The Existence of a Regular Monochromatic Square in a 2-colored Square Lattice

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Abstract

In this article we prove that every 2-coloring of the square lattice must include a monochromatic square whose corners are on the square lattice and the sides are parallel to the square grid. We generalize this result to tetrominoes of all types.

1 Introduction

When the integer points on the real line are colored by one of k colors, certain patterns begin to emerge independent of the coloring. For instance, every k coloring of integers include an arithmetic sequence of same color integers of desired length thanks to Van der Wearden's Theorem [vdW27]. We are interested in finding out whether similar patterns emerge when the square lattice is colored. We begin by asking one of the basic questions:

Does every 2-coloring of the square lattice have to contain a monochromatic square?

We answer the question as affirmative. We prove the result not only for any square, but for a square whose sides are on the square grid. We then generalize this result for all tetrominoes.

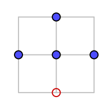
2 Theorem and Proof

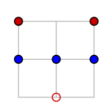
The *square lattice* is the lattice in the coordinate plane consisting of all points with integer coordinates. The *square grid* is the grid of the square lattice. A square whose corners are colored by the same color is called a *monochromatic square*. A square whose sides are on the integer grid is called a *regular square*. A regular square whose corners are n units apart is called a *regular $n \times n$ square*. Throughout the article, we only consider the regular squares.

Theorem 2.1. *Every 2-colored square lattice contains a regular monochromatic square.*

We prove the result by using a method which we refer to as the *color chasing method*. It is an iterative process of coloring a new set of points from an old set, where each point is colored by its unique color to avoid a regular monochromatic square. The process stops when it can be iterated no more. It is noted that the color chasing stops when the process necessarily constructs a regular monochromatic square in the next step.

We find it convenient to illustrate each color chasing procedure by a figure. Throughout the article, we assume the two colors are red and blue. First, we introduce the following simple patterns:

We define an *X-pattern*, visually , a coloring pattern where 3 consecutive points and the top neighbor of the middle point are blue, or any rotations, reflections or color alterations of the described pattern. We note that an X-pattern must have a red point under the middle blue point to avoid a regular monochromatic square, since otherwise would imply the existence of either a regular 1×1 blue square or a regular 2×2 red square.

We define a *Y-pattern*, visually , a coloring pattern on the lattice where 3 consecutive blue points followed by 2 red points on the top of the $1st$ and the $3rd$, or any rotations, reflections or color alterations of the described pattern. We note that a Y-pattern must have a red point under the middle blue point to avoid a regular monochromatic square, since otherwise would imply the existence of either a regular 1×1 blue square or a regular 2×2 red square.

Next, we introduce the following labeling system: In a figure, a point is either unlabeled and belongs to the starting pattern, or labeled by a ‘number’ or a ‘number symbol’ combination, where

number:	the number of the step at which its color uniquely emerges
W:	colored without loss of generality
X:	colored after an X-pattern
Y:	colored after a Y-pattern
a:	colored based on an assumption
b:	colored based on the complement of a previous assumption
$\diamond$:	corner of a regular monochromatic square independent of its color

(1)

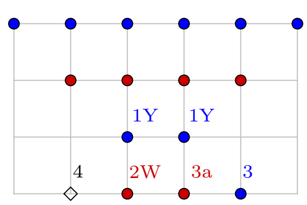
When we investigate the complement of a previous assumption, we include the points whose colors emerge independent of the previous assumption in the starting pattern and we reset the label counter to zero.

Proof. The Van der Wearden Theorem implies the existence of an arithmetic sequence of monochromatic points of any length on the x-axis. Without loss of generality, assume these points are blue and call this line the *blue line*. Also, we assume that the lattice is zoomed out so that two consecutive blue points on the blue line are 1 unit apart.

We investigate the number of consecutive red points that can appear on the horizontal line that is one unit below the blue line. We call this line the *red line* and the number n. We prove our claim for the cases a) $n \geq 4$, b) $n = 3$, c) $n = 2$, d) $n = 1$. The case $n = 0$ is trivial, since it produces a regular 1×1 blue square.

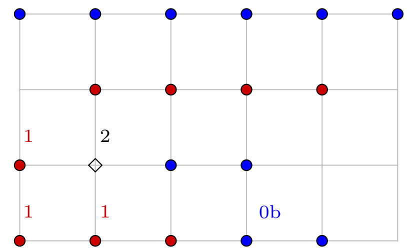
1. $n \geq 4$ case:

   (a) A straightforward color chasing procedure, an assumption and (1) lead to:

   Observe that at least one of the two points under the two 1Y points must be red, as otherwise would imply a regular 1×1 blue square, hence the point 2W is assumed to be red without loss of generality. Then, we assume the other point under 1Y is also red (point 3a). The color chasing process stops at the 3*rd* step, as the point 4 is a corner of either a regular 2×2 red square or a regular 3×3 blue square.

   (b) When we alter the previous assumption, a straightforward color chasing procedure leads to:

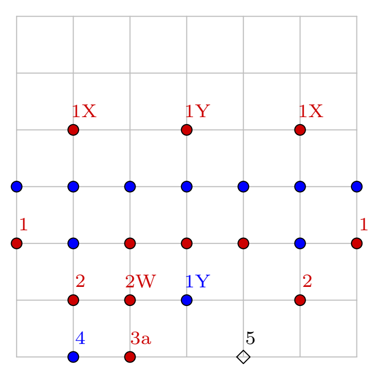
   

   and the chasing stops, as the point labeled by 2 is a corner of either a regular 1×1 red square or a regular 2×2 blue square.

   Hence, there exists a regular monochromatic square when $n \geq 4$.

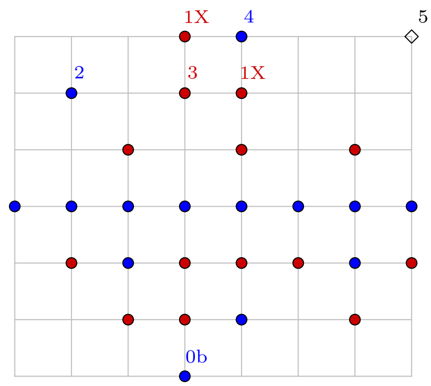
2. $n = 3$ case:

   (a) A straightforward color chasing procedure, an assumption and (1) lead to

   Notice that the point 2W is colored red without loss of generality, as the immediate left and right of the blue point 1Y cannot be blue simultaneously to avoid a regular 2×2 blue square. We assume that the point below 2W is red. Then, the color chasing process stops at step 4, since the point labeled 5 is a corner of a regular 2×2 red square or a regular 3×3 blue square.

   (b) When we alter the previous assumption, the color chasing procedure evolves as follows:

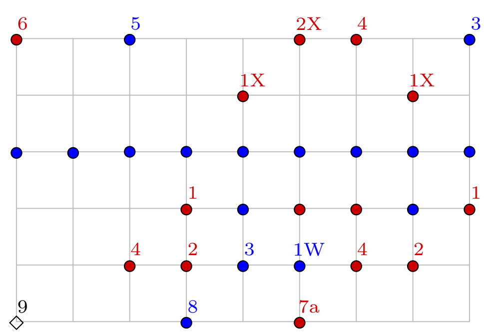


Color chasing stops at step 4 since the point labeled by 5 is a corner of either a regular 3×3 blue square or a regular 4×4 red square.

Therefore, there exists a regular monochromatic square when $n = 3$.

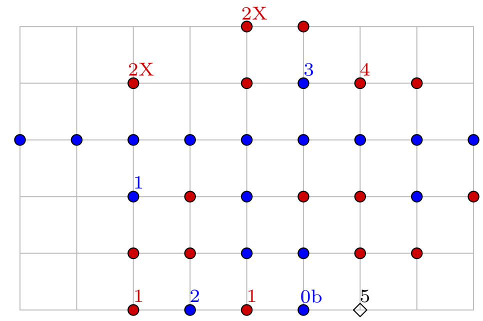
3. $n = 2$ case: At least one the points under the 2 consecutive red points on the red line needs to be blue to avoid a regular 1×1 red square. Without loss of generality assume the left one is blue.

   (a) A straightforward color chasing procedure, an assumption and (1) lead to:



   When we assume the point under 1W is red, the color chasing stops, since the point labeled 9 is a corner of either a regular 3×3 blue square or a regular 5×5 red square.

   (b) When we alter the assumption, the following coloring pattern emerges:

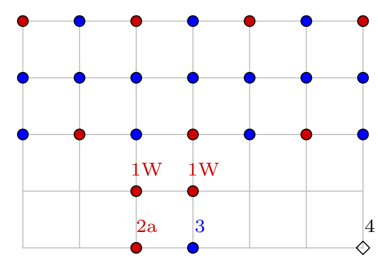


   The point labeled by 5 is a corner of either a regular 3×3 blue square or a regular 4×4 red square.

   Thus, there exists a regular monochromatic square when $n = 2$.

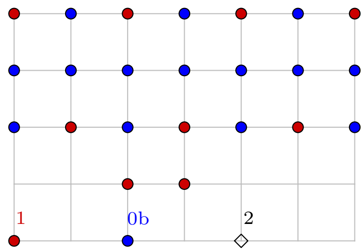
4. $n = 1$ case:

   (a) A straightforward color chasing procedure, an assumption and (1) lead to:



   since $n = 1$ implies the indicated alternating coloring pattern on the red line and its symmetric line with respect to the blue line. Two consecutive red points must occur under the red line since otherwise would imply the existence of three consecutive blue or one red with two blue neighbors, both of which lead to a regular 2×2 blue square. Without loss of generality we denote these points by 1W. Our assumption is that the point below the point 1W which is below a blue point is red (point 2a). Then, the color chasing stops at step 3 since the point 4 is a corner of either a regular 3×3 blue square or a regular 4×4 red square.

   (b) When we alter the assumption, the following coloring pattern emerges:



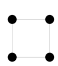
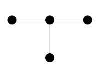
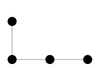
The point labeled 2 is a corner of either a regular 2×2 blue square or a regular 4×4 red square.

Hence, there exists a regular monochromatic square when $n = 1$.

In all cases, there exists a regular monochromatic square and the proof is complete. □

3 Generalization to Tetrominoes

A polyomino is a shape made by connecting certain number of equal-sized squares. A tetromino is a polyomino consisting of 4 squares. In this context, a *tetromino* is a shape composed of 4 neighboring lattice points. Each tetromino, up to translation, rotation, or reflection, is one of the following 5 different tetrominoes: [SWG96]

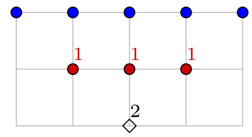
1. straight tetromino,
2. square tetromino, 
3. T-tetromino, 
4. skew tetromino,
5. L-tetromino, 

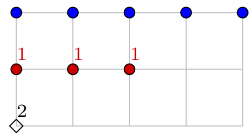
A k *unit tetromino* is a tetromino whose neighboring points are k units apart.

In this section, we generalize the main result for each type of tetromino.

Theorem 3.1. *Every 2-colored square lattice contains a monochromatic tetromino of each type, with all possible rotations and reflections.*

Proof. The existence of a monochromatic straight tetromino follows immediately from the Van der Wearden's theorem, and the existence of a monochromatic square tetromino follows from the previous result. The existence of a monochromatic T-tetromino and a monochromatic L-tetromino is obtained trivially from the following color chasing patterns after starting with 5 consecutive blue points on the blue line:





as each point labeled by 2 belongs to either a 2 unit blue L/T-tetromino or a 1 unit red L/T-tetromino.

We note that the proof shows the existence of specific type of L/T tetromino. The existence of each rotation and reflection of an L/T tetromino can be proven by repeating the same proof with 90, 180 and 270 degrees rotations of the 2-colored square lattice, obtaining the specific L/T tetromino, and reversing it back.

The skew tetromino case is obtained from the existence of a square tetromino in the following way: We observe each of the transformations $(x, y) \to (x - y, y)$, $(x, y) \to (x, x + y)$, $(x, y) \to (x, y + x)$ and $(x, y) \to (x, y - x)$ maps a 2-colored square lattice into a 2-colored square lattice, each of whose inverse transformation maps a square tetromino into a skew tetromino. Aslo note that the resulting skew tetrominoes span all possible rotations and reflections of skew tetrominoes. Hence, the existence of a square tetromino in each of the transformed lattices implies the existence of skew tetrominoes in all possible rotations and reflections in the original square lattice.

All the cases are shown and the proof is complete. □

4 Corollaries

The main result immediately implies the following results:

Corollary 4.1. *Let m, n be integers. Any 2-coloring of the square lattice must contain a regular rectangle whose sides have $m : n$ ratio.*

Proof. When x and y coordinates are zoomed out by factors m and n, respectively, our result implies the existence of a regular monochromatic square in the new lattice, and hence a regular monochromatic rectangle whose sides have $m : n$ ratio in the original square lattice. $\square$

Corollary 4.2. *Any 2-coloring of the triangular lattice contains a monochromatic rhombus.*

Proof. This result follows immediately when a square lattice is formed from rotating the right slanted axis of the 2-colored triangular lattice by 30 degrees left, and using the main result. This transformation maps a (right-slanted) regular rhombus in the triangular grid to a regular square in the square grid. $\square$

Corollary 4.3. *Any 2-coloring of the triangular lattice contains a monochromatic right slanted, a left slanted and a diamond rhombus.*

Proof. The 60 and 120 degree rotations of a 2-colored triangular lattice is a 2-colored triangular lattice. The result follows from this fact and from the previous result. $\square$

Corollary 4.4. *Let m, n be positive integers. Then, any 2-coloring of the triangular lattice must contain a monochromatic parallelogram whose sides have $m : n$ ratio.*

Proof. This result follows immediately from Corollary 4.1 and Corollary 4.2. $\square$

5 Possible Extensions

Similar results may exist in contexts regarding to k coloring of lattices. We find the following questions interesting, since the answers have potential to unveil novel monochromatic patterns.

1. A pentomino is a polyomino consisting of 5 points. It is known that there are 12 different types of pentominoes. Does any 2-coloring of the square lattice necessarily include a monochromatic pentomino of each type?
2. Does any 2-coloring of the square lattice necessarily include a 2×3 monochromatic sub-lattice? How about an $2 \times n$ monochromatic sub-lattice? Or an $m \times n$ sub-lattice?
3. Does any 2-coloring of the triangular lattice necessarily have a monochromatic regular hexagon?
4. Does any 2-coloring of the triangular lattice necessarily have a monochromatic equilateral triangle and the midpoints of its sides?
5. Does any 2-coloring of the hexagonal lattice necessarily have a monochromatic hexagon?
6. Does any 2-coloring of the lattice in 3-D necessarily include a monochromatic cube? How about a regular monochromatic cube?
7. Does any 3-coloring of the square lattice necessarily include a monochromatic square? How about a regular monochromatic square?
8. Let k be a positive integer. Does any k-coloring of the lattice necessarily include a monochromatic square, or is there a counterexample? How about a regular monochromatic square?
9. Does any k-coloring of the triangular lattice necessarily have a monochromatic equilateral triangle? What about a monochromatic rhombus?

The definitive answers to these questions, whether positive or negative, may prove to be very useful in a number of contexts.

References

[SWG96] Warren Lushbaugh Solomon W. Golomb. *Polyominoes: puzzles, patterns, problems, and packings.* Princeton University Press, 2nd edition, 1996.

[vdW27] Bartel Leendert van der Waerden. Beweis einer baudetschen vermutung. *Nieuw. Arch. Wisk.*, 15:212–216, 1927.