Road to Reality
Chapters 7-8

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Chapter 7

Review of complex analysis

- If a complex derivative exists then it is continuous.
- If the first derivative exists, then all derivatives exist, and a Taylor expansion exists.
- A function with a Taylor expansion is called *analytic*. The definition of analytic goes beyond complex functions to describe functions in other domains that have Taylor expansions. A complex analytic function is also called *holomorphic*.
- Write $f(z) = f(x+iy) = u(x,y) + iv(x,y)$. Then u and v are conjugate harmonic and satisfy the Cauchy Riemann equations. (Problem 10.12)

7.1

For nonzero n, z^n has an anti-derivative $z^{n+1}/(n+1)$ Integrated over the unit circle we have $e^{i(n+1)\theta}/(n+1)$ evaluated at 2π and 0, which yields 0.

7.2

Substituting the McLaurin series, all terms other than 1/z integrate to 0 (by 7.1) leaving

$$\frac{1}{2\pi i}\oint \frac{f(0)}{z} dz = f(0)$$

because

$$\oint \frac{dz}{z} = 2\pi i$$

7.3

Using the following relationship as the "definition" of the derivative of an analytic function

$$f^{(n)} := \frac{n!}{2\pi i} \oint \frac{f(z)}{z^{n+1}} dz \tag{1}$$

show that the Taylor expansion sums up to $f(x)$:

$$\sum_{n=0}^{\infty} a_n z^n \tag{2}$$

where $a_n = f^{(n)}(0)/n!$.
Solution Evaluate the series at $z = p$

$$\sum_{n=0}^{\infty} a_n p^n = \frac{n!}{2\pi i} \sum \oint \frac{f(z)}{z^{n+1}} p^n dz \tag{3}$$

I am going to assume without proof that I can interchange integration and summation

$$\sum_{n=0}^{\infty} a_n p^n = \frac{1}{2\pi i} \oint \frac{f(z)}{z} \sum (\frac{p}{z})^n dz \tag{4}$$

$$= \frac{1}{2\pi i} \oint \frac{f(z)}{z} \frac{z}{z-p} dz \tag{5}$$

$$= \frac{1}{2\pi i} \oint \frac{f(z)}{z-p} dz \tag{6}$$

$$= f(p) \tag{7}$$

where the last step follows from the shifted version of the Cauchy formula.

7.4

This amounts to showing that a contour integral can be broken into a sum of contour integrals surrounding each pole. Then the result follows from the Cauchy formula:

$$\oint \frac{h(z)}{(z-p)^n} dz = \frac{2\pi i}{(n-1)!} h^{(n-1)}(p)$$

7.5

Show that $\int_0^\infty x^{-1} sin(x) dx = \pi/2$

Solution: Following the hint, I will integrate the function $z^{-1}e^{iz}$ along the suggested path.

The function has a pole at 0 and the value of the residue there is 1.
Therefore we have

$$\int_{-R}^{-\epsilon} \frac{e^{ix}}{x} dx + \int_{\epsilon-semicircle} \frac{e^{iz}}{z} dz + \tag{8}$$

$$\int_{\epsilon}^{R} \frac{e^{ix}}{x} dx + \int_{R-semicircle} \frac{e^{iz}}{z} dz \quad = 0 \tag{9}$$

First we need to show that the integral over the R-semicircle goes to 0 as R goes to infinity. This can be shown by direct calculation that shows the anti-derivative of $e^{iRe^{i\theta}}$ is a bounded function divided by R (TODO: supply details)

The integral over the ϵ semicircle is half of the integral around the pole (going around clockwise), so it evaluates to $-i * pi$

Therefore we have:

$$\int_{-R}^{-\epsilon} \frac{e^{ix}}{x} dx + \int_{\epsilon}^{R} \frac{e^{ix}}{x} dx = i\pi \tag{10}$$

Now we have $frace^{ix}x = \frac{cos(x)}{x} + i\frac{sin(x)}{x}$.

The first function is anti-symmetric, so the two opposite integrals cancel each other. The other function is symmetric so the two integrals are the same. Putting this together we get:

$$2i \int_{0}^{\infty} \frac{sin(x)}{x} dx = i\pi \tag{11}$$

7.6

Sow that $\sum_{n=0}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

Solution: TODO

7.7

What is the power series for $1/z$ around p

Solution

$$\sum_{n=0}^{\infty} (-1)^n p^{-(n+1)} (z-p)^n$$

7.8

Derive the power series for lnz around 1

Solution: $dlogz/dz = 1/z$. Subsequent powers of $1/z$ yield $(-1)^{n-1}\frac{z^n}{n}$. All these evaluated at $z = 1$ yield the coefficients $(-1)^{n-1}\frac{1}{n}$.

Chapter 8

8.1

Show that the Riemann surface z^a joins back after n turns when $a = m/n$ is rational.

Solution? z^a has a branch point at 0. If $a = m/n$ then $(z^a)^{kn} = z^{km}$ which is analytic on the whole complex plane.

8.2

Figure out the topology of the Riemann surface $(1 - z^4)^{1/2}$

TODO

8.3

The Riemann surface for $logz$ is topologically equivalent to the Riemann sphere with a single missing point. The missing point can be unambiguously replaced to yield the entire sphere. Can you see how this comes about?

Hint: Think of the Riemann sphere of the variable $w = logz$.

TODO

8.4

Show that $z \to z^{-1}$ maps circles to circles.

Solution Consider a circle of radius 1 centered at the real number r. This circle is given by $r + e^{i\theta}$. We can generalize to any radius and any offset by simply multiplying with a complex number c thus causing a rotation and scaling.

The conformal mapping z^{-1} sends this circle to $\frac{1}{r+e^{i\theta}}$.

After a lot of manipulations we find that this expression is equivalent to the circle given by

$$\frac{r}{r^2 - 1} + \frac{e^{-i\theta}}{r^2 - 1}$$

8.5

Verify that any Mobius transformation can be obtained by a sequence of linear $\to$ inversion $\to$ linear transformations.

Solution Straightforward

8.6

Check that the two stereographic projections from the north and south poles are related by the transformation $z \to 1/z$.

Solution It suffices to show it on a 2D circle. The result can be extended to the whole sphere by a rotation.

Fix a point $\bar{x}$ on the real line. The line connecting it to the north pole of the sphere (circle) is given by the equation $y = \frac{-1}{\bar{x}} + 1$. The circle equation is $x^2 + y^2 = 1$. Solving this system we obtain the point on the circle corresponding to $\bar{x}$:

$$x^* = \frac{2\bar{x}}{\bar{x}^2 + 1} \tag{12}$$

$$y^* = \frac{\bar{x}^2 - 1}{\bar{x}^2 + 1} \tag{13}$$

The antipodal point on the southern hemisphere is given by $(x^*, -y^*)$.

We seek the line that connects this point to the northern hemisphere and where it intersects the real line.

We can easily compute the slope of this line to be $-\bar{x}$ and the intercept 1. The line intersects the real axis at $0 = -\bar{x} * x + 1$. Solving for x we get $x = \frac{1}{\bar{x}}$ which is the desired result.

8.7

Show that the transformation

$$t = \frac{z - 1}{iz + i}$$

sends lines parallel to the x-axis to circles in the t-plane.

Solution TODO: Note that this is the same transform as in 9.5

8.8

Form a parallelogram on the complex plane by connecting the points (0,1,p,1+p). Then identify opposite edges to form a torus. For different values of p, all such surfaces are topologically equivalent, but may not be *holomorphically* equivalent, in the sense that no holomorhpic mapping can be constructed from one to another.

This question is asking to verify that certain transformations of p, such as 1+p, -p, 1/p, yield holomorphically equivalent surfaces. Furthermore, it asks to find all the special values of p that lead to additional discrete symmetries of the Riemann surface.

Discussion of possible Solution

Must construct a holomorphic mapping (Mobius?) that maps one torus to another. It is not clear to me how to define these mappings along the identified edges.

For the second part, I suspect choosing p such that the result is a square or a line would lead to additional symmetries, whatever this is supposed to mean.