Random variables - Exercises

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1 Bivariate discrete distributions

Exercise 1. *Consider the tables from last week's Psychophysiology example:*

Table A

	R_1	R_2	R_3	R_4
S_1	*3/4*	*1/4*	*0*	*0*
S_2	*1/5*	*3/5*	*1/5*	*0*
S_3	*0*	*0*	*4/5*	*1/5*
S_4	*0*	*0*	*1/6*	*5/6*

Table B

	R_1	R_2	R_3	R_4	
S_1	*3/16*	*1/16*	*0*	*0*	*1/4*
S_2	*1/20*	*3/20*	*1/20*	*0*	*1/4*
S_3	*0*	*0*	*4/20*	*1/20*	*1/4*
S_4	*0*	*0*	*1/24*	*5/24*	*1/4*
	19/80	*17/80*	*7/24*	*31/120*	*1*

For the purpose of this exercise we will consider S and R as two r.v. with values on the set $\{1, 2, 3, 4\}$. Table B contains the joint pmf of (S, R) and both marginal univariate pmf's, $f = (1/4, 1/4, 1/4, 1/4)'$, pmf of S and $g = (19/80, 17/80, 7/120, 31/120)$, pmf of R. Table A is the table of conditional pmf's of R with respect to S, i.e., each i-th row in Table B is the pmf of R conditioned to $[S = i]$, $(1 \leq i \leq 4)$.

1. *Compute the table of conditional pmf's of S with respect to R.*
2. *Compute the covariance and the correlation coefficient for (S, R).*
3. *Are S and R independent r.v. ?*

Exercise 2. *Consider the following matrix (actually a sudoku):*

$$h = \frac{1}{9 \cdot 9 \cdot 5} \begin{pmatrix} 7 & 6 & 8 & 4 & 5 & 9 & 2 & 1 & 3 \\ 9 & 5 & 4 & 1 & 3 & 2 & 8 & 6 & 7 \\ 2 & 3 & 1 & 6 & 7 & 8 & 4 & 5 & 9 \\ 6 & 8 & 7 & 9 & 4 & 5 & 3 & 2 & 1 \\ 5 & 4 & 9 & 2 & 1 & 3 & 7 & 8 & 6 \\ 3 & 1 & 2 & 8 & 6 & 7 & 9 & 4 & 5 \\ 1 & 2 & 3 & 7 & 8 & 6 & 5 & 9 & 4 \\ 8 & 7 & 6 & 5 & 9 & 4 & 1 & 3 & 2 \\ 4 & 9 & 5 & 3 & 2 & 1 & 6 & 7 & 8 \end{pmatrix}$$

1. *Check that h can represent the bivariate pmf of a pair (X, Y) of discrete r.v. 's.*
2. *Compute both marginal pmf's, verifying that they are both discrete unifom (as it should be for a sudoku).*
3. *Compute the independence table h_0 with the same marginals as h.*
4. *Compute both tables of conditional probabilities.*
5. *Assuming the vector of values of both X and Y is $(1, 2, 3, 4, 5, 6, 7, 8, 9)$, compute the covariance and the correlation coefficient for (X, Y).*

2 Bivariate continuous distributions

Exercise 3. *The joint cdf of a pair (X, Y) of continuous r.v. is:*

$$H(x, y) = \begin{cases} 1 - e^{-x} - e^{-y} + e^{-(x+y)}, & \text{if} \quad x \geq 0,\ y \geq 0, \\ 0, & \text{if} \quad x < 0 \text{ or } y < 0. \end{cases}$$

1. *Obtain both marginal cdf's, $F(x)$ and $G(y)$. Assess the possible independence of (X, Y).*
2. *Compute the joint pdf of (X, Y), $h(x, y)$, and both marginal pdf's, $f(x)$ and $g(y)$.*
3. *Compute $E(X)$, $E(Y)$, $\mathrm{var}(X)$, $\mathrm{var}(Y)$, $cov(X, Y)$, $cor(X, Y)$.*
4. *Compute the pdf of Y, conditional to $[X = 1]$.*
5. *Compute $\mathrm{P}\left([0 < X < 1] \cap [1 < Y < 2]\right)$.*

Exercise 4. *A pair (X, Y) of r.v. has a* normal bivariate distribution *with parameters $(\mu_x, \mu_y, \sigma_x, \sigma_y, \rho)$ if their joint distribution is absolutely continuous, with joint pdf:*

$$\begin{aligned} f_{(X,Y)}(x, y) &= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)} \times \right. \\ &\qquad \left. \left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right]\right\} \end{aligned}$$

1. *Verify that both marginal pdf's are univariate normal.*
2. *Verify that the correlation coefficient of (X, Y) is equal to ρ.*
3. *Given $x \in \mathbb{R}$, compute the conditional pdf of $(Y|X = x)$, verifying it is a univariate normal with parameters:*

$$\begin{aligned} \mu_{y|x} &= \mu_y + \frac{\rho\,\sigma_y}{\sigma_x}\,(x - \mu_x), \\ \sigma^2_{y|x} &= \sigma^2_y\,(1 - \rho^2). \end{aligned}$$

4. *Similarly, given $y \in \mathbb{R}$, obtain the conditional pdf of $(X|Y = y)$.*
