Parcial #2

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$$U^\mu \partial_\mu \langle V, W\rangle = \langle U^\mu \boldsymbol{\nabla}_\mu V, W\rangle + \langle V, U^\mu \boldsymbol{\nabla}_\mu W\rangle \tag{1}$$

$$U^\mu \partial_\mu (g_{\lambda\beta} V^\alpha W^\beta) = U^\mu (\partial_\mu g_{\lambda\beta}) V^\lambda W^\beta + U^\mu g_{\lambda\beta} (\partial_\mu V^x) W^\beta + U^\mu g_{\lambda\beta} (\partial_\mu W^\beta) V^\lambda \tag{2}$$

$$\text{Tenemos que:} \tag{3}$$

$$\boldsymbol{\nabla}_\mu g_{\lambda\beta} - \Gamma^\rho_{\mu\lambda} g_{\rho\beta} - \Gamma_{\mu\beta} g_{\lambda\rho} \qquad \boldsymbol{\nabla}_\mu T^\lambda = \partial_\mu T^\lambda + \Gamma^\lambda_{\rho\mu} T^\rho \tag{4}$$

$$\text{Entonces:} \tag{5}$$

$$= U^\mu (\boldsymbol{\nabla}_\mu g_{\lambda\beta} + \Gamma^\rho_{\mu\lambda} g_{\rho\beta} + \Gamma_{\mu\beta} g_{\lambda\rho}) V^\lambda W^\beta + U^\mu g_{\lambda\beta} (\boldsymbol{\nabla}_\mu V^\lambda - \Gamma_{\rho\mu} V^\rho) W^\beta + U^\mu g_{\lambda\beta} (\boldsymbol{\nabla}_\mu W^\beta - \Gamma^\beta_{\rho\mu} W^\rho) V^\lambda \tag{6}$$

$$= U^\mu (g_{\lambda\beta} (\boldsymbol{\nabla}_\mu V)^\lambda W^\beta + g_{\lambda\beta} V^\lambda (\boldsymbol{\nabla}_\mu W^\beta)) \tag{7}$$

$$= \langle U^\mu \boldsymbol{\nabla}_\mu V, W\rangle + \langle V, U^\mu \boldsymbol{\nabla}_\mu W\rangle \tag{8}$$

2

$$\omega = 4xydx + (x^2 - 5)dy \qquad \beta = e^{2xy} dx - dy \tag{9}$$

$$x = 2\frac{d}{dx} + y\frac{d}{dy} \tag{10}$$

- $$\begin{aligned} d\omega &= d(4xydx) + d((x^2-5)dy) \\ &= (4ydx + 4xdy) \wedge dx + (2xdx) \wedge dy \\ &= (-4x + 2x)(dx \wedge dy) \end{aligned}$$

- $$\begin{aligned} d\beta &= d(e^{2y} dx) + d(-dy) \\ &= (2xe^{2xy} dy + 2ye^{2xy} dx) \wedge dx \\ &= 2xe^{2xy} dy \wedge dx \end{aligned}$$

- $$\begin{aligned} \omega \wedge \beta &= (4xydx + (x^2-5)dy) \wedge (e^{2xy}dx - dy) \\ &= 4xydx \wedge (e^{2xy}dx - dy) + (x^2-5)dy \wedge (e^{2xy}dx - dy) \\ &= -4xydx \wedge dy + (x^2-5)e^{2xy} dy \wedge dx \\ &= -(4xy + (x^2-5)e^{2xy}) dx \wedge dy \end{aligned}$$

- $$\begin{aligned} i_x d\beta &= i_x(2xe^{2xy}dy \wedge dx) = 2xe^{2xy}(dy(y\frac{\partial}{\partial y}) \otimes dx - dx(2\frac{\partial}{\partial x})) \otimes dy \\ &= 2xye^{2xy}dy - 4xe^{2xy}dy \end{aligned}$$

- $$\begin{aligned} d\beta \wedge \omega &= (2xe^{2xy}dy \wedge dx) \wedge (4xydx + (x^2-5)dy) \\ &= 2xe^{2xy}(x^2-5)dy \wedge dx \wedge dy = 0 \end{aligned}$$

- $$d\omega \wedge \beta = (-2xdx \wedge dy) \wedge (e^{2xy}dx - dy) = 0$$

3

$$\begin{aligned} \Delta\Gamma^\alpha_{\beta\gamma} &= \Gamma^\alpha_{\beta\gamma} - \bar{\Gamma}^\alpha_{\beta\gamma} \quad \in \mathcal{T}(M) \\ (\Delta\Gamma^\alpha_{\beta\gamma})' &= (\Gamma^\alpha_{\beta\gamma})' - (\bar{\Gamma}^\alpha_{\beta\gamma})' \\ &= \left(\frac{y^\alpha}{\partial x^\beta}\frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial y^\sigma}{\partial y^\mu}\Gamma^\beta_{\alpha\sigma} - \frac{y^\alpha}{\partial x^\beta}\frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial y^\sigma}{\partial y^\mu}\bar{\Gamma}^\beta_{\alpha\sigma} \right) \\ &= \frac{y^\alpha}{\partial x^\beta}\frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial y^\sigma}{\partial y^\mu}(\Gamma^\beta_{\alpha\sigma} - \bar{\Gamma}^\beta_{\alpha\sigma}) \\ &= \frac{y^\alpha}{\partial x^\beta}\frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial y^\sigma}{\partial y^\mu}\Delta\Gamma^\beta_{\alpha\sigma} \end{aligned}$$

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$$\begin{aligned} R_{\mu\nu\alpha\beta} &= R_{\mu\alpha}g_{\nu\beta} - R_{\mu\beta}g_{\nu\alpha} + R_{\nu\beta}g_{\mu\alpha} - R_{\nu\alpha}g_{\mu\beta} + \frac{R}{2}(g_{\mu\beta}g_{\nu\alpha} - g_{\mu\alpha}g_{\nu\beta}) \\ g^{\nu\beta}R_{\mu\nu\alpha\beta} = R_{\mu\alpha} &= R_{\mu\alpha}g_{\nu\beta}g^{\nu\beta} - R_{\mu\beta}g_{\nu\alpha}g^{\nu\beta} + R_{\nu\beta}g^{\nu\beta}g_{\mu\alpha} - R_{\nu\alpha}g^{\nu\beta}g_{\mu\alpha} + \frac{R}{2}(g_{\mu\beta}g_{\nu\alpha}g^{\nu\beta} - g_{\mu\alpha}g_{\nu\beta}g^{\nu\beta}) \\ &= R_{\mu\alpha}g^\nu_\nu - R_{\mu\beta}g^\beta_\alpha + Rg_{\mu\alpha} - R_{\nu\alpha}g^\nu_\mu + \frac{R}{2}(g_{\mu\beta}g^\beta_\alpha - g_{\mu\alpha}g^\nu_\nu) = R_{\mu\alpha} \end{aligned}$$