Clase 19

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1 Transporte paralelo y geodésicas

Ecuacion de las geodésicas

$$\nabla_V V = 0 \quad \rightarrow \quad \frac{d^2 x^{\mu}}{dt^2} + \Gamma^{\mu}_{\nu\lambda} \frac{dx^{\nu}}{dt} \frac{dx^{\lambda}}{dt} = 0$$

Podemos relajar la condición $\nabla_V V = 0$ por la condicion $\nabla_V V = fV$, donde $f \in \mathbb{F}(M)$

$$\nabla_V V = 0 \quad \rightarrow \quad \frac{d^2 x^{\mu}}{dt^2} + \Gamma^{\mu}_{\nu\lambda} \frac{dx^{\nu}}{dt} \frac{dx^{\lambda}}{dt} = f \frac{dx^{\mu}}{dt}$$

2 Derivada covariante de tensores

$$\boldsymbol{\nabla}_{\alpha}T^{\mu_{1}\cdots\mu_{p}}_{\nu_{1}\cdots\nu_{q}}\equiv T^{\mu_{1}\cdots\mu_{p}}_{\nu_{1}\cdots\nu_{q};\alpha}=\frac{\partial T^{\mu_{1}\cdots\mu_{p}}_{\nu_{1}\cdots\nu_{q}}}{x^{\alpha}}+\sum_{i=1}^{p}\Gamma^{\mu_{i}}_{\alpha\beta}T^{\mu_{1}\cdots\beta\cdots\mu_{p}}_{\nu_{1}\cdots\nu_{q}}-\sum_{i=1}^{q}\Gamma^{\beta}_{\alpha\nu_{i}}T^{\mu_{1}\cdots\mu_{p}}_{\nu_{1}\cdots\beta\cdots\nu_{q}}$$

Conexion y derivada convariante

$$\nabla_{e_{\mu}} e_{v} \equiv \Gamma^{\lambda}_{\mu\nu} e_{\lambda} \quad \rightarrow \quad \nabla_{f_{\mu}} f_{v} = \bar{\Gamma}^{\lambda}_{\mu\nu} f_{\lambda}$$
$$f_{\mu} = \frac{\partial x^{\beta}}{\partial y^{\mu}} e_{\beta}$$

Las bases transforman como vectores ante el cambio y(x).

En la parte izquierda:

$$\nabla_{\left(\frac{\partial x^{\alpha}}{\partial y^{\mu}}e_{\alpha}\right)}\left(\frac{\partial x^{\beta}}{\partial y^{\mu}}\right) = \frac{\partial x^{\alpha}}{\partial y^{\mu}}\nabla_{e_{\alpha}}\left[\left(\frac{\partial x^{\beta}}{\partial y^{\nu}}\right)e_{\beta}\right]
= \frac{\partial x^{\alpha}}{\partial y^{\mu}}\left[e_{\alpha}\left(\frac{\partial x^{\beta}}{\partial y^{\nu}}\right)e_{\beta} + \frac{\partial x^{\beta}}{\partial y^{\nu}}\nabla_{e_{\alpha}}e_{\beta}\right]
= \frac{\partial x^{\alpha}}{\partial y^{\mu}}\left(\frac{\partial^{2}x^{\beta}}{\partial x^{\alpha}\partial y^{\nu}} + \frac{\partial x^{\sigma}}{\partial y^{\nu}}\Gamma^{\beta}_{\alpha\sigma}\right)e_{\beta}
= \left(\frac{\partial^{2}x^{\beta}}{\partial y^{\mu}\partial y^{\nu}} + \frac{\partial x^{\alpha}}{\partial y^{\mu}}\frac{\partial x^{\sigma}}{\partial y^{\nu}}\Gamma^{\beta}_{\alpha\sigma}\right)e_{\beta}$$

Para la parte derecha:

$$\bar{\Gamma}^{\lambda}_{\mu\nu} \frac{\partial x^{\beta}}{\partial u^{\lambda}} e_{\beta}$$

Regla de tranformacion de Γ

$$\bar{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\beta}_{\alpha\sigma} \frac{\partial y^{\lambda}}{\partial x^{\beta}} \frac{\partial x^{\alpha}}{\partial y^{\mu}} \frac{\partial x^{\sigma}}{\partial y^{\nu}} + \frac{\partial y^{\lambda}}{\partial x^{\beta}} \frac{\partial^{2} x^{\beta}}{\partial y^{\mu} \partial y^{\nu}}$$

Tensor tipo (1, 0). Por definición de tensor y por su expresión en coordenadas euclidianas x:
$$T'^{\sigma} = T^{\mu} \frac{\partial z^{\sigma}}{\partial x^{\mu}}, \qquad T^{\mu}_{;\alpha} = \frac{\partial T^{\mu}}{\partial x^{\alpha}}.$$

$$T'^{\sigma}_{;\beta} = T^{\mu}_{;\alpha} \frac{\partial z^{\sigma}}{\partial x^{\mu}} \frac{\partial x^{\alpha}}{\partial z^{\beta}} = \frac{\partial T^{\mu}}{\partial x^{\alpha}} \frac{\partial z^{\sigma}}{\partial x^{\mu}} \frac{\partial z^{\sigma}}{\partial z^{\beta}} = \frac{\partial T^{\mu}}{\partial z^{\beta}} \frac{\partial z^{\sigma}}{\partial x^{\mu}}. \qquad T'^{\sigma}_{;\beta}$$

$$T'^{\sigma} = T^{\mu} \frac{\partial z^{\sigma}}{\partial x^{\mu}}, \qquad \rightarrow \qquad \frac{\partial T'^{\sigma}}{\partial z^{\beta}} = \frac{\partial}{\partial z^{\beta}} \left(T^{\mu} \frac{\partial z^{\sigma}}{\partial x^{\mu}}\right) = \frac{\partial T^{\mu}}{\partial z^{\beta}} \frac{\partial z^{\sigma}}{\partial x^{\mu}} + T^{\mu} \frac{\partial^{2} z^{\sigma}}{\partial z^{\beta} \partial x^{\mu}}$$

$$\Rightarrow \qquad \frac{\partial T^{\mu}}{\partial z^{\beta}} \frac{\partial z^{\sigma}}{\partial x^{\mu}} = \frac{\partial T'^{\sigma}}{\partial z^{\beta}} - T^{\mu} \frac{\partial^{2} z^{\sigma}}{\partial z^{\beta} \partial x^{\mu}} = \frac{\partial T'^{\sigma}}{\partial z^{\beta}} - T'^{\rho} \frac{\partial x^{\mu}}{\partial z^{\beta}} \frac{\partial^{2} z^{\sigma}}{\partial z^{\beta}} + T'^{\rho} \frac{\partial x^{\mu}}{\partial z^{\beta}} \frac{\partial^{2} z^{\sigma}}{\partial z^{\beta}} + T'^{\rho} \frac{\partial x^{\mu}}{\partial z^{\beta}} \frac{\partial^{2} z^{\sigma}}{\partial z^{\beta}} \frac{\partial^{2} z^{\sigma}}{\partial z^{\beta}} + T'^{\rho} \frac{\partial x^{\mu}}{\partial z^{\beta}} \frac{\partial^{2} z^{\sigma}}{\partial z^{\beta}} \frac{\partial^{2} z^{\sigma}}{\partial z^{\beta}} + T'^{\rho} \frac{\partial x^{\mu}}{\partial z^{\beta}} \frac{\partial^{2} z^{\sigma}}{\partial z^{\beta}} \frac{\partial^{2} z^{\sigma}}{\partial z^{\beta}} + T'^{\rho} \frac{\partial x^{\mu}}{\partial z^{\beta}} \frac{\partial x^{\mu}}{\partial z^{\beta}} \frac{\partial x^{\mu}}{\partial z^{\beta}} \frac{\partial^{2} z^{\sigma}}{\partial z^{\beta}} \frac{\partial^{2} z^$$

3 Curvatura y torsión

En \mathbb{R}^n en coordenadas cartesianas:

$$(\nabla_{\alpha}\nabla_{\beta} - \nabla_{\beta}\nabla_{\alpha})T_{(\nu)}^{(\mu)} = 0$$

En una variedad general con una conexión arbitraria en general tendremos:

$$(\nabla_{\alpha}\nabla_{\beta} - \nabla_{\beta}\nabla_{\alpha})T_{(\nu)}^{(\mu)} \neq 0$$

Tensor de torsión

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

Tensor de ...