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Reference Sheet for Discrete Maths

Propositional Calculus

Order of decreasing binding power: $=$, $\neg$, $\wedge/\vee$, $\Rightarrow/\Leftarrow$, $\equiv/\not\equiv$.

Equivales is the only equivalence relation that is associative $((p \equiv q) \equiv r) \quad \equiv \quad (p \equiv (q \equiv r))$, and it is symmetric and has identity true.

Discrepancy (difference) '$\not\equiv$' is symmetric, associative, has identity 'false', mutually associates with equivales $((p \not\equiv q) \equiv r) \quad \equiv \quad (p \not\equiv (q \equiv r))$, and mutually interchanges with it as well $(p \not\equiv q \equiv r) \quad \equiv \quad (p \equiv q \not\equiv r)$. Finally, negation commutes with difference: $\neg(p \equiv q) \equiv \neg p \equiv q$.

Implication has the alternative definition $p \Rightarrow q \;\equiv\; \neg p \vee q$, thus having true as both left identity and right zero; it distributes over $\equiv$ in the second argument, and is self-distributive; and has the properties:

Shunting $p \wedge q \Rightarrow r \;\equiv\; p \Rightarrow (q \Rightarrow r)$

Contrapositive $p \Rightarrow q \quad \equiv \quad \neg q \Rightarrow \neg p$

Leibniz $e = f \;\Rightarrow\; E[z \coloneqq e] = E[z := f]$

Modus Ponens

$$\begin{array}{lcl} p \wedge (p \Rightarrow q) & \equiv & p \wedge q \\ p \wedge (q \Rightarrow p) & \equiv & p \\ p \wedge (p \Rightarrow q) & \Rightarrow & q \end{array}$$

It is a *linear* order relation generated by '$\mathsf{false} \Rightarrow \mathsf{true}$'; whence "from false, follows anything": $\mathsf{false} \Rightarrow p$. Moreover it has the useful properties "(3.62) Contextualisation": $p \Rightarrow (q \equiv r) \quad \equiv \quad p \wedge q \quad \equiv \quad p \wedge r$ —we *have* the context p in each side of the equivalence— and $p \Rightarrow (q \Rightarrow r) \quad \equiv \quad p \wedge q \;\Rightarrow\; p \wedge r$. Implication is "Sub-associative": $((p \Rightarrow q) \Rightarrow r) \;\Rightarrow\; (p \Rightarrow (q \Rightarrow r))$. Finally, we have "$\equiv$-$\equiv$ Elimination": $(p \equiv q \equiv r) \Rightarrow s \;\equiv\; p \Rightarrow s \;\equiv\; q \Rightarrow s \;\equiv\; r \Rightarrow s$.

Conjunction and disjunction distribute over one another, are both associative and symmetric, $\vee$ has identity false and zero true whereas $\wedge$ has identity true and zero false, $\vee$ distributes over $\vee, \equiv, \wedge, \Rightarrow, \Leftarrow$ whereas $\wedge$ distributes over $\equiv - \equiv$ in that $p \wedge (q \equiv r \equiv s) \;\equiv\; p \wedge q \quad \equiv \quad p \wedge r \quad \equiv \quad p \wedge s$, and they satisfy,

Excluded Middle	**Contradiction**	**Absorption**	**De Morgan**
$p \vee \neg p$	$p \wedge \neg p \equiv \mathsf{false}$	$p \wedge (q \vee \neg p) \equiv p \wedge q$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$
		$p \vee (q \vee \neg p) \equiv p \vee q$	$\neg(p \vee q) \equiv \neg p \wedge \neg q$

Most importantly, they satisfy the **"Golden Rule"**: $p \wedge q \equiv p \equiv q \equiv p \vee q$.

The many other properties of these operations —such as weakening laws and other absorption laws and case-analysis ($\sqcup$-char)— can be found by looking at the list of *lattice properties* —since the Booleans are a lattice.

Orders

An *order* is a relation $\_ \sqsubseteq \_ : \tau \to \tau \to \mathbb{B}$ satisfying the following three properties:

Reflexivity	**Transitivity**	**Mutual Inclusion**
$a \sqsubseteq a$	$a \sqsubseteq b \wedge b \sqsubseteq c \Rightarrow a \sqsubseteq c$	$a \sqsubseteq b \wedge b \sqsubseteq a \;\equiv\; a = b$

Indirect Inclusion is like 'set inclusion' and Indirect Equality is like 'set extensionality'.

Indirect Equality (from above)	**Indirect Inclusion (from above)**
$x = y \;\equiv\; (\forall z \bullet x \sqsubseteq z \equiv y \sqsubseteq z)$	$x \sqsubseteq y \;\equiv\; (\forall z \bullet y \sqsubseteq z \Rightarrow x \sqsubseteq z)$
Indirect Equality (from below)	**Indirect Inclusion (from below)**
$x = y \;\equiv\; (\forall z \bullet z \sqsubseteq x \equiv z \sqsubseteq y)$	$x \sqsubseteq y \;\equiv\; (\forall z \bullet z \sqsubseteq x \Rightarrow z \sqsubseteq y)$

An order is *bounded* if there are elements $\top, \bot : \tau$ being the lower and upper bounds of all other elements:

Top Element	$a \sqsubseteq \top$	**Bottom Element**	$\bot \sqsubseteq a$
Top is maximal	$\top \sqsubseteq a \;\equiv\; a = \top$	**Bottom is minimal**	$a \sqsubseteq \bot \;\equiv\; a = \top$

Lattices

A *lattice* is a pair of operations $\_ \sqcap \_, \_ \sqcup \_ : \tau \to \tau \to \tau$ specified by the properties:

$\sqcup$-Characterisation	**$\sqcap$-Characterisation**
$a \sqsubseteq c \wedge b \sqsubseteq c \;\equiv\; a \sqcup b \;\sqsubseteq\; c$	$c \sqsubseteq a \wedge c \sqsubseteq b \;\equiv\; c \;\sqsubseteq\; a \sqcap b$

The operations act as providing the greatest lower bound, 'glb', 'supremum', or 'meet', by $\sqcap$; and the least upper bound, 'lub', 'infimum', or 'join', by $\sqcup$.

Let $\square$ be one of $\sqcap$ or $\sqcup$, then:

Symmetry of $\square$	**Associativity of $\square$**	**Idempotency of $\square$**
$a \square b \;=\; b \square a$	$(a \square b) \square c \;=\; a \square (b \square c)$	$a \square a \;=\; a$

Zero of $\square$	**Identity of $\square$**	**Absorption**	**Self-Distributivity of $\square$**
$a \sqcup \top = \top$	$a \sqcup \bot = a$	$a \sqcap (b \sqcup a) = a$	$a \square (b \square c) \;=\; (a \square b) \square (a \square c)$
$a \sqcap \bot = \bot$	$a \sqcap \top = a$	$a \sqcup (b \sqcap a) = a$	

Weakening / Strengthening	**Induced Defs. of Inclusion**	**Golden Rule**
$a \;\sqsubseteq\; a \sqcup b$	$a \sqsubseteq b \;\equiv\; a \sqcup b \;=\; b$	$a \sqcap b \;=\; a \quad \equiv \quad b = a \sqcup b$
$a \sqcap b \;\sqsubseteq\; a$	$a \sqsubseteq b \;\equiv\; a \sqcap b \;=\; a$	$a \sqcap b \;=\; a \sqcup b \quad \equiv \quad a = b$
$a \sqcap b \;\sqsubseteq\; a \sqcup b$		$a \sqcup b \;\sqsubseteq\; a \sqcap b \quad \equiv \quad a = b$

Monotonicity of $\square$

$a \sqsubseteq b \wedge c \sqsubseteq d \;\Rightarrow\; a \square c \;\sqsubseteq\; b \square d$

Duality Principle:
If a statement S is a theorem, then so is $S[(\sqsubseteq, \sqcap, \sqcup, \top, \bot) := (\sqsupseteq, \sqcup, \sqcap, \bot, \top)]$.

Conditionals

"If to $\wedge$" may be taken as axiom from which we may prove the remaining 'alternative definitions' "if to $\cdots$".

$$\begin{array}{lllllllll}
\textbf{if to } \wedge & P[z := \mathsf{if}\ b\ \mathsf{then}\ x\ \mathsf{else}\ y\ \mathsf{fi}] & \equiv & (b & \Rightarrow & P[z := x]) & \wedge & (\neg b & \Rightarrow\ P[z := x]) \\
\textbf{if to } \vee & P[z := \mathsf{if}\ b\ \mathsf{then}\ x\ \mathsf{else}\ y\ \mathsf{fi}] & \equiv & (b & \wedge & P[z := x]) & \vee & (\neg b & \wedge\ P[z := x]) \\
\textbf{if to } \not\equiv & P[z := \mathsf{if}\ b\ \mathsf{then}\ x\ \mathsf{else}\ y\ \mathsf{fi}] & \equiv & b & \wedge & P[z := x] & \not\equiv & \neg b & \wedge\ P[z := x] \\
\textbf{if to } \equiv & P[z := \mathsf{if}\ b\ \mathsf{then}\ x\ \mathsf{else}\ y\ \mathsf{fi}] & \equiv & b & \Rightarrow & P[z := x] & \equiv & \neg b & \Rightarrow\ P[z := x]
\end{array}$$

Note that the "$\equiv$" and "$\not\equiv$" rules can be parsed in multiple ways since '$\equiv$' is associative, and '$\equiv$' mutually associates with '$\not\equiv$'.

if true	$\mathsf{if\ true\ then}\ x\ \mathsf{else}\ y\ \mathsf{fi} = x$
if false	$\mathsf{if\ false\ then}\ x\ \mathsf{else}\ y\ \mathsf{fi} = y$
then true	$\mathsf{if}\ R\ \mathsf{then\ true\ else}\ P\ \mathsf{fi} = R \vee P$
then false	$\mathsf{if}\ R\ \mathsf{then\ false\ else}\ P\ \mathsf{fi} = \neg R \wedge P$
else true	$\mathsf{if}\ R\ \mathsf{then}\ P\ \mathsf{else\ true\ fi} = R \Rightarrow P$
else false	$\mathsf{if}\ R\ \mathsf{then}\ P\ \mathsf{else\ false\ fi} = R \wedge P$

if swap	$\mathsf{if}\ b\ \mathsf{then}\ x\ \mathsf{else}\ y\ \mathsf{fi} = \mathsf{if}\ \neg b\ \mathsf{then}\ y\ \mathsf{else}\ x\ \mathsf{fi}$
if idempotency	$\mathsf{if}\ b\ \mathsf{then}\ x\ \mathsf{else}\ x\ \mathsf{fi} = x$
if guard strengthening	$\mathsf{if}\ b\ \mathsf{then}\ x\ \mathsf{else}\ y\ \mathsf{fi} = \mathsf{if}\ b \wedge x \neq y\ \mathsf{then}\ x\ \mathsf{else}\ y\ \mathsf{fi}$
if Context	$\mathsf{if}\ b\ \mathsf{then}\ E\ \mathsf{else}\ F\ \mathsf{fi} = \mathsf{if}\ b\ \mathsf{then}\ E[b := \mathsf{true}]\ \mathsf{else}\ F[b := \mathsf{false}]\ \mathsf{fi}$
if Distributivity	$P[z := \mathsf{if}\ b\ \mathsf{then}\ x\ \mathsf{else}\ y\ \mathsf{fi}] = \mathsf{if}\ b\ \mathsf{then}\ P[z := x]\ \mathsf{else}\ P[z := y]\ \mathsf{fi}$
if junctivity	$(\mathsf{if}\ b\ \mathsf{then}\ x\ \mathsf{else}\ y\ \mathsf{fi}) \oplus (\mathsf{if}\ b\ \mathsf{then}\ x'\ \mathsf{else}\ y'\ \mathsf{fi}) = \mathsf{if}\ b\ \mathsf{then}\ (x \oplus x')\ \mathsf{else}\ (y \oplus y')\ \mathsf{fi}$

Quantification

Let $\_\oplus\_$ be an associative and symmetric operation with identity Id.

Abbreviation	$(\oplus x \bullet P) = (\oplus x \mid \mathsf{true} \bullet P)$
Empty range	$(\oplus x \mid \mathsf{false} \bullet P) = \mathsf{Id}$
One-point rule	$(\oplus x \mid x = E \bullet P) = P[x := E]$
Distributivity	$(\oplus x \mid R \bullet P \oplus Q) = (\oplus x \mid R \bullet P) \oplus (\oplus x \mid R \bullet Q)$
Nesting	$(\oplus x, y \mid X \wedge Y \bullet P) = (\oplus x \mid X \bullet (\oplus y \mid Y \bullet P))$
Dummy renaming	$(\oplus x \mid R \bullet P) = (\oplus y \mid R[x := y] \bullet P[x := y])$
Disjoint Range split	$(\oplus x \mid R \vee S \bullet P) = (\oplus x \mid R \bullet P) \oplus (\oplus x \mid S \bullet Q)$ *provided* $R \wedge S \equiv \mathsf{false}$
Range split	$(\oplus x \mid R \vee S \bullet P) \oplus (\oplus x \mid R \wedge S \bullet P) = (\oplus x \mid R \bullet P) \oplus (\oplus x \mid S \bullet Q)$
Idempotent Range split	$(\oplus x \mid R \vee S \bullet P) = (\oplus x \mid R \bullet P) \oplus (\oplus x \mid S \bullet Q)$ *provided* $\oplus$ is idempotent

Set Theory

The set theoretic symbols $\in$, $=$, $\subseteq$, are defined as follows.

Axiom, Set Membership: $F \in \{x \mid R \bullet E\} \quad \equiv \quad (\exists x \mid R \bullet F = E)$

Axiom, Extensionality: $S = T \quad \equiv \quad (\forall x \bullet x \in S \equiv x \in T)$

Axiom, Subset: $S \subseteq T \quad \equiv \quad (\forall x \bullet x \in S \Rightarrow x \in T)$

As witnessed by the following definitions, it is the $\in$ relation that *translates set theory to propositional logic.*

Universe	$x \in \mathbf{U}$	$\equiv$	$true$
Empty set	$x \in \emptyset$	$\equiv$	$false$
Union	$x \in S \cup T$	$\equiv$	$x \in S \vee x \in T$
Intersection	$x \in S \cap T$	$\equiv$	$x \in S \wedge x \in T$
Complement	$x \in \sim S$	$\equiv$	$x \notin S$
Difference	$x \in S - T$	$\equiv$	$x \in S \wedge x \notin T$
Power set	$S \in \mathbb{P}T$	$\equiv$	$S \subseteq T$

The pairs $\emptyset$/false, $\mathbf{U}$/true, $\cup/\vee$, $\cap/\wedge$, $\subseteq/\Rightarrow$, $\tilde{}/\neg$ are related by $\in$ and so all equational theorems of propositional logic also hold for set theory —indeed, that is because both are Boolean algebras.
$\rightarrow$ Set difference is a residual wrt $\cup$, and so satisfies the division properties below.
$\rightarrow$ Subset is an order and so satisfies the aforementioned order properties. It is bounded below by $\emptyset$ and above by $\mathbf{U}$.

The relationship between set comprehension and quantifier notation is:

Set comprehension as union $\quad \{x \mid R \bullet P\} \quad = \quad (\cup x \mid R \bullet \{P\})$

Combinatorics

Axiom, Size:	$\#S = (\Sigma x \mid x \in S \bullet 1)$
Axiom, Interval:	$m..n = \{x : \mathbb{Z} \mid m \leq x \leq n\}$

The following theorems serve to define '$\#$' for the usual set theory operators.

Positive definite	$\#S \subseteq 0 \quad \equiv \quad S = \emptyset$
Power set size	$\#\mathbb{P}S = 2^{\#S}$
Principle of Inclusion-Exclusion	$\#(S \cup T) = \#S + \#T - \#(S \cap T)$
Monotonicity	$S \subseteq T \Rightarrow \#S \leq \#T$
Difference rule	$S \subseteq T \Rightarrow \#(T - S) = \#T - \#S$
Complement size	$\#(\sim S) = \#\mathbf{U} - \#S$
Range size	$(\Sigma x : \mathbf{U} \mid x \notin S \bullet 1) = \#\mathbf{U} - \#S$
Interval size	$\#(m..n) = n - m + 1$ for $m \leq n$
Pigeonhole Principle	$(\Sigma i : 1..n \bullet E)/n \leq (\uparrow i : 1..n \bullet E)$
(*"min* $\leq$ *avg* $\leq$ *max"*)	$(\downarrow i : 1..n \bullet E) \leq (\Sigma i : 1..n \bullet E)/n$

Rule of sum: $\#(\cup i \mid R\,i \bullet P) = (\Sigma i \mid R\,i \bullet \#P)$
provided the range is pairwise disjoint: $\forall i, j \bullet R\,i \wedge R\,j \equiv i = j$.

Rule of product: $\#(\times i \mid R\,i \bullet P) = (\Pi i \mid R\,i \bullet \#P)$

Residuals, Division

Suppose we have an associative operation $\_ \fatsemi \_$ with identity Id and two operations "under $\backslash$" and "over $/$" specified as follows.

Characterisation of $/$	**Characterisation of $\backslash$**
$a \fatsemi b \sqsubseteq c \quad \equiv \quad a \sqsubseteq c/b$	$a \fatsemi b \sqsubseteq c \quad \equiv \quad b \sqsubseteq a\backslash c$

For the special scenario with $\fatsemi = \sqcap$, the divisions coincide and are called *the relative pseudocomplement*: Denoted $x \rightarrowtriangle y$, it is *the largest piece 'outside' of x that is still included in y.* The relative pseudocomplement *internalises inclusion*, $z \sqsubseteq (x \rightarrowtriangle y) \;\Rightarrow\; z \sqsubseteq x \Rightarrow z \sqsubseteq y$, and more generally: $x \sqsubseteq y \;\equiv\; \mathsf{Id} \sqsubseteq x\backslash y$.

Cancellation	$(a/b) \fatsemi b \sqsubseteq a$	$a \fatsemi (a\backslash b) \sqsubseteq b$
Dividing a division	$(a/b)/c \;=\; a/(c \fatsemi b)$	$a\backslash(b\backslash c) \;=\; (b \fatsemi a)\backslash c$
Division of multiples	$a \sqsubseteq (a \fatsemi b)/b$	$b \sqsubseteq a\backslash(a \fatsemi b)$

Monotonicity of $\fatsemi$ $\quad a \sqsubseteq a' \wedge b \sqsubseteq b' \Rightarrow a \fatsemi b \sqsubseteq a' \fatsemi b'$

Numerator monotonicity	$b \sqsubseteq b' \Rightarrow a\backslash b \sqsubseteq a\backslash b'$	$b \sqsubseteq b' \Rightarrow b/a \sqsubseteq b'/a$
Denominator antitonicity	$a' \sqsubseteq a \Rightarrow a\backslash b \sqsubseteq a'\backslash b$	$a' \sqsubseteq a \Rightarrow b/a \sqsubseteq b/a'$
Self-reflexivity	$\mathsf{Id} \sqsubseteq a\backslash a$	$\mathsf{Id} \sqsubseteq a/a$
Denominator Identity	$\mathsf{Id}\backslash a = a$	$a/\mathsf{Id} = a$
Numerator Zero	$a\backslash\top = \top$	$\top/a = \top$
Wraparound rule	$\bot\backslash a = \top$	$a/\bot = \top$

Exact division:

$$\begin{array}{lcl} (\exists z \bullet y = x \fatsemi z) & \equiv & x \fatsemi (x\backslash y) = y \\ (\exists z \bullet y = x\backslash z) & \equiv & x\backslash(x \fatsemi y) = y \end{array}$$

Converse

Axioms,

Co-distributivity	**$\breve{\ }\breve{\ }$, Involutive**	**Monotonicity**
$(x \fatsemi y)\breve{\ } = y\breve{\ } \fatsemi x\breve{\ }$	$x\breve{\ }\breve{\ } = x$	$x \sqsubseteq y \Rightarrow x\breve{\ } \sqsubseteq y\breve{\ }$

Theorems,

Identity	**Connection**	**Elimination**
$\mathsf{Id}\breve{\ } = \mathsf{Id}$	$a\breve{\ } \sqsubseteq b \;\equiv\; a \sqsubseteq b\breve{\ }$	$x\breve{\ } = y\breve{\ } \;\equiv\; x = y$

Isotonicity: $x \sqsubseteq y \;\equiv\; x\breve{\ } \sqsubseteq y\breve{\ }$

Named Properties

univalent	x	$\equiv$	$x\breve{\ } \fatsemi x \sqsubseteq \mathsf{Id}$
total	x	$\equiv$	$\mathsf{Id} \sqsubseteq x \fatsemi x\breve{\ }$
mapping	x	$\equiv$	$\mathsf{total}\, x \;\wedge\; \mathsf{univalent}\, x$
iso	x	$\equiv$	$\mathsf{mapping}\, x \;\wedge\; \mathsf{bijective}\, x$

injective	x	$\equiv$	$x \fatsemi x\breve{\ } \sqsubseteq \mathsf{Id}$
surjective	x	$\equiv$	$\mathsf{Id} \sqsubseteq x\breve{\ } \fatsemi x$
bijective	x	$\equiv$	$\mathsf{surjective}\, x \;\wedge\; \mathsf{injective}\, x$

Duality theorems

univalent	$(x\breve{\ })$	$\equiv$	injective x
total	$(x\breve{\ })$	$\equiv$	surjective x
mapping	$(x\breve{\ })$	$\equiv$	bijective x
iso	$(x\breve{\ })$	$\equiv$	iso x

Invertiblility theorems

$$\begin{array}{l} \mathsf{total}\, x \wedge \mathsf{injective}\, x \Rightarrow x \fatsemi x\breve{\ } = \mathsf{Id} \\ \mathsf{iso}\, x \quad\equiv\quad x \fatsemi x\breve{\ } = \mathsf{Id} \;\wedge\; x\breve{\ } \fatsemi x = \mathsf{Id} \\ \mathsf{iso}\, x \quad\Rightarrow\quad (\exists g \bullet\; x \fatsemi g = \mathsf{Id} = g \fatsemi x) \end{array}$$

Shunting laws:

$$\begin{array}{lcl} \mathsf{univalent}\, f & \Rightarrow & (x \fatsemi f \sqsubseteq y \;\Leftarrow\; x \sqsubseteq y \fatsemi f\breve{\ }) \\ \mathsf{total}\, f & \Rightarrow & (x \fatsemi f \sqsubseteq y \;\Rightarrow\; x \sqsubseteq y \fatsemi f\breve{\ }) \\ \mathsf{mapping}\, f & \Rightarrow & (x \fatsemi f \sqsubseteq y \;\equiv\; x \sqsubseteq y \fatsemi f\breve{\ }) \end{array}$$

Relations

Relations are sets of pairs ...

Tortoise	$x \mathbin{(\!\mid} R \mathbin{\mid\!)} y$	$\equiv$	$\langle x, y\rangle \in R$
Extensionality	$R = S$	$\equiv$	$(\forall\, x, y \;\bullet\; x \mathbin{(\!\mid} R \mathbin{\mid\!)} y \equiv x \mathbin{(\!\mid} S \mathbin{\mid\!)} y)$
Inclusion	$R \subseteq S$	$\equiv$	$(\forall\, x, y \;\bullet\; x \mathbin{(\!\mid} R \mathbin{\mid\!)} y \Rightarrow x \mathbin{(\!\mid} S \mathbin{\mid\!)} y)$
Empty	$u \mathbin{(\!\mid} \emptyset \mathbin{\mid\!)} v$	$\equiv$	false
Universe	$u \mathbin{(\!\mid} A \times B \mathbin{\mid\!)} v$	$\equiv$	$u \in A \;\wedge\; v \in B$
Complement	$u \mathbin{(\!\mid} \sim S \mathbin{\mid\!)} v$	$\equiv$	$\neg(u \mathbin{(\!\mid} S \mathbin{\mid\!)} v)$
Union	$u \mathbin{(\!\mid} S \cup T \mathbin{\mid\!)} v$	$\equiv$	$u \mathbin{(\!\mid} S \mathbin{\mid\!)} v \;\vee\; u \mathbin{(\!\mid} T \mathbin{\mid\!)} v$
Intersection	$u \mathbin{(\!\mid} S \cap T \mathbin{\mid\!)} v$	$\equiv$	$u \mathbin{(\!\mid} S \mathbin{\mid\!)} v \;\wedge\; u \mathbin{(\!\mid} T \mathbin{\mid\!)} v$
Difference	$u \mathbin{(\!\mid} S - T \mathbin{\mid\!)} v$	$\equiv$	$u \mathbin{(\!\mid} S \mathbin{\mid\!)} v \;\wedge\; \neg(u \mathbin{(\!\mid} T \mathbin{\mid\!)} v)$
PseudoComplement	$u \mathbin{(\!\mid} S \rightarrowtriangle T \mathbin{\mid\!)} v$	$\equiv$	$u \mathbin{(\!\mid} S \mathbin{\mid\!)} v \;\Rightarrow\; u \mathbin{(\!\mid} T \mathbin{\mid\!)} v$
An Identity	$u \mathbin{(\!\mid} \mathbb{I}\, A \mathbin{\mid\!)} v$	$\equiv$	$u = v \in A$
The Identity	$u \mathbin{(\!\mid} \mathsf{Id} \mathbin{\mid\!)} v$	$\equiv$	$u = v$
Converse	$u \mathbin{(\!\mid} R\breve{\ } \mathbin{\mid\!)} v$	$\equiv$	$v \mathbin{(\!\mid} R \mathbin{\mid\!)} u$
Composition	$u \mathbin{(\!\mid} R \fatsemi S \mathbin{\mid\!)} v$	$\equiv$	$(\exists\, x \;\bullet\; u \mathbin{(\!\mid} R \mathbin{\mid\!)} x \;\wedge\; x \mathbin{(\!\mid} S \mathbin{\mid\!)} v)$
Over Division	$u \mathbin{(\!\mid} R/S \mathbin{\mid\!)} v$	$\equiv$	$(\forall\, x \;\bullet\; x \mathbin{(\!\mid} R \mathbin{\mid\!)} u \;\Rightarrow\; x \mathbin{(\!\mid} S \mathbin{\mid\!)} v)$
Under Division	$u \mathbin{(\!\mid} S\backslash R \mathbin{\mid\!)} v$	$\equiv$	$(\forall\, x \;\bullet\; v \mathbin{(\!\mid} R \mathbin{\mid\!)} x \;\Rightarrow\; u \mathbin{(\!\mid} S \mathbin{\mid\!)} x)$