# Taylor Series and Polynomials

### 1 Motivations

The purpose of Taylor series is to approximate a function with a polynomial; not only we want to be able to approximate, but we also want to know how good the approximation is. We are already familiar with first order approximations. The limit:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

tells us that when x is very close to zero  $\sin x$  can be approximated with x. Using this result we are able to conclude that the integral:

$$\int_0^1 \frac{\sin x}{x} \, dx$$

converges if and only if

$$\int_0^1 1 \ dx$$

is convergent; and we can also say that the series:

$$\sum_{n=1}^{+\infty} n \sin \frac{1}{n}$$

is convergent if and only if:

$$\sum_{i=1}^{+\infty} 1$$

is convergent. The first case is a convergent integral and the second case is a divergent series. In both these two cases the precision provided by the first order approximation of sine suffices.

Let's analyze the following example:

$$\int_0^1 \frac{\sin x}{x^5} - \frac{1}{x^4} \, dx$$

We want to determine if the improper integral is convergent or divergent. As in the previous example we can try to use limit comparison and we obtain the following:

$$\frac{\sin x}{x^5} - \frac{1}{x^4} = \frac{1}{x^4} \left( \frac{\sin x}{x} - 1 \right) \underset{x \to 0}{\approx} \frac{1}{x^4} \cdot 0 = 0$$

When the limit is zero, limit comparison implies convergence if the integral of the function we are comparing with is convergent, but it doesn't imply divergence if the integral we compare with is divergent. In this specific case the limit is zero and the integral we are comparing with is:

$$\int_0^1 \frac{1}{x^4} \, dx$$

which is divergent. Limit comparison in this case is inconclusive.

We have used that  $\sin x - x$  is approximately zero when x is small, and because of this approximation limit comparison is inconclusive. Is there a more precise way of approximating this function?

Since we want to approximate with a polynomial we can try to find out if there is a monomial  $ax^b$  such that:

$$\lim_{x \to 0} \frac{\sin x - x}{ax^b} = 1$$

where a is a real number and b is a natural number. This limit is a 0/0 case, so we can apply de l'Hôpital:

$$\lim_{x \to 0} \frac{\sin x - x}{ax^b} = \lim_{x \to 0} \frac{\cos x - 1}{abx^{b-1}}$$

Remember now the fundamental limit:

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

By comparing the two limits we can conclude that b=3 and  $a=-\frac{1}{6}$ , so finally:

$$\lim_{x \to 0} \frac{\sin x - x}{\frac{-x^3}{6}} = 1$$

or equivalently:

$$\sin x \underset{x \to 0}{\approx} x - \frac{x^3}{6}$$

We can now use this more refined approximation to solve the improper integral:

$$\frac{\sin x}{x^5} - \frac{1}{x^4} = \frac{-x^2}{6x^4} \left( \frac{\sin x - x}{-\frac{x^3}{6}} \right) \underset{x \to 0}{\approx} \frac{1}{x} \cdot \left( \frac{-1}{6} \right)$$

We are now allowed to use limit comparison in both the two directions; since the integral of  $\frac{1}{x}$  is divergent in [0,1] we can conclude that the integral we are studying is actually divergent.

We could iterate this method and have better approximations of sin. The function:

$$\sin x - x + \frac{x^3}{6}$$

approaches to zero as x goes to zero. We can determine how fast by comparing with a monomial  $cx^d$ . We want to find c, d such that the following limit is 1:

$$\lim_{x \to 0} \frac{\sin x - x + \frac{x^3}{6}}{cx^d} = 1$$

We can apply de l'Hôpital three times and find out that d=5 and  $c=\frac{1}{5!}$  which means that  $\sin x$  is approximated by:

$$\sin x \underset{x \to 0}{\approx} x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

This kind of approach opens two questions:

- Can we iterate the method infinitely many times?
- What is the precision of the approximation at each step?

If the answer to the first question is positive, we can write the function  $\sin x$  as a power series. We call this kind of power series the Taylor series of  $\sin x$  centered at 0.

In the next section we will try to answer these two questions.

## 2 Taylor's theorem

It is clear from the example that we have just studied, that in order to determine an approximation of a function we need to calculate some of its derivatives (we have used de l'Hôpital!!!). The more the approximation is precise the higher is the number of derivatives that we need to calculate.

The following definition comes in handy when we need to quantify how many times a function is differentiable:

**Definition 2.1.** Let f be a real function defined on a domain D. If the function is continuous at every point in D we say that it belongs to  $C^0(D)$ . If the function is differentiable n times at each point of D (excluding the boundary) and the n-th derivative is continuous, we say that the function is in  $C^n(D)$ . If the function can be differentiated infinitely many times we say that it is in  $C^\infty(D)$ .

Remark 2.2. If a function is in  $C^n(D)$  it is also in  $C^k(D)$  for every  $k \leq n$ .

Taylor's theorem says that if a function f(x) belongs to  $C^{n+1}(D)$  and  $\alpha \in D$ , then the function can be approximated with a degree n polynomial of this kind:

$$P_{n,\alpha}(x) = \sum_{i=0}^{n} \frac{f^{(i)}(\alpha)}{i!} (x - \alpha)^{i}$$

**Theorem 2.3.** Suppose f is a real function in  $C^{n+1}([a,b])$ , then for every point  $\alpha \in (a,b)$  there is a function  $h_n(x)$  such that:

$$f(x) = P_{n,\alpha}(x) + h_n(x)(x - \alpha)^n$$

and:

$$\lim_{x \to 0} h_n(x) = 0$$

Remark 2.4. We should think of  $h_n(x)(x-\alpha)^n$  as a remainder of our approximation of f(x) with the polynomial  $P_{n,\alpha}(x)$ .

We can say something more precise about the remainder:

**Theorem 2.5.** In the setup of the previous theorem, for every  $x \in (a,b)$  there is a point  $\zeta$  between x and  $\alpha$  such that:

$$h_n(x)(x-\alpha)^n = \frac{f^{(n+1)}(\zeta)}{(n+1)!}(x-a)^{n+1}$$

Patching the two results together we have that:

$$f(x) = P_{n,\alpha}(x) + \frac{f^{(n+1)}(\zeta)}{(n+1)!}(x-a)^{n+1}$$

Remark 2.6. Notice that in the previous formula the value of  $\zeta$  depends on x; if this were not the case every function would a polynomial!

proof of theorem 2.3. We can prove the theorem by induction starting from n = 0. When n = 0 the formula becomes:

$$f(x) = f(\alpha) + h_0(x)$$

The function  $h_0(x)$  converges to zero when x goes to  $\alpha$  because f(x) is continuous. Assume that the statement is true for some number n:

$$f(x) = P_{n,\alpha}(x) + h_n(x)(x - \alpha)^n$$

We prove that the statement is true for n+1. Manipulating the previous identity we have:

$$f(x) = P_{n+1,\alpha}(x) - \frac{f^{(n+1)}(\alpha)}{(n+1)!}(x-\alpha)^{n+1} + h(x)(x-\alpha)^n = P_{n+1,\alpha}(x) + h_{n+1}(x)(x-\alpha)^{n+1}$$

The new remainder is the function:

$$h_{n+1}(x) = \frac{h_n(x)}{x-\alpha} - \frac{f^{(n+1)}(\alpha)}{(n+1)!}$$

In order to prove the statement for n+1, we have to prove that:

$$\lim_{x \to \alpha} h_{n+1}(x) = 0$$

or equivalently:

$$\lim_{x \to \alpha} \frac{h_n(x)}{x - \alpha} = \frac{f^{(n+1)}(\alpha)}{(n+1)!}$$

or even:

$$\lim_{x \to \alpha} \frac{f(x) - P_{n,\alpha}(x)}{(x - \alpha)^{n+1}} = \frac{f^{(n+1)}(\alpha)}{(n+1)!}$$

In order to calculate this last limit we apply de l'Hôpital n+1 times. The (n+1)-th derivative of  $(x-\alpha)^{n+1}$  is (n+1)!, the (n+1)-th derivative of  $P_{n,\alpha}(x)$  is zero because the polynomial has degree n so the limit is precisely  $\frac{f^{(n+1)}(\alpha)}{(n+1)!}$  since the (n+1)-th derivative of f(x) is continuous. This completes the inductive argument.

Remark 2.7. At a closer look, the previous proof is just the method that we have used to approximate  $\sin x$  turned into an inductive argument.

proof of theorem 2.5. Fix a value of x once and for all. Denote with g(t) the following function:

$$g(t) = f(t) - P_{n,\alpha}(t) - \frac{h_n(x)}{x - \alpha} (t - \alpha)^{n+1}$$

By construction g(x) = 0, moreover all the derivatives of g(t) at the point  $\alpha$  are zero up to the *n*-th one:

$$g(\alpha) = g^{(1)}(\alpha) = g^{(2)}(\alpha) = \dots = g^{(n)}(\alpha) = 0$$

and for the (n + 1)-th derivative we have:

$$g^{(n+1)}(t) = f^{(n+1)}(t) - \frac{h_n(x)}{x - \alpha}(n+1)!$$

since the polynomial  $P_{n,\alpha}(t)$  has degree n. In order to prove the statement, it's enough to prove that there is a point  $\zeta$  such that  $g^{(n+1)}(\zeta) = 0$ .

If f(t) is in  $C^{n+1}([a,b])$ , the function g(t) is also in  $C^{n+1}([a,b])$ . Since  $g(\alpha) = g(x) = 0$  there is a point  $x_1$  between  $\alpha$  and x such that  $g^{(1)}(x_1) = 0$  because of the mean value theorem. Since  $g^{(1)}(\alpha) = g^{(1)}(x_1) = 0$  there must be a point  $x_2$  between  $\alpha$  and  $x_1$  such that  $g^{(2)}(x_1) = 0$ . We can repeat this method up to the n-th derivative of g and find a point  $x_n$  such that  $g^{(n+1)}(x_n) = 0$ . The point  $x_n$  is the point  $\zeta$  that we are looking for.

These two results provide a partial answer to the two questions that we have formulated at the beginning:

if a function can be differentiated n+1 times we can approximate it with a degree n polynomial and the error can be calculated in terms of the (n+1)-th derivative

Providing a complete answer to the first question is subtler. Theorem 2.3 might lead us to think that a  $C^{\infty}(D)$  function f(x) is equal to its Taylor series for every point in the domain D:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!} (x - \alpha)^n; \quad \alpha \in D$$

This is not true. The statement that we have proven is true for any number n, but it doesn't imply anything about the possibility of an infinite series. There are  $C^{\infty}$  functions that are not equal to the associated Taylor series and as a matter of fact they are not equal to any power series at all. For example let F(x) be the following function:

$$F(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0\\ 0 & \text{if } x = 0 \end{cases}$$

It's not hard to check that this function is in  $C^{\infty}(\mathbb{R})$ . In doing the calculation of the derivatives just remember that the exponential goes to zero faster than any polynomial. Every derivative of F(x) is zero at the origin:

$$F^{(n)}(0) = 0$$
 if  $n \ge 0$ 

This means that the Taylor series of F(x) is zero everywhere and its radius of convergence is  $+\infty$ . But the function F(x) is zero only at the origin, so it cannot be equal to its Taylor series.

This kind of surprising result is perfectly compatible with theorem 2.3. The theorem just says that for every n the function can be approximated with zero and the remainder  $h_n(x)$  is just  $\frac{F(x)}{x^n}$ :

$$\lim_{x \to 0} h_n(x) = \lim_{x \to 0} \frac{F(x)}{x^n} = 0 \quad \forall n \in \mathbb{N}$$

# 3 Taylor series

If a function f(x) is  $C^{+\infty}$  over some interval [a,b] it makes sense to write the Taylor series centered at some point  $\alpha \in (a,b)$ :

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(\alpha)}{n!} (x - \alpha)^n$$

Theorem 2.3 guarantees that the function has a Taylor polynomial of an arbitrarily high degree, but this doesn't imply anything about the convergence of the series. The series that we have written is just some formal writing at this stage, and it might not have any practical purpose. We certainly are in one of the following scenarios:

- The series is not convergent at all
- The series is convergent in an interval that is at most [a, b]
- The series is convergent in some interval but it doesn't converge to f(x) (the function is not analytic)

### 3.1 The point at infinity

If a function f(x) belongs to  $C^{+\infty}(\mathbb{R})$  it might or might not have a Taylor series centered at  $\infty$ . Necessary condition to have an expansion at  $\infty$  is:

$$\lim_{x \to +\infty} f(x) = L < +\infty$$

which means that the function has an horizontal asymptote. If this condition is not satisfied the Taylor series would be certainly divergent. This condition doesn't suffice though, since we have already seen that a function can be non analytic at some point.

**Example 3.1.** The function  $\arctan x$  has an horizontal asymptote at  $\pm \infty$  which is  $\pm \frac{\pi}{2}$ . If we want to try to expand the function at positive infinity we can expand the function  $\arctan y^{-1}$  at zero instead, since  $\frac{1}{y}$  goes to  $+\infty$  as y approaches  $0^+$ . For this function we can apply the usual Taylor formula and we obtain:

$$\arctan \frac{1}{y} = \frac{\pi}{2} + \sum_{n=0}^{+\infty} (-1)^{n+1} \frac{y^{2n+1}}{2n+1}$$

This is a Taylor series in a right neighborhood of zero and the radius of convergence is one. In order to calculate the expansion at  $+\infty$  of  $\arctan x$  we just make the substitution  $x = \frac{1}{u}$ :

$$\arctan x = \frac{\pi}{2} + \sum_{n=0}^{+\infty} (-1)^{n+1} \frac{1}{(2n+1)x^{2n+1}}$$

and with the same method at negative infinity we have:

$$\arctan x = \frac{-\pi}{2} + \sum_{n=0}^{+\infty} (-1)^{n+1} \frac{1}{(2n+1)x^{2n+1}}$$

The function  $\arctan x$  is analytic both at positive and negative infinity.

**Example 3.2.** We can apply the same method to  $(1+x)^{-1}$  which has an horizontal asymptote at zero. We expand  $(1+y^{-1})^{-1}$  at zero and we obtain:

$$\frac{y}{1+y} = \sum_{n=0}^{+\infty} (-1)^n y^{n+1}$$

and after the substitution  $x = \frac{1}{y}$  we have the expansion at infinity:

$$\frac{1}{1+x} = \sum_{n=0}^{+\infty} (-1)^n \frac{1}{x^{n+1}}$$

**Example 3.3.** The function  $e^{-x}$  has an horizontal asymptote at  $+\infty$ . This function is analytic at every point in  $\mathbb{R}$ , but is it analytic at  $+\infty$ ?

## 4 Landau symbols: big O

In this section we introduce one of the Landau symbols: big O. This notation provides a convenient way of writing the remainder when we approximate a function with a Taylor polynomial. The notation is particularly helpful when we need to study how the error due to the approximation spreads within a certain calculation.

For example suppose that we have a product of two functions f(x)g(x) and we approximate each one of them with a Taylor polynomial up to a certain precision. We need to know what the precision of the product is, and the big O notation provides a very convenient way of solving this problem.

The definition of big O may look kind of obscure at a first sight, but in practice it is very intuitive:

**Definition 4.1.** Let f(x), g(x) be two real functions. The function f(x) is a big O of g(x) as x approaches to a if there are two positive constants  $M, \delta$  such that:

$$|f(x)| \le M|g(x)|$$
 for all x satisfying  $|x-a| < \delta$ 

The meaning of big O is clarified by the following proposition:

**Proposition 4.2.** If the limit:

$$\lim_{x \to a} \left| \frac{f(x)}{g(x)} \right| = C < +\infty$$

then f(x) = O(g(x)).

Remark 4.3. This proposition is not equivalent to the definition!!! But the difference is kind of irrelevant for the scenarios that we will consider in these notes.

Using the proposition we can calculate the following examples:

$$5 = O(1)$$
 at any point  $3x^2 = O(x^2)$  at any point  $\sin x = O(1)$  as  $x \to 0$   $\sin x = O(x)$  as  $x \to 0$   $\sin x \neq O(x^2)$  as  $x \to 0$   $\sin(x) - x = O(x^2)$  as  $x \to 0$   $1 - \cos x = O(x^2)$  as  $x \to 0$   $1 - \cos x \neq O(x^3)$  as  $x \to 0$   $n! = O\left(\left(\frac{n}{e}\right)^n \sqrt{n}\right)$  as  $n \to +\infty$ , (Stirling)

In order to use the big O notation it is essential to understand how the O symbol behaves within a formula; this is a list of its properties:

$$O(f(x))O(g(x)) = O(f(x)g(x))$$
(1)

$$O(f(x)) + O(g(x)) = O(|f(x)| + |g(x)|)$$
(2)

$$f(x) + O(g(x)) = O(|f(x)| + |g(x)|)$$
(3)

$$f(x)O(g(x)) = O(f(x)g(x)) \tag{4}$$

$$O(cf(x)) = O(f(x)) \quad c \in \mathbb{R}, \ c \neq 0 \tag{5}$$

if 
$$\lim_{x \to a} |g(x)| < +\infty$$
 then  $O(f(x)g(x)) = O(f(x))$  (6)

Remark 4.4. Suppose that  $\lim_{x\to a} \left| \frac{g(x)}{f(x)} \right| < +\infty$  then we can combine the second and the last property to obtain:

$$O(f(x)) + O(g(x)) = O(|f(x)| + |g(x)|) = O\left(|f(x)|\left(1 + \frac{|g(x)|}{|f(x)|}\right)\right) = O(f(x))$$

In doing a calculation with big O we should always try to keep the function within O as simple as possible by using the properties that we have listed. For instance, suppose  $n \to +\infty$ :

$$O\Big(\frac{2}{n^2}\Big) + O\Big(\frac{\log(n)}{3n}\Big) = O\Big(\frac{2}{n^2} + \frac{\log(n)}{3n}\Big) = O\Big(\frac{\log(n)}{n}\Big(\frac{2}{n\log(n)} + \frac{1}{3}\Big)\Big) = O\Big(\frac{\log(n)}{n}\Big)$$

At the same time we don't want to lose precision: if f(x) is  $O(x^n)$  it is also  $O(x^i)$  for  $0 \le i \le n$  but we don't want to replace  $O(x^n)$  with a lower exponent. More in general: even if O(f(x)g(x)) = O(f(x)) whenever  $\lim_{x\to a} |g(x)| < +\infty$ , if the limit of g(x) is zero we lose precision by replacing O(f(x)g(x)) with O(f(x)).

Remark 4.5. In order to complete calculations involving one or more big O's, it is fundamental to remember that if:

$$\lim_{x \to a} f(x) = 0$$

then:

$$\lim_{x \to a} O(f(x)) = 0$$

but we cannot say anything about the following:

$$\lim_{x \to a} O(1) = ???$$

If at the end of a calculation we cannot get rid of the O(1)'s, this usually means that we have lost too much precision in the calculation. This is the reason why we should be concerned about losing precision!

Remark 4.6. Something similar applies to series and improper integrals. We cannot say anything about the convergence of the following series:

$$\sum_{n=1}^{+\infty} O(1)$$

but if a certain series:

$$\sum_{n=1}^{+\infty} a_n$$

is absolutely convergent, then also:

$$\sum_{n=1}^{+\infty} O(a_n)$$

is absolutely convergent even if we don't know exactly which sequence  $O(a_n)$  is. This follows by comparison. If  $b_n = O(a_n)$ , this means that there is a constant M such that:

$$|b_n| \leq M|a_n|$$

when n is large enough. Then by comparison if  $\sum |a_n|$  is convergent, also  $\sum |b_n|$  is convergent.

This method works with absolute convergence only and it cannot be used to prove divergence. For example:

$$\sum_{n=1}^{\infty} O\left(\frac{1}{n^2}\right) < +\infty$$

but we cannot say anything about:

$$\sum_{n=1}^{+\infty} O\left(\frac{1}{n}\right)$$

since both  $\frac{1}{n}$  and  $\frac{(-1)^n}{n}$  are  $O\left(\frac{1}{n}\right)$ .

# 5 Remainder of a Taylor polynomial as a big O

**Proposition 5.1.** Let f(x) be a function in  $C^{n+1}([a,b])$  and  $\alpha$  a point in (a,b), the Taylor expansion can be written in big O notation:

$$f(x) = P_{n,\alpha}(x) + O((x - \alpha)^{n+1})$$

*Proof.* As a consequence of theorem 2.5 we have:

$$\lim_{x \to \alpha} \frac{f(x) - P_{n,\alpha}(x)}{(x - \alpha)^{n+1}} = \frac{f^{(n+1)}(\alpha)}{(n+1)!} < +\infty$$

which implies that the remainder is  $O((x-\alpha)^{n+1})$ .

Remark 5.2. Even if  $O((x-\alpha)^{n+1})$  is an unknown function, it is a function in its own right, so we can compose it with anything we want. For example:

$$e^x = 1 + x + \frac{x^2}{2} + O(x^3)$$

and

$$e^{3x^2} = 1 + 3x^2 + \frac{9x^4}{2} + O(x^6)$$

The same thing applies to series: if we know that f(x) is equal to its Taylor series for  $|x - \alpha| < R$  and g(x) is a function such that  $|g(x) - \alpha| < R$  we have:

$$f(g(x)) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(\alpha)}{n!} (g(x) - \alpha)^n$$

For example, if |x| < 1 we have:

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

we can compose it with ax and we obtain the expansion:

$$\frac{1}{1+ax} = \sum_{n=0}^{\infty} (-1)^n a^n x^n$$

but it holds for  $|x| < \frac{1}{|a|}$ 

## 6 Applications

The first application of Taylor polynomials is to the calculation of limits of functions and sequences. Through *limit comparison* for improper integrals and series and the root/ratio test, Taylor polynomials become a very powerful tool to study the convergence of improper integrals and series.

**Example 6.1.** We start from the motivating example of the first section:

$$\lim_{x \to 0} \frac{\sin x - x + 2x^3}{x^2} = \lim_{x \to 0} \frac{x + O(x^2) - x + 2x^3}{x^2} = \lim_{x \to 0} \frac{O(x^2) + 2x^3}{x^2} = \lim_{x \to 0} O(1) + 2x = \lim_{x \to 0} O(1) = ???$$

We have ended up with an O(1), this means that the precision doesn't suffice. We try to take a better approximation of  $\sin x$ :

$$\lim_{x \to 0} \frac{\sin x - x + 2x^3}{x^2} = \lim_{x \to 0} \frac{x - \frac{x^3}{6} + O(x^4) - x + 2x^3}{x^2} = \lim_{x \to 0} \frac{O(x^4) + \frac{11}{6}x^3}{x^2} = \lim_{x \to 0} O(x^2) + \frac{11}{6}x = 0$$

Note that in the previous example we could have used de l'Hôpital twice and get the same result:

$$\lim_{x \to 0} \frac{\sin x - x + 2x^3}{x^2} = \lim_{x \to 0} \frac{\cos x - 1 + 6x}{2x} = \lim_{x \to 0} \frac{-\sin x + 6}{2} = 0$$

Sometimes, however, the method of de l'Hôpital will lead to very complicated formulas. The following example illustrates this:

#### Example 6.2. Consider the limit

$$\lim_{x \to 0} \frac{\cos(x)\sin(x\ln(1+x))}{x^2}.$$

The derivative of the numerator alone is:

$$-\sin(x)\sin(x\ln(1+x)) + \cos(x)\cos(x\ln(1+x))\left(\ln(1+x) + \frac{x}{1+x}\right)$$

and this still tends to 0 as x approaches 0. So to compute the original limit we would have to differentiate the above formula once more, making it very easy to do mistakes. Instead, we can use the big O notation as follows:

$$\cos(x)\sin(x\ln(1+x)) = (1+O(x^2))\left(x\ln(1+x) + O(x^3\ln^3(1+x))\right) =$$

$$(1 + O(x^2)) \left( x(x + O(x^2)) + O(x^3) \right) = (1 + O(x^2))(x^2 + O(x^3)) = x^2 + O(x^3),$$

so we obtain:

$$\lim_{x \to 0} \frac{x^2 + O(x^3)}{x^2} = \lim_{x \to 0} 1 + O(x) = 1.$$

**Caveat:** expanding a composition of functions can be tricky. In the previous example we had to calculate  $\sin(x \ln(1+x))$ . In the example we have decided to expand sine first, and then log. In general it is faster to expand the inner function first, but it's also more delicate. Suppose we wanted to expand  $\sin(\ln(1+x))$  up to  $O(x^4)$ :

$$\sin(\ln(1+x)) = \sin(x + O(x^2)) = x + O(x^2) + \dots$$

no metter how much you expand sine you cannot improve  $O(x^2)$ . To reach the precision of  $O(x^4)$  we first need to expand log up to  $O(x^4)$ :

$$\sin(\ln(1+x)) = \sin\left(x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4)\right) = x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4) - \frac{1}{6}\left(x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4)\right)^3 = x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4) - \frac{1}{6}x^3$$

In the expansion of sine we cannot stop at the first order, we have to calculate any order that contributes monomials up to degree four. In this specific case we need to calculate the third order but we don't need to take any higher order.

### Example 6.3.

$$\lim_{x \to 0} \frac{\log(1 + x \arctan x) - e^{x^2} + 1}{\sqrt{1 + x^4} - 1}$$

We can try to solve the limit just by taking Taylor polynomials of length 1:

$$= \lim_{x \to 0} \frac{\log(1 + x(x + O(x^3))) - 1 - x^2 + O(x^4) + 1}{\frac{1}{2}x^4 + O(x^8)} =$$

$$= \lim_{x \to 0} \frac{\log(1 + x^2 + O(x^4))) - x^2 + O(x^4)}{\frac{1}{2}x^4 + O(x^8)} =$$

$$= \lim_{x \to 0} \frac{x^2 + O(x^4) + O((x^2 + O(x^4))^2) - x^2 + O(x^4)}{\frac{1}{2}x^4 + O(x^8)} =$$

$$= \lim_{x \to 0} \frac{O(x^4)}{\frac{1}{2}x^4 + O(x^8)} = \lim_{x \to 0} \frac{O(1)}{\frac{1}{2} + O(x^4)} = ????$$

The calculation is inconclusive because we have O(1) at the numerator. This means that we have to increase the precision of the numerator:

$$= \lim_{x \to 0} \frac{\log(1 + x(x - \frac{1}{3}x^3 + O(x^5))) - x^2 - \frac{1}{2}x^4 + O(x^6)}{\frac{1}{2}x^4 + O(x^8)} =$$

$$= \lim_{x \to 0} \frac{x^2 - \frac{1}{3}x^4 + O(x^6) - \frac{1}{2}(x^2 - \frac{1}{3}x^4 + O(x^6))^2 + O(x^6) - x^2 - \frac{1}{2}x^4 + O(x^6)}{\frac{1}{2}x^4 + O(x^8)} =$$

$$= \lim_{x \to 0} \frac{-\frac{1}{3}x^4 - \frac{1}{2}x^4 - \frac{1}{2}x^4 + O(x^6)}{\frac{1}{2}x^4 + O(x^8)} =$$

$$= \lim_{x \to 0} \frac{-\frac{4}{3} + O(x^2)}{\frac{1}{2} + O(x^4)} = -\frac{8}{3}$$

In the calculation we have used the following expansions:

$$\arctan x = x - \frac{1}{3}x^3 + O(x^5)$$

$$e^{x^2} = 1 + x^2 + \frac{1}{2}x^4 + O(x^6)$$

$$\log(1+x) = x - \frac{1}{2}x^2 + O(x^3)$$

$$\sqrt{1+x^4} = 1 + \frac{1}{2}x^4 + O(x^8)$$

**Caveat:** If, within a calculation, you have both  $O(x^n)$  and a monomial  $ax^m$  with  $m \ge n$ , the monomial can and must be discarded:

$$ax^{m} + O(x^{n}) = O(|ax^{m}| + |x^{n}|) = O(|x^{n}|(1 + |ax^{m-n}|)) = O(x^{n})$$

**Example 6.4.** Determine if the following series is convergent or divergent:

$$\sum_{n=1}^{+\infty} \left( n^8 \left( \cos \frac{1}{n} - 1 \right) \left( \sin \frac{1}{n} - \frac{1}{n} + \frac{1}{3n^3} \right)^2 \right)^n$$

As a preliminary step we apply the root test for absolute convergence of the series and reduce the calculation to the study of the following limit:

$$\lim_{n \to +\infty} n^8 \left( \cos \frac{1}{n} - 1 \right) \left( \sin \frac{1}{n} - \frac{1}{n} + \frac{1}{3n^3} \right)^2 =$$

$$= \left| n^8 \left( 1 - \frac{1}{2n^2} + O\left(\frac{1}{n^4}\right) - 1 \right) \left( \frac{1}{n} - \frac{1}{6n^3} + O\left(\frac{1}{n^5}\right) - \frac{1}{n} + \frac{1}{3n^3} \right)^2 \right| =$$

$$= n^8 \left( \frac{1}{2n^2} + O\left(\frac{1}{n^4}\right) \right) \left( \frac{1}{6n^3} + O\left(\frac{1}{n^5}\right) \right)^2 =$$

$$= n^6 \left( \frac{1}{2} + O\left(\frac{1}{n^2}\right) \right) \left( \frac{1}{36n^6} + O\left(\frac{1}{n^8}\right) \right) =$$

$$= \left( \frac{1}{2} + O\left(\frac{1}{n^2}\right) \right) \left( \frac{1}{36} + O\left(\frac{1}{n^2}\right) \right) = \frac{1}{72} < 1$$

The limit is less then 1, so according to the root test the series is absolutely convergent.

**Strategy:** We have approximated cosine at the second order but sine at the third one, and it is not hard to check that a less precise approximation of sine produces an inconclusive output. The intuitive reason why we need higher precision for sine is the following: the term  $-\frac{1}{n}$  cancels the first order approximation of sine, but the term 1 only cancels the zero order approximation of cosine.

## The general strategy is to expand a function up to the first term that is not canceled by some other component in the formula.

This is just a guideline and not a rule: sometimes it might not be necessary to expand a function this much, sometimes it might not suffice! Figuring out at first sight the length of a Taylor polynomial that suffices in a certain calculation is an art; till the day you master the art, trial and error is the only way! If the polynomial is too short the method is inconclusive; if the polynomial is too long, the calculation might get exponentially messy!

**Example 6.5.** Determine if the following series is convergent/absolutely convergent:

$$\sum_{n=1}^{+\infty} (-1)^n \frac{e^{\frac{1}{n^2}} - 3 + 2\cos\frac{1}{n}}{\sin\frac{1}{n} - \frac{1}{n}} =$$

$$= \sum_{n=1}^{+\infty} (-1)^n \frac{1 + \frac{1}{n^2} + \frac{1}{2n^4} + O\left(\frac{1}{n^6}\right) - 3 + 2 - \frac{1}{n^2} + \frac{2}{4!n^4}}{\frac{1}{n} - \frac{1}{6n^3} + O\left(\frac{1}{n^5}\right) - \frac{1}{n}} =$$

$$= \sum_{n=1}^{+\infty} (-1)^n \frac{\frac{7}{12n^4} + O\left(\frac{1}{n^6}\right)}{-\frac{1}{6n^3} + O\left(\frac{1}{n^5}\right)} =$$

$$= \sum_{n=1}^{+\infty} (-1)^n \left(\frac{1}{n}\right) \frac{\frac{7}{12} + O\left(\frac{1}{n^2}\right)}{-\frac{1}{6} + O\left(\frac{1}{n^2}\right)} =$$

At this point we might be tempted to use limit comparison and conclude that the series is convergent; but limit comparison cannot be applied to an alternating series. Instead of using limit comparison we try to separate the part of the series that converges but not absolutely, from the part that converges absolutely:

$$= \sum_{n=1}^{+\infty} (-1)^n \frac{1}{n} \left( \frac{-7}{2} + O\left(\frac{1}{n^2}\right) \right) \left( 1 + O\left(\frac{1}{n^2}\right) \right) =$$

$$= \sum_{n=1}^{+\infty} (-1)^n \frac{1}{n} \left( \frac{-7}{2} + O\left(\frac{1}{n^2}\right) \right) =$$

$$= \sum_{n=1}^{+\infty} (-1)^n \left( \frac{-7}{2n} \right) + \sum_{n=1}^{+\infty} O\left(\frac{1}{n^3}\right)$$

The first series is convergent and the second one is absolutely convergent because of remark 4.6, and this implies that their sum is convergent. However the entire series is not absolutely convergent, if we get rid of  $(-1)^n$  by taking the absolute value, we can actually use limit comparison and conclude that the series behaves like  $\frac{1}{n}$  which is divergent.

#### Example 6.6.

$$\lim_{x \to 0^{+}} \frac{\sin(x^{x} - x \log(x) - 1)}{1 - \cos(x^{x} - 1)} =$$

$$= \lim_{x \to 0^{+}} \frac{\sin(1 + x \log(x) + \frac{1}{2}x^{2} \log^{2}(x) + O(x^{3} \log^{3}(x)) - x \log(x) - 1)}{1 - \cos(1 + x \log(x) + O(x^{2} \log^{2}(x)) - 1)} =$$

$$= \lim_{x \to 0^{+}} \frac{\frac{1}{2}x^{2} \log^{2}(x) + O(x^{3} \log^{3}(x))}{1 - 1 + \frac{1}{2}(x \log(x) + O(x^{2} \log^{2}(x))^{2})} =$$

$$= \lim_{x \to 0^{+}} \frac{\frac{1}{2}x^{2} \log^{2}(x) + O(x^{3} \log^{3}(x))}{\frac{1}{2}x^{2} \log^{2}(x) + O(x^{3} \log^{3}(x))} = \lim_{x \to 0^{+}} \frac{\frac{1}{2} + O(x \log(x))}{\frac{1}{2} + O(x \log(x))} = 1$$

In this calculation we have used that  $x^x = e^{x \log(x)}$  and that  $x \log(x) \to 0$  as  $x \to 0^+$ .

#### Example 6.7.

$$\lim_{x \to 0^+} \frac{\left(\frac{x}{x^2+3} - \frac{x}{3} - x^{(x^4)} + 1\right)(\cos^3(x) - 1)}{(\sin^3(x) - x^3)\cosh(x)} =$$

$$\begin{split} &=\lim_{x\to 0^+} \frac{\left(\frac{x}{3}(1-\frac{x^2}{3}+O(x^4))-\frac{x}{3}-1-x^4\log(x)+O(x^8\log^2(x))+1)((1-\frac{x^2}{2}+O(x^4))^3-1)\right)}{((x-\frac{x^3}{6}+O(x^5))^3-x^3)\cdot 1} =\\ &=\lim_{x\to 0^+} \frac{\left(-\frac{1}{9}x^3+O(x^5)-x^4\log(x)\right)(O(x^4)-\frac{3}{2}x^2)}{-\frac{3}{6}x^5+O(x^7)} =\\ &=\lim_{x\to 0^+} \frac{\left(-\frac{1}{9}-x\log(x)+O(x^2)\right)(-\frac{3}{2}+O(x^2))}{-\frac{1}{2}+O(x^2)} = \frac{\frac{1\cdot 3}{9\cdot 2}}{-\frac{1}{2}} = -\frac{1}{3} \end{split}$$

A few remarks about this calculation:

- the hyperbolic cosine is completely irrelevant since it converges to one. Approximating functions with polynomials is not an algorithm to solve limits, it's just one of many tools. It's usually a good idea to simplify a limit with any other method before doing any approximation.
- This calculation would get very long and messy if we did the algebra completely. But this is not necessary since we are working with big O. For instance the calculation of the cube:

$$\left(x - \frac{x^3}{6} + O(x^5)\right)^3$$

would be very long and tedious. It is a good idea to calculate the smallest big O first. In this case it is  $3x^2O(x^5) = O(x^7)$ . The smallest big O represents the maximum precision that we can achieve. Any monomial or big O of a monomial with degree higher or equal to 7, doesn't carry any information and it can be absorbed in  $O(x^7)$ . It turns out that the only significant terms in the cube are:  $x^3$  and  $3x^2(-\frac{x^3}{6}) = \frac{-x^5}{2}$ .

• In the third step we have used:

$$O(x^5) + O(x^8 \log^2(x)) = O(|x^5|(1 + |x^3 \log^2(x)|)) = O(x^5)$$

and the fact that  $x^3 \log^2(x) \to 0$  as  $x \to 0^+$ .

## 7 An advanced application: Stirling's formula

In the section about sequences we saw that integral comparison provides a good estimate for the factorial of a number:

$$\frac{n^n}{e^{n-1}} \le n! \le \frac{(n+1)^{n+1}}{e^n}$$

This estimate suggests that the factorial might behave more or less like  $n^n e^{-n}$ ; but this is not completely true since the limit:

$$\lim_{n \to +\infty} \frac{n!}{n^n e^{-n}} = +\infty$$

is infinity and not one. This means that the factorial is faster than  $n^n e^{-n}$ ; we would like to know how faster it goes to infinity. Instead of working with the factorial we will use the gamma function  $\Gamma(x+1) = \int_0^{+\infty} t^x e^{-t} dt$ , which is related to the factorial in the following way:  $\Gamma(n+1) = n!$ .

Theorem 7.1 (Stirling).

$$\lim_{n \to +\infty} \frac{n!}{n^n e^{-n} \sqrt{2\pi n}} = \lim_{x \to +\infty} \frac{\Gamma(x+1)}{x^x e^{-x} \sqrt{2\pi x}} = 1$$

*Proof.* We can think of  $\Gamma(x+1)$  has an integral depending on a parameter x, and we want to study its asymptotic behavior. In this kind of setup it's usually convenient to write the integral in the following way:

$$\int_0^{+\infty} e^{xh(t)} dt$$

for some function h(t). In the case of  $\Gamma$  we can do the following manipulation:

$$\Gamma(x+1) = \int_0^{+\infty} t^x e^{-t} dt = \int_0^{+\infty} e^{x \log t - t} dt = x^{x+1} \int_0^{+\infty} e^{x(\log u - u)} du$$

where t = xu. Since we cannot calculate the integral as it is, we have to approximate the function  $\log u - u$ . This function is increasing from 0 to 1, it attains its maximum at 1 and its decreasing for u > 1. It goes to  $-\infty$  both at zero and infinity. Since we want to approximate the function in a neighborhood of the maximum it is convenient to shift the function so that the maximum is at the origin:

$$x^{x+1}e^{-x}\int_{-1}^{+\infty} e^{x(\log(v+1)-v)} dv$$

where u = v + 1. We can now expand the function  $h(v) = \log(v + 1) - v$  at the origin:

$$h(v) = \log(v+1) - v = v - \frac{v^2}{2} + O(v^3) - v = -\frac{v^2}{2} + O(v^3)$$

and we can also expand the exponential:

$$e^{xh(v)} = e^{-x\frac{v^2}{2}}(1+xO(v^3))$$

Since we are integrating from -1 to  $+\infty$  it's not enough to use an approximation that is good in a small neighborhood of the origin only. This approximation doesn't work at -1 where the logarithm and its derivatives are divergent and it doesn't work for an arbitrarily large v; however it holds in any closed interval  $[-1 + \epsilon, M]$  where both  $\epsilon$  and M are positive numbers. We can rewrite the integral in the following way:

$$x^{x+1}e^{-x}\left(\int_{-1}^{-1+\epsilon}e^{x(\log(v+1)-v)}\ dv + \int_{-1+\epsilon}^{M}e^{-x\frac{v^2}{2}}(1+xO(v^3))\ dv + \int_{M}^{+\infty}e^{x(\log(v+1)-v)}\ dv\right)$$

We will treat the three integrals separately. In the first integral we use that  $\log(v+1) - v$  is increasing for v < 0 and we can choose  $\epsilon$  such that for  $v \in (-1, -1 + \epsilon)$  we have:

$$\log(v+1) - v \le -1$$

If x > 1 the previous inequality implies:

$$x(\log(v+1) - v + 1) < \log(v+1) - v + 1$$

or equivalently:

$$x(\log(v+1) - v) \le -(x-1) + \log(v+1) - v$$

By comparison we have:

$$\int_{-1}^{-1+\epsilon} e^{x(\log(v+1)-v)} dv \le e^{-(x-1)} \int_{-1}^{-1+\epsilon} e^{\log(v+1)-v} dv$$

When x goes to infinity the right hand side converges to zero, and this implies that the left hand side converges to zero as well. This means that the first integral doesn't give any contribution to the calculation. Now we can take care of the second one. We separate the two summands:

$$\int_{-1+\epsilon}^{M} e^{-x\frac{v^2}{2}} dv + \int_{-1+\epsilon}^{M} O(e^{-x\frac{v^2}{2}}xv^3) dv$$

In the first of the two integrals make the substitution  $s = \sqrt{\frac{x}{2}}v$  and the integral becomes:

$$\sqrt{\frac{2}{x}} \int_{-\sqrt{\frac{x}{2}}+\epsilon}^{M\sqrt{\frac{x}{2}}} e^{-s^2} ds$$

For what concerns the second integral we can conclude by comparison that:

$$\int_{-1+\epsilon}^{M} O(e^{-x\frac{v^2}{2}}xv^3) \ dv = O\left(\int_{-1+\epsilon}^{M} e^{-x\frac{v^2}{2}}xv^3 \ dv\right)$$

The integral within the big O is something we can actually integrate explicitly:

$$\int_{-1+\epsilon}^{M} e^{-x\frac{v^2}{2}} x v^3 dv = e^{-x\frac{v^2}{2}} \left( v^2 + \frac{2}{x} \right) \Big|_{-1+\epsilon}^{M}$$

When x goes to infinity this function goes to zero so we can safely ignore it. To calculate the third integral we use the same trick we have used for the first one. Since the function  $\log(v+1) - v$  is decreasing for v > 0 we can choose M such that  $\log(v+1) - v < -1$  for every v > M. This implies that for any x > 1 the following inequality holds:

$$x(\log(v+1) - v) \le -(x-1) + \log(v+1) - v$$

By comparison we have:

$$\int_{M}^{+\infty} e^{x(\log(v+1)-v)} dv \le e^{-(x-1)} \int_{M}^{+\infty} e^{\log(v+1)-v} dv$$

The right hand side of the inequality certainly converges to zero when  $x \to +\infty$  which implies that the left hand side also converges to zero. Even this last piece can be safely ignored in the calculation.

Putting all the pieces together we have:

$$\lim_{x \to +\infty} \frac{\Gamma(x+1)}{x^x e^{-x} \sqrt{2x}} = \lim_{x \to +\infty} \frac{x^{x+1} e^{-x} \sqrt{\frac{2}{x}} \int_{-\sqrt{\frac{x}{2}} + \epsilon}^{M\sqrt{\frac{x}{2}}} e^{-s^2} ds}{x^x e^{-x} \sqrt{2\pi x}} = \lim_{x \to +\infty} \frac{1}{\sqrt{\pi}} \int_{-\sqrt{\frac{x}{2}} + \epsilon}^{M\sqrt{\frac{x}{2}}} e^{-s^2} ds = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-s^2} ds = 1$$

In this proof we have actually shown that the asymptotic behavior of the integral of  $e^{xh(v)}$  is determined only by the behavior of h(v) in neighborhood of the maximum when h(v) = log(v+1) - v. This is actually a very general phenomenon, and the technique that we have used is a baby version

of a very general and powerful method used to determine the asymptotic behavior of integrals depending on a parameter known as the *stationary phase method* or *steepest descent method* or *Laplace method*. This method is widely used in the study of Laplace transforms.

This calculation is also a clear demonstration that the mechanical usage of Taylor polynomials is not the ultimate technique to calculate limits; in this example we had to combine Taylor polynomials with a very careful study of the behavior of the integrand in different domains. This is certainly not an uncommon scenario in analysis.

## 8 Some more examples

7.

Here are more examples to practice with. Within brackets you find the numerical solution of limits, and the approximation of integrands and summands.

1. 
$$\lim_{x \to 0} \frac{2x \arctan(x) - \ln(1 + x^2) - x^2}{x^2 (1 - \cos(x))} = \left[ = -\frac{1}{3} \right]$$

2. 
$$\lim_{x \to 0} \frac{\sin(x^2 - x) + x - x^2}{2\ln(x + 1) + 2e^{-x^2} - 2 - 2x + 3x^2} \qquad \left[ = \frac{1}{4} \right]$$

3. 
$$\lim_{x \to 0} \frac{2x + x^2 + 2(1 - x^2)\ln(1 - x)}{\ln(1 - x) + e^x - 1}$$
 [= -8]

4. 
$$\lim_{x \to 0} \frac{24 - 24\cos(\sin(x)) - 12x^2 + 5x^4}{\sin(1 - \cos(x)) - 1 + \cos(x)} = \left[ = -\frac{296}{5} \right]$$

5. 
$$\sum_{n=2}^{+\infty} \sqrt{\ln\left(n\sin\frac{1}{n} + \frac{1}{6n^2}\right)} \qquad \left[\approx \frac{1}{n^2}, \text{ convergent}\right]$$

6. 
$$\int_0^1 \frac{\sqrt{1+x^2} - \frac{1}{2}x^2 - 1 + 3x^4}{(\sin x - x)^2} dx \qquad \left[ \approx \frac{1}{x^2}, \text{ divergent} \right]$$

$$\int_{3}^{+\infty} \left( \left( \frac{1}{x} \right)^{\frac{1}{x}} - 1 - \frac{1}{x} \ln \frac{1}{x} + \left( \frac{3}{x} \log \frac{1}{x} \right)^{4} \right) \ln(x + x^{2}) dx \qquad \left[ \approx \frac{\ln^{3}(x)}{x^{2}}, \text{ convergent} \right]$$

8. 
$$\sum_{n=2}^{+\infty} \frac{\left(\frac{1}{2}\sin\frac{1}{n^2} + \cos\frac{1}{n} - 1\right)^2}{\ln^3\left(1 + \frac{1}{n}\right)} \qquad \left[\approx \frac{1}{n^5}, \text{ convergent}\right]$$

9. 
$$\sum_{n=2}^{+\infty} \sin\left(\ln\left(1 - \frac{1}{n^2}\right)\right) + \sin n^2 \sin\frac{1}{n^6} + \frac{1}{n^2} \qquad \left[\approx \frac{1}{n^4}, \text{ convergent}\right]$$

10.

$$\lim_{x \to 0^+} (\ln x)^3 \left( \arctan(\ln(x + x^2)) + \frac{\pi}{2} \right) + (\ln x)^2 \qquad \left[ \frac{1}{3} \right]$$

### 9 Formulas

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^{n}}{n!} = 1 + x + \frac{x^{2}}{2!} + \frac{x^{3}}{3!} + \dots \text{ for all } x$$
 (7)

$$\log(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \text{ for } -1 \le x < 1$$
 (8)

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \text{ for } -1 < x \le 1$$
(9)

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ for } |x| < 1 \tag{10}$$

$$\sqrt{1+x} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(1-2n)(n!)^2 (4^n)} x^n = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \dots \text{ for } |x| \le 1$$
 (11)

$$(1+x)^{\alpha} = \sum_{n=0}^{\infty} {\alpha \choose n} x^n$$
 for all  $|x| < 1$  and all real  $\alpha$  (12)

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \text{ for all } x$$
 (13)

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \text{ for all } x$$
 (14)

$$\tan x = \sum_{n=1}^{\infty} \frac{B_{2n}(-4)^n (1-4^n)}{(2n)!} x^{2n-1} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots \text{ for } |x| < \frac{\pi}{2}$$
 (15)

$$\sec x = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n} \text{ for } |x| < \frac{\pi}{2}$$
 (16)

$$\arcsin x = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1} \text{ for } |x| \le 1$$
 (17)

$$\arccos x = \frac{\pi}{2} - \arcsin x = \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1} \text{ for } |x| \le 1$$
 (18)

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \text{ for } |x| \le 1$$
 (19)

$$\arctan x = \pm \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \quad \text{for } x \to \pm \infty \quad \text{radius of convergence 1}$$
 (20)

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \text{ for all } x$$
 (21)

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \text{ for all } x$$
 (22)

$$\tanh x = \sum_{n=1}^{\infty} \frac{B_{2n} 4^n (4^n - 1)}{(2n)!} x^{2n-1} = x - \frac{1}{3} x^3 + \frac{2}{15} x^5 - \frac{17}{315} x^7 + \dots \text{ for } |x| < \frac{\pi}{2}$$
 (23)

$$\operatorname{arsinh}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1} \text{ for } |x| \le 1$$
 (24)

$$\operatorname{artanh}(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} \text{ for } |x| < 1$$
 (25)

The symbol  $B_n$  in the formulas stands for the *n*-th Bernoulli number, and  $E_n$  is the *n*-th Euler number.