Clase 16

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1 Derivada

Ejemplo: Derivar $f(z) = \bar{z}$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$\lim_{\Delta z \to 0} \frac{\overline{z + \Delta z} - \bar{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\bar{\Delta} z}{\Delta z}$$

$$= \lim_{\Delta x, \Delta y \to 0,0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1$$

Por otro lado

$$\lim_{\Delta y, \Delta x \to 0,0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

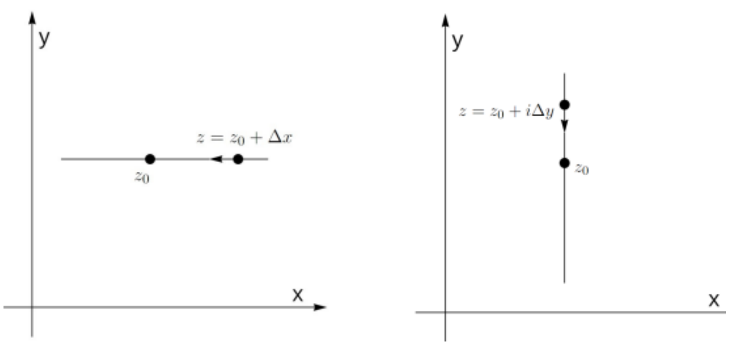
Por ambos lados el limite es diferente por lo tanto no existe.

Ejemplo: Derivar $f(z) = \left(2z^2 + 6z + 5\right)^3$ evaluada en $z = 1 + 2i$. Se puede utilizar la **derivada compuesta** $(f \circ g)' = 3(3z^2 + 6z + 5)^2(6z + 6)$

$$f'(z) = 3(3z^2 + 6z + 5)^2(6z + 6) = -17136i - 14048$$

2 Ecuaciones de Cauchy-Riemann

Sea $f(z) = u + iv$, vamos a calcular el límite que define a la derivada por dos caminos especificos:



(a) Primer Camino **(b)** Segundo camino

$$
\begin{aligned}
f'(z_0) &= \lim_{\Delta x \to 0} \frac{f(x_0+iy+\Delta x)-f(x_0+iy_0)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0)+iv(x_0+\Delta x, y_0)-u(x_0,y_0)+iv(x_0,y_0)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0)-u(x_0,y_0)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x_0+\Delta x, y_0)-v(x_0,y_0)}{\Delta x} \\
&= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}
\end{aligned}
$$

Por el otro camino:

$$
\begin{aligned}
f'(z_0) &= \lim_{\Delta y \to 0} \frac{f(z_0+iy_0+i\Delta y)-f(x_0+iy_0)}{i\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{u(x_0,y_0+\Delta y)-u(x_0,y_0)+i[v(x_0,y_0+\Delta y)-v(x_0,y_0)]}{i\Delta y} \\
&= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}
\end{aligned}
$$

Necesitamos que la parte real de la ecuacion por el primer lado sea igual que en la del segundo lado, y lo mismo con la parte compleja. De esta forma obtenemos las ecuaciones de Cauchy-Riemann:

Ec. Cauchy-Riemann

- $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$
- $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$

Ejemplo: derivar $f(z) = (z^2+5iz+8)^2$

$$
\begin{aligned}
f(z) &= z^4 - 25z^2 + 64 + 10iz^3 + 16z^2 + 80iz \\
&= (x+iy)^4 - 25(x+iy)^2 + 64 + 10i(x+iy)^2 + 80i(x+iy) \\
&= x^4 + 4x^3(iy) + 6x^2(iy)^2 + 4x(iy)^3 + (iy)^4 - 9(x^2+2ixy+(iy)^2) + 64 + 10i(x^3+3x^2(iy)+3x(iy)^2 \\
&+ (iy)^3) + 80i(x+iy) \\
&= x^4 + 4ix^3y - 6x^2y^2 - 4ixy^3 + y^4 - 9x^2 - 18ixy + 9y^2 + 64 + 10ix^3 - 30x^2y - 30ixy^2 + 10y^3 + 80ix - 80y
\end{aligned}
$$

Tenemos que:

$$
\begin{aligned}
u(x,y) &= x^4 - 6x^2y^2 + y^4 - 9x^2 + 9y^2 + 64 - 30x^2y + 10y^3 - 80y \\
v(x,y) &= 4x^3y - 4xy^3 - 18xy + 10x^3 - 30xy^2 + 80x
\end{aligned}
$$

Haciendo las derivadas:

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= 4x^3 - 12xy^2 - 18x - 60xy \\
\frac{\partial v}{\partial x} &= 12x^2y - 4y^3 - 18y + 30x^2 - 30y^2 + 80 \\
\frac{\partial u}{\partial y} &= -12x^2y + 4y^3 + 18y - 30x^2 + 30y^2 - 80 \\
\frac{\partial v}{\partial y} &= 4x^3 - 12xy^2 - 18x - 60xy
\end{aligned}
$$

Como cumple las ecuacion de cauchy-riemann no podemos decir que la derivada existe pero si no cumpliera las ec. entonces no existiria la derivada.

Ejemplo: Compobrar las ecuacion de Cauchy-Riemann para $f(z) = i\bar{z}^2 + 2\bar{z}$.

$$\begin{aligned} f(z) &= i\bar{z}^2 + 2\bar{z} \\ &= i(x^2 - 2xyi - y^2) + 2x - 2iy \end{aligned}$$

$$u(x,y) = 2xy + 2x \qquad v(x,y) = -2y + x^2 - y^2$$

$$\frac{\partial u}{\partial x} = 2y + 2 \qquad \frac{\partial v}{\partial x} = 2x$$

$$\frac{\partial u}{\partial y} = 2x \qquad \frac{\partial v}{\partial y} = -2 - 2y$$

No cumple las ecuaciones de Cauchy-Riemann.

2.1 Cauchy-Riemann en polares

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} \qquad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x} \qquad \frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y}$$

Ademas podemos comprobar que:

$$\frac{\partial r}{\partial x} = \cos\theta \qquad \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}$$

$$\frac{\partial r}{\partial y} = \sin\theta \qquad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$$

Reemplazando en las derivadas parciales de u y de v:

$$u_x = u_r\cos\theta - u_\theta\frac{\sin\theta}{r} \qquad u_y = u_r\sin\theta + u_\theta\frac{\cos\theta}{r}$$

$$v_x = v_r\cos\theta - v_\theta\frac{\sin\theta}{r} \qquad v_y = v_r\sin\theta + v_\theta\frac{\cos\theta}{r}$$

Reemplazando en las ec. de Cauchy-Riemann:

$$v_r\cos\theta - v_\theta\frac{\sin\theta}{r} = -u_r\sin\theta - u_\theta\frac{\cos\theta}{r} \qquad u_r\cos\theta - u_\theta\frac{\sin\theta}{r} = v_r\sin\theta + v_\theta\frac{\cos\theta}{r}$$

Multiplicando por $\cos\theta$ Multiplicando por $\sin\theta$

$$v_r\cos^2\theta - v_\theta\frac{\sin\theta\cos\theta}{r} = -u_r\sin\theta\cos\theta - u_\theta\frac{\cos^2\theta}{r} \qquad v_r\sin^2\theta + v_\theta\frac{\sin\theta\cos\theta}{r} = u_r\cos\theta\sin\theta - u_\theta\frac{\sin^2\theta}{r}$$

Entonces:

$$v_r = -\frac{u_\theta}{r} \qquad u_r = \frac{v_\theta}{r}$$