# Simulation 03

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## Basic acceptance-rejection algorithm

A straightforward implementation of the basic acceptance-rejection algorithm. Code for generating a sequence of random numbers following a given probability distribution, with parameters: - `n` is the sequence length, - `f` is the target pdf, - `g` is the candidate pdf, - `rg` is a function that generates random numbers following `g`. - `c` is the constant such that `f(x)<=c * g(x)` for all `x`.

The function returns a list with `z`, the sequence of random numbers, and `Proportion` of accepted values.

```
Accept.Reject<-function(f,g,rg,c,n=1.0e4){
    z<-rep(0,n)
    N<-0
    for (i in 1:n){
        accept<-FALSE
        while (accept == FALSE){
            u <- runif(1)
            v <- rg(1)
            N<-N+1
            if (c*u <= f(v)/g(v)){
                z[i] <- v
                accept <- TRUE
                }
            }
        }
    Proportion<-n/N
    return(list(z=z,Proportion=Proportion))
    }
```

### Exercise 1

Generate a sequence of random numbers following a $\mathrm{Beta}(2,2)$ using a $\mathrm{Unif}(0,1)$ candidate pdf.

The target pdf `f(x)`is:

$$f(x) = 6\,x\,(1-x) = 6\,(x-x^2), \quad x \in (0,1).$$

The candidate $\mathrm{Unif}(0,1)$ pdf is:

$$g(x) = 1, \quad x \in (0,1).$$

The steps are:

1. Calculate `c` such that `f(x)<=c * g(x)`, *[Hint: plot both `f(x)`and `g(x)`]*,
2. Use the `Accept.Reject` function above (or some improvement thereof) for several `n` values,
3. Plot histograms of the resulting sequences, superimposing the graph of `f(x)`,
4. Compare the actual acceptance rate to the thoretical value `1/c`.

```
#
#    Insert here your code
#
```

**Exercise 2**

Generate a sequence of random numbers following a $\mathrm{Beta}(2,3)$ using a $\mathrm{Unif}(0,1)$ candidate pdf.

The target pdf `f(x)`is:

$$f(x) = 12\,x\,(1-x)^2, \quad x \in (0,1).$$

The candidate $\mathrm{Unif}(0,1)$ pdf is:

$$g(x) = 1, \quad x \in (0,1).$$

The steps are:

1. Calculate `c` such that `f(x)<= c * g(x)`,

   *Hint: as above, plot both `f(x)`and `g(x)`, then locate the maximum of `f(x)` using code such as:* "' optimize(f=function(x){dbeta(x,2,3)},interval=c(0,1),maximum=TRUE)

   "'

2. Use the `Accept.Reject` function above (or some improvement thereof) for several `n` values,
3. Plot histograms of the resulting sequences, superimposing the graph of `f(x)`,
4. Compare the actual acceptance rate to the thoretical value `1/c`.

```
#
#    Insert here your code
#
```

## Acceptance-rejection algorithm with a candidate pdf

**Exercise 3 - Beta(2,3) with a triangular pdf candidate**

Generate a sequence of random numbers following a $\mathrm{Beta}(2,3)$ using a triangular candidate pdf.

The target pdf `f(x)`is:

$$f(x) = 12\,x\,(1-x)^2, \quad x \in (0,1).$$

The triangular pdf `g(x)` is constructed along the following steps:

1. Plot `f(x)` and the tangent line `L1` to `f(x)` at `(0,0)`

   `![Alt text](./Beta.2.3.jpg)`

2. Find the equation of the line `L2`through the point `(1,0)` and tangent to `f(x)`
3. The piecewise linear function with domain `(0,1)` defined by `L1` and `L2` will be the majorizing function:

$$c \cdot g(x) = \begin{cases} 12\,x, & \text{if} \quad x \in (0, 1/5), \\ 3 - 3\,x, & \text{if} \quad x \in (1/5, 1). \end{cases}$$

4. Compute $c$ by imposing the condition that $g$ must be a pdf, that is, its integral must be equal to 1. *Hint: compute $c \cdot G(x)$, the integral of $c \cdot g$ on $(-\infty, x]$, since anyway you need the cdf $G(x)$ to obtain the quantile function*

Result is: $c = 6/5$. Thus, the candidate pdf is:

$$g(x) = \left\{ \begin{array}{lll} 10\,x, & \text{if} & x \in (0, 1/5), \\ \frac{5}{2}\,(1-x), & \text{if} & x \in (1/5, 1). \end{array} \right.$$

5. Compute the quantile function $Q(y) = G^{-1}(y)$ and use it to write a function `rg` implementing a random number generator for the candidate $g$ density by the inverse transformation method. Check that it works OK (by comparing the `hist` of its output with the triangular pdf `g`)

6. Use the `Accept.Reject` function above (or some improvement thereof) for several `n` values,

7. Plot histograms of the resulting sequences, superimposing the graph of `f(x)`,

8. Compare the actual acceptance rate to the thoretical value `1/c`.

```
#
#    Insert here your code
#
```

**Exercise 4 - Normal random numbers from a Laplace distribution**

Normal $\sim \mathrm{N}(0, 1)$ from a Laplace($\alpha$)

Assume we want to generate RN following the $\mathrm{N}(0, 1)$ distribution, whose pdf is:

$$f(x) = \frac{1}{\sqrt{2\,\pi}}\,\exp\left\{-\frac{1}{2}\,x^2\right\}.$$

Our candidate pdf is in the Laplace($\alpha$) family:

$$g(x|\alpha) = \frac{\alpha}{2}\,\exp\left\{-\alpha\,x\right\}.$$

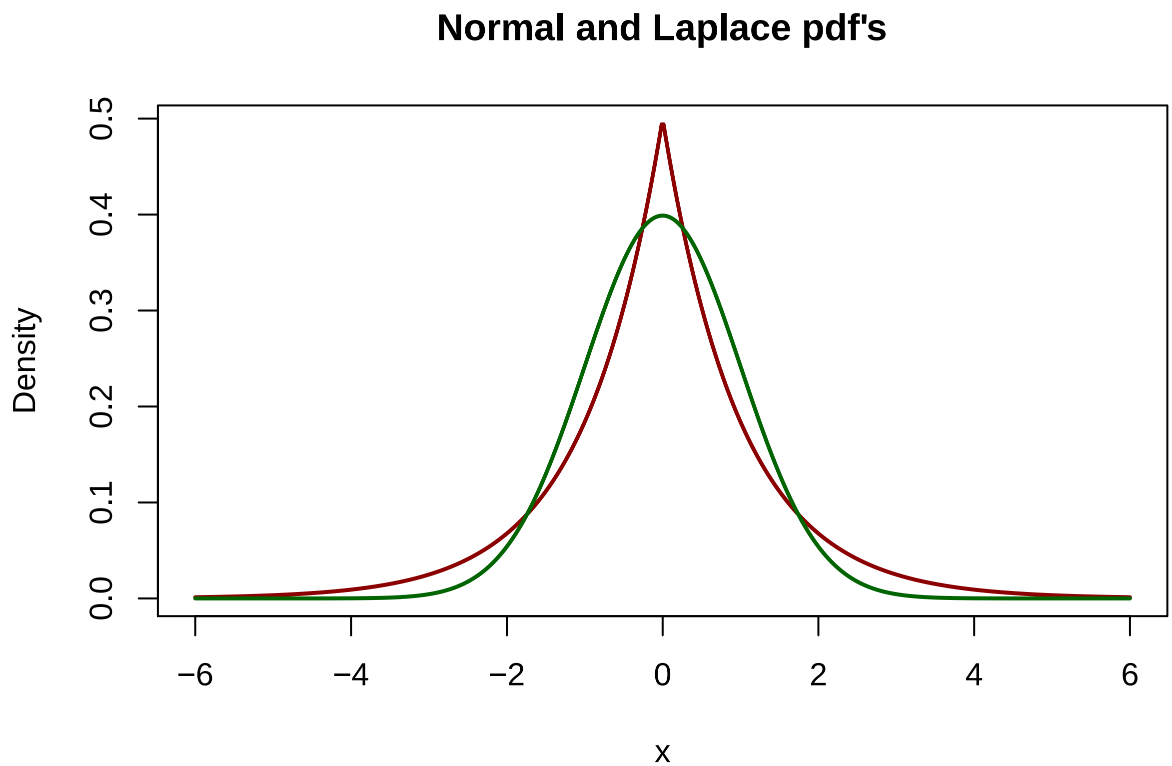
**Remark:**

The `extraDistr` package contains the Laplace distribution (and many more).

```
#install.packages("extraDistr",dependencies=TRUE,repos="https://cloud.r-project.org")
require(extraDistr)
```

```
## Loading required package: extraDistr
```

```
x<-seq(-6,6,length=500)
alpha<-1
plot(x,dlaplace(x,mu=0,sigma=1),type="l",lwd=2.0,col="DarkRed",xlab="x",ylab="Density",main="Normal and
lines(x,dnorm(x),lwd=2.0,col="DarkGreen")
```




**cdf for a** Laplace($\alpha$)

$$G(x|\alpha) = \begin{cases} \frac{1}{2}\exp(\alpha\, x), & \text{if} \quad x < 0, \\ 1 - \frac{1}{2}\exp(-\alpha\, x), & \text{if} \quad 0 \leq x. \end{cases}$$

**Quantile function for a** Laplace($\alpha$)

$$Q(y|\alpha) = \begin{cases} \frac{1}{\alpha}\log(2\, y), & \text{if} \quad y \in (0, 0.5), \\ -\frac{1}{\alpha}\log(2(1-y)), & \text{if} \quad y \in (0.5, 1). \end{cases}$$

Write a RNG to generate Laplace($\alpha$)-distributed random numbers by the inverse cdf method

```
#
#    Insert here your code
#
```

**Find a suitable $\alpha$ and the proportion $M$ for the acceptance-rejection algorithm**

Compute the quotient:

$$\frac{f(x)}{g(x)} = \sqrt{\frac{2}{\pi}}\,\frac{1}{\alpha}\,\exp\left\{-\frac{1}{2}\,x^2 + \alpha\,|x|\right\}$$

For $x > 0$, the exponent is $> 0$ when $x \in (0, 2\,\alpha)$, with maximum $\alpha^2/2$ at $x = \alpha$.

Symmetrically for $x < 0$.

Substituting, we get a function of $\alpha$:

$$K(\alpha) = \frac{f(\alpha)}{g(\alpha)} = \sqrt{\frac{2}{\pi}}\,\frac{1}{\alpha}\,e^{\alpha^2/2}.$$

The maximum of $K(\alpha)$ is attained for $\alpha = 1$. Then,

$$K(\alpha) \leq K_{\max} = \sqrt{\frac{2\,e}{\pi}} = 1.315489.$$

We take this value as $M$.

Generate a sequence of $\mathrm{N}(0,1)$-distributed random numbers with the acceptance-rejection algorithm, using a Laplace($\alpha = 1$) candidate pdf and proportionality constant $M = 1.315489$. Use several sample sizes $n$.

Plot a histogram of the resulting numbers. Superimpose to it the $\mathrm{N}(0,1)$ pdf.

Evaluate the acceptance rate from the theoretical formula. Check that the actual observed proportion of accepted values approaches this value for large $n$.

```
#
#    Insert here your code
#
```