

## Reference Sheet for Discrete Maths

### Propositional Calculus

Order of decreasing binding power:  $=, \neg, \wedge/\vee, \Rightarrow/\Leftarrow, \equiv/\neq$ .

**Equivalence** is the only equivalence relation that is associative  
 $((p \equiv q) \equiv r) \equiv (p \equiv (q \equiv r))$ , and it is symmetric and has identity **true**.

**Discrepancy** (difference) ' $\neq$ ' is symmetric, associative, has identity '**false**', mutually associates with equivalence  $((p \neq q) \equiv r) \equiv (p \neq (q \equiv r))$ , and mutually interchanges with it as well  $(p \neq q \equiv r) \equiv (p \equiv q \neq r)$ . Finally, negation commutes with difference:  $\neg(p \equiv q) \equiv \neg p \equiv q$ .

**Implication** has the alternative definition  $p \Rightarrow q \equiv \neg p \vee q$ , thus having **true** as both left identity and right zero; it distributes over  $\equiv$  in the second argument, and is self-distributive; and has the properties:

**Shunting**  $p \wedge q \Rightarrow r \equiv p \Rightarrow (q \Rightarrow r)$

**Contrapositive**  $p \Rightarrow q \equiv \neg q \Rightarrow \neg p$

**Leibniz**  $e = f \Rightarrow E[z \asymp e] = E[z := f]$

**Modus Ponens**

$$\begin{aligned} p \wedge (p \Rightarrow q) &\equiv p \wedge q \\ p \wedge (q \Rightarrow p) &\equiv p \\ p \wedge (p \Rightarrow q) &\Rightarrow q \end{aligned}$$

It is a *linear* order relation generated by '**false**  $\Rightarrow$  **true**'; whence "from false, follows anything": **false**  $\Rightarrow p$ . Moreover it has the useful properties "(3.62) Contextualisation":  $p \Rightarrow (q \equiv r) \equiv p \wedge q \equiv p \wedge r$ —we *have* the context  $p$  in each side of the equivalence—and  $p \Rightarrow (q \Rightarrow r) \equiv p \wedge q \Rightarrow p \wedge r$ . Implication is "Sub-associative":  $((p \Rightarrow q) \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))$ . Finally, we have " $\equiv$ -Elimination":  $(p \equiv q \equiv r) \Rightarrow s \equiv p \Rightarrow s \equiv q \Rightarrow s \equiv r \Rightarrow s$ .

**Conjunction and disjunction** distribute over one another, are both associative and symmetric,  $\vee$  has identity **false** and zero **true** whereas  $\wedge$  has identity **true** and zero **false**,  $\vee$  distributes over  $\vee, \equiv, \wedge, \Rightarrow, \Leftarrow$  whereas  $\wedge$  distributes over  $\equiv - \equiv$  in that  $p \wedge (q \equiv r \equiv s) \equiv p \wedge q \equiv p \wedge r \equiv p \wedge s$ , and they satisfy,

<b>Excluded Middle</b>	<b>Contradiction</b>	<b>Absorption</b>	<b>De Morgan</b>
$p \vee \neg p$	$p \wedge \neg p \equiv \text{false}$	$p \wedge (q \vee \neg p) \equiv p \wedge q$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$
		$p \vee (q \vee \neg p) \equiv p \vee q$	$\neg(p \vee q) \equiv \neg p \wedge \neg q$

Most importantly, they satisfy the "**Golden Rule**":  $p \wedge q \equiv p \equiv q \equiv p \vee q$ .

The many other properties of these operations—such as weakening laws and other absorption laws and case-analysis ( $\sqcup$ -char)—can be found by looking at the list of *lattice properties*—since the Booleans are a lattice.

### Orders

An *order* is a relation  $\sqsubseteq : \tau \rightarrow \tau \rightarrow \mathbb{B}$  satisfying the following three properties:

<b>Reflexivity</b>	<b>Transitivity</b>	<b>Mutual Inclusion</b>
$a \sqsubseteq a$	$a \sqsubseteq b \wedge b \sqsubseteq c \Rightarrow a \sqsubseteq c$	$a \sqsubseteq b \wedge b \sqsubseteq a \equiv a = b$

Indirect Inclusion is like 'set inclusion' and Indirect Equality is like 'set extensionality'.

<b>Indirect Equality (from above)</b>	<b>Indirect Inclusion (from above)</b>
$x = y \equiv (\forall z \bullet x \sqsubseteq z \equiv y \sqsubseteq z)$	$x \sqsubseteq y \equiv (\forall z \bullet y \sqsubseteq z \Rightarrow x \sqsubseteq z)$

<b>Indirect Equality (from below)</b>	<b>Indirect Inclusion (from below)</b>
$x = y \equiv (\forall z \bullet z \sqsubseteq x \equiv z \sqsubseteq y)$	$x \sqsubseteq y \equiv (\forall z \bullet z \sqsubseteq x \Rightarrow z \sqsubseteq y)$

An order is *bounded* if there are elements  $\top, \perp : \tau$  being the lower and upper bounds of all other elements:

<b>Top Element</b>	$a \sqsubseteq \top$	<b>Bottom Element</b>	$\perp \sqsubseteq a$
<b>Top is maximal</b>	$\top \sqsubseteq a \equiv a = \top$	<b>Bottom is minimal</b>	$a \sqsubseteq \perp \equiv a = \perp$

### Lattices

A *lattice* is a pair of operations  $\sqcap, \sqcup : \tau \rightarrow \tau \rightarrow \tau$  specified by the properties:

<b><math>\sqcup</math>-Characterisation</b>	<b><math>\sqcap</math>-Characterisation</b>
$a \sqsubseteq c \wedge b \sqsubseteq c \equiv a \sqcup b \sqsubseteq c$	$c \sqsubseteq a \wedge c \sqsubseteq b \equiv c \sqsubseteq a \sqcap b$

The operations act as providing the greatest lower bound, 'glb', 'supremum', or 'meet', by  $\sqcap$ ; and the least upper bound, 'lub', 'infimum', or 'join', by  $\sqcup$ .

Let  $\square$  be one of  $\sqcap$  or  $\sqcup$ , then:

<b>Symmetry of <math>\square</math></b>	<b>Associativity of <math>\square</math></b>	<b>Idempotency of <math>\square</math></b>
$a \square b = b \square a$	$(a \square b) \square c = a \square (b \square c)$	$a \square a = a$

<b>Zero of <math>\square</math></b>	<b>Identity of <math>\square</math></b>	<b>Absorption</b>	<b>Self-Distributivity of <math>\square</math></b>
$a \sqcup \top = \top$	$a \sqcup \perp = a$	$a \sqcap (b \sqcup a) = a$	$a \square (b \square c) = (a \square b) \square (a \square c)$
$a \sqcap \perp = \perp$	$a \sqcap \top = a$	$a \sqcup (b \sqcap a) = a$	

<b>Weakening</b>	<b>Induced Defs. of Inclusion</b>	<b>Golden Rule</b>
<b>/ Strengthening</b>	$a \sqsubseteq b \equiv a \sqcup b = b$	$a \sqcap b = a \equiv b = a \sqcup b$
$a \sqsubseteq a \sqcup b$	$a \sqsubseteq b \equiv a \sqcap b = a$	$a \sqcap b = a \sqcup b \equiv a = b$
$a \sqcap b \sqsubseteq a$		$a \sqcup b \sqsubseteq a \sqcap b \equiv a = b$
$a \sqcap b \sqsubseteq a \sqcup b$	<b>Monotonicity of <math>\square</math></b>	
	$a \sqsubseteq b \wedge c \sqsubseteq d \Rightarrow a \square c \sqsubseteq b \square d$	

### Duality Principle:

If a statement  $S$  is a theorem, then so is  $S[(\sqsubseteq, \sqcap, \sqcup, \top, \perp) := (\supseteq, \sqcup, \sqcap, \perp, \top)]$ .

## Conditionals

“If to  $\wedge$ ” may be taken as axiom from which we may prove the remaining ‘alternative definitions’ “if to  $\dots$ ”.

<b>if to <math>\wedge</math></b>	$P[z = \text{if } b \text{ then } x \text{ else } y \text{ fi}] \equiv (b \Rightarrow P[z = x]) \wedge (\neg b \Rightarrow P[z = x])$
<b>if to <math>\vee</math></b>	$P[z = \text{if } b \text{ then } x \text{ else } y \text{ fi}] \equiv (b \wedge P[z = x]) \vee (\neg b \wedge P[z = x])$
<b>if to <math>\neq</math></b>	$P[z = \text{if } b \text{ then } x \text{ else } y \text{ fi}] \equiv b \wedge P[z = x] \neq \neg b \wedge P[z = x]$
<b>if to <math>\equiv</math></b>	$P[z = \text{if } b \text{ then } x \text{ else } y \text{ fi}] \equiv b \Rightarrow P[z = x] \equiv \neg b \Rightarrow P[z = x]$

Note that the “ $\equiv$ ” and “ $\neq$ ” rules can be parsed in multiple ways since ‘ $\equiv$ ’ is associative, and ‘ $\neq$ ’ mutually associates with ‘ $\neq$ ’.

<b>if true</b>	$\text{if true then } x \text{ else } y \text{ fi} = x$
<b>if false</b>	$\text{if false then } x \text{ else } y \text{ fi} = y$
<b>then true</b>	$\text{if } R \text{ then true else } P \text{ fi} = R \vee P$
<b>then false</b>	$\text{if } R \text{ then false else } P \text{ fi} = \neg R \wedge P$
<b>else true</b>	$\text{if } R \text{ then } P \text{ else true fi} = R \Rightarrow P$
<b>else false</b>	$\text{if } R \text{ then } P \text{ else false fi} = R \wedge P$

<b>if swap</b>	$\text{if } b \text{ then } x \text{ else } y \text{ fi} = \text{if } \neg b \text{ then } y \text{ else } x \text{ fi}$
<b>if idempotency</b>	$\text{if } b \text{ then } x \text{ else } x \text{ fi} = x$
<b>if guard strengthening</b>	$\text{if } b \text{ then } x \text{ else } y \text{ fi} = \text{if } b \wedge x \neq y \text{ then } x \text{ else } y \text{ fi}$
<b>if Context</b>	$\text{if } b \text{ then } E \text{ else } F \text{ fi} = \text{if } b \text{ then } E[b = \text{true}] \text{ else } F[b = \text{false}] \text{ fi}$
<b>if Distributivity</b>	$P[z = \text{if } b \text{ then } x \text{ else } y \text{ fi}] = \text{if } b \text{ then } P[z = x] \text{ else } P[z = y] \text{ fi}$
<b>if junctivity</b>	$(\text{if } b \text{ then } x \text{ else } y \text{ fi}) \oplus (\text{if } b \text{ then } x' \text{ else } y' \text{ fi})$ $= \text{if } b \text{ then } (x \oplus x') \text{ else } (y \oplus y') \text{ fi}$

## Quantification

Let  $\_ \oplus \_$  be an associative and symmetric operation with identity  $\text{Id}$ .

<b>Abbreviation</b>	$(\oplus x \bullet P) = (\oplus x \mid \text{true} \bullet P)$
<b>Empty range</b>	$(\oplus x \mid \text{false} \bullet P) = \text{Id}$
<b>One-point rule</b>	$(\oplus x \mid x = E \bullet P) = P[x = E]$
<b>Distributivity</b>	$(\oplus x \mid R \bullet P \oplus Q) = (\oplus x \mid R \bullet P) \oplus (\oplus x \mid R \bullet Q)$
<b>Nesting</b>	$(\oplus x, y \mid X \wedge Y \bullet P) = (\oplus x \mid X \bullet (\oplus y \mid Y \bullet P))$
<b>Dummy renaming</b>	$(\oplus x \mid R \bullet P) = (\oplus y \mid R[x = y] \bullet P[x = y])$
<b>Disjoint Range split</b>	$(\oplus x \mid R \vee S \bullet P) = (\oplus x \mid R \bullet P) \oplus (\oplus x \mid S \bullet P)$ <i>provided <math>R \wedge S \equiv \text{false}</math></i>
<b>Range split</b>	$(\oplus x \mid R \vee S \bullet P) \oplus (\oplus x \mid R \wedge S \bullet P)$ $= (\oplus x \mid R \bullet P) \oplus (\oplus x \mid S \bullet P)$
<b>Idempotent Range split</b>	$(\oplus x \mid R \vee S \bullet P) = (\oplus x \mid R \bullet P) \oplus (\oplus x \mid S \bullet P)$ <i>provided <math>\oplus</math> is idempotent</i>

## Set Theory

The set theoretic symbols  $\in, =, \subseteq$ , are defined as follows.

**Axiom, Set Membership:**  $F \in \{x \mid R \bullet E\} \equiv (\exists x \mid R \bullet F = E)$

**Axiom, Extensionality:**  $S = T \equiv (\forall x \bullet x \in S \equiv x \in T)$

**Axiom, Subset:**  $S \subseteq T \equiv (\forall x \bullet x \in S \Rightarrow x \in T)$

As witnessed by the following definitions, it is the  $\in$  relation that *translates set theory to propositional logic*.

<b>Universe</b>	$x \in \mathbf{U}$	$\equiv \text{true}$
<b>Empty set</b>	$x \in \emptyset$	$\equiv \text{false}$
<b>Union</b>	$x \in S \cup T$	$\equiv x \in S \vee x \in T$
<b>Intersection</b>	$x \in S \cap T$	$\equiv x \in S \wedge x \in T$
<b>Complement</b>	$x \in \sim S$	$\equiv x \notin S$
<b>Difference</b>	$x \in S - T$	$\equiv x \in S \wedge x \notin T$
<b>Power set</b>	$S \in \mathbb{P}T$	$\equiv S \subseteq T$

The pairs  $\emptyset/\text{false}$ ,  $\mathbf{U}/\text{true}$ ,  $\cup/\vee$ ,  $\cap/\wedge$ ,  $\subseteq/\Rightarrow$ ,  $\sim/\neg$  are related by  $\in$  and so all equational theorems of propositional logic also hold for set theory —indeed, that is because both are Boolean algebras.

→ Set difference is a residual wrt  $\cup$ , and so satisfies the division properties below.

→ Subset is an order and so satisfies the aforementioned order properties. It is bounded below by  $\emptyset$  and above by  $\mathbf{U}$ .

The relationship between set comprehension and quantifier notation is:

**Set comprehension as union**  $\{x \mid R \bullet P\} = (\cup x \mid R \bullet \{P\})$

## Combinatorics

<b>Axiom, Size:</b>	$\#S = (\Sigma x \mid x \in S \bullet 1)$
<b>Axiom, Interval:</b>	$m..n = \{x : \mathbb{Z} \mid m \leq x \leq n\}$

The following theorems serve to define ‘ $\#$ ’ for the usual set theory operators.

<b>Positive definite</b>	$\#S \subseteq 0 \equiv S = \emptyset$
<b>Power set size</b>	$\#\mathbb{P}S = 2^{\#S}$
<b>Principle of Inclusion-Exclusion</b>	$\#(S \cup T) = \#S + \#T - \#(S \cap T)$
<b>Monotonicity</b>	$S \subseteq T \Rightarrow \#S \leq \#T$
<b>Difference rule</b>	$S \subseteq T \Rightarrow \#(T - S) = \#T - \#S$
<b>Complement size</b>	$\#(\sim S) = \#\mathbf{U} - \#S$
<b>Range size</b>	$(\Sigma x : \mathbf{U} \mid x \notin S \bullet 1) = \#\mathbf{U} - \#S$
<b>Interval size</b>	$\#(m..n) = n - m + 1 \text{ for } m \leq n$
<b>Pigeonhole Principle</b>	$(\Sigma i : 1..n \bullet E)/n \leq (\uparrow i : 1..n \bullet E)$
(“ $\text{min} \leq \text{avg} \leq \text{max}$ ”)	$(\downarrow i : 1..n \bullet E) \leq (\Sigma i : 1..n \bullet E)/n$

**Rule of sum:**  $\#(\cup i \mid R i \bullet P) = (\Sigma i \mid R i \bullet \#P)$   
*provided the range is pairwise disjoint:  $\forall i, j \bullet R i \wedge R j \equiv i = j$ .*

**Rule of product:**  $\#(\times i \mid R i \bullet P) = (\Pi i \mid R i \bullet \#P)$

## Residuals, Division

Suppose we have an associative operation  $\_;\_$  with identity  $\text{ld}$  and two operations “under  $\backslash$ ” and “over  $/$ ” specified as follows.

$$\begin{array}{ll} \text{Characterisation of } / & \text{Characterisation of } \backslash \\ a \dot{;} b \sqsubseteq c \equiv a \sqsubseteq c/b & a \dot{;} b \sqsubseteq c \equiv b \sqsubseteq a \backslash c \end{array}$$

When  $\dot{;}$  is symmetric, as in the special cases  $\dot{;} = \sqcap$ , the divisions coincide:  $x/y = y \backslash x$ .

$$\begin{array}{lll} \text{Cancellation} & (a/b) \dot{;} b \sqsubseteq a & a \dot{;} (a \backslash b) \sqsubseteq b \\ \text{Dividing a division} & (a/b)/c = a/(c \dot{;} b) & a \backslash (b \backslash c) = (b \dot{;} a) \backslash c \\ \text{Division of multiples} & a \sqsubseteq (a \dot{;} b)/b & b \sqsubseteq a \backslash (a \dot{;} b) \end{array}$$

$$\text{Monotonicity of } \dot{;} \quad a \sqsubseteq a' \wedge b \sqsubseteq b' \Rightarrow a \dot{;} b \sqsubseteq a' \dot{;} b'$$

$$\begin{array}{lll} \text{Numerator monotonicity} & b \sqsubseteq b' \Rightarrow a \backslash b \sqsubseteq a \backslash b' & b \sqsubseteq b' \Rightarrow b/a \sqsubseteq b'/a \\ \text{Denominator antitonicity} & a' \sqsubseteq a \Rightarrow a \backslash b \sqsubseteq a' \backslash b & a' \sqsubseteq a \Rightarrow b/a \sqsubseteq b/a' \\ \text{Self-reflexivity} & \text{ld} \sqsubseteq a \backslash a & \text{ld} \sqsubseteq a/a \\ \text{Denominator Identity} & \text{ld} \backslash a = a & a/\text{ld} = a \\ \text{Numerator Zero} & a \backslash \top = \top & \top/a = \top \\ \text{Wraparound rule} & \perp \backslash a = \top & a/\perp = \top \end{array}$$

Exact division:

$$\begin{array}{ll} (\exists z \bullet y = x \dot{;} z) & \equiv x \dot{;} (x \backslash y) = y \\ (\exists z \bullet y = x \backslash z) & \equiv x \backslash (x \dot{;} y) = y \end{array}$$

Division for the special case  $\dot{;} = \sqcap$  is known *the relative pseudo-complement*: Denoted  $x \rightarrow y$  (“ $x$  implies  $y$ ”), it is *the largest piece ‘outside’ of  $x$  that is still included in  $y$* . The relative pseudocomplement *internalises inclusion*,  $z \sqsubseteq (x \rightarrow y) \Rightarrow (z \sqsubseteq x \Rightarrow z \sqsubseteq y)$ ; more generally:  $x \sqsubseteq y \equiv \text{ld} \sqsubseteq x \rightarrow y$ .

$$\begin{array}{ll} \text{Pseudo-complement} & \text{Semi-complement} \\ x \sqcap a \sqsubseteq b \equiv x \sqsubseteq a \rightarrow b & a - b \sqsubseteq x \equiv a \sqsubseteq b \sqcup x \\ \\ \text{Strong modus ponens} & \text{Absorption} \\ a \sqcap (a \rightarrow b) = a \sqcap b & (x \sqcup b) - b = x - b \\ a \rightarrow (x \sqcap a) = a \rightarrow x & (a - b) \sqcup b = a \sqcup b \end{array}$$

Division for the special case  $\dot{;} = \sqcup$  in the *dual order* ( $\sqsupset$ ) is known as *the difference* or *relative semi-complement*: Denoted  $x - y$  (“ $x$  without  $y$ ”), it is *the smallest piece that along with  $y$  ‘covers’  $x$* ; i.e., it is the least value that ‘complements’ (“fill up together”)  $y$  to include  $x$ . (Possibly for this reason, set difference is sometimes denoted  $S \backslash T$  in other books!)

## Converse

Axioms,

$$\begin{array}{lll} \text{Co-distributivity} & \sim, \text{Involutive} & \text{Monotonicity} \\ (x \dot{;} y)^\sim = y^\sim \dot{;} x^\sim & x^{\sim\sim} = x & x \sqsubseteq y \Rightarrow x^\sim \sqsubseteq y^\sim \end{array}$$

Theorems,

$$\begin{array}{llll} \text{Identity} & \text{Isotonicity} & \text{Connection} & \text{Elimination} \\ \text{ld}^\sim = \text{ld} & x \sqsubseteq y \equiv x^\sim \sqsubseteq y^\sim & a^\sim \sqsubseteq b \equiv a \sqsubseteq b^\sim & x^\sim = y^\sim \equiv x = y \end{array}$$

## Named Properties

$$\begin{array}{lll} \text{reflexive} & x \equiv \text{ld} \sqsubseteq x & \text{symmetric} \quad x \equiv x^\sim = x \\ \text{irreflexive} & x \equiv \text{ld} \sqcap x = \perp & \text{antisymmetric} \quad x \equiv x \sqcap x^\sim \sqsubseteq \text{ld} \\ \text{transitive} & x \equiv x \dot{;} x \sqsubseteq x & \text{asymmetric} \quad x \equiv x \sqcap x^\sim = \perp \\ \text{idempotent} & x \equiv x \dot{;} x = \text{ld} & \end{array}$$

The above properties are preserved by converse: Let  $P$  be any of the above properties, then  $Px \equiv P(x^\sim)$ .

$$\begin{array}{lll} \text{univalent} & x \equiv x^\sim \dot{;} x \sqsubseteq \text{ld} & \text{injective} \quad x \equiv x \dot{;} x^\sim \sqsubseteq \text{ld} \\ \text{total} & x \equiv \text{ld} \sqsubseteq x \dot{;} x^\sim & \text{surjective} \quad x \equiv \text{ld} \sqsubseteq x^\sim \dot{;} x \\ \text{mapping} & x \equiv \text{total } x \wedge \text{univalent } x & \text{bijective} \quad x \equiv \text{surjective } x \wedge \text{injective } x \\ \text{iso} & x \equiv \text{mapping } x \wedge \text{bijective } x & \end{array}$$

## Duality theorems

$$\begin{array}{lll} \text{univalent } (x^\sim) & \equiv \text{injective } x \\ \text{total } (x^\sim) & \equiv \text{surjective } x \\ \text{mapping } (x^\sim) & \equiv \text{bijective } x \\ \text{iso } (x^\sim) & \equiv \text{iso } x \end{array}$$

## Invertibility theorems

$$\begin{array}{ll} \text{total } x \wedge \text{injective } x \Rightarrow x \dot{;} x^\sim = \text{ld} & \\ \text{iso } x \equiv x \dot{;} x^\sim = \text{ld} \wedge x^\sim \dot{;} x = \text{ld} & \\ \text{iso } x \Rightarrow (\exists g \bullet x \dot{;} g = \text{ld} = g \dot{;} x) & \end{array}$$

Shunting laws:

$$\begin{array}{lll} \text{univalent } f & \Rightarrow (x \dot{;} f \sqsubseteq y \Leftarrow x \sqsubseteq y \dot{;} f^\sim) \\ \text{total } f & \Rightarrow (x \dot{;} f \sqsubseteq y \Rightarrow x \sqsubseteq y \dot{;} f^\sim) \\ \text{mapping } f & \Rightarrow (x \dot{;} f \sqsubseteq y \equiv x \sqsubseteq y \dot{;} f^\sim) \end{array}$$

## Relations

Relations are sets of pairs ...

$$\begin{array}{lll} \text{Tortoise} & x \langle R \rangle y & \equiv \langle x, y \rangle \in R \\ \text{Extensionality} & R = S & \equiv (\forall x, y \bullet x \langle R \rangle y \equiv x \langle S \rangle y) \\ \text{Inclusion} & R \subseteq S & \equiv (\forall x, y \bullet x \langle R \rangle y \Rightarrow x \langle S \rangle y) \\ \text{Empty} & u \langle \emptyset \rangle v & \equiv \text{false} \\ \text{Universe} & u \langle A \times B \rangle v & \equiv u \in A \wedge v \in B \\ \text{Complement} & u \langle \sim S \rangle v & \equiv \neg(u \langle S \rangle v) \\ \text{Union} & u \langle S \cup T \rangle v & \equiv u \langle S \rangle v \vee u \langle T \rangle v \\ \text{Intersection} & u \langle S \cap T \rangle v & \equiv u \langle S \rangle v \wedge u \langle T \rangle v \\ \text{Difference} & u \langle S - T \rangle v & \equiv u \langle S \rangle v \wedge \neg(u \langle T \rangle v) \\ \text{PseudoComplement} & u \langle S \rightarrow T \rangle v & \equiv u \langle S \rangle v \Rightarrow u \langle T \rangle v \\ \text{An Identity} & u \langle \text{Id} \rangle v & \equiv u = v \\ \text{The Identity} & u \langle \text{ld} \rangle v & \equiv u = v \\ \text{Converse} & u \langle R^\sim \rangle v & \equiv v \langle R \rangle u \\ \text{Composition} & u \langle R \dot{;} S \rangle v & \equiv (\exists x \bullet u \langle R \rangle x \wedge x \langle S \rangle v) \\ \text{Over Division} & u \langle R/S \rangle v & \equiv (\forall x \bullet x \langle R \rangle u \Rightarrow x \langle S \rangle v) \\ \text{Under Division} & u \langle S \backslash R \rangle v & \equiv (\forall x \bullet v \langle R \rangle x \Rightarrow u \langle S \rangle x) \end{array}$$