THE QUASI SOLUTION METHOD

ALI ADALI

ABSTRACT. The Quasi-Solution method (QSM), a way to solve nonlinearity $\mathcal{N}(x) = 0$ between Banach spaces, is presented. When $\mathcal{N}: \mathcal{X} \to \mathcal{Y}$ it is proved that under i) the existence of an explicit near solution x_0, ii) the existence of an explicit near left inverse $\mathcal{L}^I$ of the Frechet derivative $\mathcal{N}'(x_0)$ and iii) the existence of a neighborhood S of x_0 for which $\|\mathcal{N}''\|$ is uniformly bounded, if the magnitudes of $\mathcal{N}(x_0)$, $\mathcal{L}^I\mathcal{N}'(x_0) - \mathcal{I}$ and $\mathcal{N}''$ are in a certain balance then there exists unique solution of $\mathcal{N}$ in S. Particularly, it is proved that if

$$4\|\mathcal{N}''\| \, \|\mathcal{N}(x_0)\| \frac{\|\mathcal{L}^I\|^2}{(1 - \|\mathcal{E}\|)^2} \leq 1$$

where $\mathcal{E} = \mathcal{L}^I\mathcal{N}'(x_0) - \mathcal{I}$ is the deviation from identity, then there exists a unique solution of $\mathcal{N}(x) = 0$ in S. This result is generalized to obtain a unified treatment for

$$\mathcal{N}_i(x) = 0 \text{ for } i = 1, 2, \cdots, n$$

for the nonlinear operators $\mathcal{N}_i: \mathcal{X} \to \mathcal{Y}_i$.

1. INTRODUCTION

The methods of solving

$$\mathcal{N}(x) = 0$$

are limited even in the most simple settings. The exact solutions are in general very challenging to find and in many cases are even impossible to represent with the current representation systems. Nonetheless, in many problems it is possible to obtain extremely accurate approximate solutions by computers through a Newton iteration process if $\mathcal{N}$ is regular enough and if the first guess is reasonably close to a solution. Today, the power of computers and the availability of computational softwares made it easy to experiment and to find very accurate solutions to many nonlinear equations, and countless such experimental results are published. What is left is the rigorous proof of the existance and uniqueness of exact solutions in a neighborhood of the approximate solutions. The Quasi Solution method is developed to do that. Its core idea is to modify an arithmetic based on the behavior of the nonlinearity such that the Banach Fixed Point theorem can be implemented to obtain the result.

The quasi solution idea not new. In fact it has been used in many articles. What is new in this article is a neat description of a very general method and a set of theorems which ensures the existence and uniqueness result if a certain inequality is satisfied. This is particularly interesting because all the terms in the inequalities can be estimated once the elements of the quasi solution are determined, hence the rigorous proof part is simplified to estimating a number of quantities and checking whether the inequalities are met.

2. Part 1 - Theory

2.1. Preliminaries.

Definition 2.1. *Let* V *be a vector space. Let* $A \subset V$. *We define the convex hull of* A *in* V *as*

$$\mathsf{Conv}\ A := \{(1-t)x + ty \colon x, y \in A, t \in [0,1]\}, \tag{1}$$

we define the line segment connecting $x, y \in V$ *as*

$$[x, y] := \mathsf{Conv}\ \{x, y\} \tag{2}$$

Let V be a Banach space, let $x \in V$ *and* $r > 0$. *We define the ball centered at* x *with radius* r *as*

$$\mathcal{B}_{x,r} := \left\{\|x' - x\| \le r \colon x' \in V\right\} \tag{3}$$

Let V_1, V_2 *be vector spaces. We denote the set of linear operators from* V_1 *to* V_2 *by*

$$\mathfrak{L}(V_1, V_2) := \{\mathcal{L} \colon V_1 \to V_2,\ \mathcal{L} \text{ is linear}\} \tag{4}$$

In this section we assume that $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, and $\mathcal{N} \colon \mathcal{X} \to \mathcal{Y}$ is a nonlinear operator. We fix an $x_0 \in \mathcal{X}$ and an open and convex neighborhood S of x_0. We denote the first and the second order Frechet derivatives of $\mathcal{N}$ by $\mathcal{N}'$ and $\mathcal{N}''$, respectively, wherever they exist. We denote the identity map on $\mathcal{X}$ by $\mathcal{I}$.

Lemma 2.2. *Let* $\mathcal{L}^I \in \mathfrak{L}(\mathcal{Y}, \mathcal{X})$ *such that*

$$\|\mathcal{I} - \mathcal{L}^I \mathcal{N}'(x_0)\| < 1 \tag{5}$$

then

$$\left[\mathcal{L}^I \mathcal{N}'(x_0)\right]^{-1} = \sum_{j=0}^{\infty} \left(\mathcal{I} - \mathcal{L}^I \mathcal{N}'(x_0)\right)^j$$

Proof. Straightforward. □

Assume that $\mathcal{L}^I \in \mathfrak{L}(\mathcal{Y}, \mathcal{X})$ is a fixed operator and satisfies (5). We define

$$\begin{aligned} l \colon &= \left[\mathcal{L}^I \mathcal{N}'(x_0)\right]^{-1} \mathcal{L}^I, \\ \mathcal{M}(x) \colon &= x - l\mathcal{N}(x) \end{aligned} \tag{6}$$

Lemma 2.3. *If* $\mathcal{N}'$ *and* $\mathcal{N}''$ *are continuous linear operators on* S *then for all* $x, y \in S$ *there exist* $z_1 \in [y, x]$ *and* $z_2 \in [x_0, z_1]$ *such that*

$$\mathcal{M}(x) - \mathcal{M}(y) = -l\mathcal{N}''(z_2)[z_1 - x_0][x - y] \tag{7}$$

Proof. Lagrange Mean Value theorem [1] implies

$$
\begin{aligned}
\mathcal{M}(x) - \mathcal{M}(y) &= x - y - \left[\mathcal{L}^I \mathcal{N}'(x_0)\right]^{-1} \mathcal{L}^I \left[\mathcal{N}(x) - \mathcal{N}(y)\right] \\
&= x - y - \left[\mathcal{L}^I \mathcal{N}'(x_0)\right]^{-1} \mathcal{L}^I \mathcal{N}'(z_1)[x - y] \\
&= x - y - \left[\mathcal{L}^I \mathcal{N}'(x_0)\right]^{-1} \left[\mathcal{L}^I \mathcal{N}'(x_0) + \mathcal{L}^I \mathcal{N}'(z_1) - \mathcal{L}^I \mathcal{N}'(x_0)\right] [x - y] \\
&= - \left[\mathcal{L}^I \mathcal{N}'(x_0)\right]^{-1} \mathcal{L}^I \mathcal{N}''(z_2)[z_1 - x_0][x - y]
\end{aligned}
\tag{8}
$$

where $z_1 \in [y, x]$ and $z_2 \in [x_0, z_1]$, and the proof is complete. □

From this point onwards we assume that $\mathcal{N}'$ and $\mathcal{N}''$ are continuous linear operators on S.

Theorem 2.4. *Let $e > 0$ be a real number such that $e \geq \|l\mathcal{N}(x_0)\|$ and let $r > 0$ be such that $\mathcal{B}_{\mathcal{M}(x_0),r} \subset S$. If*

$$\sup_{x \in S} \left\|l\mathcal{N}''(x)\right\| \leq \frac{r}{(r+e)^2}, \tag{9}$$

and if

$$\mathcal{L}^I[y] = 0 \iff y = 0, \tag{10}$$

then there exists a unique solution to

$$\mathcal{N}(x) = 0, \tag{11}$$

in S, and this solution belongs to $\mathcal{B}_{\mathcal{M}(x_0),r}$.

Proof. Set $x_1 = \mathcal{M}(x_0)$. Note that $\|x_1 - x_0\| = \|l\mathcal{N}(x_0)\| \leq e$ from (6). Let $x, y \in S$, then Lemma 2.3, (9) and $\|z_1 - x_0\| \leq \|z_1 - x_1\| + \|x_1 - x_0\| \leq r + e$ imply

$$\|\mathcal{M}(x) - \mathcal{M}(y)\| \leq \left\|l\mathcal{N}''(z_2)\right\| \|z_1 - x_0\| \|x - y\| \leq \frac{r}{(r+e)}\|x - y\| \tag{12}$$

Hence $\mathcal{M}$ is a contractive on S. When $y = x_0$ and $x \in \mathcal{B}_{\mathcal{M}(x_0),r}$, (12) and $\|x - x_0\| \leq \|x - x_1\| + \|x_1 - x_0\| \leq r + e$ imply

$$\|\mathcal{M}(x) - \mathcal{M}(x_0)\| \leq \frac{r}{(r+e)}\|x - x_0\| \leq r\ , \tag{13}$$

Hence $\mathcal{M}\left(\mathcal{B}_{\mathcal{M}(x_0),r}\right) \subset \mathcal{B}_{\mathcal{M}(x_0),r}$. Banach Fixed Point theorem implies the existence of a unique fixed points of $\mathcal{M}$ in $\mathcal{B}_{\mathcal{M}(x_0),r}$. Let x and $\tilde{x}$ be fixed point of $\mathcal{M}$ in S. On the one hand $x = \mathcal{M}(x)$ and $\tilde{x} = \mathcal{M}(\tilde{x})$ imply $x - \tilde{x} = \mathcal{M}(x) - \mathcal{M}(\tilde{x})$, and on the other hand the contraction property of $\mathcal{M}$ on S implies

$$\|\mathcal{M}(x) - \mathcal{M}(\tilde{x})\| < \|x - \tilde{x}\| \tag{14}$$

which is a contradiction. Hence, the fixed point in S must be unique. (6) and (10) imply

$$x = \mathcal{M}(x) \iff \left[\mathcal{L}^I \mathcal{N}'(x_0)\right]^{-1} \mathcal{L}^I \mathcal{N}(x) = 0 \iff \mathcal{L}^I \mathcal{N}(x) = 0 \iff \mathcal{N}(x) = 0$$

and the proof is complete. □

When $r = e$, immediately obtain the following result :

Corollary 2.5. *Under the conditions of Theorem 2.4, if $\mathcal{B}_{\mathcal{M}(x_0),e} \subset S$ and if*

$$4e \sup_{x \in S} \left\| l \mathcal{N}''(x) \right\| \leq 1 \tag{15}$$

then there exists a unique solution of

$$\mathcal{N}(x) = 0$$

in S, and it begins to $\mathcal{B}_{\mathcal{M}(x_0),e}$.

Corollary 2.6. *Let*

$$\mathcal{E} = \mathcal{I} - \mathcal{L}^I \mathcal{N}'(x_0), \tag{16}$$

and let

$$r_0 = \frac{\|\mathcal{N}(x_0)\| \cdot \|\mathcal{L}^I\|}{1 - \|\mathcal{E}\|}. \tag{17}$$

If $\mathcal{B}_{\mathcal{M}(x_0),r_0} \subset S$, and if

$$\sup_{x \in S} \|\mathcal{N}''(x)\| \leq \frac{(1 - \|\mathcal{E}\|)^2}{4\|\mathcal{N}(x_0)\| \cdot \|\mathcal{L}^I\|^2} \tag{18}$$

then there exists a unique solution to

$$\mathcal{N}(x) = 0$$

in S and it belongs to $\mathcal{B}_{\mathcal{M}(x_0),r_0}$.

Proof. The result follows from Corollary 2.5, Lemma 5 and from the inequality

$$\|l\| \leq \left\| \left[\mathcal{L}^I \mathcal{N}'(x_0)\right]^{-1} \right\| \|\mathcal{L}^I\| = \left\| \sum_{j=0}^{\infty} \mathcal{E}^j \right\| \|\mathcal{L}^I\| \leq \sum_{j=0}^{\infty} \|\mathcal{E}\|^j \|\mathcal{L}^I\| = \frac{\|\mathcal{L}^I\|}{1 - \|\mathcal{E}\|} \tag{19}$$

□

3. A Unified treatment for Operators on Multiple Spaces

Throughout the section we assume that $n \geq 1$ is a fixed integer,

$$\mathcal{N}_i \colon \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \to \mathcal{Y}_i$$

are nonlinear operators and $(\mathcal{X}_i, \|\|_{\mathcal{X}_i})$, $(\mathcal{Y}_i, \|\|_{\mathcal{Y}_i})$ are Banach spaces for $i = 1, \cdots, n$. Our aim is to solve

$$\mathcal{N}_i(x_1, \cdots, x_n) = 0 \tag{20}$$

simultaneously for $i = 1, \cdots, n$. The problem can be reframed as

$$\mathcal{N}(\mathbf{x}) = 0$$

for

$$\begin{aligned} \mathcal{X} &= \mathcal{X}_1 \times \cdots \times \mathcal{X}_n, \\ \mathcal{Y} &= \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n, \\ \mathcal{N} &= (\mathcal{N}_1, \cdots, \mathcal{N}_n), \end{aligned} \tag{21}$$

There are many ways to construct norms on $\mathcal{X}$ and $\mathcal{Y}$ by using the norms $\|\|_{\mathcal{X}_i}$ and $\|\|_{\mathcal{Y}_i}$. However, the solutions of (20) are independent of the norms defined on $\mathcal{X}$ and $\mathcal{Y}$. We exploit this freedom to obtain general results with robust conditions.

3.1. **Preliminaries.** We introduce the following preliminary results from linear algebra. To maintain the flow, we skip the proofs here and present them in the Appendix.

Definition 3.1. *We denote by $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_{++}$ the sets of all, non-negative and positive real numbers, respectively. We denote by $\mathbb{R}^n$, $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ the n dimensional vector spaces over $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_{++}$, respectively. We denote the conventional dot product in $\mathbb{R}^n$ by the symbol $\langle \, , \, \rangle$.*

Theorem 3.2 (Perron-Frobenius). *Let M be an $n \times n$ matrix over $\mathbb{R}_{++}$. Then,*

$$M\alpha = \lambda\alpha \ , \ \langle \alpha, \alpha \rangle = 1 \tag{22}$$

has a unique solution in $(\lambda, \alpha) \in \mathbb{R}_{++} \times \mathbb{R}^n_{++}$. It is called the Perron-Frobenius (P-F) solution of M.

Lemma 3.3. *Let Q be an $n \times n$ symmetric matrix over $\mathbb{R}_{++}$ and let (λ, α) be its P-F solution. Then*

$$\langle \mathbf{x}, Q\mathbf{x} \rangle \leq \lambda \langle \mathbf{x}, \alpha \rangle^2 \tag{23}$$

for all $\mathbf{x} \in \mathbb{R}^n_+$.

Lemma 3.4. *Let Q be an $n \times n$ symmetric matrix over $\mathbb{R}_{++}$ and let (λ, α) be its P-F solution. Then*

$$\langle \mathbf{x}, Q\mathbf{y} \rangle^2 \leq \langle \mathbf{x}, Q\mathbf{x} \rangle \langle \mathbf{y}, Q\mathbf{y} \rangle \tag{24}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$.

An immediate corollary of Lemma 3.3 and Lemma 3.4 is:

Corollary 3.5.

$$\langle \mathbf{x}, Q\mathbf{y}\rangle \leq \lambda\langle \mathbf{x}, \alpha\rangle \langle \mathbf{y}, \alpha\rangle \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+ \tag{25}$$

From this point on, we denote $\mathbf{x} \in \mathcal{X}$ by

$$\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_n)$$

Definition 3.6. *Let $\alpha \in \mathbb{R}^n_{++}$ and $p > 0$. We define the α-weighted and L_p norms, respectively, as*

$$\begin{aligned} \|\mathbf{x}\|_\alpha \colon &= \sum_{i=1}^{n} \alpha_i \|\mathbf{x}_i\|_{\mathcal{X}_i}, \\ \|\mathbf{x}\|_p \colon &= \left(\sum_{i=1}^{n} \|\mathbf{x}_i\|^p_{\mathcal{X}_i}\right)^{\frac{1}{p}} \end{aligned} \tag{26}$$

We define

$$\mathrm{abs}(\mathbf{x})\colon = (\|\mathbf{x}_1\|_{\mathcal{X}_1}, \cdots, \|\mathbf{x}_n\|_{\mathcal{X}_n}) \tag{27}$$

When $\mathcal{Z}$ is a Banach space and when $f\colon \mathcal{X} \to \mathcal{Z}$, we denote by

$$\partial_j f = \frac{\partial}{\partial \mathbf{x}_j} f \ , \ f' = \frac{\partial}{\partial \mathbf{x}} f$$

the Fréchet derivative of f with respect to the variable corresponding to $\mathcal{X}_j$ and with respect to $\mathbf{x}$, respectively.

Remark 3.7. *The α-weighted and the 2-norms in (26) are*

$$\begin{aligned} \|\mathbf{x}\|_\alpha &= \langle \mathrm{abs}(\mathbf{x}), \alpha\rangle, \\ \|\mathbf{x}\|_2 &= \langle \mathrm{abs}(\mathbf{x}), \mathrm{abs}(\mathbf{x})\rangle^{\frac{1}{2}}, \end{aligned} \tag{28}$$

Moreover,

$$\begin{aligned} \mathcal{N}'(\mathbf{x}) &= \left[\partial_j \mathcal{N}_i(\mathbf{x})\right]^n_{i,j=1}, \\ \mathcal{N}''_i(\mathbf{x}) &= \left[\partial_j \partial_k \mathcal{N}_i(\mathbf{x})\right]^n_{k,j=1}, \end{aligned}$$

for $i = 1, \cdots, n$.

Similar to the previous section, we assume that $\mathbf{x}_0 \in \mathcal{X}$, $S \subset \mathcal{X}$ is a convex neighborhood of $\mathbf{x}_0$, $\mathcal{N}'$ and $\mathcal{N}''$ continuously exist on S, an explicit $\mathcal{L}^I \in \mathfrak{L}(\mathcal{Y}, \mathcal{X})$ satisfying (5) and (10) is obtained, and l and $\mathcal{M}$ are defined as in (6).

Remark 3.8. *We note that all of these quantities, except perhaps from* S*, are independent of the norms chosen, and for* S *the norms play role only to ensure that* S *is open. Throughout the section we are going to work only with norms that induce the same topology, hence we may assume that* S *is also independent of the norms chosen.*

From this point on we assume that

$$\sup_{x\in S} \|l\,\partial_j\partial_k\mathcal{N}_i(\mathbf{x})\|_{op} < \infty \tag{29}$$

for all $i, j, k \in \{1, \cdots, n\}$.

Definition 3.9. *We define*

$$Q_{j,k}(S) := \left(\sum_{i=1}^{n} \sup_{x\in S} \|l\,\partial_j\partial_k\mathcal{N}_i(\mathbf{x})\|_{op}^2\right)^{\frac{1}{2}} \tag{30}$$

for $j, k \in \{1, \cdots, n\}$*, and*

$$Q(S) = \left[Q_{j,k}(S)\right]_{j,k=1}^{n}.$$

We denote the P-S solution of $Q(S)$ *by* (λ_S, α_S).

Lemma 3.10. *Let* $\beta \in \mathbb{R}^n_{++}$ *with* $\langle \beta, \beta\rangle = 1$*. Then, for each* $\mathbf{x}, \mathbf{y} \in S$ *there exists some* $\mathbf{z} \in [\mathbf{y}, \mathbf{x}]$ *such that*

$$\|\mathcal{M}(\mathbf{x}) - \mathcal{M}(\mathbf{y})\|_\beta \le \lambda\, \|\mathbf{z} - \mathbf{x}_0\|_{\alpha_S}\, \|\mathbf{x} - \mathbf{y}\|_{\alpha_S} \tag{31}$$

Proof. Lemma 2.3 implies

$$\mathcal{M}(\mathbf{x}) - \mathcal{M}(\mathbf{y}) = -l\mathcal{N}''(\mathbf{z}_1)[\mathbf{z} - \mathbf{x}_0][\mathbf{x} - \mathbf{y}]$$

for some $\mathbf{z} \in [\mathbf{y}, \mathbf{x}]$ and $\mathbf{z}_1 \in [\mathbf{x}_0, \mathbf{z}]$. That is

$$\begin{aligned}
\left[\mathcal{M}(\mathbf{x}) - \mathcal{M}(\mathbf{y})\right]_i &= -l\mathcal{N}_i''(\mathbf{z}_1)[\mathbf{z} - \mathbf{x}_0][\mathbf{x} - \mathbf{y}] \\
&= -\sum_{j=1}^{n}\sum_{k=1}^{n} l\,\partial_j\partial_k\mathcal{N}_i(\mathbf{z}_1)\left[\mathbf{z} - \mathbf{x}_0\right]_j\left[\mathbf{x} - \mathbf{y}\right]_k
\end{aligned}$$

for $i = 1, \cdots, n$, and hence,

$$\begin{aligned}
\|\mathcal{M}(\mathbf{x}) - \mathcal{M}(\mathbf{y})\|_\beta &= \langle \mathrm{abs}[\mathcal{M}(\mathbf{x}) - \mathcal{M}(\mathbf{y})], \beta\rangle \\
&= \sum_{i=1}^{n} \beta_i \left\| \sum_{j=1}^{n}\sum_{k=1}^{n} l\,\partial_j\partial_k\mathcal{N}_i(\mathbf{z}_1)\left[\mathbf{z} - \mathbf{x}_0\right]_j \left[\mathbf{x} - \mathbf{y}\right]_k \right\|_{\mathcal{X}_i} \\
&\le \sum_{j=1}^{n}\sum_{k=1}^{n} \left(\sum_{k=i}^{n} \beta_i \|l\,\partial_j\partial_k\mathcal{N}_i(\mathbf{z}_1)\|_{op}\right) \left\|\left[\mathbf{z} - \mathbf{x}_0\right]_j\right\|_{\mathcal{X}_j} \left\|\left[\mathbf{x} - \mathbf{y}\right]_k\right\|_{\mathcal{X}_k}
\end{aligned} \tag{32}$$

Note that the Cauchy Schwartz inequality, $\langle \beta, \beta \rangle = 1$ and (30) imply

$$\sum_{k=i}^{n} \beta_i \left\| \mathfrak{l}\, \partial_j \partial_k \mathcal{N}_i(\mathbf{z}_1) \right\|_{op} \leq \left(\sum_{i=1}^{n} \beta_i^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} \left\| \mathfrak{l}\, \partial_j \partial_k \mathcal{N}_i(\mathbf{z}_1) \right\|_{op}^2 \right)^{\frac{1}{2}} \leq Q_{j,k}(S) \tag{33}$$

Hence, (32), (33) and Corollary 3.5 imply

$$\begin{aligned} \|\mathcal{M}(\mathbf{x}) - \mathcal{M}(\mathbf{y})\|_\beta &\leq \sum_{j=1}^{n} \sum_{k=1}^{n} Q_{j,k}(S) \left\| [\mathbf{z} - \mathbf{x}_0]_j \right\| \left\| [\mathbf{x} - \mathbf{y}]_k \right\| \\ &= \langle \mathrm{abs}(\mathbf{z} - \mathbf{x}_0), Q(S)\, \mathrm{abs}(\mathbf{x} - \mathbf{y}) \rangle \\ &\leq \lambda \langle \mathrm{abs}(\mathbf{z} - \mathbf{x}_0), \alpha_S \rangle \langle \mathrm{abs}(\mathbf{x} - \mathbf{y}), \alpha_S \rangle \\ &= \lambda \|\mathbf{z} - \mathbf{x}_0\|_{\alpha_S} \|\mathbf{x} - \mathbf{y}\|_{\alpha_S} \end{aligned}$$

and the result follows. □

Theorem 3.11. *Let $\mathcal{X}$ be the Banach Space endowed with the norm $\|\ \|_{\alpha_S}$. Let $e > 0$ be a real number such that*

$$\|\mathfrak{l}\mathcal{N}(\mathbf{x}_0)\|_{\alpha_S} \leq e \tag{34}$$

and let $r > 0$ be a real number such that $\mathcal{B}_{\mathcal{M}(\mathbf{x}_0),r} \subset S$. If

$$\lambda_S \leq \frac{r}{(r+e)^2}, \tag{35}$$

then there exists a unique solution of

$$\mathcal{N}(\mathbf{x}) = 0, \tag{36}$$

in S, and this solution belongs to $\mathcal{B}_{\mathcal{M}(\mathbf{x}_0),r}$.

Proof. We obtain $\|\mathcal{M}(\mathbf{x}) - \mathcal{M}(\mathbf{y})\|_{\alpha_S} \leq \lambda \|\mathbf{z} - \mathbf{x}_0\|_{\alpha_S} \|\mathbf{x} - \mathbf{y}\|_{\alpha_S}$ by setting $\beta = \alpha_S$ in Lemma 3.10 and the rest is similar to the proof of the Theorem 2.4. □

Remark 3.12 (Precision test for the Quasi Solution)**.** *It is evident that $\frac{r}{(r+e)^2}$ attains its maximum when $r = e$ where $\frac{r}{(r+e)^2} = \frac{1}{4e}$. Hence, a simple precision test for the quasi solution to check whether*

$$4\lambda_S \|\mathfrak{l}\mathcal{N}(\mathbf{x}_0)\|_{\alpha_S} \leq 1$$

when $S = \mathcal{B}_{\mathcal{M}(\mathbf{x}_0),r}$. If this holds, the result follows from Theorem 3.11 in the case when $r = e$ and $e = \|\mathfrak{l}\mathcal{N}(\mathbf{x}_0)\|_{\alpha_S}$. If not, the precision of the quasi solution needs to be increased. This test, with a compromise, can be reduced to checking

$$4\lambda_S \frac{\|\mathcal{L}^I\|_{\alpha_S}^2 \|\mathcal{N}(x_0)\|_{\alpha_S}}{(1 - \|\mathcal{E}\|_{\alpha_S})^2} \leq 1, \tag{37}$$

when

$$r = \frac{\|\mathcal{L}^{I}\|_{\alpha_S}}{1 - \|\mathcal{E}\|_{\alpha_S}} \|\mathcal{N}(\mathbf{x}_0)\|_{\alpha_S}$$

where

$$\mathcal{E} = \mathcal{I} - \mathcal{L}^{I}\mathcal{N}'(\mathbf{x}_0), \ \mathcal{I}: \ \textit{identity} \tag{38}$$

3.2. **Results for** $(\mathcal{X}, \|\|_2)$.

Theorem 3.13. *Let* $\mathcal{X}$ *be the Banach Space endowed with the norm* $\| \|_2$. *Let* $e > 0$ *be a real number such that*

$$\|l\mathcal{N}(\mathbf{x}_0)\|_2 \leq e \tag{39}$$

and let $r > 0$ *be a real number such that* $\mathcal{B}_{\mathcal{M}(\mathbf{x}_0),r} \subset S$. *If*

$$\lambda_S \leq \frac{r}{(r+e)^2}, \tag{40}$$

then there exists a unique solution to

$$\mathcal{N}(x) = 0, \tag{41}$$

in S, *and this solution belongs to* $\mathcal{B}_{\mathcal{M}(\mathbf{x}_0),r}$.

Proof. When $w \in \mathcal{X}$ Cauchy Schwarz inequality implies

$$\|w\|_{\alpha_S} = \langle \mathrm{abs}(w), \alpha_S \rangle \leq \|w\|_2 \|\alpha_S\|_2 = \|w\|_2 \tag{42}$$

Lemma 3.10 and setting $w = \mathbf{z} - \mathbf{x}_0$ and $w = \mathbf{x} - \mathbf{y}$ in (42) yield

$$\|\mathcal{M}(\mathbf{x}) - \mathcal{M}(\mathbf{y})\|_\beta \leq \lambda_S \|\mathbf{z} - \mathbf{x}_0\|_2 \ \|\mathbf{x} - \mathbf{y}\|_2$$

for all β such that $\langle \beta, \beta \rangle = 1$, and setting $\beta = \frac{\mathrm{abs}(\mathcal{M}(x) - \mathcal{M}(y))}{\|\mathcal{M}(x) - \mathcal{M}(y)\|_2}$ implies

$$\|\mathcal{M}(\mathbf{x}) - \mathcal{M}(\mathbf{y})\|_2 \leq \lambda_S \|\mathbf{z} - \mathbf{x}_0\|_2 \ \|\mathbf{x} - \mathbf{y}\|_2$$

and The rest of the proof can be obtained by following the steps the proof of the Theorem 2.4. □

Corollary 3.14. *Under the conditions of Theorem 3.13, if* $\mathcal{B}_2(\mathcal{M}(x_0), e) \subset S \subset (\mathcal{X}, \|\|_2)$ *and*

$$4\lambda e \leq 1 \tag{43}$$

then there exists a unique solution to $\mathcal{N}(x) = 0$ *in* S, *and this solution is in the ball* $\mathcal{B}_2(\mathcal{M}(x_0), e)$.

Proof. The result is obtained by setting $r = e$ in Theorem 3.11. □

Corollary 3.15 (The 'One Condition' Corollary)**.** *Under the conditions of Theorem 3.11, if* $\mathcal{B}_2\left(\mathcal{M}(x_0), \|\mathfrak{l}\mathcal{N}(x_0)\|_2\right) \subset S \subset (\mathcal{X}, \|\|_2)$ *and*

$$4\lambda \, \|\mathfrak{l}\mathcal{N}(x_0)\|_2 \leq 1 \tag{44}$$

then there exists a unique solution to $\mathcal{N}(x) = 0$ *in* S*, and this solution is in the ball* $\mathcal{B}_2\left(\mathcal{M}(x_0), \|\mathfrak{l}\mathcal{N}(x_0)\|_2\right)$

Corollary 3.16 (The 'One Condition' Corollary version 2)**.** *Under the conditions of Theorem 3.13 and Corollary 3.15 let* $r_1 = \frac{\|\mathcal{L}^{\mathfrak{l}}\|_2}{1-\|\mathcal{E}\|_2} \|\mathcal{N}(x_0)\|_2$*. If* $\mathcal{B}_2\left(\mathcal{M}(x_0), r_1\right) \subset S$ *and*

$$4\tilde{\lambda} \, \frac{\|\mathcal{L}^{\mathfrak{l}}\|_2^2}{(1 - \|\mathcal{E}\|_2)^2} \|\mathcal{N}(x_0)\|_2 \leq 1, \tag{45}$$

then there exists a unique solution to $\mathcal{N}(x) = 0$ *in* S*, and this solution is in the ball* $\mathcal{B}_2\left(\mathcal{M}(x_0), r_1\right)$.

4. Appendix

Proofs:

Proof of Lemma 3.3. The claim holds when $\mathbf{x} = 0$. When $\mathbf{x} \neq 0$, $\langle \alpha, \mathbf{x} \rangle > 0$ as $\alpha \in \mathbb{R}^n_{++}$. The substitution $\mathbf{x} \to \frac{\mathbf{x}}{\langle \alpha, \mathbf{x} \rangle}$ in (23) converts problem into proving

$$\langle \mathbf{x}, Q\mathbf{x} \rangle \leq \lambda$$

when $\langle \alpha, \mathbf{x} \rangle = 1$. Define $f(\mathbf{x}) = \langle \mathbf{x}, Q\mathbf{x} \rangle$ and $P_\alpha = \{\mathbf{x} \in \mathbb{R}^n_+ \colon \langle \alpha, \mathbf{x} \rangle = 1\}$. First we prove f has to attain its maximum at some $\mathbf{x} \in P_\alpha$. Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\mathbf{x} = (x_1, \cdots, x_n)$. $\mathbf{x} \in \mathbb{R}^n_+$ and $\alpha \in \mathbb{R}^n_{++}$ imply

$$1 = \langle \alpha, \mathbf{x} \rangle = \sum_{i=1}^{n} \alpha_i x_i \geq \left(\min_{i=1,\cdots,n} \alpha_i \right) \sum_{i=1}^{n} x_i \geq \left(\min_{i=1,\cdots,n} \alpha_i \right) x_j$$

for all $j = 1, \cdots, n$. $\min_{i=1,\cdots,n} \alpha_i > 0$ as $\alpha \in \mathbb{R}^n_{++}$, hence,

$$0 \leq f(\mathbf{x}) = \sum_{i,j=1}^{n} Q_{i,j} x_i x_j \leq \sum_{i,j=1}^{n} \frac{Q_{i,j}}{\left(\min_{i=1,\cdots,n} \alpha_i\right)^2} < \infty$$

so $f(P_\alpha)$ is bounded. $f(P_\alpha)$ is a closed set in $\mathbb{R}$ since its the image of a continuous function f of a closed set P_α. Hence f attains its maximum for some value $\mathbf{x} \in P_\alpha$. This is on the one hand. On the other hand when the Langrange multipliers method is used for finding the extrema of $f(\mathbf{x})$ under the constraint $\langle \alpha, \mathbf{x} \rangle = 1$, the maximizer(s) should satisfy

$$\frac{\partial}{\partial x_i} f(\mathbf{x}) = 2\tilde{\lambda} \frac{\partial}{\partial x_i} \langle \alpha, \mathbf{x} \rangle \text{ for all } i = 1, \cdots, n \text{ and } \langle \alpha, \mathbf{x} \rangle = 1, \tag{46}$$

for some constant $\tilde{\lambda}$. Note that $f(\mathbf{x}) = \langle \mathbf{x}, Q\mathbf{x}\rangle = \sum_{j=1}^{n}\sum_{k=1}^{n} Q_{j,k}x_j x_k$, hence

$$\frac{\partial}{\partial x_i} f(\mathbf{x}) = 2Q_{i,i}x_i + \sum_{j=1, j\neq i}^{n} (Q_{i,j} + Q_{j,i})x_j = 2\sum_{j=1}^{n} Q_{i,j}x_j \tag{47}$$

due to the symmetry of Q. Likewise, $\frac{\partial}{\partial x_i}\langle \alpha, \mathbf{x}\rangle = \alpha_i$. When these identities are combined one observes that the maximizier(s) should satisfy

$$Q\mathbf{x} = \tilde{\lambda}\alpha \text{ and } \langle \alpha, \mathbf{x}\rangle = 1, \tag{48}$$

and the maximum value is attained at $f(\mathbf{x}) = \langle \mathbf{x}, Q\mathbf{x}\rangle = \langle \mathbf{x}, \tilde{\lambda}\alpha\rangle = \tilde{\lambda}$. This is on the one hand. On the other hand the symmetry of Q implies

$$\tilde{\lambda} = \tilde{\lambda}\langle \alpha, \mathbf{x}\rangle = \tilde{\lambda}\langle \frac{1}{\lambda}Q\alpha, \mathbf{x}\rangle = \frac{\tilde{\lambda}}{\lambda}\langle \alpha, Q^T\mathbf{x}\rangle = \frac{\tilde{\lambda}}{\lambda}\langle \alpha, Q\mathbf{x}\rangle = \frac{\tilde{\lambda}}{\lambda}\langle \alpha, \tilde{\lambda}\alpha\rangle = \frac{\tilde{\lambda}^2}{\lambda} \tag{49}$$

hence $\tilde{\lambda} = \lambda$. Thus,

$$\langle \mathbf{x}, Q\mathbf{x}\rangle \leq \lambda \text{ when } \langle \alpha, \mathbf{x}\rangle = 1. \tag{50}$$

and the proof is complete. □

Proof of Lemma 3.4. If at least one of $\mathbf{x}$ or $\mathbf{y}$ is 0 than the claim holds. We assume $\mathbf{x} \neq 0$ and $\mathbf{y} \neq 0$. Then, $\langle \mathbf{x}, Q\mathbf{x}\rangle$ and $\langle \mathbf{y}, Q\mathbf{y}\rangle$ are positive. The substitution $\mathbf{x} \to \frac{\mathbf{x}}{\langle \mathbf{x}, Q\mathbf{x}\rangle}$ and $\mathbf{y} \to \frac{\mathbf{y}}{\langle \mathbf{y}, Q\mathbf{y}\rangle}$ converts problem into proving

$$\langle \mathbf{x}, Q\mathbf{y}\rangle \leq 1, \tag{51}$$

when $\langle \mathbf{x}, Q\mathbf{x}\rangle = 1$ and $\langle \mathbf{y}, Q\mathbf{y}\rangle = 1$. Let's assume that $\mathbf{y}$ is fixed. We set $f(\mathbf{x}) = \langle \mathbf{x}, Q\mathbf{y}\rangle$ and $P = \{\mathbf{x} \in \mathbb{R}_+^n \colon \langle \mathbf{x}, Q\mathbf{x}\rangle = 1\}$. Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \cdots, y_n)$. Note that $1 = \langle \mathbf{x}, Q\mathbf{x}\rangle > \sum_{i=1}^{n} Q_{ii}x_i^2 > Q_{j,j}x_j^2$ for all $j = 1, \cdots, n$, hence $x_j < \frac{1}{\sqrt{Q_{jj}}}$ for all j. Thus,

$$f(\mathbf{x}) = \langle \mathbf{x}, Q\mathbf{y}\rangle = \sum_{i,j=1}^{n} Q_{i,j}x_i y_j \leq \sum_{i,j=1}^{n} \frac{Q_{i,j}y_j}{\sqrt{Q_{i,i}}} \tag{52}$$

f is continuous and P is a closed set imply $f(P)$ is a closed set in $\mathbb{R}$ and (52) implies $f(P)$ is bounded. Hence f attains its maximum in P. When Lagrange multipliers is implemented, the maximizer(s) should satisfy

$$\frac{\partial}{\partial x_i} f = \frac{\chi}{2}\frac{\partial}{\partial x_i}\langle \mathbf{x}, Q\mathbf{x}\rangle \text{ and } \langle \mathbf{x}, Q\mathbf{x}\rangle = 1$$

for some $\chi > 0$. Note that $\frac{\partial}{\partial x_i} f = \sum_{j=1}^{n} Q_{i,j}y_j$, and that $\frac{\partial}{\partial x_i}\langle \mathbf{x}, Q\mathbf{x}\rangle = 2\sum_{j=1}^{n} Q_{i,j}x_j$ from (47). When these identities are combined, one observes that the maximizer(s) should satisfy

$$Q\mathbf{y} = \chi Q\mathbf{x}$$

and the maximum value is $\mathsf{f}(\mathbf{x}) = \langle \mathbf{x}, \mathsf{Q}\mathbf{y}\rangle = \langle \mathbf{x}, \chi \mathsf{Q}\mathbf{x}\rangle = \chi$. This is on one hand, on the other hand the symmetry of Q implies

$$\chi = \mathsf{f}(\mathbf{x}) = \langle \mathbf{x}, \mathsf{Q}\mathbf{y}\rangle = \langle \mathsf{Q}\mathbf{x}, \mathbf{y}\rangle = \langle \frac{1}{\chi}\mathsf{Q}\mathbf{y}, \mathbf{y}\rangle = \frac{1}{\chi}$$

hence $\chi = 1$ and (51) holds. Since $\mathbf{y}$ is arbitrary, the proof is complete. □

References

[1] P.P. Zabrdjko and. The mean value theorem for differentiable mappings in banach spaces. *Integral Transforms and Special Functions*, 4(1-2):153–162, 1996.