Selected Solutions to
Loring W. Tu's
An Introduction to Manifolds (2nd ed.)

Prepared by Richard G. Ligo

Chapter 1

Problem 1.1: Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(t) = \int_0^t f(s)dt = \int_0^t s^{1/3}dt = \frac{3}{4}t^{4/3}.$$

Show that the function $h(x) = \int_0^x g(t)dt$ is C^2 but not C^3 at $x = 0$.

Proof: Note that $h''(x) = g'(x) = f(x) = x^{1/3}$. As $f(x)$ is C^0, it follows that $h''(x)$ is C^0, so $h(x)$ is C^2. Furthermore, we have that

$$h'''(x) = g''(x) = f'(x) = \begin{cases} \frac{1}{3}x^{-2/3} & \text{for } x \neq 0, \\ \text{undefined} & \text{for } x = 0. \end{cases}$$

This implies that $h'''(x)$ is not C^0 at $x = 0$, so $h(x)$ is not C^3. $\square$

Problem 1.2: Let $f(x)$ be the function on $\mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

(a) Show by induction the for $x > 0$ and $k \geq 0$, the kth derivative $f^{(k)}(x)$ is of the form $p_{2k}(1/x)e^{-1/x}$ for some polynomial $p_{2k}(y)$ of degree $2k$ in y.

Proof: Base case:

Let $k = 0$. Then we immediately have that $f^{(k)}(x) = f^{(0)}(x) = e^{-1/x}$, so $f^{(k)} = p_{2k}(1/x)e^{-1/x}$, where $p_{2k}(1/x) = 1$. Thus, the base case holds.

Inductive step:

Assume that $f^{(k)}(x)$ is of the form $p_{2k}(1/x)e^{-1/x}$ for some polynomial $p_{2k}(y)$. Now consider the following:

$$\begin{aligned}
f^{(k+1)}(x) &= \frac{d}{dx}(f^{(k)}(x)) && \text{Equivalent notation} \\
&= \frac{d}{dx}(p_{2k}(1/x)e^{-1/x}) && \text{Inductive hypothesis} \\
&= \frac{d}{dx}(p_{2k}(1/x))e^{-1/x} + p_{2k}(1/x)\frac{d}{dx}(e^{-1/x}) && \text{Product rule} \\
&= \frac{d}{dx}\left(a_{2k}\left(\frac{1}{x}\right)^{2k} + \cdots\right)e^{-1/x} + \left(a_{2k}\left(\frac{1}{x}\right)^{2k} + \cdots\right)\frac{d}{dx}(e^{-1/x}) && \text{Form of } p_{2k} \\
&= \frac{d}{dx}\left(a_{2k}\left(\frac{1}{x^{2k}}\right) + \cdots\right)e^{-1/x} + \left(a_{2k}\left(\frac{1}{x^{2k}}\right) + \cdots\right)\frac{d}{dx}(e^{-1/x}) && \text{Simplify}
\end{aligned}$$

$$
\begin{aligned}
&= \left(-2ka_{2k}\left(\frac{1}{x^{2k+1}}\right)+\cdots\right)e^{-1/x} + \left(a_{2k}\left(\frac{1}{x^{2k}}\right)+\cdots\right)\frac{1}{x^2}e^{-1/x} && \text{Evaluate derivative}\\
&= \left(-2ka_{2k}\left(\frac{1}{x^{2k+1}}\right)+\cdots\right)e^{-1/x} + \left(a_{2k}\left(\frac{1}{x^{2k+2}}\right)+\cdots\right)e^{-1/x} && \text{Distribute}\\
&= \left(-2ka_{2k}\left(\frac{1}{x^{2k+1}}\right)+\cdots + a_{2k}\left(\frac{1}{x^{2k+2}}\right)+\cdots\right)e^{-1/x} && \text{Factor}\\
&= \left(a_{2k}\left(\frac{1}{x^{2k+2}}\right)-2ka_{2k}\left(\frac{1}{x^{2k+1}}\right)+\cdots\right)e^{-1/x} && \text{Rearrange terms}
\end{aligned}
$$

Thus, we have that $f^{(k+1)}(x) = q_{2k}(1/x)e^{-1/x}$ for some polynomial $q_{2k}(y)$, as desired.

As a result, the induction holds. $\square$

(b) Prove that f is C^∞ on $\mathbb{R}$ and that $f^{(k)}(0) = 0$ for all $k \geq 0$.
Proof: In order to show that f is C^∞, we must show that $f^{(k)}$ is continuous for all $k \geq 0$. As we know that both 0 and $e^{-1/x}$ are both C^∞, we must simply show that $\lim_{x\to 0^+} \frac{d^k}{dx^k}e^{-1/x} = 0$. We know from (a) that $\frac{d^k}{dx^k}e^{-1/x} = p_{2k}(\frac{1}{x})e^{-1/x}$, where $p_{2k}(y)$ is a polynomial of even degree. We know that the exponential decay of $e^{-1/x}$ toward 0 will crush any growth coming from the rational expressions in $p_{2k}(\frac{1}{x})$, so the limit as x approaches 0 from the right will indeed be 0. If one desires to see this more concretely, it could be accomplished (albeit somewhat messily) using a repeated application of l'Hôpital's rule. As k was arbitrary, we then have that f is ineed C^∞. It is also follows immediately from the work above that $f^{(k)}(0) = 0$ for all $k \geq 0$ from the definition of f. $\square$

Problem 1.3: Let $U \in \mathbb{R}^n$ and $V \in \mathbb{R}^n$ be open subsets. A C^∞ map $F : U \to V$ is called an *diffeomorphism* if it is bijective and has a C^∞ inverse $F^{-1} : V \to U$.
(a) Show that the function $f : (-\pi/2, \pi/2) \to \mathbb{R}$, $f(x) = \tan x$, is a diffeomorphism.
Proof: Recall that $\tan x = \frac{\sin x}{\cos x}$, and note that $\cos x > 0$ for all $x \in (-\pi/2, \pi/2)$. Let k be arbitrary, then by repeated application of the quotient rule we have that

$$\frac{d^k}{dx^k}\tan x = \frac{p_k(x)}{\cos^{2k}(x)}, \text{ where } p_k(x) \text{ is products and sums of } \sin x \text{ and } \cos x.$$

As $\cos x > 0$ on $(-\pi/2, \pi/2)$, the denominator will always be non-zero. Furthermore, as $\sin x$ and $\cos x$ are C^∞ we know that $p_k(x)$ will always be defined. Thus, $\frac{d^k}{dx^k}\tan x$ is defined on $(-\pi/2, \pi/2)$ for all k, so $\tan x$ is C^∞.

It can immediately be seen that $\tan x$ is bijective on $(-\pi/2, \pi/2)$ and has the inverse $\tan^{-1} x$.

As a result, we have that $\tan x$ is diffeomorphism on $(-\pi/2, \pi/2)$. $\square$

(b) Let a, b be real numbers with $a < b$. Find a linear function $h : (a, b) \to (-1, 1)$, thus proving that any two finite open sets are diffeomorphic.

Proof: Define $h(x) = \frac{2}{b-a}x + (1 - \frac{2b}{b-a})$, and note that h is a linear function such that $h(a) = -1$ and $h(b) = 1$. As h is linear, we immediately have that h is infinitely differentiable, bijective, and invertible, so h is a diffeomorphism from (a, b) to $(-1, 1)$. $\square$

(c) The exponential function $\exp : \mathbb{R} \to (0, \infty)$ is a diffeomorphism. Use it to show that for any real numbers a and b, the intervals $\mathbb{R}$, (a, ∞), and (∞, b) are diffeomorphic.

Proof: Define the function $\exp_a : \mathbb{R} \to (a, \infty)$ by $x \mapsto \exp(x) + a$, and note that the image of $\exp_a(\mathbb{R}) = (a, \infty)$. As adding the real number a to exp does not affect differentiability, invertibility, or bijectivity (onto the new image), $\exp_a$ will be a diffeomorphism from $\mathbb{R}$ to (a, ∞).

Likewise, we may define the function $-\exp_b : \mathbb{R} \to (-\infty, b)$ by $x \mapsto -\exp(x) + b$, and note that the image of $-\exp_b(\mathbb{R}) = (-\infty, b)$. As multiplying exp by -1 and adding the real number b do not affect diffentiability, , invertibility, or bijectivity (onto the new image), $-\exp_b$ will be a diffeomorphism from $\mathbb{R}$ to $(-\infty, b)$.

As we know that compositions of diffeomorphisms are diffeomorphisms, it follows that $-\exp_b \circ (\exp_a)^{-1} : (a, \infty) \to (-\infty, b)$ is a diffeomorphism, so the intervals (a, ∞) and $(-\infty, b)$ are diffeomorphic. $\square$

Problem 1.4: Show that the map

$$f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)^n \to \mathbb{R}^n,\ f(x_1, \ldots, x_n) = (\tan x_1, \ldots, \tan x_n),$$

is a diffeomorphism.

Proof: As $\tan x$ is bijective, f is bijective coordinate-wise, so f itself is bijective.

We know that if each of the component functions of f is C^∞, then f is C^∞. As each of f component functions is identical, we need only show that an arbitrary component function f_k is C^∞. An abitrary rth partial derivative of f_k will be given by

$$\frac{\partial^r f_k}{\partial x_{i_1} \cdots \partial x_{i_r}} = \begin{cases} 0 & \text{If } \exists \text{ some } i_j \neq k, \\ \frac{\partial^r (\tan x_k)}{(\partial x_k)^r} & \text{If } i_j = k\ \forall j. \end{cases}$$

We know from Problem 1.3 that $\frac{\partial^r (\tan x_k)}{(\partial x_k)^r}$ will be continuous. As 0 is also continuous, it then follows that any rth partial derivative of f_k is continuous. Because our choices of r and k were arbitrary, f_k is C^∞, so f is C^∞.

$f^{-1}(x_1, \ldots, x_n)$ is given by $f^{-1}(x_1, \ldots, x_n) = (\tan^{-1}(x_1), \ldots, \tan^{-1}(x_n))$. Similar to the previous part, we must only show that an arbitrary component function f_k^{-1} is C^∞. An arbitrary rth partial derivative of f_k^{-1} will be given by

$$\frac{\partial^r f_k^{-1}}{\partial x_{i_1} \cdots \partial x_{i_r}} = \begin{cases} 0 & \text{If } \exists \text{ some } i_j \neq k, \\ \frac{\partial^r (\tan^{-1} x_k)}{(\partial x_k)^r} & \text{If } i_j = k \ \forall j. \end{cases}$$

Recall that $\frac{d}{dx} \tan x = \frac{1}{1+x^2}$ and note that this rational function is infinitely differentiable. As 0 is also continuous, we then have that any rth partial derivative of f_k^{-1} is continuous. Because our choices of r and k were arbitrary, f_k^{-1} is C^∞, so f^{-1} is C^∞.

As the above properties hold, it follows that f is a diffeomorphism. $\square$

Problem 1.5: Let $\mathbf{0} = (0, 0)$ be the origin and $B(\mathbf{0}, 1)$ be the open unit disk in $\mathbb{R}^2$. To find a diffeomorphism between $B(\mathbf{0}, 1)$ and $\mathbb{R}^2$, we identify $\mathbb{R}^2$ with the xy-plane in $\mathbb{R}^3$ and introduce the lower open hemisphere

$$S : x^2 + y^2 + (z - 1)^2 = 1,\ z < 1,$$

in $\mathbb{R}^3$ as an intermediate space (Figure 1.4). First note that the map

$$f : B(\mathbf{0}, 1) \to S,\ (a, b) \mapsto (a, b, 1 - \sqrt{1 - a^2 - b^2}),$$

is a bijection.

(a) The *sterographic projection* $g : S \to \mathbb{R}^2$ from $(0, 0, 1)$ is the map that sends a point $(a, b, c) \in S$ to the intersection of the line through $(0, 0, 1)$ and (a, b, c) with the xy-plane. Show that it is given by

$$(a, b, c) \mapsto \left(\frac{a}{1 - c}, \frac{b}{1 - c} \right),\ c = 1 - \sqrt{1 - a^2 - b^2},$$

with inverse

$$(u, v) \mapsto \left(\frac{u}{\sqrt{1 + u^2 + v^2}}, \frac{v}{\sqrt{1 + u^2 + v^2}}, 1 - \frac{1}{\sqrt{1 + u^2 + v^2}} \right).$$

Proof: Let $\gamma(t)$ represent the line through $(0, 0, 1)$ and (a, b, c) projecting onto the xy-plane. The parameterized equation is given as $(ta, tb, 1 + t(c - 1))$. As we desire to determine the values of the first two coordinates when $\gamma(t)$ intersects the xy-plane in $\mathbb{R}^3$, we may set $0 = 1 + t(c - 1)$ to determine the value of t at which this occurs. Solving for t yields

$t = \frac{1}{1-c}$, which when substituted back into the equation gives us $\frac{a}{1-c}$ and $\frac{b}{1-c}$ for the first two coordinates, as desired.

Now let $u = \frac{a}{1-c}$ and $v = \frac{b}{1-c}$. Consider the following manipulation of the expression from the third coordinate:

$$\begin{aligned}
&1 - \frac{1}{\sqrt{1+u^2+v^2}} && \\
&= 1 - \frac{1}{\sqrt{1 + \left(\frac{a}{1-c}\right)^2 + \left(\frac{b}{1-c}\right)^2}} && \text{Substitution} \\
&= 1 - \frac{1}{\sqrt{\frac{(1-c)^2+a^2+b^2}{(1-c)^2}}} && \text{Common denominator} \\
&= 1 - \frac{1-c}{\sqrt{(1-c)^2+a^2+b^2}} && \text{Simplify} \\
&= 1 - \frac{1-c}{1} && \text{Given} \\
&= c && \text{Simplify}
\end{aligned}$$

Thus, we have that the third coordinate equals c, as desired. We may then solve the equation

$$1 - \frac{1}{\sqrt{1+u^2+v^2}} = c \text{ for } v^2, \text{ obtaining } v^2 = \frac{1}{1-c}^{2} - u^2 - 1.$$

Substituting this expression for v^2 into the first coordinate allows us to perform the following manipulation:

$$\begin{aligned}
&\frac{u}{\sqrt{1+u^2+v^2}} && \\
&= \frac{\frac{a}{1-c}}{\sqrt{1 + \left(\frac{a}{1-c}\right)^2 + \frac{1}{(1-c)^2} - \left(\frac{a}{1-c}\right)^2 - 1}} && \text{Substitution} \\
&= \frac{\frac{a}{1-c}}{\sqrt{\frac{1}{(1-c)^2}}} = \frac{\frac{a}{1-c}}{\frac{1}{1-c}} = a && \text{Simplify}
\end{aligned}$$

Thus, we have that the first coordinate equals a, as desired. Similarly, we have that the second coordinate equals b.

As a result, we have verified that the given inverse is correct. $\square$

(b) Composing the two maps f and g gives the map

$$h = g \circ f : B(\mathbf{0}, 1) \to \mathbb{R}^2,\ h(a,b) = \left(\frac{a}{\sqrt{1-a^2-b^2}}, \frac{b}{\sqrt{1-a^2-b^2}} \right).$$

Find a formula for $h^{-1}(u,v) = (f^{-1} \circ g^{-1})(u,v)$ and conclude that h is a diffeomorphism of the open disk $B(\mathbf{0},1)$ with $\mathbb{R}^2$.

Proof: We know that $h^{-1}(a,b)$ will be given by the following:

$$\begin{aligned} h^{-1}(a,b) &= (g \circ f)^{-1}(a,b) = (f^{-1} \circ g^{-1})(a,b) = f^{-1}(g^{-1}(a,b)) \\ &= f^{-1}\left(\frac{a}{\sqrt{1+a^2+b^2}}, \frac{b}{\sqrt{1+a^2+b^2}}, 1 - \frac{1}{\sqrt{1+a^2+b^2}}\right) = \left(\frac{a}{\sqrt{1+a^2+b^2}}, \frac{b}{\sqrt{1+a^2+b^2}}\right). \end{aligned}$$

As the composition of two diffeomorphisms is a diffeomorphism and f and g are diffeomorphisms, we have that h will also be a diffeomorphism. □

(c) Generalize part (b) to $\mathbb{R}^n$.

Proof: All of the methods used in parts (a) and (b) can be extended to more dimensions by simply adding more coordinates to our equations and functions. In this manner we can generalize this problem to $\mathbb{R}^n$. □

Problem 1.6: Prove that if $f : \mathbb{R}^2 \to \mathbb{R}$ is C^∞, then there exist C^∞ functions g_{11}, g_{12}, g_{22} on $\mathbb{R}^2$ such that

$$f(x,y) = f(0,0) + \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial y}(0,0)y + x^2 g_{11}(x,y) + xy g_{12}(x,y) + y^2 g_{22}(x,y).$$

Proof: As f is C^∞ on $\mathbb{R}^2$ (which is star-shaped), we may apply Taylor's theorem at $(0,0)$ to obtain

$$f(x,y) = f(0,0) + x(f_1(x,y)) + y(f_2(x,y)), \text{ where } f_1(x,y) = \frac{\partial f}{\partial x}(x,y) \text{ and } f_2(x,y) = \frac{\partial f}{\partial y}(x,y).$$

As f is C^∞ on $\mathbb{R}^2$, both f_1 and f_2 will be C^∞ on $\mathbb{R}^2$, so we may apply Taylor's theorem again to obtain

$$f_1(x,y) = f_1(0,0) + x f_{11}(x,y) + y f_{12}(x,y) \text{ and } f_2(x,y) = f_2(0,0) + x f_{21}(x,y) + y f_{22}(x,y).$$

The work above then allows us to perform the following manipulation:

$$\begin{aligned} f(x,y) &= f(0,0) + x(f_1(x,y)) + y(f_2(x,y)) \\ &= f(0,0) + x(f_1(0,0) + x f_{11}(x,y) + y f_{12}(x,y)) + y(f_2(0,0) + x f_{21}(x,y) + y f_{22}(x,y)) \\ &= f(0,0) + x f_1(0,0) + x^2 f_{11}(x,y) + xy f_{12}(x,y)) + y f_2(0,0) + xy f_{21}(x,y) + y^2 f_{22}(x,y) \\ &= f(0,0) + x f_1(0,0) + y f_2(0,0) + x^2 f_{11}(x,y) + 2xy f_{12}(x,y)) + y^2 f_{22}(x,y) \\ &= f(0,0) + x \frac{\partial f}{\partial x}(0,0) + y \frac{\partial f}{\partial x}(0,0) + x^2 f_{11}(x,y) + 2xy f_{12}(x,y)) + y^2 f_{22}(x,y) \end{aligned}$$

Note that $x^2f_{11}(x, y)$, $2xyf_{12}(x, y))$, and $y^2f_{22}(x, y)$ are all C^∞, as f is C^∞. By then defining $g_{11} = x^2f_{11}(x, y)$, $g_{12} = 2xyf_{12}(x, y))$, and $g_{22} = y^2f_{22}(x, y)$, we have the desired result

$$f(x, y) = f(0, 0) + \frac{\partial f}{\partial x}(0, 0)x + \frac{\partial f}{\partial y}(0, 0)y + x^2g_{11}(x, y) + xyg_{12}(x, y) + y^2g_{22}(x, y),$$

where g_{11}, g_{12}, and g_{22} are all C^∞. $\square$

Problem 1.7: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^∞ function with $f(0, 0) = \partial f/\partial x(0, 0) = \partial f/\partial y(0, 0) = 0$. Define

$$g(t, u) = \begin{cases} \frac{f(t,tu)}{t} & \text{for } t \neq 0, \\ 0 & \text{for } t = 0. \end{cases}$$

Prove that $g(t, u)$ is C^∞ for $(t, u) \in \mathbb{R}^2$.

Proof: Let $x = t$ and $y = tu$, then apply the result of Problem 1.6 to $f(t, tu)$ to obtain

$$f(t, tu) = f(0, 0) + \frac{\partial f}{\partial x}(0, 0)t + \frac{\partial f}{\partial y}(0, 0)tu + t^2g_{11}(t, tu) + t^2ug_{12}(t, tu) + t^2u^2g_{22}(t, tu),$$

where g_{11}, g_{12}, g_{22} are C^∞ functions on $\mathbb{R}^2$. As we are given $f(0, 0) = \frac{\partial f}{\partial x}(0, 0) = \frac{\partial f}{\partial y}(0, 0) = 0$, we may simplify this expression to

$$f(t, tu) = t^2g_{11}(t, tu) + t^2ug_{12}(t, tu) + t^2u^2g_{22}(t, tu).$$

It then follows that

$$\frac{f(t, tu)}{t} = tg_{11}(t, tu) + tug_{12}(t, tu) + tu^2g_{22}(t, tu).$$

As g_{11}, g_{12}, g_{22} are C^∞ functions for (t, tu) it follows that $\frac{f(t,tu)}{t}$ will be C^∞ on (t, tu) except for at $t = 0$. However, as $tg_{11}(t, tu) + tug_{12}(t, tu) + tu^2g_{22}(t, tu) = 0$ at $t = 0$ and $g(0, u) = 0$ by definition, we still have that $g(t, u)$ will be C^∞ for (t, u). $\square$

Problem 1.8: Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$. Show that f is a bijective C^∞ map, but that f^{-1} is not C^∞.

Proof: We immediately know that f is bijective (this can be seen graphically). Additionally, f is C^∞, as repeated differentiation yields $x^3 \overset{d/dx}{\to} 3x^2 \overset{d/dx}{\to} 6x \overset{d/dx}{\to} 6 \overset{d/dx}{\to} 0 \overset{d/dx}{\to} 0 \cdots$.

The inverse of f can be determined to be $f^{-1}(x) = x^{1/3}$. This function is sill bijective and continuous, but it is not C^∞. This can be seen by observing that $(f^{-1})'(x) = \frac{1}{3x^{2/3}}$, which is discontinuous at $x = 0$. $\square$

Chapter 2

Problem 2.1: Let X be the vector field $x\partial/\partial x + y\partial/\partial y$ and $f(x,y,z)$ the function $x^2+y^2+z^2$ on $\mathbb{R}^3$. Compute Xf.

Proof: Xf is given by the following computation:

$$Xf = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = x(2x) + y(2y) = 2x^2 + 2y^2.$$

$\square$

Problem 2.3: Let D and D' be derivations at p in $\mathbb{R}^n$, and $c \in \mathbb{R}$. Prove that

(a) the sum $D + D'$ is a derivation at p.

Proof: As D and D' are derivations, we know that both D and D' are linear and satisfy the Leibniz Rule. It is then immediate that $D + D'$ will be linear, so it remains to show that $D + D'$ satisfies the Leibniz Rule. Now consider the following:

$$\begin{aligned}
(D + D')(fg) &= D(fg) + D'(fg) && \text{Properties of sums} \\
&= D(f)g(p) + D(g)f(p) + D'(f)g(p) + D'(g)f(p) && D \text{ and } D' \text{ are derivations} \\
&= (D(f) + D'(f))g(p) + (D(g) + D'(g))f(p) && \text{Factor} \\
&= (D + D')(f)g(p) + (D + D')(g)f(p) && \text{Properties of sums}
\end{aligned}$$

Thus, we have that $D + D'$ satisfies the Leibniz Rule, so $D + D'$ is a derivation at p. $\square$

(b) The scalar multiple cD is a derivation at p.

Proof: As D is a derivation, D is linear, so cD will also be linear. It then remains to show that cD satisfies the Leibniz Rule. By the properties of functions and the fact that D is a derivation we have that

$$cD(fg) = c(D(fg)) = c(D(f)g(p) + D(g)f(p)) = cD(f)g(p) + cD(g)f(p),$$

so it follows that cD is a derivation at p. $\square$

Problem 2.4: Let A be an algebra over a field K. If D_1 and D_2 are derivations of A, show that $D_1 \circ D_2$ is not necessarily a derivation (it is if D_1 or $D_2 = 0$), but $D_1 \circ D_2 - D_2 \circ D_1$ is always a derivation of A.

Proof: It an effort to maintain simplicity, we shall use the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$. Furthermore, we shall use the ordinary derivative as our derivation: $D_1 = D_2 = \frac{d}{dx}$.

We will first show that $D_1 \circ D_2$ violates the Leibniz Rule for some arbitrary point p. By way of contradiction assume that the Leibniz Rule does hold for $D_1 \circ D_2$. Then we that

$$(D_1 \circ D_2)(ff) = \frac{d}{dx}(\frac{d}{dx}(x^2)) = \frac{d}{dx}(2x) = 2$$

and

$$(D_1 \circ D_2)(f)f(p) + (D_1 \circ D_2)(f)f(p) = \frac{d}{dx}\left(\frac{d}{dx}(x)\right) f(p) + \frac{d}{dx}\left(\frac{d}{dx}(x)\right) f(p) = 0 + 0 = 0,$$

implying that $2 = 0$, which is a contradiction. Thus, the Leibniz Rule must not hold for $D_1 \circ D_2$, so $D_1 \circ D_2$ is not in general a derivation for any D_1 and D_2.

We can immediately see that $D_1 \circ D_2 - D_2 \circ D_1$ will maintain its linearity, as both D_1 and D_2 are linear. Using this property and the fact that the Leibniz Rule holds for D_1 and D_2 we can observe the following for an arbitrary point p:

$$\begin{aligned}
&(D_1 \circ D_2 - D_2 \circ D_1)(fg) \\
&= (D_1 \circ D_2)(fg) - (D_2 \circ D_1)(fg) \\
&= D_1(D_2(f)g(p) + D_2(g)f(p)) - D_2(D_1(f)g(p) + D_1(g)f(p)) \\
&= D_1(D_2(f)g(p)) + D_1(D_2(g)f(p)) - D_2(D_1(f)g(p)) - D_2(D_1(g)f(p)) \\
&= D_1(D_2(f)g(p)) - D_2(D_1(f)g(p)) + D_1(D_2(g)f(p)) - D_2(D_1(g)f(p)) \\
&= (D_1(D_2(f))g(p) + D_1(g(p))D_2(f(p))) - (D_2(D_1(f))g(p) + D_2(g(p))D_1(f(p))) \\
&\qquad + (D_1(D_2(g))f(p) + D_1(f(p))D_2(g(p))) - (D_2(D_1(g))f(p) + D_2(f(p))D_1(g(p))) \\
&= (D_1(D_2(f))g(p) - D_2(D_1(f))g(p)) + (D_1(D_2(g))f(p) - D_2(D_1(g))f(p)) \\
&= (D_1(D_2(f)) - D_2(D_1(f)))g(p) + (D_1(D_2(g)) - D_2(D_1(g)))f(p) \\
&= (D_1 \circ D_2 - D_2 \circ D_1)(f)g(p) + (D_1 \circ D_2 - D_2 \circ D_1)(g)f(p)
\end{aligned}$$

Thus, the Leibniz rule holds for $D_1 \circ D_2 - D_2 \circ D_1$ at an arbitrary point p, so $D_1 \circ D_2 - D_2 \circ D_1$ is indeed a derivation. $\square$

Chapter 3

Problem 3.1: Let $e_1, \ldots, e_n$ be a basis for a vector space V and let $\alpha^1, \ldots, \alpha^n$ be its dual basis in $V^\vee$. Suppose $[g_{ij}] \in \mathbb{R}^{n \times n}$ is an $n \times n$ matrix. Define a bilinear function $f : V \times V \to \mathbb{R}$ by

$$f(v, w) = \sum_{i \leq i,j \leq n} g_{ij} v^i w^j \text{ for } v = \sum v^i e_i \text{ and } w = \sum w^j e_j \text{ in } V.$$

Describe f in terms of the tensor products of α^i and α^j, $1 \leq i, j \leq n$.

Proof: We know that $v^i = \alpha^i(v)$ and $w^j = \alpha^j(v)$, and we can use this fact along with the properties of α to see that

$$f(v, w) = \sum_{1 \leq i,j \leq n} g_{ij} v^i w^j = \sum_{1 \leq i,j \leq n} g_{ij} \alpha^i(v) \alpha^j(w) = \sum_{1 \leq i,j \leq n} g_{ij} (\alpha^i \otimes \alpha^j)(v, w),$$

as desired. $\square$

Problem 3.2:

(a) Let V be a vector space of dimension n and $f : V \to \mathbb{R}$ a nonzero linear functional. Show that $\dim \ker f = n - 1$. A linear subspace of V of dimension $n - 1$ is called a *hyperplane* in V.

Proof: Recall the fact that $\dim \ker f + \dim \operatorname{im} f = \dim V$. As we know that $\dim \mathbb{R} = 1$ and f is nonzero, we immediately have that $\dim \operatorname{im} f = 1$. Because $\dim V = n$, it then follows immediately that $\dim \ker f = n - 1$. $\square$

(b) Show that a nonzero linear function on a vector space V is determined up to a multiplicative constant by its kernel, a hyperplane in V. In other words, if f and $g : V \to \mathbb{R}$ are nonzero linear functionals and $\ker f = \ker g$, then $g = cf$ for some constant $c \in \mathbb{R}$.

Proof: Let $f, g : V \to \mathbb{R}$ be nonzero linear functionals and assume that $\ker f = \ker g$. As f is nonzero, there exists some $v_0' \in V$ such that $f(v_0') = b \neq 0$. Let $v_0 = \frac{v_0'}{b}$; it then follows from the linearity of f that $f(v_0) = f(\frac{v_0'}{b}) = \frac{1}{b} f(v_0') = \frac{1}{b} b = 1$. As we have assumed that $\ker f = \ker g$ and $f(v_0) \neq 0$, it follows that $g(v_0) \neq 0$. Now let $v \in V$, $a = f(v)$, and $w = v - av_0$. Because f is linear we have that

$$f(w) = f(v - av_0) = f(v) - f(av_0) = f(v) - af(v_0) = a - a(1) = 0,$$

so then $w \in \ker f$ and therefore $w \in \ker g$. As a result, we know from the linearity of g and the work above that

$$0 = g(w) = g(v - av_0) = g(v) - g(av_0) = g(v) - ag(v_0) = g(v) - f(v)g(v_0).$$

Thus, we have that $g(v) = g(v_0)f(v)$ for all $v \in V$. We may then let $c = g(v_0)$ to observe $g(v) = cf(v)$, as desired. $\square$

Problem 3.3: Let V be a vector space of dimension n with basis $e_1, \dots, e_n$. Let $\alpha^1, \dots, \alpha^n$ be the dual basis for $V^\vee$. Show that a basis for the space $L_k(V)$ of k-linear functions on V is $\{\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}\}$ for all multi-indices $(i_1, \dots, i_k)$ (not just the strictly ascending multi-indices as for $A_k(L)$). In particular, this show that $\dim L_k(V) = n^k$.

Proof: Let $T : V^k \to \mathbb{R}$ and $T(e_{j_1}, \dots, e_{j_k}) = T_{j_1,\dots j_k}$. Now construct the function

$$T' = \sum_{1 \le i_1, \dots, i_k \le k} T_{i_1,\dots,i_k} \alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}$$

and consider the following:

$$\begin{aligned}
T'(e_{j_1}, \dots, e_{j_k}) &= \sum_{1 \le i_1, \dots, i_k \le k} T_{i_1,\dots,i_k} \alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}(e_{j_1}, \dots, e_{j_k}) && \text{Definition of } T' \\
&= \sum_{1 \le i_1, \dots, i_k \le k} T_{i_1,\dots,i_k} (\alpha^{i_1}(e_{j_1}) \cdots \alpha^{i_k}(e_{j_k})) && \text{Defintion of } \otimes \\
&= \sum_{1 \le i_1, \dots, i_k \le k} T_{i_1,\dots,i_k} (\delta^{i_1}_{j_1} \cdots \delta^{i_k}_{j_k}) && \text{Defintion of } \delta^i_j \\
&= T_{j_1,\dots,j_k} && \text{Evaluate sum} \\
&= T(e_{j_1}, \dots, e_{j_k}) && \text{Definition of } T
\end{aligned}$$

Thus, we have that $T'(e_{j_1}, \dots, e_{j_k}) = T(e_{j_1}, \dots, e_{j_k})$. As $e_{j_1}, \dots, e_{j_k}$ was an arbitrary list of elements from the basis $e_1, \dots, e_k$, it follows that $T' = T$ on all the basis elements, so $T' = T$. As a result, we have that $\{\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}\}$ spans $L_k(V)$.

Now say that $0 = \sum_{1 \le i_1, \dots i_k \le n} T_{i_1,\dots,i_k}(\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k})$, and consider the following:

$$\begin{aligned}
0 &= \sum_{1 \le i_1, \dots i_k \le n} T_{i_1,\dots,i_k}(\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}) && \text{Initial assumption} \\
&= \sum_{1 \le i_1, \dots i_k \le n} T_{i_1,\dots,i_k}(\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k})(e_{j_1}, \dots, e_{j_k}) && \text{Evalaute at a point} \\
&= \sum_{1 \le i_1, \dots i_k \le n} T_{i_1,\dots,i_k}(\alpha^{i_1}(e_{j_1}) \cdots \alpha^{i_k}(e_{j_k}) && \text{Definition of } \otimes \\
&= \sum_{1 \le i_1, \dots i_k \le n} T_{i_1,\dots,i_k}(\delta^{i_1}_{j_1} \cdots \delta^{i_k}_{j_k} && \text{Definition of } \delta^i_j \\
&= T_{j_1,\dots,j_k} && \text{Evaluate sum}
\end{aligned}$$

Thus, we have that $0 = T_{j_1,\dots,j_k}$. As $j_1, \dots, j_k$ was arbitrary, we have that $T_{i_1,\dots,i_k} = 0$ for all $i_1, \dots, i_k$, so it follows that $\{\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}\}$ is a linearly independent set.

As a result, we have that $\{\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}\}$ is a basis for $L_k(V)$. $\square$

Problem 3.4: Let f be a k-tensor on a vector space V. Prove that f is alternating if and only if f changes sign whenever two successive arguments are interchanged:

$$f(\dots, v_{i+1}, v_i, \dots) = -f(\dots, v_i, v_{i+1}, \dots) \text{ for } i = 1, \dots, k-1.$$

Proof: ($\Rightarrow$) Assume that f is alternating. Note that the interchange of the ith and $(i+1)$th entries is given by the permutation $\sigma = (i\ i+1)$, and further note that sgn $\sigma = -1$. Now consider the following:

$$\begin{aligned} f(v_1 \dots, v_{i+1}, v_i, \dots, v_n) &= (\text{sgn } \sigma) f(v_{\sigma(1)} \dots, v_{\sigma(i+1)}, v_{\sigma(i)}, \dots v_{\sigma(n)}) && f \text{ is even} \\ &= -f(v_1, \dots, v_i, v_{i+1}, \dots, v_n) && \text{Evaluate sgn and } \sigma \end{aligned}$$

Thus, we have $f(v_1 \dots, v_{i+1}, v_i, \dots, v_n) = -f(v_1, \dots, v_i, v_{i+1}, \dots, v_n)$, as desired.

($\Leftarrow$) Assume that f changes signs wheneverever two successive arguments are interchanged. Let $\sigma \in S_n$ be an arbitrary permutation, and recall that σ can be generated successive transpositions of adjacent elements, denote these transpositions as $\tau_1, \dots, \tau_r$. We then have two cases:

Case 1: r is even.
In the case that r is even, then previous results in algebra tell us that σ will be even, so sgn $\sigma = 1$. Furthermore, we know from our assumption that f changes signs with each interchange that

$$f(v_1, \dots, v_n) = (-1)^r f(v_{\sigma(1)}, \dots, v_{\sigma(i)}) = f(v_{\sigma(1)}, \dots, v_{\sigma(i)}).$$

Case 2: r is odd.
In the case that r is odd, then previous results in algebra tell us that σ will be odd, so sgn $\sigma = -1$. Furthermore, we know from our assumption that f changes signs with each interchange that

$$f(v_1, \dots, v_n) = (-1)^r f(v_{\sigma(1)}, \dots, v_{\sigma(i)}) = -f(v_{\sigma(1)}, \dots, v_{\sigma(i)}).$$

In either case we have that

$$(-1)^r f(v_{\sigma(1)}, \dots, v_{\sigma(i)}) = (\text{sgn } \sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(i)}),$$

so it follows that f is alternating. $\square$

Problem 3.5: Let f be a k-tensor on a vector space V. Prove that f is alternating if and only if $f(v_1, \ldots, v_k) = 0$ whenever two of the vectors $v_1, \ldots, v_k$ are equal.

Proof: ($\Rightarrow$) Assume that f is alternating, and let $v_i = v_j$ and $\sigma = (i\ j)$. Note that $\text{sgn}\ \sigma = -1$. Because f is alternating we have that

$$f(\ldots, v_i, \ldots, v_j, \ldots) = (\text{sgn}\ \sigma) f(\ldots, v_{\sigma(i)}, \ldots, v_{\sigma(j)}, \ldots) = -f(\ldots, v_j, \ldots, v_i, \ldots),$$

so $f(\ldots, v_i, \ldots, v_j, \ldots) = -f(\ldots, v_j, \ldots, v_i, \ldots)$. However, as $v_i = v_j$, we also immediately have that

$$f(\ldots, v_i, \ldots, v_j, \ldots) = f(\ldots, v_j, \ldots, v_i, \ldots),$$

which implies that $f(\ldots, v_j, \ldots, v_i, \ldots) = -f(\ldots, v_j, \ldots, v_i, \ldots)$. As a result, we have that $f(\ldots, v_i, \ldots, v_j, \ldots) = 0$.

($\Leftarrow$) Assume that $f(v_1, \ldots, v_k) = 0$ whenever two of the vectors $v_1, \ldots, v_k$ are equal. It follows from this assumption that

$$f(v_1, \ldots, v_i + v_{i+1}, v_{i+1} + v_i, \ldots, v_k) = 0.$$

Now consider the following manipulation:

$$\begin{aligned}
0 &= f(v_1, \ldots, v_i + v_{i+1}, v_{i+1} + v_i, \ldots, v_k) && \text{Given} \\
&= f(v_1, \ldots, v_i, v_{i+1} + v_i, \ldots, v_k) + f(v_1, \ldots, v_{i+1}, v_{i+1} + v_i, \ldots, v_k) && f \text{ is } k\text{-linear} \\
&= f(v_1, \ldots, v_i, v_{i+1}, \ldots, v_k) + f(v_1, \ldots, v_i, v_i, \ldots, v_k) && f \text{ is } k\text{-linear} \\
&\qquad + f(v_1, \ldots, v_{i+1}, v_{i+1}, \ldots, v_k) + f(v_1, \ldots, v_{i+1}, v_i, \ldots, v_k) \\
&= f(v_1, \ldots, v_i, v_{i+1}, \ldots, v_k) + 0 + 0 + f(v_1, \ldots, v_{i+1}, v_i, \ldots, v_k) && \text{Given} \\
&= f(v_1, \ldots, v_i, v_{i+1}, \ldots, v_k) + f(v_1, \ldots, v_{i+1}, v_i, \ldots, v_k) && \text{Simplify}
\end{aligned}$$

As a result, we have that $f(v_1, \ldots, v_i, v_{i+1}, \ldots, v_k) = -f(v_1, \ldots, v_{i+1}, v_i, \ldots, v_k)$, i.e. that interchanging two successive arguments changes the sign of f. It then follows from Problem 3.4 that f is alternating. $\square$

Problem 3.6: Let V be a vector space. For $a, b \in \mathbb{R}$, $f \in A_k(V)$, and $g \in A_\ell(V)$, show that $af \wedge bg = (ab) f \wedge g$.

Proof: Consider the following:

$$af \wedge bg = \frac{1}{k!\ell!} A(af \otimes bg) \qquad \text{Definition of } \wedge$$

$$\begin{aligned}
&= \frac{1}{k!\ell!}A(af(v_1,\ldots,v_k)bg(v_{k+1},\ldots,v_{k+\ell})) && \text{Definition of } \otimes \\
&= \frac{1}{k!\ell!}\sum_{\sigma\in S_{k+\ell}}(\text{sgn } \sigma)\sigma(af(v_1,\ldots,v_k)bg(v_{k+1},\ldots,v_{k+\ell})) && \text{Definition of } A \\
&= ab\frac{1}{k!\ell!}\sum_{\sigma\in S_{k+\ell}}(\text{sgn } \sigma)\sigma(f(v_1,\ldots,v_k)g(v_{k+1},\ldots,v_{k+\ell})) && \text{Pull scalars through} \\
&= ab\frac{1}{k!\ell!}A(f(v_1,\ldots,v_k)g(v_{k+1},\ldots,v_{k+\ell})) && \text{Definition of } A \\
&= ab\frac{1}{k!\ell!}A(f\otimes g) && \text{Definition of } \otimes \\
&= (ab)f\wedge g && \text{Defintion of } \wedge
\end{aligned}$$

Thus, we have $af \wedge bg = (ab)f \wedge g$, as desired. $\square$

Problem 3.7: Suppose two sets of covectors on a vector space V, $\beta^1,\ldots,\beta^k$, and $\gamma^1,\ldots,\gamma^k$ are related by

$$\beta^i = \sum_{j=1}^{k} a^i_j\gamma^j,\ i = 1,\ldots,k \text{ for a } k\times k \text{ matrix } A = [a^i_j].$$

Show that $\beta^1\wedge\cdots\wedge\beta^k = (\det A)\gamma^1\wedge\cdots\wedge\gamma^k$.

Proof: Consider the following:

$$\begin{aligned}
\beta^1\wedge\cdots\wedge\beta^k &= \sum_{j=1}^{k} a^1_j\gamma^j \wedge\cdots\wedge \sum_{j=1}^{k} a^k_j\gamma^j && \text{Given} \\
&= \sum_{1\le j_1,\ldots,j_k\le k} a^1_{j_1}\cdots a^k_{j_k}\gamma^{j_1}\wedge\cdots\wedge\gamma^{j_k} && \wedge \text{ is distributive} \\
&= \sum_{\sigma\in S_k} a^1_{\sigma(1)}\cdots a^k_{\sigma(k)}\gamma^{\sigma(1)}\wedge\cdots\wedge\gamma^{\sigma(k)} && \text{Rewrite indices} \\
&= \sum_{\sigma\in S_k} a^1_{\sigma(1)}\cdots a^k_{\sigma(k)}(\text{sgn } \sigma)\gamma^1\wedge\cdots\wedge\gamma^k && \text{Rearrange } \gamma^i\text{s} \\
&= \left(\sum_{\sigma\in S_k}(\text{sgn } \sigma)a^1_{\sigma(1)}\cdots a^k_{\sigma(k)}\right)\gamma^1\wedge\cdots\wedge\gamma^k && \text{Factor} \\
&= \det A\gamma^1\wedge\cdots\wedge\gamma^k && \text{Definition of } \det A
\end{aligned}$$

Thus, we have $\beta^1\wedge\cdots\wedge\beta^k = (\det A)\gamma^1\wedge\cdots\wedge\gamma^k$, as desired. $\square$

Problem 3.8: Let f be a k-covector on a vector space V. Suppose two sets of vectors

$u_1, \ldots, u_k$ and $v_1, \ldots, v_k$ in V are related by

$$u_j = \sum_{i=1}^{k} a_j^i v_i,\ j = 1, \ldots, k \text{ for a } k \times k \text{ matrix } A = [a_j^i].$$

Show that $f(u_1, \ldots, u_k) = (\det A) f(v_1, \ldots, v_k)$.

Proof: Consider the following:

$$\begin{aligned}
f(u_1, \ldots, u_k) &= f\left(\sum_{i=1}^{k} a_1^i v_i, \ldots, \sum_{i=1}^{k} a_k^i v_i\right) && \text{Defintion of } u_i \\
&= \sum_{1 \le i_1, \ldots, i_k \le k} a_{i_1}^1, \ldots, a_{i_k}^k f(v_{i_1}, \ldots, v_{i_k}) && f \text{ is } k\text{-linear} \\
&= \sum_{\sigma \in S_k} a_{\sigma(1)}^1 \cdots a_{\sigma(k)}^k f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) && \text{Rewrite indices} \\
&= \sum_{\sigma \in S_k} a_{\sigma(1)}^1 \cdots a_{\sigma(k)}^k (\text{sgn } \sigma) f(v_1, \ldots, v_k) && \text{Rearrange arguments} \\
&= \left(\sum_{\sigma \in S_k} (\text{sgn } \sigma) a_{\sigma(1)}^1 \cdots a_{\sigma(k)}^k\right) f(v_1, \ldots, v_k) && \text{Factor} \\
&= (\det A) f(v_1, \ldots, v_k) && \text{Defintion of } \det A
\end{aligned}$$

Thus, we have $f(u_1, \ldots, u_k) = (\det A) f(v_1, \ldots, v_k)$, as desired. □

Problem 3.9: Let V be a vector space of dimension n. Prove that if an n-covector ω vanishes on a basis $e_1, \ldots, e_n$ for V, then ω is the zero covector on V.

Proof: Let $v_1, \ldots, v_n \in V$, and note that $v_j = \sum_{i=1}^{n} e_i a_j^i$ for all j and some $n \times n$ matrix $A = [a_j^i]$, as $e_1, \ldots, e_n$ is a basis for V. It then follows from Problem 3.8 and the properties of ω that

$$\omega(v_1, \ldots, v_n) = \det A \omega(e_1, \ldots, e_n) = \det A(0) = 0,$$

so $\omega(v_1, \ldots, v_n) = 0$ for any $v_1, \ldots, v_n \in V$. As a result, we may conclude that ω is the zero covector on V. □

Problem 3.10: Let $\alpha^1, \ldots, \alpha^k$ be 1-covectors on a vector space V. Show that $\alpha^1 \wedge \cdots \wedge \alpha^k \neq 0$ if and only if $\alpha^1, \ldots, \alpha^k$ are linearly independent in the dual space $V^\vee$.

Proof: ($\Rightarrow$) We shall proceed contrapositively, so assume that $\alpha^1, \ldots, \alpha^k$ are not linearly independent. This implies that without loss of generality we may write α^k as a linear combination of the other covectors, that is $\alpha^k = \sum_{i=1}^{k-1} c_i \alpha^i$ for some scalars $c_1, \ldots, c_{k-1}$.

This implies that $\alpha^1 \wedge \cdots \wedge \alpha^k = \alpha^1 \wedge \cdots \wedge \sum_{i=1}^{k-1} c_i \alpha^i$; furthermore, note that every term has a repeated α^i. As a result, we have that $\alpha^1 \wedge \cdots \wedge \alpha^k \neq 0$, so we have shown contrapositively that $\alpha^1 \wedge \cdots \wedge \alpha^k = 0$ implies that $\alpha^1, \ldots, \alpha^k$ are linearly independent.

($\Leftarrow$) Assume that $\alpha^1, \ldots, \alpha^k$ are linearly independent. As they are linearly independent, we may extend them to some basis $\alpha^1, \ldots, \alpha^k, \ldots, \alpha^n$ for the dual space $V^\vee$. Let $v_1, \ldots, v_n$ represent the dual basis for V. It then follows from proposition 3.27 that

$$(\alpha^1 \wedge \cdots \wedge \alpha^k)(v_1, \ldots, v_k) = \det[\alpha^i(v_j)] = \det[\delta^i_j] = 1,$$

so we have that $\alpha^1 \wedge \cdots \wedge \alpha^k \neq 0$.

As a result, we have that $\alpha^1 \wedge \cdots \wedge \alpha^k \neq 0$ if and only if $\alpha^1, \ldots, \alpha^k$ are linearly independent in the dual space $V^\vee$. $\square$

Problem 3.11: Let α be a nonzero 1-covector and γ a k-covector on a fintie dimensional vector space V. Show that $\alpha \wedge \gamma = 0$ if and only if $\gamma = \alpha \wedge \beta$ for some $(k-1)$-covector β on V.

Proof: ($\Rightarrow$) Assume that $\alpha \wedge \gamma = 0$. We may extend α to a basis $\alpha^1, \ldots, \alpha^n$ for $V^\vee$, where $\alpha^1 = \alpha$. We may then say that $\gamma = \sum c_J \alpha^J$, where J runs over all strictly ascending multi-indices $1 \leq j_1 < \cdots < j_k \leq n$. In the sum $\alpha \wedge \gamma = \sum c_J \alpha \wedge \alpha^J$, all the terms $\alpha \wedge \alpha^J$ with $j_1 = 1$ vanish, since $\alpha = \alpha^1$. As a result we have that

$$0 = \alpha \wedge \gamma = \sum_{j_1 \neq 1} c_J \alpha \wedge \alpha^J.$$

As $\{\alpha \wedge \alpha^J\}_{j_1 \neq 1}$ is a subset of a basis for $A_{k+1}(V)$, it is linearly independent, and so all c_J are 0 if $j_1 \neq 1$. It then follows that

$$\gamma = \sum_{j_1 = 1} c_J \alpha^J = \alpha \wedge \left(\sum_{j_1 = 1} c_J \alpha^{j_2} \wedge \cdots \wedge \alpha^{j_k} \right),$$

where $\sum_{j_1=1} c_J \alpha^{j_2} \wedge \cdots \wedge \alpha^{j_k}$ is the desired β.

($\Leftarrow$) Assume that $\gamma = \alpha \wedge \beta$ for some $(k-1)$-covector β on V. Consider the following:

$$\begin{aligned}
\alpha \wedge \gamma &= \alpha \wedge (\alpha \wedge \beta) && \text{Initial assumption} \\
&= (\alpha \wedge \alpha) \wedge \alpha && \text{Associativity of } \wedge \\
&= 0 \wedge \alpha && \text{Corollary 3.23} \\
&= 0 && \text{Simplify}
\end{aligned}$$

Thus, we have that $\alpha \wedge \gamma = 0$.

As a result, we have $\alpha \wedge \gamma = 0$ if and only if $\gamma = \alpha \wedge \beta$ for some $(k-1)$-covector β on V. $\square$

Chapter 4

Problem 4.1: Let ω be the 1-form $zdx - dz$ and let X be the vector field $y\partial/\partial x + x\partial/\partial y$ on $\mathbb{R}^3$. Compute $\omega(X)$ and $d\omega$.

Proof: Recall that di is the covector for $\frac{\partial}{\partial i}$ and observe that

$$\begin{aligned}
\omega(X) &= (zdx - dz)\left(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) \\
&= zydx\left(\frac{\partial}{\partial x}\right) + zxdx\left(\frac{\partial}{\partial y}\right) - ydz\left(\frac{\partial}{\partial x}\right) - xdz\left(\frac{\partial}{\partial y}\right) \\
&= zy(1) + 0 - 0 - 0 = zy.
\end{aligned}$$

Thus, we have that $\omega(X) = zy$.

We may also observe that

$$d\omega = d(zdx - dz) = d(zdx) - d(dz) = dz \wedge dx + zd(dx) - d(dz) = dz \wedge dx + 0 - 0 = dz \wedge dx.$$

Thus, we have that $d\omega = dz \wedge dx$. $\square$

Problem 4.3: Suppose the standard coordinates on $\mathbb{R}^2$ are called r and θ. (this $\mathbb{R}^2$ is the (r,θ)-plane, not the (x,y)-plane). If $x = r\cos\theta$ and $y = r\sin\theta$, calculate dx, dy, and $dx \wedge dy$ in terms of dr and $d\theta$.

Proof: Observe that $dx = d(rcos\theta) = \cos\theta dr - r\sin\theta d\theta$. Likewise, we have that $dy = d(r\sin\theta) = \sin\theta dr + r\cos\theta d\theta$. We can then also see that

$$\begin{aligned}
dx \wedge dy &= (\cos\theta dr - r\sin\theta d\theta) \wedge (\sin\theta dr + r\cos\theta d\theta) \\
&= \cos\theta\sin\theta dr \wedge dr + \cos^2\theta r dr \wedge d\theta - r\sin^2\theta d\theta \wedge dr - r^2\sin\theta\cos\theta d\theta \wedge d\theta \\
&= 0 + \cos^2\theta r dr \wedge d\theta + \sin^2 r dr \wedge d\theta - 0 \\
&= (\cos^2\theta + \sin^2\theta) r dr \wedge d\theta = (1) r dr \wedge d\theta = r dr \wedge d\theta.
\end{aligned}$$

Thus, we have that $dx \wedge dy = rdr \wedge d\theta$. $\square$

Problem 4.4: Suppose the standard coordinates on $\mathbb{R}^3$ are called ρ, ϕ, and θ. If $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$, calculated dx, dy, dz, and $dx \wedge dy \wedge dz$ in terms of $d\rho$, $d\phi$, and $d\theta$.

Proof: We can immediately observe the following:

$$\begin{aligned}
dx &= d(\rho\sin\phi\cos\theta) = (\sin\phi\cos\theta)d\rho + (\rho\cos\phi\cos\theta)d\phi + (-\rho\sin\phi\sin\theta)d\theta \\
dy &= d(\rho\sin\phi\sin\theta) = (\sin\phi\sin\theta)d\rho + (\rho\cos\phi\sin\theta)d\phi + (\rho\sin\phi\cos\theta)d\theta
\end{aligned}$$

$$dz = d(\rho\cos\phi) = (\cos\phi)d\rho + (-\rho\sin\phi)d\phi$$

We can then continue to our final result:

$$\begin{aligned}
&dx \wedge dy \wedge dz \\
&= (((\sin\phi\cos\theta)d\rho + (\rho\cos\phi\cos\theta)d\phi + (-\rho\sin\phi\sin\theta)d\theta) \\
&\quad \wedge ((\sin\phi\sin\theta)d\rho + (\rho\cos\phi\sin\theta)d\phi + (\rho\sin\phi\cos\theta)d\theta)) \wedge dz \\
&= ((\sin^2\phi\sin\theta\cos\theta)d\rho\wedge d\rho + (\rho\sin\phi\cos\phi\sin\theta\cos\theta)d\rho\wedge d\phi + (\rho\sin^2\phi\cos^2\theta)d\rho\wedge d\theta \\
&\quad + (\rho\sin\phi\cos\phi\sin\theta\cos\theta)d\phi\wedge d\rho + (\rho^2\cos^2\phi\sin\theta\cos\theta)d\phi\wedge d\phi + (\rho^2\sin\phi\cos\phi\cos^2\theta)d\phi\wedge d\theta \\
&\quad + (-\rho\sin^2\phi\sin^2\theta)d\theta\wedge d\rho + (-\rho^2\sin\phi\cos\phi\sin^2\theta)d\theta\wedge d\phi + (-\rho^2\sin^2\phi\sin\theta\cos\theta)d\theta\wedge d\theta) \wedge dz \\
&= (0 + (\rho\sin\phi\cos\phi\sin\theta\cos\theta)d\rho\wedge d\phi + (\rho\sin^2\phi\cos^2\theta)d\rho\wedge d\theta + (\rho\sin\phi\cos\phi\sin\theta\cos\theta)d\phi\wedge d\rho \\
&\quad + 0 + (\rho^2\sin\phi\cos\phi\cos^2\theta)d\phi\wedge d\theta + (-\rho\sin^2\phi\sin^2\theta)d\theta\wedge d\rho + (-\rho^2\sin\phi\cos\phi\sin^2\theta)d\theta\wedge d\phi + 0) \wedge dz \\
&= ((\rho\sin\phi\cos\phi\sin\theta\cos\theta)d\rho\wedge d\phi + (\rho\sin^2\phi\cos^2\theta)d\rho\wedge d\theta + (-\rho\sin\phi\cos\phi\sin\theta\cos\theta)d\rho\wedge d\phi \\
&\quad + (\rho^2\sin\phi\cos\phi\cos^2\theta)d\phi\wedge d\theta + (-\rho\sin^2\phi\sin^2\theta)d\theta\wedge d\rho + (-\rho^2\sin\phi\cos\phi\sin^2\theta)d\theta\wedge d\phi) \wedge dz \\
&= ((\rho\sin^2\phi\cos^2\theta)d\rho\wedge d\theta + (\rho^2\sin\phi\cos\phi\cos^2\theta)d\phi\wedge d\theta + (-\rho\sin^2\phi\sin^2\theta)d\theta\wedge d\rho \\
&\quad + (-\rho^2\sin\phi\cos\phi\sin^2\theta)d\theta\wedge d\phi) \wedge ((\cos\phi)d\rho + (-\rho\sin\phi)d\phi) \\
&= (\rho\sin^2\phi\cos\phi\cos^2\theta)d\rho\wedge d\theta\wedge d\rho + (\rho^2\sin\phi\cos^2\phi\cos^2\theta)d\phi\wedge d\theta\wedge d\rho \\
&\quad + (-\rho\sin^2\phi\cos\phi\sin^2\theta)d\theta\wedge d\rho\wedge d\rho + (-\rho^2\sin\phi\cos^2\phi\sin^2\theta)d\theta\wedge d\phi\wedge d\rho \\
&\quad + (-\rho^2\sin^3\phi\cos^2\theta)d\rho\wedge d\theta\wedge d\phi + (-\rho^3\sin^2\phi\cos\phi\cos^2\theta)d\phi\wedge d\theta\wedge d\phi \\
&\quad + (\rho^2\sin^3\phi\sin^2\theta)d\theta\wedge d\rho\wedge d\phi + (\rho^3\sin^2\phi\cos\phi\sin^2\theta)d\theta\wedge d\phi\wedge d\phi \\
&= 0 + (\rho^2\sin\phi\cos^2\phi\cos^2\theta)d\phi\wedge d\theta\wedge d\rho + 0 + (-\rho^2\sin\phi\cos^2\phi\sin^2\theta)d\theta\wedge d\phi\wedge d\rho \\
&\quad + (-\rho^2\sin^3\phi\cos^2\theta)d\rho\wedge d\theta\wedge d\phi + 0 + (\rho^2\sin^3\phi\sin^2\theta)d\theta\wedge d\rho\wedge d\phi + 0 \\
&= (\rho^2\sin\phi\cos^2\phi\cos^2\theta)d\rho\wedge d\phi\wedge d\theta + (\rho^2\sin\phi\cos^2\phi\sin^2\theta)d\rho\wedge d\phi\wedge d\theta \\
&\quad + (\rho^2\sin^3\phi\cos^2\theta)d\rho\wedge d\phi\wedge d\theta + (\rho^2\sin^3\phi\sin^2\theta)d\rho\wedge d\phi\wedge d\theta \\
&= ((\rho^2\sin\phi\cos^2\phi\cos^2\theta) + (\rho^2\sin\phi\cos^2\phi\sin^2\theta) + (\rho^2\sin^3\phi\cos^2\theta) + (\rho^2\sin^3\phi\sin^2\theta))d\rho\wedge d\phi\wedge d\theta \\
&= ((\rho^2\sin\phi\cos^2\phi)(\cos^2\theta + \sin^2\theta) + (\rho^2\sin^3\phi)(\cos^2\theta + \sin^2\theta))d\rho\wedge d\phi\wedge d\theta \\
&= (\rho^2\sin\phi\cos^2\phi + \rho^2\sin^3\phi)d\rho\wedge d\phi\wedge d\theta \\
&= (\rho^2\sin\phi(\cos^2\phi + \sin^2\phi))d\rho\wedge d\phi\wedge d\theta \\
&= (\rho^2\sin\phi)d\rho\wedge d\phi\wedge d\theta
\end{aligned}$$

Thus, we have that $dx \wedge dy \wedge dz = \rho^2\sin\phi\ d\rho\wedge d\phi\wedge d\theta$. $\square$

Problem 4.5: Let α be a 1-form and β a 2-form on $\mathbb{R}^3$. Then

$$\begin{aligned}\alpha &= a_1 dx^1 + a_2 dx^2 + a_3 dx^3 \\ \beta &= b_1 dx^2 \wedge dx^3 + b_2 dx^3 \wedge dx^1 + b_3 dx^1 \wedge dx^2.\end{aligned}$$

Simplify the expression $\alpha \wedge \beta$ as much as possible.

Proof: Observe the following:

$$\begin{aligned}
&\alpha \wedge \beta \\
&= (a_1 dx^1 + a_2 dx^2 + a_3 dx^3) \wedge (b_1 dx^2 \wedge dx^3 + b_2 dx^3 \wedge dx^1 + b_3 dx^1 \wedge dx^2) \\
&= (a_1 b_1) dx^1 \wedge dx^2 \wedge dx^3 + (a_1 b_2) dx^1 \wedge dx^3 \wedge dx^1 + (a_1 b_3) dx^1 \wedge dx^1 \wedge dx^2 \\
&\quad (a_2 b_1) dx^2 \wedge dx^2 \wedge dx^3 + (a_2 b_2) dx^2 \wedge dx^3 \wedge dx^1 + (a_2 b_3) dx^2 \wedge dx^1 \wedge dx^2 \\
&\quad (a_3 b_1) dx^3 \wedge dx^2 \wedge dx^3 + (a_3 b_2) dx^3 \wedge dx^3 \wedge dx^1 + (a_3 b_3) dx^3 \wedge dx^1 \wedge dx^2 \\
&= (a_1 b_1) dx^1 \wedge dx^2 \wedge dx^3 + 0 + 0 + 0 + (a_2 b_2) dx^2 \wedge dx^3 \wedge dx^1 + 0 + 0 + 0 + (a_3 b_3) dx^3 \wedge dx^1 \wedge dx^2 \\
&= (a_1 b_1) dx^1 \wedge dx^2 \wedge dx^3 + (a_2 b_2) dx^1 \wedge dx^2 \wedge dx^3 + (a_3 b_3) dx^1 \wedge dx^2 \wedge dx^3 \\
&= (a_1 b_1 + a_2 b_2 + a_3 b_3) dx^1 \wedge dx^2 \wedge dx^3
\end{aligned}$$

Thus, we have that $\alpha \wedge \beta = (a_1 b_1 + a_2 b_2 + a_3 b_3) dx^1 \wedge dx^2 \wedge dx^3$. $\square$

Problem 4.6: The corresondence between differential forms and vector fields on an open subset of $\mathbb{R}^3$ in Subsection 4.6 also makes sense pointwise. Let V be a vector space of dimension 3 with basis e_1, e_2, e_3, and dual basis $\alpha^1, \alpha^2, \alpha^3$. To a 1-covector $\alpha = a_1\alpha^1 + a_2\alpha^2 + a_3\alpha^3$ on V, we associate the vector $v_\alpha = \langle a_1, a_2, a_3 \rangle \in \mathbb{R}^3$. To the 2-covector

$$\gamma = c_1 \alpha^2 \wedge \alpha^3 + c_2 \alpha^3 \wedge \alpha^1 + c_3 \alpha^1 \wedge \alpha^2$$

on V, we associate the vector $v_\gamma = \langle c_1, c_2, c_3 \rangle \in \mathbb{R}^3$. Show that under this correspondence, the wedge product of 1-covectors correspons to the cross product of vectors in $\mathbb{R}^3$: if $\alpha = a_1\alpha^1 + a_2\alpha^2 + a_3\alpha^3$ and $\beta = b_1\alpha^1 + b_2\alpha^2 + b_3\alpha^3$, then $v_{\alpha\wedge\beta} = v_\alpha \times v_\beta$.

Proof: Observe the following:

$$\begin{aligned}
\alpha \wedge \beta &= (a_1 b_1)\alpha^1 \wedge \alpha^1 + (a_1 b_2)\alpha^1 \wedge \alpha^2 + (a_1 b_3)\alpha^1 \wedge \alpha^3 + (a_2 b_1)\alpha^2 \wedge \alpha^1 + (a_2 b_2)\alpha^2 \wedge \alpha^2 \\
&\quad + (a_2 b_3)\alpha^2 \wedge \alpha^3 + (a_3 b_1)\alpha^3 \wedge \alpha^1 + (a_3 b_2)\alpha^3 \wedge \alpha^2 + (a_3 b_3)\alpha^3 \wedge \alpha^3 \\
&= 0 + (a_1 b_2)\alpha^1 \wedge \alpha^2 + (a_1 b_3)\alpha^1 \wedge \alpha^3 + (a_2 b_1)\alpha^2 \wedge \alpha^1 + 0 \\
&\quad + (a_2 b_3)\alpha^2 \wedge \alpha^3 + (a_3 b_1)\alpha^3 \wedge \alpha^1 + (a_3 b_2)\alpha^3 \wedge \alpha^2 + 0 \\
&= (a_1 b_2)\alpha^1 \wedge \alpha^2 + (a_1 b_3)\alpha^1 \wedge \alpha^3 + (a_2 b_1)\alpha^2 \wedge \alpha^1 + (a_2 b_3)\alpha^2 \wedge \alpha^3 + (a_3 b_1)\alpha^3 \wedge \alpha^1 + (a_3 b_2)\alpha^3 \wedge \alpha^2
\end{aligned}$$

$$= (a_2b_3 - a_3b_2)\alpha^2 \wedge \alpha^3 + (a_3b_1 - a_1b_3)\alpha^3 \wedge \alpha^1 + (a_1b_2 - a_2b_1)\alpha^1 \wedge \alpha^2$$

As a result, we have that $\mathbf{v}_{\alpha\wedge\beta} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$.

Note that $\mathbf{v}_\alpha = \langle a_1, a_2, a_3\rangle$ and $\mathbf{v}_\beta = \langle b_1, b_2, b_3\rangle$; it then follows immediately that $\mathbf{v}_\alpha \times \mathbf{v}_\beta = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$.

Thus, we have that $\mathbf{v}_{\alpha\wedge\beta} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle = \mathbf{v}_\alpha \times \mathbf{v}_\beta$, as desired. $\square$

Problem 4.7: Let $A = \bigoplus_{k=-\infty}^{\infty} A_k$ be a graded algebra over a field K with $A^k = 0$ for $k < 0$. Let m be an integer. A *superderivation of A of degree m* is a K-linear map $D : A \to A$ such that for all k, $D(A^k) \subset A^{k+m}$ and for all $a \in A^k$ and $b \in A^\ell$,

$$D(ab) = (Da)b + (-1)^{km}a(Db).$$

If D_1 and D_2 are two superderivations of A of respective degrees m_1 and m_2, define their *commutator* to be

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{m_1m_2} D_2 \circ D_1.$$

Show that $[D_1, D_2]$ is a superderivation of degree $m_1 + m_2$.

Proof: As D_1 and D_2 are superderivations, they are k-linear, so $[D_1, D_2]$ is immediately k-linear.

Now consider the following:

$$\begin{aligned}
&[D_1, D_2](ab) \\
&= (D_1 \circ D_2 - (-1)^{m_1m_2} D_2 \circ D_1)(ab) \\
&= (D_1 \circ D_2)(ab) - (-1)^{m_1m_2}(D_2 \circ D_1)(ab) \\
&= D_1(D_2(ab)) - (-1)^{m_1m_2}(D_2(D_1(ab))) \\
&= D_1((D_2a)b + (-1)^{km_2}(a(D_2b))) - (-1)^{m_1m_2} D_2((D_1a)b + (-1)^{km_1}(a(D_1b))) \\
&= D_1((D_2a)b) + (-1)^{km_2} D_1(a(D_2b)) - (-1)^{m_1m_2}(D_2((D_1a)b) + (-1)^{km_1} D_2(a(D_1b))) \\
&= ((D_1(D_2a))b + (-1)^{(k+m_2)m_1}(D_2a)(D_1b)) + (-1)^{km_2}((D_1a)(D - 2b) + (-1)^{km_1}(a)(D_1(D_2b))) \\
&\quad - (-1)^{m_1m_2}(((D_2(D_1a))b + (-1)^{(k+m_1)m_2}(D_1a)(D_2b)) + (-1)^{km_1}((D_2a)(D_1b) + (-1)^{km_2}(a)(D_2(D_1b)))) \\
&= (D_1(D_2a))b + (-1)^{km_1+m_1m_2}(D_2a)(D_1b) + (-1)^{km_2}(D_1a)(D_2b) + (-1)^{km_1+km_2}(a)(D_1(D_2b)) \\
&\quad - (-1)^{m_1m_2}(D_2(D_1a))b - (-1)^{km_2+m_1m_2+m_1m_2}(D_1a)(D_2b) \\
&\quad - (-1)^{km_1+m_1m_2}(D_2a)(D_1b) - (-1)^{km_1+km_2+m_1m_2}(a)(D_2(D_1b)) \\
&= (D_1(D_2a))b - (-1)^{m_1m_2}(D_2(D_1a))b
\end{aligned}$$

$$
\begin{aligned}
&+ (-1)^{km_1+m_1m_2}(D_2a)(D_1b) - (-1)^{km_1+m_1m_2}(D_2a)(D_1b)\\
&+ (-1)^{km_2}(D_1a)(D_2b) - (-1)^{km_2+2m_1m_2}(D_1a)(D_2b)\\
&+ (-1)^{km_1+km_2}(a)(D_1(D_2b)) - (-1)^{km_1+km_2+m_1m_2}(a)(D_2(D_1b))\\
&= b((D_1(D_2a)) - (-1)^{m_1m_2}(D_2(D_1a))) + 0 + (-1)^{km_2}(D_1a)(D_2b)(1 - (-1)^{2m_1m_2})\\
&+ (-1)^{k(m_1+m_2)}(a)((D_1(D_2b)) - (-1)^{m_1m_2}(D_2(D_1b)))\\
&= b((D_1(D_2a)) - (-1)^{m_1m_2}(D_2(D_1a))) + 0 + (-1)^{k(m_1+m_2)}(a)((D_1(D_2b)) - (-1)^{m_1m_2}(D_2(D_1b)))\\
&= b(D_1 \circ D_2 - (-1)^{m_1m_2}D_2 \circ D_1)(a) + (-1)^{k(m_1+m_2)}a(D_1 \circ D_2 - (-1)^{m_1m_2}D_2 \circ D_1)(b)\\
&= ([D_1, D_2]a)b - (-1)^{k(m_1+m_2)}a([D_1, D_2]b)
\end{aligned}
$$

Thus, we have that $[D_1, D_2]$ satisfies the rule outlined above, so $[D_1, D_2]$ is a superderivation of degree $m_1 + m_2$. □

Chapter 5

Problem 5.1: Let A and B be two points not on the real line $\mathbb{R}$. Consider the set $S = (\mathbb{R} - \{0\}) \cup \{A, B\}$. For any two positive real numbers c, d, define

$$I_A(-c, d) = (-c, 0) \cup \{A\} \cup (0, d)$$

and similarly for $I_B(-c, d)$, with B instead of A. Define a topology on S as follows: On $(\mathbb{R} - \{0\})$, use the subspace topology inherited from $\mathbb{R}$, with open intervals as a basis. A basis of neighborhoods at A is the set $\{I_A(-c, d) \mid c, d > 0\}$; similarly, a basis of neighborhoods at B is $\{I_B(-c, d) \mid c, d > 0\}$.
(a) Prove that the map $h : I_A(-c, d) \to (-c, d)$ defined by

$$h(x) = x \text{ for } x \in (-c, 0) \cup (0, d) \text{ and } h(A) = 0$$

is a homeomorphism.
Proof: (1) h is injective.
Let $x, y \in I_A(-c, d)$ and say that $h(x) = h(y)$. There are two cases:

Case i: $h(x) \neq 0$. Then $h(y) \neq 0$, so we have that $h(x) = x$ and $h(y) = y$, so $x = y$.

Case ii: $h(x) = 0$. Then $h(y) = 0$, so we have that $x = A = y$, so $x = y$.

As both cases hold, we have that h is injective.

(2) h is surjective.
Let $y \in (-c, d)$. There are two cases:

Case i: $y \neq 0$. Then $y \in I_A(-c, d)$ and $h(y) = y$.

Case ii: $y = 0$. Then $h(A) = y$.

As both cases hold, we have that h is surjective.

(3) h is continuous.
Note that we may prove continuity by showing that the preimage of basis elements in $(-c, d)$ are open in $I_A(-c, d)$. As all open intervals in $\mathbb{R}$ serve as a basis for $\mathbb{R}$, the intersection of all open intervals in $\mathbb{R}$ with $(-c, d)$ will serve as a basis for $(-c, d)$; in other words, a basis for $(-c, d)$ is given by $\{(x, y) \mid (x, y) \subseteq (-c, d)\}$. We then have two cases:

Case i: $0 \notin (x, y)$. Then we know that $h^{-1}((x, y)) = (x, y) \subseteq I_A(-c, d)$, so $h^{-1}((x, y))$ is open in $I_A(-c, d)$.

Case ii: $0 \in (x, y)$. Then we know that $h^{-1}((x, y)) = I_A(x, y) \subseteq I_A(-c, d)$, so $h^{-1}((x, y))$ is open in $I_A(-c, d)$.

As both cases hold, we have that h is continuous.

(4) h^{-1} is continuous. Note that the basis elements of $I_A(-c,d)$ are given by (x,y) when $0 \notin (x,y)$ and $I_A(x,y)$ when $x \leq 0 \leq y$. Thus, there are two cases:

Case i: $h((x,y))$. Then we know that $h((x,y)) = (x,y) \subseteq (-c,d)$, so $h((x,y))$ is open in $(-c,d)$.

Case ii: $h(I_A(x,y))$. Then we know that $h(I_A(x,y)) = (x,y) \subseteq (-c,d)$, so $h(I_A(x,y))$ is open in $(-c,d)$.

As both cases hold, we have that h^{-1} is continuous.

As the above four properties hold, it follows that h is a homeomorphism. $\square$

(b) Show that S is locally Euclidean and second countable, but not Hausdorff.
Proof: Let $p \in S$ and $U \subseteq S$ be an open set containing p. Then there is some basis element U_p containing p contained in U. We know from (a) that U_p is homeomorphic to an open subset of $\mathbb{R}$, so it follows that S is locally Euclidean.

Recall that we may construct a basis $\mathcal{B}$ for $\mathbb{R}$ using only ball with rational radius centered at rational points (this was shown last semester). As S inherits its topology from $\mathbb{R}$, we need only change those intervals (a,b) where $0 \in (a,b)$ to either $I_A(a,b)$ or $I_B(a,b)$. This does not change the countablility inherited from $\mathcal{B}$, we know that S is second countable.

Consider the points $A, B \in S$, clearly $A \neq B$. Let $A \in U$ and $B \in V$, where U and V are open in S. As we know that any open set containing A will contain a basis element containing A, we may think of U as being of the form $U = I_A(a_1,a_2)$. Similarly, we may say $V = I_B(b_1,b_2)$. Furthermore, note that this implies that $a_1, b_1 < 0 < a_2, b_2$. Let $c_1 = \max\{a_1,b_1\}$ and $c_2 = \min\{a_2,b_2\}$, and note that $(a_1,a_2) \cap (b_1,b_2) = (c_1,c_2)$. As a result, it follows that $((c_1,0) \cup (0,c_2)) \subseteq (I_A(a_1,a_2) \cap I_B(b_1,b_2))$, so we have that $I_A(a_1,a_2) \cap I_B(b_1,b_2) \neq \emptyset$ for any $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Thus, S cannot be Hausdorff. $\square$

Problem 5.2 A fundamental theorem of topology, the theorem on invariance of dimension, states that if two nonempty open sets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are homeomorphic, then $n = m$. Use the idea of Example 5.4 as well as the theorem on invariance of dimension to prove that the sphere with a hair in $\mathbb{R}^3$ is not locally Euclidean at q. Hence it cannot be a topological manifold.
Proof: Suppose the sphere with a hair is locally Euclidean of dimension n at the point q. Then q has a neighborhood U homeomorphic to an open ball $B = B(0,\epsilon) \subseteq \mathbb{R}^n$ with q map-

ping to 0. The homeomorphism $U \to B$ restricts to a homeomorphism $U \setminus \{q\} \to B \setminus \{0\}$. Now $B \setminus \{0\}$ is either connected if $n \geq 2$ or has two connected components if $n = 1$. Since $U \setminus \{q\}$ has two connected components, the only possible homeomorphism would be from U to an open ball in $\mathbb{R}$. However, we know that a neighborhood on the sphere will have dimension 2; as dimension is unvariant under homeomorphism, it is not possible that a homeomorphism exists between U and and an open ball in $\mathbb{R}$. As a result, there cannot exist a homeomorphism between U and an open ball in $\mathbb{R}^n$ for any n, so the sphere with a hair is not locally Euclidean at q. It then follows that the sphere with a hair cannot be a topological manifold. $\square$

Problem 5.3 Let S^2 be the unit sphere

$$x^2 + y^2 + z^2 = 1$$

in $\mathbb{R}^3$. Define in S^2 the six charts corresponding to the six hemispheres–the front, rear, right, left, upper, and lower hemispheres:

$$\begin{aligned}
U_1 &= \{(x, y, z) \in S^2 \mid x > 0\}, & \phi_1(x, y, z) &= (y, z),\\
U_2 &= \{(x, y, z) \in S^2 \mid x < 0\}, & \phi_2(x, y, z) &= (y, z),\\
U_3 &= \{(x, y, z) \in S^2 \mid y > 0\}, & \phi_3(x, y, z) &= (x, z),\\
U_4 &= \{(x, y, z) \in S^2 \mid y < 0\}, & \phi_4(x, y, z) &= (x, z),\\
U_5 &= \{(x, y, z) \in S^2 \mid z > 0\}, & \phi_5(x, y, z) &= (x, y),\\
U_6 &= \{(x, y, z) \in S^2 \mid z < 0\}, & \phi_6(x, y, z) &= (x, y),
\end{aligned}$$

Describe the domain $\phi_4(U_{14})$ of $\phi_1 \circ \phi_4^{-1}$ and show that $\phi_1 \circ \phi_4^{-1}$ is C^∞ on $\phi_4(U_{14})$. Do the same for $\phi_6 \circ \phi_1^{-1}$.

Proof: We know that $U_{14} = U_1 \cap U_4$, so $\phi_4(U_{14}) = \phi_4(U_1 \cap U_4) = \{(x, z) \mid x^2 + z^2 < 1, x > 0\}$ is our domain. Note that $\phi_1 \circ \phi_4^{-1}$ will be defined by $(x, z) \mapsto (-\sqrt{1 - z^2 - x^2}, z)$. Observe that

$$\frac{\partial}{\partial x} - \sqrt{1 - z^2 - x^2} = \frac{-1}{2} \frac{1}{(1 - z^2 - x^2)^{1/2}} (-2x).$$

Furthermore, note that the denominator of the rational expression $\frac{1}{(1-z^2-x^2)^{1/2}}$ is restricted to only positive numbers by our domain, so this partial derivative will be C^∞. Similarly, the partial derivative with respect to z will also be C^∞. As a result, we have that the function describing the map in the first component is C^∞. As the function describing the map in the second component is the identity function, it is immediately C^∞. Then because our $\phi_1 \circ \phi_4^{-1}$ is C^∞ in both the first and second component, $\phi_1 \circ \phi_4^{-1}$ is C^∞.

We know that $U_{61} = U_6 \cap U_1$, so $\phi_1(U_{61}) = \phi_6(U_6 \cap U_1) = \{(y, z) \mid y^2 + z^2 < 1, z < 0\}$ is our domain. Note that $\phi_6 \circ \phi_1^{-1}$ will be defined by $(y, z) \mapsto (\sqrt{1 - z^2 - y^2}, y)$. Observe that

$$\frac{\partial}{\partial y}\sqrt{1 - z^2 - y^2} = \frac{1}{2}\frac{1}{(1 - z^2 - y^2)^{1/2}}(-2y).$$

Furthermore, note that the denominator of the rational expression $\frac{1}{(1-z^2-y^2)^{1/2}}$ is restricted to only positive numbers by our domain, so this partial derivative will be C^∞. Similarly, the partial derivative with respect to z will also be C^∞. As a result, we have that the function describing the map in the first component is C^∞. As the function describing the map in the second component is the identity function, it is immediately C^∞. Then because our $\phi_6 \circ \phi_1^{-1}$ is C^∞ in both the first and second component, $\phi_6 \circ \phi_1^{-1}$ is C^∞. $\square$

Problem 5.4 Let $\{(U_\alpha, \phi_\alpha)\}$ be the maximal atlas on a manifold M. For any open set U in M and a point $p \in U$, prove the existence of a coordinate open set U_α such that $p \in U_\alpha \subset U$.

Proof: Let U_β be any coordinate neighborhood of p in the maximal atlas. Any open subset U_β is again in the maximal atlas, because it is C^∞ compatible with all the open sets in the maximal atlas. Thus $U_\alpha = U_\beta \cap U$ is a coordinate neighborhood such that $p \in U_\alpha \subset U$. $\square$

Chapter 6

Problem 6.1: Let $\mathbb{R}$ be the real line with the differentiable structure given by the maximal atlas of the chart $(\mathbb{R}, \phi = \mathbb{1} : \mathbb{R} \to \mathbb{R})$, and let $\mathbb{R}'$ be the real line with the differentiable structure given by the maximal atlas of the chart $(\mathbb{R}, \psi : \mathbb{R} \to \mathbb{R})$, where $\psi(x) = x^{1/3}$.

(a) Show that these two differentiable structures are distinct.

Proof: The manifolds described give rise to the following maps:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{F} & \mathbb{R}' \\ {\scriptstyle \mathbb{1}}\downarrow & & {\scriptstyle \psi}\downarrow \\ \mathbb{R} & & \mathbb{R} \end{array}$$

We know that if $\mathbb{R}$ and $\mathbb{R}'$ have the same differential structure, then F must be the identity map and $\psi \circ F \circ \mathbb{1}^{-1} : \mathbb{R} \to \mathbb{R}$ must be a diffeomorphism. However, when we let $F = \text{id}$ it can immediately be seen for $x \in \mathbb{R}$ that

$$\psi \circ \text{id} \circ \mathbb{1}^{-1}(x) = \psi \circ \text{id}(x) = \psi(x) = x^{1/3}.$$

As we have shown in previous work that $x^{1/3}$ is not C^∞, it follows that $\psi \circ F \circ \mathbb{1}^{-1}$ cannot be a diffeomorphism. As a result, $\mathbb{R}$ and $\mathbb{R}'$ cannot have the same differential structure. $\square$

(b) Show that there is a diffeomorphism between $\mathbb{R}$ and $\mathbb{R}'$.

Proof: We now let $F : \mathbb{R} \to \mathbb{R}'$ be defined by $F(x) = x^3$. It can then immediately be seen for $x \in \mathbb{R}$ that

$$(\psi \circ F \circ \mathbb{1}^{-1}(x))^{-1} = (\psi \circ F(x))^{-1} = (\psi(x^3))^{-1} = ((x^3)^{1/3})^{-1} = x^{-1}.$$

As we know from previous work that x is a diffeomorophism, it follows that $\psi \circ F \circ \mathbb{1}^{-1}$ is a diffeomorphism, so F is a diffeomorphism between $\mathbb{R}$ and $\mathbb{R}'$. $\square$

Problem 6.2: Let M and N be manifolds and let q_0 be a point in N. Prove that the inclusion map $i_{q_0} : M \to M \times N$ defined by $i_{q_0}(p) = (p, q_0)$ is C^∞.

Proof: Let (U_α, ϕ_α) and (V_β, ψ_β) be charts for M and N repsectively; this implies that $(U_\alpha \times V_\beta, \phi_\alpha \times \psi_\beta)$ will be a chart for $M \times N$. Then the manifolds described give rise to the following maps:

$$\begin{array}{ccc} M & \xrightarrow{i_{q_0}} & M \times N \\ {\scriptstyle \phi_\alpha}\downarrow & & {\scriptstyle \phi_\gamma \times \psi_\beta}\downarrow \\ \phi_\alpha(U_\alpha) & & \phi_\gamma(U_\gamma) \times \psi_\beta(V_\beta) \end{array}$$

(Note that we must pick α, γ so that $U_\alpha \cap U_\gamma \neq \emptyset$ for this diagram to make sense.) In order to show that i_{q_0} is C^∞, we must show that $(\phi_\gamma \times \psi_\beta) \circ i_{q_0} \circ \phi_\alpha^{-1}(x)$ is C^∞. It can immediately be seen that

$$(\phi_\gamma \times \psi_\beta) \circ i_{q_0} \circ \phi_\alpha^{-1}(x) = (\phi_\gamma \times \psi_\beta)(\phi_\alpha^{-1}(x), q_0) = (\phi_\gamma \circ \phi_\alpha^{-1}(x), \psi_\beta(q_0)).$$

We can then show that $(\phi_\gamma \circ \phi_\alpha^{-1}(x), \psi_\beta(q_0))$ is C^∞ by showing that its components are C^∞. We immediately have that $\phi_\gamma \circ \phi_\alpha^{-1}(x)$ is C^∞ for all x, as this is a transition map. Furthermore, $\psi_\beta(q_0)$ is a constant map, so it is also C^∞ for all x. As a result, we have that $(\phi_\gamma \circ \phi_\alpha^{-1}(x), \psi_\beta(q_0))$ is C^∞, so i_{q_0} is C^∞. $\square$

Problem 6.4: Find all points in $\mathbb{R}^3$ in a neighborhood of which the functions $x, x^2 + y^2 + z^2 - 1, z$ can serve as a local coordinate system.

Proof: Define $F : \mathbb{R}^3 \to \mathbb{R}^3$ by $F(x, y, z) = (x, x^2 + y^2 + z^2 - 1, z)$. The map F can serve as a coordinate map in a neighborhood of some point $p \in \mathbb{R}^3$ if and only if it is a local diffeomorphism at p. The Jacobian determinant of F is

$$\frac{\partial(F^1, F^2, F^3}{\partial(x, y, z)} = \det \begin{bmatrix} 1 & 0 & 0 \\ 2x & 2y & 2z \\ 0 & 0 & 1 \end{bmatrix} = 1 \begin{vmatrix} 2y & 2z \\ 0 & 1 \end{vmatrix} = 2y.$$

By the inverse function theorem, F is a local diffeomorphism at $p = (x, y)$ if and only if $y \neq 0$; thus, F can serve as a coordinate system at any point not in the xz-plane. $\square$

Chapter 7

Problem 7.5: Suppose a right action of a topological group G on a topological space S is continuous; this simply means that the map $S \times G \to S$ describing the action is continuous. Define two points x, y of S to be equivalent if they are in the same orbit; i.e., there is an element $g \in G$ such that $y = xg$. Let S/G be the quotient space; it is called the *orbit space* of the action. Prove that the projection map $\pi : S \to S/G$ is an open map.

Proof: Let U be an open subset of S. For each $g \in G$ we know that Ug will be open because right multiplication is a homeomorphism from S to S. We also know that $\pi{-}1(\pi(U)) = \bigcup_{g \in G} Ug$ is open, as it is a union of open sets. It then follows from the definition of the quotient topology that $\pi(U)$ is open. $\square$

Problem 7.6: Let the additive group $2\pi\mathbb{Z}$ act on $\mathbb{R}$ on the right by $x \cdot 2\pi n = x + 2\pi n$, where n is an integer. Show that the orbit space $\mathbb{R}/2\pi\mathbb{Z}$ is a smooth manifold.

Proof: Let $\pi : \mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z}$ represent the projection map. Also define $\pi_1 : \mathbb{R}/2\pi\mathbb{Z} \to (-\pi, \pi)$ by $[t] \mapsto t \in (-\pi, \pi)$ and $\pi_2 : \mathbb{R}/2\pi\mathbb{Z} \to (0, 2\pi)$ by $[t] \mapsto t \in (0, 2\pi)$.

$\mathbb{R}/2\pi\mathbb{Z}$ is locally Euclidean:

The maps described above give rise to the following:

$$\begin{array}{ccc} & \mathbb{R}/2\pi\mathbb{Z} & \\ \phi_1\downarrow & & \phi_2\downarrow \\ (-\pi,\pi) & \xrightarrow{\phi_2\circ\phi_1^{-1}} & (0,2\pi) \end{array}$$

We desire to show that ϕ_2 and ϕ_1 are C^∞-compatible. There are two cases:

Case 1: $x \in (0, \pi)$. Observe that

$$\phi_2 \circ \phi_1^{-1}(x) = \phi_2([x]) = x.$$

Case 2: $x \in (\pi, 2\pi)$. Observe that

$$\phi_2 \circ \phi_1^{-1}(x) = \phi_2([x]) = x - 2\pi.$$

The case for $\phi_1 \circ \phi_2^{-1}$ is similar. As all of these maps are all the identity map or translations, they are all homeomorphisms. We can then extend these charts to a maximal atlas, so it follows that $\mathbb{R}/2\pi\mathbb{Z}$ is locally Euclidean.

$\mathbb{R}/2\pi\mathbb{Z}$ is Hausdorff:

Let $[\alpha], [\beta] \in \mathbb{R}/2\pi\mathbb{Z}$ such that $[\alpha] \neq [\beta]$. Pick representatives $a \in [\alpha]$ and $b \in [\beta]$ such that either $a, b \in (-\pi, \pi)$ or $a, b \in (0, 2\pi)$; in either case, $a \neq b$. As $\mathbb{R}$ is Hausdorff, we have open

$U, V \subseteq \mathbb{R}$ such that $U \cap V = \emptyset$. We can then map U and V forward to $\pi(U)$ and $\pi(V)$, which will be open, as π is an open map. Furthermore, we have that $\pi(U) \cap \pi(V) = \emptyset$, as desired.

$\mathbb{R}/2\pi\mathbb{Z}$ is second countable:
Recall that $\mathbb{R}$ is second countable; it then follows that a quotient space of $\mathbb{R}$ will be second countable, so $\mathbb{R}/2\pi\mathbb{Z}$ is second countable.

As the above properties hold, we have that $\mathbb{R}/2\pi\mathbb{Z}$ is a smooth manifold. $\square$

Problem 7.7:
(a) Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha=1}^2$ be the atlas of the circle S^1 in Example 5.7, and let $\overline{\phi}_\alpha$ be the map ϕ_α followed by the projection $\mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z}$. On $U_1 \cap U_2 = A \sqcup B$, since ϕ_1 and ϕ_2 differ by an integer multiple of 2π, $\overline{\phi}_1 = \overline{\phi}_2$. Therefore, $\overline{\phi}_1$ and $\overline{\phi}_2$ piece together to give a well-defined map $\overline{\phi} : S^1 \to \mathbb{R}/2\pi\mathbb{Z}$. Prove that $\overline{\phi}$ is C^∞.
Proof: Let $\pi : \mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z}$ be the projection map. Furthermore, let $V_1 = (-\pi, \pi)$ and $V_2 = (0, 2\pi)$. Now define $\pi_1 : V_1 \to \mathbb{R}/2\pi\mathbb{Z}$ be the restriction $\pi\big|_{(-\pi,\pi)}$ and $\pi_2 : V_2 \to \mathbb{R}/2\pi\mathbb{Z}$ be the restriction $\pi\big|_{(0,2\pi)}$. The map $\overline{\phi}$ described in the problem statement can be described as $\overline{\phi} = \pi \circ \phi_\alpha$. These maps then give rise to the following diagram:

$$\begin{array}{ccc} S^1 & \xrightarrow{\overline{\phi}} & \mathbb{R}/2\pi\mathbb{Z} \\ {\scriptstyle \phi_\alpha}\downarrow & & {\scriptstyle \pi_\beta^{-1}}\downarrow \\ \phi_\alpha(U_\alpha) & & V_\beta \end{array}$$

We have several cases to verify for the different choices of ϕ_α and π_β. Case 1: $\alpha = \beta = 1$.
Let $x \in (-\pi, \pi)$. Then we have that

$$\pi_1^{-1} \circ \overline{\phi} \circ \phi_1^{-1}(x) = \pi_1^{-1} \circ \pi \circ \phi_\alpha \circ \phi_1^{-1}(t) = x.$$

Case 2: $\alpha = \beta = 2$.
Let $x \in (0, 2\pi)$. Then we have that

$$\pi_2^{-1} \circ \overline{\phi} \circ \phi_2^{-1}(x) = \pi_2^{-1} \circ \pi \circ \phi_\alpha \circ \phi_2^{-1}(t) = x.$$

Case 3: $\alpha = 1$ and $\beta = 2$. There are two subcases:
Subcase i: Let $x \in (0, \pi)$. Then we have that

$$\pi_2^{-1} \circ \overline{\phi} \circ \phi_1^{-1}(x) = \pi_2^{-1} \circ \pi \circ \phi_\alpha \circ \phi_1^{-1}(t) = x.$$

Subcase ii: Let $x \in (-\pi, 0)$. Then we have that

$$\pi_2^{-1} \circ \overline{\phi} \circ \phi_1^{-1}(x) = \pi_2^{-1} \circ \pi \circ \phi_\alpha \circ \phi_1^{-1}(t) = x + 2\pi.$$

Case 4: $\alpha = 2$ and $\beta = 1$.
Subcase i: Let $x \in (0, \pi)$. Then we have that

$$\pi_2^{-1} \circ \overline{\phi} \circ \phi_1^{-1}(x) = \pi_2^{-1} \circ \pi \circ \phi_\alpha \circ \phi_1^{-1}(t) = x.$$

Subcase ii: Let $x \in (\pi, 2\pi)$. Then we have that

$$\pi_2^{-1} \circ \overline{\phi} \circ \phi_1^{-1}(x) = \pi_2^{-1} \circ \pi \circ \phi_\alpha \circ \phi_1^{-1}(t) = x - 2\pi.$$

All of these maps are identities or translations, so they are C^∞; thus $\overline{\phi}$ is C^∞. $\square$

(b) The complex exponential $\mathbb{R} \to S^1$, $t \mapsto e^{it}$, is constant on each orbit of the action of $2\pi\mathbb{Z}$ on $\mathbb{R}$. Therefore, there is an induced map $F : \mathbb{R}/2\pi\mathbb{Z} \to S^1$, $F([t]) = e^{it}$. Prove that F is C^∞.

Proof: The maps described give rise to the following diagram:

$$\begin{array}{ccc} S^1 & \xleftarrow{F} & \mathbb{R}/2\pi\mathbb{Z} \\ {\scriptstyle \phi_\alpha}\downarrow & & {\scriptstyle \pi_\beta^{-1}}\downarrow \\ \phi(U_\alpha) & & V_\beta \end{array}$$

We have several cases to verify for the different choices of ϕ_α and π_β.
Case 1: $\alpha = \beta = 1$.
Let $x \in (-\pi, \pi)$. Then we have that

$$\phi_1 \circ F \circ \pi_1(x) = \phi_1 \circ F([x]) = \phi_1(e^{ix}) = x.$$

Case 2: $\alpha = \beta = 2$.
Let $x \in (0, 2\pi)$. Then we have that

$$\phi_2 \circ F \circ \pi_2(x) = \phi_2 \circ F([x]) = \phi_2(e^{ix}) = x.$$

Case 3: $\alpha = 1$ and $\beta = 2$. There are two subcases:
Subcase i: Let $x \in (0, \pi)$. Then we have that

$$\phi_2 \circ F \circ \pi_1(x) = \phi_2 \circ F([x]) = \phi_2(e^{ix}) = x.$$

Subcase ii: Let $x \in (-\pi, 0)$. Then we have that

$$\phi_2 \circ F \circ \pi_1(x) = \phi_2 \circ F([x]) = \phi_2(e^{ix}) = x + 2\pi.$$

Case 4: $\alpha = 2$ and $\beta = 1$.
Subcase i: Let $x \in (0, \pi)$. Then we have that

$$\phi_2 \circ F \circ \pi_1(x) = \phi_2 \circ F([x]) = \phi_2(e^{ix}) = x.$$

Subcase ii: Let $x \in (\pi, 2\pi)$. Then we have that

$$\phi_2 \circ F \circ \pi_1(x) = \phi_2 \circ F([x]) = \phi_2(e^{ix}) = x - 2\pi.$$

As all of these maps are C^∞, we have that $\phi_\alpha \circ F \circ \pi_\beta$ is always C^∞. It then follows that F is C^∞. $\square$

(c) Prove that $F : \mathbb{R}/2\pi\mathbb{Z} \to S^1$ is a diffeomorphism.

Proof: All of the maps described in the cases of (b) are identities or translations, so they are all diffeomorphisms. Thus, we have that F is a diffeomorphism. $\square$

Chapter 8

Problem 8.1: Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the map

$$(u, v, w) = F(x, y) = (x, y, xy).$$

Let $p = (x, y) \in \mathbb{R}^2$. Compute $F_*(\partial/\partial x\big|_p)$ as a linear combination of $\partial/\partial u$, $\partial/\partial v$, and $\partial/\partial w$ at $F(p)$.

Proof: To determine the coefficient a in $F_*(\partial/\partial x) = a\partial/\partial u + b\partial/\partial v + c\partial/\partial w$, we apply both sides to u to obtain

$$F_*\left(\frac{\partial}{\partial x}\right)u = \left(a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v} + c\frac{\partial}{\partial w}\right)u = a, \text{ so } a = F_*\left(\frac{\partial}{\partial x}\right)u = \frac{\partial}{\partial x}(u \circ F) = \frac{\partial}{\partial x}(x) = 1.$$

Similarly, we have that

$$F_*\left(\frac{\partial}{\partial x}\right)v = \left(a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v} + c\frac{\partial}{\partial w}\right)v = b, \text{ so } b = F_*\left(\frac{\partial}{\partial x}\right)v = \frac{\partial}{\partial x}(v \circ F) = \frac{\partial}{\partial x}(y) = 0,$$

as well as

$$F_*\left(\frac{\partial}{\partial x}\right)w = \left(a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v} + c\frac{\partial}{\partial w}\right)w = c, \text{ so } c = F_*\left(\frac{\partial}{\partial x}\right)w = \frac{\partial}{\partial x}(w \circ F) = \frac{\partial}{\partial x}(xy) = y.$$

Thus, it follows that $F_*(\partial/\partial x) = \partial/\partial u + y\partial/\partial w$. $\square$

Problem 8.2: Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. For any $p \in \mathbb{R}^n$, there is a canonical identification $T_p(\mathbb{R}^n) \xrightarrow{\sim} \mathbb{R}^n$ given by

$$\sum a^i \frac{\partial}{\partial x^i}\bigg|_p \mapsto \mathbf{a} = \langle a^1, \ldots, a^n\rangle.$$

Show that the differential $L_{*,p} : T_p(\mathbb{R}^n) \to T_{L(p)}(\mathbb{R}^m)$ is the map $L : \mathbb{R}^n \to \mathbb{R}^m$ itself, with the identification of the tangent spaces as above.

Proof: Let $v = \sum a^i \frac{\partial}{\partial x^i}\big|_p \mapsto \mathbf{a}$. We then must only observe that

$$L_*(v\big|_p) = \lim_{t\to 0} \frac{L(p + t\mathbf{a}) - L(p)}{t} = \lim_{t\to 0} \frac{L(p) + tL(\mathbf{a}) - L(p)}{t} = L(\mathbf{a}).$$

$\square$

Problem 8.3: Fix a real number α and define $F : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\begin{bmatrix} u \\ v \end{bmatrix} = (u, v) = F(x, y) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Let $X = -y\partial/\partial x + x\partial/\partial y$ be a vector field on $\mathbb{R}^2$. If $p = (x, y) \in \mathbb{R}^2$ and $F_*(X_p) = (a\partial/\partial u + b\partial/\partial v)\big|_{F(p)}$, find a and b in terms of x, y, and α.

Proof: First observe that

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\alpha - y\sin\alpha \\ x\sin\alpha + y\cos\alpha \end{bmatrix}.$$

We shall follow a process similar to the one used in 8.1. To determine the coefficient a in $F_*(X_p) = a\partial/\partial u + b\partial/\partial v$, we apply both sides to u to obtain

$$F_*(X_p)u = \left(a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v}\right)u = a, \text{ so}$$
$$a = F_*(X_p)u = X_p(u \circ F) = X_p(x\cos\alpha - y\sin\alpha) = -x\sin\alpha - y\cos\alpha.$$

Similarly we have that

$$F_*(X_p)v = \left(a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v}\right)v = b, \text{ so}$$
$$b = F_*(X_p)v = X_p(v \circ F) = X_p(x\sin\alpha + y\cos\alpha) = x\cos\alpha - y\sin\alpha.$$

Thus, we have that $a = -x\sin\alpha - y\cos\alpha$ and $b = x\cos\alpha - y\sin\alpha$. $\square$

Problem 8.4: Let x, y be the standard coordinates on $\mathbb{R}^2$, and let U be the open set

$$U = \mathbb{R}^2 - \{(x, 0) \mid x \geq 0\}.$$

On U the polar coordinates r, θ are uniquely defined by

$$x = r\cos\theta,$$
$$y = r\sin\theta,\ r > 0,\ 0 < \theta < 2\pi.$$

Find $\partial/\partial r$ and $\partial/\partial\theta$ in terms of $\partial/\partial x$ and $\partial/\partial y$.

Proof: Let $(U, (r, \theta))$ and $(U, (r\cos\theta, r\sin\theta))$ be charts on U. Then using Proposition 8.10 we immediately have that

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}.$$

Similarly, we have

$$\frac{\partial}{\partial\theta} = \frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial}{\partial y} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}.$$

Thus, we have $\frac{\partial}{\partial r} = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}$ and $\frac{\partial}{\partial \theta} = -r\sin\theta \frac{\partial}{\partial x} + r\cos\theta \frac{\partial}{\partial y}$. $\square$

Problem 8.5: Prove proposition 8.15: Let $c : (a,b) \to M$ be a smooth curve, and let $(U, x^1, \ldots, x^n)$ be a coordinate chart about $c(t)$. Write $c^i = x^i \circ c$ for the ith component of c in the chart. Then $c'(t)$ is given by

$$c'(t) = \sum_{i=1}^{n} \dot{c}^i(t) \frac{\partial}{\partial x^i}\bigg|_{c(t)}.$$

Thus, relative to the basis $\{\partial/\partial x\big|_p\}$ fro $T_{c(t)}M$, the velocity $c'(t)$ is represented by the column vector

$$\begin{bmatrix} \dot{c}^1(t) \\ \vdots \\ \dot{c}^n(t) \end{bmatrix}.$$

Proof: We know that $c'(t) = \sum a^j \frac{\partial}{\partial x^j}$. Now we apply this expression to x_i to obtain a_i in the following manner:

$$\begin{aligned}
c'(t)x^i &= \left(\sum a^j \frac{\partial}{\partial x^j}\right) x^i && \text{Apply to } x^i \\
c'(t)x^i &= \left(a^1 \frac{\partial}{\partial x^1} + \cdots + a^i \frac{\partial}{\partial x^i} + \cdots + a^n \frac{\partial}{\partial x^n}\right) x^i && \text{Expand sum} \\
c'(t)x^i &= a^1 \frac{\partial}{\partial x^1} x^i + \cdots + a^i \frac{\partial}{\partial x^i} x^i + \cdots + a^n \frac{\partial}{\partial x^n} x^i && \text{Distribute} \\
c'(t)x^i &= 0 + \cdots + a^i + \cdots + 0 && \text{Evaluate each } \frac{\partial}{\partial x^i} \\
c_*\left(\frac{d}{dt}\right) x^i &= a^i && \text{Definition of } c' \\
\frac{d}{dt}(x^i \circ c) &= a^i && \text{Definition of } c_* \\
\frac{d}{dt}(c^i) &= a^i && \text{Definition of } x^i \\
\dot{c}^i &= a^i && \text{Definition of } \frac{d}{dt}
\end{aligned}$$

Thus, we have that $a^i = \dot{c}^i$, so it follows that $c'(t) = \sum_{i=1}^{n} \dot{c}^i(t) \frac{\partial}{\partial x^i}\big|_{c(t)}$, as desired. $\square$

Problem 8.6: Let $p = (x, y)$ be a point in $\mathbb{R}^2$. Then

$$c_p(t) = \begin{bmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \; t \in \mathbb{R},$$

is a curve with initial point p in $\mathbb{R}^2$. Compute the velocity vector $c_p'(0)$.

Proof: First observe that

$$c_p(t) = \begin{bmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos 2t - y \sin 2t \\ x \sin 2t + y\cos 2t \end{bmatrix}.$$

Note that $c_p^1(t) = x\cos 2t - y\sin 2t$ and $c_p^2(t) = x\sin 2t + y\cos 2t$, so it follows that $\dot{c}_p^1(t) = -2x\sin 2t - 2y\cos 2t$ and $\dot{c}_p^2(t) = 2x\cos 2t - 2y\sin 2t$. By applying Proposition 8.15 we immediately obtain

$$\begin{aligned} c_p'(0) &= \dot{c}_p^1(0)\frac{\partial}{\partial x}\bigg|_{c(0)} + \dot{c}_p^2(0)\frac{\partial}{\partial y}\bigg|_{c(0)} \\ &= (-2x\sin 2(0) - 2y\cos 2(0))\frac{\partial}{\partial x}\bigg|_{c(0)} + (2x\cos 2(0) - 2y\sin 2(0))\frac{\partial}{\partial y}\bigg|_{c(0)} \\ &= (0-2y)\frac{\partial}{\partial x}\bigg|_{c(0)} + (2x-0)\frac{\partial}{\partial y}\bigg|_{c(0)} \\ &= 2y\frac{\partial}{\partial x}\bigg|_{c(0)} + 2x\frac{\partial}{\partial y}\bigg|_{c(0)}. \end{aligned}$$

Thus, we have that $c_p'(0) = 2y\frac{\partial}{\partial x}\big|_{c(0)} + 2x\frac{\partial}{\partial y}\big|_{c(0)}$. $\square$

Problem 8.7: If M and N are manifolds, let $\pi_1 : M\times N \to M$ and $\pi_2 : M\times N\to N$ be the two projections. Prove that for $(p,q)\in M\times N$,

$$(\pi_{1*}, \pi_{2*}) : T_{p,q}(M\times N) \to T_pM \times T_qN$$

is an isomorphism.

Proof: If $(U,\phi) = (U, x^1, \ldots, x^m)$ and $(V,\psi) = (V, y^1,\ldots,y^n)$ are charts about p in M and q in N repsectively it follows from Proposition 5.18 that a chart about (p,q) in $M\times N$ will be given by

$$(U\times V, \phi\times\psi) = (U\times V, (\pi_1^*\phi, \pi_2^*\psi)) = (U\times V, \overline{x}^1,\ldots,\overline{x}^n, \overline{y}^1,\ldots,\overline{y}^n),$$

where $\overline{x}^i = \pi_1^* x^i$ and $\overline{y}^i = \pi_2^* y^i$. Let $\pi_{1*}(\partial/\partial\overline{x}^j) = \sum a_j^i \partial/\partial x^i$. Then we have that

$$a_j^i = \pi_{1*}\left(\frac{\partial}{\partial \overline{x}^j}\right) x^i = \frac{\partial}{\partial\overline{x}^j}(x^i\circ\pi_1) = \frac{\partial \overline{x}^i}{\partial\overline{x}^j} = \delta_j^i.$$

As a result, it follows that

$$\pi_{1*}\left(\frac{\partial}{\partial\overline{x}^j}\right) = \sum_i \delta_j^i\frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^j}, \text{ which implies } \pi_{1*}\left(\frac{\partial}{\partial\overline{x}^j}\bigg|_{(p,q)}\right) = \frac{\partial}{\partial x^j}\bigg|_p.$$

We then similarly have that

$$\pi_{1*}\left(\frac{\partial}{\partial \overline{y}^i}\right) = 0,\ \pi_{2*}\left(\frac{\partial}{\partial \overline{x}^j}\right) = 0, \text{ and } \pi_{2*}\left(\frac{\partial}{\partial \overline{y}^j}\right) = \frac{\partial}{y^j}.$$

Thus, we have that a basis for $T_{(p,q)}(M \times N)$ will be given by

$$\left.\frac{\partial}{\partial \overline{x}^1}\right|_{(p,q)}, \ldots, \left.\frac{\partial}{\partial \overline{x}^m}\right|_{(p,q)}, \left.\frac{\partial}{\partial \overline{y}^1}\right|_{(p,q)}, \ldots, \left.\frac{\partial}{\partial \overline{y}^n}\right|_{(p,q)}.$$

And then a basis for $T_pM \times T_qN$ is given by

$$\left(\left.\frac{\partial}{\partial x^1}\right|_p, 0\right), \ldots, \left(\left.\frac{\partial}{\partial x^m}\right|_p, 0\right), \left(0, \left.\frac{\partial}{\partial y^1}\right|_q\right), \ldots, \left(0, \left.\frac{\partial}{\partial y^n}\right|_q\right).$$

It then follows that the linear map (π_{1*}, π_{2*}) maps a basis of $T_{(p,q)}(M \times N)$ to a basis of $T_pM \times T_qN$. As a result, we have that (π_{1*}, π_{2*}) is an isomorphism. $\square$

Problem 8.8: Let G be a Lie group with multiplication map $\mu : G \times G \to G$, the inverse map $\imath : G \to G$, and identity element e.

(a) Show that the differential at the identity of the multiplication map μ is addition:

$$\mu_{*,(e,e)} : T_eG \times T_eG \to T_eG, \text{ defined by } \mu_{*,(e,e)}(X_e, Y_e) = X_e + Y_e.$$

Proof: Let $c(t)$ be a curve starting at $e \in G$ such that $c'(0) = X_e$. Now define $\alpha(t) = (c(t), e)$ and note that $\alpha(t)$ is a curve starting at $(e, e) \in G \times G$ and $\alpha'(0) = (X_e, 0)$. Similarly, let $b(t)$ be a curve starting at $e \in G$ such that $b'(0) = Y_e$. Now define $\beta(t) = (e, b(t))$ and note that $\beta(t)$ is a curve starting at $(e, e) \in G \times G$ and $\beta'(0) = (0, Y_e)$.

We shall now compute $u_{*,(e,e)}(X_e, Y_e)$ using $\alpha(t)\beta(t)$:

$$\begin{aligned}\mu_{*,(e,e)}(X_e, Y_e) = \left.\frac{d}{dt}\right|_0 (u \circ \alpha\beta)(t) &= \left.\frac{d}{dt}\right|_0 (u(c(t)e, b(t)e)) = \left.\frac{d}{dt}\right|_0 c(t)b(t) \\ &= c'(0)b(0) + c(0)b'(0) = X_e(e,e) + Y_e(e,e) = X_e + Y_e.\end{aligned}$$

Thus, we have $\mu_{*,(e,e)}(X_e, Y_e) = X_e + Y_e$, as desired. $\square$

(b) Show that the differential at the identity of $\imath$ is the negative:

$$\imath_{*,e} : T_eG \to T_eG, \text{ defined by } \imath_{*,e}(X_e) = -X_e.$$

Proof: Let $c(t)$ be a curve starting at $e \in G$ such that $c'(0) = X_e$. We shall now compute $\imath_{*,e}(X_e)$ using $c(t)$:

$$i_{*,e}(X_e) = \left.\frac{d}{dt}\right|_0 (i \circ c)(t) = \left.\frac{d}{dt}\right|_0 (-c(t)) = -c'(0) = -X_e.$$

Thus, we have $i_{*,e}(X_e) = -X_e$, as desired. $\square$

Problem 8.9: Let $X_1, \ldots, X_n$ be n vector fields on an open subset U of a manifold of dimension n. Suppose that at $p \in U$, the vectors $(X_1)_p, \ldots, (X_n)_p$ are linearly independent. Show that there is a chart $(V, x^1, \ldots, x^n)$ about p such that $(X_i)_p = (\partial/\partial x^i)_p$ for $i = 1, \ldots, n$. *Proof:* Let $(V, y^1, \ldots, y^n)$ be a chart about p. Suppose that $(X_j)_p = \sum_i a^i_j \partial/\partial y^i|_p$. As we have that $(X_1)_p, \ldots, (X_n)_p$ are linearly independent, we know that the matrix $A = [a^i_j]$ is nonsingular. Now define a new coordinate system $x^1, \ldots, x^n$ by

$$y^i = \sum_{j=1}^{n} a^i_j x^j \text{ for } i = 1, \ldots, n.$$

It then follows from the chain rule that

$$\frac{\partial}{\partial x^j} = \sum_i \frac{\partial y^i}{\partial x^j} \frac{\partial}{\partial y^i} = \sum a^i_j \frac{\partial}{\partial y^i}.$$

Which when considered at the point p the above equations can be realized as

$$\left.\frac{\partial}{\partial x^j}\right|_p = \sum_i a^i_j \left.\frac{\partial}{\partial y^i}\right|_p = (X_j)_p.$$

Representing this result in matrix notation then gives

$$\begin{bmatrix} y^1 \\ \vdots \\ y^n \end{bmatrix} = A \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix}, \text{ which implies that } \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix} = A^{-1} \begin{bmatrix} y^1 \\ \vdots \\ y^n \end{bmatrix}.$$

As a result, we have that our new coordinate system described above is equivalent to $x^j = \sum_{i=1}^{n} (A^{-1})^j_i y^i$. $\square$

Problem 8.10: A real-valued function $f : M \to \mathbb{R}$ on a manifold is said to have a local maximum at $p \in M$ if there is a neighborhood U of p such that $f(p) \geq f(q)$ for all $q \in U$.
(a) Prove that if a differentiable function $f : I \to \mathbb{R}$ defined on an open interval I has a local maximum at $p \in I$, the $f'(p) = 0$. *Proof:* As f has a local maximum at $p \in I$, it follows that $f(p) \geq f(q)$ for all $q \in I$. Because f is differentiable the following limits exist:

$$f'(p) = \lim_{x \to p^-} \frac{f(x) - f(p)}{x - p} \geq 0 \text{ and } f'(p) = \lim_{x \to p^+} \frac{f(x) - f(p)}{x - p} \leq 0.$$

We now have that $0 \geq f'(p) \geq 0$, so $f'(p) = 0$. $\square$

(b) Prove that a local maximum of a C^∞ function $f : M \to \mathbb{R}$ is a critical point of f.

Proof: Let $p \in M$ be the point at which the local maximum occurs. Now let X_p be a tangent vector in T_pM and let $c(t)$ be a curve in M starting at p with an arbitrary initial vector X_p. Then it follows that $f \circ p$ will be a real valued function with a local maximum at 0. Applying (a) tells us that the value of the derivative is 0 at 0. So then we have that

$$0 = \left.\frac{d}{dt}\right|_0 (f \circ c)(t) = (f_*)_{c(0)} \circ (c_*)_0 \left.\frac{d}{dt}\right|_0 = (f_{*,p})(c'(0)) = f_{*,p}(X_p),$$

so $f_{*,p}(X_p) = 0$, which imtextbfplies that f_* is not surjective at p because X_p was arbitrary. As a result, we have that p is a critical point of M. $\square$

Chapter 9

Problem 9.1: Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = x^3 - 6xy + y^2$. Find all values $c \in \mathbb{R}$ for which the level set $f^{-1}(c)$ is a regular submanifold of $\mathbb{R}^2$.

Proof: To find the desired values, we will construct the Jacobian for this map (note that this case is somewhat degenerate):

$$J(f) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 3x^2 - 6y & -6x + 2y \end{bmatrix}.$$

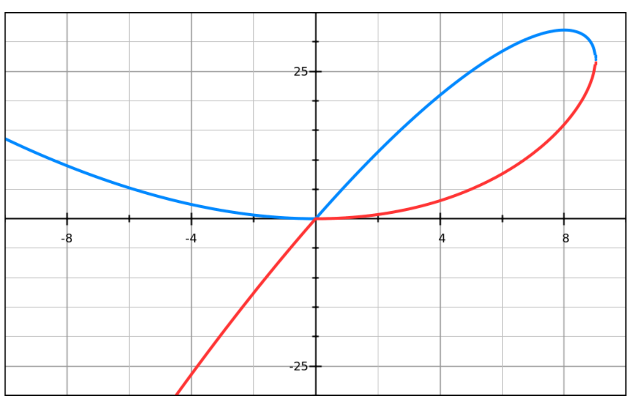
For each $(x, y) \in \mathbb{R}^2$, there are two cases for the rank of J.

Case 1: $\mathrm{rk}(J(x, y)) = 0$. Then (x, y) is a critical point of f. When $\mathrm{rk}(J(f)) = 0$ we know that $3x^2 - 6y = 0$ and $-6x + 2y = 0$. This system of equations can be solved to show that $(0, 0)$ and $(6, 18)$ are the only two points satisfying these equations. Then we have that $f(0, 0) = 0$ and $f(6, 18) = -108$, so 0 and -108 are the values of $c \in \mathbb{R}$ for which the level set will not be a regular submanifold of $\mathbb{R}^2$.

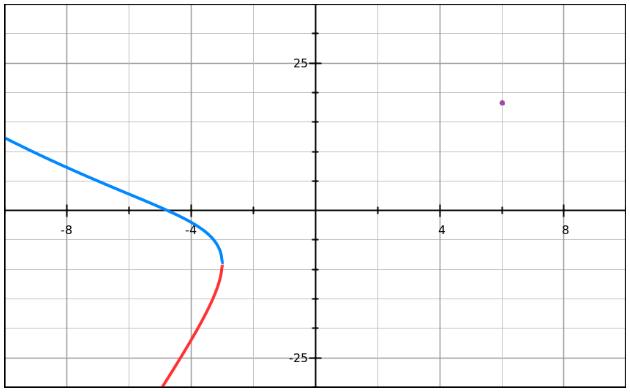
Case 2: $\mathrm{rk}(J(x, y)) = 1$. Then (x, y) is a regular point of f. As we solved for the critical points in Case 1, we know that the regular points will be given by $\mathbb{R} \setminus \{-108, 0\}$.

In conclusion, we have that the level set $f^{-1}(c)$ is a regular submanifold of $\mathbb{R}^2$ for all $c \in \mathbb{R} \setminus \{-108, 0\}$ as long as $f^{-1}(c)$ is nonempty. However, it can easily be seen that for any $x \in R$ we have $(\sqrt[3]{x}, 0) \mapsto (\sqrt[3]{x})^3 - 6(\sqrt[3]{x})(0) + 0^2 = x$, so no $f^{-1}(c)$ will be empty.

Note: Unfortunately, the regular level set theorem does not guarantee to us that $f^{-1}(0)$ and $f^{-1}(-108)$ will fail to be submanifolds. However, graphing the solutions of $x^3 - 6xy + y^2 = 0$ yields the following set in $\mathbb{R}^2$:



This solution set is not a submanifold because it fails to be locally Euclidean to $\mathbb{R}$ at $(0, 0)$. Similarly, the graph of the solutions of $x^3 - 6xy + y^2 = -108$ yields the following set in $\mathbb{R}^2$:



This solution set is not a submanifold because its connected components do not all have the same dimension; the left side of the set has dimension 1, while the isolated point at $(6, 18)$ has dimension 0. $\square$

Problem 9.2: Let x, y, z, w be the standard coordinates on $\mathbb{R}^4$. Is the solution set of $x^5 + y^5 + z^5 + w^5 = 1$ in $\mathbb{R}^4$ a smooth manifold? Explain why or why not. (Assume that the subset is given the subspace topology.)

Proof: Let $S \subseteq \mathbb{R}^4$ be the solution set of $x^5 + y^5 + z^5 + w^5 = 1$ and define $f : \mathbb{R}^4 \to \mathbb{R}$ by $(x, y, z, w) \mapsto x^5 + y^5 + z^5 + w^5$. Note that $f^{-1}(1) = S$ and $(1, 1, 1, 1) \in f^{-1}(1)$ so then $f^{-1}(1) \neq \emptyset$. To determine the answer to the given question, we will construct the Jacobian for f:

$$J(f) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \end{bmatrix} = \begin{bmatrix} 5x^4 & 5y^4 & 5z^4 & 5w^4 \end{bmatrix}.$$

As in 9.1, any 4-tuple $(x, y, z, w) \in \mathbb{R}^4$ that causes $J(f)$ to have rank zero will be a critical value of f. Note that this occurs only when $x = y = z = w = 0$; however, we can see that $(0, 0, 0, 0) \notin S$ because $0^5 + 0^5 + 0^5 + 0^5 = 0 \neq 1$. Thus, we have that S will be a smooth manifold in $\mathbb{R}^4$ because S does not contain any critical points of f. $\square$

Problem 9.3: Is the solution set of the system of equations $x^3 + y^3 + z^3 = 1$ and $z = xy$ in $\mathbb{R}^3$ a smooth manifold? Prove your answer.

Proof: Let $u(x, y, z) = x^3 + y^3 + z^3$ and $v(x, y, z) = z - xy$ and define $f : \mathbb{R}^3 \to \mathbb{R}^2$ by $(x, y, z) \mapsto (u(x, y, z), v(x, y, z))$. Now let $S \subseteq \mathbb{R}^3$ be the solution set of $f(x, y, z) = (1, 0)$. Note that $(1, 0, 0) \in f^{-1}(1, 0)$, so $f^{-1}(1, 0) \neq \emptyset$. To determine the answer to the given question, we will construct the Jacobian for this map:

$$J(f) = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{bmatrix} = \begin{bmatrix} 3x^2 & 3y^2 & 3z^2 \\ y & x & 1 \end{bmatrix}.$$

If the rank of $J(f) < 2$ for some (x, y, z), then (x, y, z) is critical point of f. The rank of $J(u, v)$ will be less than 2 if and only if the all of the 2×2 minors of $J(f)$ are zero. By setting the 2×2 minors equal to 0, we get the system

$$3x^3 - 3y^3 = 0,\ 3y^2 - 3z^2x = 0, \text{ and } 3x^2 - 3z^2y = 0.$$

The solution set of these equations is given by $Z = \{(0, 0, z) \mid x = y = 0, z \in \mathbb{R}\} \cup \{(x, y, z) \mid x = y = -z^2\}$.

Let u_1 and v_0 be the solution sets of $u = 1$ and $v = 0$ respectively, and note that $f^{-1}(1, 0) = u_1 \cap v_0$. It can immediately be seen that $Z \cap v_0 = \{(0, 0, 0), (-1, -1, 1)\}$. However, $(0, 0, 0), (-1, -1, 1) \notin u_1$, so then $u_0 \cap v_0 \cap Z = \emptyset$, so $f^{-1}(1, 0) \cap Z = \emptyset$. As a result, $f^{-1}(1, 0)$ does not contain any critical values of f, so S will be a smooth manifold. □

Problem 9.4: Suppose that a subset S of $\mathbb{R}^2$ has the property that locally on S one of the coordinates is a C^∞ function of the other coordinate. Show that S is a regular submanifold of $\mathbb{R}^2$. (Note that the unit circle defined by $x^2 + y^2 = 1$ has this property. At every point of the circle, there is a neighborhood in which y is a C^∞ function of x or x is a C^∞ function of y.)

Proof: Let $p \in S$; then there exists some open set $U \subseteq \mathbb{R}^2$ such that one of the coordinates in $U \cap S$ is a C^∞ function of the other coordinate. Without loss of generality assume that $y = f(x)$ for some C^∞ function $f : A \to B$, where $A, B \subseteq \mathbb{R}$. Now let $V = A \times B \subseteq U$ and define $F : V \to \mathbb{R}^2$ by $F(x, y) = (x, y - f(x))$. Since F is a diffeomorphism onto its image, we may use it as a coordinate map. In the chart $(V, x, y - f(x))$ we have that $V \cap S$ is defined by the vanishing of the coordinate $y - f(x)$. This then proves that S is a regular manifold. □

Problem 9.6: A polynomial $F(x_0, \dots, x_n) \in \mathbb{R}[x_0, \dots, x_n]$ is *homogeneous of degree k* if it is a linear combination of monomials $x_0^{i_0} \cdots x_n^{i_n}$ of degree $\sum_{j=0}^n i_j = k$. Let $F(x_0, \dots, x_n)$ be a homogeneous polynomial of degree k. Clearly, for any $t \in \mathbb{R}$ we have $F(tx_0, \dots, tx_n) = t^k F(x_0, \dots, x_n)$. Show that $\sum_{i=0}^n x_i \frac{\partial F}{\partial x_i} = kF$.

Proof: Define $y_i = tx_i$. Then we have from the given information that

$$F(y_0, \dots, y_n) = F(tx_0, \dots, tx_n) = t^k F(x_0, \dots, x_n).$$

Differentiating the right and left sides with respect to t yields

$$\sum_{i=0}^n \frac{\partial F}{\partial y_i} \frac{dy_i}{dt} = kt^{k-1} F(x_0, \dots, x_n).$$

But we know that $\frac{dy_i}{dt} = x_i$, so we really have that

$$\sum_{i=0}^{n} x_i \frac{\partial F}{\partial y_i} = kt^{k-1} F(x_0, \ldots, x_n).$$

As this is true for all $t \in \mathbb{R}$, we may let $t = 1$ to observe that

$$\sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i} = kF(x_0, \ldots, x_n),$$

as desired. $\square$

Problem 9.7: On the projective space $\mathbb{R}P^n$ a homogeneous polynomial $F(x_0, \ldots, x_n)$ of degree k is not a function, since its value at a point $[a_0, \ldots, a_n]$ is not unique. However, the zero set in $\mathbb{R}P^n$ of a homogeneous polynomial $F(x_0, \ldots, x_n)$ is well defined, since $F(a_0, \ldots, a_n) = 0$ if and only if

$$F(ta_0, \ldots, ta_n) = t^k F(a_0, \ldots, a_n) = 0 \text{ for all } t \in \mathbb{R}^\times := R - \{0\}.$$

The zero set of finitely many homogeneous polynomials in $\mathbb{R}P^n$ is called a *real projective variety*. A projective variety defined by a single homogeneous polynomial of degree k is called a *hypersurface* of degree k. Show that the hypersurface $Z(F)$ defined by $F(x_0, x_1, x_2) = 0$ is smooth if $\partial F/\partial x_0$, $\partial F/\partial x_1$, and $\partial F/\partial x_2$ are not simultaneously zero on $Z(F)$.

Proof: We have from Problem 9.6 that $kF = x_0 \frac{\partial F}{\partial x_0} + x_1 \frac{\partial F}{\partial x_1} + x_2 \frac{\partial F}{\partial x_2}$; furthermore we know that on $Z(F)$ we have $F = 0$. It then follows that on $Z(F)$ we have

$$0 = x_0 \frac{\partial F}{\partial x_0} + x_1 \frac{\partial F}{\partial x_1} + x_2 \frac{\partial F}{\partial x_2}.$$

By way of contradiction say that $\frac{\partial F}{\partial x_1} = \frac{\partial F}{\partial x_2} = 0$; then it follows that $\frac{\partial F}{\partial x_0} = 0$. However, this contradicts our given information that not all $\frac{\partial F}{\partial x_0}$, $\frac{\partial F}{\partial x_1}$, and $\frac{\partial F}{\partial x_2}$ are equal to zero, so we may say that without loss of generality $\frac{\partial F}{\partial x_2} \neq 0$.

Recall that $U_0 = \{[x_0, x_1, x_2] \mid x_0 \neq 0\}$, and define $x = \frac{x_1}{x_0}$, $y = \frac{x_2}{x_0}$, and $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = F(1, x, y)$. Now define $\psi : \mathbb{R}P^2 \to \mathbb{R}^2$ on U_0 by $\psi([x_0, x_1, x_2]) = (1, f(x, y))$. It then follows that

$$\psi \circ \phi_0^{-1}(x, y) = \psi\left(\left[1, \frac{x_1}{x_0}, \frac{x_2}{x_0}\right]\right) = (x, f(x, y)).$$

We also know that $F(x_0, x_1, x_2) = x_0^k f(\frac{x_1}{x_0}, \frac{x_2}{x_0})$, so $\frac{\partial F}{\partial x_2} = x_0^k \frac{\partial f}{\partial y} \frac{1}{x_0} = x_0^{k-1} \frac{\partial f}{\partial y}$. Then because $\frac{\partial F}{\partial x_2} \neq 0$ we have that $\frac{\partial f}{\partial y} \neq 0$. It can now be seen that

$$\det(J(\psi \circ \phi_0^{-1})) = \begin{vmatrix} 1 & 0 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{vmatrix} = \frac{\partial f}{\partial y} \neq 0,$$

so we know that $\psi \circ \phi_0^{-1}$ is C^1 compatible.

We can construct similar arguments on U_1 and U_2; as U_0, U_1, and U_2 cover $\mathbb{R}P^2$, we know that they will cover $Z(F)$. As a result, we have that $Z(F)$ will be a submanifold. $\square$

Problem 9.10: Let $p \in f^{-1}(S)$ and $(U, x^1, \ldots, x^m)$ be an adapted chart centered at $f(p)$ for M relative to S such that $U \cap S = Z(x^{m-k+1}, \ldots, x^m)$, the zero set of the functions $x^{m-k+1}, \ldots, x^m$. Define $g : U \to \mathbb{R}^k$ to be the map $g = (x^{m-k+1}, \ldots, x^m)$.

(a) Show that $f^{-1}(U) \cap f^{-1}(S) = (g \circ f)^{-1}(0)$.

Proof: Simply observe that

$$f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap S) = f^{-1}(g^{-1}(0)) = (g \circ f)^{-1}(0),$$

as desired. $\square$

(b) Show that $f^{-1}(U) \cap f^{-1}(S)$ is a regular level set of the function $g \circ f : f^{-1}(U) \to \mathbb{R}^k$.

Proof: Let $p \in f^{-1}(U) \cap f^{-1}(S) = f^{-1}(U \cap S)$; then $f(p) \in U \cap S$. Note that g_* will be given by the $k \times n$ matrix $[0 \mid I_k]$. Recall that S is defined by the vanishing of the last k coordinates; then any curve $\gamma(t) \in S$ will be of the form $\gamma(t) = (\gamma_1(t), \ldots, \gamma_{m-k}(t), 0, \ldots, 0)$ and $\gamma'(t) = (\gamma_1'(t), \ldots, \gamma_{m-k}'(t), 0, \ldots, 0)$. As a result, we have that

$$g_*(\gamma'(t)) = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \gamma_1'(t) \\ \vdots \\ \gamma_{m-k}'(t) \end{bmatrix} = 0.$$

As $\gamma(t)$ was arbitrary, it then follows that $g_*(T_pS) = 0$. Then because $g : U \to \mathbb{R}^k$ is a projection, we have that $g_*(T_{f(p)}M) = T_0(\mathbb{R}^k)$. By applying g_* to the transversality equation we obtain

$$\begin{aligned} g_*f_*(T_pN) + g_*(T_{f(p)}S) &= g_*(T_{f(p)}M) \\ g_*f_*(T_pN) + 0 &= T_0(\mathbb{R}^k) \\ g_*f_*(T_pN) &= T_0(\mathbb{R}^k). \end{aligned}$$

As a result, it follows that $g \circ f : f^{-1}(U) \to \mathbb{R}^k$ is a submersion at p. As $p \in f^{-1}(U) \cap f^{-1}(S) = (g \circ f)^{-1}(0)$ was arbitrary, it follows that this set is a regular level set of $g \circ f$. $\square$

(c) Prove the transervsality theorem.

Proof: It follows from the regular level set theorem that $f^{-1}(U) \cap f^{-1}(S)$ is a regular submanifold of $f^{-1}(U) \subseteq N$. As a result, every $p \in f^{-1}(S)$ has an adapted chart relative to $f^{-1}(S)$ in N. $\square$

Chapter 11

Problem 11.1: The unit sphere S^n in $\mathbb{R}^{n+1}$ is defined by the equation $\sum_{i=1}^{n+1}(x^i)^2 = 1$. For $p = (p^1, \ldots, p^{n+1}) \in S^n$, show that a necessary and sufficient condition for

$$X_p = \sum a^i \frac{\partial}{\partial x^i}\bigg|_p \in T_p(\mathbb{R}^{n+1})$$

to be tangent to S^n at p is $\sum a^k p^i = 0$.

Proof: Let $c : \mathbb{R} \to S^n$ be a curve defined by $c(t) = (x^1(t), \ldots, x^{n+1}(t))$ with $c(0) = p$ and $c'(0) = X_p$. Define $H = \{(a_1, \ldots, a_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} a^i p^i = 0\}$. Differentiating the equation $\sum_{i=1}^{n+1}(x^i(t))^2 = 1$ then yields

$$\sum_{i=1}^{n+1} 2(x'(t))(\dot{x}^i(t))\big|_{t=0} = 0$$

$$2\sum_{i=1}^{n+1} p^i a^i = 0.$$

This then implies that $T_p(S^2) \subseteq H$, but because $\dim(T_p(S^2)) = \dim(H)$ and both spaces are linear, then $T_p(S^2) = H$. As a result, we have that the condition is both necessary and sufficient, as desired. $\square$

Problem 11.2:

(a) Let $i : S^1 \hookrightarrow \mathbb{R}^2$ be the inclusion map of the unit circle. In this problem, we denote by x, y the stadnard coordinates on $\mathbb{R}^2$ and by $\overline{x}, \overline{y}$ their restrictions to S^1. Thus, $\overline{x} = i^*x$ and $\overline{y} = i^*y$. On the upper semicircle $U = \{(a, b) \in S^1 \mid b > 0\}$, $\overline{x}$ is a local coordinate, so that $\partial/\partial\overline{x}$ is defined. Prove that for $p \in U$,

$$i_*\left(\frac{\partial}{\partial \overline{x}}\bigg|_p\right) = \left(\frac{\partial}{\partial x} + \frac{\partial \overline{y}}{\partial \overline{x}}\frac{\partial}{\partial y}\right)\bigg|_p.$$

Thus, although $i_* : T_pS^1 \to T_p\mathbb{R}^2$ is injective, $\partial/\partial\overline{x}\,|_p$ cannot be identified with $\partial/\partial x\,|_p$.

Proof: It follows from the definition of pullback that $\overline{x} = i^*x = x \circ i$ and $\overline{y} = i^*y = y \circ i$. We also know that

$$i_*\left(\frac{\partial}{\partial \overline{x}}\bigg|_p\right) = \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y} \text{ for some } \alpha \text{ and } \beta.$$

We can then apply both sides of this equation to x to obtain α:

$$i_*\left(\frac{\partial}{\partial \overline{x}}\bigg|_p\right)x = \left(\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}\right)x$$

$$\begin{aligned}
\left.\frac{\partial}{\partial \overline{x}}\right|_p (x \circ i) &= \alpha \frac{\partial}{\partial x} x + \beta \frac{\partial}{\partial y} x \\
\left.\frac{\partial}{\partial \overline{x}}\right|_p (\overline{x}) &= \alpha + 0 \\
1 &= \alpha
\end{aligned}$$

We can similarly apply both sides of the original equation to y to obtain β:

$$\begin{aligned}
i_* \left(\left.\frac{\partial}{\partial \overline{x}}\right|_p \right) y &= \left(\alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y} \right) y \\
\left.\frac{\partial}{\partial \overline{x}}\right|_p (y \circ i) &= \alpha \frac{\partial}{\partial x} y + \beta \frac{\partial}{\partial y} y \\
\left.\frac{\partial}{\partial \overline{x}}\right|_p (\overline{y}) &= 0 + \beta \\
\left.\frac{\partial \overline{y}}{\partial \overline{x}}\right|_p &= \beta
\end{aligned}$$

As a result, we have that

$$i_* \left(\left.\frac{\partial}{\partial \overline{x}}\right|_p \right) = \frac{\partial}{\partial x} + \left.\frac{\partial \overline{y}}{\partial \overline{x}}\right|_p \frac{\partial}{\partial y} = \left.\left(\frac{\partial}{\partial x} + \frac{\partial \overline{y}}{\partial \overline{x}} \frac{\partial}{\partial y} \right)\right|_p ,$$

as desired. $\square$

Problem 11.3: Show that a smooth map f from a compact manifold N to $\mathbb{R}^m$ has a critical point. (Hint: Let $\pi : \mathbb{R}^m \to \mathbb{R}$ be the projection of the first factor. Consider the compositive map $\pi \circ f : N \to \mathbb{R}$ A second proof uses Corollary 11.6 and the connectedness of $\mathbb{R}^m$.)

Proof: By way of contradiction assume that f does not have a critical point; it then follows that f will be a submersion. We also know that the projection to the first factor $\pi : \mathbb{R}^m \to \mathbb{R}$ is a submersion, so then the composite function $\pi \circ f : N \to \mathbb{R}$ must be a submersion. However, as we know that N is compact, $\pi \circ f$ must have a maximum, but this implies $\pi \circ f$ has a critical point, which is a contradiction. Thus, f must have a critical point. $\square$

Problem 11.4: On the upper hemisphere of the unit sphere S^2, we have the coordinate map $\phi = (u, v)$, where

$$u(a, b, c) = a \text{ and } v(a, b, c) = b.$$

So the derivations $\partial/\partial u \mid_p, \partial/\partial v \mid_p$ are tangent vectors of S^2 at any point $p = (a, b, c)$ on the upper hemisphere. Let $i : S^2 \to \mathbb{R}^3$ be the inclusion and x, y, z the standard coordinates on

$\mathbb{R}^3$. The differential $i_* : T_pS^2 \to T_p\mathbb{R}^3$ maps $\partial/\partial u\mid_p, \partial/\partial v\mid_p$ into $T_p\mathbb{R}^3$. Thus,

$$i_*\left(\frac{\partial}{\partial u}\bigg|_p\right) = \alpha^1 \frac{\partial}{\partial x}\bigg|_p + \beta^1 \frac{\partial}{\partial y}\bigg|_p + \gamma^1 \frac{\partial}{\partial z}\bigg|_p,$$

$$i_*\left(\frac{\partial}{\partial v}\bigg|_p\right) = \alpha^2 \frac{\partial}{\partial x}\bigg|_p + \beta^2 \frac{\partial}{\partial y}\bigg|_p + \gamma^2 \frac{\partial}{\partial z}\bigg|_p,$$

for some constants $\alpha^i, \beta^i, \gamma^i$. Find $(\alpha^i, \beta^i, \gamma^i)$ for $i = 1, 2$.

Proof: Note that $x \circ i = u$, $y \circ i = v$ and $zzzzz$. Then applying the equations given above to x, y, and z respectively yields the following:

$$\alpha^1 = i_*\left(\frac{\partial}{\partial u}\bigg|_p\right)(x) = \frac{\partial}{\partial u}\bigg|_p (x \circ i) = \frac{\partial}{\partial u}\bigg|_p (u) = 1$$

$$\beta^1 = i_*\left(\frac{\partial}{\partial u}\bigg|_p\right)(y) = \frac{\partial}{\partial u}\bigg|_p (y \circ i) = \frac{\partial}{\partial u}\bigg|_p (v) = 0$$

$$\gamma^1 = i_*\left(\frac{\partial}{\partial u}\bigg|_p\right)(z) = \frac{\partial}{\partial u}\bigg|_p (z \circ i) = \frac{\partial}{\partial u}\bigg|_p (\sqrt{1-u^2-v^2}) = \frac{-2u}{2\sqrt{1-u^2-v^2}}\bigg|_p = \frac{-a}{c}$$

$$\alpha^2 = i_*\left(\frac{\partial}{\partial v}\bigg|_p\right)(x) = \frac{\partial}{\partial v}\bigg|_p (x \circ i) = \frac{\partial}{\partial v}\bigg|_p (u) = 0$$

$$\beta^2 = i_*\left(\frac{\partial}{\partial v}\bigg|_p\right)(y) = \frac{\partial}{\partial v}\bigg|_p (y \circ i) = \frac{\partial}{\partial v}\bigg|_p (v) = 1$$

$$\gamma^2 = i_*\left(\frac{\partial}{\partial v}\bigg|_p\right)(z) = \frac{\partial}{\partial v}\bigg|_p (z \circ i) = \frac{\partial}{\partial v}\bigg|_p (\sqrt{1-u^2-v^2}) = \frac{-2v}{2\sqrt{1-u^2-v^2}}\bigg|_p = \frac{-b}{c}$$

As a result, we have determined the coefficients in the expressions

$$i_*\left(\frac{\partial}{\partial u}\bigg|_p\right) = \frac{\partial}{\partial x} + \frac{-a}{c}\frac{\partial}{\partial z} \text{ and } i_*\left(\frac{\partial}{\partial v}\bigg|_p\right) = \frac{\partial}{\partial y} + \frac{-b}{c}\frac{\partial}{\partial z},$$

as desired. □

Problem 11.5: Prove that if N is a compact manifold, then a one-to-one immersion $f : N \to M$ is an embedding.

Proof: Note that f is C^∞ because f is an immersion. As N is compact and M is Hausdorff, it follows from the continuity of f that f will be a closed map, which implies that f^{-1} is continuous. Recall that a function is always surjective onto its image. As a result, we have that f is continuous in both directions and bijective onto its image, so f is a homeomorphism from N to $f(N)$. Because f is also an immersion, we have then shown that f is an embedding. □

Chapter 12

Problem 12.2: Let $(U, \phi) = (U, x^1, \dots, x^n)$ and $(V, \psi) = (V, y^1, \dots, y^n)$ be overlapping coordinate charts on a manifold M. They induce coordinate charts $(TU, \tilde{\phi})$ and $(TV, \tilde{\psi})$ on the total space TM of the tangent bundle (see equation (12.1)), with transition function $\tilde{\psi} \circ \tilde{\phi}^{-1}$:

$$(x^1, \dots, x^n, a^1, \dots, a^n) \mapsto (y^1, \dots, y^n, b^1, \dots, b^n).$$

(a) Compute the Jacobian matrix of the transition function $\tilde{\psi} \circ \tilde{\phi}^{-1}$ at $\phi(p)$.

Proof: From the given information we know that the Jacobian of $\tilde{\psi} \circ \tilde{\phi}^{-1}$ will be of the form

$$J(\tilde{\psi} \circ \tilde{\phi}^{-1}) = \begin{bmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^n} & \frac{\partial y^1}{\partial a^1} & \cdots & \frac{\partial y^1}{\partial a^n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial y^n}{\partial x^1} & \cdots & \frac{\partial y^n}{\partial x^n} & \frac{\partial y^n}{\partial a^1} & \cdots & \frac{\partial y^n}{\partial a^n} \\ \frac{\partial b^1}{\partial x^1} & \cdots & \frac{\partial b^1}{\partial x^n} & \frac{\partial b^1}{\partial a^1} & \cdots & \frac{\partial b^1}{\partial a^n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial b^n}{\partial x^1} & \cdots & \frac{\partial b^n}{\partial x^n} & \frac{\partial b^n}{\partial a^1} & \cdots & \frac{\partial b^n}{\partial a^n} \end{bmatrix}.$$

We also know that

$$\tilde{\phi}^{-1}(x^1, \dots, x^n, a^1, \dots, a^n) = (p, a_p), \text{ where } p = \phi^{-1}(x_1, \dots, x^n) \text{ and } a_p = \sum_{i=1}^n a^i \frac{\partial}{\partial x_i}\bigg|_p.$$

Furthermore, we have that

$$\frac{\partial}{\partial x^i}\bigg|_p = \sum_{j=1}^n \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^i}\bigg|_p.$$

As a result, we now have that

$$\tilde{\psi}(p, a_p) = \left(y^1(x^1, \dots, x^n), \dots, y^n(x^1, \dots, x^n), \sum_{i=1}^n \frac{\partial y^1}{\partial x^i} a^i, \dots, \sum_{i=1}^n \frac{\partial y^n}{\partial x^i} a^i \right).$$

It then follows from our above statements that

$$J(\tilde{\psi} \circ \tilde{\phi}^{-1}) = \begin{bmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^n} & \frac{\partial y^1}{\partial a^1} & \cdots & \frac{\partial y^1}{\partial a^n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial y^n}{\partial x^1} & \cdots & \frac{\partial y^n}{\partial x^n} & \frac{\partial y^n}{\partial a^1} & \cdots & \frac{\partial y^n}{\partial a^n} \\ \frac{\partial}{\partial x^1} \sum_{i=1}^n \frac{\partial y^1}{\partial x^i} a^i & \cdots & \frac{\partial}{\partial x^n} \sum_{i=1}^n \frac{\partial y^1}{\partial x^i} a^i & \frac{\partial}{\partial a^1} \sum_{i=1}^n \frac{\partial y^1}{\partial x^i} a^i & \cdots & \frac{\partial}{\partial a^n} \sum_{i=1}^n \frac{\partial y^1}{\partial x^i} a^i \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial}{\partial x^1} \sum_{i=1}^n \frac{\partial y^n}{\partial x^i} a^i & \cdots & \frac{\partial}{\partial x^n} \sum_{i=1}^n \frac{\partial y^n}{\partial x^i} a^i & \frac{\partial}{\partial a^1} \sum_{i=1}^n \frac{\partial y^n}{\partial x^i} a^i & \cdots & \frac{\partial}{\partial a^n} \sum_{i=1}^n \frac{\partial y^n}{\partial x^i} a^i \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^n} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial y^n}{\partial x^1} & \cdots & \frac{\partial y^n}{\partial x^n} & 0 & \cdots & 0 \\ \sum_{i=1}^n \frac{\partial^2 y^1}{\partial x^1 \partial x^i} a^i & \cdots & \sum_{i=1}^n \frac{\partial^2 y^1}{\partial x^n \partial x^i} a^i & \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \sum_{i=1}^n \frac{\partial^2 y^n}{\partial x^1 \partial x^i} a^i & \cdots & \sum_{i=1}^n \frac{\partial^2 y^n}{\partial x^n \partial x^i} a^i & \frac{\partial y^n}{\partial x^1} & \cdots & \frac{\partial y^n}{\partial x^n} \end{bmatrix}.$$

So we have computed the Jacobian of $\tilde{\psi} \circ \tilde{\phi}^{-1}$, as desired. $\square$

(b) Show that the Jacobian determinant of the transition function $\tilde{\psi} \circ \tilde{\phi}^{-1}$ at $\phi(p)$ is $(\det[\partial y^i/\partial x^j])^2$.

Proof: Define the $n \times n$ matrices A and B by

$$A = \begin{bmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial y^n}{\partial x^1} & \cdots & \frac{\partial y^n}{\partial x^n} \end{bmatrix} \text{ and } B = \begin{bmatrix} \sum_{i=1}^n \frac{\partial^2 y^1}{\partial x^1 \partial x^i} a^i & \cdots & \sum_{i=1}^n \frac{\partial^2 y^1}{\partial x^n \partial x^i} a^i \\ \vdots & & \vdots \\ \sum_{i=1}^n \frac{\partial^2 y^n}{\partial x^1 \partial x^i} a^i & \cdots & \sum_{i=1}^n \frac{\partial^2 y^n}{\partial x^n \partial x^i} a^i \end{bmatrix}.$$

It can then be seen that the Jacobian computed in (a) can be represented by the lower triangular block matrix

$$J(\tilde{\psi} \circ \tilde{\phi}^{-1}) = \begin{bmatrix} A & 0 \\ B & A \end{bmatrix}$$

The definition of A implies that $A = [\partial y^i/\partial x^j]$. It then follows from the properties of lower triangular block matrices and the definition of A that

$$\det(J(\tilde{\psi} \circ \tilde{\phi}^{-1})) = \begin{vmatrix} A & 0 \\ B & A \end{vmatrix} = \det(A)\det(A) = (\det(A))^2 = (\det[\partial y^i/\partial x^j])^2,$$

as desired. $\square$

Problem 12.4: Let $\pi : E \to M$ be a C^∞ vector bundle and $s_1, \ldots, s_r$ a C^∞ frame for E over an open set U in M. Then every $e \in \pi^{-1}(U)$ can be written uniquely as a linear combination

$$e = \sum_{j=1}^{r} c^j(e) s_j(p), \quad p = \pi(e) \in U.$$

Prove that $c^j : \pi^{-1}U \to \mathbb{R}$ is C^∞ for $j = 1, \ldots, r$. (Hint: First show that the coefficients of e relative to the frame $t_1, \ldots, t_r$ of a trivialization are C^∞.)

Proof: Fix $p \in U$ and choose a trivializing open set $V \subseteq U$ for E containing p, with the C^∞ trivialization $\phi : \pi^{-1}(V) \to V \times \mathbb{R}^r$, and let $t_1, \ldots, t_r$ be the C^∞ frame of the trivialization ϕ. We may now write e and s_j in terms of the frame $t_1, \ldots, t_r$ as $e = \sum_{i=1}^{r} b^i t_i$ and $s_j = \sum_{i=1}^{r} a_j^i t_i$; note that then all of the b^i and a_j^i will be C^∞ functions by Lemma 12.11. Next we express e in terms of the t_i's:

$$\sum_{i=1}^{r} b^i t_i = e = \sum_{j=1}^{r} c^j s_j = \sum_{1 \le i,j \le r} a_j^i t_i.$$

Comparing the coefficients of t_i gives $b^i = \sum_{j=1}^{r} c^j a_j^i$; represented in matrix notation as

$$b = \begin{bmatrix} b^1 \\ \vdots \\ b^r \end{bmatrix} = A \begin{bmatrix} c^1 \\ \vdots \\ c^r \end{bmatrix} = Ac.$$

At each point of V, being the transition matrix between to bases, the matrix A is invertible. By Cramer's rule, A^{-1} is a matrix of C^∞ functions on V. Hence, $c = A^{-1}b$ is a column vector of C^∞ functions on V. Thus, we have that $c^1, \ldots, c^r$ are C^∞ functions at $p \in U$. Since p is an arbitrary point of U, the coefficients c^j are C^∞ functions on U. $\square$

Chapter 14

Problem 14.1: Show that two C^∞ vector fields X and Y on a manifold M are equal if and only if for every C^∞ function f on M, we have $Xf = Yf$.

Proof: ($\Rightarrow$) Assume that X and Y are equal vector fields on M. Then it is immediate that $Xf = Yf$.

($\Leftarrow$) Let f be a C^∞ function on M and assume that $Xf = Yf$ for all f. Let $p \in M$ be arbitrary; we shall show that $X_p = Y_p$. To do this, it suffices to show that $X_p[h] = Y_p[h]$ for the germ $[h]$. Let $h : U \to \mathbb{R}$ be a C^∞ function representing the germ $[h]$. We may extend h to a C^∞ function $\tilde{h}$ by multiplying by a C^∞ bump function supported in U that is identically 1 in a neighborhood of p. It then follows from our assumption that $X\tilde{h} = Y\tilde{h}$. As a result, we have that $X_p\tilde{h} = (X\tilde{h})_p = (Y\tilde{h})_p = Y_p\tilde{h}$. As $\tilde{h} = h$ in a neighborhood of p, we have that $X_p h = X_p\tilde{h}$ and $Y_p h = Y_p\tilde{h}$ on that neighborhood. It then follows that $X_p h = Y_p h$, so $X_p = Y_p$. As p was arbitrary, it then follows that $X = Y$. $\square$

Problem 14.2: Let $x^1, y^1, \ldots, x^n, y^n$ be the standard coordinates on $\mathbb{R}^{2n}$. The unit sphere S^{2n-1} in $\mathbb{R}^{2n}$ is defined by the equation $\sum_{i=1}^n (x^i)^2 + (y^i)^2 = 1$. Show that

$$X = \sum_{i=1}^n -y^i \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial y^i}$$

is a nowhere-vanishing smooth vector field on S^{2n-1}. Since all spheres of the same dimension are diffeomorphic, this proves that on every odd-dimensional sphere there is a nowhere-vanishing smooth vector field. It is a classical theorem of differential and algebraic topology that on an even-dimensional sphere every continuous vector field must vanish somewhere (see [28, Section 5, p.31] or [16, Theorem 16.5, p.70]). (Hint: Use Problem 11.1 to show that X is tangent to S^{2n-1}.)

Proof: Let $p = (p_1, p_2, \ldots, p_{2n})$. We know that

$$X_p = \sum_{i=1}^n -y^i(p) \frac{\partial}{\partial x^i}\bigg|_p + x^i(p) \frac{\partial}{\partial y^i}\bigg|_p.$$

If we let a^i represent the ith component function we can then observe that

$$\begin{aligned}\sum_{i=1}^{2n} a^i p^i &= -y^1(p)p_1 + x^1(p)p_2 + \cdots - y^n(p)p_{2n-1} + x^n(p)p_{2n} \\ &= -p_2p_1 + p_1p_2 + \cdots - p_{2n}p_{2n-1} + p_{2n}p_{2n-1} = 0 + \cdots + 0 = 0,\end{aligned}$$

so it follows from Problem 11.1 that X_p is tangent to S^{2n-1} for all $p \in \mathbb{R}^{2n}$. As p was arbitrary, we have that X is a vector field on S^{2n-1}.

Let $p \in S^{2n-1}$ and by way of contradiction assume that $X_p = 0$. As we know that $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^n}$ are a basis, this implies that $x^1(p) = y^1(p) = \cdots = x^n(p) = y^n(p) = 0$. But then $\sum_{i=1}^n (x^i(p))^2 + (y^i(p))^2 = 0$, which is a contradiction to the fact that $\sum_{i=1}^n (x^i)^2 + (y^i)^2 = 1$, so we have that $X_p \neq 0$.

Define $t^{2i-1} = x^i$ and $t^{2i} = y^i$ for $1 \leq i \leq n$, we may then rewrite X as

$$X = \sum_{i=1}^{2n} -t^{i+1}\frac{\partial}{\partial t^i} + t^i \frac{\partial}{\partial t^{i+1}}.$$

We shall use the atlas on S^{2n-1} whose charts are given by the stereographic projection. To that effect, define $z_i = \frac{t^i}{1-t^{2n}}$ for $1 \leq i \leq 2n-1$; note that this is the stereographic projection of t^i. For each i it then follows that

$$\frac{\partial}{\partial t^i} = \sum_{k=1}^{2n-1} \frac{\partial z_k}{\partial t^i}\frac{\partial}{\partial z_k}.$$

Thus, we may rewrite X on S^{2n-1} as

$$X = \sum_{i=1}^{2n} -t^{i+1}\left(\sum_{k=1}^{2n-1} \frac{\partial z_k}{\partial t^i}\frac{\partial}{\partial z_k}\right) + t^i \left(\sum_{k=1}^{2n-1} \frac{\partial z_k}{\partial t^{i+1}}\frac{\partial}{\partial z_k}\right)$$

We can expand out this sum to see that the coefficient of each $\frac{\partial}{\partial z_k}$ will be given by a sum and product of t^i's and $\frac{\partial z_k}{\partial x^i}$'s. As we know that t^i and $\frac{\partial z_j}{\partial x^i}$ are smooth for all i, j, the coefficient function of each $\frac{\partial}{\partial z_k}$ will be smooth, so it follows that X is smooth on S^{2n-1} by Proposition 14.2.

Thus, we have that X is a nowhere-vanishing smooth vector field on S^{2n-1}. $\square$

Problem 14.3: Let M be $\mathbb{R} \setminus \{0\}$ and let X be the vector field d/dx on M. Find the maximal integral curve of X starting at $x = 1$.

Proof: We know that $c'(t) = \dot{x}(t) = 1$. Solving this differential equation with the initial condition $c(0) = 1$ yields $x = t+1$, so it follows that the maximal integral curve $c : (-1, \infty) \to \mathbb{R}$ is defined by $c(t) = t + 1$. (This domain is restricted below by -1 because $c(-1) = 0$ and $0 \notin M$. As we have by definition that the domain of an integral curve must be an interval, hence connected, we have the domain will be $(-1, \infty)$.) $\square$

Problem 14.4: Find the integral curves of the vector field

$$X_{(x,y)} = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} = \begin{bmatrix} x \\ -y \end{bmatrix} \quad \text{on } \mathbb{R}^2.$$

Proof: Let $c(t) = (x(t), y(t))$; then $c'(t) = (\dot{x}(t), \dot{y}(t))$. It follows from the given information that $\left[\begin{smallmatrix} \dot{x} \\ \dot{y} \end{smallmatrix}\right] = \left[\begin{smallmatrix} x \\ -y \end{smallmatrix}\right]$, so $\dot{x} = x$ and $\dot{y} = -y$. Solving these differential equations yields $x(t) = c_1 e^t$ and $y(t) = c_2 e^{-t}$ for some constants c_1, c_2. As a result, we have that the integral curve is given by $c : \mathbb{R} \to \mathbb{R}^2$ defined by $c(t) = (c_1 e^t, c_2 e^{-t})$. $\square$

Problem 14.5: Find the maximal integral curve $c(t)$ starting at the point $(a, b) \in \mathbb{R}^2$ of the vector field $X_{(x,y)} = \partial/\partial x + x\partial/\partial y$ on $\mathbb{R}^2$.

Proof: Let $c(t) = (x(t), y(t))$; then $c'(t) = (\dot{x}(t), \dot{y}(t))$. It follows from the given information that $\left[\begin{smallmatrix} \dot{x} \\ \dot{y} \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 \\ x \end{smallmatrix}\right]$, so $\dot{x} = 1$ and $\dot{y} = x$. Solving these differential equations yields $x(t) = t + c$ and $y(t) = \frac{t^2}{2} + ct + d$. We can then use the initial condition $c(0) = (a, b)$ to determine that $x(t) = t + a$ and $y(t) = \frac{t^2}{2} + at + b$. As a result, we have that the maximal integral curve is given by $c : \mathbb{R} \to \mathbb{R}^2$ defined by $c(t) = (t + a, \frac{t^2}{2} + at + b)$. $\square$

Problem 14.6:

(a) Suppose the smooth vector field X on a manifold M vanishes at a point $p \in M$. Show that the integral curve of X with initial point p is the constant curve $c(t) \equiv p$.

Proof: Let $c : \mathbb{R} \to M$ be defined by $c(t) = p$. We can immediately see that $c(0) = p$ and $c'(t) = 0$. Because X is given to be a smooth vector field, we have that our solution $c(t)$ to the given differential equation is unique. As a result, we have that the integral curve of X with initial point p must be the constant curve $c(t) \equiv p$. $\square$

(b) Show that if X is the zero vector field on a manifold M, and $c_t(p)$ is the maximal integral curve of X starting at p, then the one-parameter group of diffeomorphisms $c : \mathbb{R} \to \text{Diff(M)}$ is the constant map $c(t) \equiv \mathbb{1}_M$.

Proof: As $X = 0$ is smooth, we know from (a) that every integral curve on M will be constant. This implies that when fixing any t we have that the map $c_t : M \to M$ will be defined by $c_t(p) = p$. As a result, we have that $c : \mathbb{R} \to \text{Diff}(M)$ is the constant map $c(t) \equiv \mathbb{1}_M$. $\square$

Problem 14.7: Let X be the vector field $x\, d/dx$ on $\mathbb{R}$. For each p in $\mathbb{R}$, find the maximal integral curve $c(t)$ of X starting at p.

Proof: We know that the integral curve will be of the form $c : \mathbb{R} \to \mathbb{R}$ defined by $c(t) = x(t)$

and $c'(t) = \dot{x}(t)$. It follows from the given information that $\dot{x} = x$; solving this equation gives $x(t) = ae^t$ for some constant a. Using the intial condition that $c(0) = p$ then gives that $x(t) = pe^t$, so the maximal integral curve is $c : \mathbb{R} \to \mathbb{R}$ is defined by $c(t) = pe^t$. $\square$

Problem 14.8: Let X be the vector field $x^2\, d/dx$ on the real line $\mathbb{R}$. For each $p > 0$ in $\mathbb{R}$, find the maximal integral curve of X with initial point p.

Proof: We know that the integral curve will be of the form $c : \mathbb{R} \to \mathbb{R}$ defined by $c(t) = x(t)$ and $c'(t) = \dot{x}(t)$. It follows from the given information that $\dot{x} = x^2$; solving this equation gives $x(t) = \frac{-1}{t+a}$ for some constant a. Using the intial condition that $c(0) = p$ then gives that $x(t) = \frac{-1}{t+\frac{-1}{p}}$, note that this function smooth everywhere except at $t = \frac{1}{p}$. As we know that $p > 0$, it follows that $0 < \frac{1}{p}$, so the largest connected domain for $x(t)$ containing 0 is the interval $(-\infty, \frac{1}{p})$. As a result, the maximal integral curve is given by $c : (-\infty, \frac{1}{p}) \to \mathbb{R}$ defined by $c(t) = \frac{-1}{t+\frac{-1}{p}}$. $\square$

Problem 14.9: Suppose $c(a, b) \to M$ is an integral curve of the smooth vector field X on M. Show that for any real number s, the map

$$c_s : (a + s, b + s) \to M, \quad c_s(t) = c(t - s),$$

is also an integral curve of X.

Proof: We can immediately see that

$$c_s'(t) = c'(t - s) = X_{c(t-s)} = X_{c_s(t)},$$

so c_s is also an integral curve of X. $\square$

Problem 14.10: If f and g are C^∞ functions and X and Y are C^∞ vector fields on a manifold M, show that $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$.

Proof: Let h be an arbitrary C^∞ function and observe the following:

$$\begin{aligned}
[fX, gY]h &= ((fX)(gY) - (gY)(fX))h \\
&= (fX)(gY)h - (gY)(fX)h \\
&= f(XgYh) + f(gXYh) - g(YfXh) - g(fYXh) \\
&= f(gXYh) - g(fYXh) + f(XgYh) - g(YfXh) \\
&= fg(XYh) - gf(YXh) + f(XgYh) - g(YfXh)
\end{aligned}$$

$$= fg((XYh) - (YXh)) + f(XgY)h - g(YfX)h$$
$$= fg[X,Y]h + f(Xg)(Yh) - g(Yf)(Xh)$$

As h was aribtrary, it then follows that $[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X$. $\square$

Problem 14.11: Compute the Lie bracket $\left[-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right]$ on $\mathbb{R}^2$.
Proof: We can observe that

$$\begin{aligned}
\left[-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right] &= \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)\frac{\partial}{\partial x} - \frac{\partial}{\partial x}\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) \\
&= -y\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) + x\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\right) - \frac{\partial}{\partial x}\left(-y\frac{\partial}{\partial x}\right) - \frac{\partial}{\partial x}\left(x\frac{\partial}{\partial y}\right) \\
&= -y\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial y}\frac{\partial}{\partial x} + y\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial y} - x\frac{\partial}{\partial x}\frac{\partial}{\partial y} \\
&= 0 + 0 - \frac{\partial}{\partial y} = -\frac{\partial}{\partial y}.
\end{aligned}$$

$\square$

Problem 14.12: Consider two C^∞ vector fields X, Y on $\mathbb{R}^n$:

$$X = \sum a^i \frac{\partial}{\partial x^i}, \quad Y = \sum b^j \frac{\partial}{\partial x^j},$$

where a^i, b^j are C^∞ functions on $\mathbb{R}^n$. Since $[X,Y]$ is also a C^∞ vector field on $\mathbb{R}^n$,

$$[X,Y] = \sum c^k \frac{\partial}{\partial x^k}$$

for some C^∞ functions c^k. Find the formula for c^k in terms of a^i and b^j.
Proof: We may observe the following:

$$\begin{aligned}
\sum c^k \frac{\partial}{\partial x^k} = [X,Y] &= XY - YX \\
&= \sum a^i \frac{\partial}{\partial x^i} \sum b^j \frac{\partial}{\partial x^j} - \sum b^j \frac{\partial}{\partial x^j} \sum a^i \frac{\partial}{\partial x^i} \\
&= \sum_{i,j} a^i \left(\frac{\partial b^j}{\partial x^i}\frac{\partial}{\partial x^j} + b^j \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\right) - \sum_{i,j} b^j \left(\frac{\partial a^i}{\partial x^j}\frac{\partial}{\partial x^i} + a^i \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}\right) \\
&= \sum_{i,j}\left(a^i \frac{\partial b^j}{\partial x^i}\frac{\partial}{\partial x^j} - b^i \frac{\partial a^j}{\partial x^i}\frac{\partial}{\partial x^j}\right)
\end{aligned}$$

Fixing k we then have that

$$c^k = \sum a^i \frac{\partial b^k}{\partial x^i} - b^i \frac{\partial a^k}{\partial x^i}.$$

□

Problem 14.13: Let $F : N \to M$ be a C^∞ diffeomorphism of manifolds. Prove that if g is a C^∞ function and X a C^∞ vector field on N, then $F_*(gX) = (g \circ F^{-1})F_*X$.

Proof: As F is a diffeomorphism, we have that X and F_*X are F-related. This implies that $Y = F_*X$, so $(Yf) \circ F = X(f \circ F)$ for all $p \in N$. Fixing a $p \in N$, let f be an arbitrary smooth function. We then have that

$$(Yf)(F(p)) = X(f \circ F)(p) \Rightarrow (g \circ F^{-1})(Yf)(F(p)) = gX(f \circ F)(p)$$
$$\Rightarrow (g \circ F^{-1})(F(p))(Y_{F(p)}f) = F_{*,p}(gX_p)_{F(p)}f \Rightarrow (g \circ F^{-1})(F_*X)_{F(p)}f = F_*(g(X)_{F(p)}f.$$

As p and f are arbitrary, we have the desired result. □

Problem 14.14: Let $F : N \to M$ be a C^∞ diffeomorphism of manifolds. Prove that if X and Y are C^∞ vector fields on N, then $F_*[X, Y] = [F_*X, F_*Y]$.

Proof: It follows from Exercise 14.15 that X and F_*X are F-related and Y and F_*Y are F-related. Proposition 14.17 then implies that $[X, Y]$ is F-related to $[F_*X, F_*Y]$, which tells us that $F_*[X, Y] = [F_*X, F_*Y]$, as desired. □

Chapter 15

Problem 15.1: For $X \in \mathbb{R}^{n\times n}$, define the partial sum $s_m = \sum_{k=0}^{m} X^k/k!$.
(a) Show that for $\ell \geq m$,

$$\|s_\ell - s_m\| \leq \sum_{k=m+1}^{\ell} \|X\|^k/k!.$$

Proof: Recall the properties of the norm and the definition of s_m; then we may observe that

$$\|s_\ell - s_m\| = \left\|\sum_{k=0}^{\ell} \frac{X^k}{k!} - \sum_{k=0}^{m} \frac{X^k}{k!}\right\| = \left\|\sum_{k=m+1}^{\ell} \frac{X^k}{k!}\right\| \leq \sum_{k=m+1}^{\ell} \left\|\frac{X^k}{k!}\right\| \leq \sum_{k=m+1}^{\ell} \frac{\|X\|^k}{k!},$$

so $\|s_\ell - s_m\| \leq \sum_{k=m+1}^{\ell} \|X\|^k/k!$, as desired. $\square$

(b) Conclude that s_m is a Cauchy sequence in $\mathbb{R}^{n\times n}$ and therefore converges to a matrix, which we denote by e^X. This gives another way of showing that $\sum_{k=0}^{\infty} X^k/k!$ is convergent, without using the comparison test or the theorem that absolute convergence implies convergence in a complete normed vector space.

Proof: Let $\epsilon > 0$ and note that the growth of $\|X\|^k$ is at most some type of polynomial growth (because $\|X\|^k$ is based on some power of a linear combination of the columns of X). As we know that factorial growth is faster than polynomial growth, we may choose sufficently large N such that

$$\sum_{k=N}^{\infty} \frac{\|X\|^k}{k!} < \epsilon.$$

As a result, we then have from (a) that for $m, n > N$ with $m \geq n$ that

$$\|s_m - s_n\| \leq \sum_{k=n+1}^{m} \frac{\|X\|^k}{k!} < \sum_{k=N}^{\infty} \frac{\|X\|^k}{k!} < \epsilon$$

so s_m is a Cauchy sequence, as desired. $\square$

Problem 15.4: Prove that an open subgroup H of a connected Lie group G is equal to G.
Proof: By way of contradiction suppose that H is a proper subset of G. Let $g \in G$ and let $\ell_g : G \to G$ be the map representing left multiplication by g. We know that $\ell_g(H) = gH$, a left coset of H; as H is open and ℓ_g is a homeomorphism, it follows that gH is open. We also know that the set of left cosets $\{gH\}_{g\in G}$ partition G, and that this set contains more than the identity coset because we assumed that H was proper in G. However, this implies that G is the disjoint union of open, which contradicts the connectedness of G. As a result, we have that H must not be proper in G, so $H = G$. $\square$

Problem 15.5: Let G be a Lie group with multiplication map $\mu : G \times G \to G$ and let $\ell_a : G \to G$ and $r_b : G \to G$ be left and right multiplication by a and $b \in G$ respectively. Show that the differential of μ at $(a,b) \in G \times G$ is $\mu_{*,(a,b)}(X_a, Y_b) = (r_b)_*(X_a) + (\ell_a)_*(Y_b)$ for $X_a \in T_a(G)$ and $Y_b \in T_b(G)$.

Proof: Define the curve $c_1 : \mathbb{R} \to G$ such that $c_1(0) = a$ and $c_1'(0) = X_a$. Then it follows that $(c_1(t), b)$ will be a curve through (a,b). Similarly, we define $c_2 : \mathbb{R} \to G$ such that $c_2(0) = b$ and $c_2'(0) = Y_b$. Then it follows that $(a, c_2(t))$ will be a curve through (a,b). It then follows from Proposition 8.18 that

$$\mu_{*,(a,b)}(X_a, 0) = \frac{d}{dt}\bigg|_0 (\mu \circ (c_1(t), b)) = \frac{d}{dt}\bigg|_0 (c(t)b) = \frac{d}{dt}\bigg|_0 (r_b \circ c(t)) = (r_b)_*(X_a).$$

Similarly, we have that $\mu_{*,(a,b)}(0, Y_b) = (\ell_a)_*(Y_b)$. As the differential of any map is always linear, it follows that

$$\mu_{*,(a,b)}(X_a, Y_b) = \mu_{*,(a,b)}(X_a, 0) + \mu_{*,(a,b)}(0, Y_b) = (r_b)_*(X_a) + (\ell_a)_*(Y_b),$$

as desired. $\square$

Problem 15.6: Let G be a Lie group with multiplication map $\mu : G \times G \to G$, inverse map $\iota : G \to G$, and identity element e. Show that the differential of the inverse map at $a \in G$,

$$\iota_{*,a} : T_aG \to T_{a^{-1}}G,$$

is given by

$$\iota_{*,a}(Y_a) = -(r_{a^{-1}})_*(\ell_{a^{-1}})_*Y_a,$$

where $(r_{a^{-1}})_* = (r_{a^{-1}})_{*,e}$ and $(\ell a^{-1})_* = (\ell_{a^{-1}})_{*,a}$. (The differential of the inverse at the identity was calculated in Problem 8.8(b).)

Proof: Define the curve $c : \mathbb{R} \to G$ such that $c(0) = a$ and $c'(0) = Y_a$. We know that for all $t \in \mathbb{R}$ that $c(t)c^{-1}(t) = e$. Differentiating this equation then yields

$$\frac{dc}{dt}c^{-1}(t) + c(t)\frac{dc^{-1}}{dt} = 0 \Rightarrow \frac{dc^{-1}}{dt}\bigg|_t c(t) = \frac{-dc}{dt}\bigg|_t c^{-1}(t) \Rightarrow \frac{dc^{-1}}{dt}\bigg|_t = -c^{-1}(t)\frac{dc}{dt}\bigg|_t c^{-1}(t).$$

Setting $t = 0$ then gives that

$$\frac{dc^{-1}}{dt}\bigg|_0 = -c^{-1}(0)\frac{dc}{dt}\bigg|_0 c^{-1}(0) \Rightarrow (Y_a)^{-1} = -a^{-1}Y_a a^{-1} \Rightarrow \iota_{*,a}(Y_a) = -(r_{a^{-1}})_*(\ell_{a^{-1}})_*Y_a,$$

as desired. $\square$

Problem 15.7: Show that the differential of the determinant map $\det : \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}$ at $A \in \mathrm{GL}(n,\mathbb{R})$ is given by $\det_{*,A}(AX) = (\det A)\,\mathrm{tr}X$ for $X \in \mathbb{R}^{n\times n}$.

Proof: Define the curve $c : \mathbb{R} \to \mathrm{GL}(n,\mathbb{R})$ by $c(t) = Ae^{tX}$; then it follows that $c(0) = A$ and $c'(0) = AX$, where $X \in \mathbb{R}^{n\times n}$. We then have that

$$\begin{aligned}\det_{A,*}(AX) = \left.\frac{d}{dt}\right|_{t=0} \det(c(t)) &= \left.\frac{d}{dt}\right|_{t=0} (\det A) \det e^{tX} \\ &= (\det A)\left.\frac{d}{dt}\right|_{t=0} e^{t\,\mathrm{tr}X} = (\det A)\,\mathrm{tr}X e^0 = (\det A)\,\mathrm{tr}X.\end{aligned}$$

Thus, we have that $\det_{*,A}(AX) = (\det A)(\,\mathrm{tr}X)$, as desired. $\square$

Problem 15.8: Use Problem 15.7 to show that 1 is a regular value of the determinant map. This gives a quick proof that the special linear group $\mathrm{SL}(n,\mathbb{R})$ is a regular submanifold of $\mathrm{GL}(n,\mathbb{R})$.

Proof: It is immediate that $\det^{-1}(1) = \mathrm{SL}(n,\mathbb{R})$. Now let $A \in \mathrm{SL}(n,\mathbb{R})$; then $\det A = 1$. It then follows from Exercise 15.7 that $\det_{*,A}(AX) = (\det A)\,\mathrm{tr}X = \,\mathrm{tr}X$ for all $X \in \mathbb{R}^{n\times n}$. As X is any $n\times n$ matrix, $\mathrm{tr}X$ can assume any real value, so $\det_{*,A}(AX)$ is surjective for all $A \in \mathrm{SL}(n,\mathbb{R}) = \det^{-1}(1)$. As a result, 1 is a regular value of det, so $\det^{-1}(1)$ is a regular level set, hence a regular submanifold of $\mathrm{GL}(n,\mathbb{R})$. $\square$

Problem 15.9: (a) For $r \in \mathbb{R}^\times$, let M_r be the $n\times n$ matrix

$$M_r = \begin{bmatrix} r & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} = [re_1, e_2, \dots, e_n],$$

where $e_1, \dots, e_n$ is the standard basis for R^n. Prove that the map

$$f : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R}) \times \mathbb{R}^\times \text{ defined by } A \mapsto (AM_{1/\det A}, \det A),$$

is a diffeomorphism.

Proof: Before we begin, note that $M_{1/\det A}$ is invertible for all A, as M will be a diagonal matrix with no zeros on the diagonal.

(i) f is injective.

Let $A, B \in GL(n,\mathbb{R})$ such that $f(A) = f(B)$. This implies that $(AM_{1/\det A}, \det A) =$

$(BM_{1/\det B}, \det B)$. As we have that $\det A = \det B$, it follows that $M_{1/\det A} = M_{1/\det B}$, so $(M_{1/\det A})^{-1} = (M_{1/\det B})^{-1}$. As we know that $AM_{1/\det A} = BM_{1/\det B}$, we may right multiply by $(M_{1/\det A})^{-1}$ to see that

$$AM_{1/\det A} = BM_{1/\det B} \Rightarrow AM_{1/\det A}(M_{1/\det A})^{-1} = BM_{1/\det B}(M_{1/\det A})^{-1} \Rightarrow A = B.$$

As a result, f is injective.

(ii) f is surjective.
Let $(B, x) \in \mathrm{SL}(n, \mathbb{R}) \times \mathbb{R}^{\times}$, where B is the matrix $B = [b_{ij}]$ As we know that $x \neq 0$, there exists some $r \in \mathbb{R}^{\times}$ such that $x = \frac{1}{r}$. Then define $A, D \in \mathrm{GL}(n, \mathbb{R})$ by

$$A = \begin{bmatrix} \frac{b_{11}}{r} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ \frac{b_{n1}}{r} & \cdots & b_{nn} \end{bmatrix} \text{ and } D = \begin{bmatrix} r & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Note that D is in the form M_r. It then follows that

$$B = AD = \begin{bmatrix} \frac{b_{11}}{r} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ \frac{b_{n1}}{r} & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} r & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

As $B \in \mathrm{SL}(n, \mathbb{R})$, we know that $\det B = 1$. Now observe that

$$1 = \det B = \det AD = (\det A)(\det D) = r(\det A) \Rightarrow \det A = \frac{1}{r}.$$

This implies that $B = AD = AM_r$, where $r = \frac{1}{\det A}$ and $x = \frac{1}{r} = \det A$, so we have that $A \mapsto (AM_r, \frac{1}{r}) = (B, x)$. Thus, f is surjective.

(iii) f is smooth.
Recall that f is given by the map $(AM_{1/\det A}, \det A)$. The first coordinate of the image is given by multiplication of the invertible matrix A by the invertible matrix $M_{1/\det A}$, so f is C^∞ in the first coordinate. As the second coordinate is the determinant map, it is also C^∞. Thus, we have that f is C^∞.

(iv) f^{-1} is smooth.
Because we know that f is bijective, we may say that f^{-1} is given by the map $(B, x) = (AM_{1/\det A}, \det A) \mapsto AM_{1/\det A}(M_{1/\det A})^{-1} = A$. We can think then think of f^{-1} as a

projection whose remaining coordinate is defined by multiplication by an invertible matrix, so f^{-1} is C^∞.

As the above properties hold, we have that f is a diffeomorphism. $\square$

(b) The *center* $Z(G)$ of a group G is the subgroup of elements $g \in G$ that commute with all elements of G:

$$Z(G) := \{g \in G \mid gx = xg \text{ for all } x \in G\}.$$

Show that the center of $\mathrm{GL}(2, \mathbb{R})$ is isomorphic to $\mathbb{R}^\times$, corresponding to the subgroup of scalar matrices, and that the center of $\mathrm{SL}(2, \mathbb{R}) \times \mathbb{R}^\times$ is isomorphic to $\{\pm 1\} \times \mathbb{R}^\times$. The group $\mathbb{R}^\times$ has two elements of order 2, while the group $\{\pm 1\} \times \mathbb{R}^\times$ has four elements of order 2. Since their centers are not isomorphic, $\mathrm{GL}(2, \mathbb{R})$ and $SL(2, \mathbb{R}) \times \mathbb{R}^\times$ are not isomorphic as groups.

Proof: We shall first show that $Z(G)$ is given by exactly the scalar matrices in $\mathrm{GL}(2, \mathbb{R})$, that is all matrices of the form aI for $a \in \mathbb{R}^\times$.

Let $A \in Z(GL(2, \mathbb{R}))$ with $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$; then we have that A commutes with every invertible 2×2 matrix. Define the matrices $J = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and $J' = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. As we know that A will commute with J and J', it follows that

$$\left[\begin{smallmatrix} a+b & a \\ c+d & c \end{smallmatrix}\right] = AJ = JA = \left[\begin{smallmatrix} a+c & a \\ b+d & b \end{smallmatrix}\right] \text{ and } \left[\begin{smallmatrix} a & a+b \\ c & c+d \end{smallmatrix}\right] = AJ' = J'A = \left[\begin{smallmatrix} a+c & b+d \\ c & d \end{smallmatrix}\right].$$

From these identities we then have a system of equations that when solved implies that $a = d$ and $b = c = 0$. As a result, $A = \left[\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right]$, so A is a scalar matrix.

Now let A be a scalar matrix, that is $A = \left[\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right]$ for some $a \in \mathbb{R}^\times$. Then we have for any $B \in \mathrm{GL}(2, \mathbb{R})$ that $AB = aIB = aBI = BaI = BA$, so $A \in Z(\mathrm{GL}(2, \mathbb{R}))$.

Thus, we have that $Z(\mathrm{GL}(2, \mathbb{R}))$ is exactly the scalar 2×2 matrices by double containment.

Define the map $\phi : Z(\mathrm{GL}(2, \mathbb{R})) \to \mathbb{R}^\times$ to be given by $\left[\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right] \mapsto a$. Let $A, B \in Z(\mathrm{GL}(2, \mathbb{R}))$, then $A = aI$ and $B = bI$ for some $a, b \in \mathbb{R}^\times$.

(i) ϕ is a homomorphism.
We have that

$$\phi(AB) = \phi(aIbI) = \phi(abI) = ab = \phi(aI)\phi(bI) = \phi(A)\phi(B),$$

so ϕ is a homomorphism.

(ii) ϕ is injective.
Say that $\phi(A) = \phi(B)$. Then we have that $a = \phi(aI) = \phi(A) = \phi(B) = \phi(bI) = b$, so $a = b$. Thus, we have that $A = aI = bI = B$, so ϕ is injective.

(iii) ϕ is surjective.
Let $x \in \mathbb{R}^\times$. Then $xI = X \mapsto x$, so ϕ is injective.

As the above three properties hold, ϕ is an isomorphism. Thus, $Z(\mathrm{GL}(2, \mathbb{R})) \cong \mathbb{R}^\times$.

Recall that our proof that the scalar matrices were the center of $\mathrm{GL}(2, \mathbb{R})$ involved commutation with matrices whose determinants were equal to 1, specifically J and J'. As a result, $Z(\mathrm{SL}(2, \mathbb{R}))$ will be the scalar matrices of determinant 1, which are exactly I and $-I$. Then it follows immediately that $Z(\mathrm{SL}(2, \mathbb{R})) \cong \{\pm 1\}$, so $\mathrm{SL}(2, \mathbb{R}) \times \mathbb{R}^\times \cong \{\pm 1\} \times \mathbb{R}^\times$. $\square$

(c) Show that $h : \mathrm{GL}(3, \mathbb{R}) \to \mathrm{SL}(3, \mathbb{R}) \times \mathbb{R}^\times$ defined by $A \mapsto \big((\det A)^{1/3} A, \det A\big)$ is a Lie group isomorphism.
Proof: The map defined by h is only slightly different than f. We may prove injectivity and surjectivity using the same methods from (a). When showing that h and h^{-1} are smooth, we may again use the same method as shown in (a), but we must make note that $x^{1/3}$ is C^∞ away from 0 and $\det A \neq 0$ because $A \in \mathrm{GL}(3, \mathbb{R})$. Thus, we have that h is a diffeomorphism.

It remains to show that h is a homomorphism. Let $A, B \in \mathrm{GL}(3, \mathbb{R})$ and observe that

$$\begin{aligned} h(AB) = \big((\det AB)^{1/3} AB, \det(AB)\big) &= \big((\det A)^{1/3} A (\det B)^{1/3} B, \det A \det B\big) \\ &= \big((\det A)^{1/3} A, \det A\big)\big((\det B)^{1/3} B, \det B\big) = h(A)h(B). \end{aligned}$$

Thus, we have that h is a homomorphism, so h is a Lie group isomorphism. $\square$

Problem 15.10: Show that the orthogonal group $\mathrm{O}(n)$ is compact by proving the following two statements.
(a) $\mathrm{O}(n)$ is a closed subset of $\mathbb{R}^{n \times n}$.
Proof: Recall that $\mathrm{O}(n)$ is given by the set of matrices $\mathrm{O}(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid AA^T = I\}$.

Define the map $\phi : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ by $A \mapsto AA^T$ and note that ϕ is continuous and $\phi^{-1}(I) = \mathrm{O}(n)$. As we know that I is closed in $\mathrm{GL}(n, \mathbb{R})$, it then follows that $\phi^{-1}(I)$ is closed in $\mathrm{GL}(n, \mathbb{R})$, so $\mathrm{O}(n)$ is closed in $\mathrm{GL}(n, \mathbb{R})$. $\square$

(b) $\mathrm{O}(n)$ is a bounded subset of $\mathbb{R}^{n \times n}$.

Proof: Let $A \in \mathrm{O}(n)$ and say that

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \text{ then } A^T = \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nn} \end{bmatrix}.$$

It can then be seen that the jth diagonal entry of the product AA^T is given by $\sum_{i=1}^n a_{ij}^2$. As we know that $AA^T = I$, it then follows that $\sum_{i=1}^n a_{ij}^2 = 1$. Recall the definition of the matrix norm and observe that

$$\|A\| = \left(\sum_{i,j} a_{ij}^2\right)^{1/2} = \left(\sum_{j=1}^n \sum_{i=1}^n a_{ij}^2\right)^{1/2} = \left(\sum_{j=1}^n 1\right)^{1/2} = n^{1/2}.$$

As $A \in \mathrm{O}(n)$ was arbitrary, we have shown that $\|A\| = n^{1/2}$ for all $A \in \mathrm{O}(n)$, so $\mathrm{O}(n)$ is bounded. $\square$

Problem 15.11: The *special orthogonal group* $SO(n)$ is defined to be the subgroup of $\mathrm{O}(n)$ consisting of matrices of determinant 1. Show that every matrix $A \in \mathrm{SO}(2)$ can be written in the form

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

for some real number θ. Then prove that $\mathrm{SO}(2)$ is diffeomorphic to the circle S^1.

Proof: Define S to be the set of matrices $S = \left\{ \left[\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right] \mid \theta \in \mathbb{R} \right\}$

Let $A \in \mathrm{SO}(2)$ and say that $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. As $A \in \mathrm{SO}(2)$ we then have that

$$I = AA^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{bmatrix};$$

this gives us that $a^2 + b^2 = c^2 + d^2 = 1$ and $ac = -bd$. As $a^2 + b^2 = 1$, there exists some $\theta \in \mathbb{R}$ such that $a = \cos\theta$ and $b = -\sin\theta$. The remaining system of equations then becomes $(\cos\theta)c = -(-\sin\theta)d = (\sin\theta)d$ and $c^2 + d^2 = 1$, so we have that $c = \sin\theta$ and $d = \cos\theta$. Substitution back into the definition of A gives $A = \left[\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right]$ for some $\theta \in \mathbb{R}$, so $A \in S$.

Now let $A \in S$. Then it can be immediately seen that $\det A = \cos^2\theta + \sin^2\theta = 1$ and

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^T = A^T.$$

As $\det A = 1$ and $A^{-1} = A^T$, we have that $A \in \mathrm{SO}(2)$.

As a result, we have that $\mathrm{SO}(2) = S$.

Note that $\mathrm{SO}(2)$ has a unique representation as an element of S for $0 \leq \theta < 2\pi$, as $(\sin\theta, \cos\theta)$ will take unique values exactly on this interval. We shall then restrict θ to the interval $[0, 2\pi)$ for following work. Furthermore, recall that we can desribe every point in $S^1 \subseteq \mathbb{R}^2$ uniquely as $(\cos\theta, \sin\theta)$ for $\theta \in [0, 2\pi)$.

Let $\Phi : \mathrm{SO}(2) \to S^1 \subseteq \mathbb{R}^2$ be the map defined by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right] \mapsto (\cos\theta, \sin\theta)$.

(i) Φ is injective.
Let $A, B \in \mathrm{SO}(2)$ and say that $\Phi(A) = \Phi(B)$. We know that $A = \left[\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{smallmatrix}\right]$ for some $\theta, \phi \in [0, 2\pi)$. However, $\Phi(A) = \Phi(B)$ implies that $(\cos\theta, \sin\theta) = (\cos\phi, \sin\phi)$, so $\phi = \theta$. This implies that $A = B$, so Φ is injective.

(ii) Φ is surjective.
Let $(\cos\theta, \sin\theta) \in S^1$. Then it follows that the matrix $A = \left[\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right] \mapsto (\cos\theta, \sin\theta)$, so Φ is surjective.

(iii) Φ and Φ^{-1} are smooth.
As Φ can be thought of as the projection $(\cos\theta, \sin\theta, -\sin\theta, \cos\theta) \mapsto (\cos\theta, \sin\theta)$ and cos and sin are C^∞ functions, it follows that Φ and Φ^{-1} are smooth.

As the above properties holds, we have that Φ is a diffeomorphism, so $\mathrm{SO}(2)$ is diffeomorphic to S^1. $\square$

Problem 15.12: The *unitary group* $\mathrm{U}(n)$ is defined to be

$$\mathrm{U}(n) = \{A \in \mathrm{GL}(n, \mathbb{C}) \mid \overline{A}^T A = I\},$$

where $\overline{A}$ denotes the complex conjugate of A, the matrix obtained from A by conjugating every entry of A: $(\overline{A})_{ij} = \overline{a_{ij}}$. Show that $\mathrm{U}(n)$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{C})$ and that $\dim \mathrm{U}(n) = n^2$.
Proof: Let $\phi : \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$ be the map defined by $A \mapsto \overline{A}^T A$, and note that ϕ is C^∞ and I is closed in $\mathrm{GL}(n, \mathbb{C})$. As a result, we have that $\phi^{-1}(I)$ is closed in $\mathrm{GL}(n, \mathbb{C})$. It follows from the definition of ϕ that $\mathrm{U}(n) = \phi^{-1}(I)$, so we have that $\mathrm{U}(n)$ is a closed subgroup of $\mathrm{GL}(n, \mathbb{C})$. It follows from Theorem 15.12 that $\mathrm{U}(n)$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{C})$.

Let $X \in T_I\mathrm{U}(n)$ and define the curve $c : \mathbb{R} \to \mathrm{U}(n)$ such that $c(0) = I$ and $c'(0) = X$. As $c(t) \in \mathrm{U}(n)$ for all $t \in \mathbb{R}$, we have that $\overline{c}^T(t)c(t) = I$. Differentiating this equation then yields that

$$\left.\frac{d\overline{c}^T}{dt}\right|_t c(t) + \overline{c}^T \left.\frac{dc}{dt}\right|_t = 0.$$

Setting $t = 0$ then gives

$$\overline{X}^T + X = 0 \Rightarrow \overline{X}^T = -X.$$

This implies that X is a skew-Hermitian matrix. As X is arbitrary in $T_I\mathrm{U}(n)$, it then follows that every $X \in T_I\mathrm{U}(n)$ is skew-Hermitian. It follows from the definition of skew-Hermitian that X is uniquely defined exactly by the choices for its entries in the upper triangle of X. For the entries above the diagonal, we have $n(n-1)/2$ choices of complex numbers; this can then be thought of as $n(n-1)$ choices of real numbers. For the diagonal of X, we will have exactly n choices of purely imaginary numbers. This implies that the total choices uniquely defining X is $n(n-1) + n = n^2 - n + n = n^2$ choices of real numbers, so $\dim T_I\mathrm{U}(n) = n^2$. As we always have that the dimensions of a space and its tangent space are equal, we have that $\dim \mathrm{U}(n) = n^2$, as desired. $\square$

Problem 15.13: The *special unitary group* $\mathrm{SU}(n)$ is defined to be the subgroup of $\mathrm{U}(n)$ consisting of matrices of determinant 1.

(a) Show that $\mathrm{SU}(2)$ can also be described as the set

$$\mathrm{SU}(2) = \left\{ \begin{bmatrix} a & -\overline{b} \\ b & \overline{a} \end{bmatrix} \in \mathbb{C}^{2\times 2} \,\middle|\, a\overline{a} + b\overline{b} = 1 \right\}.$$

(Hint: Write out the condition $A^{-1} = \overline{A}^T$ in terms of the entries of A.)

Proof: Let S be the set described in the problem statment.

Let $A \in \mathrm{SU}(2)$, and say that $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. As $A \in \mathrm{SU}(2)$ we have that $\det A = 1$ and $A^{-1} = \overline{A}^T$. This implies that

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = A^{-1} = \overline{A}^T = \begin{bmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{bmatrix}^T = \begin{bmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{bmatrix},$$

so then we have that

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{bmatrix}.$$

This identity gives us that $d = \overline{a}$, $-b = \overline{c}$, $-c = \overline{b}$, and $a = \overline{d}$. Thus, we can substitute to define $\overline{A}^T$ as

$$\overline{A}^T = \begin{bmatrix} \overline{a} & \overline{c} \\ -c & a \end{bmatrix}, \text{ which implies that } A = \overline{\overline{A}^T}^T = \begin{bmatrix} a & -\overline{c} \\ c & \overline{a} \end{bmatrix}.$$

As the determinant is preserved under complex conjugation and transposition, we still have that $\det A = 1$, so

$$1 = \det A = \begin{vmatrix} a & -\overline{c} \\ c & \overline{a} \end{vmatrix} = a\overline{a} + c\overline{c}.$$

Thus, we have that $A \in S$.

Now let $A \in S$, say $A = \left[\begin{smallmatrix} a & -\overline{b} \\ b & \overline{a} \end{smallmatrix}\right]$. We immediately have that $\det A = 1$. Furthermore, it can be seen that

$$A\overline{A}^T = \begin{bmatrix} a & -\overline{b} \\ b & \overline{a} \end{bmatrix} \begin{bmatrix} \overline{a} & \overline{b} \\ -b & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

As a result, we have that $A \in \mathrm{SU}(2)$.

It then follows from double containment that $\mathrm{SU}(2)$ is exactly the set described above. $\square$

(b) Show that $\mathrm{SU}(2)$ is diffeomorphic to the three dimensional sphere

$$S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}.$$

Proof: As we know from (a) that $\mathrm{SU}(2)$ is defined entirely by a pair of complex numbers, we can identify each $X \in \mathrm{SU}(2)$ with a pair of complex numbers $(z, w) \in \mathbb{C}^2$. We may then say that $z = a + bi$ and $w = x + yi$ for some $a, b, x, y \in \mathbb{R}$. It also follows from (a) that

$$1 = z\overline{z} + w\overline{w} = (a + bi)(a - bi) + (x + yi)(x - yi) = a^2 + b^2 + x^2 + y^2,$$

so we have that $(a, b, x, y) \in S^3$. This means that we may then define the map $\phi : \mathrm{SU}(2) \to S^3$ by $(z, w) = (a + bi, x + yi) \mapsto (\mathrm{Re}(z), \mathrm{Im}(z), \mathrm{Re}(w), \mathrm{Im}(w)) = (a, b, x, y)$. This map is immediately injective and surjective; it is also C^∞ in both directions as its coordinates are C^∞. Thus, we have that ϕ is a diffeomorphism, so $\mathrm{SU}(2)$ is diffeomorphic to S^3. $\square$

Problem 15.14: Compute $\exp\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$.

Proof: Recall the definition of the matrix exponential and consider the following:

$$\begin{aligned}
exp \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} &= I + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{1}{2!} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2 + \frac{1}{3!} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^3 + \cdots \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{1}{2!} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{3!} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \cdots \\
&= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2!} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \cdots \right) + \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{1}{3!} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \cdots \right) \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left(1 + \frac{1}{2!} + \cdots \right) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \left(1 + \frac{1}{3!} + \cdots \right)
\end{aligned}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cosh(1) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \sinh(1)$$
$$= \begin{bmatrix} \cosh(1) & 0 \\ 0 & \cosh(1) \end{bmatrix} + \begin{bmatrix} 0 & \sinh(1) \\ \sinh(1) & 0 \end{bmatrix} = \begin{bmatrix} \cosh(1) & \sinh(1) \\ \sinh(1) & \cosh(1) \end{bmatrix}$$

Thus, we have that $\exp \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \cosh(1) & \sinh(1) \\ \sinh(1) & \cosh(1) \end{smallmatrix}\right]$. $\square$

Problem 15.15: This problem requires a knowledge of quaternions as in Appendix E. Let $\mathbb{H}$ be the skew field of quaternions. The *symplectic group* $\mathrm{Sp}(n)$ is defined to be

$$\mathrm{Sp}(n) = \{A \in \mathrm{GL}(n, \mathbb{H}) \mid \overline{A}^T A = I\},$$

where $\overline{A}$ denotes the quaternionic conjugate of A. Show that $\mathrm{Sp}(n)$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{H})$ and compute its dimension.

Proof: This exercise differs little from Exercise 15.12.

Let $\phi : \mathrm{GL}(n, \mathbb{H}) \to \mathrm{GL}(n, \mathbb{H})$ be the map defined by $A \mapsto \overline{A}^T A$, and note that ϕ is C^∞ and I is closed in $\mathrm{GL}(n, \mathbb{H})$. As a result, we have that $\phi^{-1}(I)$ is closed in $\mathrm{GL}(n, \mathbb{H})$. It follows from the definition of ϕ that $\mathrm{Sp}(n) = \phi^{-1}(I)$, so we have that $\mathrm{Sp}(n)$ is a closed subgroup of $\mathrm{GL}(n, \mathbb{H})$. It follows from Theorem 15.12 that $\mathrm{Sp}(n)$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{H})$.

Let $X \in T_I\mathrm{Sp}(n)$ and define the curve $c : \mathbb{R} \to \mathrm{Sp}(n)$ such that $c(0) = I$ and $c'(0) = X$. As $c(t) \in \mathrm{Sp}(n)$ for all $t \in \mathbb{R}$, we have that $\overline{c}^T(t)c(t) = I$. Differentiating this equation then yields that

$$\left.\frac{d\overline{c}^T}{dt}\right|_t c(t) + \overline{c}^T \left.\frac{dc}{dt}\right|_t = 0.$$

Setting $t = 0$ then gives

$$\overline{X}^T + X = 0 \Rightarrow \overline{X}^T = -X.$$

This implies that X is a skew-Hermitian matrix (this may technically be the wrong term, since it entries are now in $\mathbb{H}$, but it still communicates the main idea). As X is arbitrary in $T_I\mathrm{Sp}(n)$, it then follows that every $X \in T_I\mathrm{Sp}(n)$ is skew-Hermitian. It follows from the definition of skew-Hermitian that X is uniquely defined exactly by the choices for its entries in the upper triangle of X. For the entries above the diagonal, we have $n(n-1)/2$ choices of numbers in $\mathbb{H}$; this can then be thought of as $2n(n-1)$ choices of real numbers. For the diagonal of X, we will have exactly $3n$ choices of purely nonreal numbers. This implies that the total choices uniquely defining X is $2n(n-1) + 3n = 2n^2 - 2n + 3n = 2n^2 + n$ choices of real numbers, so $\dim T_I\mathrm{Sp}(n) = 2n^2 + n$. As we always have that the dimensions of a space and its tangent space are equal, we have that $\dim \mathrm{Sp}(n) = 2n^2 + n$. $\square$

Problem 15.16: Let J be the $2n \times 2n$ matrix $J = \left[\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right]$, where I_n denotes that $n \times n$ identity matrix. The *complex symplectic group* $\mathrm{Sp}(2n, \mathbb{C})$ is defined to be $\mathrm{Sp}(2n, \mathbb{C}) = \{A \in \mathrm{GL}(2n, \mathbb{C}) \mid A^T JA = J\}$. Show that $\mathrm{Sp}(2n, \mathbb{C})$ is a regular submanifold of $\mathrm{GL}(2n, \mathbb{C})$ and compute its dimension. (Hint: Mimic Example 15.6. It is crucial to choose the correct tangent space for the map $f(A) = A^T JA$.)

Proof: This exercise differs little from the previous exercise.

Let $\phi : \mathrm{GL}(2n, \mathbb{C}) \to \mathrm{GL}(2n, \mathbb{C})$ be the map defined by $A \mapsto \overline{A}^T JA$, and note that ϕ is C^∞ (As all the matrices involved are invertible) and I is closed in $\mathrm{GL}(2n, \mathbb{C})$. As a result, we have that $\phi^{-1}(I)$ is closed in $\mathrm{GL}(2n, \mathbb{C})$. It follows from the definition of ϕ that $\mathrm{Sp}(2n, \mathbb{C}) = \phi^{-1}(I)$, so we have that $\mathrm{Sp}(2n, \mathbb{C})$ is a closed subgroup of $\mathrm{GL}(2n, \mathbb{C})$. It follows from Theorem 15.12 that $\mathrm{Sp}(2n, \mathbb{C})$ is a regular submanifold of $\mathrm{GL}(2n, \mathbb{C})$.

Similar to the previous exercises, the tangent space $T_I\mathrm{Sp}(2n, \mathbb{C})$ will be the space of $2n \times 2n$ skew-Hermitian matrices. Let $X \in T_I\mathrm{Sp}(n)$, it then follows that every $X \in T_I\mathrm{Sp}(n)$ is skew-Hermitian. It follows from the definition of skew-Hermitian that X is uniquely defined exactly by the choices for its entries in the upper triangle of X. For the entries above the diagonal, we have $2n(2n - 1)/2$ choices of numbers in $\mathbb{C}$; this can then be thought of as $2n(2n - 1)$ choices of real numbers. For the diagonal of X, we will have exactly $2n$ choices of purely imaginary numbers. This implies that the total choices uniquely defining X is $2n(2n - 1) + 2n = 2n^2 - 2n + 2n = 2n^2$ choices of real numbers, so $\dim T_I\mathrm{Sp}(n) = 2n^2$. As we always have that the dimensions of a space and its tangent space are equal, we have that $\dim \mathrm{Sp}(2n, \mathbb{C}) = 2n^2$. $\square$

Chapter 16

Problem 16.5: Find the left-invariant vector fields on $\mathbb{R}^n$.

Proof: Let $x^1, \dots, x^n$ be the standard coordinates on $\mathbb{R}^n$. As seen in Example 16.6, the group operation on $\mathbb{R}^n$ will be addition (although it will be coordinate-wise in this case). This means that for $g, p \in \mathbb{R}^n$ with $g = (g_1, \dots, g_n)$ and $p = (p_1, \dots, p_n)$ we have

$$\ell_g(p) = g + p = (g_1 + p_1, \dots, g_n + p_n).$$

Since for each i we know that $\ell_{g*}\left(\frac{\partial}{\partial x^i}\big|_0\right)$ is a tangent vector at g, it is a linear combination of $\frac{\partial}{\partial x^i}\big|_g$:

$$\ell_{g*}\left(\left.\frac{\partial}{\partial x^i}\right|_0\right) = a_1 \left.\frac{\partial}{\partial x^1}\right|_g + \cdots + \left.\frac{\partial}{\partial x^n}\right|_g.$$

To determine the identity of a_i, we shall apply both sides of the above equation to the function $f_j(x) = (x^j)$ for each j, where $x \in \mathbb{R}^n$. We then have that when $j \neq i$ that

$$\left(a_1 \left.\frac{\partial}{\partial x^1}\right|_g + \cdots + a_n \left.\frac{\partial}{\partial x^n}\right|_g\right) f_j = a_1 \left.\frac{\partial}{\partial x^1}\right|_g x^j + \cdots + a_n \left.\frac{\partial}{\partial x^n}\right|_g x^j = a_j.$$

We then also have that

$$\ell_{g*}\left(\left.\frac{\partial}{\partial x^i}\right|_0\right) f_j = \left.\frac{\partial}{\partial x^i}\right|_0 f_j \circ \ell_g = \left.\frac{\partial}{\partial x^i}\right|_0 (g + x^j) = 0,$$

so then $a_j = 0$ for $j \neq i$. In the case in which $j = i$ we still have that $\ell_{g*}\left(\frac{\partial}{\partial x^i}\big|_0\right) f_j = a_i$, but we also have that

$$\ell_{g*}\left(\left.\frac{\partial}{\partial x^i}\right|_0\right) f_j = \left.\frac{\partial}{\partial x^i}\right|_0 f_i \circ \ell_g = \left.\frac{\partial}{\partial x^i}\right|_0 (g + x^i) = 1,$$

so $a_i = 1$. As a result, we have that $\ell_{g*}\left(\frac{\partial}{\partial x^i}\big|_0\right) = \frac{\partial}{\partial x^i}\big|_g$, so it follows that $\frac{\partial}{\partial x^i}$ will be left invariant for all i.

As we know that ℓ_{g*} is a linear map, it then follows that all vector fields X of the form $X = \sum_{i=1}^n a_i \frac{\partial}{\partial x^i}$ will be left-invariant, provided that the a^i are constants. $\square$

Problem 16.6: Find the left-invariant vector fields on S^1.

Proof: Recall that we may realize S^1 as the points e^{it}, where $t \in \mathbb{R}$, and let $g = e^{it_0}$ for some $t_0 \in \mathbb{R}$. Our coordinate map will be $f(e^{it}) = t$. It also follows that $\ell_g(t) = e^{it_0} \cdot e^{it} = e^{i(t_0+t)}$. We also know that $\ell_{g*}\left(\frac{d}{dt}\big|_0\right) = a\frac{d}{dt}\big|_g$ for some $a \in \mathbb{R}$. If we now apply both sides of this equation to our coordinate map f we may observe that

$$a = a\left.\frac{d}{dt}\right|_g f = \ell_{g*}\left(\left.\frac{d}{dt}\right|_0\right) f = \left.\frac{d}{dt}\right|_0 f \circ \ell_g = \left.\frac{d}{dt}\right|_0 (t_0 + t) = 1,$$

so $a = 1$ and then $\ell_{g*}\left(\frac{d}{dt}\big|_0\right) = \frac{d}{dt}\big|_g$. As we know that ℓ_{g*} is a linear map, the left-invariant vector fields X on S^1 will be of the form $X = a\frac{d}{dt}$, where $a \in \mathbb{R}$. $\square$

Problem 16.7: Let $A \in \mathfrak{gl}(n, \mathbb{R})$ and let $\tilde{A}$ be the left-invariant vector field on $GL(n, \mathbb{R})$ generated by A. Show that $c(t) = e^{tA}$ is the integral curve of $\tilde{A}$ starting at the identity matrix I. Find the integral curve of $\tilde{A}$ starting at $g \in GL(n, \mathbb{R})$.

Proof: It can immediately be seen that

$$c(0) = e^0 = I \quad \text{ and } \quad c'(t) = (e^{tA})A = c(t)A = \ell_{c(t)*}(A) = A_{c(t)},$$

so then $c(t) = e^{tA}$ is the integral curve of $\tilde{A}$ starting at the identity matrix I.

Now for $g \in GL(n, \mathbb{R})$ define the curve d by $d(t) = ge^{tA}$ and observe that

$$d(0) = ge^0 = I \quad \text{ and } \quad d'(t) = (ge^{tA})A = c(t)A = \ell_{d(t)*}(A) = A_{d(t)},$$

so then $d(t) = ge^{tA}$ is the integral curve of $\tilde{A}$ starting at $g \in GL(n, \mathbb{R})$, as desired. $\square$

Problem 16.9: Show that every Lie group is parallelizable.

Proof: We know that the tangent space at the identity, T_eM has some basis $X_{1,e}, \ldots, X_{n,e}$. Now let $X_1, \ldots, X_n$ be the left-invariant vector fields generated by $X_{1,e}, \ldots, X_{n,e}$. Proposition 16.8 tells us that $X_1, \ldots, X_n$ will be C^∞ because they are left-invariant. Furthermore, because for all $g \in G$ we know that ℓ_g induces an isomorphism ℓ_{g*}, then $X_1, \ldots, X_n$ will be a basis at each g. As a result, we have shown that $X_1, \ldots, X_n$ is a smooth frame on M. Thus, a smooth frame exists for any Lie group G, so it follows from Exercise 16.8 that G is parallelizable. $\square$

Problem 16.11: Let G be a Lie group of dimension n with Lie algebra $\mathfrak{g}$.

(a) For each $a \in G$, the differential at the identity of the conjugation map $c_a := \ell_a \circ r_{a^{-1}} : G \to G$ is a linear isomorphism $c_{a^*} : \mathfrak{g} \to \mathfrak{g}$. Hence $c_{a^*} \in GL(\mathfrak{g})$. Show that the map $\mathrm{Ad} : G \to GL(\mathfrak{g})$ defined by $\mathrm{Ad}(a) = c_{a^*}$ is a group homomorphism. It is called the *adjoint representation* of the Lie group G.

Proof: Let X be an arbitrary vector field, let $a, b \in G$, recall the chain rule, and consider the following:

$$\begin{aligned}\mathrm{Ad}(ab)X = c_{ab*}(X) &= \ell_{ab*} \circ r_{(ab)^{-1}*}(X) = \ell_{ab*} \circ r_{a^{-1}a^{-1}*}(X) = abXb^{-1}a^{-1} \\ &= \ell_{a*} \circ r_{a^{-1}*}(bXb^{-1}) = \ell_{a*} \circ r_{a^{-1}*} \circ \ell_{b*} \circ r_{b^{-1}*}(X) = c_{a*} \circ c_{b*}(X) = \mathrm{Ad}(a)\mathrm{Ad}(b)(X).\end{aligned}$$

Thus, $\mathrm{Ad}(ab)(X) = \mathrm{Ad}(a)\mathrm{Ad}(b)(X)$ for all X, so Ad is a group homomorphism. $\square$

(b) Show that $\mathrm{Ad} : G \to GL(\mathfrak{g})$ is C^∞.

Proof: Let $(U, x^1, \ldots, x^n)$ be a chart at $e \in G$. With repsect to this chart, c_{a*} at e is given by the Jacobian $\left[\frac{\partial(x^i \circ c_a)}{\partial x^j}\Big|_e\right]$. As $c_a(x) = axa^{-1}$ will be C^∞ for all a, all the partials $\frac{\partial(x^i \circ c_a)}{\partial x^j}\Big|_e$ are C^∞ and thus $\mathrm{Ad}(a)$ is a C^∞ function of a. $\square$

Problem 16.12: The Lie algebra $\mathfrak{o}(n)$ of the orthogonal group $\mathrm{O}(n)$ is the Lie algebra of $n \times n$ skew-symmetric real matrices, with Lie bracket $[A, B] = AB - BA$. When $n = 3$, there is a vector space isomorphism $\varphi : \mathfrak{o}(3) \to \mathbb{R}^3$,

$$\varphi(A) = \varphi\left(\begin{bmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ -a_2 \\ a_3 \end{bmatrix} = a.$$

Prove that $\varphi([A, B]) = \varphi(A) \times \varphi(B)$. Thus, $\mathbb{R}^3$ with the cross product is a Lie algebra.

Proof: This problem simply boils down to verifying an identity. To that end, we determine $\varphi(A) \times \varphi(B)$:

$$\begin{aligned}
\varphi(A) \times \varphi(B) &= \varphi\left(\begin{bmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{bmatrix}\right) \times \varphi\left(\begin{bmatrix} 0 & b_1 & b_2 \\ -b_1 & 0 & b_3 \\ -b_2 & -b_3 & 0 \end{bmatrix}\right) \\
&= \begin{bmatrix} a_1 \\ -a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ -b_2 \\ b_3 \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & -a_2 & a_3 \\ b_1 & -b_2 & b_3 \end{vmatrix} = \begin{bmatrix} -a_2b_3 + a_3b_2 \\ -a_1b_3 + a_3b_1 \\ -a_1b_2 + a_2b_1 \end{bmatrix}
\end{aligned}$$

Next, we determine $\varphi([A, B])$ for $A, B \in \mathfrak{o}(3)$:

$$\begin{aligned}
\varphi([A, B]) &= \varphi(AB - BA) \\
&= \varphi\left(\begin{bmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{bmatrix}\begin{bmatrix} 0 & b_1 & b_2 \\ -b_1 & 0 & b_3 \\ -b_2 & -b_3 & 0 \end{bmatrix} - \begin{bmatrix} 0 & b_1 & b_2 \\ -b_1 & 0 & b_3 \\ -b_2 & -b_3 & 0 \end{bmatrix}\begin{bmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{bmatrix}\right) \\
&= \varphi\left(\begin{bmatrix} -a_1b_1 - a_2b_2 & -a_2b_3 & a_1b_3 \\ -a_3b_2 & -a_1b_1 - b_3a_3 & -a_1b_2 \\ a_3b_1 & -a_2b_1 & -a_2b_2 - a_3b_3 \end{bmatrix}\right. \\
&\qquad\qquad \left. - \begin{bmatrix} -a_1b_1 - a_2b_2 & -a_3b_2 & a_3b_1 \\ -a_2b_3 & -a_1b_1 - b_3a_3 & -a_2b_1 \\ a_1b_3 & -a_1b_2 & -a_2b_2 - a_3b_3 \end{bmatrix}\right)
\end{aligned}$$

$$= \varphi \left(\begin{bmatrix} 0 & -a_2b_3 + a_3b_2 & a_1b_3 - a_3b_1 \\ -a_3b_2 + a_2b_3 & 0 & -a_1b_2 + a_2b_1 \\ a_3b_1 - a_1b_3 & -a_2b_1 + a_1b_2 & 0 \end{bmatrix} \right) = \begin{bmatrix} -a_2b_3 + a_3b_2 \\ -a_1b_3 + a_3b_1 \\ -a_1b_2 + a_2b_1 \end{bmatrix}$$

Thus, we have that

$$\varphi([A, B]) = \begin{bmatrix} -a_2b_3 + a_3b_2 \\ -a_1b_3 + a_3b_1 \\ -a_1b_2 + a_2b_1 \end{bmatrix} = \varphi(A) \times \varphi(B)$$

for all $A, B \in \mathfrak{o}(3)$, so $\varphi([A, B]) = \varphi(A) \times \varphi(B)$, as desired. $\square$

Chapter 17

Problem 17.1: Denote the standard coordinates on $\mathbb{R}^2$ by x, y and let

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \quad \text{and} \quad Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

be vector fields on $\mathbb{R}^2$. Find a 1-form ω on $\mathbb{R}^2 \setminus \{(0,0)\}$ such that $\omega(X) = 1$ and $\omega(Y) = 0$.
Proof: Let $\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ (note that this is allowed because we are on the punctured plane). We can then immediately observe that

$$\omega(X) = \left(\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy\right)\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) = \frac{y^2}{x^2+y^2} + \frac{x^2}{x^2+y^2} = 1 \text{ and}$$
$$\omega(Y) = \left(\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy\right)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) = \frac{-xy}{x^2+y^2} + \frac{xy}{x^2+y^2} = 0,$$

so $\omega(X) = 1$ and $\omega(Y) = 0$, as desired. $\square$

Problem 17.2: Suppose $(U, x^1, \dots, x^n)$ and $(V, y^1, \dots, y^n)$ are two charts on M with nonempty overlap $U \cap V$. Then a C^∞ 1-form ω on $U \cap V$ has two different local expressions:

$$\omega = \sum a_j dx^j = \sum b_j dy^i.$$

Find a formula for a_j in terms of b_i.
Proof: We can apply the equation above to $\frac{\partial}{\partial x^j}$ to find an expression for a_j in terms of the b_i in the following manner:

$$\omega\left(\frac{\partial}{\partial x^j}\right) = \left(\sum_{j=1}^n a_j dx^j\right)\left(\frac{\partial}{\partial x^j}\right) = \left(\sum_{i=1}^n b_i dy^i\right)\left(\frac{\partial}{\partial x^j}\right) \implies a_j = \sum_{i=1}^n b_i \frac{\partial y^i}{\partial x^j}$$

Thus, we have the desired result. $\square$

Problem 17.3: Multiplication in the unit circle S^1, viewed as a subset of the complex plane, is given by $e^{it} \cdot e^{iu} = e^{i(t+u)}$ where $t, u \in \mathbb{R}$. In terms of real and imaginary parts this is

$$(\cos t + i \sin t)(x + iy) = ((\cos t)x - (\sin t)y) + i((\sin t)x + (\cos t)y).$$

Hence, if $g = (\cos t, \sin t) \in S^1 \subset \mathbb{R}^2$, then the left multiplication $\ell_g : S^1 \to S^1$ is given by

$$\ell_g(x, y) = ((\cos t)x - (\sin t)y, (\sin t)x + (\cos t)y).$$

Let $\omega = -ydx + xdy$ be the 1-form from Example 17.15. Prove that $\ell_g^*\omega = \omega$ for all $g \in S^1$.
Proof: We may observe that

$\ell_g^*\omega = \ell_g^*(-ydx + xdy)$

$$= \ell_g^*(-y)\ell_g^*(dx) + \ell_g^*(x)\ell_g^*(dy)$$
$$= \ell_g^*(-y)d\ell_g^*(x) + \ell_g^*(x)d\ell_g^*(y)$$
$$= -((\sin t)x + (\cos t)y)d((\cos t)x - (\sin t)y) + ((\cos t)x - (\sin t)y)d((\sin t)x + (\cos t)y)$$
$$= -((\sin t)x + (\cos t)y)((\cos t)dx - (\sin t)dy) + ((\cos t)x - (\sin t)y)((\sin t)dx + (\cos t)dy)$$
$$= -(\cos t)(\sin t)xdx + (\sin^2 t)xdy - (\cos^2 t)ydx + (\sin t)(\cos t)ydy+$$
$$(\cos t)(\sin t)xdx + (\cos^2 t)xdy - (\sin^2 t)ydx - (\sin t)(\cos t)ydy$$
$$= (\sin^2 t + \cos^2 t) - ydx + (\sin^2 t + \cos^2 t)xdy = -ydx + xdy = \omega$$

As a result, we have that $\ell_g^*\omega = \omega$, as desired. □

Problem 17.4:
(a) Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a chart on a manifold M, and let

$$(\pi^{-1}U, \tilde{\phi}) = (\phi^{-1}U, \overline{x}^1, \ldots, \overline{x}^n, c_1, \ldots, c_n)$$

be the induced chart on the cotangent bundle T^*M. Find a formula for the Liouville form λ on $\pi^{-1}U$ in terms of the coordinates $\overline{x}^1, \ldots, \overline{x}^n, c_1, \ldots, c_n$.
Proof: Let $\omega_p = \sum_{i=1}^n c_i dx^i|_p$; then we have that

$$\lambda_{\omega_p} = \pi^*(\omega_p) = \sum_{i=1}^n c_i \pi^*(dx^i|_p) = \sum_{i=1}^n c_i(\pi^* dx^i)_{\omega_p} = \sum_{i=1}^n c_i(d\pi^* x^i)_{\omega_p} = \sum_{i=1}^n c_i(d\overline{x}^i)_{\omega_p}.$$

As a result, it follows that $\lambda = \sum c_i d\overline{x}^i$. □

(b) prove that the Liouville form λ on T^*M is C^∞. (Hint: Use (a) and Proposition 17.6.)
Proof: It follows from (a) that $\lambda = \sum c_i d\overline{x}^i$. Proposition 17.6 then tells us λ will be C^∞ because the chart we began with in (a) was arbitrary. □

Problem 17.5: Prove Proposition 17.11 by verifying both sides of each equality on a tangent vector X_p at a point p.
Proof: Let $F : N \to M$ be a C^∞ map of manifolds, $p \in N$, and $X_p \in T_pN$. Suppose $\omega, \tau \in \Omega^1(M)$ and $g \in C^\infty(M)$. Then we can observe that

$$F^*(\omega + \tau)_p(X_p) = (\omega + \tau)_{F(p)}(F_{*,p}X_p) = (\omega_{F(p)} + \tau_{F(p)})(F_{*,p}X_p)$$
$$= \omega_{F(p)}(F_{*,p}X_p) + \tau_{F(p)}(F_{*,p}X_p) = (F^*\omega)_p(X_p) + (F^*\tau)_p(X_p).$$

As X_p is arbitrary, we have that the pullback distributes over a sum. We can also see that

$$F^*(g\omega)_p(X_p) = (g\omega)_{F(p)}(F_{*,p}X_p) = (g_{F(p)})(\omega_{F(p)})(F_{*,p}X_p) = (F^*g)_p(F^*\omega)_p(X_p).$$

As X_p is arbitrary, we have that the pullback distributes over the product. $\square$

Chapter 20

Problem 20.7: Let ω be a differential form, X a vector field, and f a smooth function on a manifold. The Lie derivative $\mathcal{L}_X\omega$ is not $\mathcal{F}$-linear in either variable, but prove that it satisfies the following identity: $\mathcal{L}_{fX}\omega = f\mathcal{L}_X\omega + df \wedge \imath_X\omega$. *Proof:* Consider the following:

$$\begin{aligned}
\mathcal{L}_{fX}\omega &= d\imath_{fX}\omega + \imath_{fX}d\omega && \text{Cartan} \\
&= df\imath_X\omega + f\imath_X d\omega && \mathcal{F}\text{-linearity of } \imath \\
&= df \wedge \imath_X\omega + fd\imath_X\omega + f\imath_X d\omega && \text{Leibniz rule} \\
&= df \wedge \imath_X\omega + f(d\imath_X\omega + \imath_X d\omega) && \text{Factor} \\
&= df \wedge \imath_X\omega + f\mathcal{L}_X\omega && \text{Cartan}
\end{aligned}$$

Thus, we have $\mathcal{L}_{fX}\omega = f\mathcal{L}_X\omega + df \wedge \imath_X\omega$, as desired. $\square$

Problem 20.9: Let $\omega = dx^1 \wedge \cdots \wedge dx^n$ be the volume form and $X = \sum_{j=1}^n x^j \frac{\partial}{\partial x^j}$ the radial vector field on $\mathbb{R}^n$. Compute the contraction $\imath_X\omega$.

Proof: First observe that

$$dx^i(X) = dx^i\left(\sum_{j=1}^n x^j \frac{\partial}{\partial x^j}\right) = \sum_{j=1}^n x^j \frac{\partial x^i}{\partial x^j} = x^i.$$

It then follows from Proposition 20.7 that

$$\begin{aligned}
\imath_X\omega &= \sum_{i=1}^n (-1)^{i-1} dx^i(X) dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n \\
&= \sum_{i=1}^n (-1)^{i-1} x^i dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n. \ \square
\end{aligned}$$

Problem 20.10: Let $\omega = xdy \wedge dz - ydx \wedge dy + zdx \wedge dy$ and $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ on the unit 2-sphere S^2 in $\mathbb{R}^3$. Compute the Lie derivative $\mathcal{L}_X\omega$.

Proof: First observe that

$$\begin{aligned}
Xx &= -y\frac{\partial}{\partial x}(x) + x\frac{\partial}{\partial y}(x) = -y + 0 = -y, \\
Xy &= -y\frac{\partial}{\partial x}(y) + x\frac{\partial}{\partial y}(y) = 0 + x = x, \quad \text{and} \\
Xz &= -y\frac{\partial}{\partial x}(z) + x\frac{\partial}{\partial y}(z) = 0 + 0 = 0.
\end{aligned}$$

Now consider the following:

$\mathcal{L}_X\omega = \mathcal{L}_X(xdy \wedge dz - ydx \wedge dy + zdx \wedge dy)$

$$
\begin{aligned}
&= \mathcal{L}_X(xdy \wedge dz) - \mathcal{L}_X((y+z)dx \wedge dy) \\
&= \mathcal{L}_X(x)dy \wedge dz + x\mathcal{L}_X(dy \wedge dz) - \mathcal{L}_X(y+z)dx \wedge dy - (y+z)\mathcal{L}_X(dx \wedge dy) \\
&= \mathcal{L}_X(x)dy \wedge dz + x(\mathcal{L}_X(dy) \wedge dz + dy \wedge \mathcal{L}_X(dz)) \\
&\qquad\qquad - \mathcal{L}_X(y+z)dx \wedge dy - (y+z)(\mathcal{L}_X(dx) \wedge dy + dx \wedge \mathcal{L}_X(dy)) \\
&= Xxdy \wedge dz + x(dXy \wedge dz + dy \wedge dXz) - (Xy + Xz)dx \wedge dy - (y+z)(dXx \wedge dy + dx \wedge dXy) \\
&= -ydy \wedge dz + x(dx \wedge dz + dy \wedge 0) - (x+0)dx \wedge dy - (y+z)(dy \wedge dy + dx \wedge dx) \\
&= -ydy \wedge dz + x(dx \wedge dz + 0) - xdx \wedge dy - (y+z)(0+0) \\
&= -ydy \wedge dz + xdx \wedge dz - xdx \wedge dy
\end{aligned}
$$

Thus, we have that $\mathcal{L}_X\omega = -ydy \wedge dz + xdx \wedge dz - xdx \wedge dy$. $\square$

Chapter 21

21.7: Show that every Lie group G is orientable by constructing a nowhere-vanishing top form on G.

Proof: Say that $\dim G = n$. Let $X_1, \ldots, X_n \in T_eG$ be a basis for T_eG, and let $\alpha^1, \ldots, \alpha^n \in T_e^*G$ be the dual of this basis. We shall now outline the construction for a nowhere-vanishing top form ω. We first define $\omega_e = \alpha^1 \wedge \cdots \wedge \alpha^n$. This implies that

$$\omega_e(X_1, \ldots, X_n) = \alpha^1 \wedge \cdots \wedge \alpha^n(X_1, \ldots, X_n) = \det[\alpha^i(X_j)] = \det[\delta^i_j] = \det[I_n] = 1,$$

so ω_e is nonvanishing. On page 207 the text describes that we can uniquely determine a left-invariant n-form (hence a top form) ω_g for all $g \in G$ by $\omega_g = \ell^*_{g^{-1}}(w_e)$. Thus, we have completely defined a top form ω on G. As ω is left-invariant and nonvanishing at e, we know that ω_g will be nonvanishing for all $g \in G$. Thus, we have that ω is then nonwhere vanishing on G.

Finally, it follows from Proposition 18.14 that ω will be C^∞ as a result of its left-invariance, so we have the desired result. $\square$