Tarea #3 Relatividad General

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1 Derivada Covariante

a) Usando $\nabla T = d^* \nabla T$ y la regla de leibniz:

$$\nabla T = \sum \nabla (T^{j_1 \dots j_l}_{i_1 \dots i_k} dx^{i_1} \otimes \ldots \otimes dx^{i_k} \otimes \partial_{j_1} \otimes \ldots \otimes \partial_{j_l})$$

$$= dx^i \otimes (\partial_i T^{j_1 \dots j_s}_{i_1 \dots i_r} dx^{i_1} \otimes \cdots \otimes \partial_{j_s} - T^{i_1 \dots i_s}_{j_1 \dots j_r} (\Gamma^{i_1}_{i\lambda} dx^{\lambda} \otimes \cdots \otimes \partial_{j_s} - \cdots - \Gamma^{i_1}_{i\lambda} dx^{i_1} \otimes dx^{\lambda} \otimes \cdots \otimes \partial_{j_s}$$
$$+ \Gamma^{\lambda}_{ij_1} dx^{i_1} \otimes \cdots \otimes dx^{i_r} \otimes \partial_{\lambda} \otimes \cdots + \cdots + \Gamma^{\lambda}_{ij_s} dx^{i_1} \otimes \cdots \otimes \partial_{\lambda}))$$

Cambiando entre índices mudos y simplificando:

$$= \left(\partial_i T^{j_1 \dots j_s}_{i_1 \dots i_r} + \sum_{m=1}^{s} T^{j_1 \dots j_{m-1} \lambda_{j_{m-1}} \dots j_s}_{i_1 i_r} \Gamma^{i_\lambda}_{j_m} - \sum_{l=1}^{r} T^{j_1 \dots j_s}_{i_1 \dots i_{l-}} \Gamma^{\lambda}_{ii_l} \right) dx^i \otimes dx^{i_1} \otimes \cdots \otimes \partial_{j_s}$$

b) Asumimos compatibilidad de la metrica por lo que:

$$\nabla_X g^{ij} = 0$$

Para el producto (S, T):

$$X(g^{ij} S_{ij} T_{kl}) = \nabla_X (g^{ij} g_{ij} S_{ij} T_{kl}) = g^{ik} g^{jl} ((\nabla_X S_{ij}) T_{kl} + S_{ij} (\nabla_X T_{kl}))$$
$$= \langle \nabla_X S | T \rangle_g + \langle S | \nabla_X T) \rangle_g$$

Expresando g^{ij} con su contracción de índices ($g^{ik} g_{il} = \delta^k_l$) tenemos:

$$X(\mathrm{tr}_g T) = X(g^{ik} g^{jl} g_{kl} T_{ij}) = X(\langle g | T \rangle_g)$$

Usando la identidad demostrada en el punto anterior:

$$X(\mathrm{tr}_g T) = \langle g | \nabla_X T \rangle_g$$

Localmente aplicamos la contracción de los tensores métricos nuevamente:

$$X(g^{ij} T_{ij}) = X^{\rho} \nabla_{\rho} (g^{ik} g^{jl} g_{kl} T_{ij}) = X^{\rho} g^{ik} g^{jl} g_{kl} (\nabla_{\rho} T_{ij}) = x^{\rho} g^{ij} (\nabla_p T_{ij})$$

c) Usamos la identidad $\det(A) = e^{\mathrm{tr}(\ln A)}$:

$$\frac{\partial \ln \det(A)}{\partial x^j} = \mathrm{tr}(A^{-1} \frac{\partial A}{\partial x^j}) \tag{1}$$

$$\frac{\partial \det(A)}{\partial x^j} = \det(A) \mathrm{tr}(A^{-1} \frac{\partial A}{\partial x^j}) \tag{2}$$

Reemplazando A por el tensor métrico, obtenemos:

$$\frac{\partial \sqrt{\det(g_{\mu\nu})}}{\partial x^j} = \frac{1}{2} (\det(g_{\mu\nu}))^{-\frac{1}{2}} g^{lm} \partial_j g_{lm}$$

$$\frac{\partial \ln \det g_{\mu\nu}}{\partial x^j} = g^{\rho\sigma} \partial_j g_{\rho\sigma}$$

2 Tensor de Riemann

a) Primer Metodo Tenemos que

$$R^{\rho}_{\sigma\mu\nu}V^{\sigma} = (\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)V^\rho \qquad \nabla_\mu V^\rho = \partial_\mu V^\rho + \Gamma^{\rho}_{\mu\sigma}V^\sigma$$

Entonces

$$\begin{aligned} R^{\rho}_{\sigma\mu\nu}V^{\rho} &= \nabla_\mu(\partial_\nu V^\rho + \Gamma^{\rho}_{\nu\lambda}V^\lambda) - \nabla_\nu(\partial_\mu V^\rho + \Gamma^{\rho}_{\mu\lambda}V^\lambda) \\ &= \partial_\mu\nabla_\mu V^\rho + \Gamma^{\rho}_{\nu\lambda}\nabla_\mu V^\lambda - \partial_\mu\nabla_\nu V^\rho - \Gamma^{\rho}_{\mu\lambda}\nabla_\nu V^\lambda \\ &= \partial_\mu\partial_\nu V^\rho - \partial_\mu\partial_\nu V^\rho + \partial_\nu\Gamma^{\rho}_{\mu\lambda}V^\lambda - \partial_\mu\Gamma^{\rho}_{\nu\lambda}V^\lambda + \Gamma^{\rho}_{\nu\lambda}\partial_\mu V^\lambda \\ &\quad + \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}V^\sigma - \Gamma^{\rho}_{\mu\lambda}\partial_\nu V^\lambda - \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma}V^\sigma \\ &= \{(\partial_\nu\Gamma^{\rho}_{\mu\sigma} - \partial_\mu\Gamma^{\rho}_{\mu\sigma}) + (\Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} - \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma})\}V^\sigma \end{aligned}$$

Por lo tanto:

$$R^{\rho}_{\sigma\mu\nu} = \partial_\nu\Gamma^{\rho}_{\mu\sigma} - \partial_\mu\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} - \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma}$$

Segundo Metodo $x^1 = a, x^1 = b,$ y $x^2 = b,$ y $x^2 = b+\delta b$. Un vector $\vec{V}$ definido en A es transportado paralelamente
0 tiene la forma componente

$$\frac{\partial V^\alpha}{\partial x^1} = -\Gamma^{\alpha}_{\mu 1}V^\mu$$

Integrando esto de A a B se obtiene

$$V^\alpha(B) = V^\alpha(A) + \int_A^B \frac{\partial V^\alpha}{\partial x^1}dx^1 = V^\alpha(A) - \int_{x^2=b}\Gamma^{\alpha}_{\mu 1}V^\mu dx^1$$

Deforma similar:

$$V^\alpha(C) = V^\alpha(B) - \int_{x^1=a+\delta a}\Gamma^{\alpha}_{\mu 2}V^\mu dx^2$$

$$V^\alpha(D) = V^\alpha(C) + \int_{x^2=b+\delta b}\Gamma^{\alpha}_{\mu 1}V^\mu dx^1$$

La integral en la última ecuación tiene un signo diferente porque la dirección de transporte de C a D está en la dirección negativa de x^1. Tambien:

$$V^\alpha(A_{final}) = V^\alpha(D) + \int_{x^1=a}\Gamma^{\alpha}_{\mu 2}V^\mu dx^2$$

El cambio neto en $V^\alpha(A)$ es un vector δV^α:

$$\delta V^\alpha = V^\alpha(A_{final}) - V^\alpha(A_{inicial})$$

$$= \int_{x^1=a}\Gamma^{\alpha}_{\mu 2}V^\mu dx^2 - \int_{x^1=a+\delta a}\Gamma^{\alpha}_{\mu 2}V^\mu dx^2 + \int_{x^2=b+\delta b}\Gamma^{\alpha}_{\mu 1}V^\mu dx^1 - \int_{x^2=b}\Gamma^{\alpha}_{\mu 1}V^\mu dx^1$$

Si combinamos las integrales sobre variables de integración similares y trabajamos hasta el primer orden en la separación de los caminos, obtenemos:

$$\begin{aligned} \delta V^\alpha &\sim -\int_b^{b+\delta b}\delta a\frac{\partial}{\partial x^1}(\Gamma^{\alpha}_{\mu 2}V^\mu)dx^2 + \int_a^{a+\delta a}\delta b\frac{\partial}{\partial x^2}(\Gamma^{\alpha}_{\mu 1}V^\mu)dx^1 \\ &\approx \delta a\delta b\left[-\frac{\partial}{\partial x^1}(\Gamma^{\alpha}_{\mu 2}V^\mu) + \frac{\partial}{\partial x^2}(\Gamma^{\alpha}_{\mu 1}V^\mu)\right] \end{aligned}$$

Utilizando $\frac{\partial V^\alpha}{\partial x^1} = -\Gamma^{\alpha}_{\mu 1}V^\mu$

$$\delta V^\alpha = \delta a \delta b \left[\Gamma^\alpha_{\mu 1,2} - \Gamma^\alpha_{\mu 2,1} + \Gamma^\alpha_{\lambda 2}\Gamma^\lambda_{\mu 1} - \Gamma^\alpha_{\lambda 1}\Gamma^\lambda_{\mu 2}\right] V^\mu$$

Si usáramos coordenadas generales x^σ y x^λ:

$$\delta V^\alpha = \delta a\, \delta b \left[\Gamma^\alpha_{\sigma\mu,\lambda} - \Gamma^\alpha_{\lambda\mu,\sigma} + \Gamma^\alpha_{\nu\sigma}\Gamma^\nu_{\lambda\mu} - \Gamma^\alpha_{\nu\lambda}\Gamma^\nu_{\sigma\mu}\right] V^\mu$$

δV^α depende linealmente de $\delta a\, \epsilon_\sigma$ y $\delta b\, \epsilon_\lambda$. Además, ciertamente también depende linealmente en la Ecuación de V^α en sí mismo y en $\delta\epsilon_\sigma$, que es la forma uno base que da δV^α desde el vector $\delta\mathbf{V}$. Por lo tanto, tenemos el siguiente resultado:

$$R^\alpha_{\beta\mu\nu} := \Gamma^\alpha_{\beta\nu,\mu} - \Gamma^\alpha_{\beta\mu,\nu} + \Gamma^\alpha_{\sigma\mu}\Gamma^\sigma_{\beta\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\beta\mu}$$

b)

- Por definicion:

$$R_{\rho\sigma\mu\nu} = \partial_\mu \Gamma_{\rho\nu\sigma} - \partial_\nu \Gamma_{\rho\mu\sigma} + \Gamma_{\rho\mu\lambda}\Gamma_{\lambda\nu\sigma} - \Gamma_{\rho\nu\lambda}\Gamma_{\lambda\mu\sigma}$$

Intercambiando los ultimos dos indices:

$$R_{\rho\sigma\nu\mu} = \partial_\nu \Gamma_{\rho\mu\sigma} - \partial_\mu \Gamma_{\rho\nu\sigma} + \Gamma_{\rho\nu\lambda}\Gamma_{\lambda\mu\sigma} - \Gamma_{\rho\mu\lambda}\Gamma_{\lambda\nu\sigma} = -(\partial_\mu \Gamma_{\rho\nu\sigma} - \partial_\nu \Gamma_{\rho\mu\sigma} + \Gamma_{\rho\mu\lambda}\Gamma_{\lambda\nu\sigma} - \Gamma_{\rho\nu\lambda}\Gamma_{\lambda\mu\sigma}) = -R_{\rho\sigma\mu\nu}$$

- De igual forma:

$$R_{\sigma\rho\nu\mu} = \partial_\mu \Gamma_{\sigma\nu\rho} - \partial_\nu \Gamma_{\sigma\mu\rho} + \Gamma_{\sigma\mu\lambda}\Gamma_{\lambda\nu\rho} - \Gamma_{\sigma\nu\lambda}\Gamma_{\lambda\mu\rho} = -R_{\rho\sigma\mu\nu}$$

-

$$\begin{aligned} R_{ijkl} + R_{iklj} + R_{iljk} = {} & \partial_k \Gamma_{ilj} - \partial_l \Gamma_{ikj} + \Gamma_{ikn}\Gamma_{nlj} - \Gamma_{iln}\Gamma_{nkj} \\ & + \partial_l \Gamma_{ijk} - \partial_j \Gamma_{ilk} + \Gamma_{iln}\Gamma_{njk} - \Gamma_{ijn}\Gamma_{nlk} \\ & + \partial_j \Gamma_{ikl} - \partial_k \Gamma_{ijl} + \Gamma_{ijn}\Gamma_{nkl} - \Gamma_{ikn}\Gamma_{njl} \end{aligned}$$

Como $\Gamma_{ijl} = \Gamma_{njl}$:

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0$$

- Aplicando derivada convariante a lo anterior

$$\nabla_m R_{ijkl} + \nabla_m R_{iklj} + \nabla_m R_{iljk} = 0$$

Como m es un indice mudo:

$$\nabla_m R_{ijkl} + \nabla_n R_{iklj} + \nabla_r R_{iljk} = 0$$

-

$$\begin{aligned} R_{\alpha\beta\mu\nu} - R_{\mu\nu\alpha\beta} &= R_{\alpha\beta\mu\nu} - (-R_{\nu\alpha\beta\mu} - R_{u\beta\alpha}) \\ &= R_{\alpha\beta\mu\nu} - R_{\alpha\mu\nu\beta} - R_{\beta\mu\alpha} \\ &= R_{\alpha\beta\mu\nu} - (-R_{\alpha\beta\mu\nu} - R_{\alpha\beta\nu}) + (-(-R_{\alpha\beta\mu\nu} - R_{\alpha\beta\mu\nu} - R_{\beta\alpha\nu})) \\ &= R_{\alpha\beta\mu\nu} + R_{\alpha\beta\mu\nu} + R_{\alpha\beta\mu\nu} + R_{\alpha\beta\mu\nu} + R_{\beta\alpha\nu}) \\ &= -R_{\alpha\beta\mu\nu} + R_{\alpha\beta\mu\nu} + R_{\beta\alpha\nu}) \\ &= (-R_{\alpha\beta\mu\nu} - R_{\alpha\beta\mu\nu}) + R_{\beta\alpha\nu}) \\ &= R_{\alpha\beta\mu\nu} + R_{\beta\alpha\nu} \\ &= -R_{\alpha\beta\mu\nu} + R_{\beta\alpha\nu} \end{aligned}$$

Ahora intercambiamos los nombres de tanto $\alpha\beta$ y $\mu\nu$. A la izquierda, esto da dos signos menos, pero a la derecha solo uno:

$$
\begin{gathered}
R_{\alpha\beta\mu\nu} - R_{\mu\nu\alpha\beta} = -R_{\alpha\beta\mu\nu} + R_{\beta\alpha\nu} \\
R_{\beta\alpha\nu} - R_{\mu\nu\alpha\beta} = -R_{\beta\alpha\nu} + R_{\alpha\beta\mu\nu} \\
= (-1)^2(R_{\alpha\beta\mu\nu} - R_{\mu\nu\alpha\beta}) = +R_{\alpha\beta\mu\nu} - R_{\beta\alpha\nu} \\
= -(-R_{\alpha\beta\mu\nu} - R_{\mu\nu\alpha\beta})
\end{gathered}
$$

Esta diferencia por lo tanto desaparece, y tenemos simetría bajo el intercambio de los pares,

$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}$$

3 Ecuaciones de Einstein

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0$$

a) Multiplicando por $g^{\mu\nu}$

$$g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\mu\nu}R + \Lambda g^{\mu\nu}g_{\mu\nu} = 0$$

Como $g^{\mu\nu}g_{\mu\nu} = 4$

$$
\begin{gathered}
R - 2R + \Lambda 4 = 0 \\
R = 4\Lambda
\end{gathered}
$$

Reemplazando $R = 4\Lambda$

$$
\begin{gathered}
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(4\Lambda) + \Lambda g_{\mu\nu} = 0 \\
R_{\mu\nu} - \Lambda g_{\mu\nu} = 0 \qquad \rightarrow \qquad R_{\mu\nu} = \Lambda g_{\mu\nu}
\end{gathered}
$$

Por lo tanto

$$R_{\mu\nu} = 3kg_{\mu\nu} \qquad \text{Con} \qquad k = \frac{\Lambda}{3}$$

b)

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

i) Utilizando *SageManifolds*

```
# Definir la variedad
M = Manifold(4, 'R^4', start_index=1)
# Definir las coordenadas
c_spher.<t,r,th,ph> = M.chart(r't:(0,+oo) r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi')
# Definir la función a(t)
f = function('f')(r)

# Definir la métrica g de FRW
g = M.metric('g')
```

```
g[1,1] = -f
g[2,2] = 1/f
g[3,3] = r^2
g[4,4] = r^2 * sin(th)^2
#inversa de la metrica
ginv = g.inverse()
```

Calculamos los simbolos de christoffel por el metodo indicado:

```
# Calcular las derivadas parciales \partial_\mu g_{\nu\sigma}
partial_g = [[[[None for _ in range(4)] for _ in range(4)] for _ in range(4)] for _ in range(4)]

for mu in range(4):
    for nu in range(4):
        for sigma in range(4):
            partial_g[mu][nu][sigma] = g[nu+1, sigma+1].diff(c_spher[mu+1])

# Calcular los símbolos de Christoffel \Gamma^\lambda_{\mu\nu}
Gamma = [[[None for _ in range(4)] for _ in range(4)] for _ in range(4)]

for lam in range(4):
    for mu in range(4):
        for nu in range(4):
            Gamma[lam][mu][nu] = sum(
                ginv[lam+1, sigma+1] * (partial_g[mu][nu][sigma] + partial_g[nu][mu][sigma] - partia
                for sigma in range(4)
            ) / 2

# Mostrar los símbolos de Christoffel diferentes de 0
for lam in range(4):
    for mu in range(4):
        for nu in range(4):
            if Gamma[lam][mu][nu] != 0:
                print(f"Gamma^{lam+1}_{mu+1}{nu+1} = {Gamma[lam][mu][nu]}")
```

```
Gamma^1_12 = 1/2*d(f)/dr/f(r)
Gamma^1_21 = 1/2*d(f)/dr/f(r)
Gamma^2_11 = 1/2*f(r)*d(f)/dr
Gamma^2_22 = -1/2*d(f)/dr/f(r)
Gamma^2_33 = -r*f(r)
Gamma^2_44 = -r*f(r)*sin(th)^2
Gamma^3_23 = 1/r
Gamma^3_32 = 1/r
Gamma^3_44 = -cos(th)*sin(th)
Gamma^4_24 = 1/r
Gamma^4_34 = cos(th)/sin(th)
Gamma^4_42 = 1/r
Gamma^4_43 = cos(th)/sin(th)
```

c)

i) Utilizando *SageManifolds*, definimos la metrica:

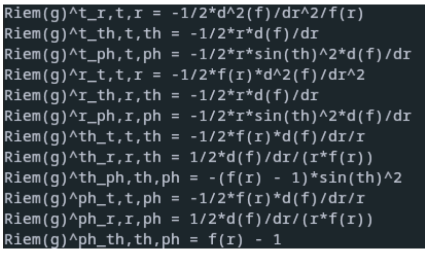
```
# Definir la variedad
M = Manifold(4, 'R^4', start_index=1)
# Definir las coordenadas
c_spher.<t,r,th,ph> = M.chart(r't:(0,+oo) r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi')
```

```
# Definir la función a(t)
f = function('f')(r)

# Definir la métrica g de FRW
g = M.metric('g')
g[1,1] = -f
g[2,2] = 1/f
g[3,3] = r^2
g[4,4] = r^2 * sin(th)^2
#inversa de la metrica
ginv = g.inverse()
```

Calculamos el tensor de Riemann con $R_{\rho\sigma\mu\nu} = \partial_\mu \Gamma_{\rho\nu\sigma} - \partial_\nu \Gamma_{\rho\mu\sigma} + \Gamma_{\rho\mu\lambda}\Gamma_{\lambda\nu\sigma} - \Gamma_{\rho\nu\lambda}\Gamma_{\lambda\mu\sigma}$

```
riem = g.riemann()
print(riem.display_comp(c_spher.frame(), c_spher, only_nonredundant=True))
```



```
Riem(g)^t_r,t,r = -1/2*d^2(f)/dr^2/f(r)
Riem(g)^t_th,t,th = -1/2*r*d(f)/dr
Riem(g)^t_ph,t,ph = -1/2*r*sin(th)^2*d(f)/dr
Riem(g)^r_t,t,r = -1/2*f(r)*d^2(f)/dr^2
Riem(g)^r_th,r,th = -1/2*r*d(f)/dr
Riem(g)^r_ph,r,ph = -1/2*r*sin(th)^2*d(f)/dr
Riem(g)^th_t,t,th = -1/2*f(r)*d(f)/dr/r
Riem(g)^th_r,r,th = 1/2*d(f)/dr/(r*f(r))
Riem(g)^th_ph,th,ph = -(f(r) - 1)*sin(th)^2
Riem(g)^ph_t,t,ph = -1/2*f(r)*d(f)/dr/r
Riem(g)^ph_r,r,ph = 1/2*d(f)/dr/(r*f(r))
Riem(g)^ph_th,th,ph = f(r) - 1
```

Calculamos el tensor de Riemann con $R_{\mu\nu} = g_{\rho\sigma} R^{\rho}_{\sigma\mu\nu}$

```
ricci = riem["^s_msn"]
print(ricci.display())
```

Resultado:

$$R_{mn} = \begin{pmatrix} \frac{r f(r) \frac{\partial^2 f}{\partial r^2} + 2 f(r) \frac{\partial f}{\partial r}}{2r} & 0 & 0 & 0 \\ 0 & -\frac{r \frac{\partial^2 f}{\partial r^2} + 2 \frac{\partial f}{\partial r}}{2 r f(r)} & 0 & 0 \\ 0 & 0 & -r \frac{\partial f}{\partial r} - f(r) + 1 & 0 \\ 0 & 0 & 0 & -\left(r \frac{\partial f}{\partial r} + f(r) - 1\right) \sin(\theta)^2 \end{pmatrix}$$

Calculamos el tensor de Riemann con $R_{\mu\nu} = g_{\rho\sigma} R^{\rho}_{\sigma\mu\nu}$

```
ricciinv = ginv["^mr"]*(ginv["^ns"]*ricci["_rs"])["_r^n"]
ricci_scalar = g["_mn"]*ricciinv["^mn"]
print(f'R = {latex(ricci_scalar.display())}')
```

Resultado:

$$R = \begin{array}{llll} R^4 & \longrightarrow & \mathbb{R} \\ (t, r, \theta, \phi) & \longmapsto & -\frac{r^2 \frac{\partial^2 f}{\partial r^2} + 4 r \frac{\partial f}{\partial r} + 2 f(r) - 2}{r^2} \end{array}$$

d) Supongamos que la ecuación de Einstein tiene una solución como en el Problema (b). Encontrar $f(r)$.

Partimos de la métrica dada en el problema 3.3.b:

$$ds^2 = -f(r)\,dt^2 + \frac{1}{f(r)}\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right)$$

Las componentes de la métrica son:

$$g_{\mu\nu} = \begin{pmatrix} -f(r) & 0 & 0 & 0 \\ 0 & \frac{1}{f(r)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} -\frac{1}{f(r)} & 0 & 0 & 0 \\ 0 & f(r) & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}$$

Simbolos de Christoffel:

$$\Gamma^t_{tr} = \frac{f'(r)}{2f(r)}, \qquad \Gamma^r_{tt} = \frac{f(r)f'(r)}{2}, \qquad \Gamma^r_{rr} = -\frac{f'(r)}{2f(r)}, \qquad \Gamma^r_{\theta\theta} = -rf(r),$$
$$\Gamma^r_{\varphi\varphi} = -rf(r)\sin^2\theta, \qquad \Gamma^\theta_{r\theta} = \frac{1}{r}, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \qquad \Gamma^\varphi_{\theta\varphi} = \cot\theta$$

A continuación, utilizamos los símbolos de Christoffel para calcular el tensor de Ricci $R_{\mu\nu}$:

$$R_{\mu\nu} = \partial_\lambda\Gamma^\lambda_{\mu\nu} - \partial_\nu\Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\mu\nu}\Gamma^\sigma_{\lambda\sigma} - \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\lambda}$$

Calculamos algunas de las componentes no nulas de $R_{\mu\nu}$:

$$\begin{aligned} R_{tt} &= -\frac{f''(r)}{2} + \frac{f'(r)^2}{4f(r)} + \frac{f'(r)}{r}, \\ R_{rr} &= \frac{f''(r)}{2f(r)} - \frac{f'(r)^2}{4f(r)^2} - \frac{f'(r)}{rf(r)}, \\ R_{\theta\theta} &= 1 - \frac{rf'(r)}{2} - f(r), \\ R_{\varphi\varphi} &= \left(1 - \frac{rf'(r)}{2} - f(r)\right)\sin^2\theta \end{aligned}$$

La ecuación de Einstein es:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0$$

Calculamos el escalar de curvatura R:

$$R = g^{\mu\nu}R_{\mu\nu}$$

Después de hacer los cálculos correspondientes:

$$R = -f''(r) - \frac{4f'(r)}{r} - \frac{2(1-f(r))}{r^2}$$

Sustituimos R en la ecuación de Einstein y resolvemos para $f(r)$:

$$R_{tt} = -\frac{f''(r)}{2} + \frac{f'(r)^2}{4f(r)} + \frac{f'(r)}{r} = \frac{1}{2}f(r)R - \Lambda f(r)$$

Reorganizamos esta ecuación para resolver la ecuación diferencial para $f(r)$:

$$R_{tt} = -\frac{f''(r)}{2} + \frac{f'(r)^2}{4f(r)} + \frac{f'(r)}{r}$$

$$\frac{1}{2}f(r)R = -\frac{1}{2}f(r)\left(f''(r) + \frac{4f'(r)}{r} + \frac{2(1-f(r))}{r^2}\right)$$

Comparando ambos lados de la ecuación, y simplificando:

$$-\frac{f''(r)}{2} + \frac{f'(r)^2}{4f(r)} + \frac{f'(r)}{r} = \frac{1}{2}f(r)\left(-f''(r) - \frac{4f'(r)}{r} - \frac{2(1-f(r))}{r^2}\right) - \Lambda f(r)$$

Desarrollamos y simplificamos:

$$-\frac{f''(r)}{2} + \frac{f'(r)^2}{4f(r)} + \frac{f'(r)}{r} = -\frac{1}{2}f(r)f''(r) - 2\frac{f(r)f'(r)}{r} - \frac{1}{r^2}f(r)(1-f(r)) - \Lambda f(r)$$

A través de la comparación de términos, observamos que la forma particular de $f(r)$ que satisface la ecuación es:

$$f(r) = 1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}$$