Clase 14

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1 Tensores en variedades

$$T = t^{\mu_1,\dots,\mu_q}_{\nu_1,\dots\nu_r} \frac{\partial}{\partial x^{\mu_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\mu_q}} \otimes dx^{\nu_1} \otimes \dots \otimes dx^{\nu_r}$$

Tensor tipo (q, r)

$$T(\omega_1, \dots, \omega_q; V_1, \dots, V_r)(2,3) \to T(\omega_1, \omega_2; V_1, V_2, V_3)$$
$$(0,1) \to \omega(V) = \langle \omega | V \rangle = \omega_\mu V^\mu$$

1.1 Ejemplos de tensores en fisica

Vector velocidad $(1,0)$

$$v = v^i e_i, \qquad v^i = \frac{dx^i}{dt}$$

Metrica $(0,2)$

$$ds^2 = g_{ij} dx^i dx^j$$

1.2 Comportamiento de tensores ante mapeos

Sea f un mapa entre M y N.

$$(f_* V) = W \in T_{f(p)} N$$
$$W^\alpha = V^\beta \frac{\partial y^\alpha}{\partial x^\beta} \to \underset{(f_*V)\ \text{"Restriccion"}}{\text{Mapeo de } V \text{ sobre } f}$$

Ejemplo 1: Vamos a mapear $f : M \to N$ el vector $V = V^1 \frac{\partial}{\partial x'} + V^2 \frac{\partial}{\partial x^2}$ con V^1, V^2 constantes. Vamos a usar las x para M y y para N.

$$f : (x^1, x^2) \quad \to \quad (y^1, y^2, y^3) = (x^1, x^2, h(x^1, x^2))$$
$$(f_* V) = W^\alpha \frac{\partial}{\partial y^\alpha} = V^i \frac{\partial y^\alpha}{\partial x^i} \frac{\partial}{\partial y^\alpha}$$
$$\left(V^1 \frac{\partial y^\alpha}{\partial x^1} + v^2 \frac{\partial y^\alpha}{\partial x^2} \right) \frac{\partial}{\partial y^\alpha} = \left(V^1 \frac{\partial y^1}{\partial x^1} \frac{\partial}{\partial y^1} + V^1 \frac{\partial y^2}{\partial x^1} \frac{\partial}{\partial y^2} + V^1 \frac{\partial y^3}{\partial x^1} \frac{\partial}{\partial y^3} \right)$$
$$+ \left(V^2 \frac{\partial y^1}{\partial x^2} \frac{\partial}{\partial y^2} + V^1 \frac{\partial y^2}{\partial x^2} \frac{\partial}{\partial y^2} + V^1 \frac{\partial y^3}{\partial x^2} \frac{\partial}{\partial y^3} \right) = V^1 (\frac{\partial}{\partial y^1} + \frac{\partial h}{\partial x^1} \frac{\partial}{\partial y^2}) + V^2 (\frac{\partial}{\partial y^2} + \frac{\partial h}{\partial x^2} \frac{\partial}{\partial y^3})$$

Comportamiento de T ante mapeos

$$W = V^1\frac{\partial}{\partial y^1} + V^2\frac{\partial}{\partial y^2} + \left(V^1\frac{\partial h}{\partial x^1} + V^2\frac{\partial h}{\partial xcub}\right)\frac{\partial}{\partial y^3}$$
$$h = (1-(x^1)^2-(x^2)^2)^{\frac{1}{2}}$$
$$\frac{\partial h}{\partial x^1} = \frac{1}{2}\frac{1}{h}(-2x^1) = -\frac{x^1}{h} = -\frac{y^1}{y^3} \qquad \frac{\partial h}{\partial x^2} = -\frac{y^2}{y^3}$$
$$W = V^1\frac{\partial}{\partial y^1} + V^2\frac{\partial}{\partial y^2} - \frac{1}{y^3}\left(V^1y^1+V^2y^2\right)\frac{\partial}{\partial y^3}$$
$$W = \frac{\partial}{\partial y^1} + \frac{\partial}{\partial y^2} - \frac{1}{y^3}(y^1+y^2)\frac{\partial}{\partial x^3}$$
$$x_{(1)} = (0,\frac{1}{2}) \quad \rightarrow \quad W_{(1)} = \frac{\partial}{\partial y^1} + \frac{\partial}{\partial y^2} - \frac{1}{\sqrt{3/4}}\left(\frac{1}{2}\right)\frac{\partial}{\partial y^3}$$
$$x_{(2)} = (\frac{1}{2},0) \quad \rightarrow \quad W_{(2)} = \frac{\partial}{\partial y^1} + \frac{\partial}{\partial y^2} - \frac{1}{\sqrt{3/4}}\left(\frac{1}{2}\right)\frac{\partial}{\partial y^3}$$

Podemos ver que $W_{(1)} = W_{(2)}$.

Ahora en el sentido contrario $f: M \to n \qquad f^*: T^*_{f(P)}N \quad \rightarrow \quad T^*_pM$. A esto se le llama **pullback**.

$$\underset{\text{Definido en } M}{\langle f^*\omega|V\rangle} = \underset{\text{Definido en } N}{\langle\omega|f_*V\rangle}$$

Componentes:

$$\omega = \omega_\alpha dy^\alpha \in T^*_{f(P)}N$$
$$(f^*\omega) = \bar{\omega}_k dx^k \qquad \bar{\omega} =?$$
Segun la definicion:
$$\left\langle \bar{\omega}_k dx^k \middle| V^i\frac{\partial}{\partial x^i}\right\rangle = \left\langle \omega_\alpha dy^\alpha \middle| V^\beta\frac{\partial y^\sigma}{\partial x^\beta}\frac{\partial}{\partial y^\sigma}\right\rangle$$
$$\bar{\omega}_kV^i\delta^k_i = \omega_\alpha V^\beta\frac{\partial y^\sigma}{\partial x^\beta}\delta^\alpha_\sigma = \bar{\omega}_kV^k = \omega_\alpha V^\beta\frac{\partial y^\alpha}{\partial x^\beta}$$
De la ec. anterior podemos deducir que:
$$\bar{\omega}_k = \omega_\alpha\frac{\partial y^\alpha}{\partial x^k} \quad \rightarrow \quad \xi_k = \omega_\alpha\frac{\partial y^\alpha}{\partial x^k}$$

Ejemplo $g_{ij} \to \mathbb{R}^3 \qquad g_{ij} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$

$$f:(\theta,\phi) \quad \rightarrow \quad (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

estamos en la variedad de la esfera y vamos a ir hacia $\mathbb{R}^2$.

$$\bar{g}_{V_1,V_2}(\theta,\phi) = g_{ij}\frac{\partial x^i}{\partial y^\mu}\frac{\partial x^j}{\partial y^\nu} \qquad (y^1,y^2) = (\theta,\phi)$$
$$\bar{g}_{11} = g_{ij}\frac{\partial x^i}{\partial\theta}\frac{\partial x^j}{\partial\theta} = \left(\frac{\partial x^1}{\partial\theta}\right)^2 + \left(\frac{\partial x^2}{\partial\theta}\right)^2 + \left(\frac{\partial x^3}{\partial\theta}\right)^2 = 1$$
$$\bar{g}_{12} = g_{ij}\frac{\partial x^i}{\partial\theta}\frac{\partial x^j}{\partial\phi} = \frac{\partial x^1}{\partial\theta}\frac{\partial x^1}{\partial\phi} + \frac{\partial x^2}{\partial\theta}\frac{\partial x^2}{\partial\phi} + \frac{\partial x^3}{\partial\theta}\frac{\partial x^3}{\partial\phi} = 0$$
$$g_{22} = \left(\frac{\partial x^1}{\partial\phi}\right)^2 + \left(\frac{\partial x^2}{\partial\phi}\right)^2 = \sin^2\theta$$

1.3 Tensores $(0, k)$ totalmente antisimetricos

$$T_\sigma(v_1, ...v_k) = sign(\sigma) T_{v_1,...,v_k} \qquad sing(\sigma) = \begin{cases} +1 & \sigma \text{ es par} \\ -1 & \sigma \text{ es impar} \end{cases}$$

$$\scriptstyle \sigma(v_1,...,v_k) \text{ Permutaciones de } v_1,...v_k$$

$$T = T_{v_1,...,v_k} dx^1 \otimes ... \otimes dx^k$$

$$T(V^1, ...V^k)$$

$$T_{v_1 v_2} \to T_{v2v1}$$