Tensor-Product Relations via a Finite Free Presentation (3×3 with Scalars)

Hand-worked lecture sheet using generators $x_{(u,w)}$ and cosets $[\,\cdot\,]$

1. Setup and Generator List

Let $V = \operatorname{span}\{e_1, e_2\}$ and $W = \operatorname{span}\{f_1, f_2\}$ over a field $\mathbb{F}$. Choose three vectors on each side and add two scalar auxiliaries:

- In V: $e_1,\ e_2,\ e_1 + e_2$ and the scaled vector $3e_2$.
- In W: $f_1,\ f_2,\ f_1 + f_2$ and the scaled vector $\frac{1}{2}f_1$.

In the free vector space $F = \mathrm{FreeVec}(V \times W)$, take the finite generator set

$$
\begin{aligned}
\mathcal{G} = \{&x_{(e_1,f_1)},\ x_{(e_1,f_2)},\ x_{(e_2,f_1)},\ x_{(e_2,f_2)},\\
&x_{(e_1+e_2,f_1)},\ x_{(e_1+e_2,f_2)},\ x_{(e_1,f_1+f_2)},\ x_{(e_2,f_1+f_2)},\ x_{(e_1+e_2,f_1+f_2)},\\
&x_{(3e_2,f_1)},\ x_{(e_2,\frac{1}{2}f_1)}\}.
\end{aligned}
$$

2. Relations (All Vanish in the Quotient)

Let $N_{\mathrm{ex}} \subset F$ be the span of the following seven relations:

$$
\begin{aligned}
&r_1:\ x_{(e_1+e_2,f_1)} - x_{(e_1,f_1)} - x_{(e_2,f_1)},\\
&r_2:\ x_{(e_1+e_2,f_2)} - x_{(e_1,f_2)} - x_{(e_2,f_2)},\\
&r_3:\ x_{(e_1,f_1+f_2)} - x_{(e_1,f_1)} - x_{(e_1,f_2)},\\
&r_4:\ x_{(e_2,f_1+f_2)} - x_{(e_2,f_1)} - x_{(e_2,f_2)},\\
&r_5:\ x_{(e_1+e_2,f_1+f_2)} - x_{(e_1,f_1)} - x_{(e_1,f_2)} - x_{(e_2,f_1)} - x_{(e_2,f_2)},\\
&r_6:\ x_{(3e_2,f_1)} - 3\,x_{(e_2,f_1)}, \qquad r_7:\ x_{(e_2,\frac{1}{2}f_1)} - \tfrac{1}{2}\,x_{(e_2,f_1)}.
\end{aligned}
$$

We work in the finite quotient $F_{\mathrm{ex}}/N_{\mathrm{ex}}$ where $F_{\mathrm{ex}} = \operatorname{span}(\mathcal{G})$.

Dimension check. $|\mathcal{G}| = 11$ generators and 7 independent relations give $\dim(F_{\mathrm{ex}}/N_{\mathrm{ex}}) = 11-7 = 4 = \dim V \cdot \dim W$, as expected.

3. Eliminations (Hand Reductions)

In the quotient, each relation reads as an equality of cosets:

$$
\begin{aligned}
&\left[x_{(e_1+e_2,f_1)}\right] = \left[x_{(e_1,f_1)}\right] + \left[x_{(e_2,f_1)}\right], && \left[x_{(e_1+e_2,f_2)}\right] = \left[x_{(e_1,f_2)}\right] + \left[x_{(e_2,f_2)}\right],\\
&\left[x_{(e_1,f_1+f_2)}\right] = \left[x_{(e_1,f_1)}\right] + \left[x_{(e_1,f_2)}\right], && \left[x_{(e_2,f_1+f_2)}\right] = \left[x_{(e_2,f_1)}\right] + \left[x_{(e_2,f_2)}\right],\\
&\left[x_{(e_1+e_2,f_1+f_2)}\right] = \left[x_{(e_1,f_1)}\right] + \left[x_{(e_1,f_2)}\right] + \left[x_{(e_2,f_1)}\right] + \left[x_{(e_2,f_2)}\right], &&\\
&\left[x_{(3e_2,f_1)}\right] = 3\,\left[x_{(e_2,f_1)}\right], && \left[x_{(e_2,\frac{1}{2}f_1)}\right] = \tfrac{1}{2}\,\left[x_{(e_2,f_1)}\right].
\end{aligned}
$$

Therefore the *survivors* are precisely

$$\boxed{\left[x_{(e_1,f_1)}\right],\ \left[x_{(e_1,f_2)}\right],\ \left[x_{(e_2,f_1)}\right],\ \left[x_{(e_2,f_2)}\right]}$$

which we identify with the tensor basis $(e_1 \otimes f_1,\ e_1 \otimes f_2,\ e_2 \otimes f_1,\ e_2 \otimes f_2)$.

4. Worked Reductions (Board-Ready)

A) Pure first-slot scaling

$\left[x_{(3e_2,f_1)}\right] = 3\left[x_{(e_2,f_1)}\right] = 3\,(e_2 \otimes f_1).$

B) Pure second-slot scaling

$\left[x_{(e_2,\frac{1}{2}f_1)}\right] = \frac{1}{2}\left[x_{(e_2,f_1)}\right] = \frac{1}{2}\,(e_2 \otimes f_1).$

C) Mixed: scale first, then add in W vs add first, then scale

$$\begin{aligned}\left[x_{(3e_2,f_1+f_2)}\right] &= \left[x_{(3e_2,f_1)}\right] + \left[x_{(3e_2,f_2)}\right] = 3\left[x_{(e_2,f_1)}\right] + 3\left[x_{(e_2,f_2)}\right],\\ \left[x_{(3e_2,f_1+f_2)}\right] &= 3\left[x_{(e_2,f_1+f_2)}\right] = 3\big(\left[x_{(e_2,f_1)}\right] + \left[x_{(e_2,f_2)}\right]\big).\end{aligned}$$

Both paths agree: $3(e_2 \otimes f_1) + 3(e_2 \otimes f_2)$.

D) Mixed: add in V, scale in W

$$\begin{aligned}\left[x_{(e_1+e_2,\frac{1}{2}f_1)}\right] &= \left[x_{(e_1,\frac{1}{2}f_1)}\right] + \left[x_{(e_2,\frac{1}{2}f_1)}\right] = \frac{1}{2}\left[x_{(e_1,f_1)}\right] + \frac{1}{2}\left[x_{(e_2,f_1)}\right],\\ \left[x_{(e_1+e_2,\frac{1}{2}f_1)}\right] &= \frac{1}{2}\left[x_{(e_1+e_2,f_1)}\right] = \frac{1}{2}\big(\left[x_{(e_1,f_1)}\right] + \left[x_{(e_2,f_1)}\right]\big).\end{aligned}$$

Again both reductions coincide.

E) Composite linear combination

Let $\Xi := \left[x_{(3e_2,f_1)}\right] + \left[x_{(e_1+e_2,\frac{1}{2}f_1)}\right] - \left[x_{(e_2,\frac{1}{2}f_1)}\right]$. Then

$$\begin{aligned}\Xi &= 3\left[x_{(e_2,f_1)}\right] + \frac{1}{2}\left[x_{(e_1,f_1)}\right] + \frac{1}{2}\left[x_{(e_2,f_1)}\right] - \frac{1}{2}\left[x_{(e_2,f_1)}\right]\\ &= \frac{1}{2}\left[x_{(e_1,f_1)}\right] + 3\left[x_{(e_2,f_1)}\right] = \frac{1}{2}(e_1 \otimes f_1) + 3(e_2 \otimes f_1).\end{aligned}$$

5. Takeaways

- The additivity and homogeneity relations together encode bilinearity.
- Any finite set of auxiliaries can be eliminated; the survivors are the canonical classes $\left[x_{(e_i,f_j)}\right] = e_i \otimes f_j$.
- Whether you add-then-scale or scale-then-add, reductions agree—a coherence check you can show live.