Tarea #3 Relatividad General

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1 Derivada Covariante

2 Tensor de Riemann

a) Primer Metodo Tenemos que

$$R^\rho_{\sigma\mu\nu}V^\sigma = (\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)V^\rho \qquad \nabla_\mu V^\rho = \partial_\mu V^\rho + \Gamma^\rho_{\mu\sigma}V^\sigma$$

Entonces

$$
\begin{aligned}
R^\rho_{\sigma\mu\nu}V^\rho &= \nabla_\mu(\partial_\nu V^\rho + \Gamma^\rho_{\nu\lambda}V^\lambda) - \nabla_\nu(\partial_\mu V^\rho + \Gamma^\rho_{\mu\lambda}V^\lambda) \\
&= \partial_\mu\nabla_\mu V^\rho + \Gamma^\rho_{\nu\lambda}\nabla_\mu V^\lambda - \partial_\mu\nabla_\nu V^\rho - \Gamma^\rho_{\mu\lambda}\nabla_\nu V^\lambda \\
&= \partial_\mu\partial_\nu V^\rho - \partial_\mu\partial_\nu V^\rho + \partial_\nu\Gamma^\rho_{\mu\lambda}V^\lambda - \partial_\mu\Gamma^\rho_{\nu\lambda}V^\lambda + \Gamma^\rho_{\nu\lambda}\partial_\mu V^\lambda \\
&\quad + \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}V^\sigma - \Gamma^\rho_{\mu\lambda}\partial_\nu V^\lambda - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma}V^\sigma \\
&= \{(\partial_\nu\Gamma^\rho_{\mu\sigma} - \partial_\mu\Gamma^\rho_{\mu\sigma}) + (\Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma})\}V^\sigma
\end{aligned}
$$

Por lo tanto:

$$R^\rho_{\sigma\mu\nu} = \partial_\nu\Gamma^\rho_{\mu\sigma} - \partial_\mu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma}$$

Segundo Metodo $x^1 = a, x^1 = b$, y $x^2 = b$, y $x^2 = b + \delta b$. Un vector $\vec{V}$ definido en A es transportado paralelamente
0 tiene la forma componente

$$\frac{\partial V^\alpha}{\partial x^1} = -\Gamma^\alpha_{\mu 1}V^\mu$$

Integrando esto de A a B se obtiene

$$V^\alpha(B) = V^\alpha(A) + \int_A^B \frac{\partial V^\alpha}{\partial x^1}dx^1 = V^\alpha(A) - \int_{x^2=b}\Gamma^\alpha_{\mu 1}V^\mu dx^1$$

Deforma similar:

$$V^\alpha(C) = V^\alpha(B) - \int_{x^1=a+\delta a}\Gamma^\alpha_{\mu 2}V^\mu dx^2$$

$$V^\alpha(D) = V^\alpha(C) + \int_{x^2=b+\delta b}\Gamma^\alpha_{\mu 1}V^\mu dx^1$$

La integral en la última ecuación tiene un signo diferente porque la dirección de transporte de C a D
está en la dirección negativa de x^1. Tambien:

$$V^\alpha(A_{final}) = V^\alpha(D) + \int_{x^1=a} \Gamma^\alpha_{\mu 2} V^\mu dx^2$$

El cambio neto en $V^\alpha(A)$ es un vector δV^α:

$$\delta V^\alpha = V^\alpha(A_{final}) - V^\alpha(A_{inicial})$$

$$= \int_{x^1=a} \Gamma^\alpha_{\mu 2} V^\mu dx^2 - \int_{x^1=a+\delta a} \Gamma^\alpha_{\mu 2} V^\mu dx^2 + \int_{x^2=b+\delta b} \Gamma^\alpha_{\mu 1} V^\mu dx^1 - \int_{x^2=b} \Gamma^\alpha_{\mu 1} V^\mu dx^1$$

Si combinamos las integrales sobre variables de integración similares y trabajamos hasta el primer orden en la separación de los caminos, obtenemos:

$$\delta V^\alpha \sim - \int_b^{b+\delta b} \delta a \frac{\partial}{\partial x^1}(\Gamma^\alpha_{\mu 2} V^\mu) dx^2 + \int_a^{a+\delta a} \delta b \frac{\partial}{\partial x^2}(\Gamma^\alpha_{\mu 1} V^\mu) dx^1$$

$$\approx \delta a \delta b \left[-\frac{\partial}{\partial x^1}(\Gamma^\alpha_{\mu 2} V^\mu) + \frac{\partial}{\partial x^2}(\Gamma^\alpha_{\mu 1} V^\mu) \right]$$

Utilizando $\frac{\partial V^\alpha}{\partial x^1} = -\Gamma^\alpha_{\mu 1} V^\mu$

$$\delta V^\alpha = \delta a \delta b \left[\Gamma^\alpha_{\mu 1,2} - \Gamma^\alpha_{\mu 2,1} + \Gamma^\alpha_{\lambda 2}\Gamma^\lambda_{\mu 1} - \Gamma^\alpha_{\lambda 1}\Gamma^\lambda_{\mu 2} \right] V^\mu$$

Si usáramos coordenadas generales x^σ y x^λ:

$$\delta V^\alpha = \delta a \, \delta b \left[\Gamma^\alpha_{\sigma\mu,\lambda} - \Gamma^\alpha_{\lambda\mu,\sigma} + \Gamma^\alpha_{\nu\sigma}\Gamma^\nu_{\lambda\mu} - \Gamma^\alpha_{\nu\lambda}\Gamma^\nu_{\sigma\mu} \right] V^\mu$$

δV^α depende linealmente de $\delta a \, \epsilon_\sigma$ y $\delta b \, \epsilon_\lambda$. Además, ciertamente también depende linealmente en la Ecuación de V^α en sí mismo y en $\delta\epsilon_\sigma$, que es la forma uno base que da δV^α desde el vector $\delta \mathbf{V}$. Por lo tanto, tenemos el siguiente resultado:

$$R^\alpha_{\beta\mu\nu} := \Gamma^\alpha_{\beta\nu,\mu} - \Gamma^\alpha_{\beta\mu,\nu} + \Gamma^\alpha_{\sigma\mu}\Gamma^\sigma_{\beta\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\beta\mu}$$

b)

- Por definicion:

$$R_{\rho\sigma\mu\nu} = \partial_\mu \Gamma_{\rho\nu\sigma} - \partial_\nu \Gamma_{\rho\mu\sigma} + \Gamma_{\rho\mu\lambda}\Gamma_{\lambda\nu\sigma} - \Gamma_{\rho\nu\lambda}\Gamma_{\lambda\mu\sigma}$$

Intercambiando los ultimos dos indices:

$$R_{\rho\sigma\nu\mu} = \partial_\nu \Gamma_{\rho\mu\sigma} - \partial_\mu \Gamma_{\rho\nu\sigma} + \Gamma_{\rho\nu\lambda}\Gamma_{\lambda\mu\sigma} - \Gamma_{\rho\mu\lambda}\Gamma_{\lambda\nu\sigma} = -(\partial_\mu \Gamma_{\rho\nu\sigma} - \partial_\nu \Gamma_{\rho\mu\sigma} + \Gamma_{\rho\mu\lambda}\Gamma_{\lambda\nu\sigma} - \Gamma_{\rho\nu\lambda}\Gamma_{\lambda\mu\sigma}) = -R_{\rho\sigma\mu\nu}$$

- De igual forma:

$$R_{\sigma\rho\nu\mu} = \partial_\mu \Gamma_{\sigma\nu\rho} - \partial_\nu \Gamma_{\sigma\mu\rho} + \Gamma_{\sigma\mu\lambda}\Gamma_{\lambda\nu\rho} - \Gamma_{\sigma\nu\lambda}\Gamma_{\lambda\mu\rho} = -R_{\rho\sigma\mu\nu}$$

-

$$R_{ijkl} + R_{iklj} + R_{iljk} = \partial_k \Gamma_{ilj} - \partial_l \Gamma_{ikj} + \Gamma_{ikn}\Gamma_{nlj} - \Gamma_{iln}\Gamma_{nkj}$$
$$+ \partial_l \Gamma_{ijk} - \partial_j \Gamma_{ilk} + \Gamma_{iln}\Gamma_{njk} - \Gamma_{ijn}\Gamma_{nlk}$$
$$+ \partial_j \Gamma_{ikl} - \partial_k \Gamma_{ijl} + \Gamma_{ijn}\Gamma_{nkl} - \Gamma_{ikn}\Gamma_{njl}$$

Como $\Gamma_{ijl} = \Gamma_{njl}$:

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0$$

- Aplicando derivada convariante a lo anterior

$$\nabla_m R_{ijkl} + \nabla_m R_{iklj} + \nabla_m R_{iljk} = 0$$

Como m es un indice mudo:

$$\nabla_m R_{ijkl} + \nabla_n R_{iklj} + \nabla_r R_{iljk} = 0$$

-

$$R_{\alpha\beta\mu\nu} - R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu} - (-R_{\nu\alpha\beta\mu} - R_{u\beta\alpha})$$
$$= R_{\alpha\beta\mu\nu} - R_{\alpha\mu\nu\beta} - R_{\beta\mu\alpha}$$
$$= R_{\alpha\beta\mu\nu} - (-R_{\alpha\beta\mu\nu} - R_{\alpha\beta\nu}) + (-(-R_{\alpha\beta\mu\nu} - R_{\alpha\beta\mu\nu} - R_{\beta\alpha\nu}))$$
$$= R_{\alpha\beta\mu\nu} + R_{\alpha\beta\mu\nu} + R_{\alpha\beta\mu\nu} + R_{\alpha\beta\mu\nu} + R_{\beta\alpha\nu})$$
$$= -R_{\alpha\beta\mu\nu} + R_{\alpha\beta\mu\nu} + R_{\beta\alpha\nu})$$
$$= (-R_{\alpha\beta\mu\nu} - R_{\alpha\beta\mu\nu}) + R_{\beta\alpha\nu})$$
$$= R_{\alpha\beta\mu\nu} + R_{\beta\alpha\nu}$$
$$= -R_{\alpha\beta\mu\nu} + R_{\beta\alpha\nu}$$

Ahora intercambiamos los nombres de tanto $\alpha\beta$ y $\mu\nu$. A la izquierda, esto da dos signos menos, pero a la derecha solo uno:

$$R_{\alpha\beta\mu\nu} - R_{\mu\nu\alpha\beta} = -R_{\alpha\beta\mu\nu} + R_{\beta\alpha\nu}$$
$$R_{\beta\alpha\nu} - R_{\mu\nu\alpha\beta} = -R_{\beta\alpha\nu} + R_{\alpha\beta\mu\nu}$$
$$= (-1)^2(R_{\alpha\beta\mu\nu} - R_{\mu\nu\alpha\beta}) = +R_{\alpha\beta\mu\nu} - R_{\beta\alpha\nu}$$
$$= -(-R_{\alpha\beta\mu\nu} - R_{\mu\nu\alpha\beta})$$

Esta diferencia por lo tanto desaparece, y tenemos simetría bajo el intercambio de los pares,

$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}$$

3 Ecuaciones de Einstein

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0$$

a) Multiplicando por $g^{\mu\nu}$

$$g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\mu\nu}R + \Lambda g^{\mu\nu}g_{\mu\nu} = 0$$

Como $g^{\mu\nu}g_{\mu\nu} = 4$

$$R - 2R + \Lambda 4 = 0$$
$$R = 4\Lambda$$

Reemplazando $R = 4\Lambda$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(4\Lambda) + \Lambda g_{\mu\nu} = 0$$
$$R_{\mu\nu} - \Lambda g_{\mu\nu} = 0 \qquad \rightarrow \qquad R_{\mu\nu} = \Lambda g_{\mu\nu}$$

Por lo tanto

$$R_{\mu\nu} = 3kg_{\mu\nu} \qquad \text{Con} \qquad k = \frac{\Lambda}{3}$$

b)

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

i) Utilizando *SageManifolds*

```
# Definir la variedad
M = Manifold(4, 'R^4', start_index=1)
# Definir las coordenadas
c_spher.<t,r,th,ph> = M.chart(r't:(0,+oo) r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi')
# Definir la función a(t)
f = function('f')(r)

# Definir la métrica g de FRW
g = M.metric('g')
g[1,1] = -f
g[2,2] = 1/f
g[3,3] = r^2
g[4,4] = r^2 * sin(th)^2
#inversa de la metrica
ginv = g.inverse()
```

Calculamos los simbolos de christoffel por el metodo indicado:

```
# Calcular las derivadas parciales \partial_\mu g_{\nu\sigma}
partial_g = [[[[None for _ in range(4)] for _ in range(4)] for _ in range(4)] for _ in range(4)]

for mu in range(4):
    for nu in range(4):
        for sigma in range(4):
            partial_g[mu][nu][sigma] = g[nu+1, sigma+1].diff(c_spher[mu+1])

# Calcular los símbolos de Christoffel \Gamma^\lambda_{\mu\nu}
Gamma = [[[None for _ in range(4)] for _ in range(4)] for _ in range(4)]

for lam in range(4):
    for mu in range(4):
        for nu in range(4):
            Gamma[lam][mu][nu] = sum(
                ginv[lam+1, sigma+1] * (partial_g[mu][nu][sigma] + partial_g[nu][mu][sigma] - partia
                for sigma in range(4)
            ) / 2

# Mostrar los símbolos de Christoffel diferentes de 0
for lam in range(4):
    for mu in range(4):
        for nu in range(4):
            if Gamma[lam][mu][nu] != 0:
                print(f"Gamma^{lam+1}_{mu+1}{nu+1} = {Gamma[lam][mu][nu]}")
```

```
Gamma^1_12 = 1/2*d(f)/dr/f(r)
Gamma^1_21 = 1/2*d(f)/dr/f(r)
Gamma^2_11 = 1/2*f(r)*d(f)/dr
Gamma^2_22 = -1/2*d(f)/dr/f(r)
Gamma^2_33 = -r*f(r)
Gamma^2_44 = -r*f(r)*sin(th)^2
Gamma^3_23 = 1/r
Gamma^3_32 = 1/r
Gamma^3_44 = -cos(th)*sin(th)
Gamma^4_24 = 1/r
Gamma^4_34 = cos(th)/sin(th)
Gamma^4_42 = 1/r
Gamma^4_43 = cos(th)/sin(th)
```

c)

i) Utilizando *SageManifolds*, definimos la metrica:

```
# Definir la variedad
M = Manifold(4, 'R^4', start_index=1)
# Definir las coordenadas
c_spher.<t,r,th,ph> = M.chart(r't:(0,+oo) r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi')
# Definir la función a(t)
f = function('f')(r)

# Definir la métrica g de FRW
g = M.metric('g')
g[1,1] = -f
g[2,2] = 1/f
g[3,3] = r^2
g[4,4] = r^2 * sin(th)^2
#inversa de la metrica
ginv = g.inverse()
```

Calculamos el tensor de Riemann con $R_{\rho\sigma\mu\nu} = \partial_\mu \Gamma_{\rho\nu\sigma} - \partial_\nu \Gamma_{\rho\mu\sigma} + \Gamma_{\rho\mu\lambda}\Gamma_{\lambda\nu\sigma} - \Gamma_{\rho\nu\lambda}\Gamma_{\lambda\mu\sigma}$

```
riem = g.riemann()
print(riem.display_comp(c_spher.frame(), c_spher, only_nonredundant=True))
```

```
Riem(g)^t_r,t,r = -1/2*d^2(f)/dr^2/f(r)
Riem(g)^t_th,t,th = -1/2*r*d(f)/dr
Riem(g)^t_ph,t,ph = -1/2*r*sin(th)^2*d(f)/dr
Riem(g)^r_t,t,r = -1/2*f(r)*d^2(f)/dr^2
Riem(g)^r_th,r,th = -1/2*r*d(f)/dr
Riem(g)^r_ph,r,ph = -1/2*r*sin(th)^2*d(f)/dr
Riem(g)^th_t,t,th = -1/2*f(r)*d(f)/dr/r
Riem(g)^th_r,r,th = 1/2*d(f)/dr/(r*f(r))
Riem(g)^th_ph,th,ph = -(f(r) - 1)*sin(th)^2
Riem(g)^ph_t,t,ph = -1/2*f(r)*d(f)/dr/r
Riem(g)^ph_r,r,ph = 1/2*d(f)/dr/(r*f(r))
Riem(g)^ph_th,th,ph = f(r) - 1
```

Calculamos el tensor de Riemann con $R_{\mu\nu} = g_{\rho\sigma}R^\rho_{\sigma\mu\nu}$

```
ricci = riem["^s_msn"]
print(ricci.display())
```

Resultado:

$$
R_{mn} = \begin{pmatrix}
\frac{rf(r)\frac{\partial^2 f}{\partial r^2} + 2f(r)\frac{\partial f}{\partial r}}{2r} & 0 & 0 & 0 \\
0 & -\frac{r\frac{\partial^2 f}{\partial r^2} + 2\frac{\partial f}{\partial r}}{2rf(r)} & 0 & 0 \\
0 & 0 & -r\frac{\partial f}{\partial r} - f(r) + 1 & 0 \\
0 & 0 & 0 & -\left(r\frac{\partial f}{\partial r} + f(r) - 1\right)\sin(\theta)^2
\end{pmatrix}
$$

Calculamos el tensor de Riemann con $R_{\mu\nu} = g_{\rho\sigma}R^{\rho}_{\sigma\mu\nu}$

```
ricciinv = ginv["^mr"]*(ginv["^ns"]*ricci["_rs"])["_r^n"]
ricci_scalar = g["_mn"]*ricciinv["^mn"]
print(f'R = {latex(ricci_scalar.display())}')
```

Resultado:

$$
R = \begin{array}{ccc}
R^4 & \longrightarrow & \mathbb{R} \\
(t, r, \theta, \phi) & \longmapsto & -\frac{r^2\frac{\partial^2 f}{\partial r^2} + 4r\frac{\partial f}{\partial r} + 2f(r) - 2}{r^2}
\end{array}
$$

d) Supongamos que la ecuación de Einstein tiene una solución como en el Problema (b). Encontrar $f(r)$.

Partimos de la métrica dada en el problema 3.3.b:

$$
ds^2 = -f(r)\,dt^2 + \frac{1}{f(r)}\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right)
$$

Las componentes de la métrica son:

$$
g_{\mu\nu} = \begin{pmatrix}
-f(r) & 0 & 0 & 0 \\
0 & \frac{1}{f(r)} & 0 & 0 \\
0 & 0 & r^2 & 0 \\
0 & 0 & 0 & r^2\sin^2\theta
\end{pmatrix}
\qquad
g^{\mu\nu} = \begin{pmatrix}
-\frac{1}{f(r)} & 0 & 0 & 0 \\
0 & f(r) & 0 & 0 \\
0 & 0 & \frac{1}{r^2} & 0 \\
0 & 0 & 0 & \frac{1}{r^2\sin^2\theta}
\end{pmatrix}
$$

Simbolos de Christoffel:

$$
\Gamma^t_{tr} = \frac{f'(r)}{2f(r)}, \qquad \Gamma^r_{tt} = \frac{f(r)f'(r)}{2}, \qquad \Gamma^r_{rr} = -\frac{f'(r)}{2f(r)}, \qquad \Gamma^r_{\theta\theta} = -rf(r),
$$

$$
\Gamma^r_{\varphi\varphi} = -rf(r)\sin^2\theta, \qquad \Gamma^\theta_{r\theta} = \frac{1}{r}, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \qquad \Gamma^\varphi_{\theta\varphi} = \cot\theta
$$

A continuación, utilizamos los símbolos de Christoffel para calcular el tensor de Ricci $R_{\mu\nu}$:

$$
R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\mu\nu}\Gamma^\sigma_{\lambda\sigma} - \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\lambda}
$$

Calculamos algunas de las componentes no nulas de $R_{\mu\nu}$:

$$
R_{tt} = -\frac{f''(r)}{2} + \frac{f'(r)^2}{4f(r)} + \frac{f'(r)}{r},
$$

$$
R_{rr} = \frac{f''(r)}{2f(r)} - \frac{f'(r)^2}{4f(r)^2} - \frac{f'(r)}{rf(r)},
$$

$$
R_{\theta\theta} = 1 - \frac{rf'(r)}{2} - f(r),
$$

$$
R_{\varphi\varphi} = \left(1 - \frac{rf'(r)}{2} - f(r)\right)\sin^2\theta
$$

La ecuación de Einstein es:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0$$

Calculamos el escalar de curvatura R:

$$R = g^{\mu\nu}R_{\mu\nu}$$

Después de hacer los cálculos correspondientes:

$$R = -f''(r) - \frac{4f'(r)}{r} - \frac{2(1-f(r))}{r^2}$$

Sustituimos R en la ecuación de Einstein y resolvemos para $f(r)$:

$$R_{tt} = -\frac{f''(r)}{2} + \frac{f'(r)^2}{4f(r)} + \frac{f'(r)}{r} = \frac{1}{2}f(r)R - \Lambda f(r)$$

Reorganizamos esta ecuación para resolver la ecuación diferencial para $f(r)$:

$$R_{tt} = -\frac{f''(r)}{2} + \frac{f'(r)^2}{4f(r)} + \frac{f'(r)}{r}$$

$$\frac{1}{2}f(r)R = -\frac{1}{2}f(r)\left(f''(r) + \frac{4f'(r)}{r} + \frac{2(1-f(r))}{r^2}\right)$$

Comparando ambos lados de la ecuación, y simplificando:

$$-\frac{f''(r)}{2} + \frac{f'(r)^2}{4f(r)} + \frac{f'(r)}{r} = \frac{1}{2}f(r)\left(-f''(r) - \frac{4f'(r)}{r} - \frac{2(1-f(r))}{r^2}\right) - \Lambda f(r)$$

Desarrollamos y simplificamos:

$$-\frac{f''(r)}{2} + \frac{f'(r)^2}{4f(r)} + \frac{f'(r)}{r} = -\frac{1}{2}f(r)f''(r) - 2\frac{f(r)f'(r)}{r} - \frac{1}{r^2}f(r)(1-f(r)) - \Lambda f(r)$$

A través de la comparación de términos, observamos que la forma particular de $f(r)$ que satisface la ecuación es:

$$f(r) = 1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}$$