Road to Reality Problems and Notes
Chapters 2 and 3

Dimitri Papaioannou

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Chapter 2

2.2

$d(A, B) := log\frac{QAPB}{QBPA}$

$$
\begin{aligned}
d(A, B) + d(B, C) &= log\frac{QAPB}{QBPA} + log\frac{QBPC}{QCPB} && (1)\\
&= log\frac{QAPBQBPC}{QBPAQCPB} && (2)\\
&= log\frac{QAPC}{PAQC} && (3)\\
&= d(A, C) && (4)
\end{aligned}
$$

Chapter 3

Continued Fractions is a way to expand rational numbers in a recursive form like this:

$$
a + \frac{1}{b + \frac{1}{c + \frac{1}{d + \ldots}}} \tag{5}
$$

Unlike decimal expansions, or any base expansion, all these expressions terminate and we don't get periodic expansions for complete fractions such as

$$
\frac{1}{3} = 0.333333....
$$

Interestingly, all *quadratic irrationals* have periodic expressions in this expansion, whereas other rationals have non-periodic expansions.

3.1

Experiment with irrational expansions: https://github.com/algorythmist/continued-fractions

3.2

Assuming, the following expansions continue being periodic, show that they correspond to the numbers on the left.

Solution: For the first one, the periodic part satisfies the equation

$$x = \tfrac{1}{2+x} \implies \tag{6}$$

$$x^2 + 2x - 1 = 0 \implies \tag{7}$$

$$x = \sqrt{2} - 1 \tag{8}$$

For the second, we need to show

$$2 - \sqrt{3} = \frac{1}{3+x} \tag{9}$$

where x is given by the recursive relation:

$$x = \tfrac{1}{1+\frac{1}{2+x}} \implies \tag{10}$$

$$x^2 + 2x - 21 = 0 \implies \tag{11}$$

$$x = \sqrt{3} - 1 \tag{12}$$

Plugging this back into **??** we get

$$2 - \sqrt{3} = \tfrac{1}{1+3+(\sqrt{3}-1)} \implies \tag{13}$$

$$2 - \sqrt{3} = \tfrac{1}{2+\sqrt{3}} \implies \tag{14}$$

which is correct.

3.3

The criterion that the ratio a:b is greater than the ratio c:d is that some positive integers M and N exist such that the length of a added to itself M times exceeds b added to itself N times, while also the length of d added to itself N times exceeds c added to itself M times. Can you see why this works?

Solution: Translating the sentence above we have $Ma > Nb$ and $Nd > Mc$ so, in other words

$$a/b > N/M > c/d$$