The Existence of a Monochromatic P-pentomino in a Bicolored Integer Lattice

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Abstract

Any 2-coloring of the integer lattice must include 5 monochromatic points which constitute a P-pentomino, that is, a square combined with one of the neighboring points. In this article we present a proof of this result.

1 Introduction

When the integer lattice points are colored by one of k colors, certain patterns begin to emerge independent of the coloring. In [] it is proved that any 2-coloring of the integer lattice must have monochromatic 4 points which constitute a square whose sides are parallel to the lattice.

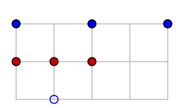
In this article we prove a stronger result: we show that any 2-coloring of the lattice must include a P-pentomino.

The proof mainly based on separating a coloring into cases and analyzing each case by a color chasing procedure.

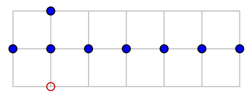
2 Theorem and the Proof

The integer lattice is the lattice in the coordinate axis which consists of all points whose both coordinates are integers. A P-pentomino is set of 5 points on the integer lattice 4 of which are corners of a square whose sides parallel to the lattice, and the fifth is a neighboring point of one points. We call a k P-pentomino when its corresponding square is a $k \times k$ P-pentomino.
Theorem 2.1. *Any 2-coloring of the integer lattice has a monochromatic P-pentomino.*

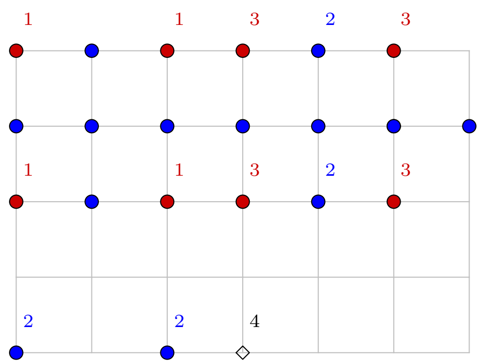
Proof. We assume that each point of the integer lattice is colored either blue or red. From Van der Wearden Theorem we know that we can get an arithmetic sequence of same colored dots of any length. Without loss of generality assume these points are blue. For convenience we will call this line that has the arithmetic sequence of blue dots of any length, the blue line and we will ignore any points in between the points that form the arithmetic sequence and adjust our grid accordingly. We look at the line that is parallel to the blue line where the distance between the lines is the same as the common distance of the arithmetic sequence of the same colored blue dots, call this the red line. We call the longest consecutive sequence of red dots on the red line by n. We separate n into cases: a) $n \geq 4$, b) $n = 3$, c) $n = 2$, d)$n = 1$ and work case by case. In each case we will present a color chasing procedure to obtain a monochromatic P-pentomino. In the case $n = 0$ we trivially have a monochromatic P-pentomino.

First, we introduce some patterns. We define a *Y-pattern*, visually , a coloring pattern on the lattice where 3 consecutive blue points of 2 units apart followed by 2 red points of one unit apart where the first red point is one unit under the first blue, or any color alterations or rotations of the described pattern.

Note that in a Y-pattern, the point under the $2nd$ red must be a blue, since otherwise would imply the existence of a red 1 P-pentomino or a blue 2 P-pentomino.

We define a *cross pattern*, an *X-pattern*, visually , a coloring pattern which consists of 8 blue points on the lattice 7 of which are consecutive points on a line and the $8th$ is 1-unit above to the $2nd$ point in the line. Any rotations, reflections or color alterations of the described pattern is also referred to as a *cross pattern*.

Note that if the reflection of this blue point across the blue line is also blue then the following pattern emerges trivially to prevent a monochromatic P-pentomino:



where the number on each point represents the step at which the point is colored in a unique way to avoid monochromatic P-pentominoes.

Then point 4 is either a part of a red 2 P-pentomino or a part of a blue 3 P-pentomino. This implies that any reflection of a blue point over the blue line must be a red point to avoid a monochromatic P-pentomino.

We are going to present the coloring patterns using figures. For convenience We also label each point either with a 'number' or a 'number letter', where

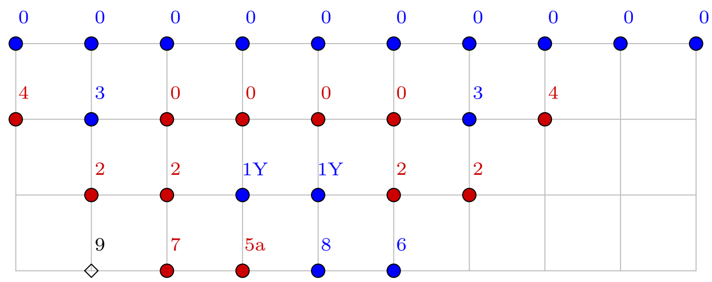
number:	the number of the step at which the coloring is done
X:	colored due to an X-pattern
Y:	colored due to a Y-pattern
W:	colored without loss of generality
a:	colored based on an assumption
b:	colored based on the complement of a previous assumption
no #:	colored based on no # consecutive monochromatic in row can occur due to the previous case for n
$\diamond$:	part of a monochromatic P-pentomino independent of its color

(1)

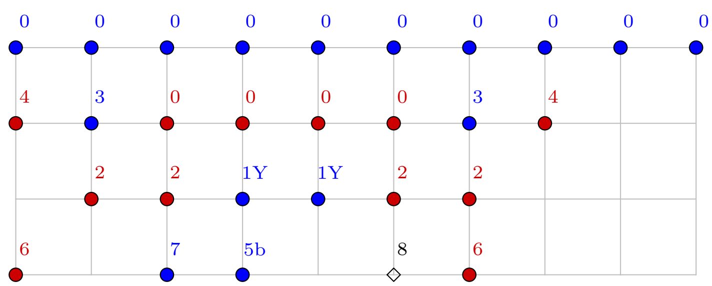
We proceed case by case:

1. $n \geq 4$ case: We assume there are at least 4 consecutive red points under the blue line. Then, the points under the middle two red points must be blue (points labeled by 1(Y) in the next figure), otherwise there exists a Y-pattern and hence a regular unicolor square. The two points under these blue points cannot be blue at the same time since this would construct a 1×1 blue regular unicolor square. Hence, at least one of these points must be red. We split this into two cases: a) both of them are red, b) one red one blue.

(a) We assume both points are red (labeled by 2(A) in the figure below). Then, the right point to these red points must be blue (labeled by 3) since otherwise would imply a regular 2×2 red square. But then the point to the left of these points (labeled by 4) is a corner of either a regular 2×2 red square or a regular 3×3 blue square.



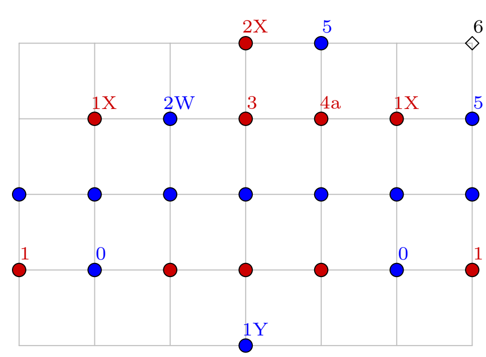
(b) We assume one of the points is blue and the other is red. By symmetry we may assume the left point is red and the right point is blue. This immediately implies the following unique coloring pattern shown in the figure when regular unicolor squares are avoided.



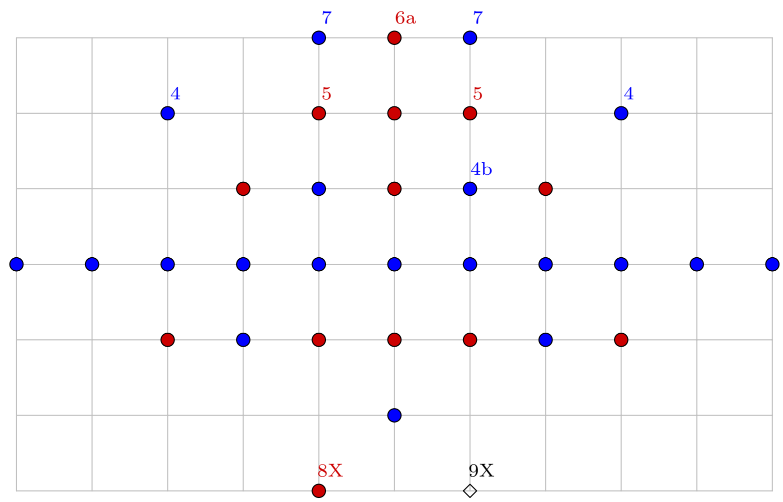
Then, the point labeled by 3 is a corner of either a regular 2×2 blue square or a regular 2×2 red square. Hence, we always get a regular unicolored square when $n \geq 4$.

2. $n = 3$ case: When there are at most 3 consecutive red points on the line below blue line the following coloring pattern emerges uniquely (points labeled by 0 through 2) thanks to a straightforward color chasing and (1). There are two possibilities, the point labeled by 3(A) is a) red, or b) blue,

(a) When it is red, straightforward color chasing implies the point labeled by 4 is blue, and hence the point labeled by 5 is a corner of either a regular 4×4 blue square or a regular 2×2 red square.

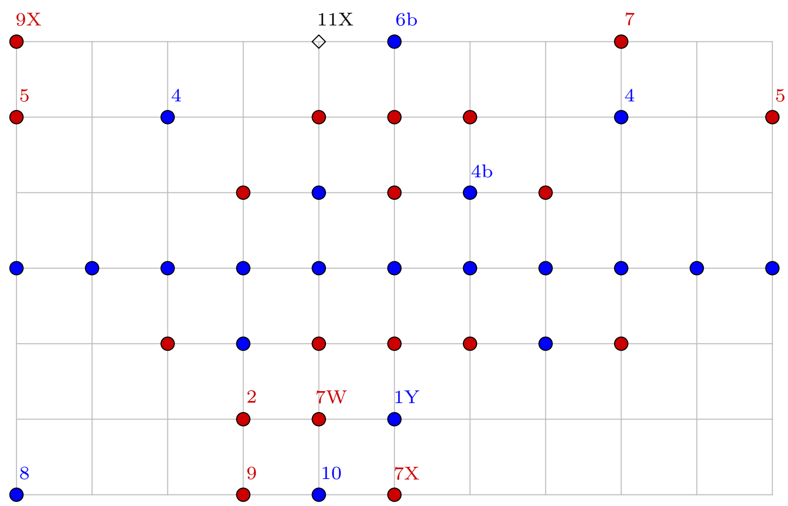


(b) When it is blue, an iterative color chasing procedure determines the colors of the points immediately until the point labeled by 8*. This point must be colored blue, since otherwise would imply a 4 reds in a row above a blue line. After this step, the remaining coloring pattern emerges immediately.



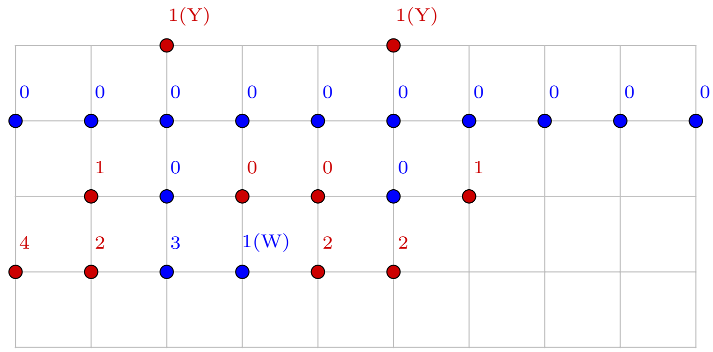
The point labeled by 10 is a corner of either a regular 1×1 red square or a regular 4×4 blue square.

(c)



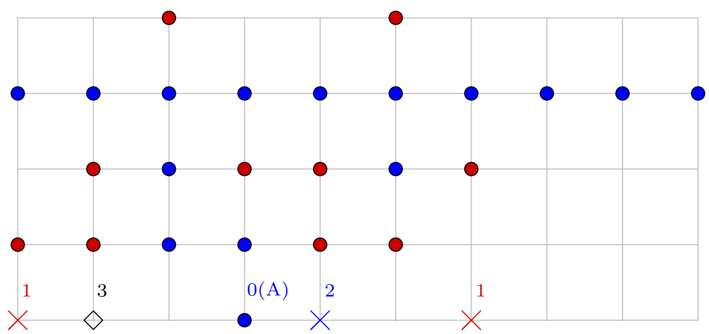
Therefore, we always get a regular unicolored square when $n = 3$.

3. $n = 2$ case: In this case the following coloring pattern emerges uniquely



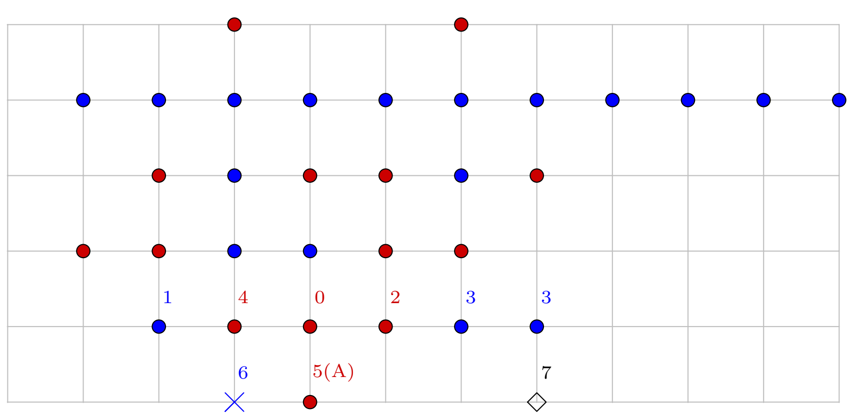
There are two possibilities for the point under 1(W); blue or red.

(a) When this point is blue; the following coloring pattern emerges. The point labeled by 3 is a corner of either a regular 3×3 blue square or a 1×1 red square.



Hence, we may assume the point is red. Then the coloring pattern follows trivially up to point labeled 4 in the following figure. A sequence of 3 consecutive red points are formed so we investigate the possibilities for the point under the middle one (0). There are two possibilities; red or blue.

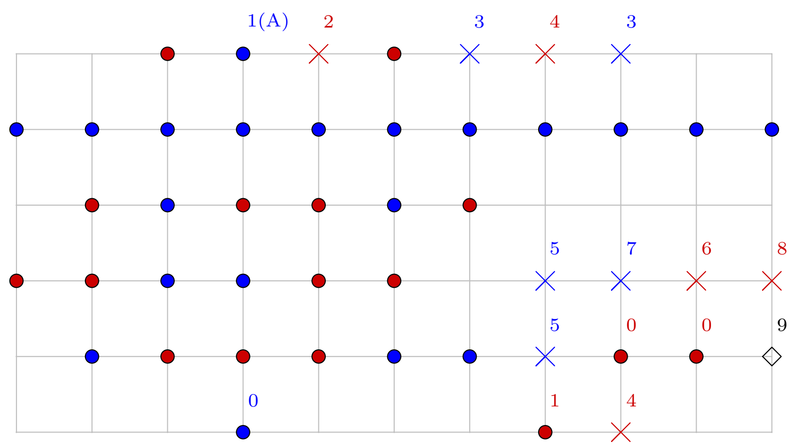
(b) When the point is red, the following coloring pattern emerges immediately;



Thus, the point labeled 7 is either a corner of a regular 3×3 red square or a regular 4×4 blue square.

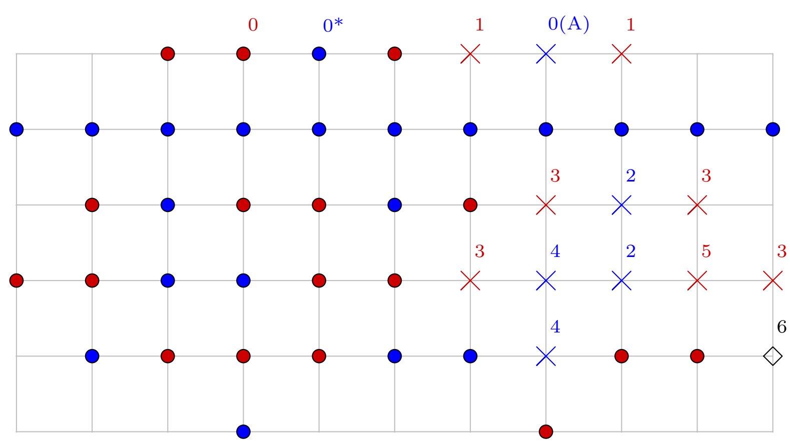
Hence, we assume that the point 5(A) in the previous figure is blue. We turn our focus to the points labeled 1(A) and 2 above the blue line in the next figure: These points cannot be both red as that would imply $n \geq 4$, and cannot be both blue as that would yield a regular 1×1 blue square. We make our assumptions on the left one labeled 1(A), it is either blue or red.

(c) When it is blue, the following coloring pattern emerges immediately;



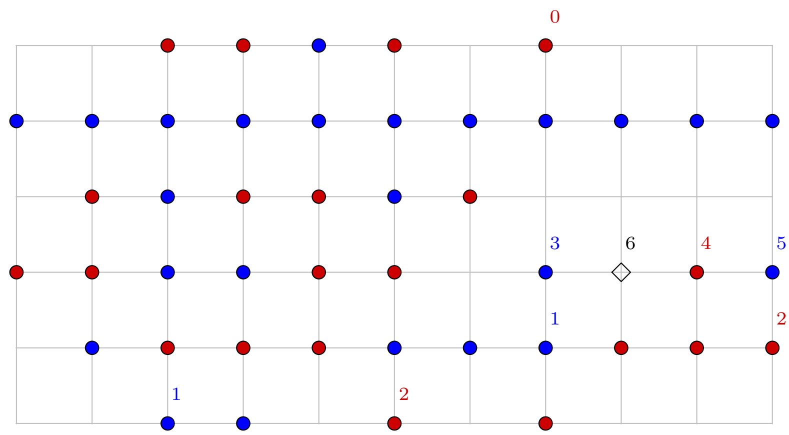
The point labeled 9 is a corner of either a regular 1×1 red square or a regular 3×3 blue square.

Hence we may assume that the point 1(A) in the previous figure is red. Next we assume the point 0(A) in the following figure is blue. Then it follows:



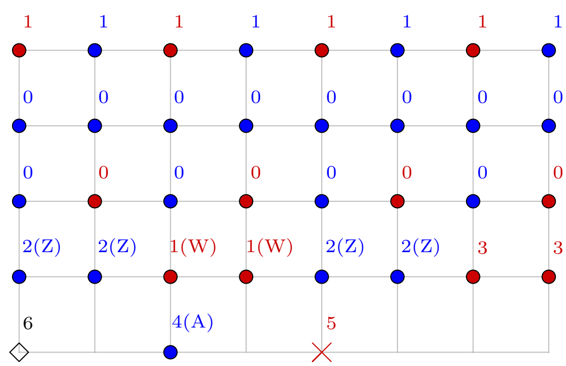
The point labeled 6 is either a corner of regular 1×1 red square or a regular 3×3 blue square.

Thus we may assume that the point 0(A) in the previous figure is red. From this assumption, the following coloring pattern emerges:



The point labeled 6 is a corner of either a regular 1×1 red square or a regular 2×2 blue square. Thus, we always get a regular unicolored square when $n = 3$.

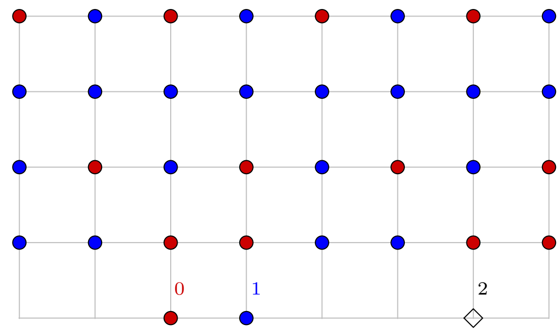
4. $n = 1$ case: In this case the line below and above the blue line must be colored in an alternating order, and two lines should have shifted patterns when regular unicolored squares are avoided. The line below the red line must contain two consecutive red points since otherwise would imply two neighboring points of a red point are blue which forms a regular 2×2 blue square. Hence, without loss of generality we assume consecutive red points under the red line as it is shown in the following figure.



The points labeled by 2(Z) must be blue, since otherwise would imply the $n = 2$ case when we zoom out the lattice by 2 units. (This step is legitimate since we can have as many blue points on the blue line as we require due to the Van Der Wearden's theorem.) After an obvious color chasing, we observe that the line under the red line has a coloring pattern where 2 blue points are followed by 2 red points so long as the patterns above the two lines continue.

(a) We investigate the point labeled 4(A) in the previous figure. If it is blue the point labeled 6 is a corner of either a regular 2×2 blue square or a regular 4×4 red square. Hence we may assume the point 4(A) in the previous figure is red.

(b) If it is red, it immediately follows that 1 is blue.



So the point 2 is a corner of either a regular 3×3 blue square or a regular 4×4 red square.

Hence we always have a regular unicolored square when $n = 1$.

In all cases there exists a regular unicolored square and the proof is complete. □

3 Possible Extensions

A similar kind of result may be true for the following questions:

Question 1: Does any 3-coloring of the lattice necessarily include a unicolor square?

Question 2: Does any 3-coloring of the lattice necessarily include a regular unicolor square?

Question 3: Does any k-coloring of the lattice necessarily include a regular unicolor square, or is there a counterexample? How about a unicolor square?

Question 4: Does any 2-coloring of the lattice always include a 2×3 unicolored sub-lattice? How about an $m \times n$ unicolored sub-lattice?

The definitive answers to these questions, positive or negative, may prove to be very useful in a number of contexts.