Probability and Bayes' rule
Solutions to Exercises

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1. *We extract a ball at random from an urn containing 2 red balls, 3 white balls and 2 black balls and deposit it into a second urn already containing 2 white balls and 3 black balls. What is the probability of extracting now a black ball from the second urn (after mixing)?*

Solution: Consider the partition of Ω into the three subsets:

$$R_1 = \{1^{st} \text{ ball is red}\},$$
$$W_1 = \{1^{st} \text{ ball is white}\},$$
$$B_1 = \{1^{st} \text{ ball is black}\}.$$

We want to compute the probability of:

$$B_2 = \{2^{nd} \text{ ball is black}\}.$$

$$\begin{aligned} \mathrm{P}(B_2) &= \mathrm{P}(B_2|R_1)\cdot \mathrm{P}(R_1) + \mathrm{P}(B_2|W_1)\cdot \mathrm{P}(W_1) + \\ &+ \mathrm{P}(B_2|B_1)\cdot \mathrm{P}(B_1) = \\ &= \frac{3}{6}\times\frac{2}{7} + \frac{3}{6}\times\frac{3}{7} + \frac{4}{6}\times\frac{2}{7} \\ &= \frac{23}{42}. \end{aligned}$$

2. *An urn contains 6 white balls labelled 1 to 6. We extract a ball and paint in black as many balls as the label of the extracted ball indicates. Then (after mixing) we extract a second ball. What is the probability that this second ball is white?*

Solution: Consider the 6 events from the 1^{st} extraction:

$$C_i = \{\text{Extract the ball labelled } i\}, \qquad 1 \leq i \leq 6.$$

They are a partition of the total Ω.

With the notation:

$$W = \{\text{The second extracted ball is white}\},$$

$$\begin{aligned} P(W) &= \sum_{i=1}^{6} P(W|C_i) \cdot P(C_i) = \\ &= \sum_{i=1}^{6} \frac{6-i}{6} \cdot \frac{1}{6} = \\ &= \frac{5}{12}. \end{aligned}$$

3. *Two urns, U_1 and U_2, contain white (w) and red (r) balls in the following proportions:*

$$U_1 = (3w, 5r), \qquad U_2 = (2w, 1r).$$

We roll a die. If a 3 or a 6 turns up then a randomly selected ball from U_2 is transferred to U_1 and then (after mixing) we extract at random a ball from U_1. In the other cases a randomly selected ball from U_1 is transferred to U_2 and then (after mixing) we extract at random a ball from U_2. Compute the probabilities:

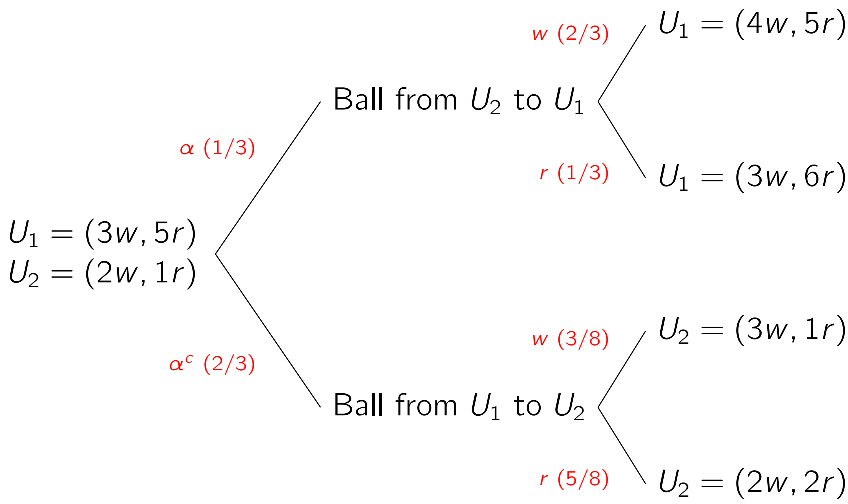
1. *That both balls are red.*
2. *That both balls are white.*

Solution:

$$\begin{aligned} \text{Notation:} \quad \alpha \quad &= \quad \text{"a 3 or a 6 turns up"}. \\ \text{Then,} \qquad P(\alpha) \quad &= \quad 1/3, \quad P(\alpha^c) = 2/3. \end{aligned}$$

We construct a tree with all possible results.

Probabilities written along the branches are conditional (on the node from which the branch sprouts).



Probability of extracting two red balls:

$$
\begin{aligned}
\mathrm{P}(2r) &= \mathrm{P}(2r|\alpha) \cdot \mathrm{P}(\alpha) + \mathrm{P}(2r|\alpha^c) \cdot \mathrm{P}(\alpha^c) \\
&= (1/3 \times 6/9) \times 1/3 + (5/8 \times 1/2) \times 2/3 \\
&= 61/216.
\end{aligned}
$$

Probability of extracting two white balls

$$
\begin{aligned}
\mathrm{P}(2w) &= \mathrm{P}(2w|\alpha) \cdot \mathrm{P}(\alpha) + \mathrm{P}(2w|\alpha^c) \cdot \mathrm{P}(\alpha^c) \\
&= (2/3 \times 4/9) \times 1/3 + (3/8 \times 3/4) \times 2/3 \\
&= 371/1296.
\end{aligned}
$$

4. *30% of the people in a city are vaccinated against flu. The probability of catching flu is* 0.01 *for vaccinated individuals and* 0.1 *for non-vaccinated individuals.*

What is the probability that a patient with flu has been vaccinated?

What is the probability that a given individual who has not caught the flu has been vaccinated?

Solution: Notation: a randomly selected individual:

$$V = \{\text{has been vaccinated}\},$$

$$F = \{\text{has caught flu}\}.$$

From the statement,

$$\mathrm{P}(V) = \frac{3}{10}, \quad \mathrm{P}(F|V) = \frac{1}{100}, \quad \mathrm{P}(F|V^c) = \frac{1}{10}.$$

$$\mathrm{P}(V|F) = \frac{\mathrm{P}(F|V)\ \mathrm{P}(V)}{\mathrm{P}(F|V)\ \mathrm{P}(V) + \mathrm{P}(F|V^c)\ \mathrm{P}(V^c)} = \frac{3}{73}.$$

$$\mathrm{P}(V|F^c) = \frac{\mathrm{P}(F^c|V)\ \mathrm{P}(V)}{\mathrm{P}(F^c|V)\ \mathrm{P}(V) + \mathrm{P}(F^c|V^c)\ \mathrm{P}(V^c)} = \frac{33}{103}.$$

5. *A bag contains 8 red, 15 white, and 5 yellow balls. The experimenter extracts a ball at random and registers its color (does not tell). If the ball is red or yellow it is returned to the bag, adding 2 more balls of the same color; if the ball is white it is kept out of the bag. Then a second ball is extracted. Compute the probability:*

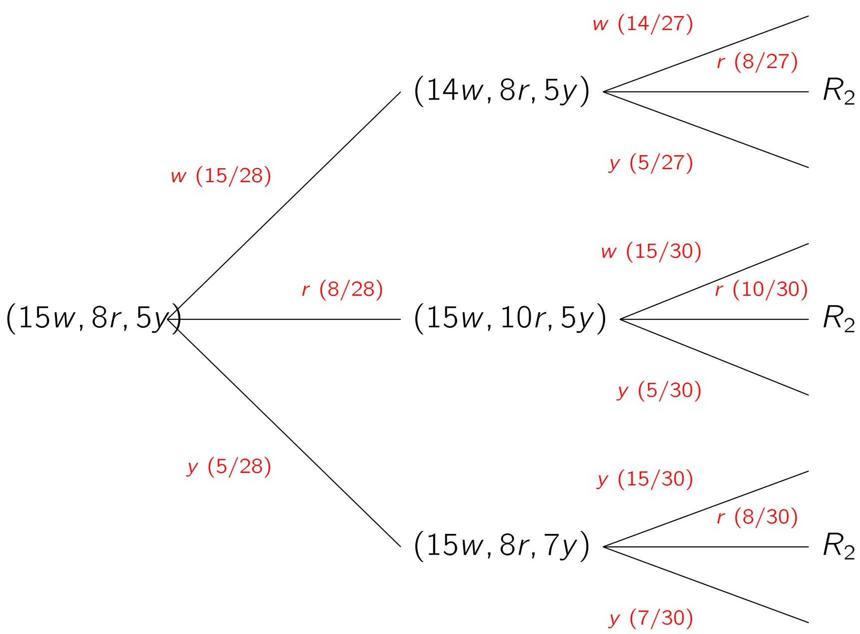
1. *That the second ball is red.*
2. *That the first ball was not red if we observe the second ball is red.*

Solution: Notations:

W_1, R_1, Y_1	$=$	White, red, yellow ball on the first extraction,
R_2	$=$	Red ball on the second extraction.

Questions are:

$$\mathrm{P}(R_2), \qquad \mathrm{P}(R_1^c|R_2).$$



$$\begin{aligned} \mathrm{P}(R_2) &= \mathrm{P}(R_2|W_1)\cdot \mathrm{P}(W_1) + \mathrm{P}(R_2|R_1)\cdot \mathrm{P}(R_1) \\ &\quad + \mathrm{P}(R_2|Y_1)\cdot \mathrm{P}(Y_1) \\ &= \frac{8}{27}\times\frac{15}{28} + \frac{10}{30}\times\frac{8}{28} + \frac{8}{30}\times\frac{5}{28} \\ &= \frac{10}{63} + \frac{6}{63} + \frac{3}{63} = \frac{19}{63}. \end{aligned}$$

$$\begin{aligned} \mathrm{P}(R_1^c|R_2) &= \frac{\mathrm{P}(R_1^c \cap R_2)}{\mathrm{P}(R_2)} = \frac{\mathrm{P}((W_1 \cup Y_1)\cap R_2)}{\mathrm{P}(R_2)} \\ &= \frac{\mathrm{P}(W_1 \cap R_2) + \mathrm{P}(Y_1 \cap R_2)}{\mathrm{P}(R_2)} \\ &= \frac{\mathrm{P}(R_2|W_1)\cdot \mathrm{P}(W_1) + \mathrm{P}(R_2|Y_1)\cdot \mathrm{P}(Y_1)}{\mathrm{P}(R_2)} \\ &= \frac{10/63 + 3/63}{19/63} = \frac{13}{19}. \end{aligned}$$

6. *A box A contains* 9 *cards numbered* 1 *to* 9 *and another box B contains* 5 *cards numbered* 1 *to* 5*. A box is chosen at random and a card is extracted from it, also at random. If the card number is even, whithout returning it, a second card is extracted from the same box. If the card number is odd, a card is extracted from the other box.*

1. *What is the probability that both card numbers are even?*
2. *If both card numbers are even, what is the probability that they came from box A?*
3. *What is the probability that both card numbers are odd?*

Solution: We write a description of all possible paths, annotating on it the transition *(= changes in configuration)* probabilities.

Instead of a tree format here we use a table format. Just an equivalent visual alternative.

e = even; o = odd.

1^{st} box selection	1^{st} card extraction	2^{nd} box selection	2^{nd} card extraction	
A (1/2)	e (4/9)	A	e (3/8)	1/12
		(3e,5o)	o (5/8)	5/36
(4e,5o)	o (5/9)	B	e (2/5)	1/9
		(2e,3o)	o (3/5)	1/6
B (1/2)	e (2/5)	B	e (1/4)	1/20
		(1e,3o)	o (3/4)	3/20
(2e,3o)	o (3/5)	A	e (4/9)	2/15
		(4e,5o)	o (5/9)	1/6

Directly from the table, the answers (1) and (3) are:

$$\mathrm{P}(ee) = \frac{1}{12} + \frac{1}{20} = \frac{2}{15},$$

$$\mathrm{P}(oo) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

In (2) we are asked the conditional probability:

$$\begin{aligned}
\mathrm{P}(AA|ee) &= \frac{\mathrm{P}(ee|AA) \cdot \mathrm{P}(AA)}{\mathrm{P}(ee|AA) \cdot \mathrm{P}(AA) + \mathrm{P}(ee|BB) \cdot \mathrm{P}(BB)} \\
&= \frac{\mathrm{P}(ee|AA) \cdot \mathrm{P}(AA)}{\mathrm{P}(ee)} \\
&= \frac{1/12}{2/15} = \frac{5}{8}.
\end{aligned}$$

7 (the Kahneman cab problem)**.** *A cab was involved in a hit and run accident at night. Two cab companies, the Green and the Blue, operate in the city. 85% of the cabs in the city are Green and 15% are Blue. A witness identified the cab as Blue. The court tested the reliability of the witness under the same circumstances that existed on the night of the accident and concluded that the witness correctly identified each one of the two colours 80% of the time and failed 20% of the time.*

What is the probability that the cab involved in the accident was Blue rather than Green knowing that this witness identified it as Blue?

[from: Kahneman, Daniel, Thinking, fast and slow, *New York: Farrar, Straus and Giroux (2013)].*

Solution:

$$G = \text{“The guilty cab is Green”}$$

$$B = \text{“The guilty cab is Blue”}$$

$$W_G = \text{“Witness says the guilty cab is Green”}$$

$$W_B = \text{“Witness says the guilty cab is Blue”}$$

A priori probabilities:

$$\begin{aligned} \mathrm{P}(G) &= 0.85, \\ \mathrm{P}(B) &= 0.15, \end{aligned}$$

Likelihood of each possible witness response, given each possible factual situation:

$$\begin{aligned} \mathrm{P}(W_B|B) &= 0.80, & \mathrm{P}(W_G|B) &= 0.20, \\ \mathrm{P}(W_B|G) &= 0.20, & \mathrm{P}(W_G|G) &= 0.80. \end{aligned}$$

Applying Bayes' rule:

$$\begin{aligned} \mathrm{P}(B|W_B) &= \frac{\mathrm{P}(B \cap W_B)}{\mathrm{P}(W_B)} \\ &= \frac{\mathrm{P}(W_B|B) \cdot \mathrm{P}(B)}{\mathrm{P}(W_B|B) \cdot \mathrm{P}(B) + \mathrm{P}(W_B|G) \cdot \mathrm{P}(G)} \\ &= \frac{0.80 \times 0.15}{0.80 \times 0.15 + 0.20 \times 0.85} \\ &= 0.4139. \end{aligned}$$
