A Language Feature to Unbundle Data at Will

(Short Paper)

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Abstract

Programming languages with sufficiently expressive type systems provide users with different means of data 'bundling'. Specifically, in dependently-typed languages such as Agda, Coq, Lean and Idris, one can choose to encode information in a record either as a parameter or a field. For example, we can speak of graphs over a particular vertex set, or speak of arbitrary graphs where the vertex set is a component. These create isomorphic types, but differ with respect to intended use. Traditionally, a library designer would make this choice (between parameters and fields); if a user wants a different variant, they are forced to build conversion utilities, as well as duplicate functionality. For a graph data type, if a library only provides a Haskell-like typeclass view of graphs over a vertex set, yet a user wishes to work with the category of graphs, they must now package a vertex set as a component in a record along with a graph over that set.

We design and implement a language feature that allows both the library designer and the user to make the choice of information exposure only when necessary, and otherwise leave the distinguishing line between parameters and fields unspecified. Our language feature is currently implemented as a prototype meta-program incorporated into Agda's Emacs ecosystem, in a way that is unobtrusive to Agda users.

CCS Concepts • Software and its engineering → Extensible languages; Modules / packages; Functional languages; Polymorphism; Source code generation; Integrated and visual development environments; Data types and structures; Design languages; Imperative languages; Patterns; Syntax; Abstract data types.

Keywords Agda, meta-program, extensible, Emacs, pacakges, modules, dependent-types

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1 Introduction — Selecting the 'Right' Perspective

Library designers want to produce software components that are useful for the perceived needs of a variety of users and usage scenarios. It is therefore natural for designers to aim for substantial generality, in the hopes of increased reusability. One such particular "choice" will occupy us here: When creating a record to bundle up certain information that "naturally" belongs together, what parts of that record should be *parameters* and what parts should be *fields*? This is analogous to whether functions are curried and so arguments may be provided partially, or otherwise must be provided all-together in one tuple.

The subtlety of what is a 'parameter' — exposed at the type level — and what is a 'field' — a component value — has led to awkward formulations and the duplication of existing types for the sole purpose of different uses. Tom Hales [5] is quite eloquent in his critique of Lean:

Structures are meaninglessly parameterized from a mathematical perspective. [...] I think of the parametric versus bundled variants as analogous to currying or not; are the arguments to a function presented in succession or as a single ordered tuple? However, there is a big difference between currying functions and currying structures. Switching between curried and uncurried functions is cheap, but it is nearly impossible in Lean to curry a structure. That is, what is bundled cannot be later opened up as a parameter. (Going the other direction towards increased bundling of structures is easily achieved with sigma types.) This means that library designers are forced to take a conservative approach and expose as a parameter anything that any user might reasonably want exposed, because once it is bundled, it is not coming back.

This is the problem we are solving.

For example, each Haskell typeclass can have only one instance per datatype; since there are several monoids with the datatype Bool as carrier, in particular those induced by conjunction and disjunction, the de-facto-standard libraries for Haskell define two isomorphic copies All and Any of Bool, only for the purpose of being able to attach the respective monoid instances to them.

But perhaps Haskell's type system does not give the programmer sufficient tools to adequately express such ideas.

As such, for the rest of this paper we will illustrate our ideas in Agda [2, 7]. For the monoid example, it seems that there are three contenders for the monoid interface:

```
record Monoid<sub>0</sub> : Set<sub>1</sub> where
   field
       Carrier : Set
                    : Carrier → Carrier → Carrier
       _9_
                     : Carrier
       assoc
                    : ∀ {x y z}
                     \rightarrow (x \stackrel{\circ}{9} y) \stackrel{\circ}{9} z \equiv x \stackrel{\circ}{9} (y \stackrel{\circ}{9} z)
       leftId : \forall \{x\} \rightarrow Id \ \ x \equiv x
       rightId : \forall \{x\} \rightarrow x \ {}_{9}^{\circ} \ Id \equiv x
record Monoid<sub>1</sub> (Carrier : Set) : Set where
   field
                     : Carrier \rightarrow Carrier \rightarrow Carrier
       _9^_
       Ιd
                     : Carrier
       assoc
                    : ∀ {x y z}
                     \rightarrow (x \stackrel{\circ}{9} y) \stackrel{\circ}{9} z \equiv x \stackrel{\circ}{9} (y \stackrel{\circ}{9} z)
       leftId : \forall \{x\} \rightarrow Id \ ^{\circ}_{9} \ x \equiv x
       rightId : \forall \{x\} \rightarrow x \  Id \equiv x
record Monoid<sub>2</sub>
                (Carrier : Set)
                (_%_ : Carrier → Carrier → Carrier)
             : Set where
   field
       Ιd
                     : Carrier
                     : ∀ {x y z}
       assoc
                     \rightarrow (x ^{\circ}_{9} y) ^{\circ}_{9} z \equiv x ^{\circ}_{9} (y ^{\circ}_{9} z)
       leftId : \forall \{x\} \rightarrow Id \ % \ x \equiv x
       rightId : \forall \{x\} \rightarrow x \  id \equiv x
```

In Monoid₀, we will call Carrier "bundled up", while we call it "exposed" in Monoid₁ and Monoid₂. The bundled-up version allows us to speak of a monoid, rather than a monoid on a given type which is captured by Monoid₁. While Monoid₂ exposes both the carrier and the composition operation, we might in some situation be interested in exposing the identity element instead — e.g., the discrepancy ' \neq ' and indistinguishability ' \equiv ' operations on the Booleans have the same identities as conjunction and disjunction, respectively. Moreover, there are other combinations of what is to be exposed and hidden, for applications that we might never think of.

Rather than code with *interface formulations we think people will likely use*, we can instead try to *commit to no particular formulation* and allow the user to select the form most convenient for their use-cases. This desire for reusability motivates a new language feature: The PackageFormer.

Moreover, it is often the case that one begins working with a record of useful semantic data, but then, say, for proof automation, may want to use the associated datatype for syntax. For example, the syntax of closed monoid terms can be expressed, using trees, as follows.

```
\begin{array}{c} \textbf{data} \  \, \textbf{Monoid}_3 \ : \  \, \textbf{Set} \  \, \textbf{where} \\ \quad \_\S\_ \ : \  \, \textbf{Monoid}_3 \  \, \rightarrow \  \, \textbf{Monoid}_3 \  \, \rightarrow \  \, \textbf{Monoid}_3 \\ \quad \, \textbf{Id} \  \, : \  \, \textbf{Monoid}_3 \end{array}
```

We can see that this can be obtained from $Monoid_0$ by discarding the fields denoting equations, then turning the remaining fields into constructors.

We show how these different presentations can be derived from a single PackageFormer declaration via a generative meta-program integrated into the most widely-used Agda "IDE", the Emacs mode for Agda. In particular, if one were to explicitly write M different bundlings of a package with N constants then one would write nearly $N \times M$ lines of code, yet this quadratic count becomes linear N+M by having a single package declaration of N constituents with M subsequent instantiations. We hope that reducing such duplication of effort, and of potential maintenance burden, will be beneficial to the software engineering of large libraries of formal code — and consider it the main contribution of our work.

2 PackageFormers — Being Non-committal as Much as Possible

We claim that the above monoid-related pieces of Agda code can be unified as a single declaration which does not distinguish between parameters and fields, where PackageFormer is a keyword with similar syntax as record:

```
PackageFormer MonoidP : Set_1 where Carrier : Set __%_ : Carrier \rightarrow Carrier \rightarrow Carrier Id : Carrier assoc : \forall {x y z} \rightarrow (x ^\circ, y) ^\circ, z \equiv x ^\circ, (y ^\circ, z) leftId : \forall {x} \rightarrow Id ^\circ, x \equiv x rightId : \forall {x} \rightarrow x ^\circ, Id \equiv x
```

Then, with various directives that let one declare what should be parameters and what should be fields, we can reproduce the above presentations. The directives can be built from the following grammar:

A package former is used via *instantiations*, written as low-precedence juxtapositions of a package former name and expression of type Variational. Notice that some variationals have arguments. For example, exposing explicitly lists the names that should be turned into parameters, in that

sequence, "unbundled n" exposes the first n names declared in the package former.

An *instantiation* juxtaposition is written PF v to indicate that the PackageFormer named PF is to be restructred according to scheme v. A *composition* of variationals is denoted using the symbol ' \oplus '; for example,

denotes the forward-composition of iterated instantiations, namely (((PF v_1) v_2) \cdots) v_n , since we take prefix instantiation application to have lower precedence that variational composition. In particular, an empty composition is the identity scheme, which performs no alteration, and has the explicit name id. Since PF id \approx PF and id is the identity of composition, we may write any *instantiation* as a sequence of $-\oplus$ -separated clauses:

```
\mathsf{PF} \xrightarrow{\bullet} v_1 \xrightarrow{\bullet} v_2 \xrightarrow{\bullet} \cdots \xrightarrow{\bullet} v_n
```

The previous presentations can be obtained as follows.

0. To make Monoid₀' the type of *arbitrary monoids* (that is, with arbitrary carrier), we declare:

```
Monoid_0' = MonoidP record
```

 We may obtain the previous formulation of Monoid₁ in two different equivalent ways:

```
Monoid₁' = MonoidP record → unbundled 1
Monoid₁'' = Monoid₀' exposing (Carrier)
```

It is interesting to note that *PackageFormer* MonoidP is treated on the same footing *record* Monoid₀': Both may be subjected to variationals.

2. As with Monoid₁, there are also different ways to obtain Monoid₂.

```
Monoid<sub>2</sub>' = MonoidP record → unbundled 2
Monoid<sub>2</sub>'' = Monoid<sub>0</sub>' exposing (Carrier; _%_)
```

3. Metaprogramming is clearly needed to produce the term language:

```
Monoid<sub>3</sub>' = MonoidP termtype "Carrier"
```

Our running example uses the theory of monoids, which is a single-sorted theory. In general, a PackageFormer may have multiple sorts —as is the case with graphs— and so one of the possibly many sorts needs to be designated as the universe of discourse, or carrier, of the resulting inductively defined term type. Such a purpose is served by the single argument to termtype.

We may also want to have terms *over* a particular variable set, and so declare:

Since a parameter's name does not matter, due to α equivalence, an arbitrary, albeit unique, name for the

variable set is introduced along with an embedding function from it to the resulting term type. For brevity, the embedding function's name is inj and the user must ensure there is no name clash. The resulting elaboration is as follows.

```
\begin{array}{lll} \textbf{data} \  \, \textbf{Monoid}_4 \  \, (\textbf{Vars} : \textbf{Set}) : \textbf{Set} \  \, \textbf{where} \\ & \textbf{inj} : \textbf{Vars} \, \rightarrow \, \textbf{Monoid}_4 \  \, \textbf{Vars} \\ & \underline{\phantom{a}}_9^\circ \_ : \  \, \textbf{Monoid}_4 \  \, \textbf{Vars} \\ & \rightarrow \, \, \textbf{Monoid}_4 \  \, \textbf{Vars} \, \rightarrow \, \textbf{Monoid}_4 \  \, \textbf{Vars} \\ & \textbf{Id} : \, \textbf{Monoid}_4 \  \, \textbf{Vars} \end{array}
```

Note that these instantiations implicitly drop equations, such as associativity from MonoidP. This is what is commonly done in Universal Algebra. If we were instead doing *n*-category theory, these would be kept, but will be the subject of future work.

We also have elaborations into nested dependent-sums, which is useful when looking at coherent substructures. Alongside unbundled, we also have infix combinators for extending an instantiation with additional fields or constructors, and the renaming of constituents according to a user provided String-to-String function. Moreover, just as syntactic datatype declarations may be derived, we also allow support for the derivation of induction principles and structure-preserving homomorphism types. Our envisioned system would be able to derive simple, tedious, uninteresting concepts; leaving difficult, interesting ones for humans to solve.

Quadratic to Linear: Notice that the previous 5 monoid presentations, Monoid₀ to Monoid₄, spanned 32 lines (8 for the original, 24 for the variants). Using MonoidP and our operators, this can be done in 7 + 6 = 13 lines. This corresponds to using a 2-part code, with the initial lines being a model, and then 1-2 lines to specify variants.

3 Variational Polymorphism

Suppose we want to produce the function concat, which folds over the elements of a list according to a compositionality scheme — examples of this include summing over a list, multiplication over a list, checking all items in a list are true, or at least one item in the list is true. Depending on the selected instantiation, the resulting function may have types such as the following:

```
\begin{array}{c} \mathsf{concat}_0 \ : \ \{\mathsf{M} \ : \ \mathsf{Monoid}_0\} \\ & \to \ \mathsf{let} \ \mathsf{C} \ = \ \mathsf{Monoid}_0 \, . \mathsf{Carrier} \ \mathsf{M} \\ & \quad \mathsf{in} \quad \mathsf{List} \ \mathsf{C} \to \mathsf{C} \\ \\ \mathsf{concat}_1 \ : \ \{\mathsf{C} \ : \ \mathsf{Set}\} \ \{\mathsf{M} \ : \ \mathsf{Monoid}_1 \ \mathsf{C}\} \\ & \to \ \mathsf{List} \ \mathsf{C} \to \mathsf{C} \\ \\ \mathsf{concat}_2 \ : \ \{\mathsf{C} \ : \ \mathsf{Set}\} \ \{\_{\ref{general}}^\circ \ : \ \mathsf{C} \to \mathsf{C} \to \mathsf{C}\} \\ & \quad \{\mathsf{M} \ : \ \mathsf{Monoid}_2 \ \mathsf{C} \ \_{\ref{general}}^\circ \} \\ & \to \ \mathsf{List} \ \mathsf{C} \to \mathsf{C} \end{array}
```

```
concat_3 : let C = Monoid_3
in List C \rightarrow C
```

Given our previous work, and providing that the variationals are already defined, we add a new declaration which, unlike the rest, comes equipped with a *definition*.

```
concat : List Carrier \rightarrow Carrier concat = foldr _{-}^{\circ}_ Id
```

This is known as a *definitional extension* (of a theory), which is known to be conservative (i.e. has the same models).

The variationals is where this power comes from. Furthermore, we have alluded to the fact that the type of variationals is extensible; this is achieved by having

```
Variational ≅ (PackageFormer → PackageFormer)
```

Indeed, our implementation relies on 5 meta-primitives to form arbitrarily complex schemes that transform abstract PackageFormers into other grouping mechanisms. The meta-primitives were arrived at by codifying a number of structuring mechanisms directly then carefully extracting the minimal ingredients that enable them to be well-defined.

4 How Does This Work?

We have implemented our system as an "editor tactic" metaprogram.

In actual use, an Agda programmer declares what they want using the combinators above (inside special Agda code comments). The comments are read by Emacs Lisp and legitimate Agda is produced in a generated file, which is then automatically imported into the current file — examples are provided in an appendix. The generated file never needs to be consulted, as the declared names are furnished with tooltips rendering the elaborated Agda form. Moreover, we also provide a feature to extract a 'bare bones' version of a file that strips out all PackageFormer annotations, leaving only Agda as well as the import to the generated file. Finally, since the elaborations are just Agda, one only needs to use the system once and future users are not forced to know about it.

The existing prototype already has the following nice properties:

Extensible Users may extend the collection of variationals by providing the intended elaboration scheme.

We have provided a number of auxiliary, derived, combinators that can be used to construct complex and common schemes. Furthermore, the user has full and direct access to the entirety of Emacs Lisp as a programming language for restructuring PackageFormers into any desired shape—the well-formedness of which is a matter the user must then worry about.

Practical The user manual demonstrates how boilerplate code for renamings, hidings, decorations, and generations of hierarchical structures can be formed; [3].

Pragmatic The prototype comes equipped with a number of menus to display the abstract PackageFormer's defined, as well as the variationals defined, and one may enable highlighting for these syntactical items, have folded away, or simply extract an Agda file that does not mention them at all.

As it can be tedious to consult generated code for highlevel PackageFormer instantiations and so every variational and PackageFormer is tagged with tooltips providing relevant information.

```
PackageFormer M-Set : Set<sub>1</sub> where
    Scalar : Set
                : Set
    Vector
                : Scalar → Vector → Vector
                : Scalar → Scalar → Scalar
               : \{v : Vector\} \rightarrow 1 \cdot v \equiv
                : \forall \{a b v\} \rightarrow (a \times b) \cdot v \equiv a \cdot (b \cdot v)
NearRIng = M-Set record ⊕ single-sorted "Scalar"
         {- NearRing = M-Set record - single-sorted "Scalar" -}
         record NearRing: Set, where
           field Scalar
                                 : Set
           field _-_
                           : Scalar → Scalar → Scalar
           field 1
                           : Scalar
           field_x
                           : Scalar → Scalar → Scalar
           field leftId
                                 : \{v : Scalar\} → 1 \cdot v \equiv
                                 \forall \{a \mid b \mid v\} \rightarrow (a \times b) \cdot v \equiv a \cdot (b \cdot v)
```

Figure 1. Hovering to show details. Notice special syntax has default colouring: Red for PackageFormer delimiters, yellow for elements, and green for variationals.

The details of the implementation and numerous common structuring mechanisms derived from the meta-primitives can be found on the prototype's homepage:

https://alhassy.github.io/next-700-module-systemsproposal/prototype/PackageFormer.html

5 Next Steps

We have outlined a new language feature that is intended to reduce duplicated effort involved in taking different perspectives on structures—and to solve Hales' problem of premature commitment to a particular encoding. Moreover, on the road to making this tractable, we have unearthed a novel form of polymorphism and demonstrated its usefulness with some examples.

We have presented our work indirectly by using examples, which we hope are sufficiently clear to indicate our intent. We next intend to provide explicit (elaboration) semantics for PackageFormer within a minimal type theory; [4].

Furthermore, there are additional pieces of future work, including:

- 1. Explain how generative modules [6] are supported by this scheme.
- 2. How do multiple default, or optional, clauses for a constituent fit into this language feature.
- 3. Explore inheritance, coercion, and transport along canonical isomorphisms.

Finally, the careful reader will have noticed that our abstract mentions graphs, yet there was no further discussion of that example. We have avoided it for simplicity; the prototype accommodates multi-sorted structures where sorts may *depend* on one another, as edge-sets depend on the vertex-set chosen. Examples can be found on the prototype's webpage.

This short paper proposes a language feature that enables users to selectively choose how information is to be organised, such as which parts are exposed as parameters, thereby reducing effort when taking different perspectives on structures. To demonstrate that this feature seems useful in practice, we have implemented a meta-program to generate Agda using special code comments that specify how package elements are to be organised, such as their selective exposure as parameters which is a common issue with libraries in dependently-typed languages.

Our variationals cannot yet be directly defined in Agda. Instead, we are making use of Emacs Lisp, a language close to the Agda ecosystem. Going forward, one of the aims of our work is to have variationals definable directly within Agda — rather than having our users learn yet another language. Our exploratory efforts suggest that we may be able to realise PackageFormer's as Agda records of 'elements' —a tuple of qualifier, name, type, and definitional clauses— and, so, the result is a conservative extension to Agda's underlying type theory. However, from a practical standpoint, it is highly likely that we will extend Agda to support the new syntax.

Our resulting system has turned hand-written instances of structuring schemes from a design pattern into full-fledged library methods. In turn, the system addresses the following extremely unsatisfactory points of hand-written instances, mentioned by the "Deriving Via" [1] group.

- 1. It is not obvious that we are instantiating a general principle.
- 2. Because the general principle is not written down in code with a name and documentation, it has to be communicated through folklore or in comments and is difficult to discover and search for. Our code has lost a connection to its origin.
- 3. There are many such rules, some quite obvious, but others more surprising and easy to overlook.
- 4. While the work required to define instances manually for Monoid—which only has 6 constituents—is perhaps acceptable, it quickly becomes extremely tedious and error-prone for packages with many constituents.

Paraphrasing [1], we believe that PackageFormers have the potential to dramatically change the way we write instances of structuring mechanisms, as it encourages giving names and documentation to recurring patterns and reusing them where needed.

References

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6 Appendix: Source code

Below is a nearly self-contained source sample for the presented fragments.

6.1 Module Header

```
open import Data.List hiding (concat)
open import Relation.Binary.PropositionalEquality
            using (_≡_)
```

module Paper0 where

open import Paper0_Generated

6.2 Plain MonoidP PackageFormer

```
{-700
PackageFormer MonoidP : Set_1 where
     Carrier : Set
          : Carrier → Carrier → Carrier
               : Carrier
     assoc : \forall \{x \ y \ z\} \rightarrow (x \ y) \ z \equiv x \ (y \ z)
    leftId : \forall \{x : Carrier\} \rightarrow Id \ x \equiv x
     rightId : \forall \{x : Carrier\} \rightarrow x \ \ d \equiv x
-}
```

6.3 The record variational and three instantiations

```
V-record = :kind record :waist-strings ("field")
Monoid_0' = MonoidP record
Monoid_1'' = MonoidP record \longrightarrow :waist 1
Monoid_2'' = MonoidP record \longrightarrow :waist 2
-}
```

In the paper proper we mentioned "unbundled", which in the prototype takes the form of the meta-primitive :waist.

6.4 Complex variationals in lisp blocks

{-700

```
{-lisp
(V termtype carrier
  = "Reify as parameterless Agda "data" type.
    CARRIER refers to the sort that is designated as the concat_3 = concat^3
    domain of discourse of the resulting single-sorted
    inductive term data type.
    :kind data
    :level dec
    :alter-elements (lambda (fs)
      (thread-last fs
        (--filter (s-contains? carrier (target (get-type it))))
        (--map (map-type (s-replace carrier $name type) it)))))
(V termtype-with-variables carrier = \cdots) -}
```

```
Monoid<sub>3</sub>' = MonoidP termtype :carrier "Carrier"
Monoid₄ = MonoidP termtvpe-with-variables :carrier "Carrier"
-}
```

6.5 PackageFormers with Equations

```
{-700
                                                                       PackageFormer MonoidPE : Set_1 where
                                                                            -- A few declarations
                                                                            Carrier : Set
                                                                                      : Carrier \rightarrow Carrier \rightarrow Carrier
                                                                            _9°_
                                                                            Td
                                                                                      : Carrier
{- Automatically generated & inserted by the prototype -} assoc : \forall \{x \ y \ z\} \rightarrow (x \ \S \ y) \ \S \ z \equiv x \ \S \ (y \ \S \ z)
                                                                            -- A few declarations with equations
                                                                            Rid : Carrier \rightarrow Carrier
                                                                            Rid x = x : Id
                                                                            concat : List Carrier \rightarrow Carrier
                                                                            concat = foldr _______ Id
                                                                            -- More declarations
                                                                            leftId : \forall \{x : Carrier\} \rightarrow Id \ \ \ \ x \equiv x
                                                                            rightId : \forall \{x : Carrier\} \rightarrow Rid \ x \equiv x
                                                                       -}
                                                                       6.6 concat<sub>0</sub> and concat<sub>3</sub>
                                                                       {-700
                                                                       V-decorated by = \cdots
```

```
Monoid<sup>0</sup> = MonoidPE decorated :by "<sup>0</sup>" → record<sub>e</sub>
Monoid^3 = MonoidPE \longrightarrow decorated :by "3"
                             \oplus termtype<sub>e</sub> :carrier "Carrier<sup>3</sup>"
-}
```

```
{- "Concatenation over an arbitrary monoid" -}
concat_0 : \{M : Monoid^0\}
            \rightarrow let C = Monoid<sup>0</sup>.Carrier<sup>0</sup> M
               in List C \rightarrow C
```

```
{- Concatenation over an arbitrary *closed* monoid term -}
concat_3 : let C = Monoid<sup>3</sup>
            in List C \rightarrow C
```

 $concat_0 \{M\} = Monoid^0.concat^0 M$