More on Group Quotients

Sept. 18/25 II

- given a group G and (normal) subgroup H, we can quotient G by H to get a new group, G/H.

$\Rightarrow$ same happen with vector spaces V and subspace W. Form a new vector space V/W.

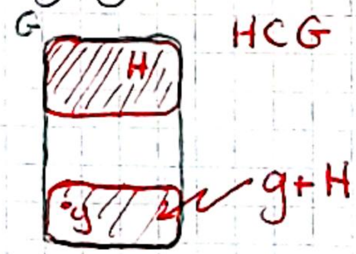
Group Example:

- take $G = \mathbb{Z}$ under addition, $+$.
  $H = 2\mathbb{Z}$, group of even numbers under $+$.

- note $H \subset G$ as a subset, but still a group in its own right.

- recall the coset of a group elt g wrt the subgroup H is:

$\boxed{[g] = g + H = \{g+h \mid \forall h \in H\}}$ "a shift of H by g"



- try $0 \in G$.

$$[0] = 0 + 2\mathbb{Z}$$
$$= \{0 + h \mid h \in 2\mathbb{Z}\}$$
$$= \{0, 2, -2, 4, -4, \cdots\} = \text{even numbers.}$$

try $1 \notin [0]$:

$$[1] = 1 + 2\mathbb{Z}$$
$$= \{1 + h \mid h \in 2\mathbb{Z}\}$$
$$= \{1, -1, 3, -3, \ldots\} = \text{odd numbers.}$$

$\Rightarrow$ that's it!

$\Rightarrow$ So $\mathbb{Z}/2\mathbb{Z}$ has two cosets: $[0]$, $[1]$

- a group $\bar{c}$ two elts. & operation

$$[x] +_c [y] = [x +_G y]$$

$$\left.\begin{array}{ll} [0]+[0]=[0] & [0]+[1]=[1] \\ [1]+[1]=[0] & [1]+[0]=[1] \end{array}\right\} \simeq \mathbb{Z}_2.$$

$+_c$	[0]	[1]
[0]	[0]	[1]
[1]	[1]	[0]

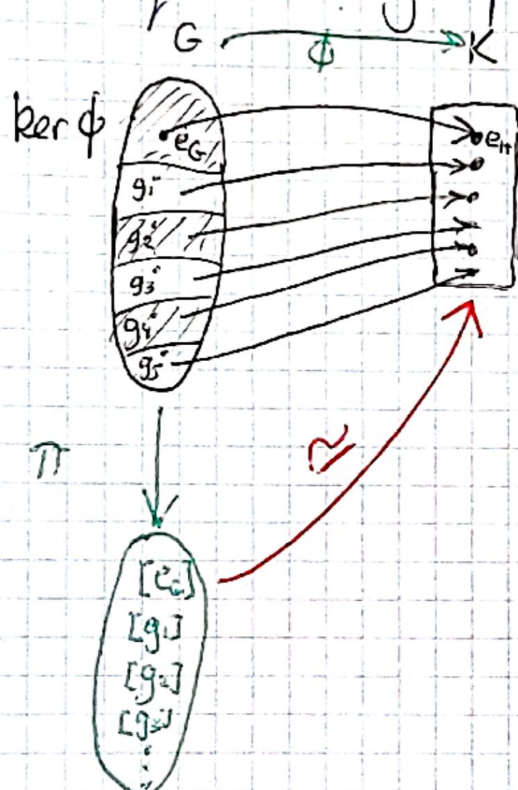
First Isomorphism Th^{m}

Say you have a group G and a homomorphism ϕ from $G \to K$, so that:

$$\boxed{\phi(g_1 +_G g_2) = \phi(g_1) +_K \phi(g_2)}$$

① The kernel of ϕ, $\{g \in G \mid \phi(g) = e_K\}$ is a (normal) subgroup of G.

② The quotient group $G/\ker\phi \simeq K$.



$$\begin{tikzcd} G \arrow[r, "\phi", "\text{homom.}"'] \arrow[d, "(\text{projection})\,\pi"'] & K \\ G/\ker\phi \arrow[ur, "\bar{\phi}"', "\text{isom.}"] & \end{tikzcd}$$

Commutation Diagram

Example: From $\mathbb{Z}/2\mathbb{Z}$.

$G = \mathbb{Z}$

$K = \mathbb{Z}_2$

- we need to choose a ϕ:

Let $\phi : \mathbb{Z} \longrightarrow \mathbb{Z}_2$

$x \longmapsto 0$ if x even

$\longmapsto 1$ " " odd

Note: $\ker\phi = \{x \in G \mid \phi(x) = 0\}$

= even numbers.

$= 2\mathbb{Z}$.

As expected from FIT:

$G/\ker\phi \simeq K$

$\Rightarrow \boxed{\mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}_2}$

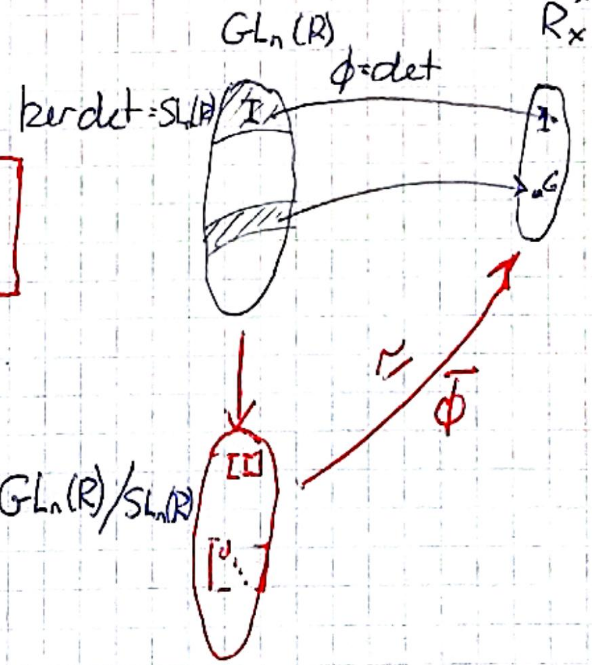
Example:

- Start with real invertible $n \times n$ matrices under matrix mul^n.
  These are a group $G = GL_n(\mathbb{R})$.
- choose a map to the real numbers : ϕ = det. function
  Take the target group of ϕ to be $\mathbb{R}^*_\times$
  → real numbers except 0 under mult^n.

$$\therefore \quad \phi = \det : GL_n(\mathbb{R}) \longrightarrow \mathbb{R}^*_\times$$
$$X \longmapsto \det X.$$

- note it's a homomorphism : $\det(X \cdot_G Y) = \det X \times_{\mathbb{R}} \det Y$
- identity I_n in G is mapped to identity 1 in $\mathbb{R}$.
- the kernel of the det function is the set
  $\ker \phi = \{ X \in GL_n(\mathbb{R}) \mid \det X = 1 \}$.
- this is $SL_n(\mathbb{R})$.
- so from the FIT:

$$\boxed{GL_n(\mathbb{R}) / SL_n(\mathbb{R}) \cong \mathbb{R}^*_\times}$$



- note dim^{n}s :

$$\boxed{\begin{aligned} \dim GL_n(\mathbb{R}) &= n^2 \\ \dim SL_n(\mathbb{R}) &= n^2 - 1 \\ \dim \mathbb{R}^* &= 1. \end{aligned}}$$

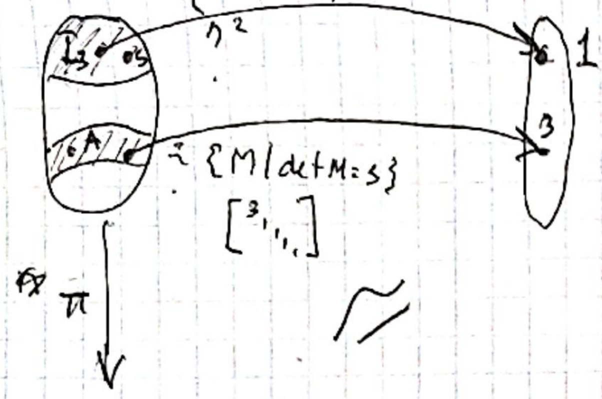
Ta. ②③④s

- can do similar $\bar{c}$ the trace function : $\phi = \mathrm{tr} : M_n(\mathbb{R}) \to \mathbb{R}_+$.
  ⇒ kernel is $sl_n(\mathbb{R})$. → all matrices $\bar{c}$ trace 1. (add've identity)

$n\times n$ $\mathbb{R}$ matrice under mult

$G = GL_n(\mathbb{R})$ $\quad \phi = \det$ $\quad \mathbb{R}^{\times}_{*}$ ~1 dim. $\quad \ker\phi = \{M \in GL_n(\mathbb{R}) \mid \det M = 1\}$

$= SL_n(\mathbb{R})$.



- why is $\phi = \det$ a homo^{A}.
- $\phi(a \circ_a b) = \phi(a) \circ_{\mathbb{R}} \phi(b)$.
- $\det(A\cdot B) = \det A \cdot_{\mathbb{R}} \det B$

$GL_n(\mathbb{R})/SL_n(\mathbb{R}) \simeq \mathbb{R}^{\times}_{*}$.

$\sim n^2-1$

$\begin{bmatrix} 3 & & \\ & 1 & \\ & & 1 \end{bmatrix} S =$ $\quad \begin{bmatrix} 6 & \\ & 1 \end{bmatrix} S = \begin{bmatrix} 2 & \\ & 3 \end{bmatrix}$

$S = \frac{1}{3}\left(\begin{bmatrix} 1/6 & \\ & 1 \end{bmatrix} \begin{bmatrix} 2 & \\ & 3 \end{bmatrix} \right) = \begin{bmatrix} 1/3 & \\ & 3 \end{bmatrix}$

$\det S = 1$.

$\Rightarrow$ try $\bar{c}$ trace fcn : $M_n(\mathbb{R}) \xrightarrow{tr} \mathbb{R}^{+}$.

$\ker tr = \{M \mid tr\, M = 0\}$

$= \mathfrak{sl}_n(\mathbb{R})$.

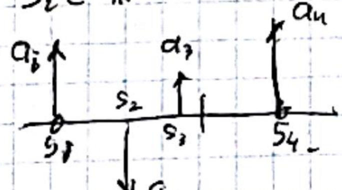
$\Rightarrow M_n(\mathbb{R})/\mathfrak{sl}_n(\mathbb{R}) \simeq \mathbb{R}^{+}$

START HERE

Recall the free vector space of a set S over $\mathbb{R}$.

$F(S) = \sum_{k=1}^{K} a_i (s_i)$ where $s_i \in \mathbb{R}$

or $\sum_{k=1}^{K} a_i \delta_{s_i}$

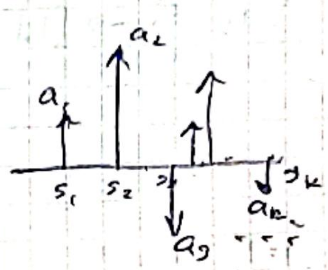


-> each $s_i \in \mathbb{R}$ is an independent basis $\bar{c}$ no relations or constraints.

$\Rightarrow$ Now introduce "relations" or constraints.

$\Rightarrow$ we want to create an n-dim^{c} vector space from this uncountbly ∞ one.

- treat the $v = \sum_{k=1}^{K} a_k \delta_{s_k}$ as a pseudo-pdf & calc. it's signed moments. via:



$m_0 = \sum_k^K a_k s_k^0 = \sum_k a_k$ → "DC"

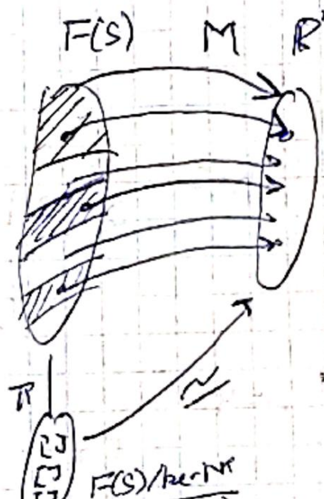
$m_1 = \sum_k a_k s_k$ → "mean"

$m_2 = \sum_k a_k s_k^2$ → "var"

⋮

$m_{n-1} = \sum_k a_k s_k^{n-1}$

- this the "moment map" $M : F(S) \to \mathbb{R}^n$.



→ NOTE: It's linear in that:

$M(av+bu) = aM(v) + bM(u)$.

· say $M\left(a \cdot \sum_i \alpha_i \delta_{v_i} + b \sum_j \beta_j \delta_{u_j}\right)$

$= M\left(\sum_i a\alpha_i \delta_{v_i} + \sum_j b\beta_j \delta_{u_j}\right) = \underline{m}$

$m_0 = \sum_i a\alpha_i + \sum_j b\beta_j = a M_0(v) + b M_0(u)$.

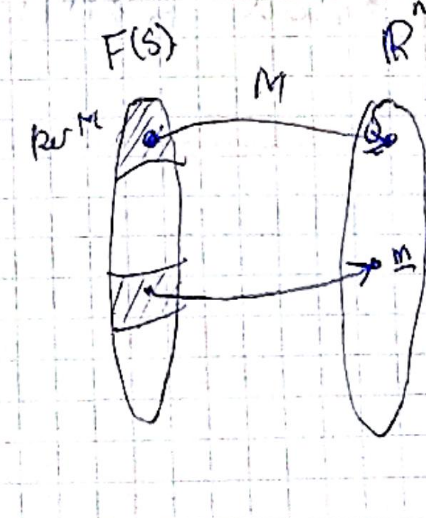
$m_1 = \sum_i a\alpha_i v_i + \sum_j b\beta_j u_j = a M_1(v) + b M_1(u)$.

$m_2 = \sim v_i^2 + \sim u_j^2 = a M_2(v) + b M_2(u)$ . .

⋮

it's linear in each co-ord.

⇒ So ~~a~~ homomorphism.



• from the FIT : $\boxed{F(S)/\ker M \simeq \mathbb{R}^n}$

• so $\ker M = \{ v \in F(S) \mid M(\sum_k \alpha_k \delta_{s_k}) = 0_{n\times 1} \}$

→ the first n "moments" equal zero.

• also $u \sim v$ in $F(S)$ iff $Mu = Mv = \underline{m}$ → they have the same n-moments.

- what does this kernel look like?

- given a $v \in F(S)$, $v = \sum_{k=1}^{K} \omega_k \delta_{s_k}$

a moment map":

$$M_n : F(S) \longrightarrow \mathbb{R}^n$$

$$v = \sum_{k}^{K} \omega_k \delta_{s_k} \longrightarrow \begin{bmatrix} \sum_k^K \omega_k \\ \sum_k \omega_k s_k \\ \sum_k \omega_k s_k^2 \\ \vdots \\ \sum_k \omega_k s_k^{n-1} \end{bmatrix}_{n\times 1} \begin{matrix} \text{DC} \\ \text{mean} \\ \text{var.} \\ \\ \\ \end{matrix} = \sum_{k=1}^{K} \omega_k \begin{bmatrix} 1 \\ s_k \\ s_k^2 \\ \vdots \\ s_k^{n-1} \end{bmatrix}.$$

- for a given K (which varies in general $< \infty$).

N_n is the set of all $\{(\omega_k, s_k)\}_{k=1}^{K}$ s.t $\begin{cases} \sum_k \omega_k = 0 \\ \sum_k \omega_k s_k = 0 \\ \vdots \\ \sum_k \omega_k s_k^{n-1} = 0. \end{cases}$

⇒ See back of pg 3 for examples.

→ so, we need $K > n$ anchor points. for non-zero, $\underset{\sim}{0}$, elements of N_n.

Note:
All V's are Vandermonde
& invertible

$n=1$: $\delta_s \to [1]$ $\qquad \sum_k w_k = 0$

$\underline{n=2}$

$$\delta_s \to \begin{bmatrix} 1 \\ s \end{bmatrix} \Rightarrow \left.\begin{array}{l} \sum w_k = 0 \\ \sum_k w_k s_k = 0 \end{array}\right\} \begin{bmatrix} 1 & 1 \\ s_1 & s_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \underline{0}$$

$n=3$: $\delta_s \to \begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 1 \\ s_1 & s_2 & s_3 \\ s_1^2 & s_2^2 & s_3^2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \underline{0}$$

FROM $\boxed{6}$

$n=4$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ s_1 & s_2 & s_3 & s_4 \\ s_1^2 & s_2^2 & s_3^2 & s_4^2 \\ s_1^3 & s_2^3 & s_3^3 & s_4^3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \underline{0}$$

- need $K \geq n+1$ to get non-$\underline{w}=0$ sol^s.

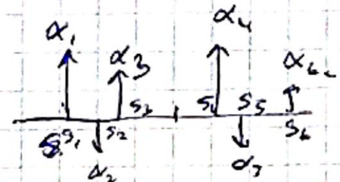
ex. $\underset{F(\mathbb{R})}{\underset{\Uparrow}{\tilde{x}}} = \sum_k^K \beta_k \delta_{s_k} \xrightarrow{M} \underset{\mathbb{R}^n}{\underset{\Uparrow}{\tilde{m}}}$

- note, any elt in $[\delta_{s_1}]$ equals $\delta_{s_1} + n$ for some $n \in N_n$

  Why: $M(\delta_{s_1} + n) = M(\delta_{s_1}) + M(n) = M(\delta_{s_1}) + \underline{0} = M(\delta_{s_1})$.

START HERE

- start w/ free vect. spc. ~~over~~ on $\mathbb{R}$ over $\mathbb{R}$, $F(S)$.
- typical elt $v = \alpha_1 \delta_{s_1} + \alpha_2 \delta_{s_2} + \cdots + \alpha_k \delta_{s_k}$.
  - $\to \alpha_i, s_i \in \mathbb{R}$
  - is unct. ∞ in dim^n.
- picture like a signed, unnorml'zd pdf:



- evaluate its "moments" via moment map:

$$\begin{array}{l} \text{"DC"} \\ \text{"mean"} \\ \text{"var}^n\text{"} \\ \vdots \\ \vdots \end{array} \begin{bmatrix} \sum_k \alpha_k \\ \sum_k \alpha_k s_k \\ \sum_k \alpha_k s_k^2 \\ \vdots \\ \sum_k \alpha_k s_k^{n-1} \end{bmatrix}_{n\times 1} \in \mathbb{R}^n = Mv$$

Note: M is linear in v.

$M: F(S) \to \mathbb{R}^n$.

- so we can get a finite v.s. by quotienting out $\ker M$ from $F(S)$: $F(S)/\ker M$
- denote $\ker M = N_n$ (null space)

  $F(S)/N_n = Q_n$ (quotient)

- we have $v_1 \sim v_2$ iff $Mx_1 = Mx_2$ → have the same moment vectors

  $\to M(x_1 - x_2) = 0$

  $\to \boxed{x_1 - x_2 \in N_n}$

  $\Rightarrow$ difference of vectors have vanishing moments after appl^n of moment map.

TO 8

(8)

- to get concrete basis to work in our $\mathbb{R}$ choose $k=n$ distinct anchor points in $\mathbb{R}$ to index basis vectors in $F(S)$, $\{s_1, s_2, \dots, s_n\}$
- create the $n \times n$ invertible Vandermode matrix

$$V_1 = [1]$$

$$\begin{bmatrix} s_1^0 & s_2^0 & \cdots & s_n^0 \\ s_1^1 & s_2^1 & \cdots & s_n^1 \\ s_1^2 & s_2^2 & \cdots & s_n^2 \\ & & \vdots & \\ s_1^{n-1} & s_2^{n-1} & \cdots & s_n^{n-1} \end{bmatrix}_{n \times n}$$

e.g. $n=2$ $\quad V_2 = \begin{bmatrix} 1 & 1 \\ s_1 & s_2 \end{bmatrix}$

$$V_3 = \begin{bmatrix} 1 & 1 & 1 \\ s_1 & s_2 & s_3 \\ s_1^2 & s_2^2 & s_3^2 \end{bmatrix}$$

- these s_i's correspond to the basis vectors $\delta_{s_1}, \delta_{s_2}, \dots, \delta_{s_n}$ in $F(S)$
- we know that ~~$x_1 \sim$~~ $v_1 \sim v_2$ iff

$$Mv_1 = Mv_2$$

$$\text{or } \underline{m}_1 = \underline{m}_2$$

- suppose we are given a moment vector $\underline{m} \in \mathbb{R}^n$ & want to find weights $w_i = \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix}$ for the δ_{s_i} so $v = \sum_{i=1}^{n} w_i \delta_i$ has $Mv = \underline{m}$
- note

$$\begin{bmatrix} 1 & 1 & 1 \\ s_1 & s_2 & s_3 \\ s_1^2 & s_2^2 & s_3^2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} w_1 + w_2 + w_3 \\ w_1 s_1 + w_2 s_2 + w_3 s_3 \\ w_1 s_1^2 + w_2 s_2^2 + w_3 s_3^2 \end{bmatrix} = \begin{bmatrix} \sum_k^n w_k \\ \sum_k^n w_k s_k \\ \sum_k w_k s_k^2 \end{bmatrix} \equiv \begin{bmatrix} m_0 \\ m_1 \\ m_2 \end{bmatrix}$$

$$\text{or } \quad V_n \underline{w} = \underline{m} \quad \Longrightarrow \quad \boxed{\underline{w} = V_n^{-1} \underline{m}}$$

- $\therefore$ if $M\underline{v} = \underline{m}$ is the moment vector for $v \in F(S)$

$$\underline{w} = \sum_{i=1}^{n} w_i \delta_{s_i} \sim v \text{ in } F(S). \in \mathbb{R}^n.$$

$$\downarrow$$

serves as a coset rep've for $[v]$ for any $v \in F(S)$

$\therefore \underline{w} \sim \text{all } v \in [\underline{w}]$

- so the $[\delta_{s_i}]_{i=1}^{n}$ serve as coset basis in $F(S)/\cancel{\text{Ker}}\,M. = Q_n$

- $m_{s_i} = \begin{bmatrix} 1 \\ s_i \\ s_i^2 \\ \vdots \end{bmatrix}$ is the ~~mean~~ moment vector. of δ_{s_i}

- note, since V_n is invertible its col^n's $[m_{s_1}\, m_{s_2} \cdots m_{s_n}]$ are linearly indep. & form a basis in $\mathbb{R}^n$.

$\Rightarrow$ Why are the $[\delta_{s_i}]_{i=1..n}$ in Q_n a basis?

For if $\sum_{j=1}^{n} \gamma_j [\delta_{s_j}] = [0]$ $\longrightarrow$ if only sol^n is $\gamma_j = 0\ \forall j$ the lin indp

i.e. $\sum_j \gamma_j \delta_{s_j} \in N_n$

i.e. $\sum_j \gamma_j s_j^m = 0 \quad \forall\ m$

$$\begin{array}{l} m=1 \\ m=2 \\ m=3 \end{array} \begin{bmatrix} \gamma_1 s_1 + \gamma_2 s_2 + \gamma_3 s_3 \\ \gamma_1 s_1^2 + \gamma_2 s_2^2 + \gamma_3 s_3^2 \\ \gamma_1 s_1^3 + \gamma_2 s_2^3 + \gamma_3 s_3^3 \end{bmatrix} = \underline{0} = \begin{bmatrix} s_1 & s_2 & s_3 \\ s_1^2 & s_2^2 & s_3^2 \\ & \vdots & \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} = V\gamma$$

But $V\gamma = 0 \rightarrow \gamma = 0 \because V$ is inv.

$\therefore$ $[\delta_{s_i}]$ are lin. indep. as a basis in Q_n.

- a general δ_s has mean vector $\underline{m}_s = \begin{bmatrix} 1 \\ s \\ s^2 \\ \vdots \end{bmatrix}$

- Converting to a basis cosets is just

$\underline{\omega}_s \rightarrow \omega_s = V_n^{-1} \begin{bmatrix} 1 \\ s \\ s^2 \\ \vdots \end{bmatrix}$ $\Rightarrow$ For $n=2$:

$V = \begin{bmatrix} 1 & 1 \\ s_1 & s_2 \end{bmatrix}$

$V^{-1} = \begin{bmatrix} s_2 & -1 \\ -s_1 & 1 \end{bmatrix} \cdot \frac{1}{s_2 - s_1}$

$\rightarrow$ For $s_1 = 0$, $s_2 = 1$

$V^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$\omega_s = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix} = \begin{bmatrix} 1-s \\ s \end{bmatrix}$

$\underline{\omega}_s = (1-s)\delta_0 + s\delta_1$

$[\delta_s] = (1-s)[\delta_0] + s[\delta_1]$

• for $n=2$ the null space N_2 is just

$\{\delta_s - (1-s)\delta_0 - s\,\delta_1\}$ for basis pts $s \neq 0,1$.

i.e. $[\delta_s] - \underbrace{(1-s)[\delta_0] - s[\delta_1]}_{-[\delta_s]} = [\delta_s] - [\delta_s] = [0]$ ✓.

→ is parameterized by s but each δ_s is an indep. dim^n.

Tensor Products:

Extend from $S = V = \{v \in V\}$ to $S = U \times V = \{(u,v) \in U \times V\}$

- say $U \cong V \cong \mathbb{R}$.