Clase 12

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September 15, 2023

1 Funcion Logaritmo

$$\omega = \log z \leftrightarrow z = e^{\omega}$$
$$\log_{\alpha} z = \log |z| + (\theta + 2\pi k)i, \qquad k = ..., -2, -1, 0, 1, 2, ...$$
$$\log_{\alpha} z : \text{Rama } \alpha \text{ del log de z}$$
$$\alpha < Arg(z) \leq \alpha + 2\pi$$

Ej $\log_{\frac{45\pi}{7}}(\sqrt{3} - i) = ?$

$$|z| = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$$
$$\tan\theta = \frac{1}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}}$$
$$\theta = -\tan^{-1}\frac{\sqrt{3}}{3} = \frac{-\pi}{6}$$
$$\alpha = \frac{45\pi}{7}$$
$$\alpha + 2\pi = \frac{45\pi}{7} + 2\pi = \frac{59\pi}{7}$$
$$\frac{45\pi}{7} < Arg(z) \leq \frac{59\pi}{7}$$
$$Arg(z) = \frac{45\pi}{7} + \frac{59\pi}{42} = \frac{329\pi}{42} \qquad \frac{59\pi}{42} \text{ es la diferencia entre } \frac{45\pi}{7} \text{ y } \frac{-\pi}{6}\text{.Reducido es } \frac{11\pi}{6} - \frac{3\pi}{7} = \frac{59\pi}{42}$$
$$\frac{45\pi}{7} - \underset{\#\text{ vueltas}}{3}(2\pi) = 3\frac{3\pi}{7}$$

Ej $\log_{\frac{75\pi}{8}} \frac{(-1-\sqrt{3}i)}{4}$

$$|z| = \frac{1}{2} \qquad \theta = -\frac{2\pi}{3}$$
$$\frac{75\pi}{8} < Arg(z) \leq \frac{75\pi}{8} + 2\pi = \frac{9\pi}{8}$$
$$\frac{75\pi}{8} \text{ da 4 vueltas } \frac{75\pi}{8} - 4(2\pi) = \frac{11\pi}{8} \text{ entonces } \frac{11\pi}{8} - \frac{4\pi}{3} - 2\pi = \frac{47\pi}{24}$$
$$Arg(z) = \frac{75\pi}{8} + \frac{47\pi}{24} = \frac{34\pi}{3}$$

2 Potencias de complejos

En general $\log z_1 z_2 \neq \log z_1 + \log z_2$ para los complejos.

$$z^a = e^{a \log z}, \quad z \neq 0$$

Como el logaritmo tiene muchas raices esta ecuacion es una ecuacion multivaluada.

Ej Vamos a calcular i^{i+1}.

$$\begin{aligned}
z = i, \quad a = i+1 \quad \rightarrow \quad i^{i+1} &= e^{(i+1)\log i} = e^{(i+1)\log e^{i(\frac{\pi}{2}+2\pi k)}} \\
i^{i+1} &= e^{(i+1)\left(i\left(\frac{\pi}{2}+2\pi k\right)\right)} \\
&= e^{-\left(\frac{\pi}{2}+2\pi k\right)} e^{i\left(\frac{\pi}{2}+2\pi k\right)} \\
&= e^{-\left(\frac{\pi}{2}+2\pi k\right)} \{\cos\frac{\pi}{2} + 2\pi k + i\sin\frac{\pi}{2} + 2\pi k\} \\
\text{(Solo el V.P)} &= e^{-\frac{\pi}{2}}\{i\} = ie^{-\frac{\pi}{2}}
\end{aligned}$$

Ej encontrar $\left(\frac{i+1}{i-1}\right)^{i+3}$.

$$\frac{i+1}{i-1}\frac{i+1}{i+1} = \frac{2i}{-2} = -i\text{Entonces: } \left(\frac{i+1}{i-1}\right)^{i+3} = (-i)^{i+3}$$

$$-i^{i+3} = e^{(i+3)log(-i)}$$

$$\text{Calculamos } \log -i = \log i + i\left(-\frac{\pi}{2} + 2\pi k\right)$$

Entonces:

$$e^{(i+3)i\left(\frac{\pi}{2}+2\pi k\right)} = e^{(-1+3i)\left(-\frac{\pi}{2}+2k\pi\right)} = e^{\left(\frac{\pi}{2}-2k\pi\right)}\left(\cos -\frac{3\pi}{2} + 6\pi k + i\sin -\frac{3\pi}{2} + 6\pi k\right)$$

Ej $i^{i^i} =?$.

$$i^{i^i} = e^{i^i \log i} = e^{i^i i\frac{\pi}{2}} = e^{i^{i+1}\frac{\pi}{2}}$$

$$\text{Teniamos que } i^{i+1} = ie^{-\pi/2} \text{ entonces:}$$

$$i^{i^i} = e^{ie^{-\frac{\pi}{2}}\frac{\pi}{2}} = \left(e^{i\frac{\pi}{2}}\right)^{e^{-\frac{\pi}{2}}} = i^{e^{-\frac{\pi}{2}}}$$

Demostrar que $\sin^{-1} z = -i\log\left(iz + (1-z^2)^{1/2}\right)$

$$\text{En nuestra definicion: } \sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \rightarrow \quad \sin z = \frac{e^{iz} - \frac{1}{e^{iz}}}{2i} = \frac{e^{2iz} - 1}{2ie^{iz}}$$

$$\text{Queremos sustituir } z \rightarrow \sin^{-1} z$$

$$\sin\sin^{-1} z = \frac{e^{2i\sin^{-1} z} - 1}{2ie^{i\sin^{-1} z}}$$

$$\text{Entonces tenemos que } z = \frac{e^{2i\sin^{-1} z} - 1}{2ie^{i\sin^{-1} z}}$$

$$\text{Despejamos } \sin^{-1} z$$

$$2ize^{i\sin^{-1} z} = e^{2i\sin^{-1} z} - 1$$

$$0 = e^{2i\sin^{-1} z} - 2ize^{i\sin^{-1} z} - 1$$

$$e^{i\sin^{-1} z} = \frac{2iz \pm \sqrt{-4z^2+4}}{2} = iz \pm \sqrt{1-z^2}$$

$$i\sin^{-1} z = \log\left(iz \pm \sqrt{1-z^2}\right)$$

$$\sin^{-1} z = -i\log\left(iz \pm \sqrt{1-z^2}\right)$$

Ejercicios

- $\sin^{-1} i =?.$
- $\sin^{-1}(-i) =?$
- $\tan^{-1}(z) =?$
- $\cos^{-1} z =?$