Finding Eigenvalues using Automatic Differentiation

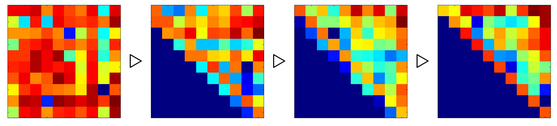
Alex Hirzel, Peter Solfest, and Ryan Bruner

Michigan Technological University

Abstract

Eigenvalues are ubiquitous in all branches of science and engineering. The dominant eigenvalue algorithm reduces input matrices to almost diagonal and then creates and chases bulges using parameters called shifts. Good parameter choices can allow shrinking of the original problem allowing for a more aggressive deflation on the subproblem. Such deflations greatly improve performance. We apply automatic differentiation tools to achieve frequent deflations.

Existing Techniques

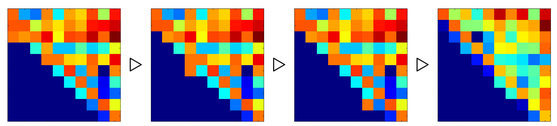


A general matrix is first reduced to Hessenberg form, then the QR algorithm is used to reduce the subdiagonal entries to zero (shown above). This exposes eigenvalues on the diagonal. The QR algorithm consists of repeated iterations of:

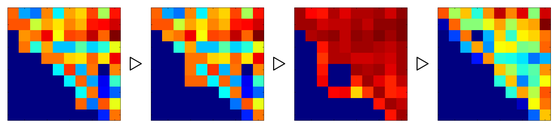
$$A_n \to Q_n R_n \tag{1}$$

$$A_{n+1} \leftarrow R_n Q_n = Q_n^{-1} A_n Q_n \tag{2}$$

As written, (1) and (2) require matrix multiplies which can be very large. For this reason, the QR algorithm is implemented *implicitly*. This results in a *bulge* moving from top-left to bottom-right:

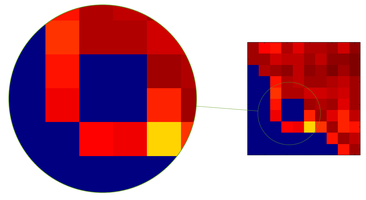


This bulge is seeded with a starting value, a *shift*. A second bulge can be started from the bottom right. When bulges are chased from each side, they meet in the middle. We can *inflate* this bulge using a Schur decomposition to form *spikes*. When one of these *spike* values is sufficiently close to zero, the problem can be subdivided into the top-left and bottom-right halves, exponentially reducing computation time for the eigenvalues. This is shown below.



Motivation: Frequent Deflations

In practice, middle deflations are improbable. They occur when the spike tips become zero.



We apply **automatic differentiation** (AD) tools to these spikes. AD tools allow *algorithms* to be differentiated in the same sense as algebraic functions. Classical calculus tells us

$$\frac{d}{dx}(\overbrace{x^2+2x+1}^{f(x)}) = \overbrace{2x+2}^{f'(x)},$$

which allows us to use Newton's method

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)} \tag{3}$$

to find the zeros of $f(x)$. **The technique for finding zeros of the spike values is analogus.** Applying AD tools to the Schur decomposition creates the gradient with respect to the input values (shifts). When the proper shifts are found to induce a zero on the spike, a deflation has been found.

Performance Improvement

Our algorithm outperforms XXX by XXXX. We show this by using a MATLAB package which counts FLOPs which shows that our algorithm performs fewer operations when compared to traditional QR on the same matrix.

(pretty pictures here)

Other text TBD

Note that this whole poster is now in English (no Latin :)).

There is a very cool-looking package that might work with MATLAB for counting FLOPS: `http://research.microsoft.com/en-us/um/people/minka/software/lightspeed/`

Methods

In this section, cover:

- Shift strategy

Future Work

In this section, cover:

- What are we going to do next? Very much depends on results that have not yet been teased out.

Conclusion

We successfully apply Automatic Differentiation techniques to induce deflations in the center of matrices. With the proper implementation, this will provide a large performance improvement over traditional algorithms in use today. Additionally, this is an enabling technique for extremely large eigenvalue problems. Our preliminary results indicate that a performance gain of XXX can be expected from proper implementations of this technique. This was determined by naively implementing each algorithm in MATLAB and comparing FLOP counts using each algorithm for the same inputs.

Additional Information

The authors are pursuing publication. For more details on this technique, please contact the authors directly.

References

[1] G. Anderson, *A procedure for differentiating perfect-foresight-model reduced-from coefficients*, Journal of Economic Dynamics and Control, 11 (1987), pp. 465 – 481.

[2] K. S. Braman, *Toward a Recursive QR Algorithm*, PhD thesis, Lawrence, KS, USA, 2003. AAI3103375.

Acknowledgements

The authors thank **Dr. Allan Struthers** (Mathematics department) for his guidance.

Contact Information

- Email: `ahirzel@mtu.edu`
- Phone: +1 (906) 231 0866

Michigan Technological University