Geodesica Alrededor de una Particula con Masa en el Espacio de Minkowski

Manuel Angel Garcia[1]   Carlos Andres Llanos[1]

[1]Facultad de Fisica
Universidad Nacional de Colombia

Clase de Metodos Geometricos

Metrica de Schwarzschild

Partiendo de la metrica de Minkowski en esfericas:

$$ds^2_{\text{Minkowski}} = -dt^2 + dr^2 + r^2 d\Omega^2$$

Necesitamos una metrical que mantenga la forma del angulo solido, así que podemos multiplicarlo por una funcion radial:

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + e^{2\gamma(r)} r^2 d\Omega^2$$

Aplicamos la siguiente transformacion:

$$\bar{r} = e^{\gamma(r)} r \qquad d\bar{r} = \left(1 + r\frac{d\gamma}{dr}\right) e^{\gamma} dr$$

La metrica nos queda:

$$ds^2 = -e^{2\alpha(r)} dt^2 + \left(1 + r\frac{d\gamma}{dr}\right)^{-2} e^{2\beta(r) - 2\gamma(r)} d\bar{r}^2 + \bar{r}^2 d\Omega^2$$

Metrica de Schwarzschild

$$ds^2 = -e^{2\alpha(r)}dt^2 + \left(1 + r\frac{d\gamma}{dr}\right)^{-2} e^{2\beta(r)-2\gamma(r)}d\bar{r}^2 + \bar{r}^2 d\Omega^2$$

Haciendo $\bar{r} \to r$ obtenemos que: $\left(1 + r\frac{d\gamma}{dr}\right)^{-2} e^{2\beta(r)-2\gamma(r)} \to e^{2\beta}$

Por lo tanto podemos escribir la metrica como:

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2$$

Metrica de Schwarzschild

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2$$

Calculamos los simbolos de Christoffel:

$$\Gamma^t_{tr} = \partial_r \alpha \qquad \Gamma^r_{tt} = e^{2(\alpha-\beta)}\partial_r \alpha \qquad \Gamma^r_{rr} = \partial_r \beta$$

$$\Gamma^\theta_{r\theta} = \frac{1}{r} \qquad \Gamma^r_{\theta\theta} = -re^{-2\beta} \qquad \Gamma^\phi_{r\phi} = \frac{1}{r}$$

$$\Gamma^r_{\phi\phi} = -re^{-2\beta}\sin^2\theta \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \qquad \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta}$$

Metrica de Schwarzschild

Con los simbolos de Christoffel podemos obtener el tensor de Riemann:

$$
\begin{aligned}
R^t_{rtr} &= \partial_r\alpha\partial_r\beta - \partial_r^2\alpha - (\partial_r\alpha)^2 \\
R^t_{\theta t\theta} &= -re^{-2\beta}\partial_r\alpha \\
R^t_{\phi t\phi} &= -re^{-2\beta}\sin^2\theta\partial_r\alpha \\
R^r_{\theta r\theta} &= re^{-2\beta}\partial_r\beta \\
R^r_{\phi r\phi} &= re^{-2\beta}\sin^2\theta\partial_r\beta \\
R^\theta_{\phi\theta\phi} &= \left(1 - e^{-2\beta}\right)\sin^2\theta
\end{aligned}
$$

Metrica de Schwarzschild

Tomando la contraccion obtenemos el tensor de Ricci:

$$
\begin{aligned}
R_{tt} &= e^{2(\alpha-\theta)}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha\right] \\
R_{rr} &= -\partial_r^2\alpha - (\partial_r\alpha)^2 + \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\beta \\
R_{\theta\theta} &= e^{-2\beta}\left[r\left(\partial_r\beta - \partial_r\alpha\right) - 1\right] + 1 \\
R_{\phi\phi} &= \sin^2\theta \quad R_{\theta\phi}
\end{aligned}
$$

Metrica de Schwarzschild

Reemplazamos el tensor de Ricci en la ecuacion de campo de Einstein en el vacio para hallar α y β:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$$

Obteniendo:

$$\begin{aligned} e^{2(\beta-\alpha)} e^{2(\alpha-\beta)} & \left[\partial_r^2 \alpha + (\partial_r \alpha)^2 - \partial_r \alpha \partial_r \beta + \frac{2}{r} \partial_r \alpha \right] \\ & - \partial_r^2 \alpha - (\partial_r \alpha)^2 + \partial_r \alpha \partial_r \beta + \frac{2}{r} \partial_r \beta = 0 \\ \implies \frac{2}{r} \partial_r \alpha + \frac{2}{r} \partial_r \beta = 0 & \end{aligned}$$

Para que se cumpla esta ecuacion necesitamos que $\alpha = -\beta + c$

Metrica de Schwarzschild

$$\alpha = -\beta$$

Reemplazando en $R_{\theta\theta} = 0$:

$$e^{2\alpha}(2r\partial_r\alpha + 1) = 1 \Longrightarrow \partial_r(re^{2\alpha}) = 1$$

Resolviendo esta ecuacion obtenemos:

$$e^{2\alpha} = 1 - \frac{R_s}{r}$$

De esta forma obtenemos que podemos escribir la metrica como:

$$ds^2 = -\left(1 - \frac{R_s}{r}\right) dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Metrica de Schwarzschild

EXPLICAR POR QUÉ EL RADIO DE Schwarzschild ES $R_S = 2GM$

Geodesicas de Schwarzschild

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Calculamos los simbolos de Christoffel

```
m = var('m')
M = Manifold(4, 'R^4', start_index=1)

c_spher.<t,r,th,ph> = M.chart(r't:(0,+oo) r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi')

g = M.metric('g')

g[1,1], g[2,2], g[3,3], g[4,4] = (-1)*(1-2*m/r), (1-2*m/r)^(-1), r^2, r^2*sin(th)^2

print(g.christoffel_symbols_display(chart=c_spher))


BL.<t,r,th,ph> = M.chart(r"t r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi")
M.default_frame() is BL.frame()
xi = BL.frame()[0]

xi_form = xi.down(g)
print(xi_form.display())
```

Geodesicas de Schwarzschild

Simbolo de Christoffel para la metrica:

$$
\begin{aligned}
\Gamma^{t}{}_{t\,r} &= -\frac{M}{2\,Mr - r^2} \\
\Gamma^{r}{}_{t\,t} &= -\frac{2\,M^2 - Mr}{r^3} \\
\Gamma^{r}{}_{r\,r} &= \frac{M}{2\,Mr - r^2} \\
\Gamma^{r}{}_{\theta\,\theta} &= 2\,M - r \\
\Gamma^{r}{}_{\phi\,\phi} &= (2\,M - r)\sin(\theta)^2 \\
\Gamma^{\theta}{}_{r\,\theta} &= \frac{1}{r} \\
\Gamma^{\theta}{}_{\phi\,\phi} &= -\cos(\theta)\sin(\theta) \\
\Gamma^{\phi}{}_{r\,\phi} &= \frac{1}{r} \\
\Gamma^{\phi}{}_{\theta\,\phi} &= \frac{\cos(\theta)}{\sin(\theta)}
\end{aligned}
$$

Geodesicas de Schwarzschild

Reemplazando los simbolos de Christoffel en la ecuacion de la geodesica:

$$\ddot{t} + \frac{2M}{r(r-2M)}\dot{r}\dot{t} = 0$$

$$\ddot{r} + \frac{M}{r^3}(r-2M)\dot{t}^2 - \frac{M\dot{r}}{r(r-2M)} - (r-2M)\dot{\phi}^2 = 0$$

$$\ddot{\phi} + \frac{2}{r}\dot{\phi}\dot{r} = 0$$