Physics with Friends – QFT

How to Transform a Spinor

Chapter 37, QFT for Gifted Amateurs

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The Lorentz group

Can be generated by the following three 1-parameter families of rotations and boosts (see §1.4, [1]):

Spatial rotations

$$R_x(\theta_x) \;:=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_x & \sin\theta_x \\ 0 & 0 & -\sin\theta_x & \cos\theta_x \end{pmatrix}$$

$$R_y(\theta_y) \;:=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 0 & 1 & 0 \\ 0 & \sin\theta_y & 0 & \cos\theta_y \end{pmatrix}$$

$$R_z(\theta_z) \;:=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_z & \sin\theta_z & 0 \\ 0 & -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Boosts

$$B_x(\phi_x) \;:=\; \begin{pmatrix} \cosh\phi_x & -\sinh\phi_x & 0 & 0 \\ -\sinh\phi_x & \cosh\phi_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B_y(\phi_y) \;:=\; \begin{pmatrix} \cosh\phi_x & 0 & -\sinh\phi_x & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh\phi_x & 0 & \cosh\phi_x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B_z(\phi_z) \;:=\; \begin{pmatrix} \cosh\phi_x & 0 & 0 & -\sinh\phi_x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\phi_x & 0 & 0 & \cosh\phi_x \end{pmatrix}$$

Generators of the Lie algebra of the Lorentz group

We follow §3.3, [1].

$$J_x := -\mathbf{i}\cdot \frac{\mathrm{d}}{\mathrm{d}\,\theta_x}\bigg|_{\theta_x=0} R_x(\theta_x) = -\mathbf{i}\cdot \frac{\mathrm{d}}{\mathrm{d}\,\theta_x}\bigg|_{\theta_x=0} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_x & \sin\theta_x \\ 0 & 0 & -\sin\theta_x & \cos\theta_x \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{i} \\ 0 & 0 & \mathbf{i} & 0 \end{pmatrix}$$

Similarly,

$$J_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{i} \\ 0 & 0 & 0 & 0 \\ 0 & -\mathbf{i} & 0 & 0 \end{pmatrix}, \qquad J_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathbf{i} & 0 \\ 0 & \mathbf{i} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_x := -\mathbf{i}\cdot \frac{\mathrm{d}}{\mathrm{d}\,\phi_x}\bigg|_{\phi_x=0} B_x(\phi_x) = \cdots = \begin{pmatrix} 0 & \mathbf{i} & 0 & 0 \\ \mathbf{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K_y = \begin{pmatrix} 0 & 0 & \mathbf{i} & 0 \\ 0 & 0 & 0 & 0 \\ \mathbf{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_z = \begin{pmatrix} 0 & 0 & 0 & \mathbf{i} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathbf{i} & 0 & 0 & 0 \end{pmatrix}$$

Commutation relations among $J_x, J_y, J_z,\ K_x, K_y, K_z$

$$
\begin{aligned}
[\, J_i \,,\, J_j \,] &= \mathbf{i} \cdot \varepsilon_{ijk} \cdot J_k \\
[\, J_i \,,\, K_j \,] &= \mathbf{i} \cdot \varepsilon_{ijk} \cdot K_k \\
[\, K_i \,,\, K_j \,] &= -\mathbf{i} \cdot \varepsilon_{ijk} \cdot J_k
\end{aligned}
$$

Define: $$N_i^{\pm} := \frac{1}{2}\left(J_i \pm \mathbf{i}\, K_i \right)$$

Then,

$$
\begin{aligned}
\left[N_i^{+} \,,\, N_j^{+} \right] &= \mathbf{i} \cdot \varepsilon_{ijk} \cdot N_k^{+} \\
\left[N_i^{-} \,,\, N_j^{-} \right] &= \mathbf{i} \cdot \varepsilon_{ijk} \cdot N_k^{-}
\end{aligned}
\qquad\qquad
\left[N_i^{-} \,,\, N_j^{+} \right] = 0
$$

Two commuting copies of $\mathfrak{su}(2)$ in $\mathfrak{so}^{\uparrow}(1,3)$: $\langle\, N_x^{+}, N_y^{+}, N_z^{+} \,\rangle$ and $\langle\, N_x^{-}, N_y^{-}, N_z^{-} \,\rangle$

Left-hand Weyl spinors, i.e., $(\frac{1}{2}, 0)$ rep'n of $\mathfrak{spin}(1,3)$

Recall:

- The finite-dimensional irreducible representations of $\mathrm{SU}(2)$ (equivalently, those of $\mathfrak{su}(2)$, due to simple-connectedness of SU(2)) have been classified.

  For each $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots,$ there exists a unique representation $\rho_s : \mathrm{SU}(2) \longrightarrow \mathrm{GL}(\mathbb{R}, 2s+1)$
- $\mathfrak{spin}(1,3) \cong \mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{su}(2) \oplus \mathbf{i}\,\mathfrak{su}(2)$ (the copies of $\mathfrak{su}(2)$ commute)

  (Skew self-adjoint matrices with trace zero plus self-adjoint matrices with trace zero gives all matrices with trace zero.)
- The irreducible representations of $\mathrm{Spin}^{\uparrow}(1,3)$ are parametrized by the ordered pairs (s_+, s_-) of non-negative multiples of $\frac{1}{2}$, where s_+ refers to $\mathfrak{su}(2) \subset \mathfrak{sl}(2,\mathbb{C})$ and s_- refers to $\mathbf{i}\,\mathfrak{su}(2) \subset \mathfrak{sl}(2,\mathbb{C})$.

  See Theorem on page 517, [2].

$(\frac{1}{2}, 0)$ **representation of $\mathfrak{spin}(1,3)$, i.e., left-handed Weyl spinors**

- Generators of $\mathfrak{spin}(1,3)$:

$$N_i^{\pm} := \frac{1}{2} \cdot (\, J_i \pm \mathbf{i}\, K_i \,)\,, \quad \left[\, N_i^+ \,,\, N_j^+ \,\right] = \mathbf{i} \cdot \varepsilon_{ijk} \cdot N_k^+, \quad \left[\, N_i^- \,,\, N_j^- \,\right] = \mathbf{i} \cdot \varepsilon_{ijk} \cdot N_k^-\,, \quad \left[\, N_i^- \,,\, N_j^+ \,\right] = 0$$

- The $(\frac{1}{2}, 0)$ representation:

$$\langle\, N_x^+, N_y^+, N_z^+ \,\rangle \cong (s = \frac{1}{2}) \text{ rep'n of } \mathrm{SU}(2)\,, \qquad \left\langle\, N_x^-, N_y^-, N_z^- \,\right\rangle \cong (s = 0) \text{ rep'n of } \mathrm{SU}(2)$$

Left-handed Weyl spinors, i.e., $\left(\frac{1}{2}, 0\right)$ rep'n of $\mathfrak{spin}(1,3)$

- Generators of $\mathfrak{spin}(1,3)$:

$$N_i^{\pm} := \frac{1}{2} \cdot (J_i \pm \mathbf{i}\, K_i), \quad \left[N_i^+ , N_j^+ \right] = \mathbf{i} \cdot \varepsilon_{ijk} \cdot N_k^+, \quad \left[N_i^- , N_j^- \right] = \mathbf{i} \cdot \varepsilon_{ijk} \cdot N_k^-, \quad \left[N_i^- , N_j^+ \right] = 0$$

- $(\frac{1}{2}, 0)$ rep'n: $\langle N_x^+, N_y^+, N_z^+ \rangle \cong (s = \frac{1}{2})$ rep'n of SU(2), $\left\langle N_x^-, N_y^-, N_z^- \right\rangle \cong (s = 0)$ rep'n of SU(2)

- Note: The representation $\rho_{1/2,0} : \mathrm{Spin}^{\uparrow}(1,3) \longrightarrow \mathrm{GL}(\mathbb{C}^2)$ is a Lie group homomorphisim, and

$$(\rho_{1/2,0})_* : T_I\, \mathrm{Spin}^{\uparrow}(1,3) = \mathfrak{spin}(1,3) \longrightarrow T_I\, \mathrm{GL}(\mathbb{C}^2) = \mathrm{End}(\mathbb{C}^2) = \mathrm{Linear}(\mathbb{C}^2 \to \mathbb{C}^2) = \mathbb{C}^{2\times 2}$$

- $\left\langle N_x^-, N_y^-, N_z^- \right\rangle \cong (s = 0)$ rep'n of SU(2) $\Longrightarrow$

$$\mathbb{C}^{2\times 2} \ni (\rho_{(1/2,0)})_* \left(N_i^- \right) = N_i^- := \frac{1}{2} \cdot (J_i - \mathbf{i}\, K_i) = 0 \quad \Longrightarrow \quad J_i = \mathbf{i}\, K_i,$$

- $\langle N_x^+, N_y^+, N_z^+ \rangle \cong (s = \frac{1}{2}) \Longrightarrow$

$$\frac{1}{2} \cdot \sigma_i = (\rho_{(1/2,0)})_* (N_i^+) = N_i^+ := \frac{1}{2} \cdot (J_i + \mathbf{i}\, K_i) = \frac{1}{2} \cdot (J_i + J_i) = J_i = \mathbf{i}\, K_i$$

Left-handed Weyl spinors, i.e., $\left(\frac{1}{2}, 0\right)$ rep'n of $\mathfrak{spin}(1,3)$

- $\langle\, N_x^+, N_y^+, N_z^+ \,\rangle \cong (s = \frac{1}{2}) \implies \frac{1}{2}\cdot\sigma_i \;=\; N_i^+ \;=\; \frac{1}{2}\cdot(\,J_i + \mathbf{i}\,K_i\,) \;=\; \frac{1}{2}\cdot(\,J_i + J_i\,) \;=\; J_i \;=\; \mathbf{i}\,K_i$

- Rotations: $R(\,\theta\,) \;=\; \exp\Big(\mathbf{i}\cdot(\theta_x J_x + \theta_y J_y + \theta_z J_z)\Big) \;=\; \exp\Big(\frac{\sqrt{-1}}{2}\cdot(\theta_x\sigma_x + \theta_y\sigma_y + \theta_z\sigma_z)\Big)$

$$
\begin{aligned}
R_x(\theta_x) \;&=\; \exp\left(\mathbf{i}\cdot\frac{\theta_x}{2}\cdot\sigma_x\right) \;=\; I_2 + \frac{\mathbf{i}\,\theta_x}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \frac{1}{2!}\left(\frac{\mathbf{i}\,\theta_x}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right)^2 + \frac{1}{3!}\left(\frac{\mathbf{i}\,\theta_x}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right)^3 + \cdots \\
&=\; I_2 + \mathbf{i}\cdot\left(\frac{\theta_x}{2}\right)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \frac{\mathbf{i}^2}{2!}\left(\frac{\theta_x}{2}\right)^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{\mathbf{i}^3}{3!}\left(\frac{\theta_x}{2}\right)^3\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \frac{\mathbf{i}^4}{4!}\left(\frac{\theta_x}{2}\right)^4\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \cdots \\
&=\; \begin{pmatrix}\cos(\theta_x/2) & \mathbf{i}\,\sin(\theta_x/2)\\ \mathbf{i}\,\sin(\theta_x/2) & \cos(\theta_x/2)\end{pmatrix}
\end{aligned}
$$

- Boosts: $B(\,\phi\,) \;=\; \exp\Big(\mathbf{i}\cdot(\phi_x K_x + \phi_y K_y + \phi_z K_z)\Big) \;=\; \exp\Big(\frac{1}{2}\cdot(\phi_x\sigma_x + \phi_y\sigma_y + \phi_z\sigma_z)\Big)$

$$
\begin{aligned}
B_x(\phi_x) \;&=\; \exp\left(\frac{\phi_x}{2}\cdot\sigma_x\right) \;=\; I_2 + \frac{\phi_x}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \frac{1}{2}\left(\frac{\phi_x}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right)^2 + \cdots \\
&=\; I_2 + \frac{\phi_x}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \frac{1}{2}\left(\frac{\phi_x}{2}\right)^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{1}{3!}\left(\frac{\phi_x}{2}\right)^3\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \frac{1}{4!}\left(\frac{\phi_x}{2}\right)^4\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \cdots \\
&=\; \begin{pmatrix}\cosh(\phi_x/2) & \sinh(\phi_x/2)\\ \sinh(\phi_x/2) & \cosh(\phi_x/2)\end{pmatrix}
\end{aligned}
$$

Left-handed Weyl spinors, i.e., $\left(\frac{1}{2}, 0\right)$ rep'n of $\mathfrak{spin}(1,3)$

Rotations

$$R_x(\theta_x) = \begin{pmatrix} \cos(\theta_x/2) & \mathbf{i}\sin(\theta_x/2) \\ \mathbf{i}\sin(\theta_x/2) & \cos(\theta_x/2) \end{pmatrix}$$

$$R_y(\theta_x) = \begin{pmatrix} \cos(\theta_y/2) & \sin(\theta_y/2) \\ -\sin(\theta_y/2) & \cos(\theta_y/2) \end{pmatrix}$$

$$R_z(\theta_x) = \begin{pmatrix} \exp(\mathbf{i}\,\theta_z/2) & 0 \\ 0 & \exp(-\mathbf{i}\,\theta_z/2) \end{pmatrix}$$

Boosts

$$B_x(\phi_x) = \begin{pmatrix} \cosh(\phi_x/2) & \sinh(\phi_x/2) \\ \sinh(\phi_x/2) & \cosh(\phi_x/2) \end{pmatrix}$$

$$B_y(\phi_x) = \begin{pmatrix} \cosh(\phi_x/2) & -\mathbf{i}\sinh(\phi_x/2) \\ \mathbf{i}\sinh(\phi_x/2) & \cosh(\phi_x/2) \end{pmatrix}$$

$$B_z(\phi_x) = \begin{pmatrix} \exp(\phi_z/2) & 0 \\ 0 & \exp(-\phi_z/2) \end{pmatrix}$$

Right-handed Weyl spinors, i.e., $(0, \frac{1}{2})$ rep'n of $\mathfrak{spin}(1,3)$

- $\langle N_x^+, N_y^+, N_z^+ \rangle \cong (s=0) \implies 0 = N_i^+ := \frac{1}{2} \cdot (J_i + \mathbf{i}\, K_i) \implies J_i = -\mathbf{i}\, K_i$
- $\left\langle N_x^-, N_y^-, N_z^- \right\rangle \cong (s = \frac{1}{2}) \implies \dfrac{1}{2} \cdot \sigma_i = N_i^- := \frac{1}{2} \cdot (J_i - \mathbf{i}\, K_i) = \frac{1}{2} \cdot (J_i + J_i) = J_i = -\mathbf{i}\, K_i$
- Rotations: $R(\theta) = \exp\Big(\mathbf{i} \cdot (\theta_x J_x + \theta_y J_y + \theta_z J_z)\Big) = \exp\Big(\frac{\mathbf{i}}{2} \cdot (\theta_x \sigma_x + \theta_y \sigma_y + \theta_z \sigma_z)\Big)$

$$
\begin{aligned}
R_x(\theta_x) &= \exp\left(\mathbf{i} \cdot \frac{\theta_x}{2} \cdot \sigma_x\right) = I_2 + \frac{\mathbf{i}\,\theta_x}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{2} \left(\frac{\mathbf{i}\,\theta_x}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)^2 + \cdots \\
&= \begin{pmatrix} \cos(\theta_x/2) & \mathbf{i}\, \sin(\theta_x/2) \\ \mathbf{i}\, \sin(\theta_x/2) & \cos(\theta_x/2) \end{pmatrix}
\end{aligned}
$$

- Boosts: $B(\phi) = \exp\Big(\mathbf{i} \cdot (\phi_x K_x + \phi_y K_y + \phi_z K_z)\Big) = \exp\Big(-\frac{1}{2} \cdot (\phi_x \sigma_x + \phi_y \sigma_y + \phi_z \sigma_z)\Big)$

$$
\begin{aligned}
B_x(\phi_x) &= \exp\left(-\frac{\phi_x}{2} \cdot \sigma_x\right) = I_2 + \left(-\frac{\phi_x}{2}\right) \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{2} \left(-\frac{\phi_x}{2}\right)^2 \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 + \cdots \\
&= I_2 + \left(-\frac{\phi_x}{2}\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{2} \left(-\frac{\phi_x}{2}\right)^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{3!} \left(-\frac{\phi_x}{2}\right)^3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \cdots \\
&= \begin{pmatrix} \cosh(\phi_x/2) & -\sinh(\phi_x/2) \\ -\sinh(\phi_x/2) & \cosh(\phi_x/2) \end{pmatrix}
\end{aligned}
$$

Left-handed Weyl, i.e., $\left(\frac{1}{2}, 0\right)$

$$R(\theta) = \exp\left(\frac{\mathbf{i}}{2}\cdot(\theta_x\sigma_x + \theta_y\sigma_y + \theta_z\sigma_z)\right)$$

$$R_x(\theta_x) = \begin{pmatrix} \cos(\theta_x/2) & \mathbf{i}\sin(\theta_x/2) \\ \mathbf{i}\sin(\theta_x/2) & \cos(\theta_x/2) \end{pmatrix}$$

$$R_y(\theta_x) = \begin{pmatrix} \cos(\theta_y/2) & \sin(\theta_y/2) \\ -\sin(\theta_y/2) & \cos(\theta_y/2) \end{pmatrix}$$

$$R_z(\theta_x) = \begin{pmatrix} \exp(\mathbf{i}\,\theta_z/2) & 0 \\ 0 & \exp(-\mathbf{i}\,\theta_z/2) \end{pmatrix}$$

$$B(\phi) = \exp\left(+\frac{1}{2}\cdot(\phi_x\sigma_x + \phi_y\sigma_y + \phi_z\sigma_z)\right)$$

$$B_x(\phi_x) = \begin{pmatrix} \cosh(\phi_x/2) & \sinh(\phi_x/2) \\ \sinh(\phi_x/2) & \cosh(\phi_x/2) \end{pmatrix}$$

$$B_y(\phi_x) = \begin{pmatrix} \cosh(\phi_x/2) & -\mathbf{i}\sinh(\phi_x/2) \\ \mathbf{i}\sinh(\phi_x/2) & \cosh(\phi_x/2) \end{pmatrix}$$

$$B_z(\phi_x) = \begin{pmatrix} \exp(\phi_z/2) & 0 \\ 0 & \exp(-\phi_z/2) \end{pmatrix}$$

Right-handed Weyl, i.e., $\left(0, \frac{1}{2}\right)$

$$R(\theta) = \exp\left(\frac{\mathbf{i}}{2}\cdot(\theta_x\sigma_x + \theta_y\sigma_y + \theta_z\sigma_z)\right)$$

$$R_x(\theta_x) = \begin{pmatrix} \cos(\theta_x/2) & \mathbf{i}\sin(\theta_x/2) \\ \mathbf{i}\sin(\theta_x/2) & \cos(\theta_x/2) \end{pmatrix}$$

$$R_y(\theta_x) = \begin{pmatrix} \cos(\theta_y/2) & \sin(\theta_y/2) \\ -\sin(\theta_y/2) & \cos(\theta_y/2) \end{pmatrix}$$

$$R_z(\theta_x) = \begin{pmatrix} \exp(\mathbf{i}\,\theta_z/2) & 0 \\ 0 & \exp(-\mathbf{i}\,\theta_z/2) \end{pmatrix}$$

$$B(\phi) = \exp\left(-\frac{1}{2}\cdot(\phi_x\sigma_x + \phi_y\sigma_y + \phi_z\sigma_z)\right)$$

$$B_x(\phi_x) = \begin{pmatrix} \cosh(\phi_x/2) & -\sinh(\phi_x/2) \\ -\sinh(\phi_x/2) & \cosh(\phi_x/2) \end{pmatrix}$$

$$B_y(\phi_x) = \begin{pmatrix} \cosh(\phi_x/2) & \mathbf{i}\sinh(\phi_x/2) \\ -\mathbf{i}\sinh(\phi_x/2) & \cosh(\phi_x/2) \end{pmatrix}$$

$$B_z(\phi_x) = \begin{pmatrix} \exp(-\phi_z/2) & 0 \\ 0 & \exp(\phi_z/2) \end{pmatrix}$$

Thank You!

[1] ROBINSON, M.
Symmetry and the Standard Model: Mathematics and Particle Physics, 1 ed.
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[2] WOIT, P.
Quantum Theory, Groups and Representations: An Introduction.
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