Clase 3

Manuel Garcia.

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1 Repaso de la clase pasada

Representacion polar

$$z = re^{i\mathrm{Arg}(z)} \equiv re^{i\theta} \tag{1}$$
$$\mathrm{Arg}(z) = arg(z) + 2\pi k, \qquad k = 0, \pm 1, .., \pm i, ... \tag{2}$$
$$i^n = i^{mod_4 n} \tag{3}$$
$$i^{25} = i^{24} * i^1 = i \tag{4}$$

2 Teorema de Moirre

$z = re^{i\theta}, \quad n \in \mathbb{Z} \qquad z^n = (re^{i\theta})^n = r^n e^{in\theta}$

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n[\cos n\theta + i\sin n\theta]] \tag{5}$$
$$[\cos\theta + i\sin\theta]^n = \cos n\theta + i\sin n\theta \tag{6}$$
$$[\cos\theta + i\sin\theta]^5 = \cos 5\theta + i\sin 5\theta = e^{i5\theta} \tag{7}$$

En el cuadrante de arriba el angulo de puede calcular como: $\pi + \tan^{-1}\frac{y}{x}$ o $\pi - |\tan^{-1}\frac{y}{x}|$

Ejercicio

$$[-1 + i\sqrt{3}]^{60} \tag{8}$$
$$z = 2e^{\frac{2\pi i}{3}} = 2 * [\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}] \tag{9}$$
$$z^{60} = 2^{60}[\cos\frac{120\pi}{3} + i\sin\frac{120\pi}{3}] = 2^{60} \tag{10}$$

Identidad

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \tag{11}$$
$$\sin 2\theta = 2\cos\theta\sin\theta \tag{12}$$

Demostrar que el polinomio $f(x))[\cos\alpha + x\sin\alpha]^n - \cos n\alpha - x\sin n\alpha$ proporcional a (x^2+1) es divisible por $x^2+1 = (x+i)(x-i)$

$$f(x)\text{proporcional } (x-a) \qquad \text{si} \quad f(a) = 0 \tag{13}$$
$$\text{Debemos demostrar que } f(-i) = 0 \quad f(i) = 0 \tag{14}$$
$$f(-i) = [\cos\alpha - i\sin\alpha]^n - \cos n\alpha + i\sin n\alpha \tag{15}$$
$$= \cos n\alpha - i\sin n\alpha - \cos n\alpha + i\sin n\alpha \text{ Q.E.D} \tag{16}$$

Tarea: terminar demostracion

Ejercicios

- Pasar a polar $(\frac{1+i\sqrt{3}}{1-i})^{40}$
- Demostrar que $f(x) = x^n\sin\alpha - \lambda^{n-1}x\sin n\alpha + \lambda^n\sin(n-1)\alpha$ es divisible por $x^2 - 2\lambda x + \lambda^2$
- identidad $\sin 3\phi$

3 Raices de un polinomio complejo

$$2 = 2e^{2\pi ki} \qquad \text{k entero} \tag{17}$$
$$\sqrt{2} = \sqrt{2}e^{2\pi ki/2} \tag{18}$$
$$= \sqrt{2}e^{\pi ki} \tag{19}$$
$$= \sqrt{2} - \sqrt{2}(\sqrt{2} - \sqrt{2} - \underbrace{\sqrt{2}}_{Repetidas}) \tag{20}$$

Raiz polinomio complejo

$$\sqrt[n]{z} = z^{1/n} \tag{21}$$
$$z^{1/n} = r^{1/n}[e^{i\alpha+2\pi k}]^{1/n} \tag{22}$$
$$= r^{1/n}[e^{i\alpha/n+2\pi k/n}] \tag{23}$$

n raices, n= 0, ...n=n-1.

$$k = 0 \qquad \sqrt[n]{z} = \sqrt[n]{r}e^{i\alpha/n} \tag{24}$$
$$...k = n-1 \qquad = \sqrt[n]{r}e^{i\alpha/n + \frac{2\pi(n-1)}{n}i} \tag{25}$$
$$k = n \qquad = \sqrt[n]{r}e^{i\alpha/n + \frac{2\pi(n)}{n}i} \tag{26}$$
$$\sqrt[n]{z} = \sqrt[n]{r}e^{i\alpha/n + \frac{2\pi k}{n}i} \tag{27}$$

Las raices de los polinomios complejos son ciclicas.

Ejemplo: $\sqrt[3]{i}$

$$z = 1e^{\frac{i\pi}{2}} \tag{28}$$
$$\sqrt[3]{z} = e^{\frac{i\pi}{6} + \frac{2\pi ki}{3}} \qquad k = 0, 1, 2 \tag{29}$$
$$k = 0 \to e^{\frac{i\pi}{6}} = \frac{\sqrt{3}}{2} + i\frac{1}{2} \tag{30}$$
$$k = 1 \to e^{i\frac{\pi}{6} + \frac{2\pi i}{3}} = \cos\frac{5\pi}{6} + i\sin 5\pi/6 \tag{31}$$
$$k = 2 \to e^{i\frac{\pi}{6} + \frac{2\pi}{3}2} = e^{\frac{3i\pi}{2}} \tag{32}$$

Con $\sqrt[3]{i}$

$$\sqrt[3]{i} \to z = e^{i\frac{\pi}{8}} \to \sqrt[3]{z} = e^{i\frac{\pi}{8}+\frac{2\pi k i}{4}} \tag{33}$$
$$k = 0 \to e^{i\frac{\pi}{8}} \tag{34}$$
$$k = 1 \to e^{i\frac{\pi}{8}+\frac{2\pi i}{4}} = e^{i\frac{5\pi}{8}} \tag{35}$$
$$k = 2 \to e^{i\frac{\pi}{8}+\frac{4\pi i}{4}} = e^{i\frac{9\pi}{8}} \tag{36}$$
$$k = 3 \to e^{i\frac{\pi}{8}+\frac{3\pi i}{2}} = e^{i\frac{13\pi}{8}} \tag{37}$$

Ejemplo polinomio

$z^2 + (2i-3)z + 5 - i = 0$
Tenemos que $z = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ Entonces:

$$z = \frac{-(2i-3) \pm \sqrt{(2i-3)^2 - 3(5-i)}}{2} \tag{38}$$
$$= \frac{3 - 2i \pm \sqrt{-15-8i}}{2} \tag{39}$$

Todo se reduce a encontrar las raices de $\sqrt{-15-8i}$:

$$z_1 = -15 - 8i \to \cos\theta = \frac{-15}{17}, \quad \frac{-8}{17} \tag{40}$$
$$|z_1| = \sqrt{15^2 + 8^2} = 17 \tag{41}$$
$$z_1 = 17[\frac{-15}{17} - \frac{8}{17}i] = 17e^{i\theta} \to \sqrt{z_1} = \sqrt{17}e^{i\frac{\theta}{2}} \tag{42}$$
$$k = 1 \to \sqrt{17}e^{i\frac{\theta}{2}+\frac{2\pi 1}{2}i} = \sqrt{17}e^{i\frac{\theta}{2}} \tag{43}$$
$$\text{Tenemos que } \Delta\theta = \frac{2\pi}{n} \tag{44}$$
$$\sqrt{z_1} = \sqrt{17}[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}] \tag{45}$$
$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}} = -\frac{1}{\sqrt{17}} \tag{46}$$
$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} = \frac{4}{\sqrt{17}} \tag{47}$$

Tarea

con $z = -1 - 1i$ encontrar:
$\sqrt{z} = \frac{2^{\frac{1}{4}}}{2}[\sqrt{2-\sqrt{2}} - i\sqrt{2+\sqrt{2}}]$

Propiedades del módulo

- $|z_1| = |\bar{z}_1|$
- $z\bar{z} = |z|^2$
- $|z_1||z_2| = |z_1 z_2|$
- $|\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|}$
- $|z^n| = |z|^n$
- $|Re(z)| \leq |z| \qquad |Im(z)| \leq |z|$
- $|z_1 + z_2| = |z_1| + |z_2|$
- $||z_1| + |z_2|| \leq |z_1 - z_2|$