A GENERALIZED CAUCHY-SCHWARTZ INEQUALITY

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Abstract. In this brief article, I present a proof of a generalized form of the Cauchy-Schwarz inequality. Notably, the argument employs the forward-backward induction technique, originally introduced by Cauchy in his proof of the arithmetic mean?geometric mean inequality. This approach highlights a natural connection between the result and its historical origins.

Theorem 0.1. *Let* K *and* n *be positive integers, let* $a_{i,k} \geq 0$ *be real numbers for* $i = 1, \cdots, n$ *and* $k = 1, \cdots, K$. *Then*

$$\prod_{i=1}^{n}\left(\sum_{k=1}^{K} a_{i,k}^{n}\right) \geq \left(\sum_{k=1}^{K}\prod_{i=1}^{n} a_{i,k}\right)^{n} \tag{1}$$

Remark 0.2. *When* $n = 2$ *the inequality is equivalent to Cauchy-Schwartz inequality.*

Proof. It is observed that when $n = 2^N$, (1) holds after applying the Cauchy-Schwartz inequality for sufficiently many times:

(2)

$$\prod_{i=1}^{2^N}\left(\sum_{k=1}^{K} a_{i,k}^{2^N}\right) = \prod_{i=1}^{2^{N-1}}\left[\left(\sum_{k=1}^{K} a_{i,k}^{2^N}\right)\left(\sum_{k=1}^{K} a_{i+2^{N-1},k}^{2^N}\right)\right] \geq \prod_{i=1}^{2^{N-1}}\left[\left(\sum_{k=1}^{K}(a_{i,k}a_{i+2^{N-1},k})^{2^{N-1}}\right)^{2}\right]$$

$$\geq \prod_{i=1}^{2^{N-2}}\left(\sum_{k=1}^{K}(a_{i,k}a_{i+2^{N-2},k}a_{i+2\cdot 2^{N-2},k}a_{i+3\cdot 2^{N-2},k})^{2^{N-2}}\right)^{2^2} \geq \cdots \geq \left(\sum_{k=1}^{K}\prod_{i=1}^{2^N} a_{i,k}\right)^{2^N}.$$

Secondly, let $2^N < n < 2^{N+1}$. Applying (2) with

$$a_{i,k} \to \begin{cases} a_{i,k}^{\frac{n}{2^{N+1}}} & \text{if } i \leq n \\ \prod_{i=1}^{n} a_{i,k}^{\frac{1}{2^{N+1}}} & \text{if } 2^{N+1} > i > n \end{cases} \tag{3}$$

one obtains

$$\prod_{i=1}^{n}\left(\sum_{k=1}^{K} a_{i,k}^{n}\right)\left[\sum_{k=1}^{K}\prod_{i=1}^{n} a_{i,k}\right]^{2^{N+1}-n} \geq \left[\sum_{k=1}^{K}\prod_{i=1}^{n} a_{i,k}\right]^{2^{N+1}} \tag{4}$$

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and the result is obtained after cancellation, completing the proof. □