sum_{j=1}^{m} |j\rangle \otimes |x_j\rangle \qquad \text{for } |x_j\rangle \in H.$$

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e.g., $H^{\oplus 2} = H_A \oplus H_B$ , $H_A = H_B$.

$$U: |u\rangle_A \oplus |v\rangle_B \longrightarrow |1\rangle \otimes |u\rangle_A + |2\rangle \otimes |v\rangle_B.$$

OR $m=3$: $|u\rangle_A \oplus |v\rangle_B + |w\rangle_C \longrightarrow |1\rangle \otimes |u\rangle_A + |2\rangle \otimes |v\rangle_B + |3\rangle \otimes |w\rangle_C$.

- dim lhs $= m \cdot \dim H$
- " rhs $= m \cdot \dim H$

✓. Checks.

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- note in the $m=2$ e.g., linearity by scaling holds:

$$U(\alpha(|u\rangle_A \oplus |v\rangle_B)) = U(\alpha|u\rangle_A + \alpha|v\rangle_B)$$
$$= |1\rangle \otimes \alpha|u\rangle_A + |2\rangle \otimes \alpha|v\rangle_B$$
$$= \alpha(|1\rangle \otimes |u\rangle_A + |2\rangle \otimes |v\rangle_B)$$
$$= \alpha \cdot U(|u\rangle_A \oplus |v\rangle_B).$$

- similar holds for sums & arb. $m$.

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- use unitary transform preserve the inner product def$^n$:

$$\langle Ux, Uy\rangle = \langle x, y\rangle$$

- take $x = \bigoplus_{i=1}^{m} x_i$ , $y = \bigoplus_{i=1}^{m} y_i$ ; $x_i, y_i \in H$.

rhs: $\langle x, y\rangle = \langle \bigoplus_{i=1}^{m} x_i, \bigoplus_{i=1}^{m} y_i\rangle = \sum_{i=1}^{n} \langle x_i, y_i\rangle$. ✓

lhs: $\langle Ux, Uy\rangle = \langle \sum_{i=1}^{n} |i\rangle \otimes x_i, \sum_{j=1}^{m} |j\rangle \otimes y_j\rangle$

$$= \sum_{ij} \underbrace{\langle i|j\rangle}_{\delta_{ij}} \cdot \langle x_i, y_j\rangle = \sum_i \langle x_i, y_i\rangle$$

$\Big\}$ lhs = rhs ✓ $\therefore$ unitary.

ii) Compute $U^\dagger$:

- being unitary the adjoint is just the inverse operation: $U^\dagger = U^{-1}$.

$$U^\dagger \sum_j |j x_j\rangle = \bigoplus_j |x_j\rangle .$$

and $U^\dagger |k x_k\rangle = \oplus_j \delta_{jk} |x_j\rangle$ → extracts the k'th direct sum component.

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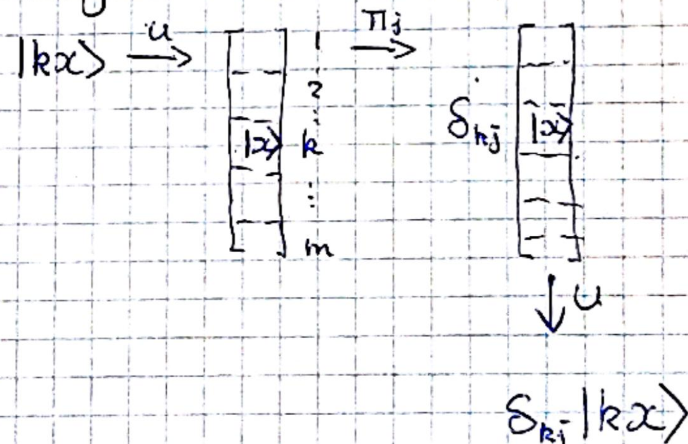
iii) Show $\Pi_j \xrightarrow{U} |j\rangle\langle j| \otimes I$. where $\Pi_j$ projects onto the j'th direct sum summand.

- under the co-ord transformation $U$, $\Pi_j \to U \Pi_j U^\dagger$

So, $\underbrace{U \Pi_j U^\dagger}\cdot |kx\rangle$

$$= U \Pi_j \bigoplus_\ell \delta_{k\ell} |x\rangle$$

$$= U \bigoplus_\ell \delta_{j\ell}\, \delta_{k\ell} |x\rangle \quad \leftarrow \text{need } j = \ell = k \text{ for non-zero.}$$




$$= \delta_{kj} |k\rangle \otimes |x\rangle$$

$$= \delta_{kj} |j\rangle \otimes |x\rangle$$

$$= |j\rangle \delta_{kj} \otimes |x\rangle$$

$$= |j\rangle\langle j|k\rangle \otimes |x\rangle$$

$$= \big(|j\rangle\langle j| \otimes I\big)\big(|k\rangle \otimes |x\rangle\big).$$

$$= \underbrace{\big(|j\rangle\langle j| \otimes I\big)}|kx\rangle .$$

$\therefore\ U \Pi_j U^\dagger = |j\rangle\langle j| \otimes I$ ✓

## (C1) Trace Cyclicity.



Let $X \in L(H_A, H_B)$, $Y \in L(H_B, H_C)$, $Z \in L(H_C, H_A)$.

- note dim's can be different but must be commensurate

e.g. $X \sim \begin{bmatrix} & & \end{bmatrix} \begin{bmatrix} \\ \\ \end{bmatrix}_{\in H_A} = \begin{bmatrix} \\ \end{bmatrix}_{\in H_B}$.

a) Show $\mathrm{tr}(XYZ) = \mathrm{tr}(ZXY) = \mathrm{tr}(YZX)$.

- note $(XYZ)_{ii} = \sum_{ab} X_{ia} Y_{ab} Z_{bi}$

- $\mathrm{tr}(XYZ) = \sum_i (XYZ)_{ii}$

$$= \sum_{iab} X_{ia} Y_{ab} Z_{bi} \qquad = \mathrm{tr}\, XYZ$$

$$= \sum_{iab} Z_{bi} X_{ia} Y_{ab} \qquad = \mathrm{tr}(ZXY)$$

$$= \sum_{iab} Y_{ab} Z_{bi} X_{ia} \qquad = \mathrm{tr}(YZX).$$

b) Show $\mathrm{tr}(X \otimes Y)(Z \otimes W) = \mathrm{tr}\, XY \cdot \mathrm{tr}\, ZW$.

- first show $\mathrm{tr}\, A \otimes B = \mathrm{tr}\, A \cdot \mathrm{tr}\, B$.
- in coordinates of the two tensored vector spaces,

$$[A \otimes B]_{ij, i'j'} = \langle ij | A \otimes B | i'j' \rangle.$$

$$\therefore \mathrm{tr}\, A \otimes B = \sum_{ij} \langle ij | A \otimes B | ij \rangle$$

$$= \sum_{ij} \langle i | \otimes \langle j |\, A \otimes B\, | i \rangle \otimes | j \rangle$$

$$= \sum_{ij} \langle i | A | i \rangle \langle j | B | j \rangle$$

$$= \sum_i A_{ii} \cdot \sum_j B_{jj}$$

$$= \mathrm{tr}\, A \cdot \mathrm{tr}\, B$$

So,

$$\mathrm{tr}\,(X \otimes Y)(Z \otimes W)$$

$$= \mathrm{tr}\, XZ \otimes YW$$

$$= \mathrm{tr}\, XZ \cdot \mathrm{tr}\, YW$$

# C1) Trace Cyclicity.

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$$= \sum_{ij} \langle i | \otimes \langle j | \, A \otimes B \, | i \rangle \otimes | j \rangle$$

$$= \sum_{ij} \langle i | A | i \rangle \langle j | B | j \rangle$$

$$= \sum_i A_{ii} \cdot \sum_j B_{jj}$$

$$= \mathrm{tr}\, A \cdot \mathrm{tr}\, B$$

$\rightarrow$ So,

$$\mathrm{tr}(X \otimes Y)(Z \otimes W)$$

$$= \mathrm{tr}\, XZ \otimes YW$$

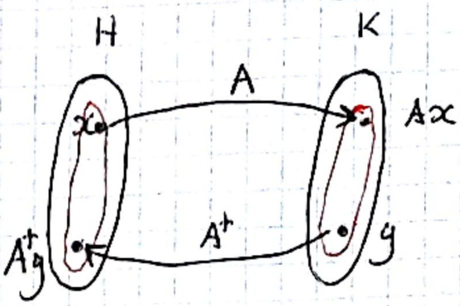
$$= \mathrm{tr}\, XZ \cdot \mathrm{tr}\, YW$$

# D1) Basis independence of the adjoint, dependence of transpose:

a.) **Adjoint**: Recall the defⁿ:

For $A: H \to K$

$$(A^\dagger y, x)_H = (y, Ax)_K \qquad (*)$$




$(A^\dagger y, x)_H = (y, Ax)_K$

→ compatible inner products

→ note: no coord system used in proving existence & uniquess ⇒ just an i.p.

- take an ONB $\{|e_i\rangle\}$ & write the matrix elts of A wrt them:

$$[A]_{ij} = \langle e_i | A | e_j \rangle.$$

- now

$$\begin{aligned} [A^\dagger]_{ij} &= \langle e_i | A^\dagger e_j \rangle \\ &= \langle A e_i | e_j \rangle \quad \text{by } (*) \\ &= \langle e_j | A e_i \rangle^* \quad \text{by conj}^{n} \\ &= [A]_{ji}^* \end{aligned}$$

⇒ So, the $(i,j)$'th elt of $[A^\dagger]$ is the complex conjugated $(j,i)$th element of $[A]$

⇒ $[A^\dagger] = [A]^\dagger$ — note: $[A^\dagger]$ = matrix of adjoint of A operator
$[A]^\dagger$ = conjugate transpose of A matrix

b.) **Co-ord Transformⁿ of Adjoint**

- under a coord transfⁿ by unitary $U$,

$$[A] \xrightarrow{U} U[A]U^\dagger \equiv [A]'$$

- the adjoint of A as a matrix goes like this as well,

$$\begin{aligned} [A^\dagger]' &= U[A^\dagger]U^\dagger \\ &= U[A]^\dagger U^\dagger \\ &= (U[A]U^\dagger)^\dagger \\ &= [A]'^\dagger \end{aligned}$$

$\therefore \boxed{[A^\dagger]' = [A]'^\dagger}$ ← this is the same formula in the new coord

- it's like how $F = ma$ holds regardless of feet, metres, $(x,y,z)$, $(r,\theta,\phi)$ units & co-ord systems.

## b.) Transpose:

- if the transpose were co-ord independent we would expect it to also satisfy the commutation for adjoint

**Adjoint:**

$$\begin{tikzcd} A \arrow[r, "U"] \arrow[d, "\dagger"'] & UAU^{\dagger} \arrow[d, "\dagger"] \\ A^{\dagger} \arrow[r, "U"] & UA^{\dagger}U \end{tikzcd}$$

**Transpose:** This breaks

$$\begin{tikzcd} A \arrow[r, "U"] \arrow[dd, "T"'] & UAU^{\dagger} \arrow[d, "T"] \\ & U^{*}A^{T}U^{T} \\ A^{T} \arrow[r, "U"] & UA^{T}U^{\dagger} \arrow[u, phantom, "\neq"] \end{tikzcd} \quad !$$

← only works if we work in real numbers, so $U$ is now real orthogonal and:

$U^{*} = U$

$U^{\dagger} = U^{T}$.

**Example:**

Choose $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$U = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$, $U^{\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$.

# D2 Transpose as a Super Operator

Show the linear map $T(\cdot) = (\cdot)^T$ has a rep^n

$$T(X) = \sum_{ij} E_{ji} X E_{ji}$$

where $E_{ij}$ are the standard operator basis ( $[E_{ij}]_{lm} = \delta_{il} \cdot \delta_{jm}$
→ a 1 at the $(i,j)$'th elt, 0's elsewhere.

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- we can write $X = \sum_{ab} x_{ab} E_{ab}$ in the operator basis expansion
- So $T(X) = T\left(\sum_{ab} x_{ab} E_{ab}\right)$.
- by linearity of $T()$ just look at an individual summand. basis elt, $E_{ab}$.
- $T(E_{ab}) = \sum_{ij} E_{ji} E_{ab} E_{ji}$ ← note: these are matrices, not matrix elements
- express the $E_{..}$ matrices as $E_{uv} = |u\rangle\langle v|$ in standard basis.
- $\Rightarrow T(E_{ab}) = \sum_{ij} |j\rangle \underbrace{\langle i| \cdot |a\rangle}_{\delta_{ia}} \underbrace{\langle b| \cdot |j\rangle}_{\delta_{bj}} \langle i|$

$$= \sum_{ij} \delta_{ia} \delta_{bj} |j\rangle\langle i|$$

$$= |b\rangle\langle a|$$

$$= E_{ba}$$

$$= E_{ab}^T.$$

$\therefore T(E_{ab}) = E_{ab}^T$

→ $T(X) = X^T$ by linearity,

Why do this?

See 6a

## Why write the transpose as a super operator?

- this will be used to model quantum channels to come.
- Vectors: quatum states $|\psi\rangle$ live in a Hilbert space $H$
  Operators: linear maps $A: H \to H$, matrices acting on vectors
- a super operator is one level up:
  $\Rightarrow$ a linear map that takes one operator and spits out another.

$$T: L(H) \longrightarrow L(H).$$
$$A \longmapsto T(A).$$

- vectors live in $H$.
  operators act on vectors
  super operators act on operators.

## Why do this?

- When we introduce probability a quantum system's state will be represented by a density operator, $\rho$.
- it will evolve under a unitary channel, $U$, as

$$\mathcal{U}(\rho) = U \rho U^{\dagger}$$

the super operator.

- this is a noiseless (deterministic) quantum channel.
- (noisey) depolarizing channel:

$$\mathcal{D}_p(\rho) = (1-p)\rho + p\frac{I}{2}$$

(can be rewritten in same form) $= \left(1-\frac{3p}{4}\right)\rho + \frac{p}{4}\left(X\rho X + Y\rho Y + Z\rho Z\right)$

where $X, Y, Z$ are the Pauli matrices

# E2 Maximally Entangled Vector Identity.

Let $|\Gamma\rangle = \sum_i |i\rangle \otimes |i\rangle$, in standard ONB.

Show: ① $(I\otimes A)|\Gamma\rangle = (A^T \otimes I)|\Gamma\rangle$.

② $\operatorname{tr} X = \langle \Gamma | (X\otimes I)|\Gamma\rangle$.

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① Expand lhs and use : a) resolution of identity in ONB
$I = \sum_i |i\rangle\langle i|$

b) matrix elts $A_{ij} = \langle i|A|j\rangle$.

So,

$$\boxed{\text{lhs} = (I\otimes A)|\Gamma\rangle} =$$

$$= (I\otimes A)\sum_i |i\rangle\otimes|i\rangle.$$

$$= \sum_i I|i\rangle \otimes A|i\rangle.$$

$$= \sum_i |i\rangle \otimes \overbrace{\sum_j |j\rangle\langle j|}^{I} \underbrace{\langle j|A|i\rangle}_{A_{ji}}$$

$$= \sum_i |i\rangle \otimes \sum_j A_{ji}|j\rangle.$$

$$= \sum_{ij} A_{ji}|i\rangle\otimes|j\rangle.$$

$$= \sum_{ij} (A^T)_{ij} |i\rangle\otimes|j\rangle.$$

$$= \sum_{ij} \underbrace{|i\rangle\langle i|}_{I} A^T|j\rangle \otimes |j\rangle$$

$$= \sum_j A^T|j\rangle\otimes|j\rangle.$$

$$= (A^T\otimes I)\sum_j |j\rangle\otimes|j\rangle$$

$$\boxed{\text{rhs} = (A^T\otimes I)|\Gamma\rangle.} ✓$$

② $\boxed{\langle\Gamma|(X\otimes I)|\Gamma\rangle.}$

$$= \sum_i \langle ii| (X\otimes I) \sum_j |jj\rangle$$

$$= \sum_i \langle i|\otimes\langle i| (X\otimes I) \sum_j |j\rangle\otimes|j\rangle$$

$$= \sum_{ij} \langle i|X|j\rangle \cdot \langle i|I|j\rangle.$$

$$\cancel{= \delta_{ij}}$$

$$= \sum_{ij} X_{ij}\langle i|j\rangle$$

$$= \sum_{ij} X_{ij}\,\delta_{ij}$$

$$= \sum_i X_{ii}$$

$$\boxed{= \operatorname{tr} X} ✓$$

## E3 State-Operator Isomorphisms:

Show: ① Every $|\psi\rangle \in H_A \otimes H_B$ can be written as

$|\psi\rangle = (I \otimes A)|\Gamma\rangle$ for some $A$; and

② $|\psi\rangle = (B \otimes I)|\Gamma\rangle$ for some $B$, with $A = T(B)$.

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① Expanding $|\psi\rangle$ in the $|i\rangle \otimes |j\rangle$ basis we have

$$|\psi\rangle = \sum_{ij} \psi_{ij} \, |i\rangle \otimes |j\rangle.$$

- now <u>define</u> an operator $A$ such that its matrix elts are $\psi_{ij}$. That is,

$$[A]_{ij} \equiv \psi_{ij}.$$

- so, $\psi_{ij} = \langle i|A|j\rangle$ now in terms of operator $A$.
- $\therefore$ $|\psi\rangle = \sum_{ij} \langle i|A|j\rangle \cdot |i\rangle \otimes |j\rangle.$ (*).

$$= \sum_j \underbrace{\sum_i |i\rangle \cdot \langle i|}_{I} A|j\rangle \otimes |j\rangle.$$

$$= \sum_j A|j\rangle \otimes |j\rangle.$$

$$= (A \otimes I) \sum_j |j\rangle|j\rangle.$$

$$\boxed{|\psi\rangle = (A \otimes I)|\Gamma\rangle}$$ ✓

② Similar to ①:

From (*): $|\psi\rangle = \sum_{ij} |i\rangle \otimes |j\rangle \cdot \langle i|A|j\rangle$ (just move the scalar i.p.)

$$= \sum_i |i\rangle \otimes \overbrace{\sum_j |j\rangle\langle j|}^{I} A^T|i\rangle$$

$$= \sum_i |i\rangle \otimes A^T|i\rangle = \sum_i (I \otimes A^T)|i\rangle \otimes |i\rangle$$

$$\boxed{= (I \otimes A^T)|\Gamma\rangle}$$ ✓

## F2 Schmidt Spectra of Partial Traces:

If $|\psi\rangle = \sum_{i=1}^{r} \sqrt{\lambda_i}\, |a_i\rangle \otimes |b_i\rangle$ show (as a Schmidt decomp$^{n}$).

$\rho_A = tr_B |\psi\rangle\langle\psi|$ and $\rho_B = tr_A |\psi\rangle\langle\psi|$ have the same eigenvalues.

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- the partial trace over the tensor product of two spaces is defined on a single element & extended linearly.

**Def$^{n}$** Given $V \in L(H_A)$ & $W \in L(H_B)$, so $V \otimes W \in L(H_A \otimes H_B)$,

and $V = |v_1\rangle\langle v_2|$

$W = |w_1\rangle\langle w_2|$

$V \otimes W = |v_1\rangle\langle v_2| \otimes |w_1\rangle\langle w_2|$,

**Def$^{n}$** $\Rightarrow$ $tr_A\, V \otimes W \equiv tr(|v_1\rangle\langle v_2|) \cdot |w_1\rangle\langle w_2|$

$= tr \langle v_2|v_1\rangle \cdot |w_1\rangle\langle w_2|$

$= \langle v_2|v_1\rangle \cdot |w_1\rangle\langle w_2|$.

and $tr_B\, V \otimes W = \langle w_2|w_1\rangle \cdot |v_1\rangle\langle v_2|$.

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**Sol$^{n}$:** Write $|\psi\rangle\langle\psi| = \sum_{ij} \sqrt{\lambda_i\lambda_j}\, (|a_i\rangle \otimes |b_i\rangle)(\langle a_j| \otimes \langle b_j|)$.

$$= \sum_{ij} \sqrt{\lambda_i\lambda_j}\; |a_i\rangle\langle a_j| \otimes |b_i\rangle\langle b_j|$$

$\therefore\ tr_A |\psi\rangle\langle\psi| = \sum_{ij} \sqrt{\lambda_i\lambda_j}\; \underbrace{\langle a_j|a_i\rangle}_{\delta_{ij}} \cdot |b_i\rangle\langle b_j|$

$$= \sum_i \lambda_i |b_i\rangle\langle b_i|$$

- $tr_B |\psi\rangle\langle\psi| = \sum_{ij} \sqrt{\lambda_i\lambda_j}\; \langle b_j|b_i\rangle\, |a_i\rangle\langle a_j|$

$$= \sum_i \lambda_i\, |a_i\rangle\langle a_i|$$

$\because$ $|a_i\rangle$'s & $|b_i\rangle$'s are an ONB from the Schmidt decomp$^{n}$, these are eigendecomposition $\bar{c}$ same eigenvalues.