# Probability and Bayes' rule
# Solutions to Exercises

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**1** (The Monty Hall problem)**.** *A TV game show. Choose one of three closed doors. One of them hides a hefty prize (a car). If you choose it you win the car. Behind each of the other two doors there is a goat. If you choose either of them you win nothing.*

*Procedure: You choose a door. For the time being, it remains closed. The game show host (Monty Hall), who knows where the car is hidden, opens one of the unselected doors, showing a goat. Then you have the opportunity to maintain your initial choice or to switch to the remaining closed door.*

*Which is the best strategy? Stick to the first choice or change? Or, perhaps, it is indifferent? Once a door has been selected, for sure at least one of the other two hides a goat, hence opening one of them supplies no new information. Thus we can conclude that switching will not affect chances of winning. Or does it?*

*Compute the probability of winning the car, depending on the strategy (keep or switch), assuming:*

- *The car is behind door $A$, written in red.*
- *Choices by M (Monty, the game show host) are as random as possible, e.g., if we select $A$ he opens B or C with equal probability 0.5; if we select another door he has no choice.*
- *Our first selection is at random, the second one depends on the adopted strategy.*

**Solution:** 1. Three hypotheses $H$:

$$H = A, \quad H = B, \quad \text{and} \quad H = C,$$

respectively representing the (unknown but possible) fact that the car is behind the namesake door.

Initially, the *prior probabilities* $\mathrm{P}(H)$ are:

$$\mathrm{P}(A) = \mathrm{P}(B) = \mathrm{P}(C) = 1/3.$$

2. Observed data

AFTER picking your door, relabel the three doors so that $A$ is your choice.

Assume the observed data is:

$$D = \boxed{\begin{array}{c}\text{Monty has opened door } B \\ \text{AND} \\ \text{inside door } B \text{ there is a goat.}\end{array}}$$

3. Likelihood

Conditional probabilities $\mathrm{P}(D|H)$ of the observed data $D$, assuming each possible hypothesis $H$ as being true:

| | Prior $\mathrm{P}(H)$ | Likelihood $\mathrm{P}(D\vert H)$ |
|---|---|---|
| A | $1/3$ | $1/2$ |
| B | $1/3$ | $0$ |
| C | $1/3$ | $1$ |

4. Joint probabilities

Multiply each $\mathrm{P}(D|H)$ times the prior $\mathrm{P}(H)$, giving the Joint or intersection probabilities $\mathrm{P}(D \cap H) \equiv \mathrm{P}(D, H)$:

| | Prior $\mathrm{P}(H)$ | Likelihood $\mathrm{P}(D\vert H)$ | Joint $\mathrm{P}(D, H)$ |
|---|---|---|---|
| A | $1/3$ | $1/2$ | $1/6$ |
| B | $1/3$ | $0$ | $0$ |
| C | $1/3$ | $1$ | $1/3$ |

5. Bayes' formula $\implies$ Posterior probabilities

Divide $\mathrm{P}(D, H)$ by their sum $\mathrm{P}(D) = 1/2$, giving the posterior probabilities $\mathrm{P}(H|D) = \mathrm{P}(D, H) / \mathrm{P}(D)$:

| | Prior $\mathrm{P}(H)$ | Likelihood $\mathrm{P}(D\vert H)$ | Joint $\mathrm{P}(D, H)$ | Posterior $\mathrm{P}(H\vert D)$ |
|---|---|---|---|---|
| A | $1/3$ | $1/2$ | $1/6$ | $1/3$ |
| B | $1/3$ | $0$ | $0$ | $0$ |
| C | $1/3$ | $1$ | $1/3$ | $2/3$ |

**2** (The double dice problem)**.** *Suppose I have a box that contains one each of 4-sided, 6-sided, 8-sided, and 12-sided dice. I choose a die at random, and roll it twice without letting you see the die or the outcome.*

*1. What is the probability of obtaining the same outcome of both rolls?*

*2. Now, I report that I got the same outcome on both rolls. What is the posterior probability that I rolled each of the dice?*

*3. If I roll the same die again, what is the probability that I get the same outcome a third time?*

**Solution:** See: Probably Overthinking It. A blog by Allen Downey.

1. Considering the results of both rolls, denoting by $S2$ the event:

$$S2 = \text{“Same outcome on both rolls”},$$

the probability of $S2$, conditional to having chosen an $m$-sided die, is:

$$\mathrm{P}(S2|m) = \frac{\text{“Favorable outcomes”}}{\text{“Total outcomes”}} = \frac{m}{m^2} = \frac{1}{m}.$$

There are $k = 4$ dice, randomly chosen with prior probabilities $\mathrm{P}(m) = \dfrac{1}{4}$, $m = 4, 6, 8, 12$, i.e., the prior distribution is:

$$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right).$$

The total probability of $S2$ is:

$$\mathrm{P}(S2) = \frac{1}{4} \times \left(\frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{12}\right) = \frac{5}{32}.$$

2. From Bayes' formula we obtain the posterior probabilities:

$$\mathrm{P}(m|S2) = \frac{\mathrm{P}(S2|m) \cdot \mathrm{P}(m)}{\mathrm{P}(S2)} = \frac{\dfrac{1}{m}}{\dfrac{5}{8}} = \frac{8}{5\,m}, \qquad m = 4, 6, 8, 12.$$

Substituting each $m = 4, 6, 8, 12$ we get the posterior distribution:

$$\begin{cases} m = 4, & \to \ 8/20 \ = \ 6/15, \\ m = 6, & \to \ 8/30 \ = \ 4/15, \\ m = 8, & \to \ 8/40 \ = \ 3/15, \\ m = 12, & \to \ 8/60 \ = \ 2/15. \end{cases}$$

3. Predicting with these posterior probabilities the probability of:

$$S3 = \text{“Same outcome a third time”},$$

we proceed as above. The probability of $S3$, conditional to having chosen an $m$-sided die, is:

$$\mathrm{P}(S3|m) = \frac{\text{“Favorable outcomes”}}{\text{“Total outcomes”}} = \frac{1}{m},$$

since now, in the third roll, there is only a single favorable outcome, namely the same result obtained in both previous rolls. The total probability of $S3$ is:

$$\begin{aligned} \mathrm{P}(S3) &= \mathrm{P}(S3|4) \cdot \mathrm{P}(4) + \mathrm{P}(S3|6) \cdot \mathrm{P}(6) + \mathrm{P}(S3|8) \cdot \mathrm{P}(8) + \mathrm{P}(S3|12) \cdot \mathrm{P}(12) \\ &= \frac{1}{4} \times \frac{6}{15} + \frac{1}{6} \times \frac{4}{15} + \frac{1}{8} \times \frac{3}{15} + \frac{1}{12} \times \frac{2}{15} = \frac{13}{72}. \end{aligned}$$