1 ejercicio 5.13

$$\begin{aligned} x &= r\cos\theta & r &= \sqrt{x^2+y^2} \\ y &= r\sin\theta & \theta &= \arctan\left(\frac{y}{x}\right) \end{aligned}$$

$$\partial_x r = \frac{x}{r} \qquad \partial_y r = \frac{y}{r}$$

$$\begin{aligned} \partial_r x &= c_\theta & \partial_\theta x &= -rs_\theta & \partial_x\theta &= \frac{\frac{-y}{x^2}}{1+\frac{y^2}{x^2}} = \frac{-y}{x^2+y^2} \\ \partial_r y &= s_\theta & \partial_\theta y &= rc_\theta & \partial_y\theta &= \frac{x}{x^2+y^2} \end{aligned}$$

$$dx\wedge dy = (c_\theta dr + -rs_\theta d\theta)\wedge(s_\theta dr + rc_\theta d\theta) = rc_\theta^2 dr\wedge d\theta + rs_\theta^2 dr\wedge d\theta = rdr\wedge d\theta$$

2 Ejercicio 5.9

Dado $X = X^\mu\frac{\partial}{\partial x^\mu}$ y $Y = Y^\mu\frac{\partial}{\partial x^\mu}$

Los brackets de lie: $[X,Y]f = X[Y[f]] - Y[X[f]]$.

$$[X,Y]f = \left[X^\nu\frac{\partial Y^\mu}{\partial x^\nu} - Y^\nu\frac{\partial X^\mu}{\partial x^\nu}\right]\frac{\partial f}{\partial x^\mu}$$

$$[X,Y] = \left[X^\mu\frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu\frac{\partial X^\nu}{\partial x^\mu}\right]\frac{\partial}{\partial x^\nu}$$

Tiene la forma de los backets de lie.

3 Ejercicio 5.10

(a) bilinearidad

Necesitamos: $\begin{aligned} &[X, c_1Y_1 + c_2Y_2] = c_1[X,Y_1] + c_2[X,Y_2] \\ &[c_1X_1 + c_2X_2, Y] = c_1[X_1,Y] + c_2[X_2,Y] \end{aligned}$

$$\begin{aligned} [X, c_1Y_1+c_2Y_2] &= X^\mu\frac{\partial}{\partial x^\mu}(c_1Y_1+c_2Y_2)^\nu - (c_1Y_1+c_2Y_2)^\mu\frac{\partial X^\nu}{\partial x^\mu} = \\ &= c_1\left(X^\mu\frac{\partial Y_1^\nu}{\partial x^\mu} - Y_1^\mu\frac{\partial X^\nu}{\partial x^\mu}\right) + c_2\left(X^\mu\frac{\partial Y_2^\nu}{\partial x^\mu} - Y_2^\mu\frac{\partial X^\nu}{\partial x^\mu}\right) = c_1[X,Y_1] + c_2[X,Y_2] \\ [c_1X_1+c_2X_2, Y] &= (c_1X_1+c_2X_2)^\mu\frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu\frac{\partial}{\partial x^\mu}(c_1X_1+c_2X_2)^\nu = \\ &= c_1\left(X_1^\mu\frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu\frac{\partial X_1^\nu}{\partial x^\mu}\right) + c_2\left(X_2^\mu\frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu\frac{\partial X_2^\nu}{\partial x^\mu}\right) = c_1[X_1,Y] + c_2[X_2,Y] \end{aligned}$$

(b)

$$[Y,X] = Y^\mu\frac{\partial X^\nu}{\partial x^\mu} - X^\mu\frac{\partial Y^\nu}{\partial x^\mu} = -\left(X^\mu\frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu\frac{\partial X^\nu}{\partial x^\mu}\right) = -[X,Y]$$

(c) Necesitamos probar que:

$$[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] = 0$$

Entonces:

$$[[X,Y],Z]$$

$$\begin{aligned} [X,Y]^\mu\frac{\partial Z^\nu}{\partial x^\mu} - Z^\mu\frac{\partial}{\partial x^\mu}[X,Y]^\nu &= (X^a\partial_aY^\mu - Y^a\partial_aX^\mu)\partial_\mu Z^\nu - Z^\mu(\partial_\mu X^a\partial_aY^\nu + X^a\partial^2_{\mu a}Y^\nu - \partial_\mu Y^a\partial_aX^\nu - Y^a\partial^2_{\mu a}X \\ &= X^a\partial_aY^\mu\partial_\mu Z^\nu - Y^a\partial_aX^\mu\partial_\mu Z^\nu - Z^\mu\partial_\mu X^a\partial_aY^\nu - Z^\mu X^a\partial^2_{\mu a}Y^\nu + Z^\mu\partial_\mu Y^a\partial_aX^\nu + Z^\mu Y^a\partial^2_{\mu a}X^\nu \end{aligned}$$

De forma analoga:

$$[[Z,X],Y]^\nu = Z^a\partial_a X^\mu \partial_\mu Y^\nu - X^a\partial_a Z^\mu\partial_\mu Y^\nu - Y^\mu\partial_\mu Z^a\partial_a X^\nu - Y^\mu Z^a\partial^2_{\mu a}X^\nu + Y^\mu\partial_\mu X^a\partial_a Z^\nu + Y^\mu X^a\partial^2_{\mu a}Z^\nu$$
$$[[Y,Z],X]^\nu = Y^a\partial_a Z^\mu \partial_\mu X^\nu - Z^a\partial_a Y^\mu\partial_\mu X^\nu - X^\mu\partial_\mu Y^a\partial_a Z^\nu - X^\mu Y^a\partial^2_{\mu a}Z^\nu + X^\mu\partial_\mu Z^a\partial_a Y^\nu + X^\mu Z^a\partial^2_{\mu a}Y^\nu$$

Todos los terminos se cancelan.