Homework1

Ali Arous

Note: for some strange reason with my windows operating system, I could not make a compressed file (.zip) containing the two files (.pdf and .ipynb). so I have uploaded the jupyter notebook on the following url: https://ali-arous.github.io/db/homework1_aliarous.ipynb

Exercise 1:

Since we know that the number X1 of heads is a binomial r.v: *X1 ~ B(n, ½),*

$$P(X1 = x1\ ) = \binom{n}{x1}\ p^{x1}\ (1-p)^{(n-x1)}$$

$$= \frac{\binom{n}{x1}}{2^n}$$

1)

a) Find the pmf of X2 conditional to a given value of X1

$$P(X2 = x2 \mid X1 = x1\ ) = \binom{n-k1}{x2}\ p^{x2}\ (1-p)^{(n-x1-x2)} = \frac{\binom{n-x1}{x2}}{2^{(n-x1)}}$$

b) Find the joint pmf of (X1, X2)

$$P(\{X1 = x1, X2 = x2\}\ ) = \ P(X1 = x1)\,.\,P(X2 = x2 \mid X1 = x1)$$

$$= \frac{\binom{n}{x1}}{2^n} \cdot \frac{\binom{n-x1}{x2}}{2^{(n-x1)}} = \frac{\binom{n}{x1}\binom{n-x1}{x2}}{2^{(2n-x1)}}$$

c) Find the joint pmf of (X1, X2, R2)

$$P(\{X1 = x1\,, X2 = x2, R2 = r2\})$$
$$= P(\{X1 = x1, X2 = x2\})\,.\,P(R2 = r2 \mid X1 = x1\,, X2 = x2)$$
$$= \frac{\binom{n}{x1}\binom{n-x1}{x2}}{2^{(2n-x1)}}\,.\,1$$

d) Find the marginal pmf of X2 and the marginal pmf of R2

$$P(X2 = x2\,) = \sum_{i=0}^{n} \frac{\binom{n}{i}\binom{n-i}{x2}}{2^{(2n-i)}}$$

$$P(R2 = r2\,) = \sum_{i=0}^{n} P(\{X1 = i, X2 = n - i - r2\}\,) = \sum_{i=0}^{n} \frac{\binom{n}{i}\binom{n-i}{n-i-r2}}{2^{(2n-i)}}$$

2)

The pmf of X3 conditional to given values of X1, X2:

$$P(X3 = x3 \mid X2 = x2, X1 = x1\,)$$
$$= \binom{n - x1 - x2}{x3} p^{x3}\,(1-p)^{(n-x1-x2-x3)} = \frac{\binom{n-x1-x2}{x3}}{2^{(n-x1-x2)}}$$

The pmf of X_k conditional to given values of X_i where i = 1,…,k-1:

$$P(X_k = x_k \mid X_1 = x_1, X_2 = x_2\,, \dots, X_{k-1} = x_{k-1})$$
$$= \binom{n - \sum_{i=1}^{k-1} x_i}{x_k} p^{x3}\,(1-p)^{(n-\sum_{i=1}^{k} x_i)} = \frac{\binom{n-\sum_{i=1}^{k-1} x_i}{x_k}}{2^{(n-\sum_{i=1}^{k-1} x_i)}}$$

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The joint pmf of (X1, X2, X3):

$$P(\{X1 = x1, \quad X2 = x2, \quad X3 = x3\})$$

$$= P(X1 = x1) \,.\, P(X2 = x2, X3 = x3 \mid X1 = x1)$$

$$= P(X1 = x1) \,.\, P(X2 = x2). P(X3 = x3 \mid X2 = x2, X3 = x3)$$

$$= \frac{\binom{n}{x1}}{2^n} \cdot \frac{\binom{n-x1}{x2}}{2^{(n-x1)}} \cdot \frac{\binom{n-x1-x2}{x3}}{2^{(n-x1-x2)}}$$

The joint pmf of (X_1, X_2,…, X_k):

$$P(\{X1 = x1, X2 = x2, \ldots, X_k = x_k\}) = \prod_{i=1}^{k} \frac{\binom{n-\sum_{j=1}^{i-1} x_j}{x_i}}{2^{(n-\sum_{j=1}^{i-1} x_j)}}$$

The joint pmf of (X_1, X_2,…, X_k, R_k):

$$P(\{X1 = x1, X2 = x2, \ldots, X_k = x_k, R_k = r_k\}) = 1. \prod_{i=1}^{k} \frac{\binom{n-\sum_{j=1}^{i-1} x_j}{x_i}}{2^{\left(n-\sum_{j=1}^{i-1} x_j\right)}}$$

While $\quad r_k = n - \sum_{i=1}^{k} x_i \quad$ holds, otherwise pmf $= 0$.

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The marginal pmf of X3 and the marginal pmf of R3

$$P(X3 = x3) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \frac{\binom{n}{i}}{2^n} \cdot \frac{\binom{n-i}{j}}{2^{(n-i)}} \cdot \frac{\binom{n-i-j}{x3}}{2^{(n-i-j)}}$$

$$P(R3 = r3) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} P(\{X1 = i, X2 = j, X3 = n - i - j - r3\})$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n-i} \frac{\binom{n}{i}}{2^n} \cdot \frac{\binom{n-i}{j}}{2^{(n-i)}} \cdot \frac{\binom{n-i-j}{n-i-j-r3}}{2^{(n-i-j)}}$$

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The marginal pmf of Xk:

$$P(X_k = x_k) = \sum_{i_1+i_2+,..+i_{k-1\leq n}}^{n} \quad \prod_{i_1+i_2+,..+i_{k\leq n}}^{n} \cdot \frac{\binom{n-\sum_{j=1}^{k-1} i_j}{i_k}}{2^{(n-\sum_{j=1}^{k-1} i_j)}} \cdot \frac{\binom{n-\sum_{j=1}^{k} i_j}{x_k}}{2^{(n-\sum_{j=1}^{k} i_j)}}$$

3) Define the r.v. Y = "Total number of tosses". Obtain the cdf of Y .

Each coin toss is a Bernoulli experiment.

The total number of tosses (m) required to get (n) heads out of (n) coins is:

A random variable Y ~ Negative Binomial Distribution:

$$f(m; n, p) = P(Y = m) = \binom{m-1}{n-1} p^n \ (1-p)^{m-n}$$

$$f\left(m; n, \frac{1}{2}\right) = P(Y = m) = \binom{m-1}{n-1} \frac{1}{2^m}$$

The cdf of Y is:

$$F(m) = P(Y \leq m) = \sum_{m=1}^{\infty} \binom{m-1}{n-1} \frac{1}{2^m}$$

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