Clase 19

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1 Transporte paralelo y geodésicas

Ecuacion de las geodésicas

$$\nabla_V V = 0 \quad \rightarrow \quad \frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{dt}\frac{dx^\lambda}{dt} = 0$$

Podemos relajar la condición $\nabla_V V = 0$ por la condicion $\nabla_V V = fV$, donde $f \in \mathbb{F}(M)$

$$\nabla_V V = 0 \quad \rightarrow \quad \frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{dt}\frac{dx^\lambda}{dt} = f\frac{dx^\mu}{dt}$$

2 Derivada covariante de tensores

$$\nabla_\alpha T^{\mu_1\cdots\mu_p}_{\nu_1\cdots v_q} \equiv T^{\mu_1\cdots\mu_p}_{\nu_1\cdots v_q;\alpha} = \frac{\partial T^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_q}}{x^\alpha} + \sum_{i=1}^{p}\Gamma^{\mu_i}_{\alpha\beta}T^{\mu_1\cdots\beta\cdots\mu_p}_{\nu_1\cdots\nu_q} - \sum_{i=1}^{q}\Gamma^{\beta}_{\alpha\nu_i}T^{\mu_1\cdots\mu_p}_{\nu_1\cdots\beta\cdots\nu_q}$$

Conexion y derivada convariante

$$\nabla_{e_\mu} e_v \equiv \Gamma^\lambda_{\mu\nu}e_\lambda \quad \rightarrow \quad \nabla_{f_\mu} f_v = \bar{\Gamma}^\lambda_{\mu\nu}f_\lambda$$
$$f_\mu = \frac{\partial x^\beta}{\partial y^\mu}e_\beta$$

Las bases transforman como vectores ante el cambio $y(x)$.

En la parte izquierda:

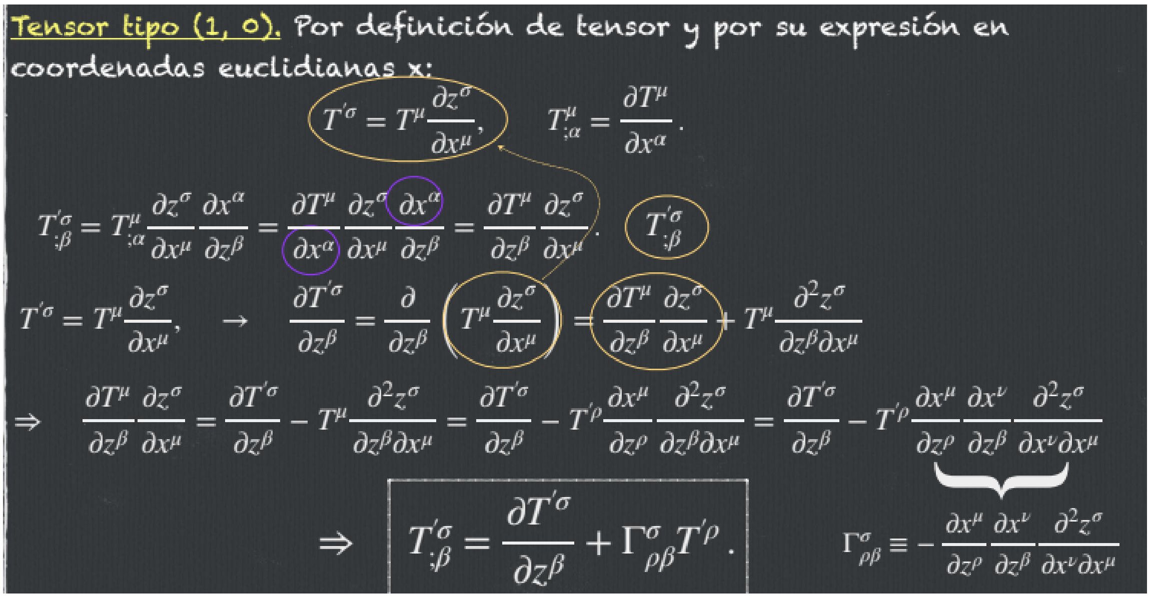
$$\begin{aligned}
\nabla_{\left(\frac{\partial x^\alpha}{\partial y^\mu}e_\alpha\right)}\left(\frac{\partial x^\beta}{\partial y^\mu}\right) &= \frac{\partial x^\alpha}{\partial y^\mu}\nabla_{e_\alpha}\left[\left(\frac{\partial x^\beta}{\partial y^\nu}\right)e_\beta\right] \\
&= \frac{\partial x^\alpha}{\partial y^\mu}\left[e_\alpha\left(\frac{\partial x^\beta}{\partial y^\nu}\right)e_\beta + \frac{\partial x^\beta}{\partial y^\nu}\nabla_{e_\alpha}e_\beta\right] \\
&= \frac{\partial x^\alpha}{\partial y^\mu}\left(\frac{\partial^2 x^\beta}{\partial x^\alpha\partial y^\nu} + \frac{\partial x^\sigma}{\partial y^\nu}\Gamma^\beta_{\alpha\sigma}\right)e_\beta \\
&= \left(\frac{\partial^2 x^\beta}{\partial y^\mu\partial y^\nu} + \frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\sigma}{\partial y^\nu}\Gamma^\beta_{\alpha\sigma}\right)e_\beta
\end{aligned}$$

Para la parte derecha:

$$\bar{\Gamma}^{\lambda}_{\mu\nu}\frac{\partial x^{\beta}}{\partial y^{\lambda}}e_{\beta}$$

Regla de tranformacion de Γ

$$\bar{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\beta}_{\alpha\sigma}\frac{\partial y^{\lambda}}{\partial x^{\beta}}\frac{\partial x^{\alpha}}{\partial y^{\mu}}\frac{\partial x^{\sigma}}{\partial y^{\nu}} + \frac{\partial y^{\lambda}}{\partial x^{\beta}}\frac{\partial^2 x^{\beta}}{\partial y^{\mu}\partial y^{\nu}}$$



3 Curvatura y torsión

En $\mathbb{R}^n$ en coordenadas cartesianas:

$$(\boldsymbol{\nabla}_{\alpha}\boldsymbol{\nabla}_{\beta} - \boldsymbol{\nabla}_{\beta}\boldsymbol{\nabla}_{\alpha})T^{(\mu)}_{(\nu)} = 0$$

En una variedad general con una conexión arbitraria en general tendremos:

$$(\boldsymbol{\nabla}_{\alpha}\boldsymbol{\nabla}_{\beta} - \boldsymbol{\nabla}_{\beta}\boldsymbol{\nabla}_{\alpha})T^{(\mu)}_{(\nu)} \neq 0$$

Tensor de torsión

$$T(X,Y) = \boldsymbol{\nabla}_X Y - \boldsymbol{\nabla}_Y X - [X,Y]$$

Tensor de ...