Do-it-yourself Module Systems

Extending Dependently-Typed Languages to Implement Module System Features In The Core Language

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PhD Thesis

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Abstract

Can parameterised records and algebraic datatypes —i.e., Π-, Σ-, and $\mathcal{W}$-types— be derived from one pragmatic declaration?

Record types give a universe of discourse, parameterised record types fix parts of that universe ahead of time, and algebraic datatypes give us first-class syntax, whence evaluators and optimisers.

The answer is in the affirmative. Besides a practical shared declaration interface, which is extensible in the language, we also find that common data structures correspond to simple theories.

A middle-path with margins

Imagine having to stop reading mid-sentence, go to the bottom of the page, read a footnote, then stumble around till you get back to where you were reading$^{\alpha}$. Even worse is when one seeks a cryptic abbreviation and must decode a world-away, in the references at the end of the document.

α No more such oppresion!

I would like you to be able to read this work *smoothly, with minimal interpretations.* As such, inspired by [9] among others, we have opted to include "mathematical graffiti" in the margins. In particular, the margins side notes may have *informal and optioniated* remarks$^{\beta}$. We're trying to avoid being too dry, and aim at being somewhat light-hearted.

[9] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik. *Concrete Mathematics: A Foundation for Computer Science, 2nd Ed.* Addison-Wesley, 1994. ISBN: 0-201-55802-5. URL: `https://www-cs-faculty.stanford.edu/%5C%7Eknuth/gkp.html`

β Professional academic writing to the left; here in the right we take a relaxed tone.

Dijkstra [4] might construe the graffiti as *mathematical politeness* that could potentially save the reader a minute. Even though a characteristic of academic writing is its terseness$^{\omega}$, we don't want to baffle or puzzle our readers, and so we use the informality of the graffiti to say what we mean bluntly, *but* it may be less accurate or not as formally justifiable as the text proper.

[4] Edsger W. Dijkstra. *The notational conventions I adopted, and why.* circulated privately. July 2000. URL: `http://www.cs.utexas.edu/users/EWD/ewd13xx/EWD1300.PDF`

> *Some consider the puzzles that are created by their omissions as spicy challenges, without which their texts would be boring; others shun clarity lest their worth is considered trivial. [...] Some authors believe that, in order to keep the reader awake, one has to tickle him with surprises. [...] essential for earning the respect of their readership.*
> *—Edsger Dijkstra [4]*

ω "It's so obvious, I won't waste time on it"; i.e., "It's an exercise to the reader to figure out what I'm really saying." Elaboration removes mystery and some authors might prefer academia be exclusive.

When there are no side remarks to be made, or a code snippet would be better viewed with greater width, we will unabashedly switch to using the full width of the page —temporarily, on the fly, and without ceremony.

A superficial cost of utilising margin space is that the overall page count may be 'over-exaggerated'$^{\gamma}$. Nonetheless, I have found long empty columns of margin space *yearning* to be filled with explanatory remarks, references, or somewhat helpful diagrams. Paraphrasing Hofstadter [10], the little pearls in the margins were so connected in my own mind with the ideas that I was writing about that for me to deprive my readers of the connection that I myself felt so strongly would be nothing less than perverse.

γ Which doesn't matter, since you're likely reading this online!

[10] Douglas R. Hofstadter. *Gödel, Escher, Bach: an Eternal Golden Braid.* Basic Books Inc., 1979

Contents

0 [0]Draft

1 The PackageFormer Prototype

From the lessons learned from spelunking in a few libraries, we concluded that metaprogramming is a reasonable road on the journey toward first-class modules in DTLs. As such, we begin by forming an 'editor extension' to Agda with an eye toward a small number of 'meta-primitives'[2] for forming combinators on modules. The extension is written in Lisp, an excellent language for rapid prototyping. The purpose of writing the editor extension is not only to show that the 'flattening' of value terms and module terms is feasible[3]; but to also show that ubiquitous packaging combinators can be generated[4] from a small number of primitives. The resulting tool resolves many of the issues discussed in section **??**.

For the interested reader, the full implementation is presented *literately* as a discussion at `https://alhassy.github.io/next-700-module-systems/prototype/package-former.html`. We will not be discussing any Lisp code in particular.

[2]Section 4.3 contains an example-driven approach

[3]Indeed, the MathScheme [3] prototype already shows this.

[4]Just as the primitive of a programming language permit arbitrarily complex programs to be written.

The core of this chapter shows how some of the problems of Chapter 3, *Examples from the wild*, can be solved using `PackageFormer`.

Chapter Contents

1.1 Why an editor extension?

The prototype[5] *rewrites* Agda phrases from an extended Agda syntax to legitimate existing syntax; it is written as an Emacs editor extension to Emacs' Agda interface, using Lisp [8]. Since Agda code

[5]A prototype's raison d'etre is a testing ground for ideas, so its ease of development may well be more important than its usability.

[8] Paul Graham. *ANSI Common Lisp*. USA: Prentice Hall Press, 1995. ISBN: 0133708756

Why Emacs?

is predominately written in Emacs, a practical and pragmatic editor extension would need to be in Agda's de-facto IDE[6] , Emacs. Moreover, Agda development involves the manipulation of Agda source code by Emacs Lisp —for example, for case splitting and term refinement tactics— and so it is natural to extend these ideas. Nonetheless, at a first glance, it is humorous[7] that a module extension for a statically dependently-typed language is written in a dynamically type checked language. However, *a lack of static types means some design decisions can be deferred as much as possible.*

[6]**IDE**: Interactive Development Environment

[7]None of my colleagues thought Lisp was at all the 'right' choice; of-course, none of them had the privilege to use the language enough to appreciate it for the wonder that it is.

Unless a language provides an extension mechanism, one is forced to either alter the language's compiler or to use a preprocessing tool —both have drawbacks. The former[8] is *dangerous*; e.g., altering the grammar of a language requires non-trivial propagated changes throughout its codebase, but even worse, it could lead to existing language features to suddenly break due to incompatibility with the added features. The latter is *tiresome*[9] : It can be a nuisance to remember always invoke a preprocessor before compilation or type-checking, and it becomes extra baggage to future users of the codebase —i.e., a further addition to the toolchain that requires regular maintenance in order to be kept up to date with the core language. A middle-road between the two is not always possible.

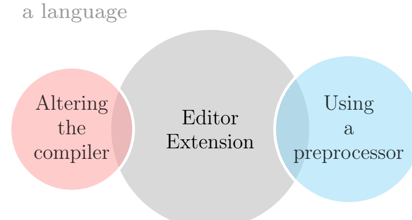
Why an editor extension? Because we quickly needed a *convenient* prototype to actually "figure out the problem".

[8]Instead of "hacking in" a new feature, one could instead carefully research, design, and implement a new feature.

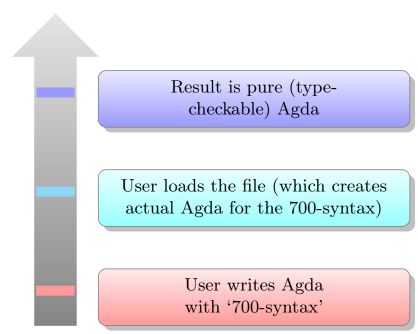
[9]Unless one uses a sufficiently flexible IDE that allows the seemless integration of preprocessing tools; which is exactly what we have done with Emacs.

However, if the language's community subscribes to *one* IDE, then a reasonable approach to extending a language would be to *plug-in* the necessary preprocessing —to transform the extended language into the pure core language— in a saliently *silent* fashion such that users need not invoke it manually.

A reasonable middle path to growing a language



Moreover, to mitigate the burden of increasing the toolchain, the salient preprocessing would *not transform user code* but instead *produce auxiliary files* containing core language code which are then *imported* by user code —furthermore, such import clauses could be automatically inserted when necessary. The benefit here is that *library users* need not know about the extended language features; since all files are in the core language with extended language feature appearing in special comments. Details can be found in section 1.2.



How does it work? All stages transpire in *one* user-written file

Why Lisp? Emacs is extensible using Elisp[10] wherein literally every key may be remapped and existing utilities could easily be altered *without* having to recompile Emacs. In some sense, Emacs is a Lisp interpreter and state machine. This means, we can hook our editor extension *seamlessly into the existing Agda interface* and even provide tooltips, among other features[11], to quickly see what our extended Agda syntax transpiles into.

Finally, Lisp uses a rather small number of constructs, such as macros and lambda, which themselves are used to build 'primitives', such as `defun` for defining top-level functions [11]. Knowing this about Lisp encourages us to emulate this expressive parsimony.

[10] Emacs Lisp is a combination of a large porition of Common Lisp and a editor language supporting, e.g., buffers, text elements, windows, fonts.

[11] E.g., since Emacs is a self-documenting editor, whenever a user of our tool wishes to see the documentation of a module combinator that they have written, or to read its Lisp elaboration, they merely need to invoke Emacs' help system —e.g., `C-h o` or `M-x describe-symbol`.

[11] Doug Hoyte. *Let Over Lambda*. Lulu.com, 2008. ISBN: 1435712757

1.2 Aim: *Scrap the Repetition*

Programming Language research is summarised, in essence, by the question: *If $\mathcal{X}$ is written manually, what information $\mathcal{Y}$ can be derived for free?* Perhaps the most popular instance is *type inference*: From the syntactic structure of an expression, its type can be derived. From a context, the `PackageFormer` editor extension can generate the many common design patterns discussed earlier in section **??**; such as unbundled variations of any number wherein fields are exposed as parameters at the type level, term types for syntactic manipulation, arbitrary renaming, extracting signatures, and forming homomorphism types. In this section we discuss how `PackageFormer` works and provide a 'real-world' use case, along with a discussion.

With the extension, Agda's usual `C-c C-l` command parses special comments containing fictitious Agda declarations, produces an auxiliary Agda file which it ensures is imported in the current file, then control is passed to the usual Agda typechecking mechanism.

Below is example code that can occur in the specially recognised comments. The first eight lines, starting at line 1, are essentially an Agda `record` declaration but the `field` qualifier is absent. The declaration is intended to name an abstract context, a sequence of "name : type" pairs as discussed at length in chapter **??**, but we use the name `PackageFormer` instead of 'context, signature, telescope', nor 'theory' since those names have existing biased connotations — besides, the new name is more 'programmer friendly'.

In the code block, the names have been chosen to stay relatively close to the real-world examples presented in chapter **??**. The name **M-Set** comes from *monoid acting on a set*; in our example, `Scalar` values may act on `Vector` values to produce new `Scalar` values. The programmer may very well appreciate this example if the names `Scalar`, 𝟙, `_×_`, `Vector`, `_·_` were chosen to be `Program`, `do-nothing`, `_⨾_`, `Input`, `run`. With this new naming, `leftId` says *running the empty program on any input, leaves the input unchanged*, whereas `assoc` says *to run a sequence of programs on an input, the input must be threaded through the programs*. Whence, **M-Sets abstract program execution**.

M-Sets are sets 'Scalar' acting '_·_' on semigroups 'Vector'

```
1  PackageFormer M-Set : Set₁ where
2     Scalar   : Set
3     Vector   : Set
4     _·_      : Scalar → Vector → Vector
5     𝟙        : Scalar
6     _×_      : Scalar → Scalar → Scalar
7     leftId   : {v : Vector}  →  𝟙 · v  ≡  v
8     assoc    : {a b : Scalar} {v : Vector} →   (a × b) · v
9                                              ≡  a · (b · v)
```

Now to actually use this context …

Different Ways to Organise ("interpret" / "use") M-Sets

```
Semantics   = M-Set ⟴ record
Semantics𝒟 = Semantics ⟴ rename (λ x → (concat x "𝒟"))
Semantics₃ =  Semantics :waist 3

Left-M-Set  = M-Set ⟴ record
Right-M-Set = Left-M-Set ⟴ flipping "_·_" :renaming "leftId
↪  to rightId"

ScalarSyntax = M-Set ⟴ primed ⟴ data "Scalar′"
Signature    = M-Set ⟴ record ⟴ signature
Sorts        = M-Set ⟴ record ⟴ sorts

𝒱-one-carrier   = renaming "Scalar to Carrier; Vector to
↪  Carrier"
𝒱-compositional = renaming "_×_ to _⨾_; _·_ to _⨾_"
𝒱-monoidal      = one-carrier ⟴ compositional ⟴ record

LeftUnitalSemigroup = M-Set ⟴ monoidal
Semigroup           = M-Set ⟴ keeping "assoc" ⟴ monoidal
Magma               = M-Set ⟴ keeping "_×_" ⟴ monoidal
```

M-Sets as records, possibly with renaming or parameters.

⋆ ⋆ ⋆

Duality; we might want to change the order of the action, say, to write `evalAt x f` instead of `run f x` —using the program-input interpretation of M-Sets above.

⋆ ⋆ ⋆

Keeping only the 'syntactic interface', say, for serialisation or automation.

⋆ ⋆ ⋆

Collapsing different features to obtain the notion of "monoid".

⋆ ⋆ ⋆

Obtaining parts of the monoid hierarchy (see chapter 3) from M-Sets

These manually written ∼25 lines elaborate into the ∼100 lines of raw, legitimate, Agda syntax below —line breaks are denoted by the symbol '↪' rather than inserted manually, since all subsequent code snippets in this section are **entirely generated** by `PackageFormer`. The result is nearly a **400% increase in size**; that is, our fictitious code will save us a lot of repetition.

Let's discuss what's actually going on here.

The first line declares the context of `M-Set`s using traditional Agda syntax " `record M-Set :` Set_1 `where` " except the we use the word `PackageFormer` to avoid confusion with the existing record concept, but[12] we also *omit* the need for a `field` keyword and *forbid* the existence of parameters. Such abstract contexts have no concrete form in Agda and so no code is generated; the second snippet above[13] shows sample declarations that result in legitimate Agda.

`PackageFormer` module combinators are called *variationals* since they provide a variation on an existing grouping mechanism. The syntax `p` ⟴ v_1 ⟴ $\cdots$ ⟴ v_n is tantamount to explicit forward function application $v_n\ (v_{n-1}\ (\cdots\ (v_1\ p)))$. With this understanding, we can explain the different ways to organise M-sets.

[12]**Conflating fields, parameters, and definitional extensions:** The lack of a `field` keyword and forbidding parameters means that arbitrary programs may 'live within' a `PackageFormer` and it is up to a variational to decide how to treat them and their optional definitions.

[13]For every (special comment) declaration $\mathcal{L}$ = $\mathcal{R}$ in the source file, the name $\mathcal{L}$ obtains a tooltip which mentions its specification $\mathcal{R}$ and the resulting legitimate Agda code. This feature is indispensable as it lets one generate grouping mechanisms and quickly ensure that they are what one intends them to be.

In line 9, the `record` variational is invoked to transform the abstract context `M-Set` into a valid Agda record declaration, with the key word `field` inserted as necessary. Later, its first 3 fields are lifted as parameters using the meta-primitive `:waist`.

The waist is the number of parameters exposed; recall $\Pi^w \Sigma$ from chapter 2.

Elaboration of lines 9-11 Record / decorated renaming / typeclass forms

```
{- Semantics           = M-Set ⟴ record -}
record Semantics : Set₁ where
    field Scalar                 : Set
    field Vector                 : Set
    field _·_           : Scalar → Vector → Vector
    field 𝟙             : Scalar
    field _×_           : Scalar → Scalar → Scalar
    field leftId                 : {v : Vector}  →  𝟙 · v  ≡  v
    field assoc         : {a b : Scalar} {v : Vector} → (a × b) · v  ≡  a · (b · v)


{- Semantics𝒟          = Semantics ⟴ rename (λ x → (concat x "𝒟")) -}
record Semantics𝒟 : Set₁ where
    field Scalar𝒟                : Set
    field Vector𝒟                : Set
    field _·𝒟_          : Scalar𝒟 → Vector𝒟 → Vector𝒟
    field 𝟙𝒟            : Scalar𝒟
    field _×𝒟_          : Scalar𝒟 → Scalar𝒟 → Scalar𝒟
    field leftId𝒟                : {v : Vector𝒟}  →  𝟙𝒟 ·𝒟 v  ≡  v
    field assoc𝒟                 : {a b : Scalar𝒟} {v : Vector𝒟} → (a ×𝒟 b) ·𝒟 v  ≡  a ·𝒟
    ↪  (b ·𝒟 v)
    toSemantics         : let View X = X in View Semantics ;    toSemantics = record {Scalar =
    ↪  Scalar𝒟;Vector = Vector𝒟;_·_ = _·𝒟_;𝟙 = 𝟙𝒟;_×_ = _×𝒟_;leftId = leftId𝒟;assoc =
    ↪  assoc𝒟}


{- Semantics₃          =  Semantics :waist 3 -}
record Semantics₃ (Scalar : Set) (Vector : Set) (_·_ : Scalar → Vector → Vector) : Set₁ where
    field 𝟙             : Scalar
    field _×_           : Scalar → Scalar → Scalar
    field leftId                 : {v : Vector}  →  𝟙 · v  ≡  v
    field assoc         : {a b : Scalar} {v : Vector} → (a × b) · v  ≡  a · (b · v)
```

Notice how `Semantics𝒟` was *built from* a concrete context, namely the `Semantics` record. As such, every instance of `Semantics𝒟` can be transformed as an instance of `Semantics`: This view[14] —see Section **??**— is automatically generated and named `toSemantics` above, by default. Likewise, `Right-M-Set` was derived from `Left-M-Set` and so we have automatically have a view `Right-M-Set` → `Left-M-Set`.

[14]It is important to remark that the mechanical construction of such views (coercions) is **not built-in**, but rather a *user-defined* variational that is constructed from `PackageFormer`'s meta-primitives.

"Arbitrary functions act on modules": When only one variational is applied to a context, the one and only sequencing operator '⟴' may be omitted. As such, the $\mathcal{D}$ecorated `Semantics𝒟` is defined as `Semantics rename f`, where `f` is the decoration function. In this form, one is tempted to believe

```
_rename_ : PackageFormer → (Name → Name) → PackageFormer
```

That is, we have a binary operation in which functions may act on modules —this is yet a new feature that Agda cannot perform.

Likewise, line 13, mentions another combinator

```
_flipping_ : PackageFormer → Name → PackageFormer
```

All combinators are demonstrated in this section and their usefulness is dicussed in the nextion section. For example, in contrast to the above 'type', the `flipping` combinator also takes an *optional keyword argument* `:renaming`, which simply renames the given pair. The notation of keyword arguments is inherited from Lisp.

More accurately, the '⟴'-based mini-language for variationals is realised as a Lisp macro and so, in general, the right side of a declaration in 700-comments is interpreted as valid Lisp modulo this mini-language: `PackageFormer` names and variationals are variables in the Emacs environment —for declaration purposes, and to avoid touching Emacs specific utilities, variationals `f` are actually named `𝒱-f`. One may quickly obtain the documentation of a variational `f` with `C-h o RET 𝒱-f` to see how it works.

Elaboration of lines 13-14 Duality: Sets can act on semigroups from the left or the right

```
{- Left-M-Set          = M-Set ⟴ record -}
record Left-M-Set : Set₁ where
    field Scalar                : Set
    field Vector                : Set
    field _·_           : Scalar → Vector → Vector
    field 𝟙             : Scalar
    field _×_           : Scalar → Scalar → Scalar
    field leftId                : {v : Vector}  →  𝟙 · v  ≡  v
    field assoc         : {a b : Scalar} {v : Vector} → (a × b) · v  ≡  a · (b · v)


{- Right-M-Set         = Left-M-Set ⟴ flipping "_·_" :renaming "leftId to rightId" -}
record Right-M-Set : Set₁ where
    field Scalar                : Set
    field Vector                : Set
    field _·_           :  Vector  → Scalar  →  Vector
    field 𝟙             : Scalar
    field _×_           : Scalar → Scalar → Scalar
    field rightId               : let _·_ = λ x y → _·_ y x in {v : Vector}  →  𝟙 · v  ≡  v
    field assoc         : let _·_ = λ x y → _·_ y x in {a b : Scalar} {v : Vector} → (a × b)
    ↪  · v  ≡  a · (b · v)
    toLeft-M-Set                : let _·_ = λ x y → _·_ y x in let View X = X in View
    ↪  Left-M-Set ;      toLeft-M-Set  = let _·_ = λ x y → _·_ y x in   record {Scalar =
    ↪  Scalar;Vector = Vector;_·_ = _·_;𝟙 = 𝟙;_×_ = _×_;leftId = rightId;assoc = assoc}
```

Next, in line 16, we view a context as such a termtype by declaring one sort of the context to act as the termtype (carrier) and then keep only the function symbols that target it —this is the **core idea** that is used when we operate on Agda `Term`s in the next chapter.

An algebraic data type is a tagged union of symbols, terms, and so is one type —see section **??**.

Recall from Chapter **??**, symbols that target `Set` are considered sorts and if we keep only the symbols targeting a sort, we have a signature. By allowing symbols to be of type `Set`, we actually have **generalised contexts**.

Elaboration of lines 16-18 Termtypes and lawless presentations

```
{- ScalarSyntax        = M-Set ⟴ primed ⟴ data "Scalar′" -}
data ScalarSyntax : Set where
    𝟙′          : ScalarSyntax
    _×′_               : ScalarSyntax → ScalarSyntax →
    ↪  ScalarSyntax


{- Signature           = M-Set ⟴ record ⟴ signature -}
record Signature : Set₁ where
    field Scalar                 : Set
    field Vector                 : Set
    field _·_           : Scalar → Vector → Vector
    field 𝟙             : Scalar
    field _×_           : Scalar → Scalar → Scalar


{- Sorts               = M-Set ⟴ record ⟴ sorts -}
record Sorts : Set₁ where
    field Scalar                 : Set
    field Vector                 : Set
```

The priming decoration in `ScalarSyntax` is needed so that the names 𝟙, `_×_` do not pollute the global name space.

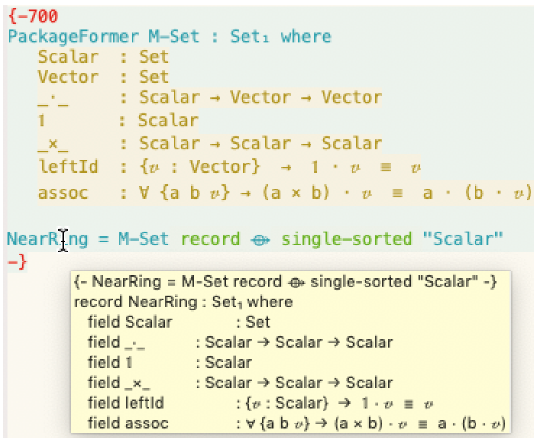
Finally, starting with line 20, declarations start with " 𝒱- " to indicate that a new variation *combinator* is to be formed, rather than a new *grouping* mechanism. For instance, the user-defined `one-carrier` variational identifies both the `Scalar` and `Vector` sorts, whereas `compositional` identifies the binary operations; then, finally, `monoidal` performs both of those operations and also produces a concrete Agda `record` formulation. Below, in the final code snippet of this section, are the elaborations of using these new new user-defined variationals.

User defined variationals are applied as if they were built-ins.

Elaboration of lines 24-26 Conflating features gives familiar structures

```
{- LeftUnitalSemigroup = M-Set ⟴ monoidal -}
record LeftUnitalSemigroup : Set₁ where
    field Carrier                : Set
    field _⨾_           : Carrier → Carrier → Carrier
    field 𝟙              : Carrier
    field leftId                 : {v : Carrier}  →  𝟙 ⨾ v  ≡  v
    field assoc         : {a b : Carrier} {v : Carrier} → (a ⨾ b) ⨾ v  ≡  a ⨾ (b ⨾ v)


{- Semigroup           = M-Set ⟴ keeping "assoc" ⟴ monoidal -}
record Semigroup : Set₁ where
    field Carrier                : Set
    field _⨾_           : Carrier → Carrier → Carrier
    field assoc         : {a b : Carrier} {v : Carrier} → (a ⨾ b) ⨾ v  ≡  a ⨾ (b ⨾ v)


{- Magma               = M-Set ⟴ keeping "_×_" ⟴ monoidal -}
record Magma : Set₁ where
    field Carrier                : Set
    field _⨾_           : Carrier → Carrier → Carrier
```

As shown in the figure below, the source file is furnished with tooltips displaying the special comment that a name is associated with, as well as the full elaboration into legitimate Agda syntax. In addition, the above generated elaborations also document the special comment that produced them. Moreover, since the editor extension results in valid code in an auxiliary file, future users of a library need not use the `PackageFormer` extension at all —thus we essentially have a static **editor tactic** similar to Agda's (Emacs interface) proof finder.



Hovering to show details. Notice special syntax has default colouring: Red for PackageFormer delimiters, yellow for elements, and green for variationals.

1.3 Practicality

Herein we demonstrate how to use this system from the perspective of *library designers*. That is to say, we will demonstrate how common desirable features encountered "in the wild" —chapter **??**— can be used with our system. The exposition here follows section 2 [2], reiterating many the ideas therein. These features are **not built-in** but instead are constructed from a small set of primitives, shown below, just as a small core set of language features give way to complex software programs. Moreover, users may combine the primitives —using Lisp— to **extend** the system to produce grouping mechanisms for any desired purpose.

[2] Jacques Carette and Russell O'Connor. "Theory Presentation Combinators". In: *Intelligent Computer Mathematics* (2012), pp. 202–215. DOI: 10.1007/978-3-642-31374-5_14

Metaprogramming Meta-primitives for Making Modules

Name	Description
`:waist`	Consider the first N elements as, possibly ill-formed, parameters.
`:kind`	Valid Agda grouping mechanisms: `record, data, module`.
`:level`	The Agda level of a PackageFormer.
`:alter-elements`	Apply a `List Element` → `List Element` function over a PackageFormer.
⟴	Compose two variational clauses in left-to-right sequence.
`map`	Map a `Element` → `Element` function over a PackageFormer.
`generated`	Keep the sub-PackageFormer whose elements satisfy a given predicate.

The few constructs demonstrated in this section not only create new grouping mechanisms from old ones, but also create morphsisms from the new, child, presentations to the old parent presentations. For example, a theory extended by new declarations comes equipped with a map that forgets the new declarations to obtain an instance of the original theory. Such morphisms are tedious to write out, and our system provides them for free. The user can implement such features using our 5 primitives —but we have implemented a few to show that the primitives are deserving of their name, as shown below.

Do-it-yourself Extendability: In order to make the editor extension immediately useful, and to substantiate the claim that **common module combinators can be defined using the system**, we have implemented a few notable ones, as described in the table below. The implementations, in the user manual, are discussed along with the associated Lisp code and use cases.

Summary of Sample Variationals Provided With The System

Name	**Description**
`record`	Reify a PackageFormer as a valid *Agda record*
`data`	Reify a PackageFormer as a valid Agda algebraic data type, $\mathcal{W}$-type
`extended-by`	Extend a PackageFormer by a string-";"-list of declaration
`union`	Union two PackageFormers into a new one, maintaining relationships
`flipping`	Dualise a binary operation or predicate
`unbundling`	Consider the first N elements, which may have definitions, as parameters
`open`	Reify a given PackageFormer as a parameterised *Agda module* declaration
`opening`	Open a record as a module exposing only the given names
`open-with-decoration`	Open a record, exposing all elements, with a given decoration
`keeping`	Largest well-formed PackageFormer consisting of a given list of elements
`sorts`	Keep only the types declared in a grouping mechanism
`signature`	Keep only the elements that target a sort, drop all else
`rename`	Apply a `Name` → `Name` function to the elements of a PackageFormer
`renaming`	Rename elements using a list of "to"-separated pairs
`decorated`	Append all element names by a given string
`codecorated`	Prepend all element names by a given string
`primed`	Prime all element names
`subscripted`$_i$	Append all element names by subscript `i : 0..9`
`hom`	Formulate the notion of homomorphism of parent PackageFormer algebras

`PackageFormer` packages are an **implementation of the idea** of packages fleshed out in Chapter **??**. Tersely put, a `PackageFormer` package is essentially a pair of tags —alterable by `:waist` to determine the height delimiting parameters from fields, and by `:kind` to determine a possible legitimate Agda representation that lives in a universe dictated by `:level`— as well as a list of declarations (elements) that can be manipulated with `:alter-elements`.

The remainder of this section is an exposition of notable *user-defined* combinators —i.e., those which can be constructed using the system's primitives and a small amount of Lisp. Along the way, for each example, we show both the terse specfication using `PackageFormer` and its elaboration into pure typecheckable Agda. In particular, since packages are essentially a list of declarations —see Chapter **??**— we begin in section 1.3.1 with the `extended-by` combinator which "grows a package". Then, in section 1.3.2, we show

Any variational v that takes an argument of type τ can be thought of as a **binary packaged-valued operator**,

```
_v_ : PackageFormer
    → τ
    → PackageFormer
```

With this perspective, the *sequencing variational combinator* '⟴' is essentially forward function composition/application. Details can be found on the associated webpage; whereas the next chapter provides an Agda function-based semantics.

how *Agda users* can **quickly**, with a *tiny* amount of Lisp[15] knowledge, make useful variationals to abbreviate commonly occurring situations, such as a method to adjoin named operation properties to a a package. After looking at a `renaming` combinator, in section 1.3.3, and its properties that make it resonable; we show the Lisp code, in section 1.3.4 required for a pushout construction on packages. Of note is how Lisp's keyword argument feature allows the *verbose* 5-argument pushout operation to be **used** *easily* as a 2-argument operation, with other arguments optional. This construction is shown to generalise set union (disjoint and otherwise) and provide support for granular hierarchies thereby solving the so-called 'diamond problem'. Afterword, in section 1.3.5, we turn to another example of *formalising common patterns* —see Chapter **??**— by showing how the idea of duality, not much used in simpler type systems, is used to mechanically produce new packages from old ones. Then, in section 1.3.6, we show how the interface segregation principle can be *applied after the fact*. Finally, we close in section 1.3.7 with a measure of the systems immediate practicality.

[15] The `PackageFormer` manual provides the expected Lisp methods one is interested in, such as `(list` x_0 `...` x_n`)` to make a list and `first, rest` to decompose it, and `(--map (···it···) xs)` to traverse it. Moreover, an Emacs Lisp cheat sheet covering the basics is provided.

1.3.1 Extension

The simplest operation on packages is when one package is included, verbatim, in another. Concretely, consider `Monoid` —which consists of a number of *parameters* and the derived result `𝟙-unique`— and `CommutativeMonoid₀` below.

One may use the call `P = Q extended-by R :adjoin-retract nil` to extend `Q` by declaration `R` but avoid having a view (coercion) `P → Q`. Of-course, `extended-by` is *user-defined* and we have simply chosen to adjoint retract views by default; the online documentation shows how users can define their own variationals.

Manually Repeating the entirety of 'Monoid' within 'CommutativeMonoid$_0$'

```
PackageFormer Monoid : Set₁ where
   Carrier : Set
   _·_     : Carrier → Carrier → Carrier
   assoc   : {x y z : Carrier} → (x · y) · z  ≡  x · (y · z)
   𝟙       : Carrier
   leftId  : {x : Carrier} → 𝟙 · x  ≡ x
   rightId : {x : Carrier} → x · 𝟙  ≡ x
   𝟙-unique : ∀ {e} (lid : ∀ {x} → e · x ≡ x) (rid : ∀ {x} →
   ↪  x · e ≡ x) → e ≡ 𝟙
   𝟙-unique lid rid = ≡.trans (≡.sym leftId) rid

PackageFormer CommutativeMonoid₀ : Set₁ where
   Carrier : Set
   _·_     : Carrier → Carrier → Carrier
   assoc   : {x y z : Carrier} → (x · y) · z  ≡  x · (y · z)
   𝟙       : Carrier
   leftId  : {x : Carrier} →  𝟙 · x  ≡ x
   rightId : {x : Carrier} →  x · 𝟙  ≡ x
   comm    : {x y : Carrier} →  x · y  ≡  y · x
   𝟙-unique : ∀ {e} (lid : ∀ {x} → e · x ≡ x) (rid : ∀ {x} →
   ↪  x · e ≡ x) → e ≡ 𝟙
   𝟙-unique lid rid = ≡.trans (≡.sym leftId) rid
```

So much repetition for an additional axiom! Eek!

As expected, the only difference is that $\texttt{CommutativeMonoid}_0$ adds a `comm`utatity axiom. Thus, given `Monoid`, it would be **more economical** to define:

Economically declaring only the new additions to 'Monoid'

```
CommutativeMonoid = Monoid extended-by "comm : {x y : Carrier} → x · y ≡ y · x"
```

As discussed in section **??**, to obtain this specification of `CommutativeMonoid` in the current implementation of Agda, one would likely declare a record with two fields —one being a `Monoid` and the other being the commutativity constraint— however, this only gives the appearance of the above specification for consumers; those who produce instances of `CommutativeMonoid` are then forced to know the particular hierarchy and must provide a `Monoid` value first. It is a happy coincidence that our system alleviates such an issue; i.e., we have **flattened extensions**.

As discussed in the previous section, mouse-hovering over the left-hand-side of this declaration gives a tooltip showing the resulting elaboration, which is identical to $\texttt{CommutativeMonoid}_0$ above —followed by forgetful operation. The tooltip shows the *expanded* version of the theory, which is **what we want to specify but not what we want to enter manually**.

1.3.2 Defining a Concept Only Once

From a library-designer's perspective, our definition of `CommutativeMonoid` has the commutativity property 'hard coded' into it. If we wish to speak of commutative magmas —types with a single commutative operation— we need to hard-code the property once again. If, at a later time, we wish to move from having arguments be implicit to being explicit then we need to track down every hard-coded instance of the property then alter them —having them in-sync then becomes an issue. Instead, as shown below, the system lets us 'build upon' the `extended-by` combinator: We make an associative list of names and properties, then string-replace the meta-names *op*, *op*′, *rel* with the provided user names.

The definition below uses functional methods and should not be inaccessible to Agda programmers.

⋆ ⋆ ⋆

Method call `(s-replace old new s)` replaces all occurrences of string `old` by `new` in the given string `s`.

⋆ ⋆ ⋆

$(\texttt{pcase e } (\texttt{x}_0\ \texttt{y}_0) \ldots (\texttt{x}_n\ \texttt{y}_n))$ pattern matches on `e` and performs the first $\texttt{y}_i$ if $\texttt{e} = \texttt{x}_i$, otherwise it returns `nil`.

Writing definitions **only once** with the 'postulating' variational

```
(𝒱 postulating bop prop (using bop) (adjoin-retract t)
 = "Adjoin a property PROP for a given binary operation BOP.

   PROP may be a string: associative, commutative, idempotent, etc.
   Some properties require another operator or a relation; which may
   be provided via USING.

   ADJOIN-RETRACT is the optional name of the resulting retract morphism.
   Provide nil if you do not want the morphism adjoined."
   extended-by
    (s-replace "op" bop (s-replace "rel" using (s-replace "op′" using
     (pcase prop
      ("associative"   "assoc : ∀ x y z → op (op x y) z ≡ op x (op y z)")
      ("commutative"   "comm  : ∀ x y   → op x y ≡ op y x")
      ("idempotent"    "idemp : ∀ x     → op x x ≡ x")
      ("left-unit"     "unitˡ : ∀ x y z → op e x ≡ e")
      ("right-unit"    "unitʳ : ∀ x y z → op x e ≡ e")
      ("absorptive"    "absorp  : ∀ x y  → op x (op′ x y) ≡ x")
      ("reflexive"     "refl    : ∀ x y  → rel x x")
      ("transitive"    "trans   : ∀ x y z → rel x y → rel y z → rel x z")
      ("antisymmetric" "antisym : ∀ x y → rel x y → rel y x → x ≡ z")
      ("congruence"    "cong    : ∀ x x′ y y′ → rel x x′ → rel y y′ → rel (op x x′) (op y y′)")
      (_ (error "𝒱-postulating does not know the property “%s”" prop))
      )))) :adjoin-retract 'adjoin-retract)
```

As such, we have a formal approach to the idea that **each piece of mathematical knowledge should be formalised only once** [7]. We can extend this database of properties as needed with relative ease. Here is an example use along with its elaboration.

[7] Adam Grabowski and Christoph Schwarzweller. "On Duplication in Mathematical Repositories". In: *Intelligent Computer Mathematics, 10th International Conference, AISC 2010, 17th Symposium, Calculemus 2010, and 9th International Conference, MKM 2010, Paris, France, July 5-10, 2010. Proceedings.* Ed. by Serge Autexier et al. Vol. 6167. Lecture Notes in Computer Science. Springer, 2010, pp. 300–314. ISBN: 978-3-642-14127-0. DOI: 10.1007/978-3-642-14128-7_26. URL: https://doi.org/10.1007/978-3-642-14128-7%5C_26

Example Use

```
PackageFormer Magma : Set₁ where
  Carrier : Set
  _·_     : Carrier → Carrier → Carrier

RawRelationalMagma = Magma extended-by "_≈_ : Carrier →
↪  Carrier → Set" ⟴ record

RelationalMagma    = RawRelationalMagma postulating "_·_"
↪  "congruence" :using "_≈_" ⟴ record
```

Associated Elaboration

```
record RawRelationalMagma : Set₁ where
    field Carrier      : Set
    field op        : Carrier → Carrier → Carrier
    toType      : let View X = X in View Type ; toType =
    ↪  record {Carrier = Carrier}
    field _≈_        : Carrier → Carrier → Set
    toMagma     : let View X = X in View Magma ;    toMagma =
    ↪  record {Carrier = Carrier;op = op}

record RelationalMagma : Set₁ where
    field Carrier      : Set
    field op        : Carrier → Carrier → Carrier
    toType      : let View X = X in View Type ; toType =
    ↪  record {Carrier = Carrier}
    field _≈_        : Carrier → Carrier → Set
    toMagma     : let View X = X in View Magma ;    toMagma =
    ↪  record {Carrier = Carrier;op = op}
    field cong      : ∀ x x′ y y′ → _≈_ x x′ → _≈_ y y′ →
    ↪  _≈_ (op x x′) (op y y′)
    toRawRelationalMagma        : let View X = X in View
    ↪  RawRelationalMagma ;   toRawRelationalMagma = record
    ↪  {Carrier = Carrier;op = op;_≈_ = _≈_}
```

The `let View X = X in View` ⋯ clauses are a part of the user implementation of `extended-by`; they are used as markers to indicate that a declaration is a *view* and so should not be an element of the current view constructed by a call to `extended-by`.

In conjunction with `postulating`, the `extended-by` variational makes it **tremendously easy to build fine-grained hierarchies** since at any stage in the hierarchy we have views to parent stages (unless requested otherwise) *and* the hierarchy structure is *hidden* from end-users. That is to say, ignoring the views, the above initial declaration of `CommutativeMonoid₀` is identical to the `CommutativeMonoid` package obtained by using variationals, as follows.

Building fine-grained hierarchies with ease

```
PackageFormer Empty : Set1 where {- No elements -}
Type                = Empty                  extended-by "Carrier : Set"
Magma               = Type                   extended-by "_·_ : Carrier → Carrier → Carrier"
Semigroup           = Magma                  postulating "_·_" "associative"
LeftUnitalSemigroup = Semigroup              postulating "_·_" "left-unit"  :using "𝟙"
Monoid              = LeftUnitalSemigroup postulating "_·_" "right-unit" :using "𝟙"
CommutativeMonoid   = Monoid                 postulating "_·_" "commutative"
```

Of-course, one can continue to build packages in a monolithic fashion, as shown below.

```
Group = Monoid extended-by "_⁻¹ : Carrier → Carrier; left⁻¹ : ∀ {x} → (x ⁻¹) · x ≡ 𝟙;
↪  right⁻¹ : ∀ {x} → x · (x ⁻¹) ≡ 𝟙" ⟴ record
```

After discussing renaming, we return to discuss the loss of relationships when we augment `Group` with a commutativity axiom —commutative groups are commutative monoids!

1.3.3 Renaming

From an end-user perspective, our `CommutativeMonoid` has one flaw: Such monoids are frequently written *additively* rather than multiplicatively. Such a change can be rendered conveniently:

Renaming Example

```
AbealianMonoid = CommutativeMonoid renaming "_·_ to _+_"
```

An Abealian monoid is *both* a commutative monoid and also, simply, a monoid. The above declaration freely maintains these relationships: The resulting record comes with a new projection `toCommutativeMonoid`, and still has the *inherited* projection `toMonoid`.

There are a few reasonable properties that a renaming construction should support. Let us briefly look at the (operational) properties of `renaming`.

Relationship to Parent Packages. Dual to `extended-by` which can construct (retract) views *to parent* modules mechanically, `renaming` constructs (coretract) views *from parent* packages.

That is, it has an optional argument `:adjoin-coretract` which can be provided with `t` to use a default name or provided with a string to use a desired name for the inverse part of a projection, `fromMagma` below.

Adjoining coretracts —views from parent packages

```
Sequential = Magma renaming "op to _⨾_" :adjoin-coretract t
```

Sequential elaboration

```
record Sequential : Set₁ where
    field Carrier : Set
    field _⨾_     : Carrier → Carrier → Carrier

    toType : let View X = X in View Type
    toType = record {Carrier = Carrier}

    toMagma : let View X = X in View Magma
    toMagma = record {Carrier = Carrier;op = _⨾_}

    fromMagma : let View X = X in Magma → View
    ↪  Sequential
    fromMagma = λ g227742 → record {Carrier =
    ↪  Magma.Carrier g227742;_⨾_ = Magma.op g227742}
```

This user implementation of `renaming` avoid name clashes for λ-arguments by using *gensyms* —generated symbolic names, "fresh variable names".

Commutativity. Since `renaming` and `postulating` both adjoin retract morphisms, by default, we are led to wonder about the result of performing these operations in sequence 'on the fly', rather than naming each application. Since `P renaming X ⟴ postulating Y` comes with a retract `toP` via the `renaming` and another, distinctly defined, `toP` via `postulating`, we have that the operations commute if *only* the first permits the creation of a retract[16] .

[16] For instance, we may define idempotent magmas with

```
renaming "_·_ to _⊔_"
⟴ postulating "_⊔_" "idempotent"
:adjoin-retract nil
```

or, equivalently (up to reordering of constituents), with

```
postulating "_⊔_" "idempotent"
⟴ renaming "_·_ to _⊔_"
:adjoin-retract nil
```

It is important to realise that the renaming and postulating combinators are *user-defined*, and could have been defined without adjoining a retract by default; consequently, we would have **unconditional commutativity of these combinators**. The user can make these alternative combinators as follows:

Alternative 'renaming' and 'postulating' —with an example use

```
𝒱-renaming′ by = renaming 'by :adjoin-retract nil
𝒱-postulating′ p bop (using) = postulating 'p 'bop :using 'using :adjoin-retract nil

IdempotentMagma = Magma postulating′ "_⊔_" "idempotent" ⟴ renaming′ "_·_ to _⊔_"
```

Finally, as expected, simultaneous renaming works too, and renaming is an invertible operation —e.g., below `Magmaʳʳ` is identical to `Magma`.

(Recall `renaming′` performs renaming but does not adjoin retract views.)

```
Magmaʳ  = Magma  renaming′ "_·_  to op"
Magmaʳʳ = Magmaʳ renaming′ "op   to _·_"
```

`TwoR` is just `Two` but as an Agda `record`, so it typechecks.

Simultaneous textual substitution example

```
PackageFormer Two : Set₁ where
  Carrier : Set
  𝟘       : Carrier
  𝟙       : Carrier

TwoR = Two record ⟴ renaming′ "𝟘 to 𝟙; 𝟙 to 𝟘"
```

Do-it-yourself. Finally, to demonstrate the accessibility of the system, we show how a generic renaming operation can be defined swiftly using the primitives mentioned listed in the first table of this section. Instead of `renaming` elements *one at a time*, suppose we want to be able to uniformly `rename` all elements in a package. That is, given a function `f` on strings, we want to map over the name component of each element in the package. This is easily done with the following declaration.

Tersely forming a new variational

```
𝒱-rename f = map (λ element → (map-name (λ nom → (funcall f nom))) element)
```

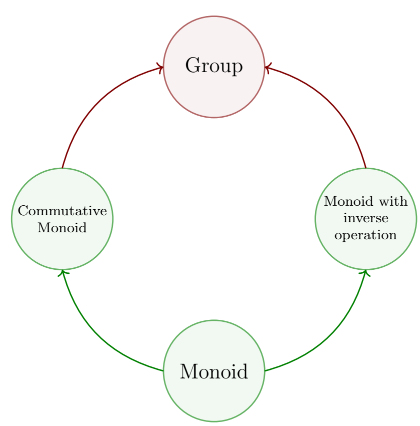
1.3.4 Unions/Pushouts (and intersections)

But even with these features, using `Group` from above, we would find ourselves writing:

```
CommutativeGroup₀ = Group extended-by "comm : {x y : Carrier}
↪  →  x · y  ≡  y · x" ⟴ record
```

This is **problematic**: We lose the *relationship* that every commutative group is a commutative monoid. This is not an issue of erroneous hierarchical design: From `Monoid`, we could orthogonally add a commutativity property or inverse operation; `CommutativeGroup₀` then closes this diamond-loop by adding both features, as shown in the figure to the right. The simplest way to share structure is to union two presentations:

Given green, require red



Unions of packages

```
CommutativeGroup = Group union CommutativeMonoid ⟴ record
```

The resulting record, `CommutativeMonoidR`, comes with three[17] derived fields —`toMonoidR`, `toGroupR`, `toCommutativeMonoidR`— that retain the results relationships with its hierarchical construction. This approach "works" to build a sizeable library, say of the order of 500 concepts, in a fairly economical way [2]. The union operation is an instance of a *pushout* operation, which consists of 5 arguments —three objects and two morphisms— which may be included into the `union` operation

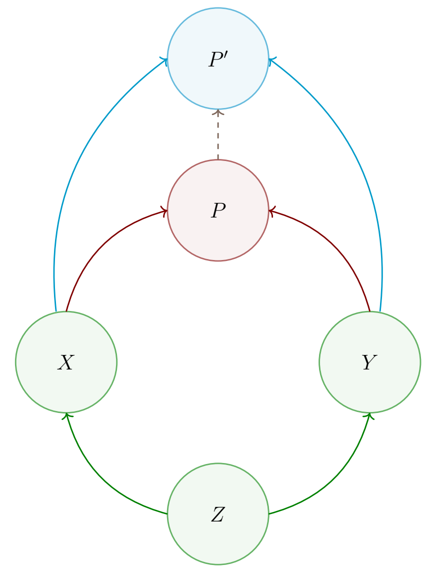
[17] The three green arrows in the diagram above!

[2] Jacques Carette and Russell O'Connor. "Theory Presentation Combinators". In: *Intelligent Computer Mathematics* (2012), pp. 202–215. DOI: 10.1007/978-3-642-31374-5_14

as optional keyword arguments. The more general notion of pushout is required if we were to combine[18] `Group` with `AbealianMonoid`, which have non-identical syntactic copies of `Monoid`.

[18] For example, to make rings!

What is a pushout?



Given green, require red, such that every candidate cyan has a unique umber

The pushout of morphisms $f : Z \to X$ and $g : Z \to Y$ is, essentially, the disjoint sum of contexts X and Y where embedded elements are considered 'indistinguishable' when they share the same origin in Z via the 'paths' f and g —the pushout generalises the notion of *least upper bound* as shown in the figure to the right, by treating each '$\to$' as a '$\leq$'. Unfortunately, the resulting 'indistinguishable' elements $f(z) \approx g(z)$ are **actually distinguishable**: They may be the f-name or the g-name and a choice must be made as to which name is preferred since users actually want to refer to them later on. Hence, to be useful for library construction, the pushout construction actually requires at least another input function that provides canonical names to the supposedly 'indistinguishable' elements. Hence, 6 inputs are actually needed for forming a *usable* pushout object.

At first, a pushout construction needs 5 inputs, to be practical it further needs a function for canonical names for a total of 6 inputs. However, a pushout of $f : Z \to X$ and $g : Z \to Y$ is intended to be the 'smallest object P that contains a copy of X and of Y sharing the common substructure X', and as such it outputs two functions $\mathsf{inj}_1 : X \to P$, $\mathsf{inj}_2 : Y \to P$ that inject the names of X and Y into P. If we realise P as a record —a type of models— then the embedding functions are *reversed*, to obtain projections $P \to X$ and $P \to Y$: If we have a model of P, then we can forget some structure and rename via f and g to obtain models of X and Y. For the resulting construction to be useful, these names could be automated such as $\mathsf{toX} : P \to X$ and $\mathsf{toY} : P \to Y$ but such a naming scheme does not scale —but we shall use it for default names. As such, we need two more inputs to the pushout construction so the names of the resulting output functions can be used later on. *Hence, a practical choice of pushout needs 8 inputs!*

By changing perspective, we half the number of inputs to the pushout construction!

Since a `PackageFormer` is essentially just a *signature* —a collection of typed names—, we can make a 'partial choice of pushout' to reduce the number of arguments from 6 to 4 by letting the typed-names object Z be 'inferred' and encoding the canonical names function into the operations f and g. The input functions f, g are necessarily *signature morphisms* —mappings of names that preserve types— and so are simply lists associating names of Z to names of X and Y. If we instead consider $f' : Z' \leftarrow X$ and $g' : Z' \leftarrow Y$, in the *opposite direction*, then we may reconstruct a pushout by setting Z to be common image of f', g', and set f, g to be inclusions. In-particular, the full identity of Z' is not necessarily relevant for the pushout reconstruction and so it may be omitted. Moreover, the issue of canonical names is resolved: *If $x \in X$ is intended to be identified with $y \in Y$ such that the resulting element has z as the chosen canonical name,* *then we simply require* $f'\,x = z = g'\,y$.

That is, *this particular user implementation* realises

X_1 union X_2 :renaming$_1$ f' :renaming$_2$ g'

as the pushout of the inclusions

$$f'\ X_1 \cap g'\ X_2 \hookrightarrow X_i$$

where the source is the set-wise intersection of *names*. Moreover, when either renaming$_i$ is omitted, it defaults to the identity function.

In Lisp, optional keyword arguments are passed with the syntax `:arg val`.

Incidentally, using the reversed directions of f, g via f', g', we can infer the shared structure Z and the canonical name function. Likewise, by using toChild : $P \to$ Child default-naming scheme, we may omit the names of the retract functions. If we wish to rename these retracts or simply omit them altogether, we make them *optional* arguments.

Before we show the implementation of `union`, let us showcase an example that mentions all arguments, optional and otherwise —i.e., test-driven development. Besides the elaboration The **commutative** diagram, to the right, *informally* carries out the `union` construction that results in the elaborated code below.

⋆ ⋆ ⋆

Invoke `union` with `:adjoin-retract`$_i$ `"new-function-name"` to use a new name, or `nil` instead of a string to omit the retract —as was done for `extended-by` earlier.

⋆ ⋆ ⋆

Whew, a worked-out example!

The user manual contains full details and an implementation of intersection, pullback, as well.

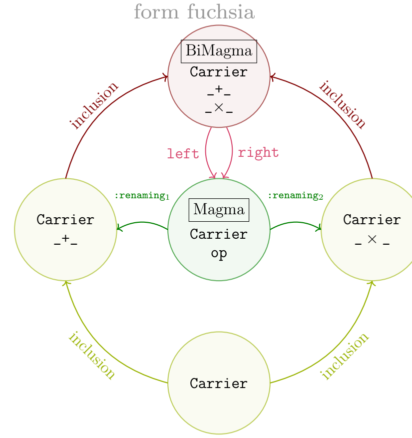
Bimagmas: Two magmas sharing the same carrier

```
BiMagma = Magma union Magma :renaming₁ "op to _+_" :renaming₂
↪  "op to _×_"  :adjoin-retract₁ "left" :adjoin-retract₂
↪  "right"
```

Elaboration

```
record BiMagma : Set₁ where
    field Carrier : Set
    field _+_     : Carrier → Carrier → Carrier

    toType : let View X = X in View Type
    toType = record {Carrier = Carrier}

    field _×_     : Carrier → Carrier → Carrier

    left : let View X = X in View Magma
    left = record {Carrier = Carrier;op = _+_}

    right : let View X = X in View Magma
    right = record {Carrier = Carrier;op = _×_}
```

Given green, yield yellow, require red, form fuchsia



Idempotence. The main reason that the construction is named ‘union’ instead of ‘pushout’ is that, modulo adjoined retracts, it is idempotent. For example, `Magma union Magma` ≈ `Magma` —this is essentially the previous bi-magma example *but* we are not distinguishing (via `:renaming`$_i$) the two instances of `Magma`.

MagmaAgain = Magma union Magma

```
record MagmaAgain : Set₁ where
    field Carrier : Set
    field op      : Carrier → Carrier → Carrier

    toType : let View X = X in View Type
    toType = record {Carrier = Carrier}

    toMagma : let View X = X in View Magma
    toMagma = record {Carrier = Carrier;op = op}
```

Disjointness. On the other extreme, distinguishing all the names of one of the input objects, we have disjoint sums. In contrast to the above bi-magma, in the example below, we are not distinguishing the two instances of `Magma` 'on the fly' via `:renaming`$_i$ but instead making them disjoint beforehand using `primed` —which is specified informally as `p primed ≈ p :renaming (λ name → name ++ "′")`.

```
Magma′     = Magma primed ⟴ record
SumMagmas = Magma union Magma′ :adjoin-retract₁ nil ⟴ record
```

Elaboration

```
record SumMagmas : Set₁ where
    field Carrier  : Set
    field op       : Carrier → Carrier → Carrier

    toType         : let View X = X in View Type
    toType = record {Carrier = Carrier}

    field Carrier′ : Set
    field op′      : Carrier′ → Carrier′ → Carrier′

    toType′ : let View X = X in View Type
    toType′ = record {Carrier = Carrier′}

    toMagma : let View X = X in View Magma
    toMagma = record {Carrier = Carrier′;op = op′}

    toMagma′ : let View X = X in View Magma′
    toMagma′ = record {Carrier′ = Carrier′;op′ = op′}
```

Before returning to the diamond problem, we show an implementation not so that the reader can see some cleverness —not that we even expect the reader to understand it— but instead to showcase that a sufficiently complicated combinator, which is *not built-in*, can be defined without much difficulty.

(Abridged) Pushout combinator with 4 optional arguments

```
(𝒱 union pf (renaming₁ "") (renaming₂ "") (adjoin-retract₁ t) (adjoin-retract₂ t)

 = "Union the elements of the parent PackageFormer with those of
   the provided PF symbolic name, then adorn the result with two views:
   One to the parent and one to the provided PF.

   If an identifer is shared but has different types, then crash.

   ADJOIN-RETRACTᵢ, for i : 1..2, are the optional names of the resulting
   views. Provide NIL if you do not want the morphisms adjoined."
  :alter-elements (λ es →
    (let* ((p (symbol-name 'pf))
           (es₁ (alter-elements es renaming renaming₁ :adjoin-retract nil))
           (es₂ (alter-elements ($elements-of p) renaming renaming₂
                                :adjoin-retract nil))
           (es′ (-concat es₁ es₂))
           (name-clashes (loop for n in (find-duplicates (mapcar #'element-name
           ↪  es′))
                               for e = (--filter (equal n (element-name it))
                               ↪  es′)
                               unless (--all-p (equal (car e) it) e)
                               collect e))
           (er₁ (if (equal t adjoin-retract₁) (format "to%s" $parent)
                  adjoin-retract₁))
           (er₂ (if (equal t adjoin-retract₂) (format "to%s" p)
                  adjoin-retract₂)))

      (if name-clashes
           (-let [debug-on-error nil]
             (error "%s = %s union %s \n\n\t\t → Error:
                     Elements ‘%s’ conflict!\n\n\t\t\t%s"
                     $name $parent p (element-name (caar name-clashes))
                     (s-join "\n\t\t\t" (mapcar #'show-element (car
                     ↪  name-clashes))))))

  ;; return value
  (-concat es′
           (and adjoin-retract₁ (not er₁) (list (element-retract $parent es :new
           ↪  es₁ :name adjoin-retract₁)))
           (and adjoin-retract₂ (not er₂) (list (element-retract p ($elements-of
           ↪  p) :new es₂ :name adjoin-retract₂)))))))
```

Indeed, the core of the construction lies in the first 12 lines of the `let*` clause; the rest are extra bells-and-whistles —which could have been omitted, by the user, for a faster implementation.

The unabridged definition, on the PackageFormer webpage, has more features. In particular, it accepts additional keyword toggles that dictate how it should behave when name clashes occur; e.g., whether it should halt and report the name clash or whether it should silently perform a name change, according to another provided argument. The additional flexibility is useful for rapid experimentation.

Support for Diamond Hierarchies

A common scenario is extending a structure, say Magma, into orthogonal directions, such as by making its operation associative or idempotent, then closing the resulting diamond by combining them, to obtain a semilattice. However, the orthogonal extensions may involve different names and so the resulting semilattice presentation can only be formed via pushout; below are three ways to form it.

Three ways to get to SemiLattice

```
Semigroup          = Magma postulating "_·_" "associative"
IdempotentMagma    = Magma renaming "_·_ to _⊔_" ⟴ postulating "_⊔_" "idempotent"
↪  :adjoin-retract nil

⊔-SemiLattice     = Semigroup union IdempotentMagma :renaming₁ "_·_ to _⊔_"
·-SemiLattice     = Semigroup union IdempotentMagma :renaming₂ "_⊔_ to _·_"
↑-SemiLattice     = Semigroup union IdempotentMagma :renaming₁ "_·_ to _↑_" :renaming₂ "_⊔_ to
↪  _↑_"
```

Application: Granular (Modular) Hierarchy for Rings

We will close with the classic example of forming a ring structure by combining two monoidal structures. This example also serves to further showcase how using postulating can make for more granular, modular, developments.

```
Additive            = Magma renaming "_·_ to _+_" ⟴
↪  postulating "_+_" "commutative"  :adjoin-retract nil ⟴
↪  record

Multiplicative      = Magma renaming "_·_ to _×_"
↪  :adjoin-retract nil ⟴ record

AddMult             = Additive union Multiplicative ⟴ record

AlmostNearSemiRing = AddMult ⟴ postulating "_×_"
↪  "distributiveˡ" :using "_+_" ⟴ record
```

Elaboration

```
record AlmostNearSemiRing : Set₁ where
    field Carrier : Set
    field _+_     : Carrier → Carrier → Carrier

    toType : let View X = X in View Type
    toType = record {Carrier = Carrier}

    toMagma : let View X = X in View Magma
    toMagma = record {Carrier = Carrier;op = _+_}

    field comm       : ∀ x y   → _+_ x y ≡ _+_ y x
    field _×_        : Carrier → Carrier → Carrier

    toAdditive : let View X = X in View Additive
    toAdditive = record {Carrier = Carrier;_+_ =
    ↪  _+_;comm = comm}

    toMultiplicative : let View X = X in View
    ↪  Multiplicative
    toMultiplicative = record {Carrier = Carrier;_×_ =
    ↪  _×_}

    field distˡ      : ∀ x y z → _×_ x (_+_ y z) ≡ _+_
    ↪  (_×_ x y) (_×_ x z)
```

This example, as well as mitigating diamond problems, show that the implementation outlined is reasonably well-behaved.

1.3.5 Duality

Maps between grouping mechanisms are sometimes called *views*, which are essentially an internalisation of the *variationals* in our system. A useful view is that of capturing the heuristic of *dual concepts*, e.g., by changing the order of arguments in an operation. Classically in Agda, duality is *utilised* as follows:

The **dual**, or opposite, of a binary operation `_·_ : X → Y → Z` is the operation `_·ᵒᵖ_ : Y → X → Z` defined by `x ·ᵒᵖ y = y · x`.

1. Define a *parameterised* module `R _·_` for the desired ideas **on** the operation `_·_`.

Example

```
module R (_·_ : X → Y → Z) where
    ·-isLeftId : X → Set
    ·-isLeftId e = ∀ {x} → e · x ≡ x
```

2. Define a shallow (parameterised) module `R`^{op} `_·_` that essentially only opens `R _·`^{op}`_` and renames the concepts in `R` with dual names.

Continuing...

```
module Rᵒᵖ (_·_ : X → Y → Z) where
    public open R _·_
        renaming (·-isLeftId to ·-isRightId)
```

The RATH-Agda [12] library performs essentially this approach, for example for obtaining `UpperBounds` from `LowerBounds` in the context of an ordered set. Moreover, since category theory can serve as a foundational system of reasoning (logic) and implementation (programming), the idea of duality immediately applies to produce "two for one" theorems and programs.

The ubiquity of duality!

[12] Wolfram Kahl. *Relation-Algebraic Theories in Agda.* 2018. URL: `http://relmics.mcmaster.ca/RATH-Agda/` (visited on 10/12/2018)

Unfortunately, this means that any record definitions in `R` must have their field names be sufficiently generic to play *both* roles of the original and the dual concept. However, well-chosen names come at an upfront cost: One must take care to provide sufficiently generic names and account for duality at the outset, irrespective of whether one *currently* cares about the dual or not; otherwise when the dual is later formalised, then the names of the original concept must be refactored throughout a library and its users. This is not the case using `PackageFormer`.

Admittedly, RATH-Agda's names are well-chosen; e.g., `value`, `bound`$_i$, `universal` to denote a `value` that is a lower/upper `bound` of two given elements, satisfying a least upper bound or greatest lower bound `universal` property.

Consider the following heterogeneous algebra —which is essentially the main example of section 1.2 but missing the associativity field.

Left unital actions

```
PackageFormer LeftUnitalAction : Set₁ where
  Scalar : Set
  Vector : Set
  _·_      : Scalar → Vector → Vector
  𝟙        : Scalar
  leftId   : {x : Vector} → 𝟙 · x ≡ x

-- Let's reify this as a valid Agda record declaration
LeftUnitalActionR  = LeftUnitalAction ⟴ record
```

Informally, one now 'defines' a right unital action by duality, flipping the binary operation and renaming `leftId` to be `rightId`. Such informal parlance is in-fact nearly formally, as the following:

Right unital actions —mechanically by duality

```
RightUnitalActionR = LeftUnitalActionR flipping "_·_" :renaming "leftId to rightId" ⟴ record
```

Of-course the resulting representation is semantically identical to the previous one, and so it is furnished with a to*Parent* mapping:

```
forget : RightUnitalActionR → LeftUnitalActionR
forget = RightUnitalActionR.toLeftUnitalActionR
```

Likewise, for the RATH-Agda library's example from above, to define semi-lattice structures by duality:

import Data.Product as P

```
PackageFormer JoinSemiLattice : Set₁ where
  Carrier : Set
  _⊑_     : Carrier → Carrier → Set

  refl    : ∀ {x}     → x ⊑ x
  trans   : ∀ {x y z} → x ⊑ y → y ⊑ z → x ⊑ z
  antisym : ∀ {x y}   → x ⊑ y → y ⊑ x → x ≡ y

  _⊔_     : Carrier → Carrier → Carrier
  ⊔-lub   : ∀ {x y z} → x ⊑ z → y ⊑ z → (x ⊔ y) ⊑ z
  ⊔-lub˘  : ∀ {x y z} → (x ⊔ y) ⊑ z  →  x ⊑ z  P.×  y ⊑ z

JoinSemiLatticeR = JoinSemiLattice record
MeetSemiLatticeR = JoinSemiLatticeR flipping "_⊑_" :renaming "_⊔_ to _⊓_; ⊔-lub to ⊓-glb"
```

In this example, besides the map from meet semi-lattices to join semi-lattices, the types of the dualised names, such as `⊓-glb`, are what one would expect were the definition written out explicitly:

Checking the types of the duals

```
module woah (M : MeetSemiLatticeR) where
  open MeetSemiLatticeR M

  lub_dual_type : ∀ {x y z} → z ⊑ x → z ⊑ y → z ⊑ (x ⊓ y)
  lub_dual_type = ⊓-glb

  trans_dual_type : let _⊒_ = λ x y → y ⊑ x
                    in ∀ {x y z} → x ⊒ y → y ⊒ z → x ⊒ z
  trans_dual_type = trans
```

1.3.6 Extracting Little Theories

The `extended-by` variational allows Agda users to easily employ the *tiny theories* [5] approach to library design: New structures are built from old ones by augmenting one concept at a time —as shown below— then one uses mixins such as `union` to obtain a complex structure. This approach lets us write a program, or proof, in a context that only provides what is *necessary* for that program-proof and nothing more. In this way, we obtain *maximal generality* for re-use! This approach can be construed as *the interface segregation principle* [14, 6] : *No client should be forced to depend on methods it does not use.*

Tiny Theories Example

```
PackageFormer Empty : Set₁ where {- No elements -}
Type  = Empty extended-by "Carrier : Set"
Magma = Type  extended-by "_·_ : Carrier → Carrier → Carrier"
CommutativeMagma = Magma extended-by "comm : {x y : Carrier} →  x · y  ≡  y · x"
```

[5] William M. Farmer, Joshua D. Guttman, and F. Javier Thayer. "Little theories". In: *Automated Deduction—CADE-11*. Ed. by Deepak Kapur. Berlin, Heidelberg: Springer Berlin Heidelberg, 1992, pp. 567–581. ISBN: 978-3-540-47252-0

[14] Robert C. Martin. *Design Principles and Design Patterns*. Ed. by Deepak Kapur. 1992. URL: https://fi.ort.edu.uy/innovaportal/file/2032/1/design_principles.pdf (visited on 10/19/2018)

[6] Eric Freeman and Elisabeth Robson. *Head first design patterns - your brain on design patterns*. O'Reilly, 2014. ISBN: 978-0-596-00712-6. URL: http://www.oreilly.de/catalog/hfdesignpat/index.html

However, life is messy and sometimes one may hurriedly create a structure, then later realise that they are being forced to depend on unused methods. Rather than throw a `not implemented` exception or leave them undefined, we may use the `keeping` variational to **extract the smallest well-formed sub-PackageFormer that mentions a given list of identifiers**. For example, suppose we quickly formed `Monoid` **monolithicaly** as presented at the start of section 1.3.1, but later wished to utilise other substrata. This is easily achieved with the following declarations.

Extracting Substrata from a Monolithic Construction

```
Empty′        = Monoid keeping ""
Type′         = Monoid keeping "Carrier"
Magma′        = Monoid keeping "_·_"
Semigroup′    = Monoid keeping "assoc"
PointedMagma′ = Monoid keeping "▯; _·_"
                 -- This is just “ keeping: Carrier; _·_; ▯ ”
```

Even better, we may go about deriving results —such as theorems or algorithms— in familiar settings, such as `Monoid`, only to realise that they are written in **settings more expressive than necessary**. Such an observation no longer need to be found by inspection, instead it may be derived mechanically.

Specialising a result from an expressive setting to the minimal necessary setting

```
LeftUnitalMagma = Monoid keeping "▯-unique" ⟴ record
```

This expands to the following theory, minimal enough to derive `▯-unique`.

Elaboration

```
record LeftUnitalMagma : Set₁ where

  field
    Carrier : Set
    _·_     : Carrier → Carrier → Carrier
    ▯       : Carrier
    leftId  : {x : Carrier} → ▯ · x  ≡ x

  ▯-unique     : ∀ {e} (lid : ∀ {x} → e · x ≡ x) (rid : ∀ {x} → x · e ≡ x) → e ≡ ▯
  ▯-unique lid rid = ≡.trans (≡.sym leftId) rid
```

Surprisingly, in some sense, `keeping` let's us apply the interface segregation principle, or 'little theories', **after the fact** —this is also known as *reverse mathematics*.

1.3.7 200+ theories —one line for each

In order to demonstrate the **immediate practicality** of the ideas embodied by `PackageFormer`, we have implemented a list of mathematical concepts from universal algebra —which is useful to computer science in the setting of specifications. The list of structures is adapted from the source of a MathScheme library, which in turn was inspired

People should enter terse, readable, specifications that expand into useful, typecheckable, code that may be dauntingly larger in textual size.

by web lists of Peter Jipsen, John Halleck, and many others from Wikipedia and nLab [2, 3] . Totalling over 200 theories which elaborate into nearly 1500 lines of typechecked Agda, this demonstrates that our systems works; the **750% efficiency savings** speak for themselves.

[2] Jacques Carette and Russell O'Connor. "Theory Presentation Combinators". In: *Intelligent Computer Mathematics* (2012), pp. 202–215. DOI: 10.1007/978-3-642-31374-5_14

[3] Jacques Carette et al. *The MathScheme Library: Some Preliminary Experiments.* 2011. arXiv: 1106.1862v1 [cs.MS]

The 200+ one line specifications and their ~1500 lines of elaborated typechecked Agda can be found on PackageFormer's webpage.

https://alhassy.github.io/next-700-module-systems

If anything, this elaboration demonstrates our tool as a useful engineering result. The main novelty being the ability for library users to extend the collection of operations on packages, modules, and then have it immediately applicable to Agda, an **executable** programming language.

Unlike other systems, PackageFormer does not come with a static set of module operators —it grows dynamically, possibly by you, the user.

Since the resulting **expanded code is typechecked** by Agda, we encountered a number of places where non-trivial assumptions accidentally got-by the MathScheme team. For example, in a number of places, an arbitrary binary operation occurred multiple times leading to ambiguous terms, since no associativity was declared. Even if there was an implicit associativity criterion, one would then expect multiple copies of such structures, one axiomatisation for each parenthesisation. Nonetheless, we are grateful for the source file provided by the MathScheme team.

MathScheme's design hierarchy raised certain semantic concerns that we think are out-of-place, but we chose to leave them as is —e.g., one would think that a "partially ordered magma" would consist of a set, an order relation, and a binary operation that is monotonic in both arguments; however, `PartiallyOrderedMagma` instead comes with a single monotonicity axiom which is only equivalent to the two monotonicity claims in the setting of a monoidal operation.

1.4 Contributions: From Theory to Practice

The `PackageFormer` implements the ideas of Chapters **??** and **??**. As such, as an editor extension, it is mostly **language agnostic** and could be altered to work with other languages such as Coq, Idris [1] , and even Haskell [13]. The `PackageFormer` implementation has the following useful properties.

[1] Edwin Brady. *Type-driven Development With Idris.* Manning, 2016. ISBN: 9781617293023. URL: http://www.worldcat.org/isbn/9781617293023

[13] Sam Lindley and Conor McBride. "Hasochism: the pleasure and pain of dependently typed haskell programming". In: *Proceedings of the 2013 ACM SIGPLAN Symposium on Haskell, Boston, MA, USA, September 23-24, 2013.* Ed. by Chung-chieh Shan. ACM, 2013, pp. 81–92. ISBN: 978-1-4503-2383-3. DOI: 10.1145/2503778.2503786. URL: https://doi.org/10.1145/2503778.2503786

1. Expressive & extendable specification language for the library developer.

   ◇ Our meta-primitives give way to the ubiquitous module combinators of Table **??**.

   ◇ E.g., from a theory we can derive its homomorphism type, signature, its termtype, etc; we generate useful constructions inspired from universal algebra and seen in the wild —see Chapter ??.

- ⋄ An example of the freedom allotted by the extensible nature of the system is that combinators defined by library developers can, say, utilise auto-generated names when names are irrelevant, use 'clever' default names, and allow end-users to supply desirable names on demand using Lisps' keyword argument feature —see section 1.3.4.

2. Unobtrusive and a tremendously simple interface to the end user.

   - ⋄ Once a library is developed using (the current implementation of) `PackageFormer`, the end user only needs to reference the resulting generated Agda, without any knowledge of the existence of `PackageFormer`.
   - ⋄ We demonstrates how end-users can build upon a library by using *one line* specifications, by reducing over 1500 lines of Agda code to nearly 200 specifications using `PackageFormer` syntax.

Generated modules are necessarily 'flattened' for typechecking with Agda —see section 1.3.1.

3. Efficient: Our current implementation processes over 200 specifications in ~ 3 seconds; yielding typechecked Agda code *which* is what consumes the majority of the time.

Moreover, all of this happens in the *background* preceeding the ussual typechecking command, `C-c C-l`.

4. Pragmatic: Common combinators can be defined for library developers, and be furnished with concrete syntax for use by end-users.

5. Minimal: The system is essentially invariant over the underlying type system; with the exception of the meta-primitive `:waist` which requires a dependent type theory to express 'unbundling' component fields as parameters.

6. Demonstrated expressive power *and* use-cases.

   Over 200 modules are formalised as one-line specifications!

   - ⋄ Common boiler-plate idioms in the standard Agda library, and other places, are provided with terse solutions using the `PackageFormer` system.
     - ∘ E.g., automatically generating homomorphism types and wholesale renaming fields using a single function —see section .

7. Immediately useable to end-users *and* library developers.

   - ⋄ We have provided a large library to experiment with — thanks to the MathScheme group for providing an adaptable source file.

In the online user manual, we show how to formulate module combinators using a simple and straightforward subset of Emacs Lisp —a terse introduction to Lisp is provided.

Recall that we alluded —in the introduction to section 1.3— that we have a categorical structure consisting of `PackageFormers` as objects and those variationals that are signature morphisms. While this can be a starting point for a semantics for `PackageFormer`, we will instead pursue a *mechanised semantics*. That is, we shall encode (part of) the syntax of `PackageFormer` as Agda functions, thereby giving it not only a semantics but rather a life in a familar setting and lifting it from the status of *editor extension* to *language library*.

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