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Reference Sheet for Discrete Maths

Propositional Calculus

Order of decreasing binding power: $=, \neg, \wedge/\vee, \Rightarrow / \Leftarrow, \equiv / \not\equiv$.

Equivales is the only equivalence relation that is associative $((p \equiv q) \equiv r) \quad \equiv \quad (p \equiv (q \equiv r))$, and it is symmetric and has identity true.

Discrepancy (difference) '$\not\equiv$' is symmetric, associative, has identity 'false', mutually associates with equivales $((p \not\equiv q) \equiv r) \quad \equiv \quad (p \not\equiv (q \equiv r))$, and mutually interchanges with it as well $(p \not\equiv q \equiv r) \quad \equiv \quad (p \equiv q \not\equiv r)$. Finally, negation commutes with difference: $\neg(p \equiv q) \equiv \neg p \equiv q$.

Implication has the alternative definition $p \Rightarrow q \;\equiv\; \neg p \vee q$, thus having true as both left identity and right zero; it distributes over $\equiv$ in the second argument, and is self-distributive; and has the properties:

Shunting $p \wedge q \Rightarrow r \;\equiv\; p \Rightarrow (q \Rightarrow r)$

Contrapositive $p \Rightarrow q \quad\equiv\quad \neg q \Rightarrow \neg p$

Leibniz $e = f \;\Rightarrow\; E[z \coloneqq e] = E[z := f]$

Modus Ponens

$$\begin{array}{lcl} p \wedge (p \Rightarrow q) & \equiv & p \wedge q \\ p \wedge (q \Rightarrow p) & \equiv & p \\ p \wedge (p \Rightarrow q) & \Rightarrow & q \end{array}$$

It is a *linear* order relation generated by '$\mathsf{false} \Rightarrow \mathsf{true}$'; whence "from false, follows anything": $\mathsf{false} \Rightarrow p$. Moreover it has the useful properties "(3.62) Contextualisation": $p \Rightarrow (q \equiv r) \quad\equiv\quad p \wedge q \quad\equiv\quad p \wedge r$ —we *have* the context p in each side of the equivalence— and $p \Rightarrow (q \Rightarrow r) \quad\equiv\quad p \wedge q \Rightarrow p \wedge r$. Implication is "Sub-associative": $((p \Rightarrow q) \Rightarrow r) \quad\Rightarrow\quad (p \Rightarrow (q \Rightarrow r))$. Finally, we have "$\equiv$-$\equiv$ Elimination": $(p \equiv q \equiv r) \Rightarrow s \;\equiv\; p \Rightarrow s \;\equiv\; q \Rightarrow s \;\equiv\; r \Rightarrow s$.

Conjunction and disjunction distribute over one another, are both associative and symmetric, $\vee$ has identity false and zero true whereas $\wedge$ has identity true and zero false, $\vee$ distributes over $\vee, \equiv, \wedge, \Rightarrow, \;\Leftarrow$ whereas $\wedge$ distributes over $\equiv - \equiv$ in that $p \wedge (q \equiv r \equiv s) \;\equiv\; p \wedge q \quad\equiv\quad p \wedge r \quad\equiv\quad p \wedge s$, and they satisfy,

Excluded Middle	Contradiction	Absorption	De Morgan
$p \vee \neg p$	$p \wedge \neg p \equiv \mathsf{false}$	$p \wedge (q \vee \neg p) \equiv p \wedge q$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$
		$p \vee (q \wedge \neg p) \equiv p \vee q$	$\neg(p \vee q) \equiv \neg p \wedge \neg q$

Most importantly, they satisfy the **"Golden Rule"**: $p \wedge q \equiv p \equiv q \equiv p \vee q$.

Max $\uparrow$ and Min $\downarrow$ each distribute over the other, addition distributes over both, subtraction acts like De Morgans, the operators are selective, and non-negative multiplication distributes over both. (*Tropical mathematics* is math with '$\uparrow$, $+$' instead of '$+$, $\times$'.)

The many other properties of these operations —such as weakening laws and other absorption laws and case-analysis ($\sqcup$-char)— can be found by looking at the list of *lattice properties* —since *both* the Booleans ($\Rightarrow, \wedge, \vee$) and numbers ($\leq, \downarrow, \uparrow$) are lattices.

Orders

An *order* is a relation $\_ \sqsubseteq \_ \;:\; \tau \to \tau \to \mathbb{B}$ satisfying the following three properties:

Reflexivity	Transitivity	Mutual Inclusion
$a \sqsubseteq a$	$a \sqsubseteq b \wedge b \sqsubseteq c \Rightarrow a \sqsubseteq c$	$a \sqsubseteq b \wedge b \sqsubseteq a \;\equiv\; a = b$

Indirect Inclusion is like 'set inclusion' and Indirect Equality is like 'set extensionality'.

Indirect Equality (from above)
$x = y \;\equiv\; (\forall z \bullet x \sqsubseteq z \equiv y \sqsubseteq z)$

Indirect Inclusion (from above)
$x \sqsubseteq y \;\equiv\; (\forall z \bullet y \sqsubseteq z \Rightarrow x \sqsubseteq z)$

Indirect Equality (from below)
$x = y \;\equiv\; (\forall z \bullet z \sqsubseteq x \equiv z \sqsubseteq y)$

Indirect Inclusion (from below)
$x \sqsubseteq y \;\equiv\; (\forall z \bullet z \sqsubseteq x \Rightarrow z \sqsubseteq y)$

An order is *bounded* if there are elements $\top, \bot : \tau$ being the lower and upper bounds of all other elements:

Top Element	$a \sqsubseteq \top$	**Bottom Element**	$\bot \sqsubseteq a$
Top is maximal	$\top \sqsubseteq a \;\equiv\; a = \top$	**Bottom is minimal**	$a \sqsubseteq \bot \;\equiv\; a = \top$

Lattices

A *lattice* is a pair of operations $\_ \sqcap \_, \; \_ \sqcup \_ \;:\; \tau \to \tau \to \tau$ specified by the properties:

$\sqcup$-Characterisation
$a \sqsubseteq c \wedge b \sqsubseteq c \;\equiv\; a \sqcup b \;\sqsubseteq\; c$

$\sqcap$-Characterisation
$c \sqsubseteq a \wedge c \sqsubseteq b \;\equiv\; c \;\sqsubseteq\; a \sqcap b$

The operations act as providing the greatest lower bound, 'glb', 'supremum', or 'meet', by $\sqcap$; and the least upper bound, 'lub', 'infimum', or 'join', by $\sqcup$.

Let $\square$ be one of $\sqcap$ or $\sqcup$, then:

Symmetry of $\square$	Associativity of $\square$	Idempotency of $\square$
$a \square b \;=\; b \square a$	$(a \square b) \square c \;=\; a \square (b \square c)$	$a \square a \;=\; a$

Zero of $\square$	Identity of $\square$	Absorption	Self-Distributivity of $\square$
$a \sqcup \top = \top$	$a \sqcup \bot = a$	$a \sqcap (b \sqcup a) = a$	$a \square (b \square c) \;=\; (a \square b) \square (a \square c)$
$a \sqcap \bot = \bot$	$a \sqcap \top = a$	$a \sqcup (b \sqcap a) = a$	

Weakening / Strengthening
$a \;\sqsubseteq\; a \sqcup b$
$a \sqcap b \;\sqsubseteq\; a$
$a \sqcap b \;\sqsubseteq\; a \sqcup b$

Induced Defs. of Inclusion
$a \sqsubseteq b \;\equiv\; a \sqcup b \;=\; b$
$a \sqsubseteq b \;\equiv\; a \sqcap b \;=\; a$

Monotonicity of $\square$
$a \sqsubseteq b \wedge c \sqsubseteq d \;\Rightarrow\; a \square c \;\sqsubseteq\; b \square d$

Golden Rule
$a \sqcap b \;=\; a \quad\equiv\quad b = a \sqcup b$
$a \sqcap b \;=\; a \sqcup b \quad\equiv\quad a = b$
$a \sqcup b \;\sqsubseteq\; a \sqcap b \quad\equiv\quad a = b$

The following four properties are all equivalent:

$\sqcap$-Selective :: $\forall a, b \bullet a \sqcap b = a \vee a \sqcap b = a$

$\sqcup$-Selective :: $\forall a, b \bullet a \sqcup b = a \vee a \sqcup b = a$

Linearity :: $\forall a, b \bullet a \sqsubseteq b \vee b \sqsubseteq a$

Order Complement :: $\neg(a \sqsubseteq b) \;\equiv\; b \sqsubset a$

Duality Principle:
If a statement S is a theorem, then so is $S[(\sqsubseteq, \sqcap, \sqcup, \top, \bot) := (\sqsupseteq, \sqcup, \sqcap, \bot, \top)]$.

Conditionals

"If to $\wedge$" may be taken as axiom from which we may prove the remaining 'alternative definitions' "if to $\cdots$".

if to $\wedge$	$P[z \coloneqq \text{if } b \text{ then } x \text{ else } y \text{ fi}]$	$\equiv$	$(b \Rightarrow P[z \coloneqq x])$	$\wedge$	$(\neg b \Rightarrow P[z := x])$
if to $\vee$	$P[z \coloneqq \text{if } b \text{ then } x \text{ else } y \text{ fi}]$	$\equiv$	$(b \wedge P[z \coloneqq x])$	$\vee$	$(\neg b \wedge P[z := x])$
if to $\not\equiv$	$P[z \coloneqq \text{if } b \text{ then } x \text{ else } y \text{ fi}]$	$\equiv$	$b \wedge P[z \coloneqq x]$	$\not\equiv$	$\neg b \wedge P[z := x]$
if to $\equiv$	$P[z \coloneqq \text{if } b \text{ then } x \text{ else } y \text{ fi}]$	$\equiv$	$b \Rightarrow P[z \coloneqq x]$	$\equiv$	$\neg b \Rightarrow P[z := x]$

Note that the "$\equiv$" and "$\not\equiv$" rules can be parsed in multiple ways since '$\equiv$' is associative, and '$\equiv$' mutually associates with '$\not\equiv$'.

if true	$\text{if true then } x \text{ else } y \text{ fi} = x$
if false	$\text{if false then } x \text{ else } y \text{ fi} = y$
then true	$\text{if } R \text{ then true else } P \text{ fi} = R \vee P$
then false	$\text{if } R \text{ then false else } P \text{ fi} = \neg R \wedge P$
else true	$\text{if } R \text{ then } P \text{ else true fi} = R \Rightarrow P$
else false	$\text{if } R \text{ then } P \text{ else false fi} = R \wedge P$

if swap	$\text{if } b \text{ then } x \text{ else } y \text{ fi} = \text{if } \neg b \text{ then } y \text{ else } x \text{ fi}$
if idempotency	$\text{if } b \text{ then } x \text{ else } x \text{ fi} = x$
if guard strengthening	$\text{if } b \text{ then } x \text{ else } y \text{ fi} = \text{if } b \wedge x \neq y \text{ then } x \text{ else } y \text{ fi}$
if Context	$\text{if } b \text{ then } E \text{ else } F \text{ fi} = \text{if } b \text{ then } E[b \coloneqq \text{true}] \text{ else } F[b \coloneqq \text{false}] \text{ fi}$
if Distributivity	$P[z \coloneqq \text{if } b \text{ then } x \text{ else } y \text{ fi}] = \text{if } b \text{ then } P[z \coloneqq x] \text{ else } P[z \coloneqq y] \text{ fi}$
if junctivity	$\big(\text{if } b \text{ then } x \text{ else } y \text{ fi}\big) \oplus \big(\text{if } b \text{ then } x' \text{ else } y' \text{ fi}\big)$ $= \text{if } b \text{ then } (x \oplus x') \text{ else } (y \oplus y') \text{ fi}$

Quantification

Let $\_\oplus\_$ be an associative and symmetric operation with identity Id.

Abbreviation	$(\oplus x \bullet P) = (\oplus x \mid \text{true} \bullet P)$
Empty range	$(\oplus x \mid \text{false} \bullet P) = \mathsf{Id}$
One-point rule	$(\oplus x \mid x = E \bullet P) = P[x \coloneqq E]$
Distributivity	$(\oplus x \mid R \bullet P \oplus Q) = (\oplus x \mid R \bullet P) \oplus (\oplus x \mid R \bullet Q)$
Nesting	$(\oplus x, y \mid X \wedge Y \bullet P) = (\oplus x \mid X \bullet (\oplus y \mid Y \bullet P))$
Dummy renaming	$(\oplus x \mid R \bullet P) = (\oplus y \mid R[x \coloneqq y] \bullet P[x \coloneqq y])$
Disjoint Range split	$(\oplus x \mid R \vee S \bullet P) = (\oplus x \mid R \bullet P) \oplus (\oplus x \mid S \bullet Q)$ *provided* $R \wedge S \equiv \text{false}$
Range split	$(\oplus x \mid R \vee S \bullet P) \oplus (\oplus x \mid R \wedge S \bullet P)$ $= (\oplus x \mid R \bullet P) \oplus (\oplus x \mid S \bullet Q)$
Idempotent Range split	$(\oplus x \mid R \vee S \bullet P) = (\oplus x \mid R \bullet P) \oplus (\oplus x \mid S \bullet Q)$ *provided* $\oplus$ is idempotent

Set Theory

The set theoretic symbols $\in$, $=$, $\subseteq$, are defined as follows.

Axiom, Set Membership: $F \in \{x \mid R \bullet E\} \equiv (\exists x \mid R \bullet F = E)$

Axiom, Extensionality: $S = T \equiv (\forall x \bullet x \in S \equiv x \in T)$

Axiom, Subset: $S \subseteq T \equiv (\forall x \bullet x \in S \Rightarrow x \in T)$

As witnessed by the following definitions, it is the $\in$ relation that *translates set theory to propositional logic.*

Universe	$x \in \mathbf{U}$	$\equiv$	$true$
Empty set	$x \in \emptyset$	$\equiv$	$false$
Complement	$x \in {\sim}S$	$\equiv$	$x \notin S$
Union	$x \in S \cup T$	$\equiv$	$x \in S \vee x \in T$
Intersection	$x \in S \cap T$	$\equiv$	$x \in S \wedge x \in T$
PseudoComplement	$x \in S \rightarrow T$	$\equiv$	$x \in S \Rightarrow x \in T$
Difference	$x \in S - T$	$\equiv$	$x \in S \wedge x \notin T$
Power set	$S \in \mathbb{P}T$	$\equiv$	$S \subseteq T$

The pairs $\emptyset$|false, $\mathbf{U}$|true, $\cup|\vee$, $\cap|\wedge$, $\subseteq|\Rightarrow$, $\sim|\neg$ are related by $\in$ and so all equational theorems of propositional logic also hold for set theory —indeed, that is because both are Boolean algebras.

→ Set difference is a residual wrt $\cup$, and so satisfies the division properties below.

→ Subset is an order and so satisfies the aforementioned order properties. It is bounded below by $\emptyset$ and above by $\mathbf{U}$.

The relationship between set comprehension and quantifier notation is:

Set comprehension as union	$\{x \mid R \bullet P\} = (\cup x \mid R \bullet \{P\})$
Membership as inclusion	$x \in S \equiv \{x\} \subseteq S$
Equality as membership	$x = y \equiv x \in \{y\}$

Combinatorics

Axiom, Size:	$\#S = (\Sigma x \mid x \in S \bullet 1)$
Axiom, Interval:	$m..n = \{x : \mathbb{Z} \mid m \leq x \leq n\}$

The following theorems serve to define '#' for the usual set theory operators.

Positive definite	$\#S \leq 0 \equiv S = \emptyset$
Power set size	$\#\mathbb{P}S = 2^{\#S}$
Principle of Inclusion-Exclusion	$\#(S \cup T) = \#S + \#T - \#(S \cap T)$
Monotonicity	$S \subseteq T \Rightarrow \#S \leq \#T$
Difference rule	$S \subseteq T \Rightarrow \#(T - S) = \#T - \#S$
Complement size	$\#({\sim}S) = \#\mathbf{U} - \#S$
Range size	$(\Sigma x : \mathbf{U} \mid x \notin S \bullet 1) = \#\mathbf{U} - \#S$
Interval size	$\#(m..n) = n - m + 1$ for $m \leq n$
Pigeonhole Principle	$(\Sigma i : 1..n \bullet E)/n \leq (\uparrow i : 1..n \bullet E)$
(*"min $\leq$ avg $\leq$ max"*)	$(\downarrow i : 1..n \bullet E) \leq (\Sigma i : 1..n \bullet E)/n$

Rule of sum: $\#(\cup i \mid R\, i \bullet P) = (\Sigma i \mid R\, i \bullet \#P)$
provided the range is pairwise disjoint: $\forall i, j \bullet R\, i \wedge R\, j \equiv i = j$.

Rule of product: $\#(\times i \mid R\, i \bullet P) = (\Pi i \mid R\, i \bullet \#P)$

Converse —an over-approximation of inverse (A4)

Co-distributivity	$\smile\smile$, Involutive	Monotonicity
$(x \fatsemi y)^\smile = y^\smile \fatsemi x$	$x^{\smile\smile} = x$	$x \sqsubseteq y \Rightarrow x^\smile \sqsubseteq y^\smile$

Identity	Isotonicity	Connection	Elimination
$\mathsf{Id}^\smile = \mathsf{Id}$	$x \sqsubseteq y \;\equiv\; x^\smile \sqsubseteq y^\smile$	$a^\smile \sqsubseteq b \;\equiv\; a \sqsubseteq b^\smile$	$x^\smile = y^\smile \;\equiv\; x = y$

Residuals, Division

Suppose we have an associative operation $\_\fatsemi\_$ with identity Id and two operations "under \" and "over /" specified as follows.

Characterisation of /	Characterisation of \
$a \fatsemi b \sqsubseteq c \quad\equiv\quad a \sqsubseteq c/b$	$a \fatsemi b \sqsubseteq c \quad\equiv\quad b \sqsubseteq a\backslash c$

When $\fatsemi$ is symmetric, as in the special cases $\fatsemi = \sqcap$, the divisions coincide: $x/y \;=\; y\backslash x$.

Cancellation	$(a/b) \fatsemi b \sqsubseteq a$	$a \fatsemi (a\backslash b) \sqsubseteq b$
Dividing a division	$(a/b)/c \;=\; a/(c \fatsemi b)$	$a\backslash(b\backslash c) \;=\; (b \fatsemi a)\backslash c$
Division of multiples	$a \sqsubseteq (a \fatsemi b)/b$	$b \sqsubseteq a\backslash(a \fatsemi b)$

Monotonicity of $\fatsemi$	$a \sqsubseteq a' \wedge b \sqsubseteq b' \Rightarrow a \fatsemi b \sqsubseteq a' \fatsemi b'$
Subdistributivity of $\fatsemi$ over $\sqcap$	$a \fatsemi (b \sqcap c) \;\sqsubseteq\; a \fatsemi b \sqcap a \fatsemi c$

Numerator monotonicity	$b \sqsubseteq b' \Rightarrow a\backslash b \sqsubseteq a\backslash b'$	$b \sqsubseteq b' \Rightarrow b/a \sqsubseteq b'/a$
Denominator antitonicity	$a' \sqsubseteq a \Rightarrow a\backslash b \sqsubseteq a'\backslash b$	$a' \sqsubseteq a \Rightarrow b/a \sqsubseteq b/a'$

Exact division	$(\exists z \bullet y = x \fatsemi z) \quad\equiv\quad x \fatsemi (x\backslash y) = y$
Exact division	$(\exists z \bullet y = x\backslash z) \quad\equiv\quad x\backslash(x \fatsemi y) = y$

Modal and Dedekind rules:

(Axioms)	(Theorems)
$a \fatsemi b \sqcap c \;\sqsubseteq\; a \fatsemi (b \sqcap a^\smile \fatsemi c)$	$a\backslash b \sqcap c \;\sqsubseteq\; a\backslash(b \sqcap a \fatsemi c)$
$a \fatsemi b \sqcap c \;\sqsubseteq\; (a \sqcap c \fatsemi b^\smile) \fatsemi b$	$a\backslash b \sqcap c \;\sqsubseteq\; (a \sqcap c\backslash b)\backslash b$
$a \fatsemi b \sqcap c \;\sqsubseteq\; (a \sqcap c \fatsemi b^\smile) \fatsemi (b \sqcap a^\smile \fatsemi c)$	$a\backslash b \sqcap c \;\sqsubseteq\; (a \sqcap c\backslash b)\backslash(b \sqcap a \fatsemi c)$

Division for the special case $\fatsemi = \sqcap$ is known as *the relative pseudo-complement*: Denoted $x \twoheadrightarrow y$ ("*x implies y*"), it is *the largest piece 'outside' of x that is still included in y*. The relative pseudocomplement *internalises inclusion*, $z \sqsubseteq (x \twoheadrightarrow y) \;\Rightarrow\; (z \sqsubseteq x \Rightarrow z \sqsubseteq y)$; more generally: $x \sqsubseteq y \;\equiv\; \mathsf{Id} \sqsubseteq x\backslash y$.

Pseudo-complement	Semi-complement
$x \sqcap a \sqsubseteq b \;\equiv\; x \sqsubseteq a \twoheadrightarrow b$	$a - b \sqsubseteq x \;\equiv\; a \sqsubseteq b \sqcup x$

Strong modus ponens	Absorption
$a \sqcap (a \twoheadrightarrow b) \;=\; a \sqcap b$	$(x \sqcup b) - b \;=\; x - b$
$a \twoheadrightarrow (x \sqcap a) \;=\; a \twoheadrightarrow x$	$(a - b) \sqcup b \;=\; a \sqcup b$

Division for the special case $\fatsemi = \sqcup$ in the *dual order* ($\sqsupseteq$) is known as *the difference* or *relative semi-complement*: Denoted $x - y$ ("*x without y*"), it is *the smallest piece that along with y 'covers' x*; i.e., it is the least value that 'complements' ("fill up together") y to include x. (Possibly for this reason, set difference is sometimes denoted $S\backslash T$ in other books!)

Named Properties

reflexive	x	$\equiv$	$\mathsf{Id} \sqsubseteq x$	
irreflexive	x	$\equiv$	$\mathsf{Id} \sqcap x = \bot$	
transitive	x	$\equiv$	$x \fatsemi x \sqsubseteq x$	
idempotent	x	$\equiv$	$x \fatsemi x = x$	
symmetric	x	$\equiv$	$x^\smile = x$	
antisymmetric	x	$\equiv$	$x \sqcap x^\smile \sqsubseteq \mathsf{Id}$	
asymmetric	x	$\equiv$	$x \sqcap x^\smile = \bot$	

The above properties are preserved by converse: Let P be any of the above properties, then $P\,x \;\equiv\; P(x^\smile)$.

univalent	x	$\equiv$	$x^\smile \fatsemi x \sqsubseteq \mathsf{Id}$
total	x	$\equiv$	$\mathsf{Id} \sqsubseteq x \fatsemi x^\smile$
mapping	x	$\equiv$	total x $\wedge$ univalent x
iso	x	$\equiv$	mapping x $\wedge$ bijective x
injective	x	$\equiv$	$x \fatsemi x^\smile \sqsubseteq \mathsf{Id}$
surjective	x	$\equiv$	$\mathsf{Id} \sqsubseteq x^\smile \fatsemi x$
bijective	x	$\equiv$	surjective x $\wedge$ injective x

Duality theorems

univalent $(x^\smile)$	$\equiv$	injective x
total $(x^\smile)$	$\equiv$	surjective x
mapping $(x^\smile)$	$\equiv$	bijective x
iso $(x^\smile)$	$\equiv$	iso x

Invertiblility theorems

total $x \wedge$ injective $x \Rightarrow x \fatsemi x^\smile = \mathsf{Id}$
iso $x \quad\equiv\quad x \fatsemi x^\smile = \mathsf{Id} \;\wedge\; x^\smile \fatsemi x = \mathsf{Id}$
iso $x \quad\Rightarrow\quad (\exists g \bullet\; x \fatsemi g = \mathsf{Id} = g \fatsemi x)$

Shunting laws:

univalent f	$\Rightarrow$	$(x \fatsemi f \sqsubseteq y \;\Leftarrow\; x \sqsubseteq y \fatsemi f^\smile)$
total f	$\Rightarrow$	$(x \fatsemi f \sqsubseteq y \;\Rightarrow\; x \sqsubseteq y \fatsemi f^\smile)$
mapping f	$\Rightarrow$	$(x \fatsemi f \sqsubseteq y \;\equiv\; x \sqsubseteq y \fatsemi f^\smile)$

Relations

Relations are sets of pairs . . .

Tortoise	$x \llparenthesis R \rrparenthesis y$	$\equiv$	$\langle x, y\rangle \in R$
Extensionality	$R = S$	$\equiv$	$(\forall\, x, y \;\bullet\; x \llparenthesis R \rrparenthesis y \equiv x \llparenthesis S \rrparenthesis y)$
Inclusion	$R \subseteq S$	$\equiv$	$(\forall\, x, y \;\bullet\; x \llparenthesis R \rrparenthesis y \Rightarrow x \llparenthesis S \rrparenthesis y)$
Empty	$u \llparenthesis \emptyset \rrparenthesis v$	$\equiv$	false
Universe	$u \llparenthesis A \times B \rrparenthesis v$	$\equiv$	$u \in A \;\wedge\; v \in B$
Complement	$u \llparenthesis {\sim}S \rrparenthesis v$	$\equiv$	$\neg(u \llparenthesis S \rrparenthesis v)$
Union	$u \llparenthesis S \cup T \rrparenthesis v$	$\equiv$	$u \llparenthesis S \rrparenthesis v \;\vee\; u \llparenthesis T \rrparenthesis v$
Intersection	$u \llparenthesis S \cap T \rrparenthesis v$	$\equiv$	$u \llparenthesis S \rrparenthesis v \;\wedge\; u \llparenthesis T \rrparenthesis v$
Difference	$u \llparenthesis S - T \rrparenthesis v$	$\equiv$	$u \llparenthesis S \rrparenthesis v \;\wedge\; \neg(u \llparenthesis T \rrparenthesis v)$
PseudoComplement	$u \llparenthesis S \twoheadrightarrow T \rrparenthesis v$	$\equiv$	$u \llparenthesis S \rrparenthesis v \;\Rightarrow\; u \llparenthesis T \rrparenthesis v$
An Identity	$u \llparenthesis \mathbb{I}\, A \rrparenthesis v$	$\equiv$	$u = v \in A$
The Identity	$u \llparenthesis \mathsf{Id} \rrparenthesis v$	$\equiv$	$u = v$
Converse	$u \llparenthesis R^\smile \rrparenthesis v$	$\equiv$	$v \llparenthesis R \rrparenthesis u$
Composition	$u \llparenthesis R \fatsemi S \rrparenthesis v$	$\equiv$	$(\exists\, x \;\bullet\; u \llparenthesis R \rrparenthesis x \;\wedge\; x \llparenthesis S \rrparenthesis v)$
Under Division	$u \llparenthesis S\backslash R \rrparenthesis v$	$\equiv$	$(\forall x \bullet x \llparenthesis S \rrparenthesis u \Rightarrow x \llparenthesis R \rrparenthesis v)$
Over Division	$u \llparenthesis R/S \rrparenthesis v$	$\equiv$	$(\forall y \bullet v \llparenthesis S \rrparenthesis y \Rightarrow u \llparenthesis R \rrparenthesis y)$

"u is related by 'S ***under*** *R' to v precisely when everything S-****under*** *u is also R-****under*** *v."* This generalises extensional subset inclusion and indirect reasoning for orders.

Example: Define E via $x \llparenthesis \mathsf{E} \rrparenthesis X \;\equiv\; x \in X$, then $A \llparenthesis \mathsf{E}\backslash\mathsf{E} \rrparenthesis B \;\equiv\; A \subseteq B$.

Example (Indirect inclusion): Define L via $x \llparenthesis \mathsf{L} \rrparenthesis y \;\equiv\; x \sqsubseteq y$, then $\mathsf{L}\backslash\mathsf{L} = \mathsf{L}/\mathsf{L} = \mathsf{L}$.

Interpreting Named Properties

We will interpret the named properties using

- Relations: Formulae on sets of pairs; “ $\forall x \bullet \ldots$ ”
- Graphs: Dots and lines on a page
- Matrices: 1s and 0s on a grid
- Programs: Transformations of inputs to outputs

Properties of a relationship flavour

reflexive	R	$\equiv$	$(\forall\, b \bullet\ b \,⦇\, R \,⦈\, b)$ Every node in a graph has a ‘loop’, a line to itself (Thus, paths can always be increased in length: $R \subseteq R ⨾ R$) The diagonal of a matrix is all 1s
irreflexive	R	$\equiv$	$(\forall\, b \bullet\ \neg(b \,⦇\, R \,⦈\, b))$ No node in a graph has a loop The diagonal of a matrix is all 0s
symmetric	R	$\equiv$	$(\forall\, b, c \bullet\ b \,⦇\, R \,⦈\, c \equiv c \,⦇\, R \,⦈\, b)$ The graph is undirected; we have a symmetric matrix
antisymmetric	R	$\equiv$	$(\forall\, b, c \bullet\ b \,⦇\, R \,⦈\, c \wedge c \,⦇\, R \,⦈\, b \ \Rightarrow\ b = c)$ Mutually related nodes are necessarily self-loops “Mutually related items are necessarily indistinguishable”
asymmetric	R	$\equiv$	$(\forall\, b, c \bullet\ b \,⦇\, R \,⦈\, c \ \Rightarrow\ \neg(c \,⦇\, R \,⦈\, b))$ At most 1 edge (regardless of direction) relating any 2 nodes
transitive	R	$\equiv$	$(\forall b, c, d \bullet\ b \,⦇\, R \,⦈\, c \,⦇\, R \,⦈\, d \ \Rightarrow\ b \,⦇\, R \,⦈\, d)$ Paths can always be shortened (but nonempty)
idempotent	R	$\equiv$	Lengths of paths can be changed arbitrarily (nonzero)

Intuitively, by considering the interpretations only, we find

$$\mathsf{reflexive}\, R \ \wedge\ \mathsf{transitive}\, R \ \Rightarrow\ \mathsf{idempotent}\, R$$

Super cool stuff!

“Relations are simple graphs”

Relations directly represent *simple graphs*: Dots (*nodes*) and at most 1 line (*edge*) between any two. E.g., cities and highways (ignoring multiple highways).

Treating R as a graph:

R	A bunch of dots on a page and an arrow from x to y when $x \,⦇\, R \,⦈\, y$
$R^{\smile}$	Flip the arrows in the graph
$\mathsf{Dom}\, R$	The nodes that have an outgoing edge
$\mathsf{Ran}\, R$	The nodes that have an incoming edge
$x \,⦇\, R \,⦈\, y$	A path of length 1 (an edge) from x to y
$x \,⦇\, R ⨾ R \,⦈\, y$	A path of length 2 from x to y
$R \cup R^{\smile}$	The associated undirected graph (“symmetric closure”)

Properties of an operational flavour

univalent	R	$\equiv$	$(\forall b, c, c' \bullet\ b \,⦇\, R \,⦈\, c \wedge b \,⦇\, R \,⦈\, c' \ \Rightarrow\ c = c')$ —aka “partial function” Graph: Every node has at most one outgoing edge Matrix: Every row has at most one 1 Prog: The program is deterministic, same-input yields same-output
injective	R	$\equiv$	$(\forall b, b', c \bullet\ b \,⦇\, R \,⦈\, c \wedge b' \,⦇\, R \,⦈\, c \ \Rightarrow\ b = b')$ Graph: Every node has at most one incoming edge Matrix: Every column has at most one 1 Prog: The program preserves distinctness (by contraposition)
total	R	$\equiv$	$(\forall b \bullet\ \exists c \bullet\ b \,⦇\, R \,⦈\, c)$ Graph: Every node has at least one outgoing edge Matrix: Every row has at least one 1 Prog: The program terminates; has at least one output for each input
surjective	R	$\equiv$	$(\forall c \bullet\ \exists b \bullet\ b \,⦇\, R \,⦈\, c)$ Graph: Every node has at least one incoming edge Matrix: Every column has at least one 1 Prog: All possible outputs arise from some input
mapping	R	$\equiv$	$\mathsf{total}\, R \ \wedge\ \mathsf{univalent}\, R$ —also known as a “(total) function” Graph: Every node has exactly one outgoing edge Matrix: Every row has exactly one 1 Prog: The program always terminates with a unique output
bijective	R	$\equiv$	$\mathsf{surjective}\, R \ \wedge\ \mathsf{injective}\, R$ Graph: Every node has exactly one incoming edge Matrix: Every column has exactly one 1 Prog: Every output arises from a unique input
iso	R	$\equiv$	$\mathsf{mapping}\, R \ \wedge\ \mathsf{bijective}\, R$ Graph: It’s a bunch of ‘circles’ Matrix: It’s a permutation; a re-arrangement of the identity matrix Prog: A non-lossy protocol associating inputs to outputs