Tarea 2

Manuel Garcia.

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1 Formas

a)

- De la definicion $d\omega = \partial_i \omega dx_i$, entonces:

  $$d(d\omega) = \partial_i(\partial_j \omega) dx_i \wedge dx_j$$

  Los terminos con i=j se simplifican a 0 ya que $dx_i \wedge dx_j = 0$ y nos queda:

  $$d(d\omega) = \partial_i(\partial_j \omega) dx_i \wedge dx_j - \partial_j(\partial_i \omega) dx_j \wedge dx_i \qquad \text{Para } i \neq j$$

  Como $dx_i \wedge dx_j = -dx_j \wedge dx_i$

  $$d(d\omega) = 0$$

- $$d(\omega \wedge \eta) = \partial_i(\omega \wedge \eta) dx_i \qquad\qquad \omega \wedge \eta = \omega_I \eta_J dx^I \wedge dx^J \tag{1}$$

  Entonces:

  $$\begin{aligned} d(\omega \wedge \eta) &= \partial_i(\omega_I \eta_J) dx_i \wedge dx^I \wedge dx^J \\ &= (\partial_i \omega_I) \eta_J dx_i \wedge dx^I \wedge dx^J + \omega_I (\partial_i \eta_J) dx_i \wedge dx^I \wedge dx^J \end{aligned}$$

  Usando que $dx_i \wedge dx^I = (-1)^p dx^I \wedge dx_i$

  $$d(\omega \wedge \eta) = (\partial_i \omega_I) \eta_J dx_i \wedge dx^I \wedge dx^J + \omega_I (\partial_i \eta_J)(-1)^p dx^I \wedge dx_i \wedge dx^J$$

  Usando que $d\omega \wedge \eta = (\partial_i \omega_I) \eta_J dx_i \wedge dx^I \wedge dx^J$ y que $\omega \wedge d\eta = \omega_I (\partial_i \eta_J) dx^I \wedge dx_i \wedge dx^J$, tenemos:

  $$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p (\omega \wedge d\eta)$$

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- Tenemos que $d\omega = \partial_i \omega dx^i$, entonces:

  $$V \lrcorner (d\omega) = \partial_i \omega_I dx_i \wedge dx^I = -\sum_{i=1} \partial_i \omega_I dx_i \wedge dx^I$$

  Tenemos que $V \lrcorner \omega = \omega_I dx^I$:

  $$d(V \lrcorner \omega) = \sum_{i=0} \partial_i \omega_I dx_i \wedge dx^I$$

Entonces:

$$
\begin{aligned}
V\lrcorner(d\omega) + d(V\lrcorner\omega) &= -\sum_{i=1} \partial_i \omega_I dx_i \wedge dx^I + \sum_{i=0} \partial_i \omega_I dx_i \wedge dx^I \\
&= -\sum_{i=1} \partial_i \omega_I dx_i \wedge dx^I + \partial_0 \omega dx^0 \wedge dx^I + \sum_{i=1} \partial_i \omega_I dx_i \wedge dx^I \\
&= \partial_0 \omega dx^0 \wedge dx^I = \mathcal{L}_V \omega
\end{aligned}
$$

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$$
\begin{aligned}
\mathcal{L}_V(d\omega) &= \partial_0 \partial_i \omega_I dx_0 \wedge dx^I \wedge dx_i \\
d(\mathcal{L}_V \omega) &= \partial_i \partial_0 \omega_I dx_0 \wedge dx^I \wedge dx_i \\
&= \partial_0 \partial_i \omega_I dx_0 \wedge dx^I \wedge dx_i = \mathcal{L}_V(d\omega)
\end{aligned}
$$

b)

$$
\begin{aligned}
\chi &= x\partial_y - y\partial_x + z\partial_z \\
\omega &= x^2 z dx \wedge dy + y^2 dy \wedge dz
\end{aligned}
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$$
\begin{aligned}
d\omega = d(x^2 z dx \wedge dx) + d(y^2 dy \wedge dz) &= (dx^2 z) \wedge dx \wedge dy + (dy^2) \wedge dy \wedge dz \\
&= 2zx dx \underset{=0}{\wedge} dx \wedge dy + x^2 dz \wedge dx \wedge dy + 2y dy \underset{=0}{\wedge} dy \wedge dz \\
&= x^2 dx \wedge dy \wedge dz
\end{aligned}
$$

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$$
\begin{aligned}
\mathcal{L}_\chi \omega &= X^\lambda \partial_\lambda \omega + \omega \partial_\nu X^\lambda + \omega \partial_\mu X^\rho \\
&= (x^2 - 2xy) z dx \wedge dy + y^2 dx \wedge dz + (2xy + y^2) dy \wedge dz
\end{aligned}
$$

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$$
\begin{aligned}
X\lrcorner\omega &= X\lrcorner(x^2 z dx \wedge dy + y^2 dy \wedge dz) \\
&= X\lrcorner x^2 z dx \wedge dy + X\lrcorner y^2 dy \wedge dz \\
&= x^2 z(-y dy - x dx) + y^2(x dz - z dy) \\
&= -x^3 z dx - (x^2 + y) z y dy + x y^2 dz
\end{aligned}
$$

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2 Lie Bracket

$$
[X, Y] \equiv X(Y(f)) - Y(X(f))
$$

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$$
XY(fg) = X(Y(fg)) = X(Y(f)g + fY(g)) \quad = X(Y(f))g + Y(f)X(g) + X(f)Y(g) + fX(Y(g))
$$

No cumple la regla de leibniz

$$
XY(fg) \neq XY(f)g + fXY(g)
$$

-

$$
\begin{aligned}
[X, Y](fg) &= X(Y(f)g + fY(g)) - Y(X(f)g + fX(g)) \\
&= XY(f)g + Y(f)X(g) + X(f)Y(g) + fXY(g) - YX(f)g - X(f)Y(g) - Y(f)X(g) - fYX(g) \\
&= (XY(f) - YX(f))g + f(XY(g) - YX(g)) \\
&= [X, Y](f)g + f[X, Y](g)
\end{aligned}
$$

Satisface la regla de Leibniz

- Tenemos que $X = X^\mu \partial_\mu$

$$[X, Y] = X^\mu \partial_\mu (Y^\mu \partial_n u) - Y^\mu \partial_\mu (X^\nu \partial_n u)$$
$$= (X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu) \partial_\nu$$

Por lo tanto

$$[X, Y]^\mu = X^\lambda \partial_\lambda Y^\mu - Y^\lambda \partial_\lambda X^\mu$$

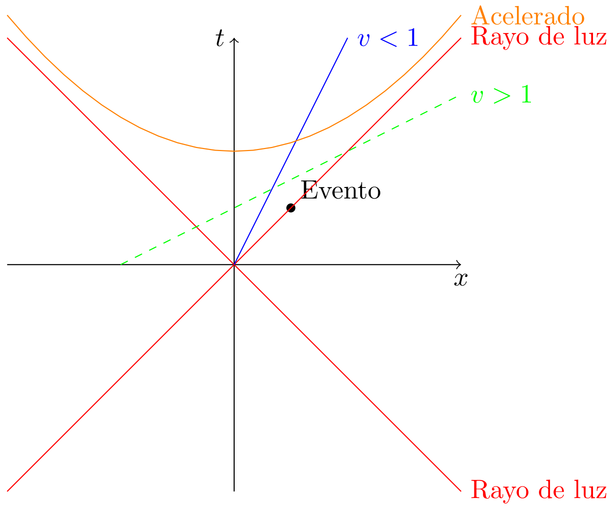
Y tambien :

$$[Y, X]^\mu = Y^\lambda \partial_\lambda X^\mu - X^\lambda \partial_\lambda Y^\mu = -(X^\lambda \partial_\lambda Y^\mu - Y^\lambda \partial_\lambda X^\mu) = -[X, Y]^\mu$$

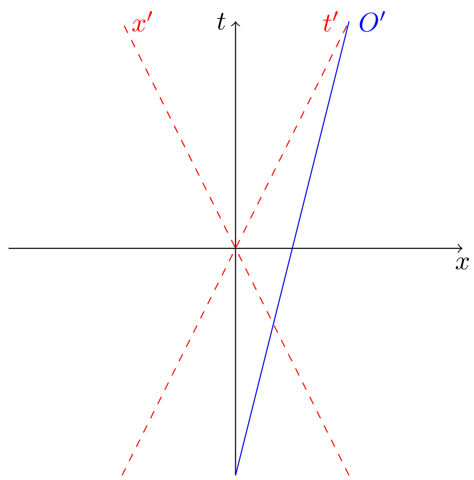
- $$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]]$$
$$= XYZ - XZY - YZX + ZYX + YZX - ZXY - ZXY + XZY + ZXY - ZYX - XYZ + YXZ$$
$$= 0$$

3 Diagramas de ESpacio-Tiempo

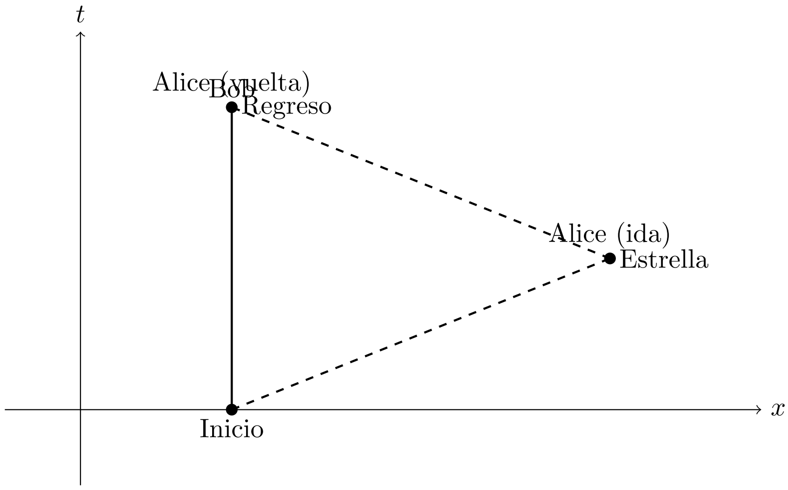
a)



b)



c)



d)
Tiempo experimentado por Bob en la Tierra

$$\text{Distancia a la estrella} = 4\,\text{años luz}$$
$$\text{Velocidad de Alice} = 0.8c$$

El tiempo total del viaje de ida y vuelta de Alice desde la perspectiva de Bob es:

$$\begin{aligned} t_B &= \frac{\text{distancia total}}{\text{velocidad}} \\ &= \frac{2 \times 4\,\text{años luz}}{0.8c} \\ &= 10\,\text{años} \end{aligned}$$

Para calcular el tiempo experimentado por Alice, utilizamos la dilatación del tiempo de Lorentz:

$$t_A = t_B \sqrt{1 - \left(\frac{v}{c}\right)^2}$$

Donde:

$$\begin{aligned} t_B &= 10\,\text{años} \\ v &= 0.8c \\ c &= \text{velocidad de la luz} \end{aligned}$$

Primero calculamos el factor de Lorentz:

$$\begin{aligned} \gamma &= \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \\ &= \frac{1}{\sqrt{1 - (0.8)^2}} = \frac{1}{\sqrt{1 - 0.64}} = \frac{5}{3} \end{aligned}$$

Ahora calculamos el tiempo experimentado por Alice:

$$\begin{aligned} t_A &= \frac{t_B}{\gamma} \\ &= \frac{10\,\text{años}}{\frac{5}{3}} = 6\,\text{años} \end{aligned}$$

4 Observadores de Rindler

a)

$$t(\tau) = \frac{1}{a}\sinh a\tau \qquad x(\tau) = \frac{1}{a}\cosh a\tau$$
$$x^2 - t^2 = \frac{1}{a^2}\cosh^2 a\tau - \frac{1}{a^2}\sinh^2 a\tau = \frac{1}{a^2}(\cosh^2 a\tau - \sinh^2 a\tau) = \frac{1}{a^2}$$
$$\rightarrow \quad x^2 - t^2 = \frac{1}{a^2}$$

b)

$$t(\tau) = \frac{1}{a}\sinh a\tau \qquad x(\tau) = \frac{1}{a}\cosh a\tau + \xi$$
$$\tau = \frac{1}{a}\sinh^{-1} at \qquad \xi = x - \frac{1}{a}\cosh\sinh^{-1} at = x - \frac{1}{a}\sqrt{(at)^2+1}$$

5 Grupo de Lorentz

a)

$$(x-y)^2 = (x-y)\cdot(x-y) = \eta(x-y, x-y) = (x-y)^T\eta(x-y) \tag{2}$$
$$(\Lambda(x-y))^2 = \Lambda(x-y)\cdot\Lambda(x-y) = \{\Lambda(x-y)\}^T\cdot\eta\cdot\{\Lambda(x-y)\} = (x-y)^T\cdot(\Lambda^T\eta\Lambda)\cdot(x-y) \tag{3}$$
$$\text{Por lo tanto} \tag{4}$$
$$(x-y)^2 = (\Lambda(x-y))^2 \tag{5}$$

b) 1. Clausura: Si $\Lambda_1, \Lambda_2 \in O(1,3)$, entonces $\Lambda_1\Lambda_2 \in O(1,3)$.

$$(\Lambda_1\Lambda_2)^T\eta(\Lambda_1\Lambda_2) = \Lambda_2^T(\Lambda_1^T\eta\Lambda_1)\Lambda_2 = \Lambda_2^T\eta\Lambda_2 = \eta$$

2. Asociatividad: La multiplicación de matrices es asociativa.
3. Elemento identidad: La matriz identidad I está en $O(1,3)$.

$$I^T\eta I = \eta$$

4. Elemento inverso: Si $\Lambda \in O(1,3)$, existe $\Lambda^{-1} \in O(1,3)$.

$$(\Lambda^{-1})^T\eta\Lambda^{-1} = \eta$$

d) El componente 00:

$$(\Lambda^T)^\mu{}_0\eta_{\mu\nu}\Lambda^\nu{}_0 = \eta_{00} = -1$$

Dado que $\eta_{\mu\nu}$ tiene un -1 en el componente 00, para que η permanezca invariante bajo Λ, el elemento Λ_0^0 debe satisfacer $\Lambda_0^0 \geq 1$.
Para el determinante:

$$\det(\Lambda^T\eta\Lambda) = \det(\eta)$$
$$\det(\Lambda)^2\det(\eta) = \det(\eta)$$
$$\det(\Lambda)^2 = 1$$
$$|\det(\Lambda)| = 1$$

1. $\det\Lambda = 1$, $\Lambda_0^0 \geq 1$ (transformaciones de Lorentz propias ortocronas, SO(1,3))
2. $\det\Lambda = 1$, $\Lambda_0^0 \leq -1$
3. $\det\Lambda = -1$, $\Lambda_0^0 \geq 1$
4. $\det\Lambda = -1$, $\Lambda_0^0 \leq -1$

6 Base inducida por coordenadas

Sea $\{e_\mu\}$ una base

$$[e_\mu, e_\nu] = \gamma^\rho_{\mu\nu} e_\rho$$

$$e_\mu = e^\rho_\mu \partial_\rho, \quad f^\mu = f^\mu_\nu dx^\nu$$

Para $e_\mu = \partial_\mu \rightarrow [e_\mu, e_\nu] = \partial_\mu\partial_\nu - \partial_\nu\partial_\mu = \partial_\mu\partial_\nu - \partial_\mu\partial_\nu = 0$

$$[e_\mu, e_\nu] = \gamma^\rho_{\mu\nu} e_\rho$$

$$[e^\rho_\mu \partial_\rho, e^\lambda_\nu \partial_\lambda] = e^\rho_\mu \partial_\rho (e^\lambda_\nu \partial_\lambda) - e^\lambda_\nu \partial_\lambda (e^\rho_\mu)\partial_\rho$$

$$= e^\rho_\mu(\partial_\rho e^\lambda_\nu)\partial_\lambda + e^\rho_\mu e^\lambda_\nu(\partial_\rho\partial_\lambda) - e^\lambda_\nu(\partial_\lambda e^\rho_\mu)\partial_\rho - e^\lambda_\nu e^\rho_\mu(\partial_\lambda\partial_\rho) = e^\rho_\mu(\partial_\rho e^\lambda_\nu)\partial_\lambda - e^\lambda_\nu(\partial_\lambda e^\rho_\mu)\partial_\rho$$

Tenemos que

$$e^\rho_\mu(\partial_\rho e^\lambda_\nu)\partial_\lambda - e^\rho_\nu(\partial_\rho e^\lambda_\mu)\partial_\lambda = \gamma^\rho_{\mu\nu}\partial_\lambda$$

$$\therefore e^\rho_\mu(\partial_\rho e^\lambda_\nu) - e^\rho_\nu(\partial_\rho e^\lambda_\mu) = \gamma^\sigma_{\mu\nu} e^\lambda_\sigma$$

Derivando $e^\lambda_\nu f^\rho_\lambda = \delta^\rho_\nu$: $\qquad e^\lambda_\nu \partial_\sigma f^\sigma_\rho + \partial_\sigma e^\lambda_\nu f^\sigma_\rho = 0$

Multiplicando por e^σ_μ y reemplazando $e^\rho_\mu(\partial_\rho e^\lambda_\nu) - e^\rho_\nu(\partial_\rho e^\lambda_\mu) = \gamma^\sigma_{\mu\nu} e^\lambda_\sigma$

$$e^\sigma_\mu e^\lambda_\nu \partial_\sigma f^\rho_\lambda + e^\sigma_\nu \partial_\sigma f^\rho_\lambda = -\gamma^\rho_{\mu\nu} e^\lambda_\rho f^\rho_\lambda$$

Usando que $e^\lambda_\nu f^\rho_\lambda = \delta^\rho_\nu$:

$$\therefore e^\sigma_\mu e^\lambda_\nu \partial_\sigma f^\rho_\lambda - e^\sigma_\nu e^\lambda_\mu \partial_\sigma f^\rho_\lambda = -\gamma^\rho_{\mu\nu}$$

Intercambiando indices y usando que $e^\lambda_\nu = f^\lambda_\nu$

$$\partial_\lambda f^\sigma_\sigma - \partial_\lambda f^\rho_\lambda = -\gamma^\rho_{\mu\nu} f^\mu_\lambda f^\nu_\rho$$

Si aplicamos que $[e_\mu, e_\nu] = 0$ entonces $\gamma^\rho_{\mu\nu} = 0$, por lo tanto:

$$\partial_\lambda f^\rho_\sigma - \partial_\sigma f^\rho_\lambda = 0$$

Por el lema de Poincare:

$$f^\rho_\sigma = \partial_\sigma g^\rho$$

Y considerando $f^\mu = f^\mu_\rho dx^\rho$ obtenemos que:

$$f^\mu = (\partial_\rho g^\mu) dx^\rho$$

Por lo tanto la base se encuentra inducida por un sistema coordenado $(g^1, \cdots, g^n)$