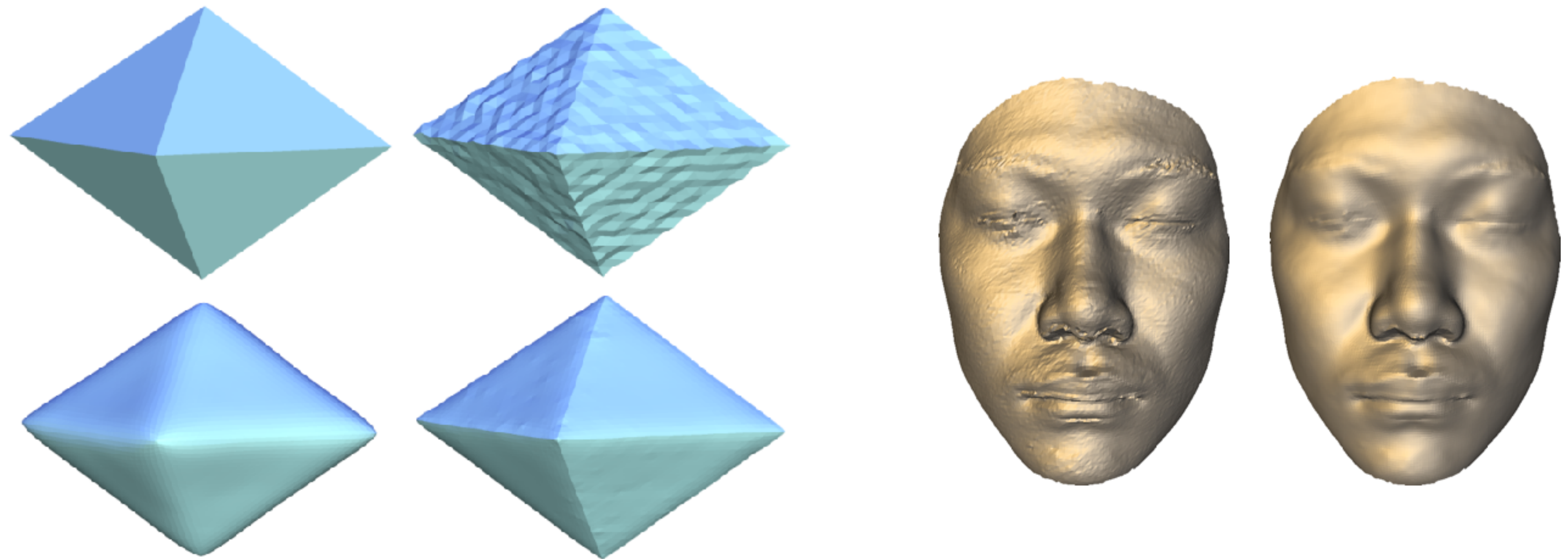


Elementary Differential Geometry of Curves

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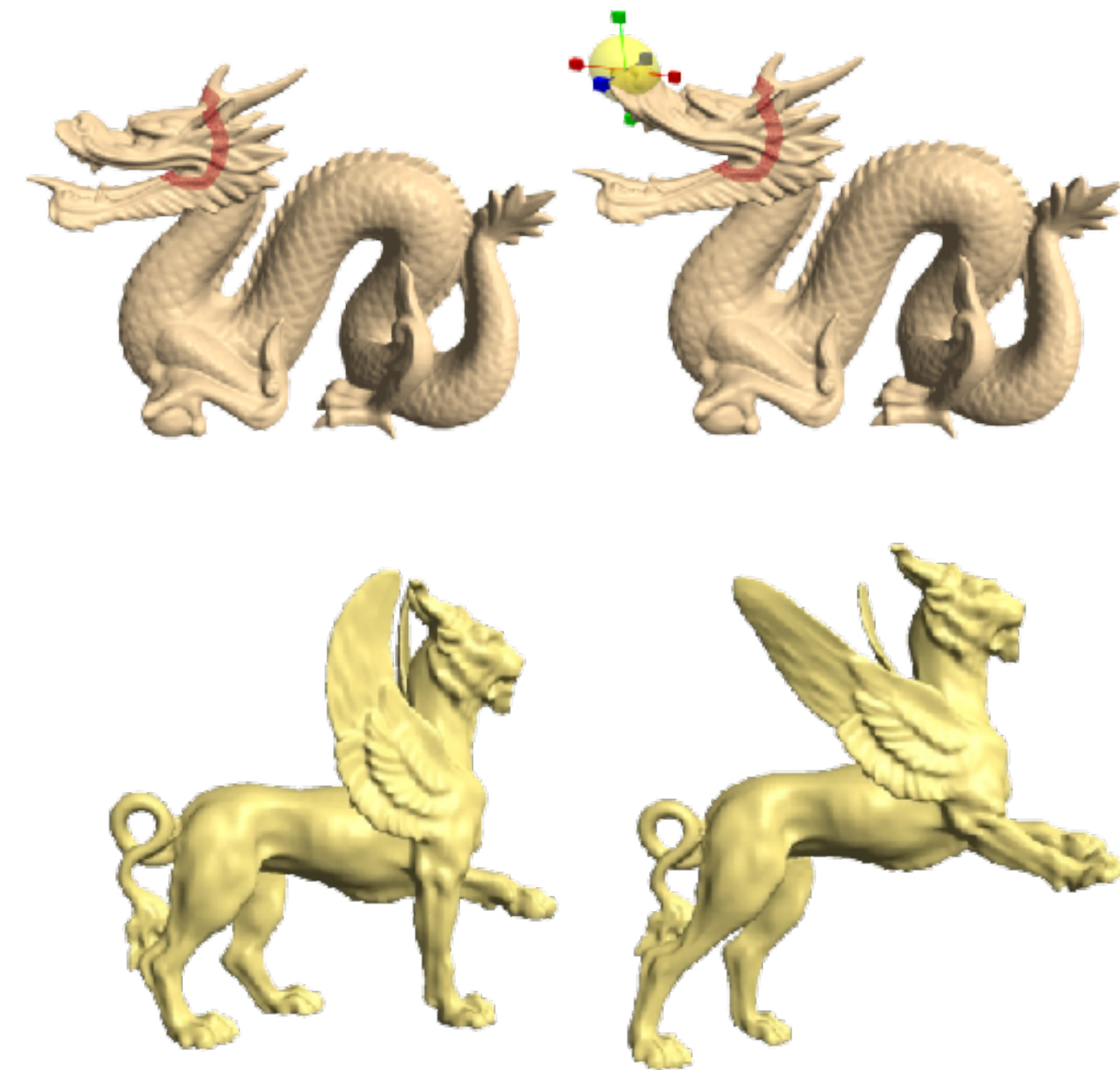
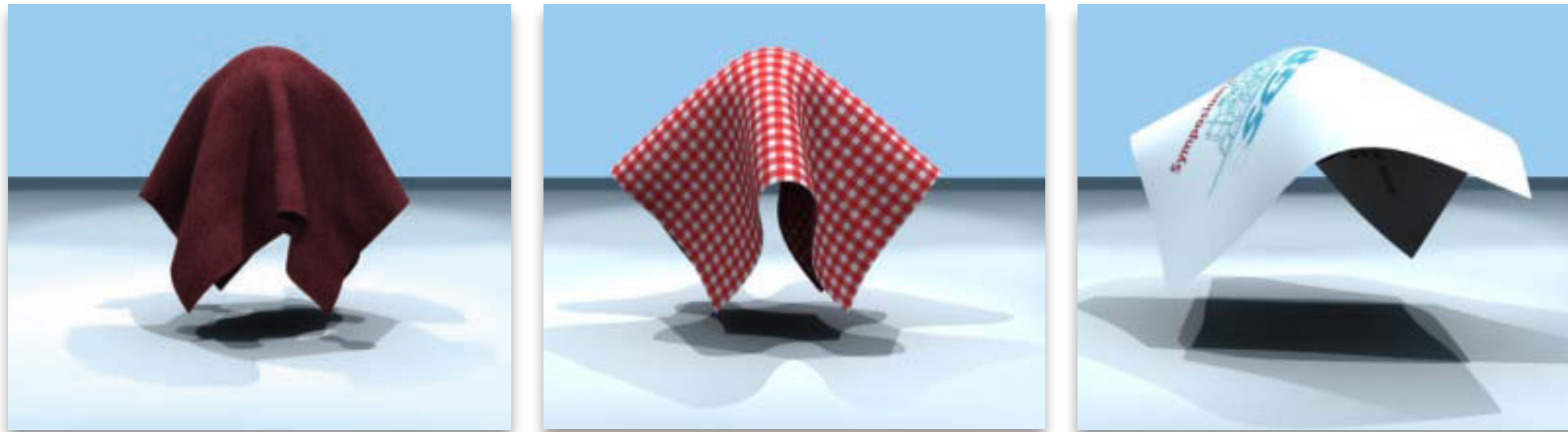
Differential Geometry – Motivation

- Describe and analyze geometric characteristics of shapes
 - e.g. how smooth?



Differential Geometry – Motivation

- Describe and analyze geometric characteristics of shapes
 - e.g. how smooth?
 - how shapes deform



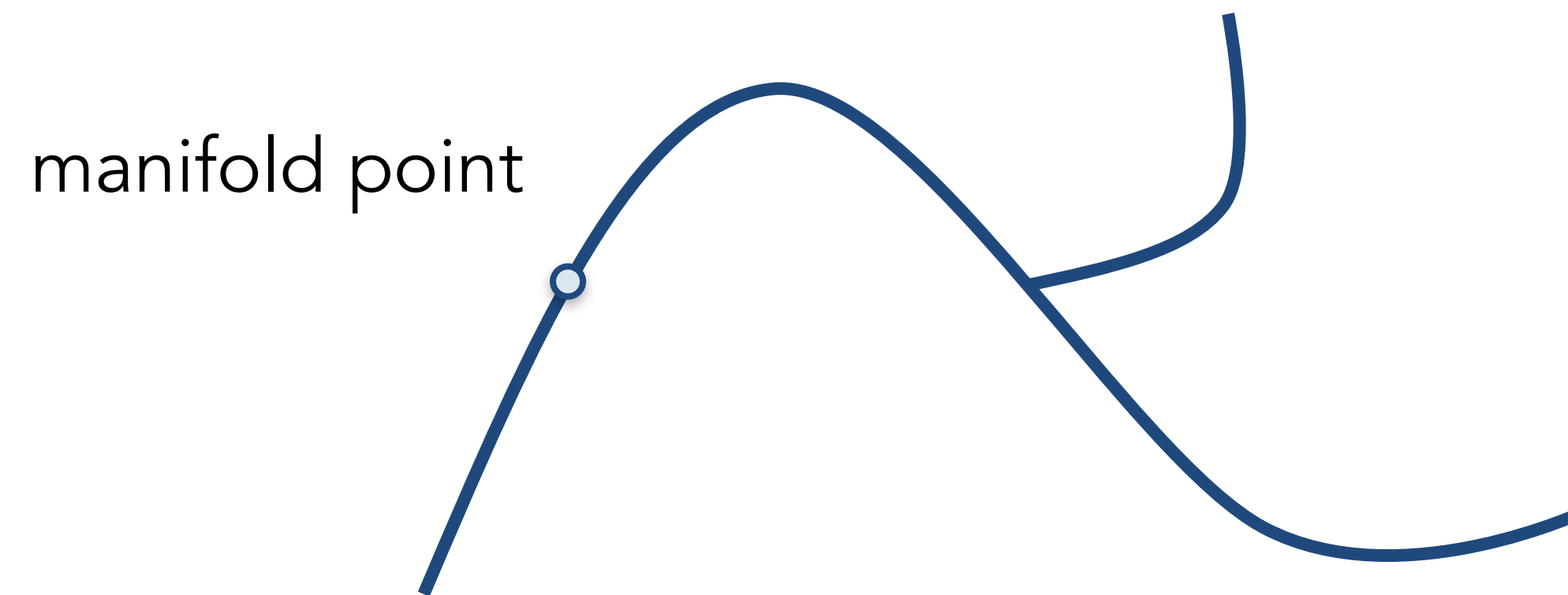
Differential Geometry Basics

- Geometry of manifolds
- Things that can be discovered by local observation: point + neighborhood



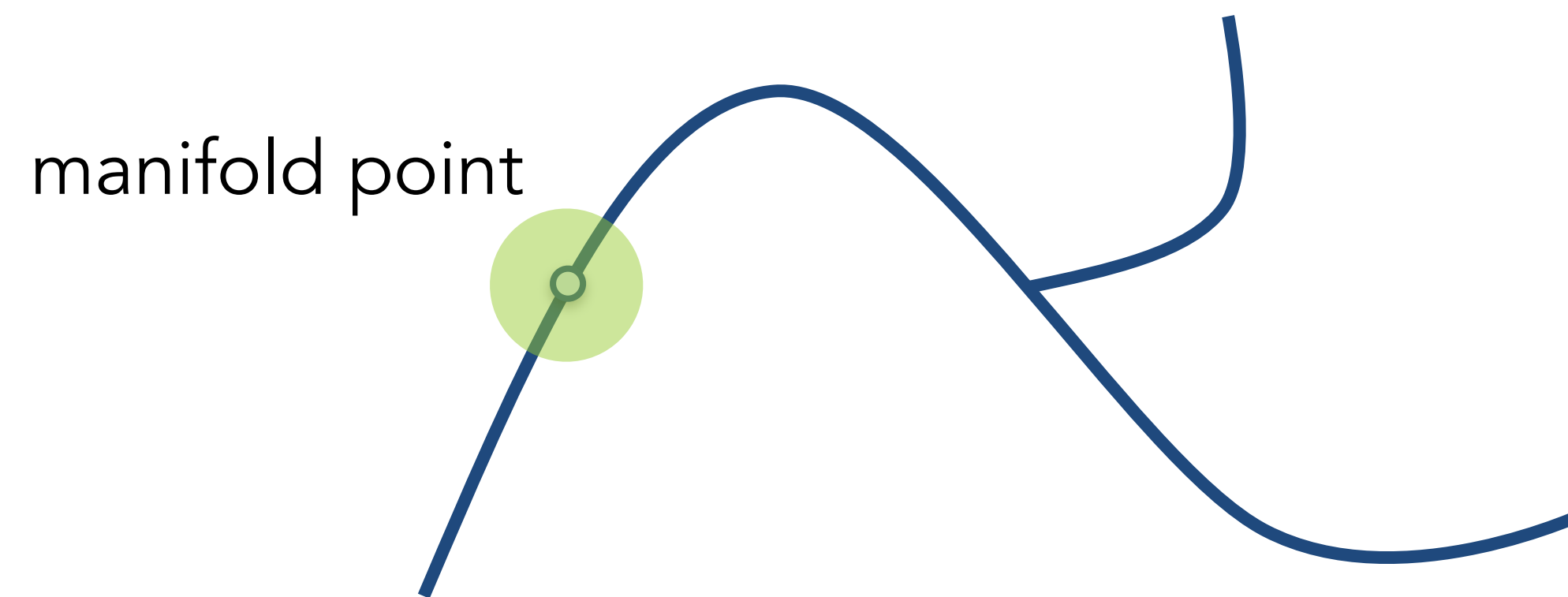
Differential Geometry Basics

- Geometry of manifolds
- Things that can be discovered by local observation: point + neighborhood



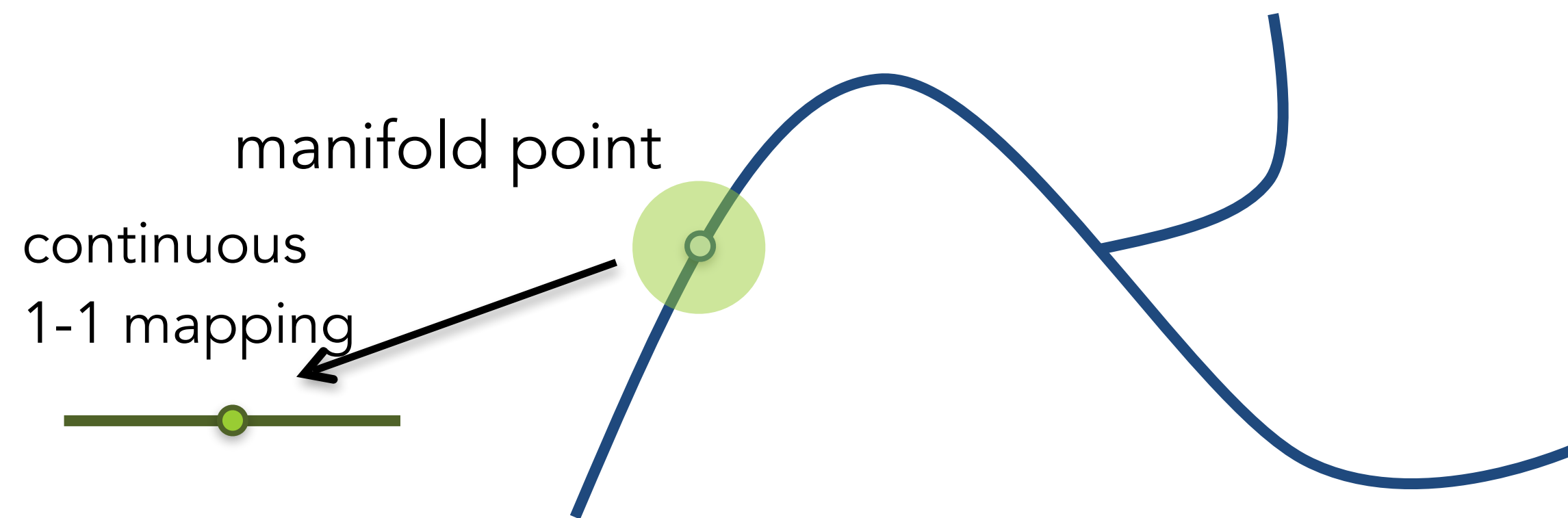
Differential Geometry Basics

- Geometry of manifolds
- Things that can be discovered by local observation: point + neighborhood



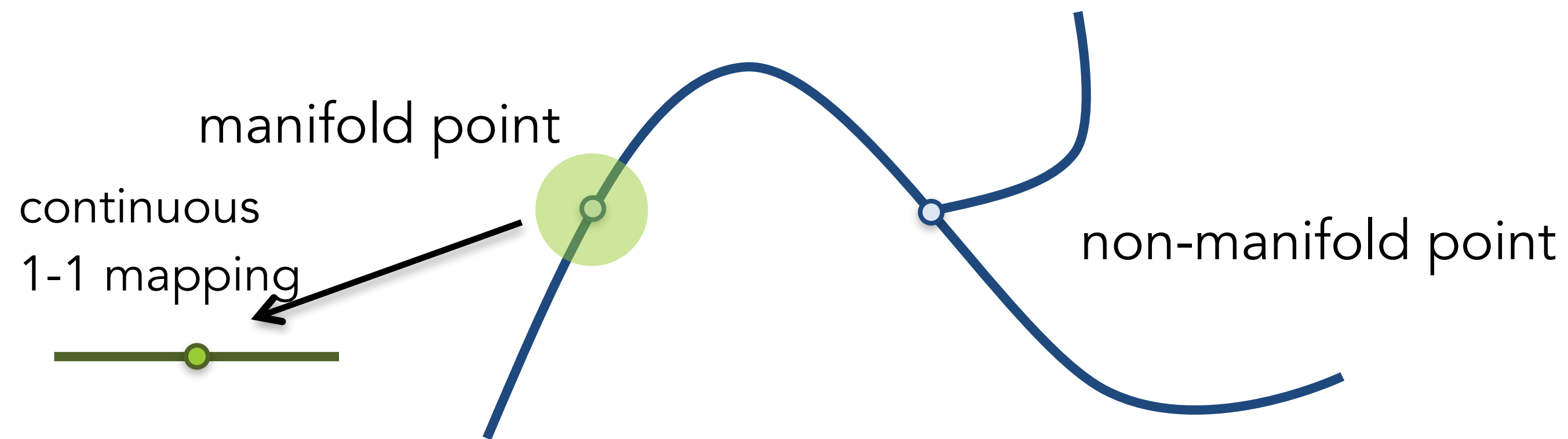
Differential Geometry Basics

- Geometry of manifolds
- Things that can be discovered by local observation: point + neighborhood



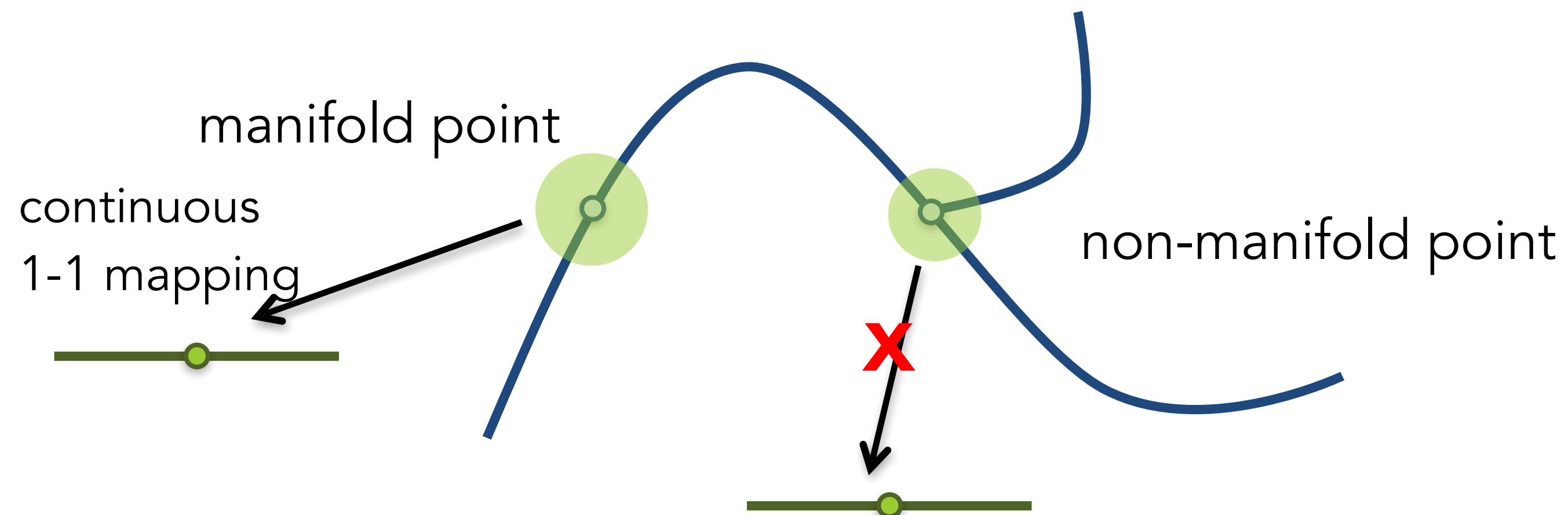
Differential Geometry Basics

- Geometry of manifolds
- Things that can be discovered by local observation: point + neighborhood



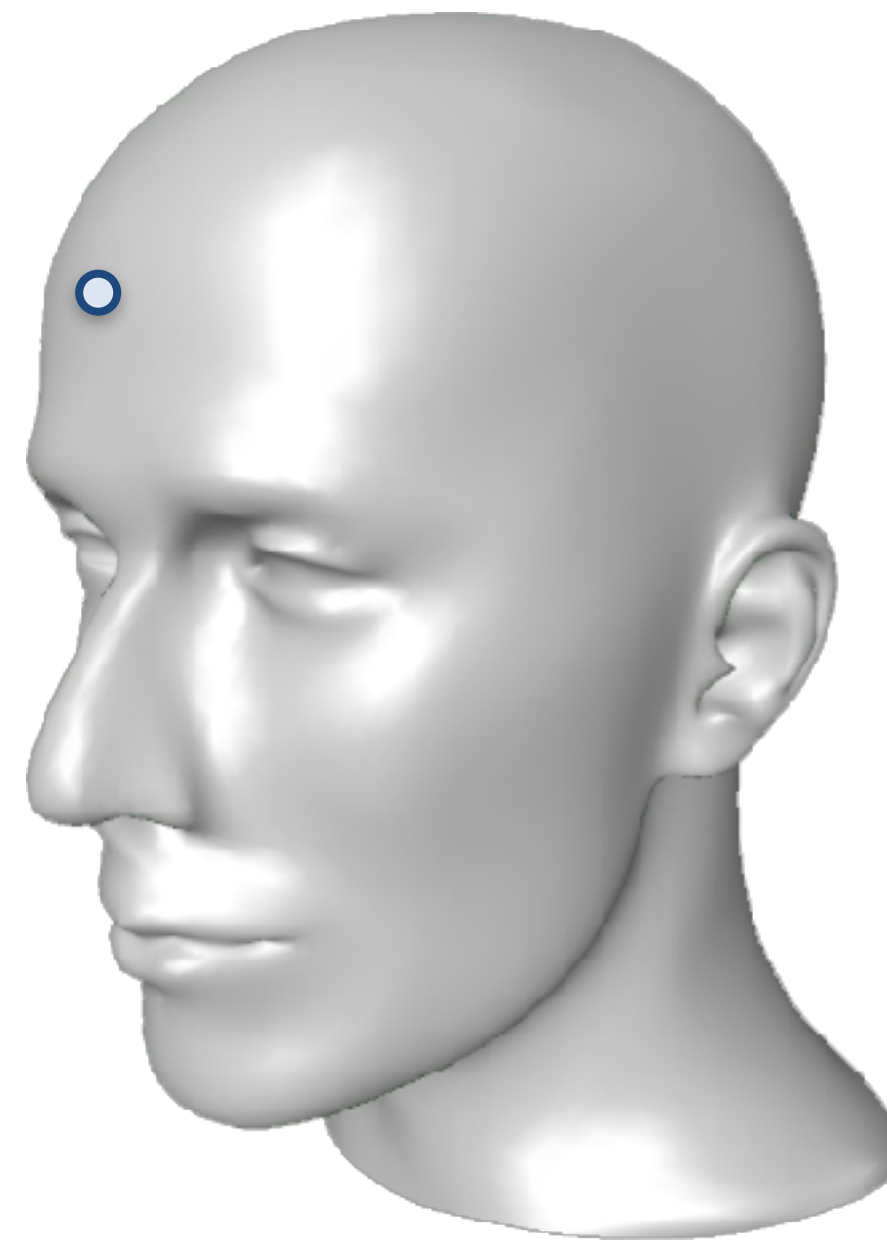
Differential Geometry Basics

- Geometry of manifolds
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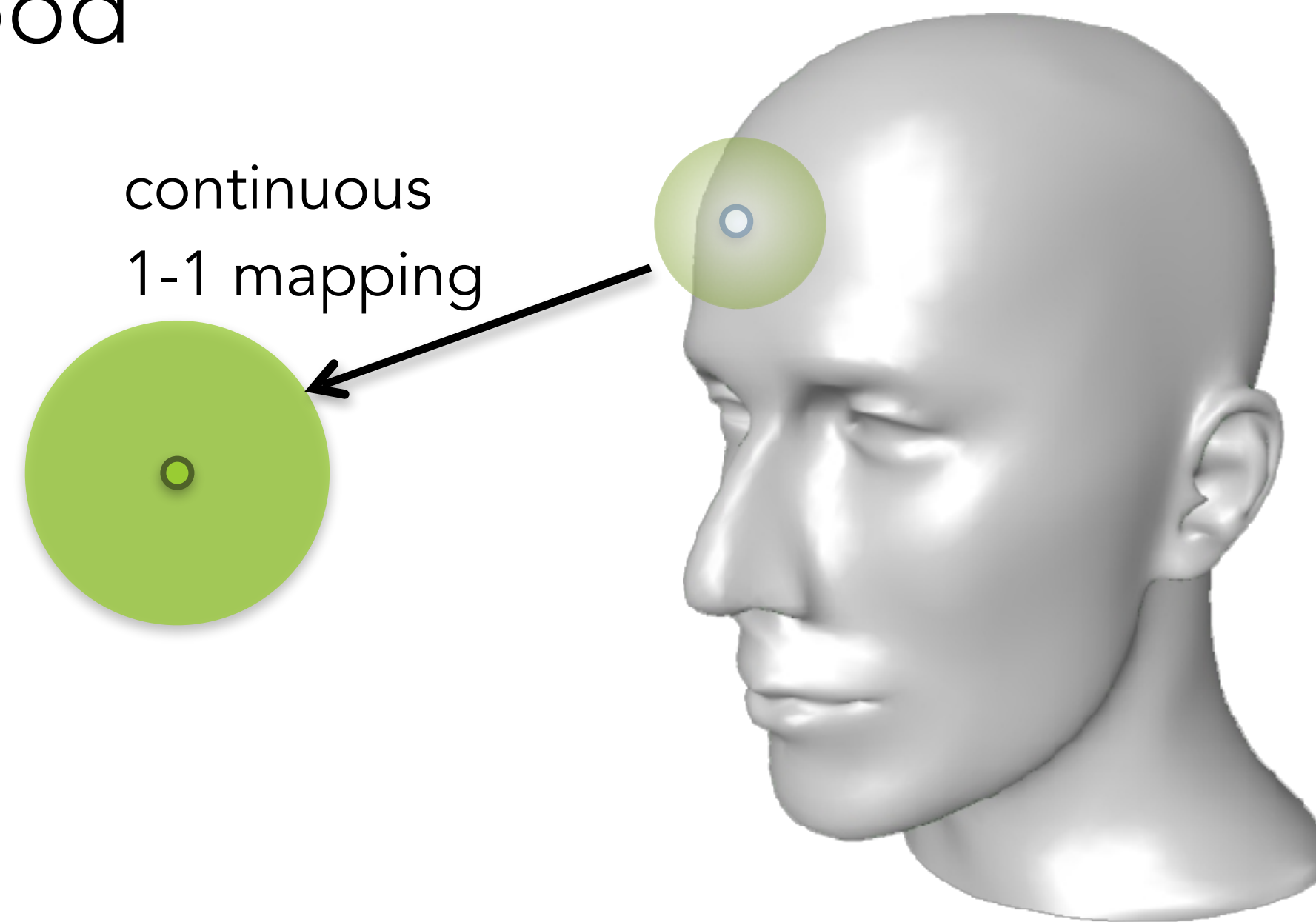
Differential Geometry Basics

- Geometry of manifolds
- Things that can be discovered by local observation: point + neighborhood



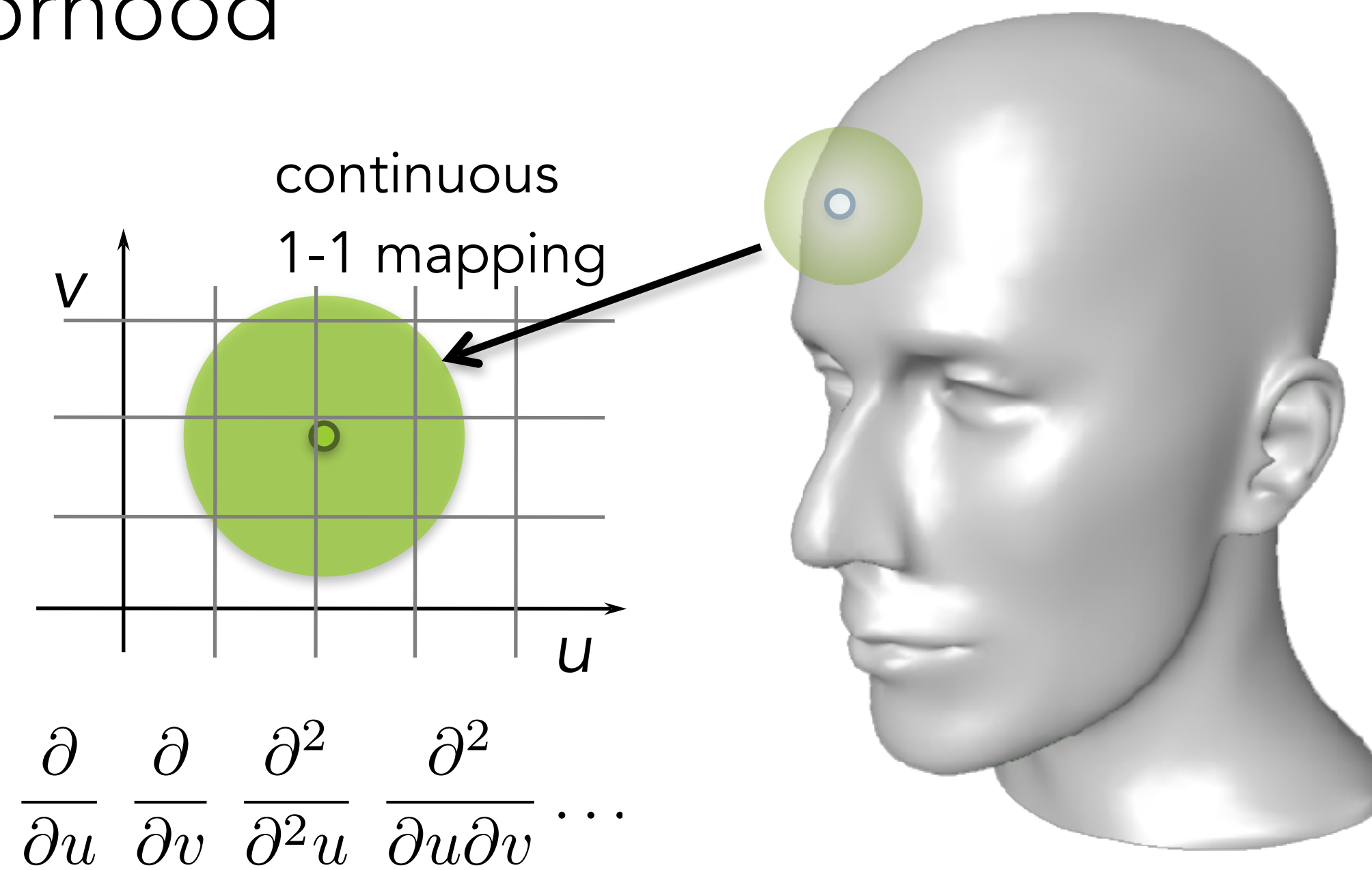
Differential Geometry Basics

- Geometry of manifolds
- Things that can be discovered by local observation: point + neighborhood



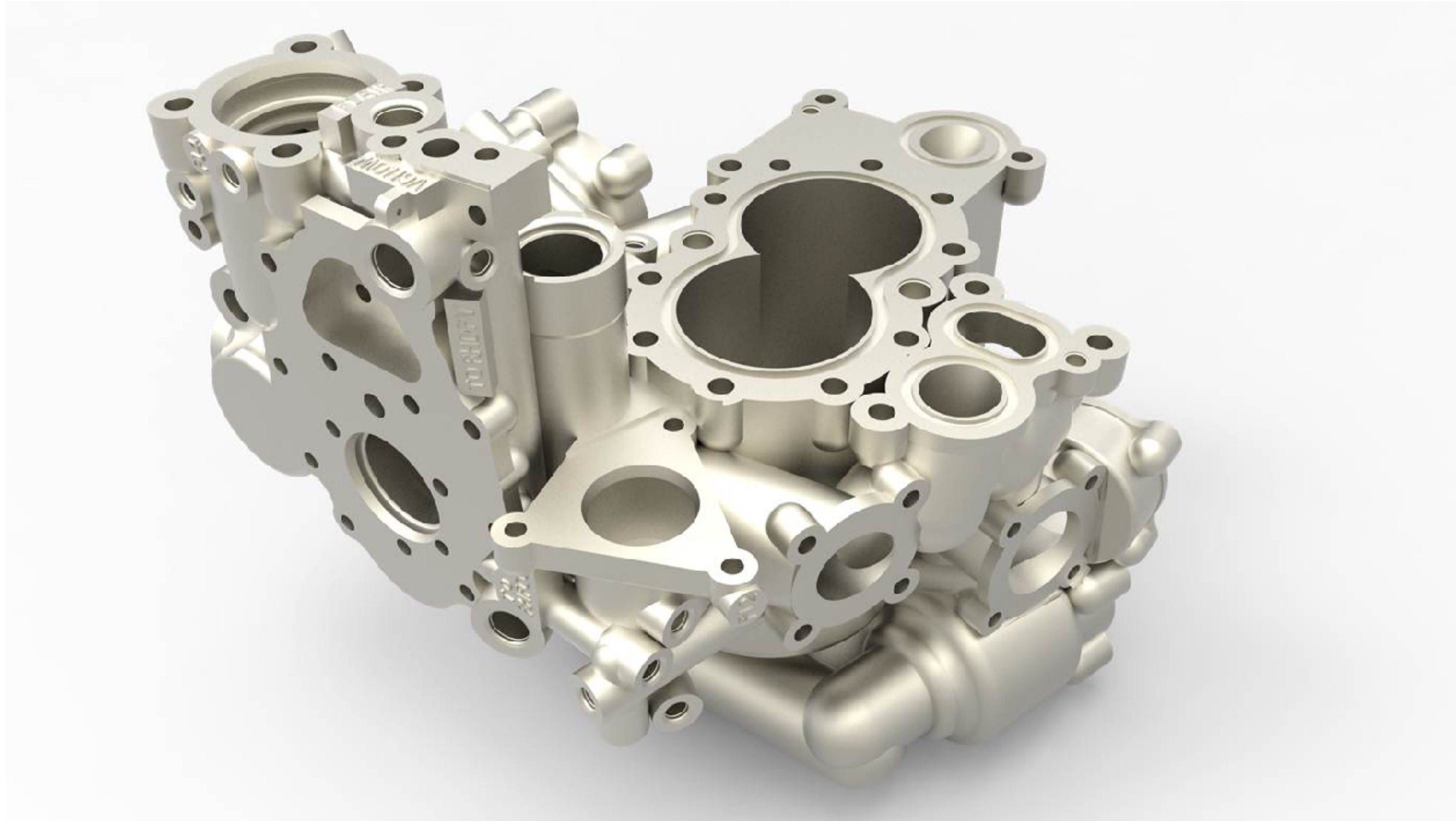
Differential Geometry Basics

- Geometry of manifolds
- Things that can be discovered by local observation: point + neighborhood



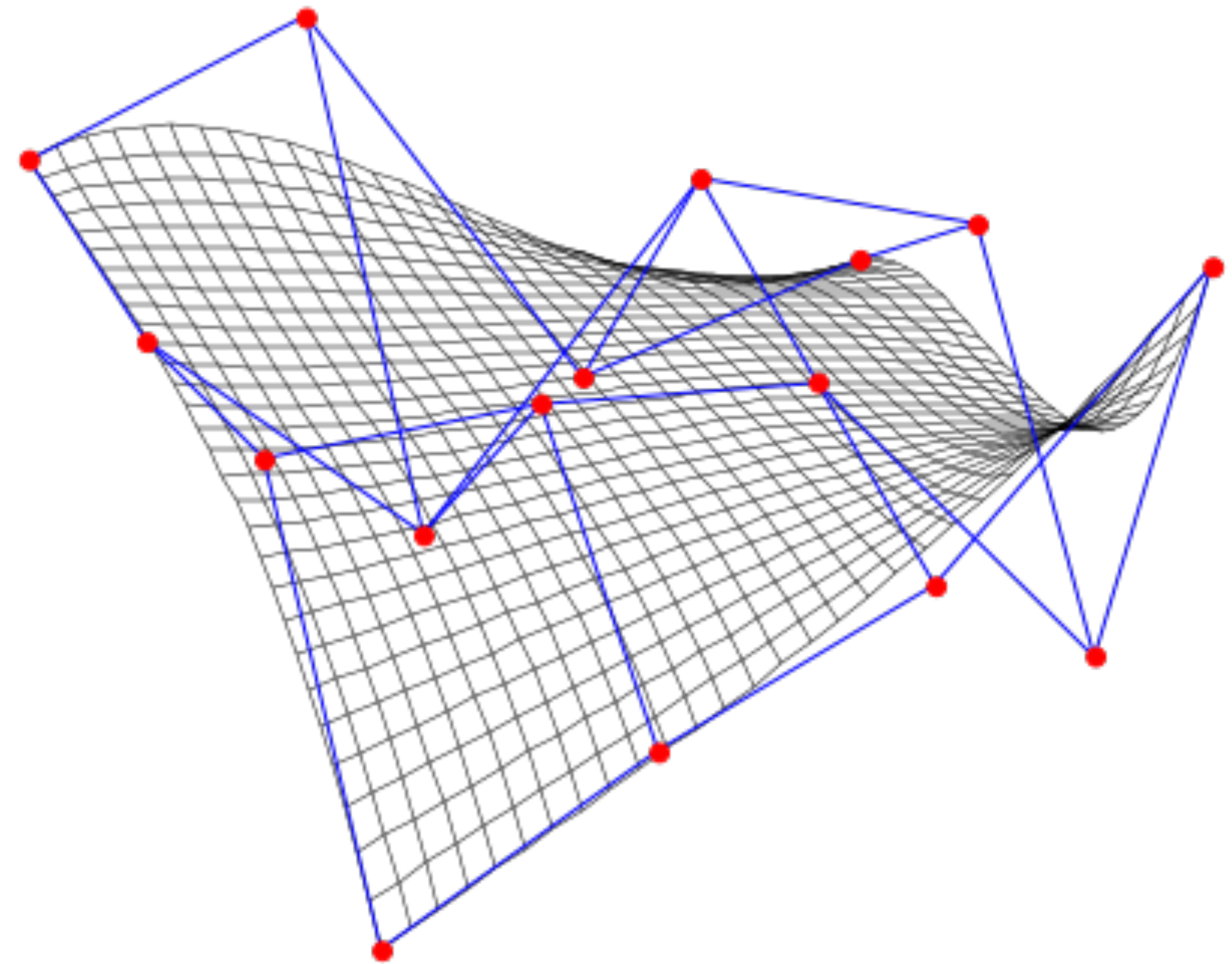
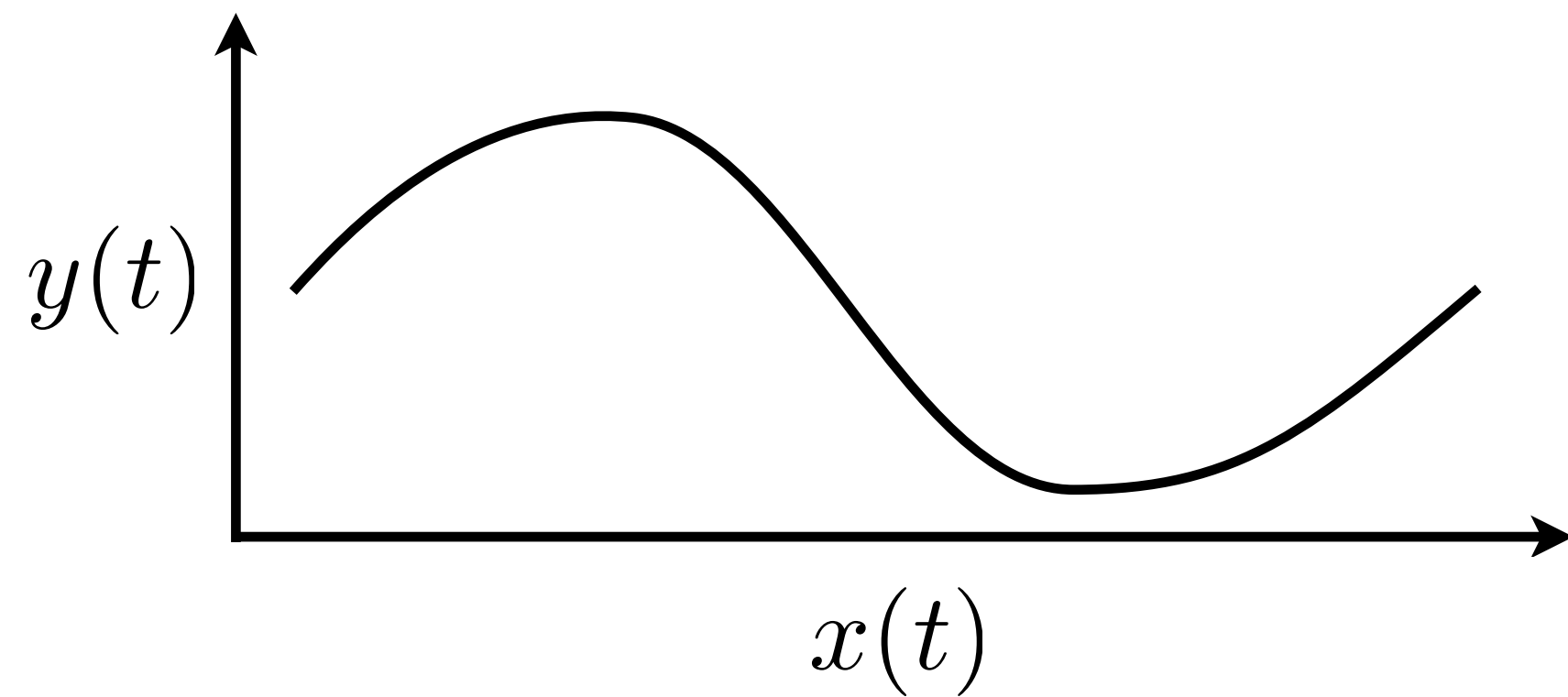
If a sufficiently smooth mapping can be constructed, we can look at its first and second derivatives
tangents, normals, curvature

How do we model shapes?



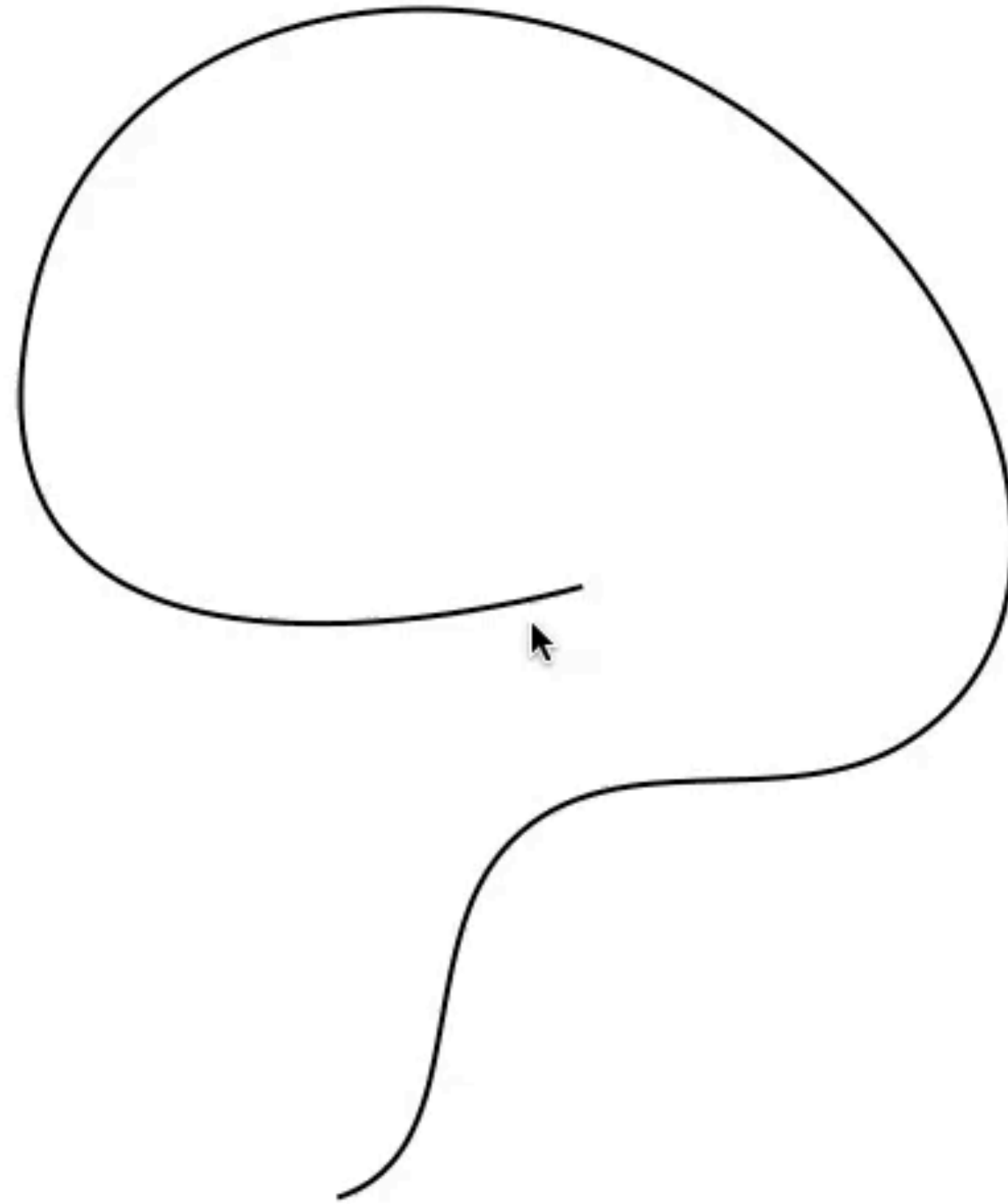
Delcam Plc. [CC BY-SA 3.0 (<http://creativecommons.org/licenses/by-sa/3.0>) or GFDL (<http://www.gnu.org/copyleft/fdl.html>)], via Wikimedia Commons

Building blocks: curves and surfaces

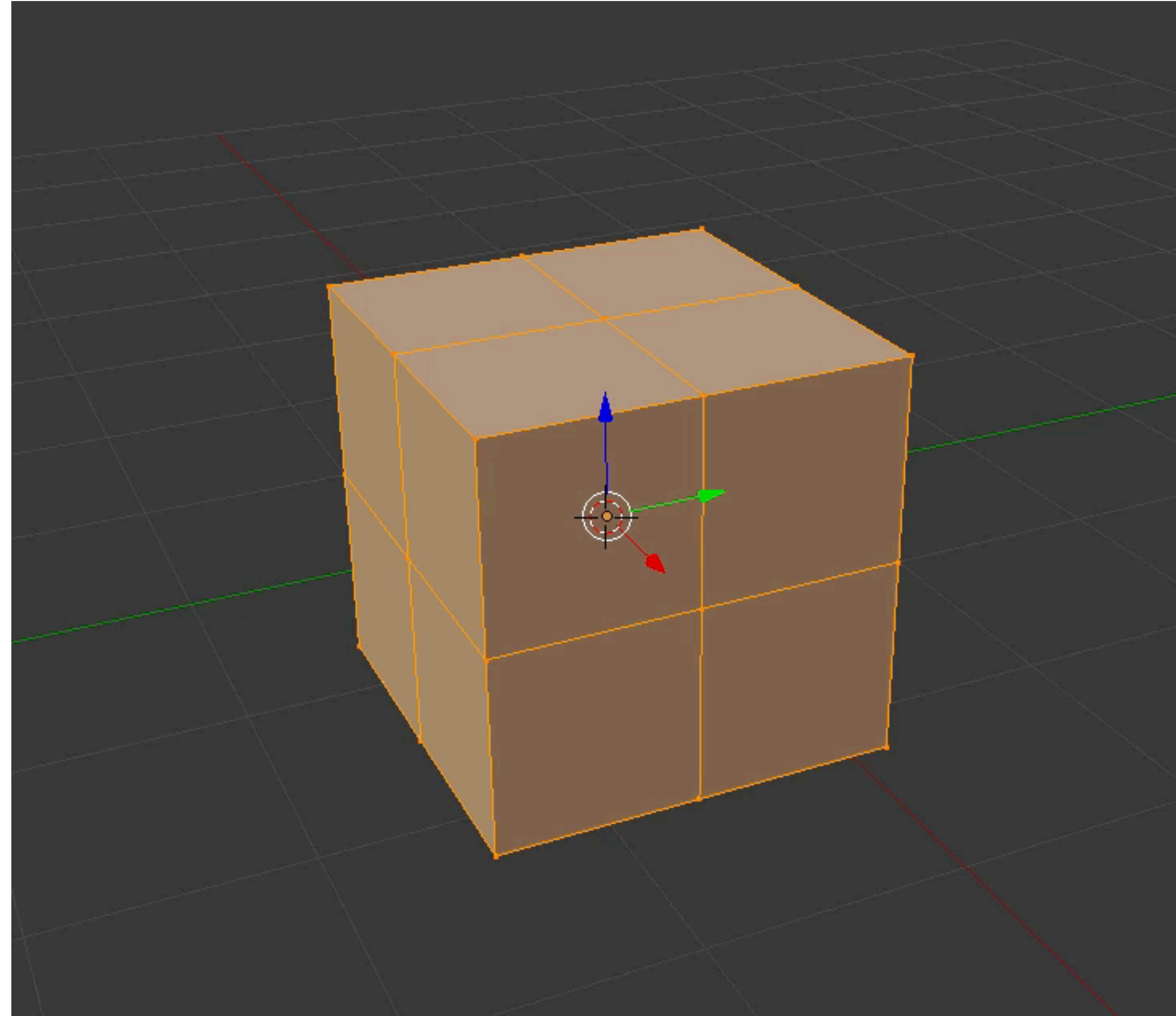


By Wojciech mula at Polish Wikipedia, CC BY-SA 3.0, <https://commons.wikimedia.org/w/index.php?curid=9853555>

Keynote Demo



Blender Demo



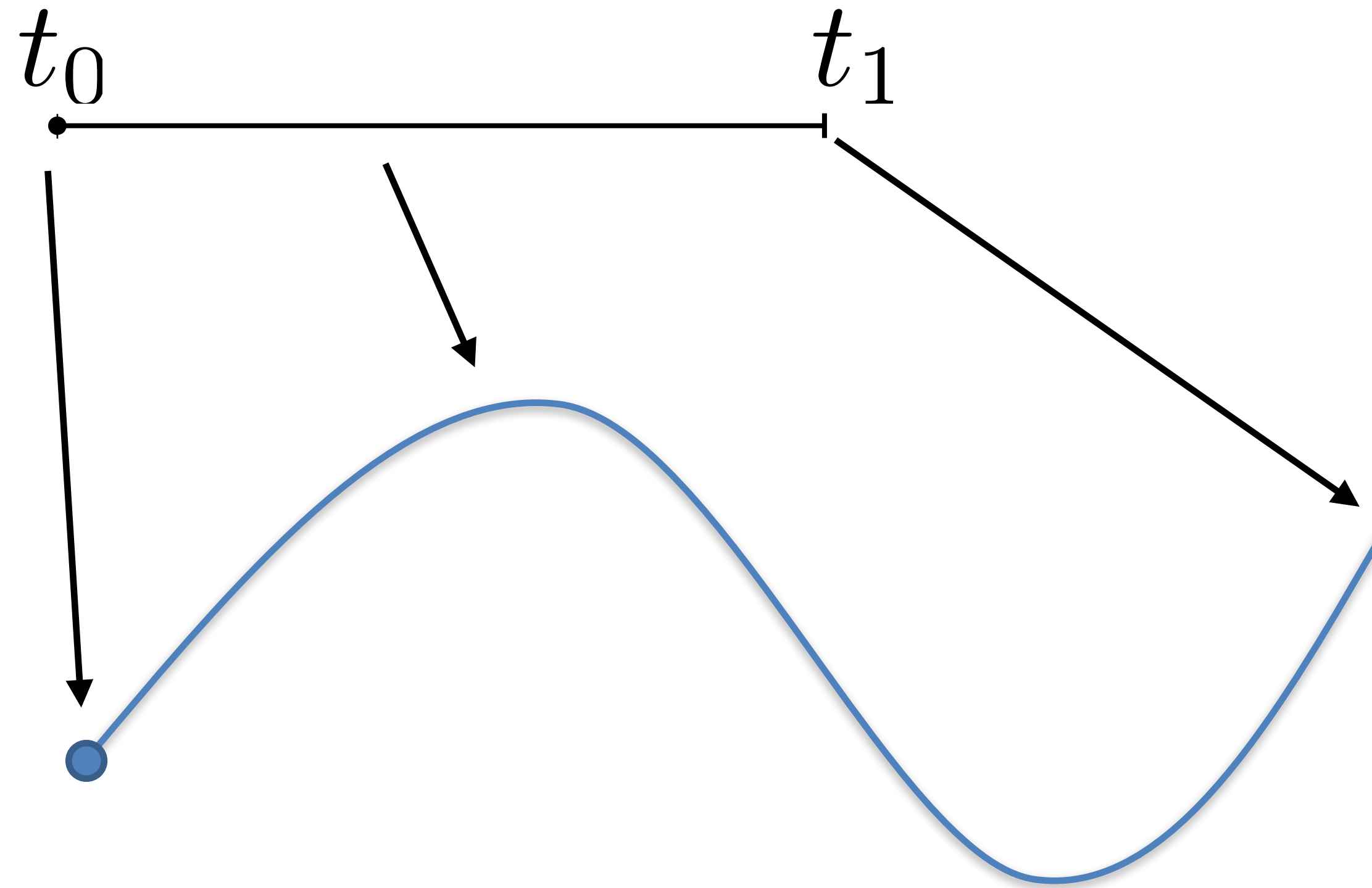
Modeling curves

- We need **mathematical concepts** to characterize the desired curve properties
- Notions from **curve geometry** help with designing user interfaces for curve creation and editing

2D parametric curve

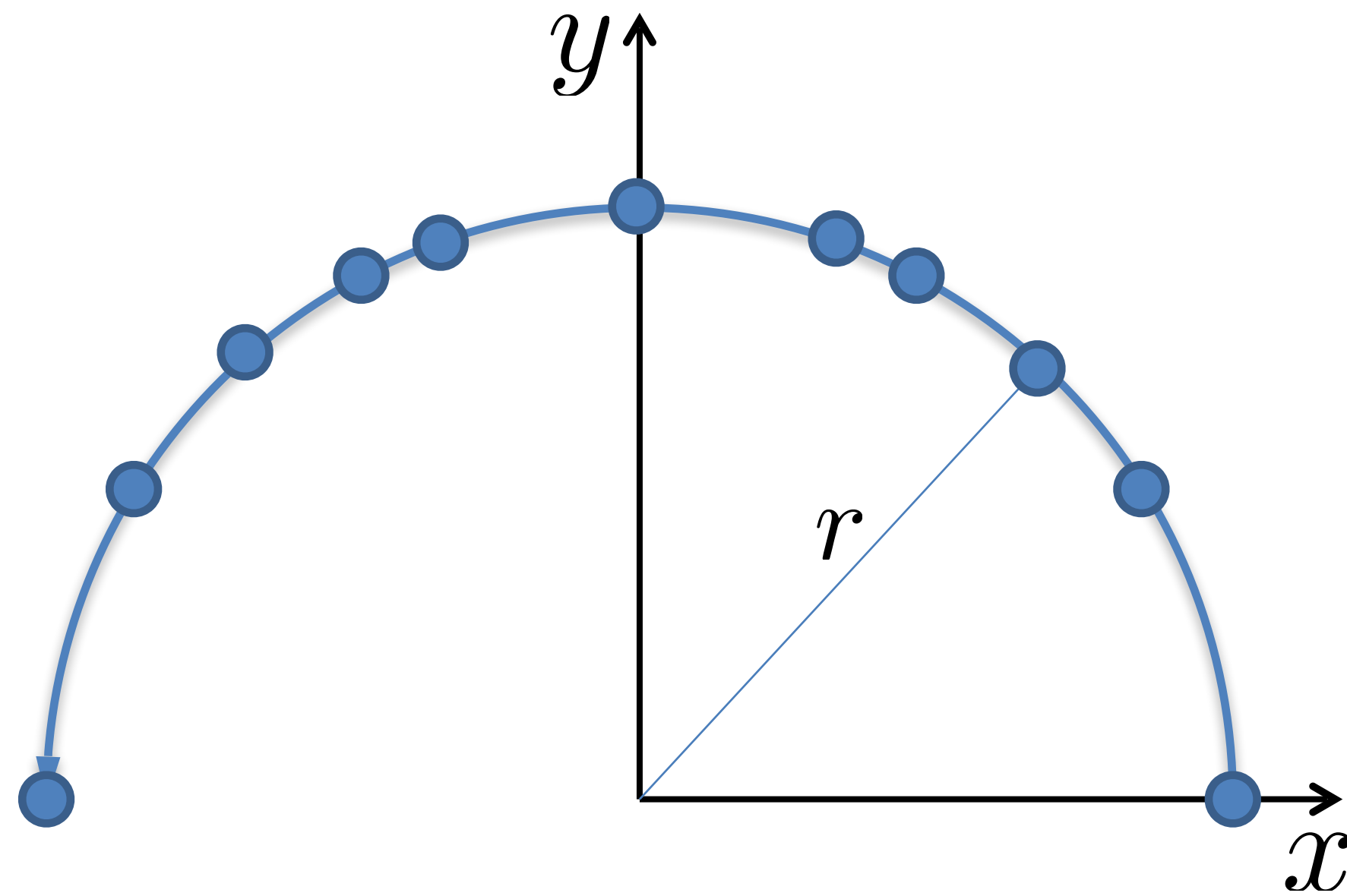
$$\mathbf{p}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad t \in [t_0, t_1]$$

• $\mathbf{p}(t)$ must be continuous

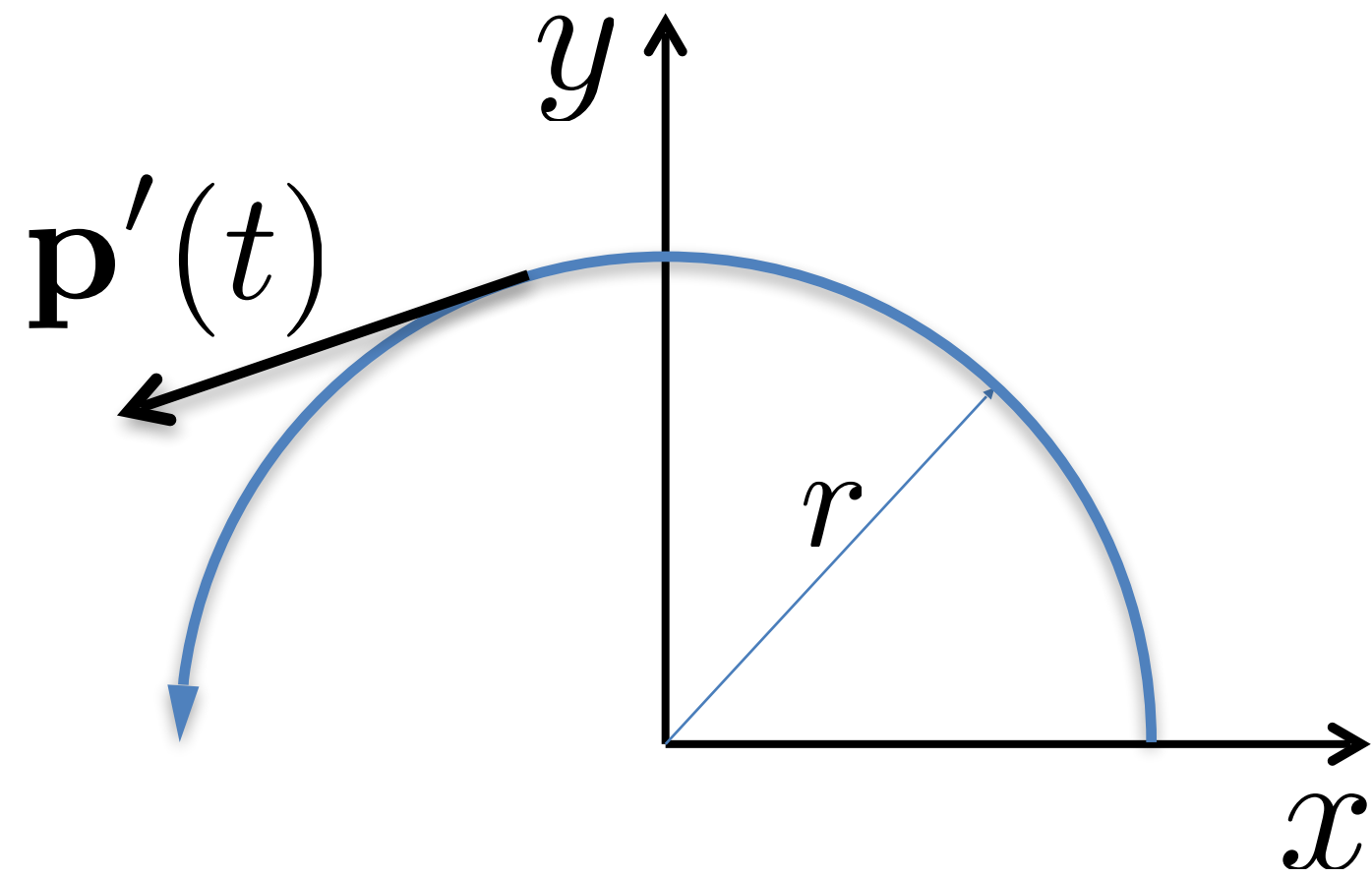


A curve can be parameterized in many different ways

$$\begin{pmatrix} r \cos t \\ r \sin t \end{pmatrix}, t \in [0, \pi] \qquad \begin{pmatrix} -rt \\ r\sqrt{1-t^2} \end{pmatrix}, t \in [-1, 1]$$



Tangent vector



$$\mathbf{p}(t) = \begin{pmatrix} r \cos t \\ r \sin t \end{pmatrix}$$

$$\mathbf{p}'(t) = \begin{pmatrix} -r \sin t \\ r \cos t \end{pmatrix}$$

$$\|\mathbf{p}'(t)\| = \text{speed}$$

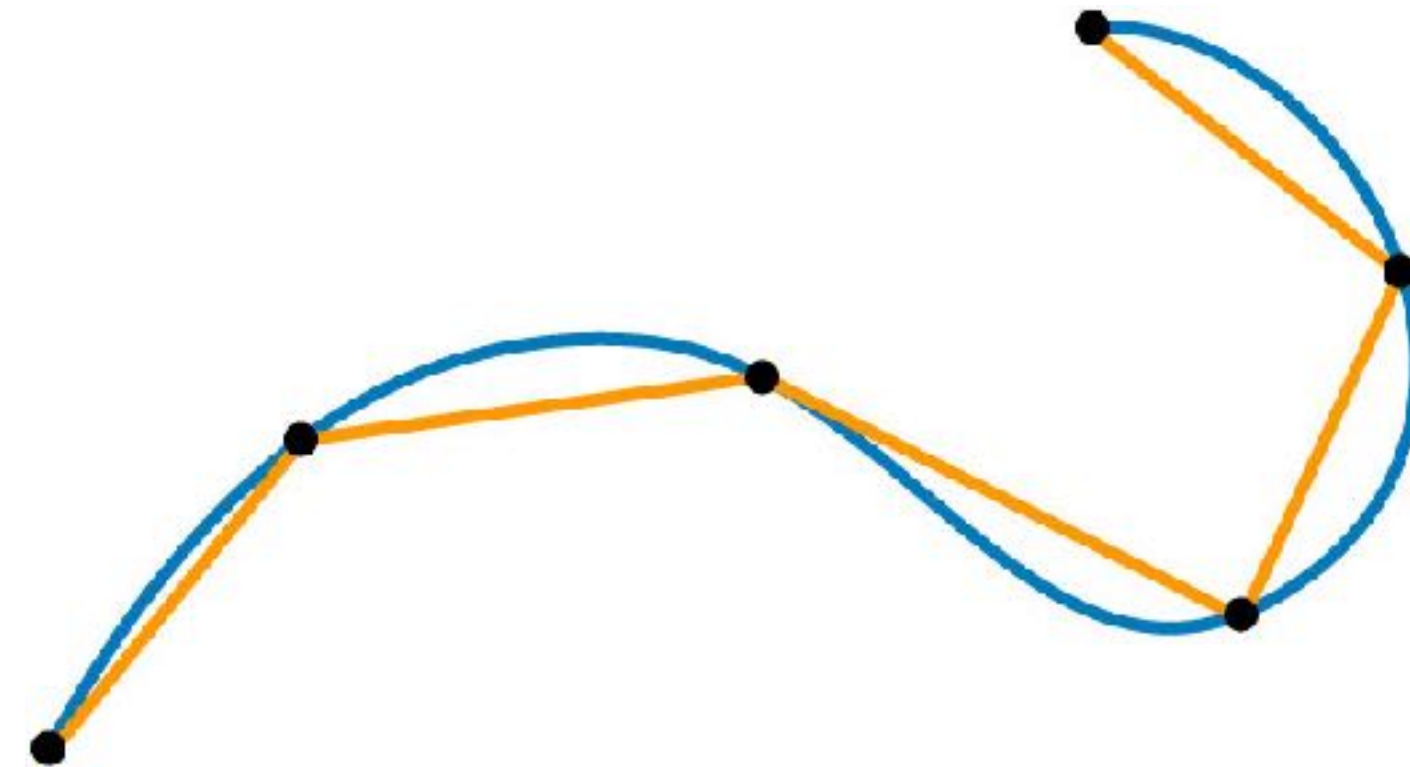
$$\frac{\mathbf{p}'(t)}{\|\mathbf{p}'(t)\|} = \mathbf{T}(t) = \text{unit tangent}$$

Parametrization-independent!

Arc length

- How long is the curve between t_0 and t ? How far does the particle travel?
- Speed is $\|\mathbf{p}'(t)\|$, so:

$$s(t) = \int_{t_0}^t \|\mathbf{p}'(t)\| dt$$

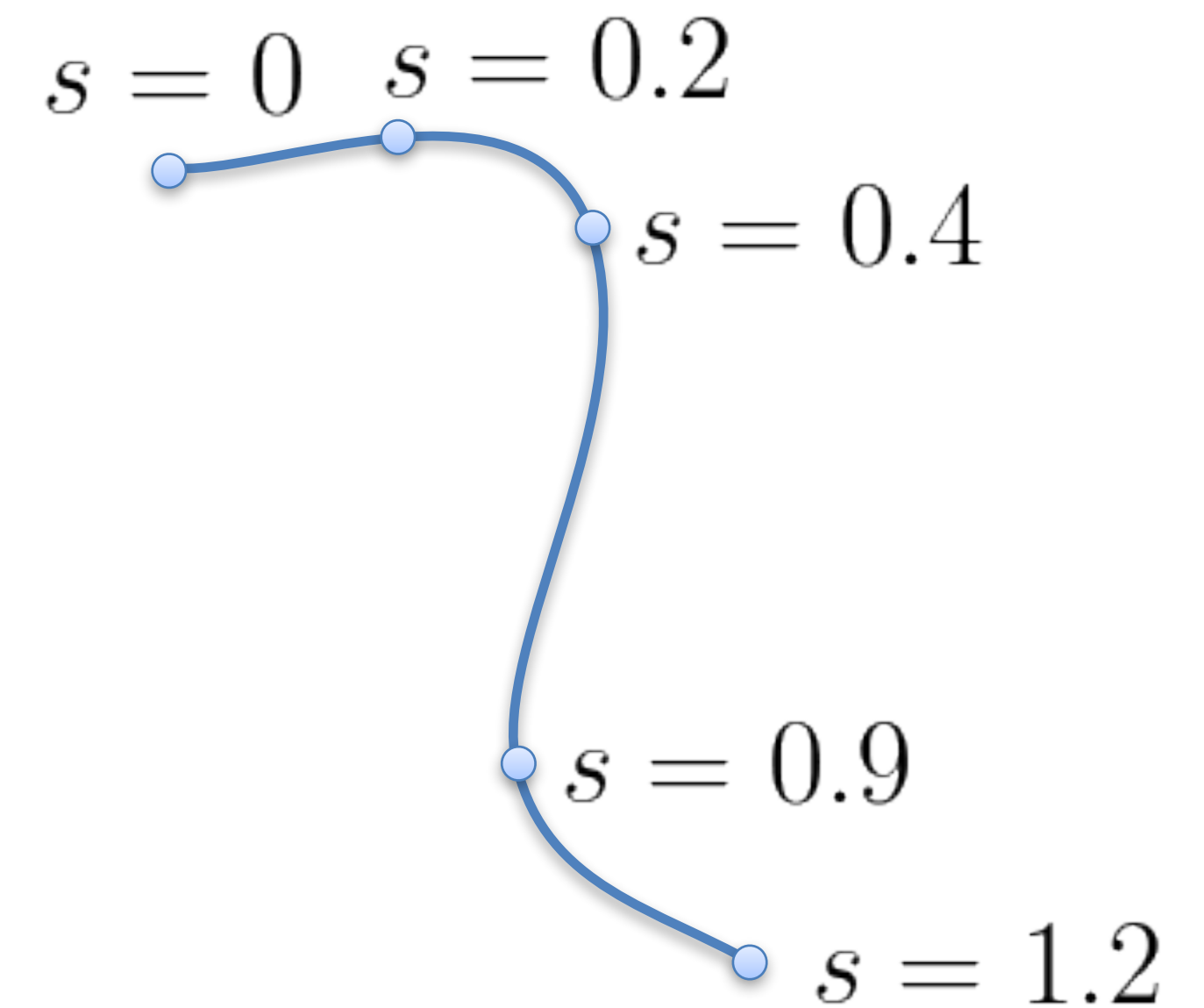


- Speed is nonnegative, so $s(t)$ is non-decreasing

Arc length parameterization

- Every curve has a natural parameterization:

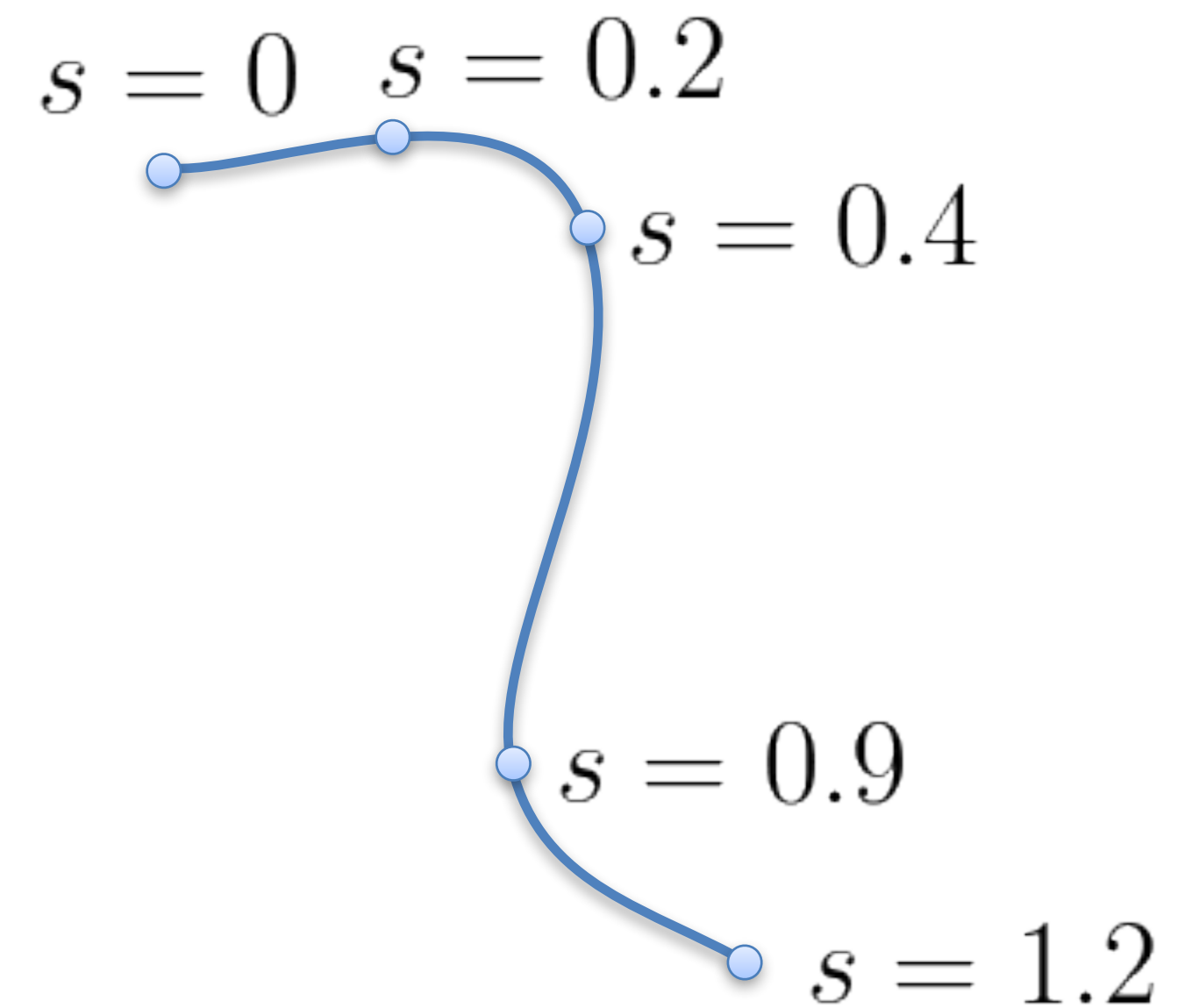
$\mathbf{p}(s)$, such that $\|\mathbf{p}'(s)\| = 1$



Arc length parameterization

- Every curve has a natural parameterization:

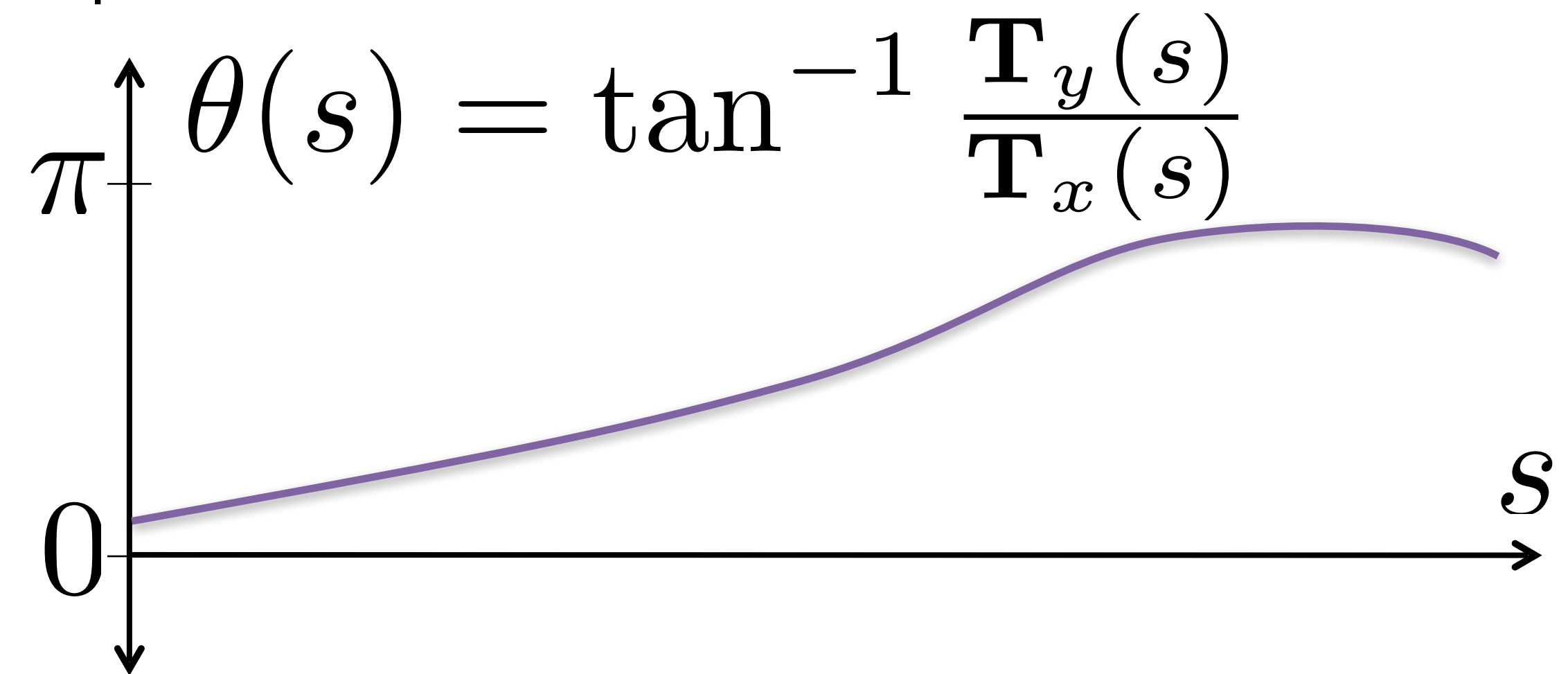
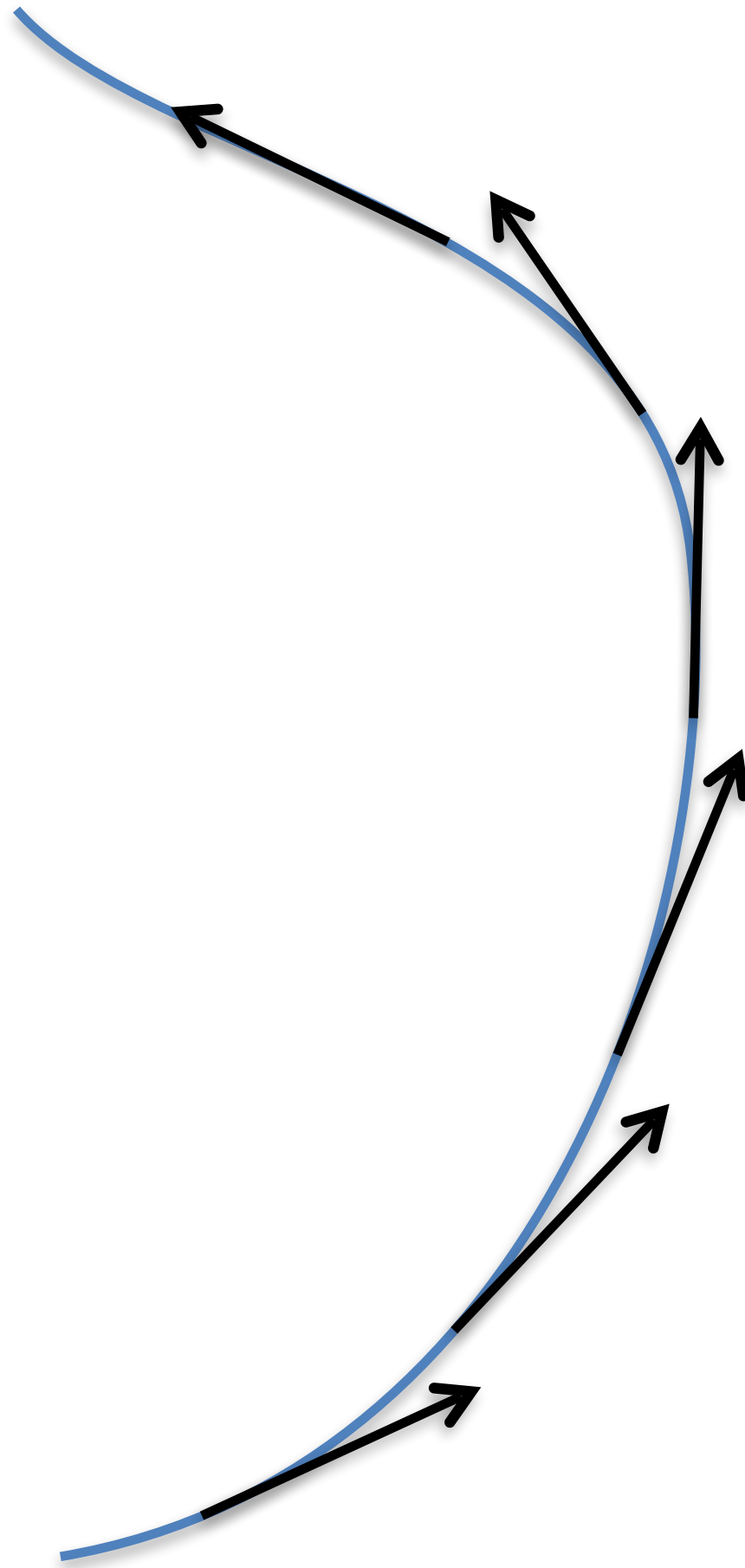
$$\mathbf{p}(s), \text{ such that } \|\mathbf{p}'(s)\| = 1$$



- Isometry between parameter domain and curve
- Tangent vector is unit-length: $\mathbf{p}'(s) = \mathbf{T}(s)$

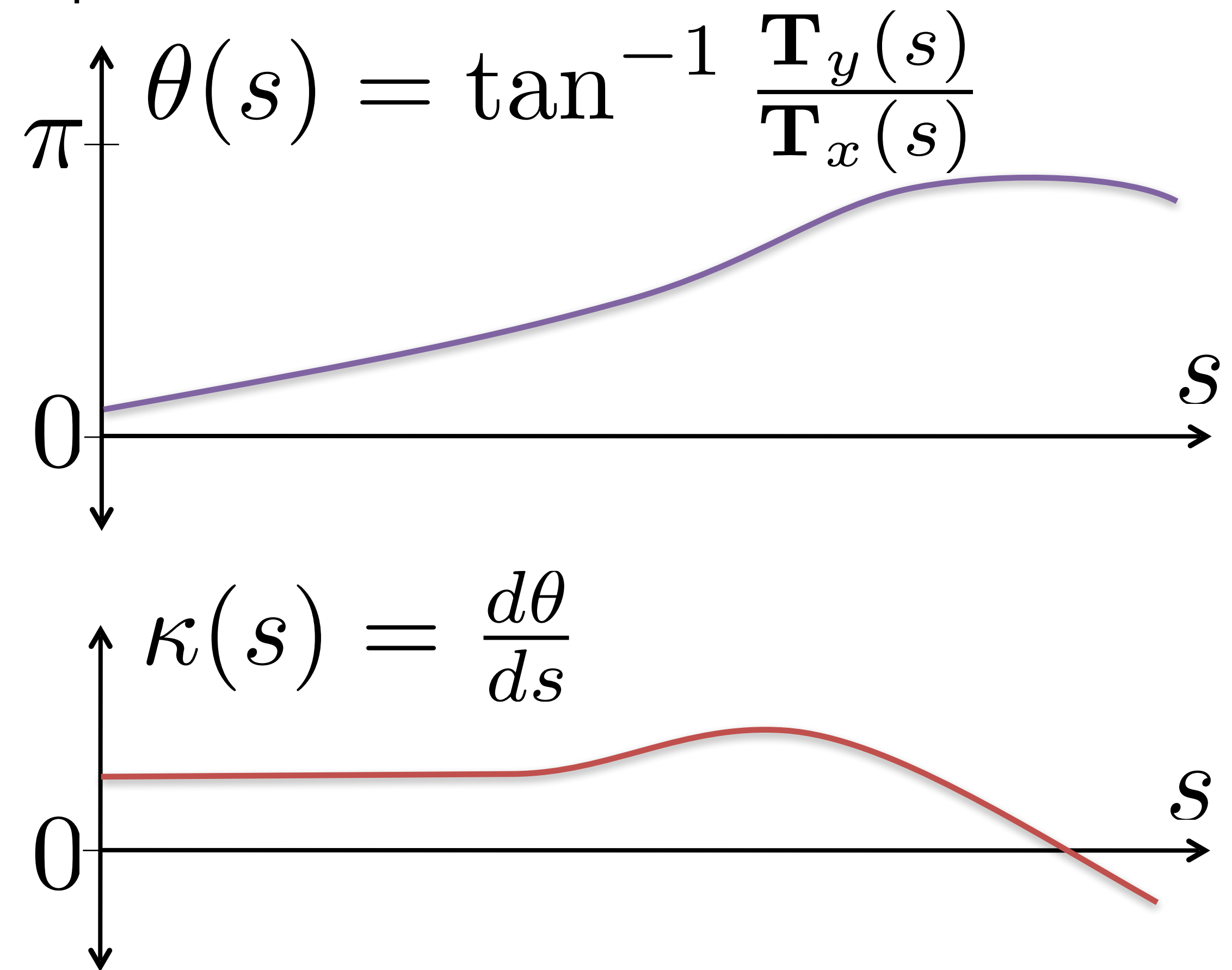
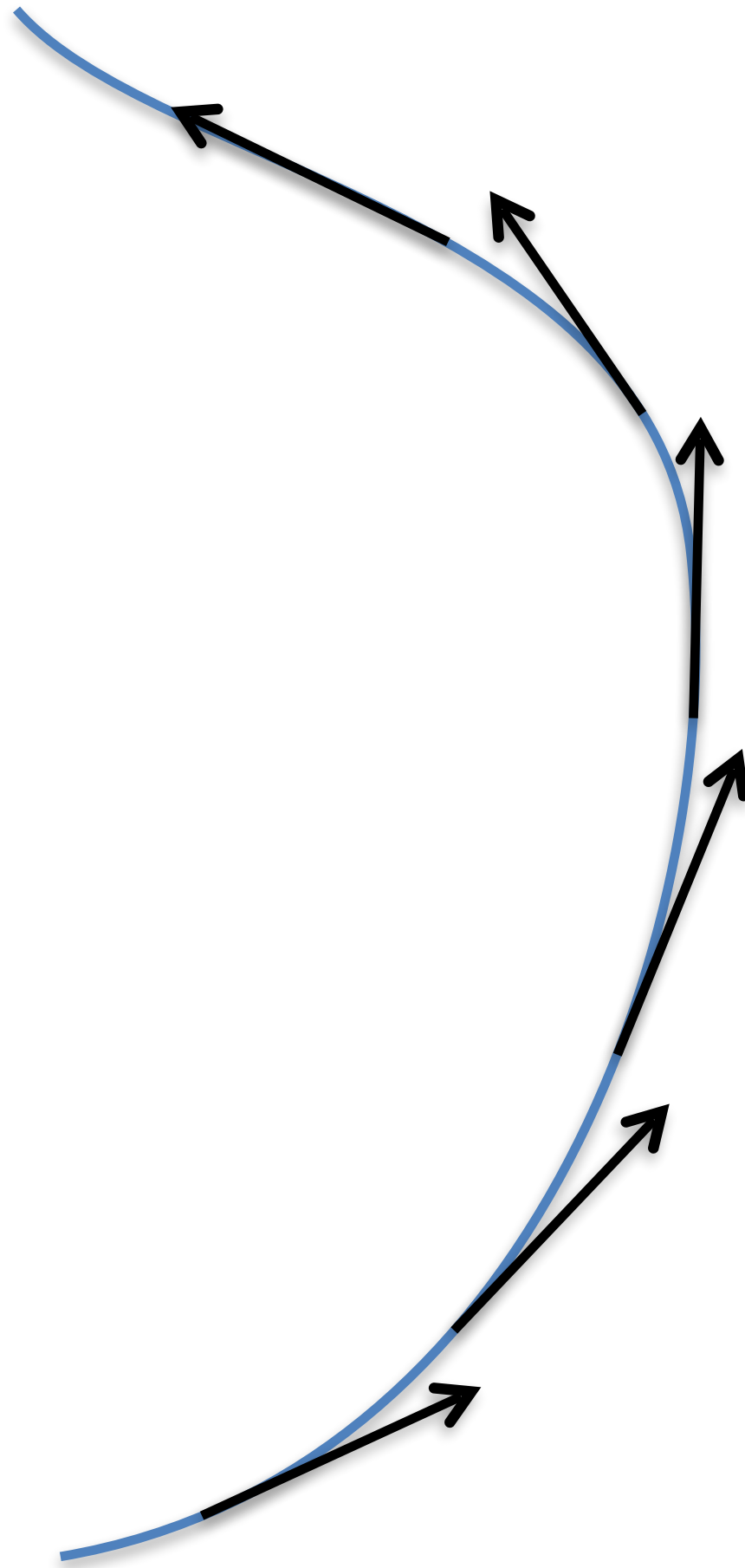
Curvature

- How much does the curve turn per unit s ?



Curvature

- How much does the curve turn per unit s ?



Curvature profile

- Given $\kappa(s)$, we can get $\theta(s)$ up to a constant by integration.
Integrating

$$\mathbf{p}(s) = \mathbf{p}_0 + \int_{s_0}^{s_1} \begin{pmatrix} \cos \theta(s) \\ \sin \theta(s) \end{pmatrix} ds$$

reconstructs the curve up to rigid motion.

Curvature of a circle

- Curvature of a circle:

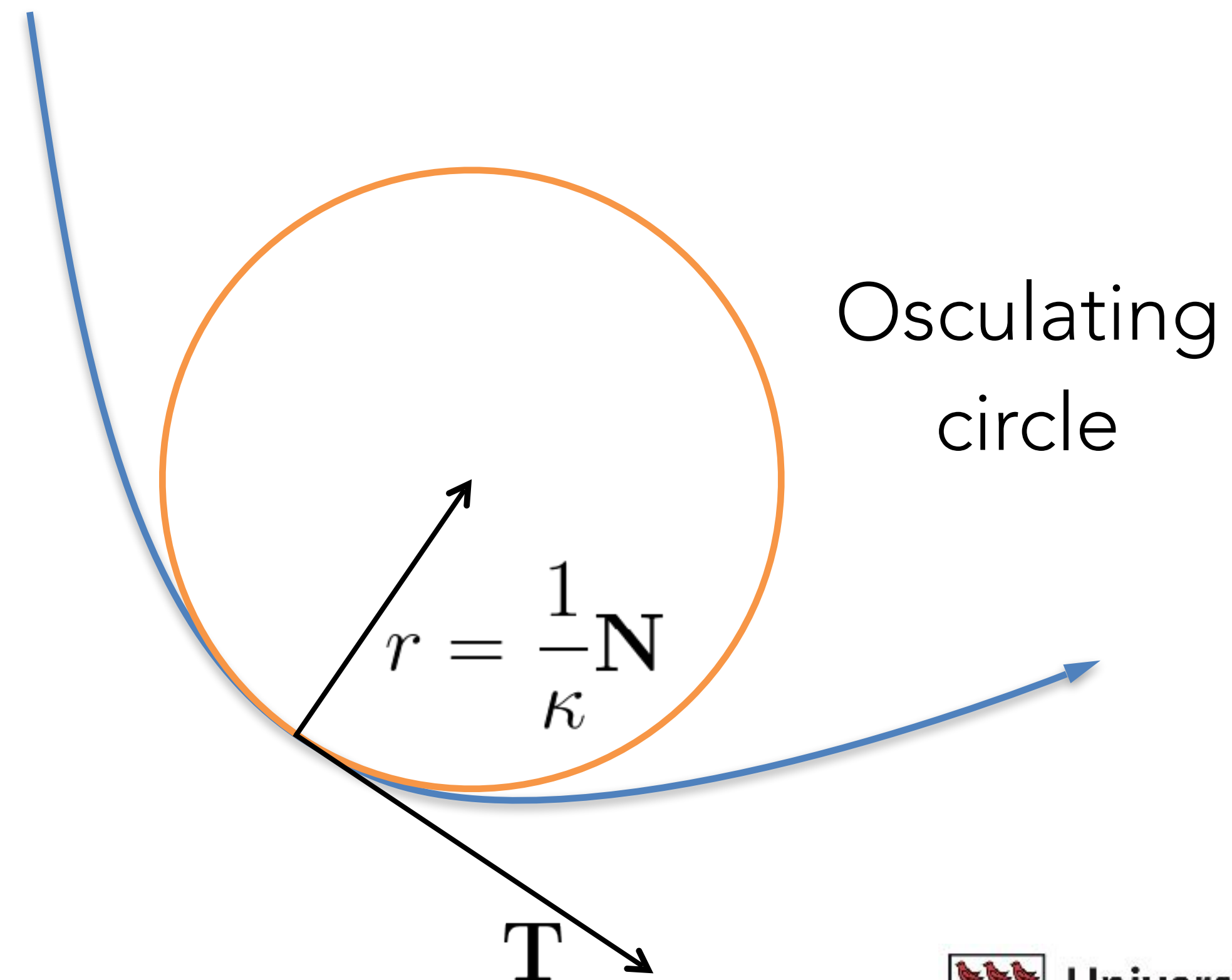
$$\mathbf{p}(s) = \begin{pmatrix} r \cos(s/r) \\ r \sin(s/r) \end{pmatrix}$$

$$\mathbf{p}'(s) = \begin{pmatrix} -\sin(s/r) \\ \cos(s/r) \end{pmatrix}$$

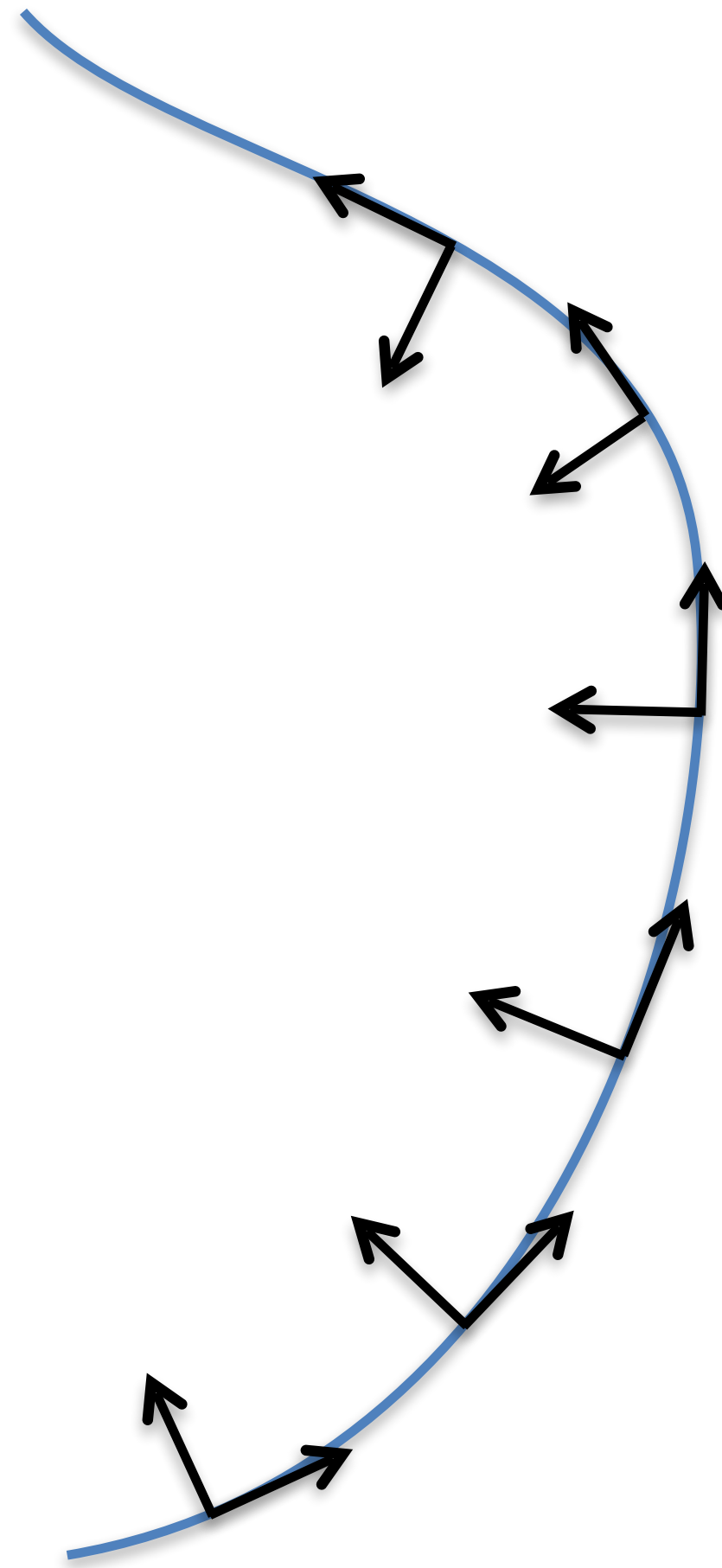
$$\begin{aligned} \theta(s) &= \tan^{-1} \frac{\cos(s/r)}{-\sin(s/r)} \\ &= s/r - \pi/2 \end{aligned}$$

$$\kappa(s) = 1/r$$

$$\mathbf{N}(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{T}(t) = \text{Unit Normal}$$



Frenet Frame



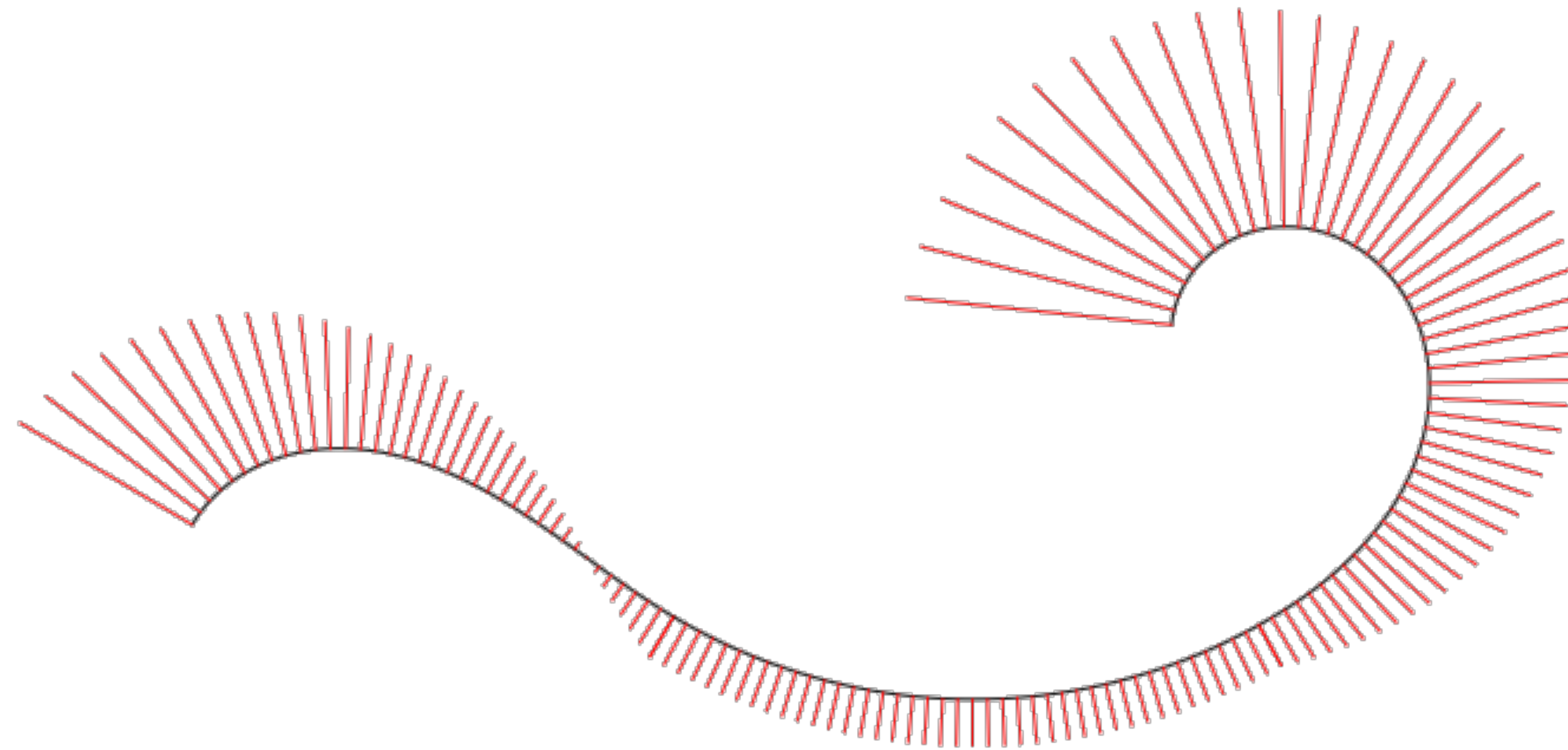
$$\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s)$$

$$\mathbf{N}'(s) = -\kappa(s)\mathbf{T}(s)$$

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}$$

Curvature normal

- Points inward $\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s)$
- $-\kappa(s)\mathbf{N}(s)$ useful for evaluating curve quality



Smoothness

Two kinds, parametric and geometric:

C^1 : $\mathbf{p}(t)$ is continuously differentiable

G^1 : $\mathbf{p}(s)$ is continuously differentiable



Parametrization-Independent

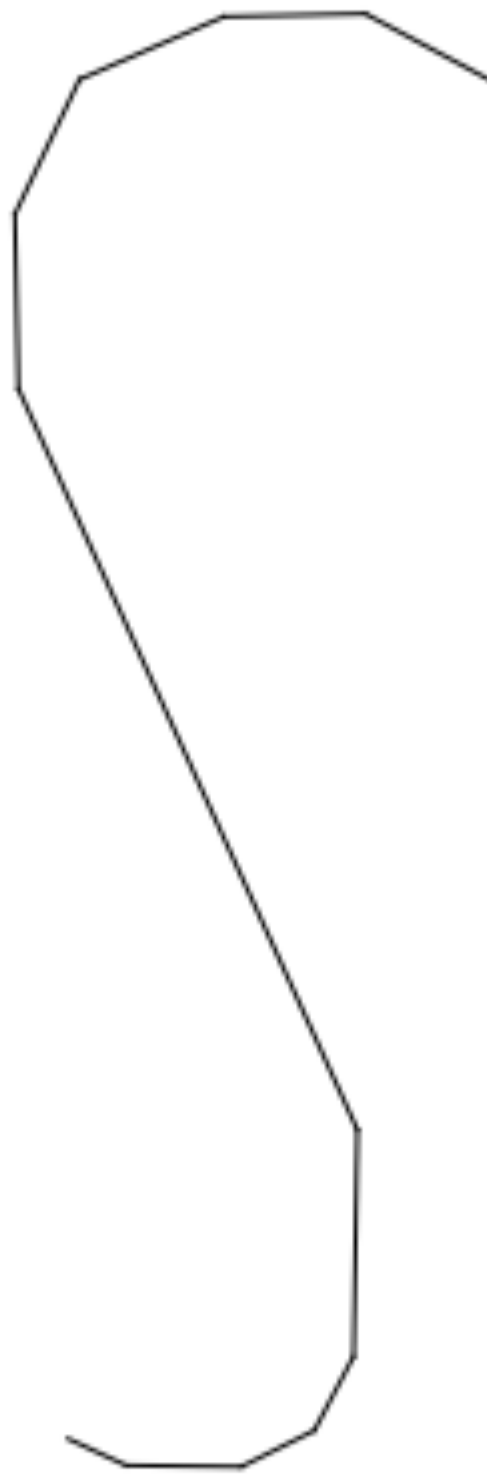
$$C^1 \quad \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

$$G^1 \quad \begin{pmatrix} \cos \hat{t} \\ \sin \hat{t} \end{pmatrix}, \quad \hat{t} = \begin{cases} t + 1 & \text{if } t < 1 \\ 2t & \text{if } t \geq 1 \end{cases}$$

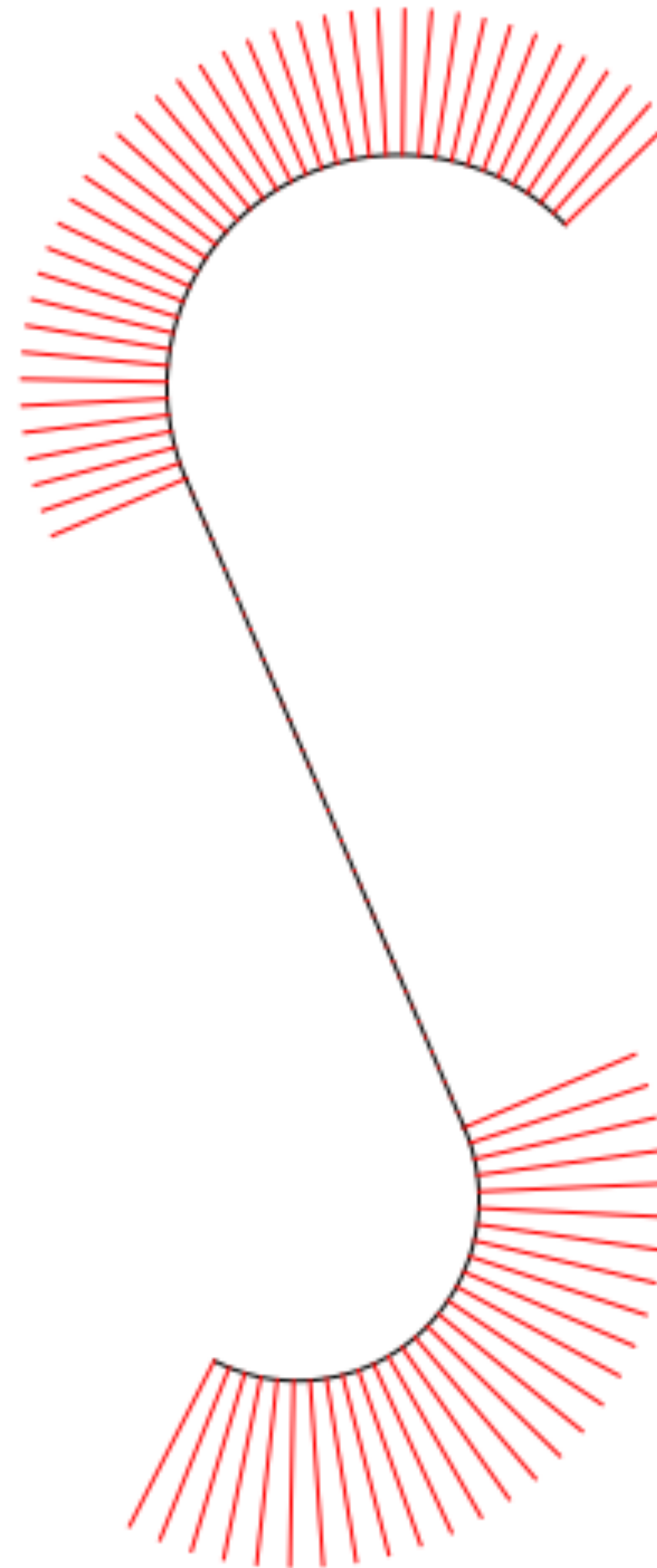


Smoothness example

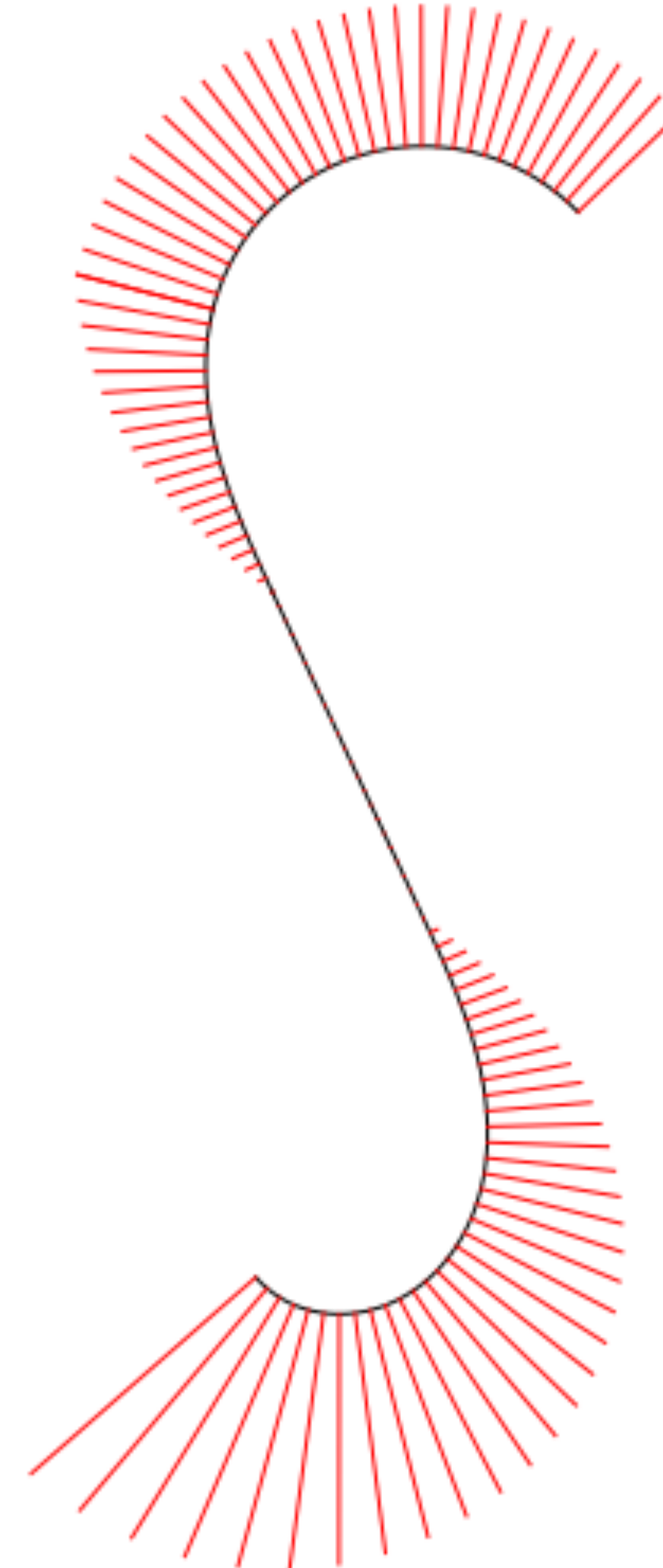
G^0



G^1

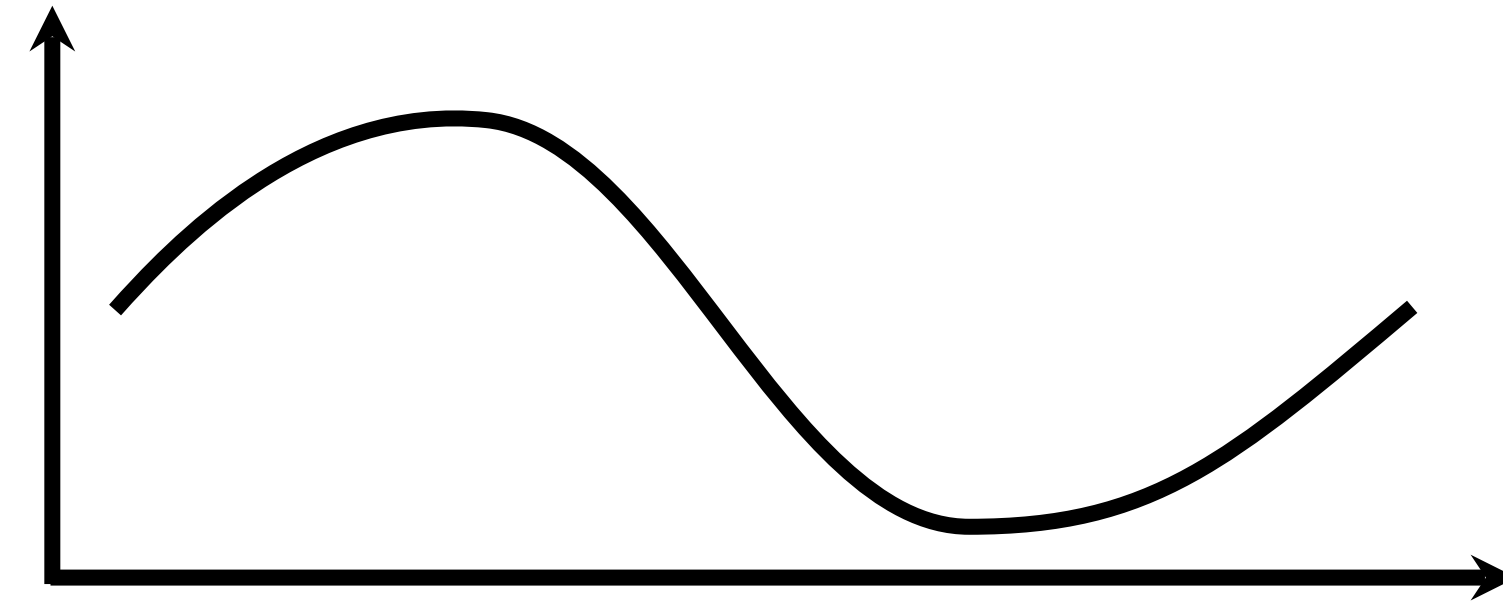


G^2



Recap on parametric curves

$$\mathbf{p}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad t \in [t_0, t_1]$$



$$\|\mathbf{p}'(t)\| = \text{speed}$$

$$s(t) = \int_{t_0}^t \|\mathbf{p}'(t)\| dt$$

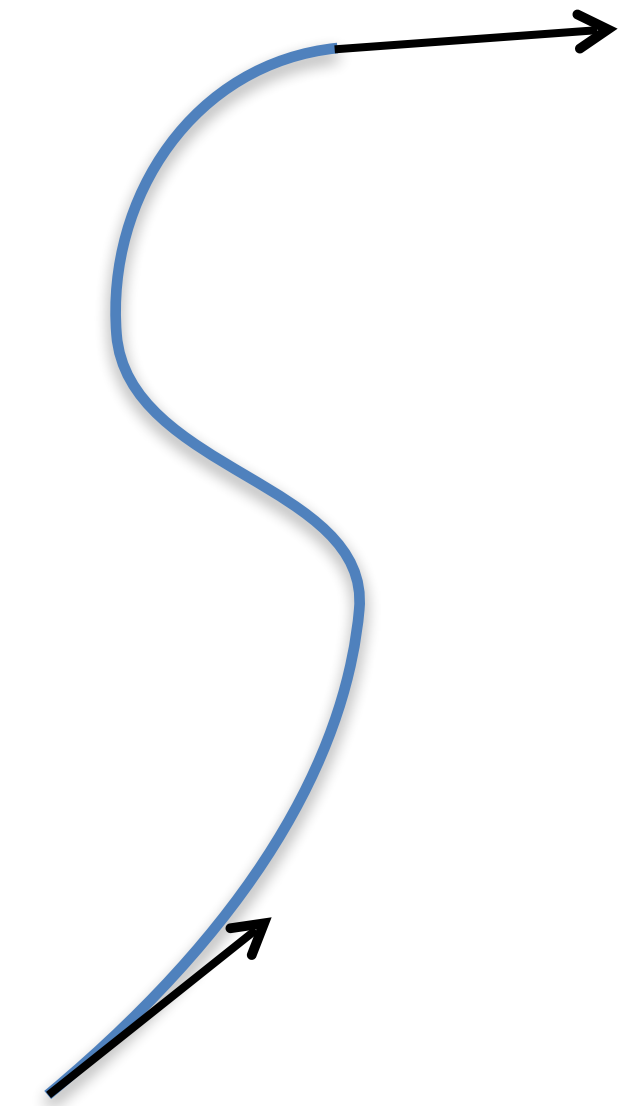
$$\kappa(s) = \frac{d\theta}{ds} = \mathbf{T}'(s) \cdot \mathbf{N}(s)$$

$$\frac{\mathbf{p}'(t)}{\|\mathbf{p}'(t)\|} = \mathbf{T}(t)$$

Turning

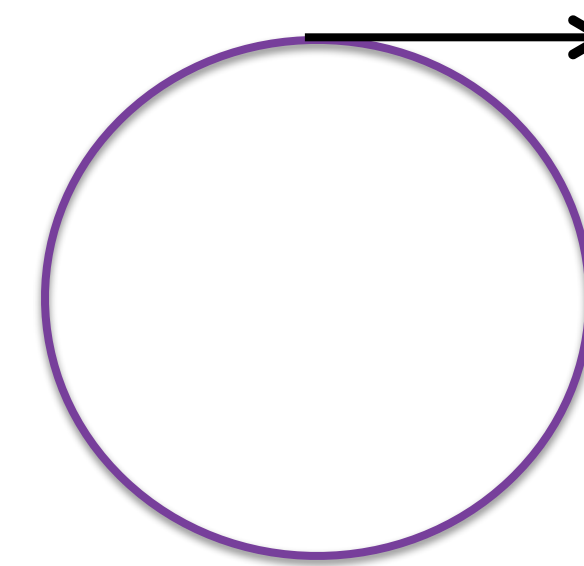
- Angle from start tangent to end tangent:

$$\int_{s_0}^{s_1} \kappa(s) ds = \int_{t_0}^{t_1} \kappa(t) \|\mathbf{p}'(t)\| dt$$



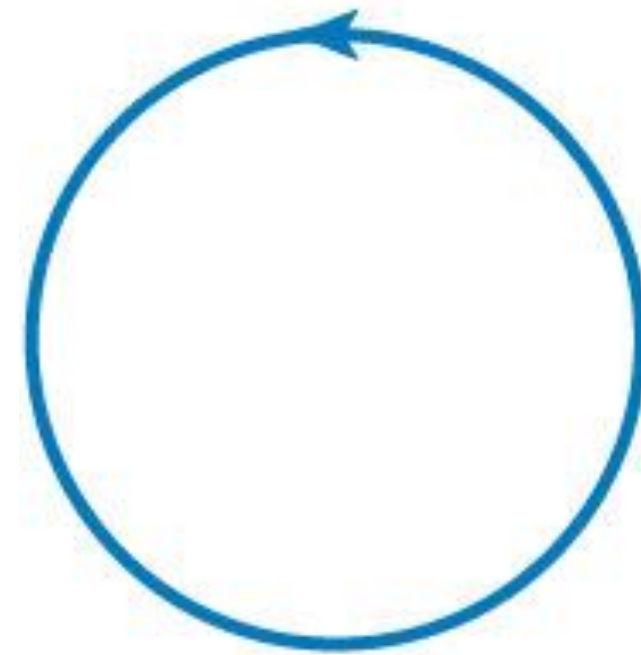
- If curve is closed, the tangent at the beginning is the same as the tangent at the end

$$\oint_{\mathbf{p}} \kappa(s) ds = 2\pi n$$

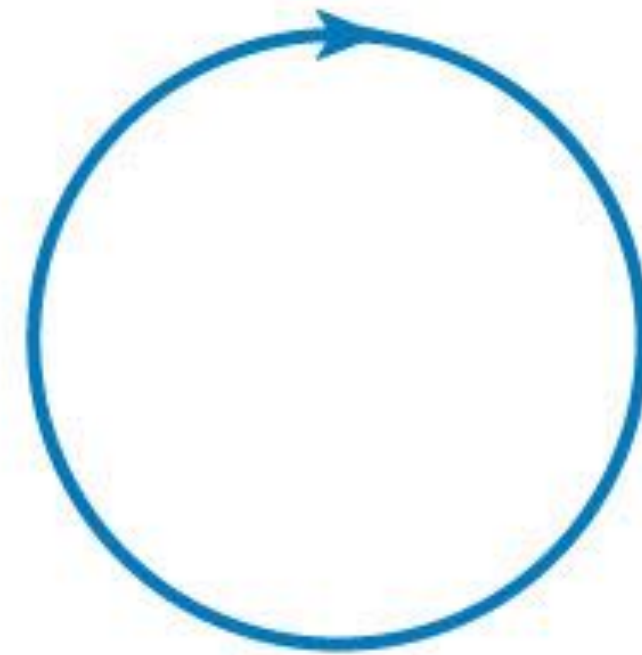


Turning numbers

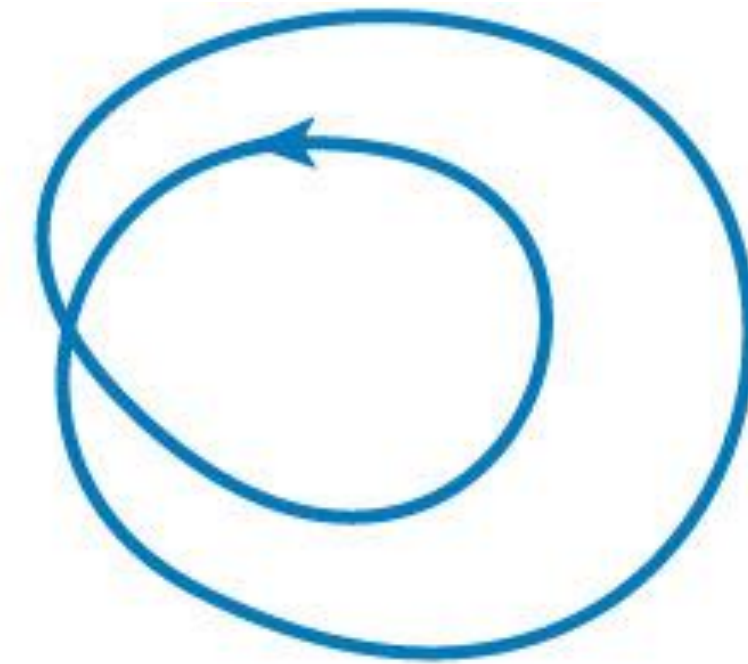
$$\oint_{\mathbf{p}} \kappa(s) ds = 2\pi n$$



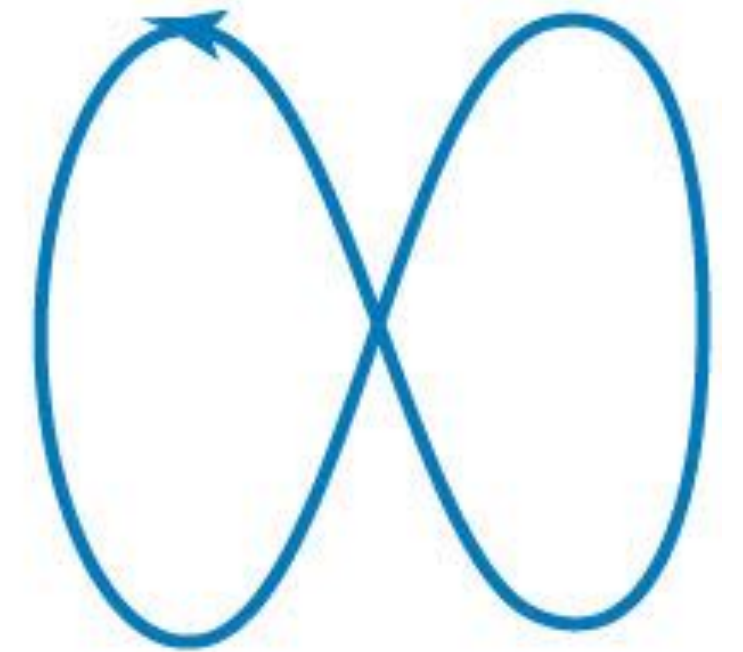
1



-1



2

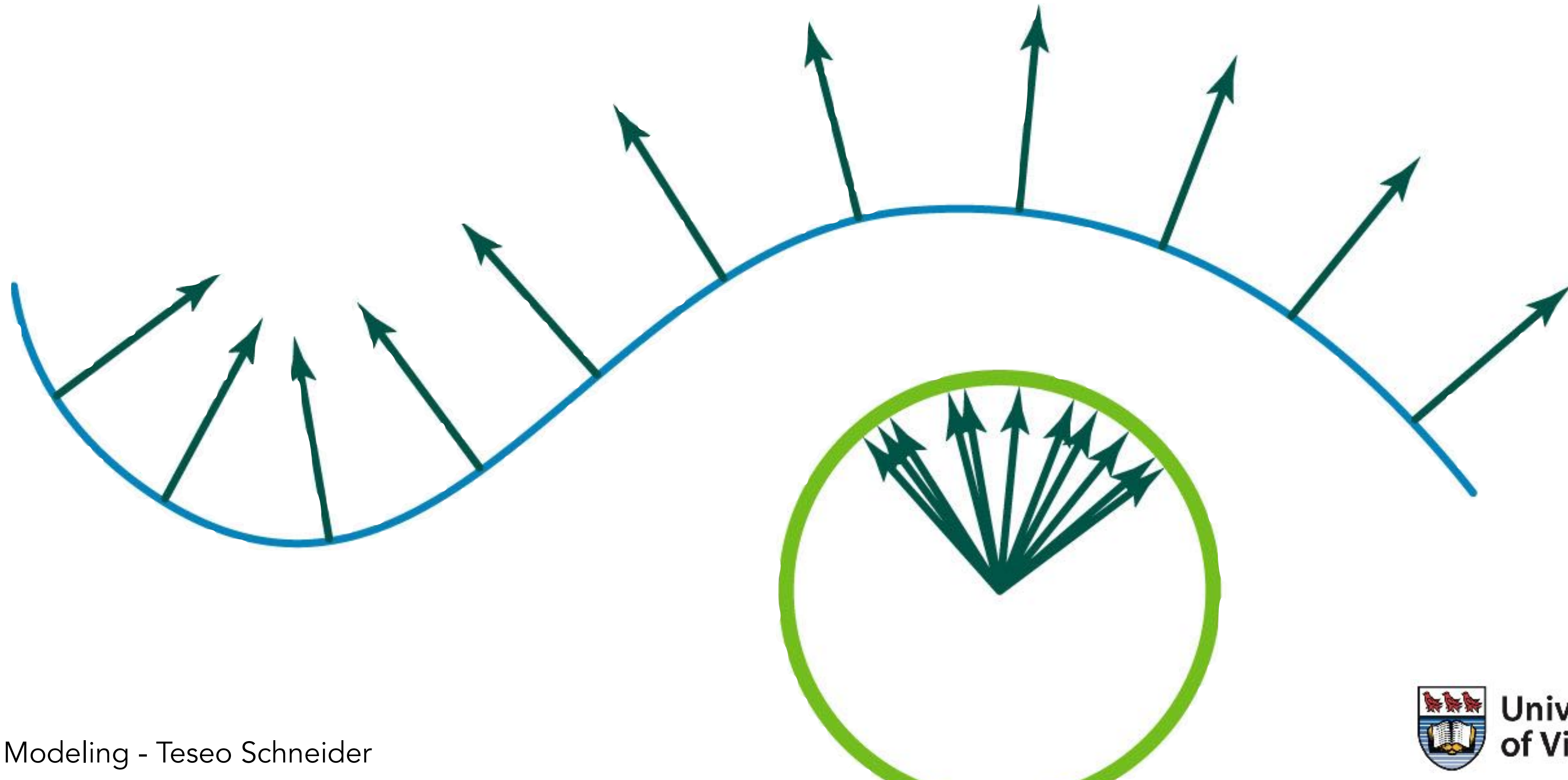


0

- n measures how many full turns the tangent makes.

Gauss map $\hat{n}(\mathbf{p})$

- Point on curve maps to point on unit circle.



Space curves (3D)

- In 3D, many vectors are orthogonal to \mathbf{T}

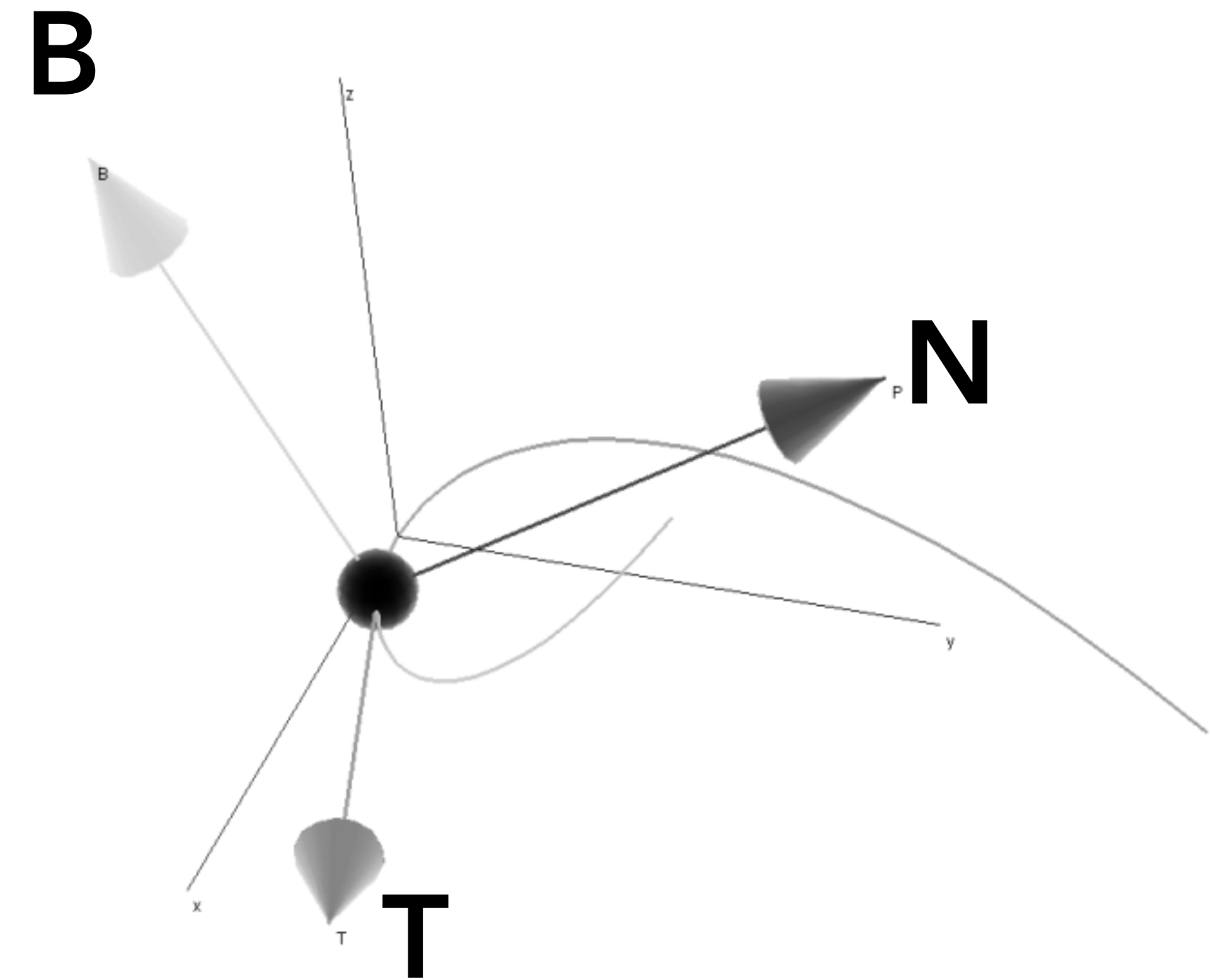
$$\mathbf{N}(s) := \mathbf{T}'(s) / \|\mathbf{T}'(s)\|$$

$$\mathbf{B}(s) := \mathbf{T}(s) \times \mathbf{N}(s)$$

- $\mathbf{T}, \mathbf{N}, \mathbf{B}$ are the “Frenet frame”

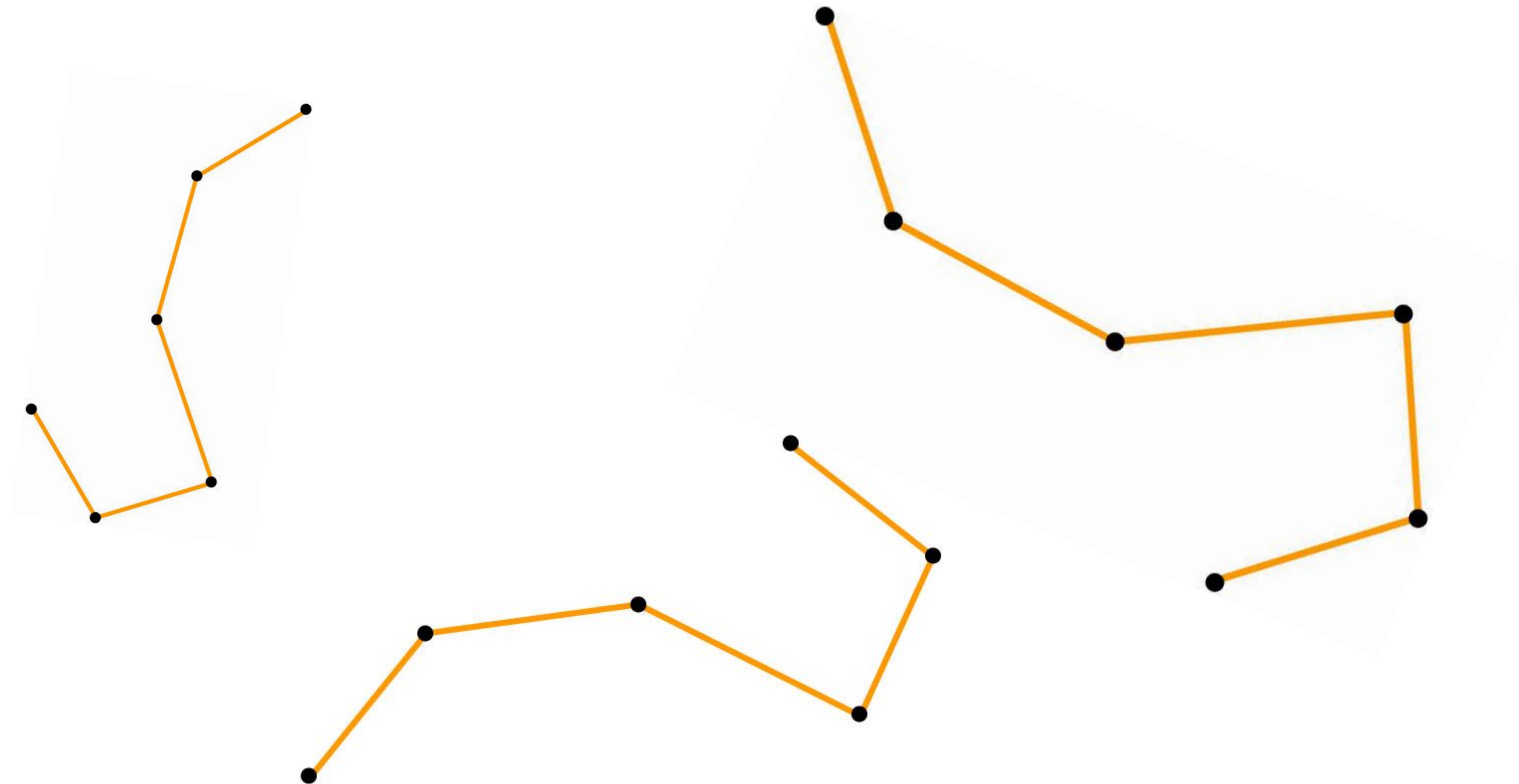
- τ is torsion: non-planarity

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} & \kappa & \\ -\kappa & & \\ & -\tau & \tau \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}$$



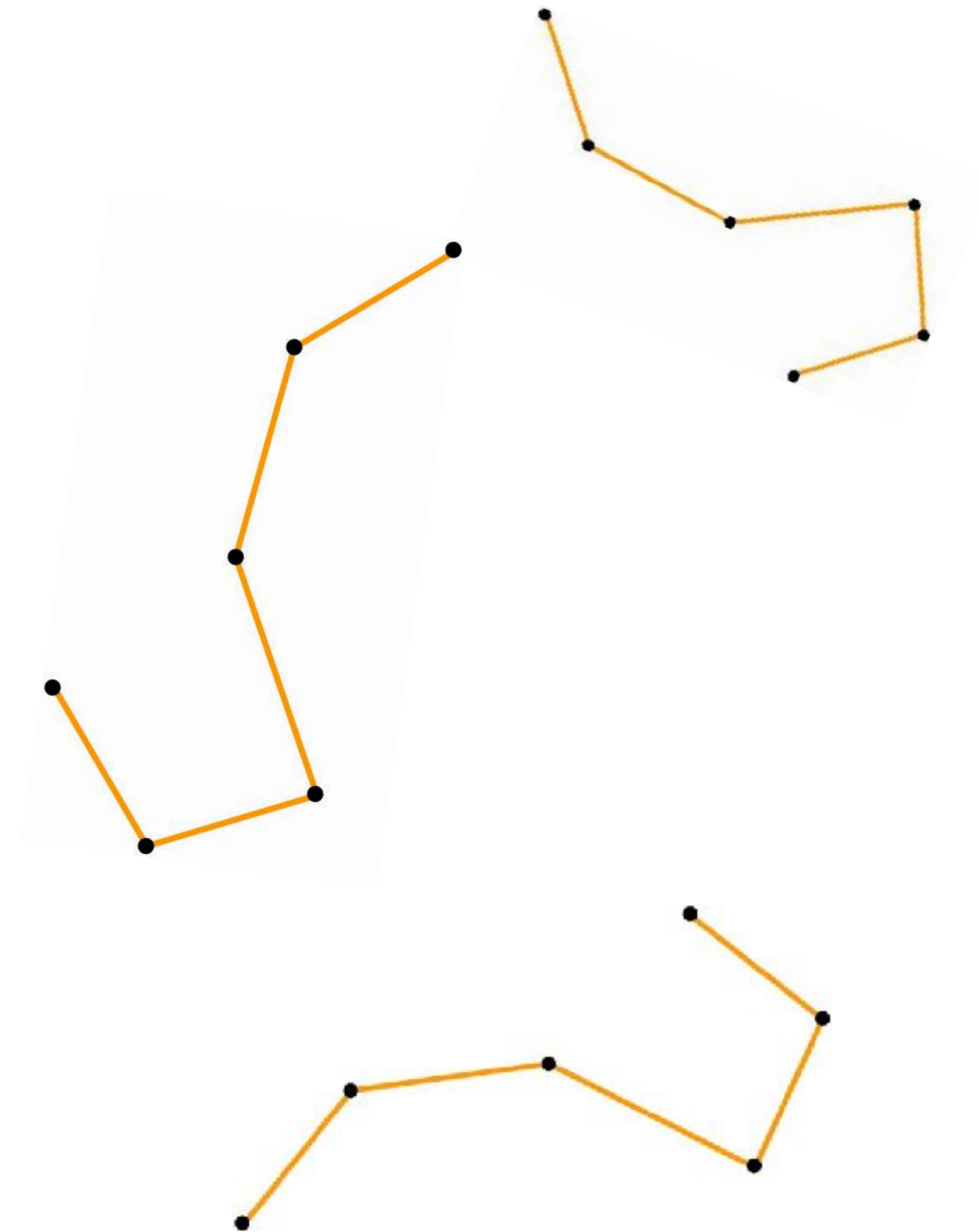
Discrete Differential Geometry of Curves

Discrete Planar Curves



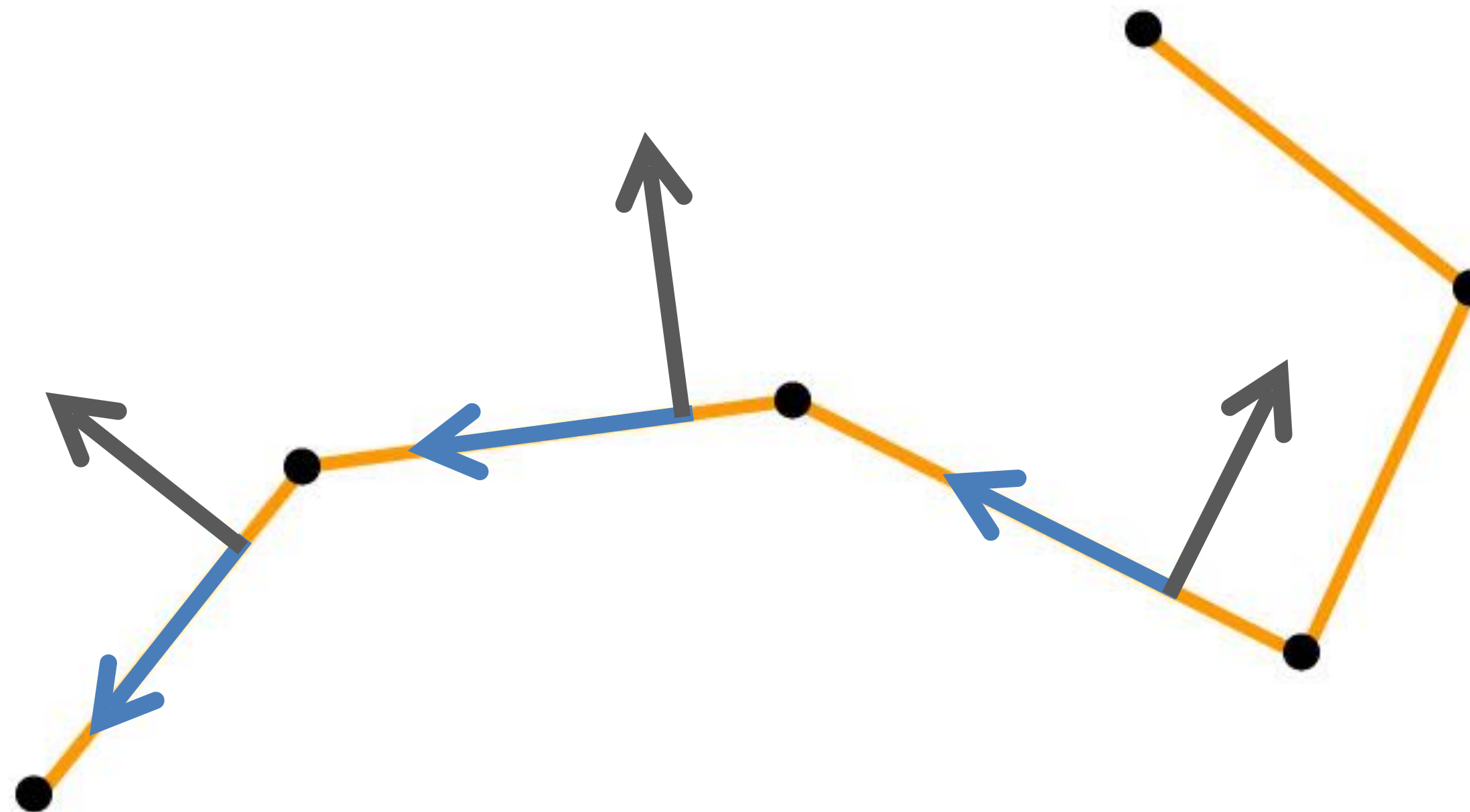
Discrete Planar Curves

- Piecewise linear curves
- Not smooth at vertices
- Can't take derivatives
- Generalize notions from the smooth world for the discrete case!
- **There is no one single way**



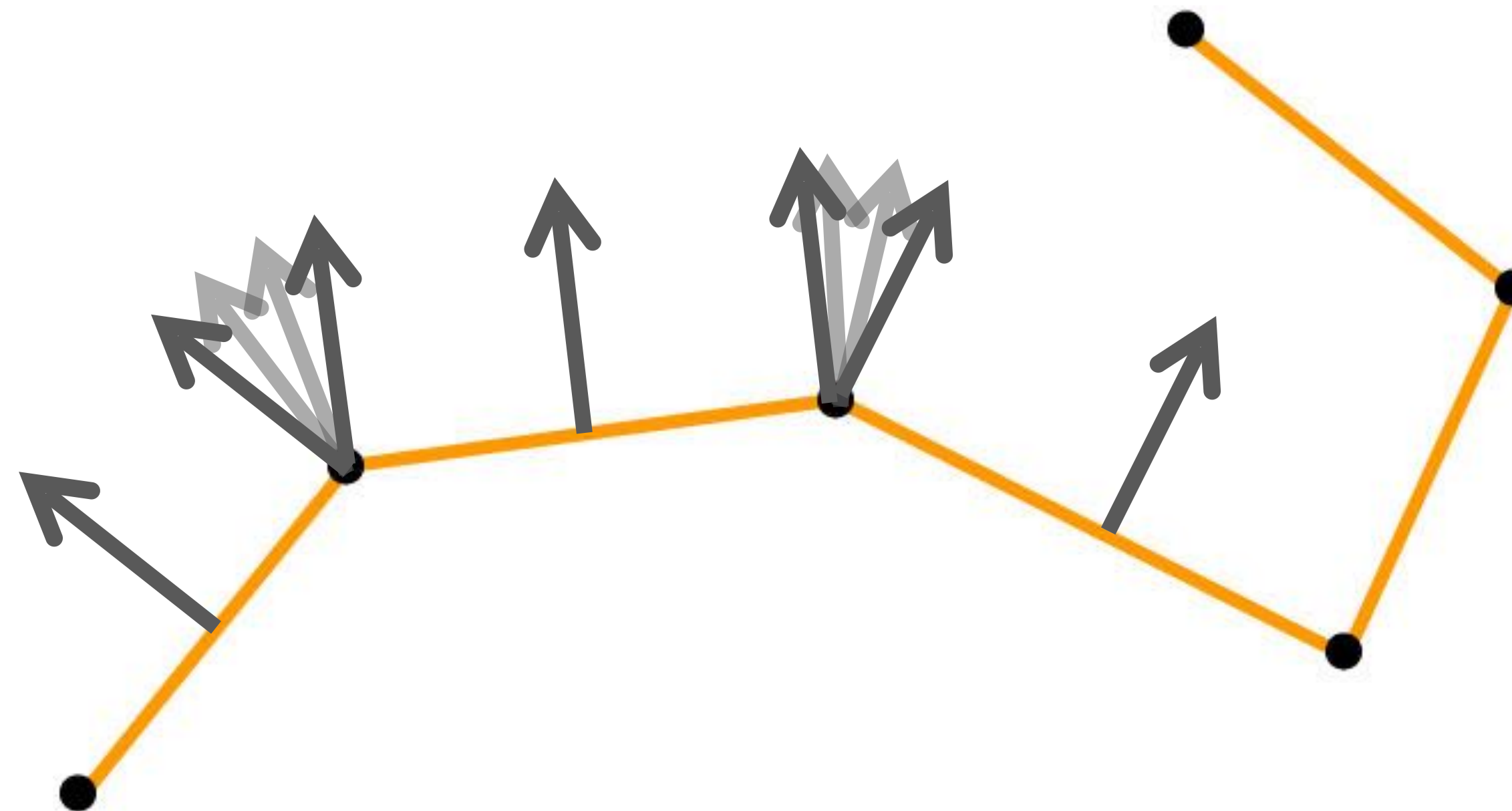
Tangents, Normals

- For any point on the edge, the tangent is simply the unit vector along the edge and the normal is the perpendicular vector



Tangents, Normals

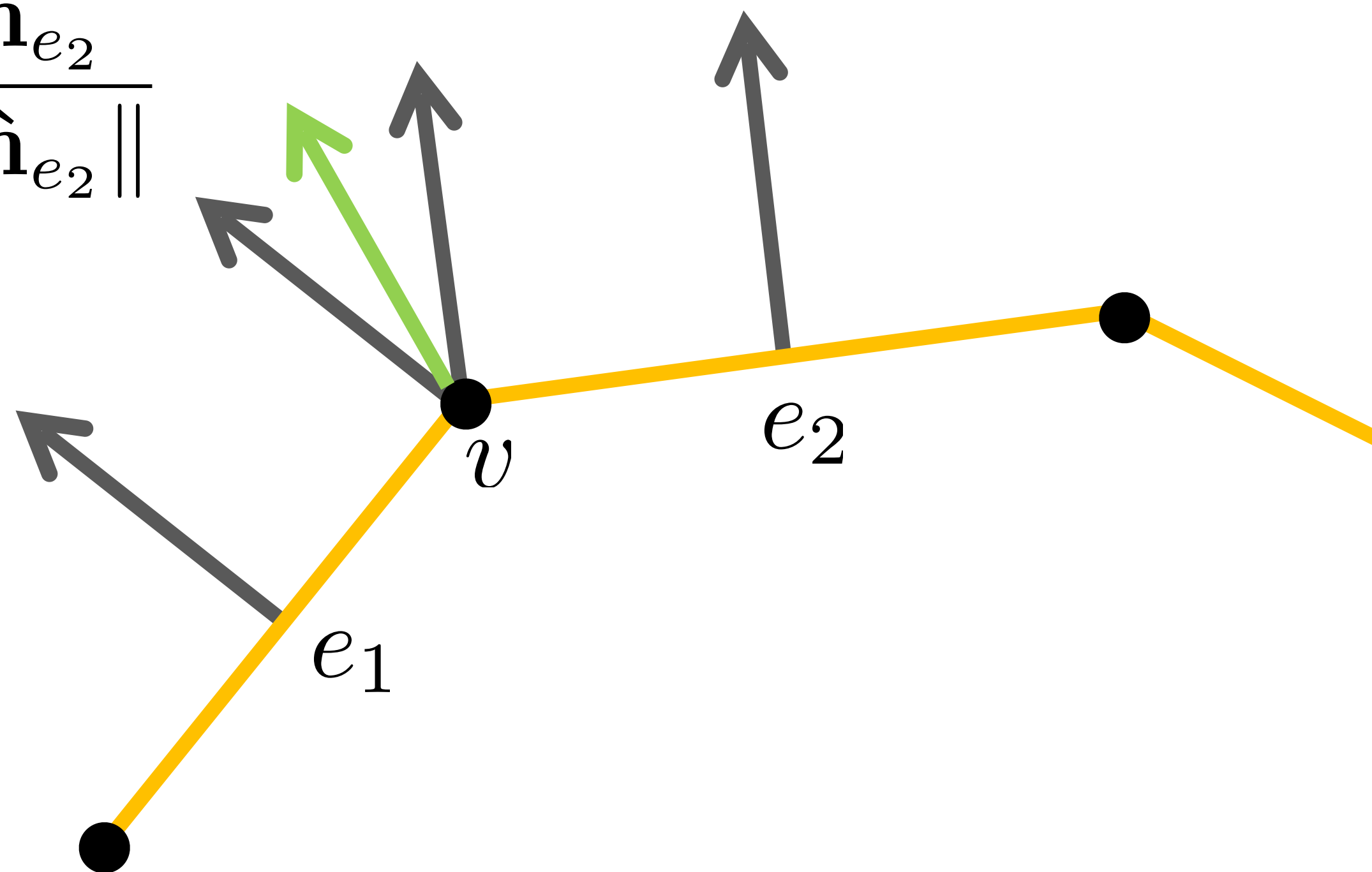
- For vertices, we have many options



Tangents, Normals

- Can choose to average the adjacent edge normals

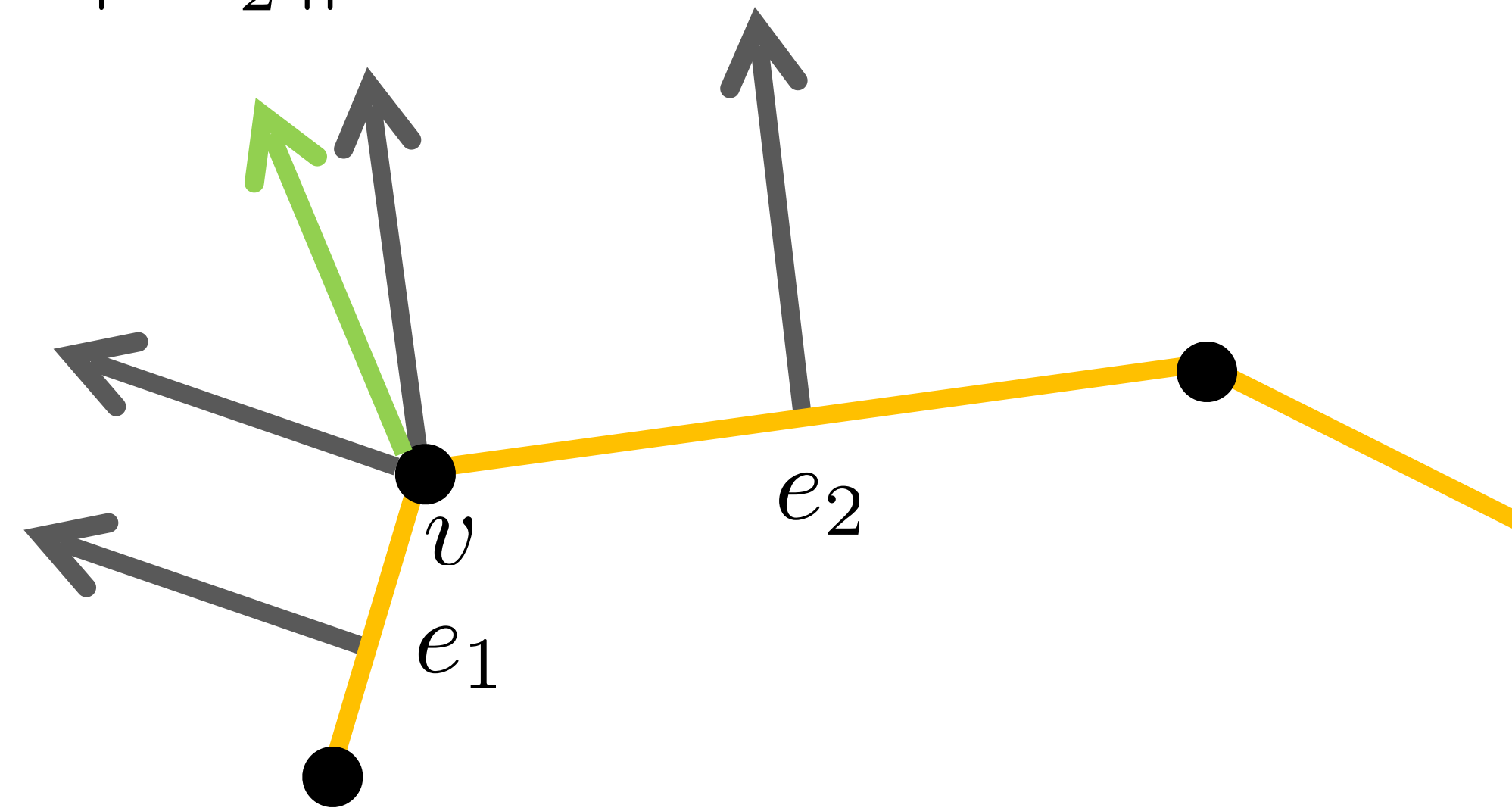
$$\hat{\mathbf{n}}_v = \frac{\hat{\mathbf{n}}_{e_1} + \hat{\mathbf{n}}_{e_2}}{\|\hat{\mathbf{n}}_{e_1} + \hat{\mathbf{n}}_{e_2}\|}$$



Tangents, Normals

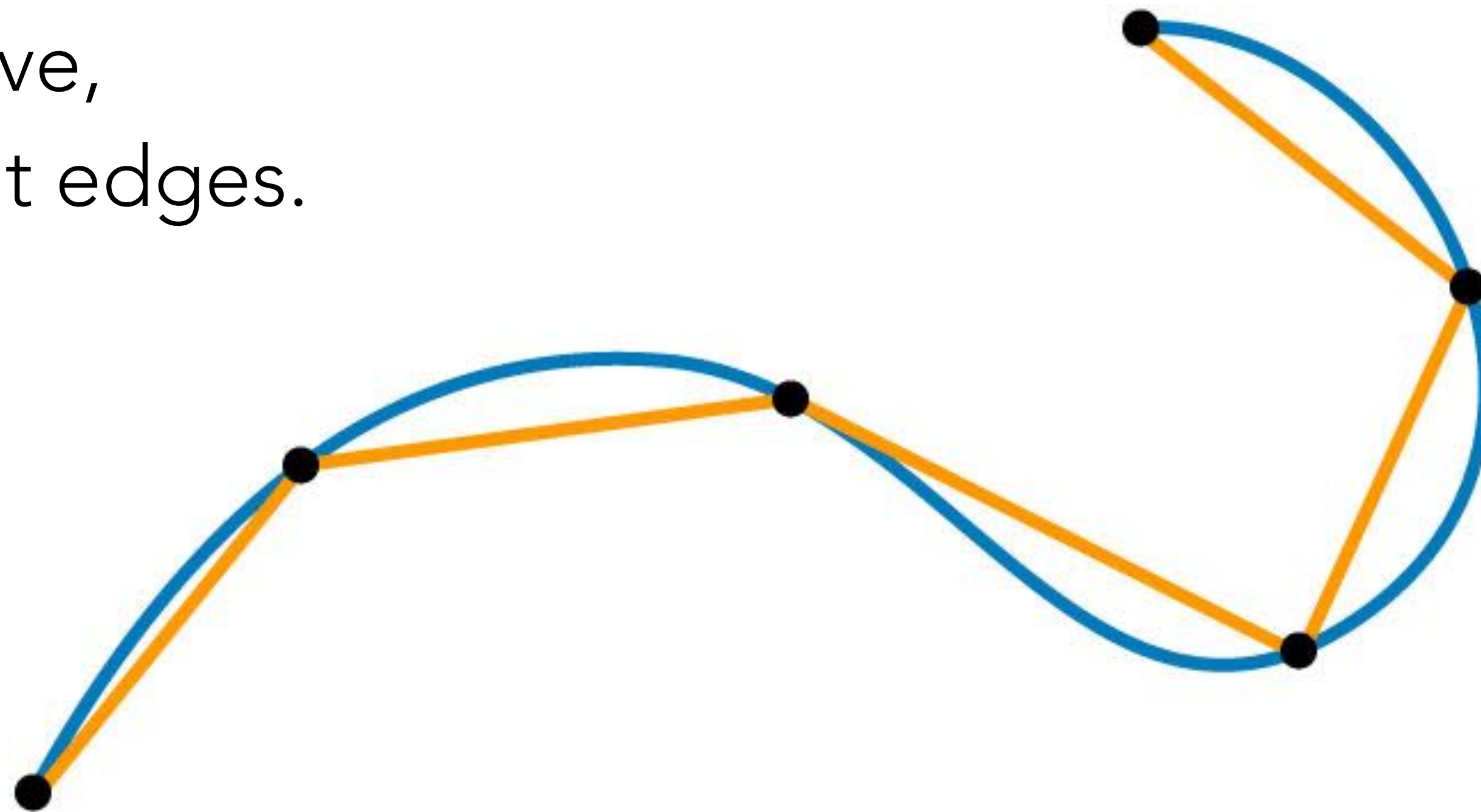
- Weight by edge lengths

$$\hat{\mathbf{n}}_v = \frac{|e_1| \hat{\mathbf{n}}_{e_1} + |e_2| \hat{\mathbf{n}}_{e_2}}{\| |e_1| \hat{\mathbf{n}}_{e_1} + |e_2| \hat{\mathbf{n}}_{e_2} \|}$$



Inscribed Polygon, p

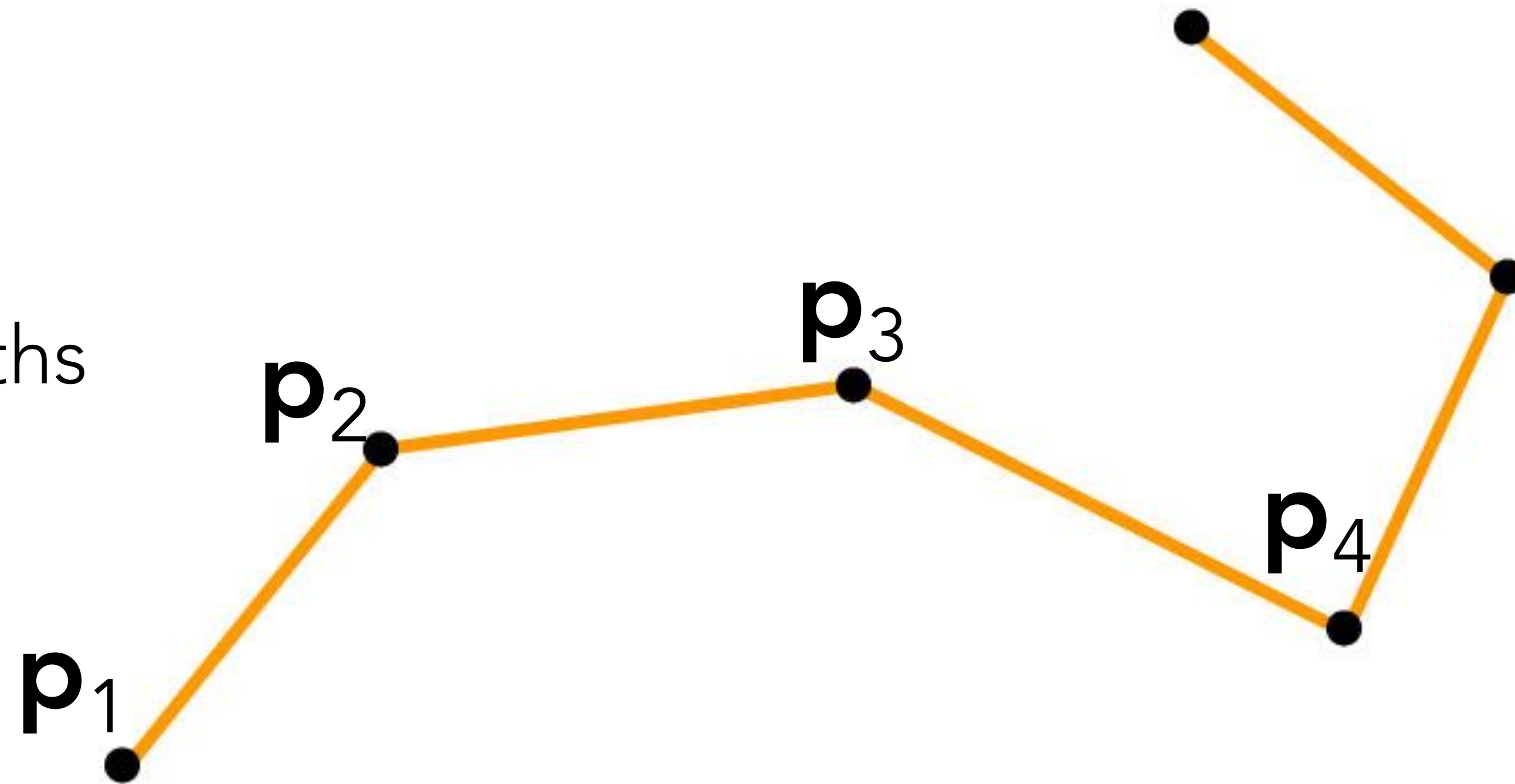
- Connection between discrete and smooth
- Finite number of vertices each lying on the curve, connected by straight edges.



The Length of a Discrete Curve

$$\text{len}(p) = \sum_{i=1}^{n-1} \|\mathbf{p}_{i+1} - \mathbf{p}_i\|$$

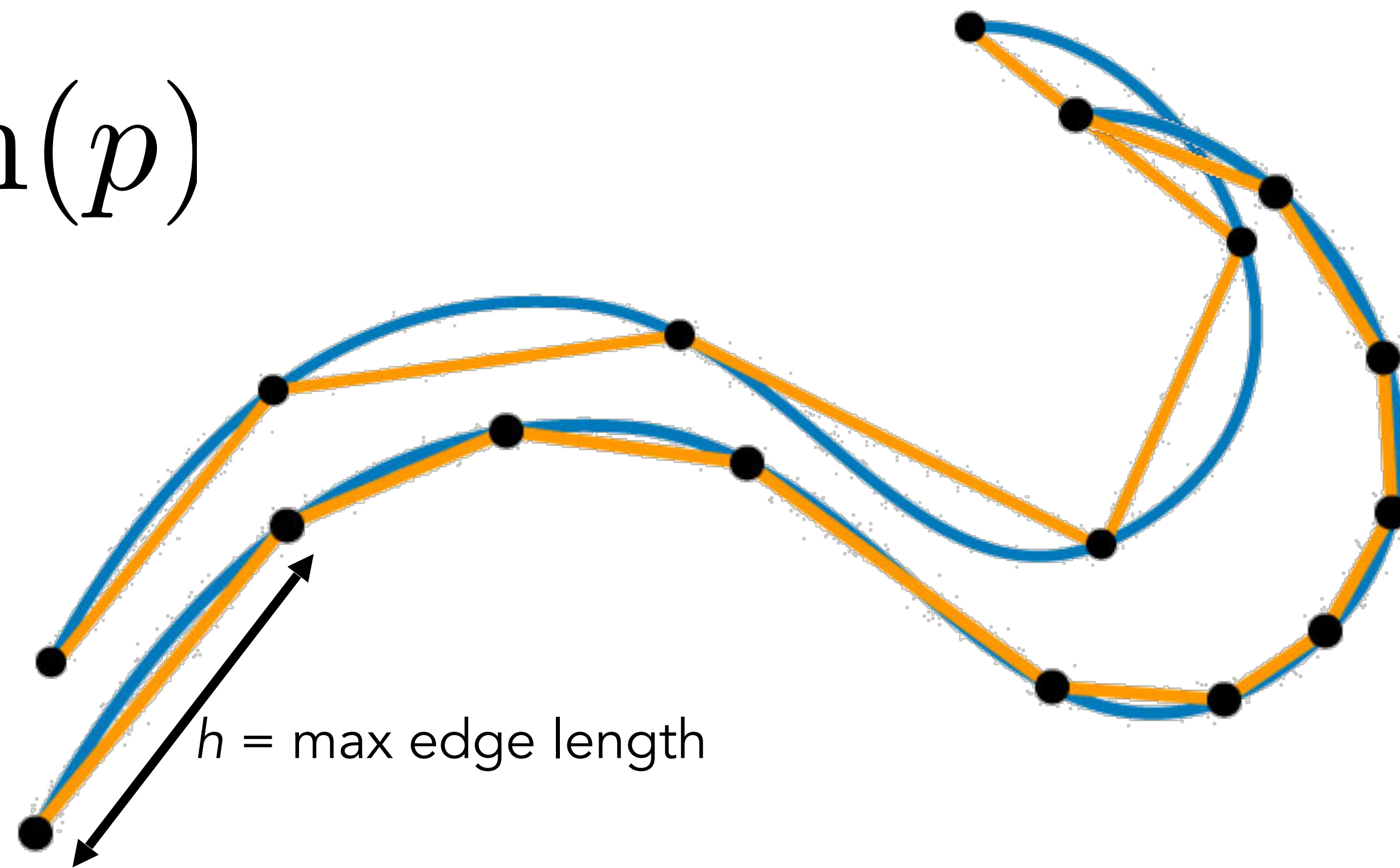
- Sum of edge lengths



The Length of a Continuous Curve

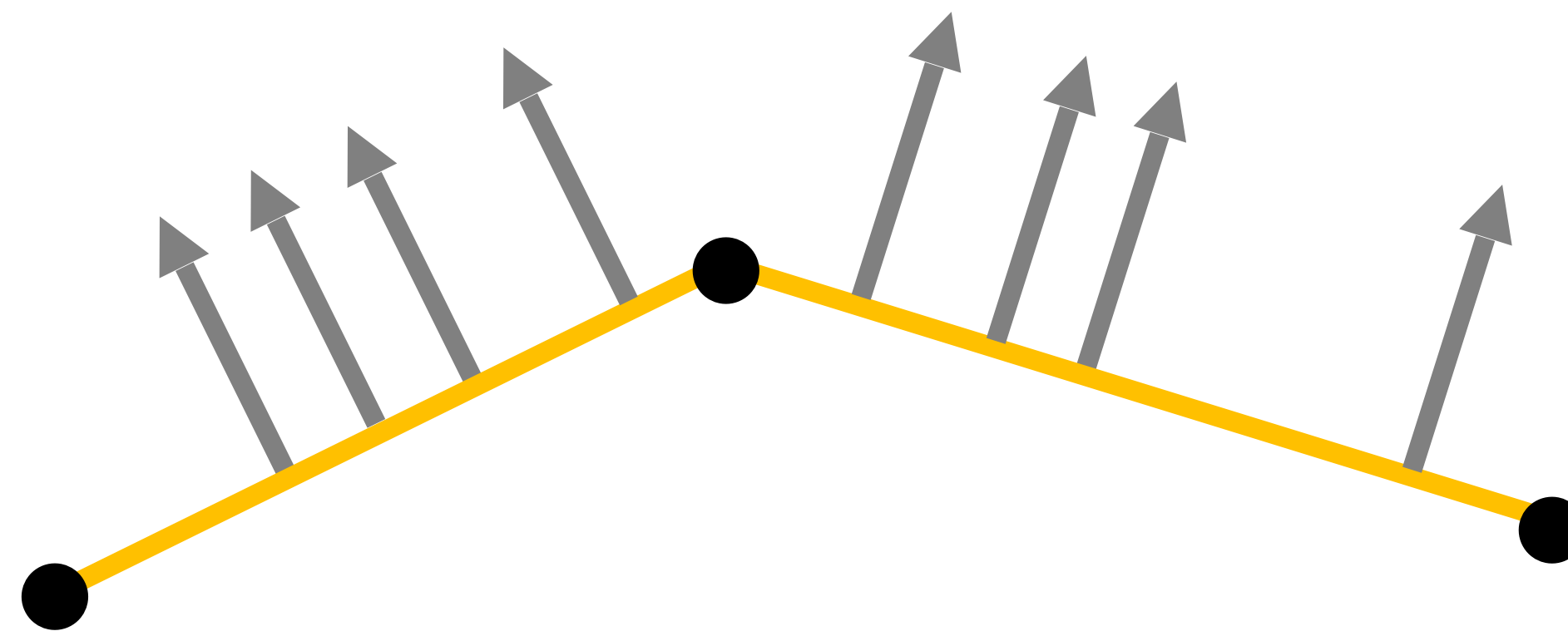
- Take limit over a refinement sequence

$$\lim_{h \rightarrow 0} \text{len}(p)$$



Curvature of a Discrete Curve

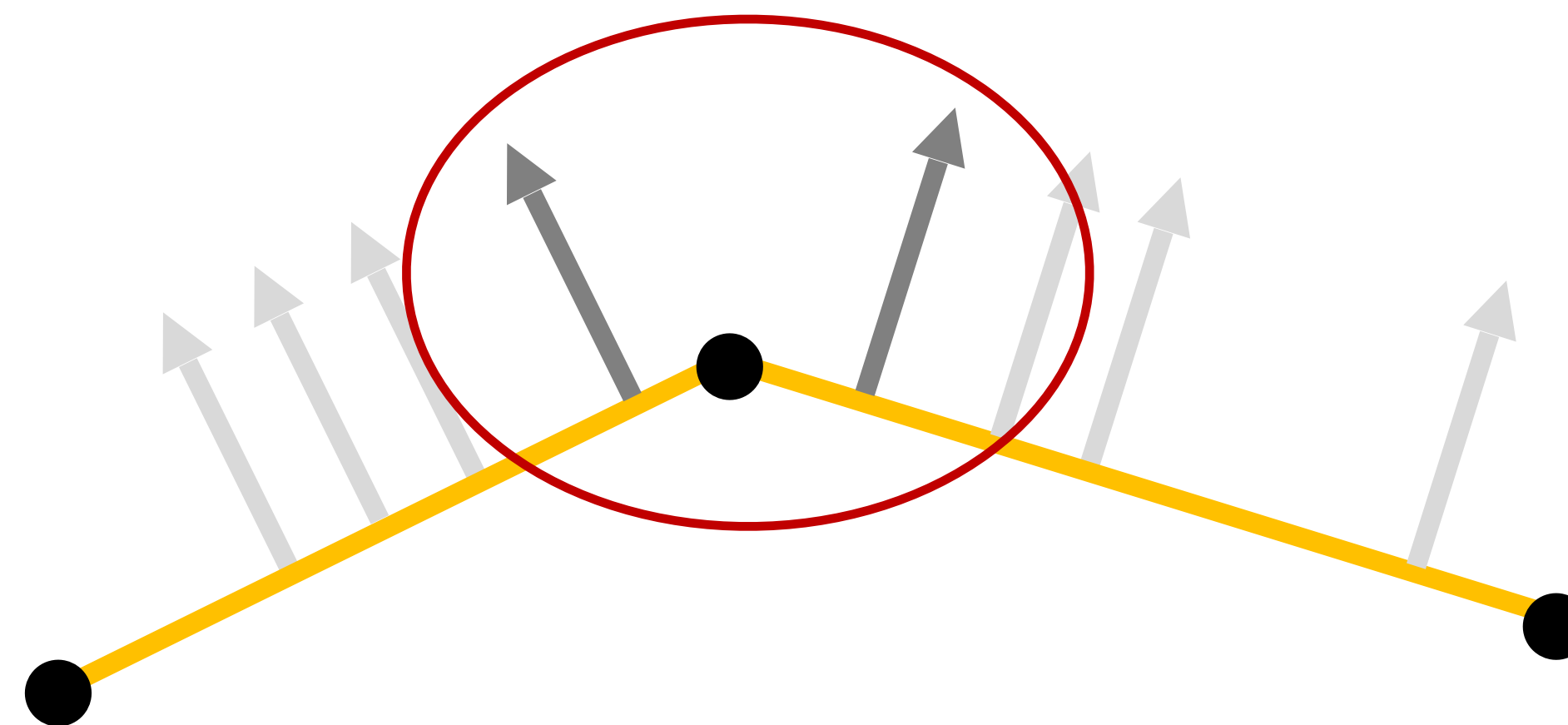
- Curvature is the change in normal direction as we travel along the curve



no change along each edge –
curvature is zero along edges

Curvature of a Discrete Curve

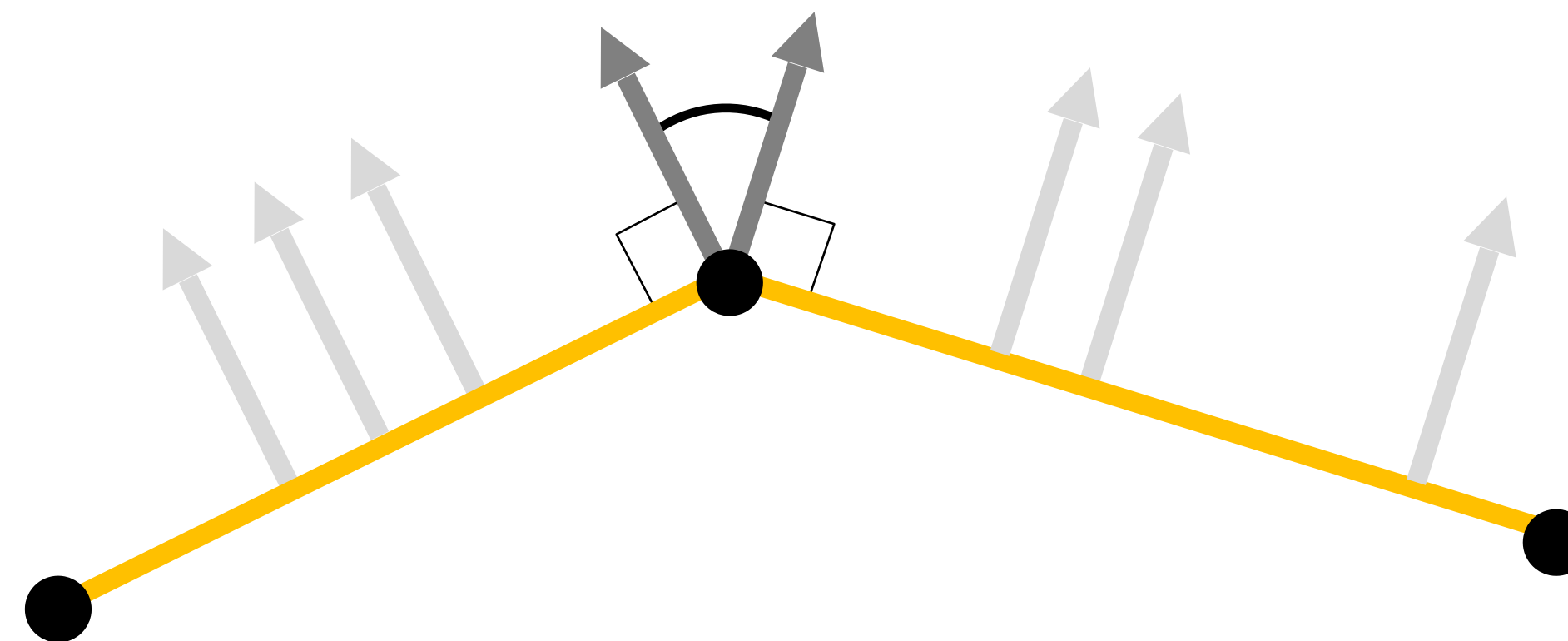
- Curvature is the change in normal direction as we travel along the curve



normal changes at vertices –
record the turning angle!

Curvature of a Discrete Curve

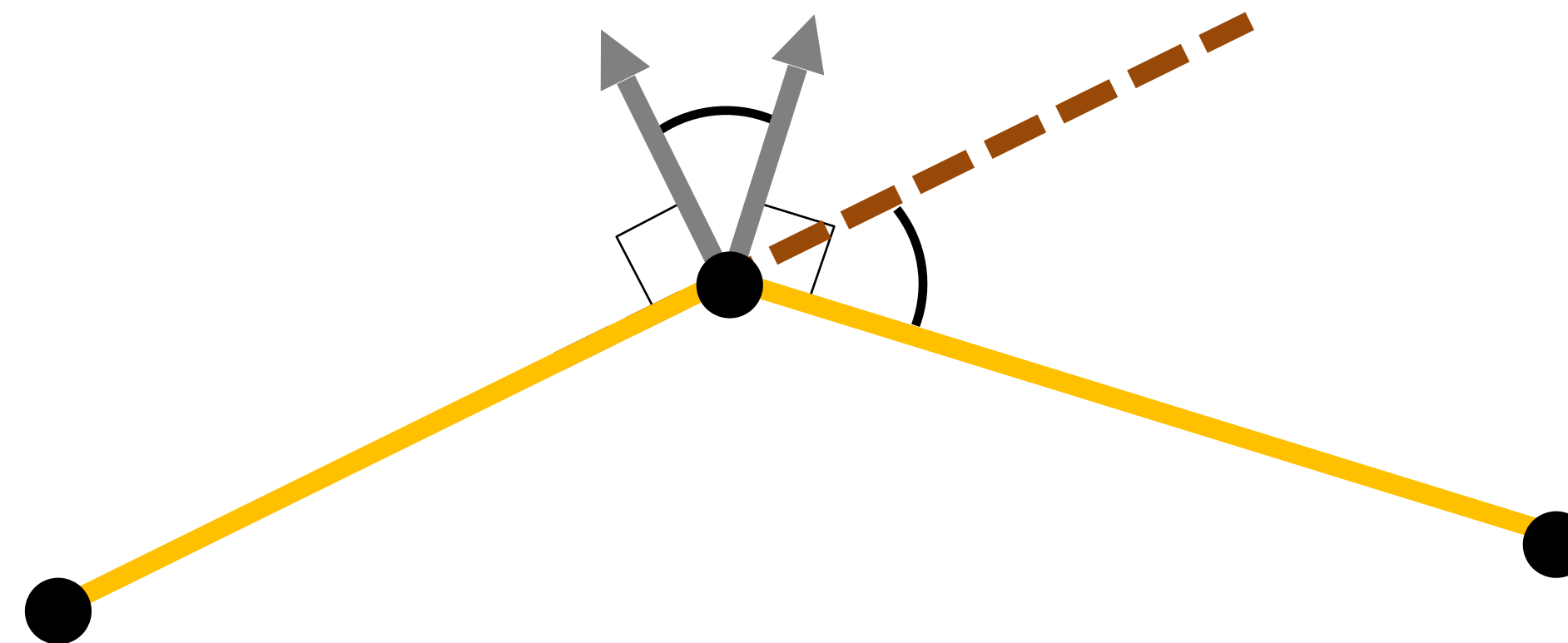
- Curvature is the change in normal direction as we travel along the curve



normal changes at vertices –
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Curvature of a Discrete Curve

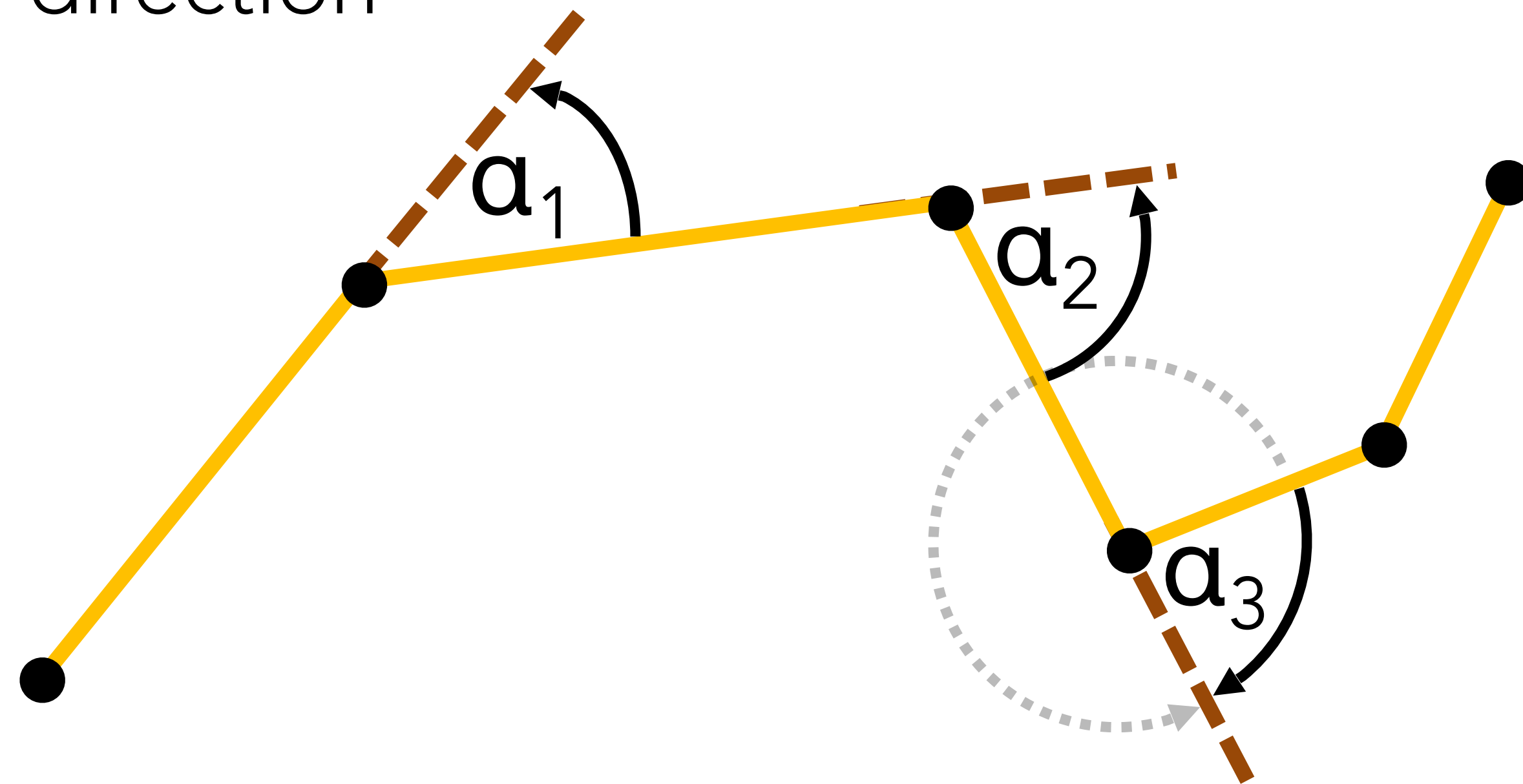
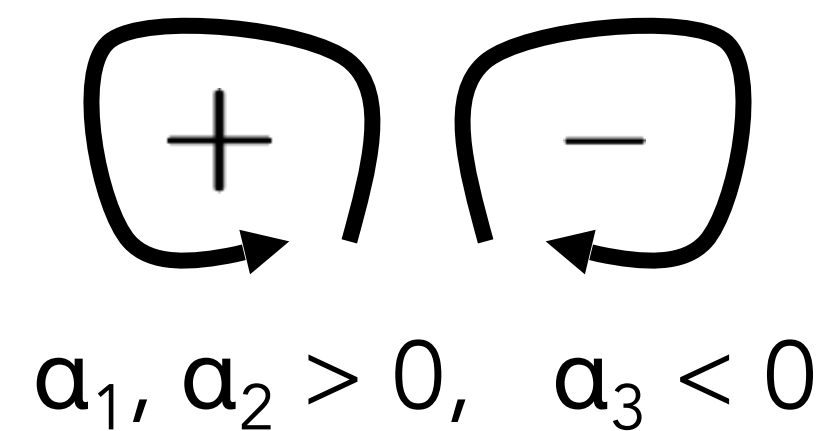
- Curvature is the change in normal direction as we travel along the curve



same as the turning angle
between the edges

Curvature of a Discrete Curve

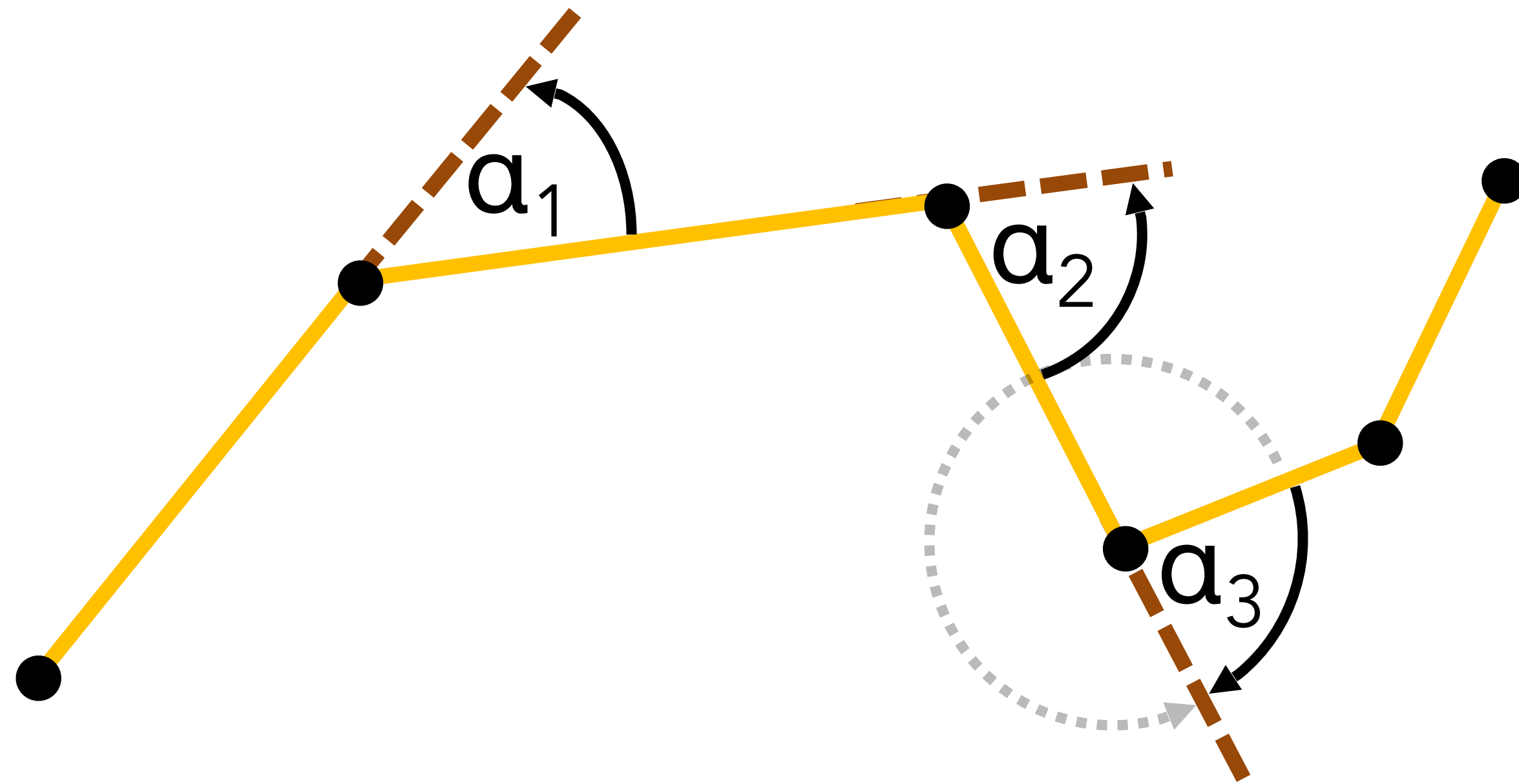
- Zero along the edges
- Turning angle at the vertices
= the change in normal direction



Total Signed Curvature

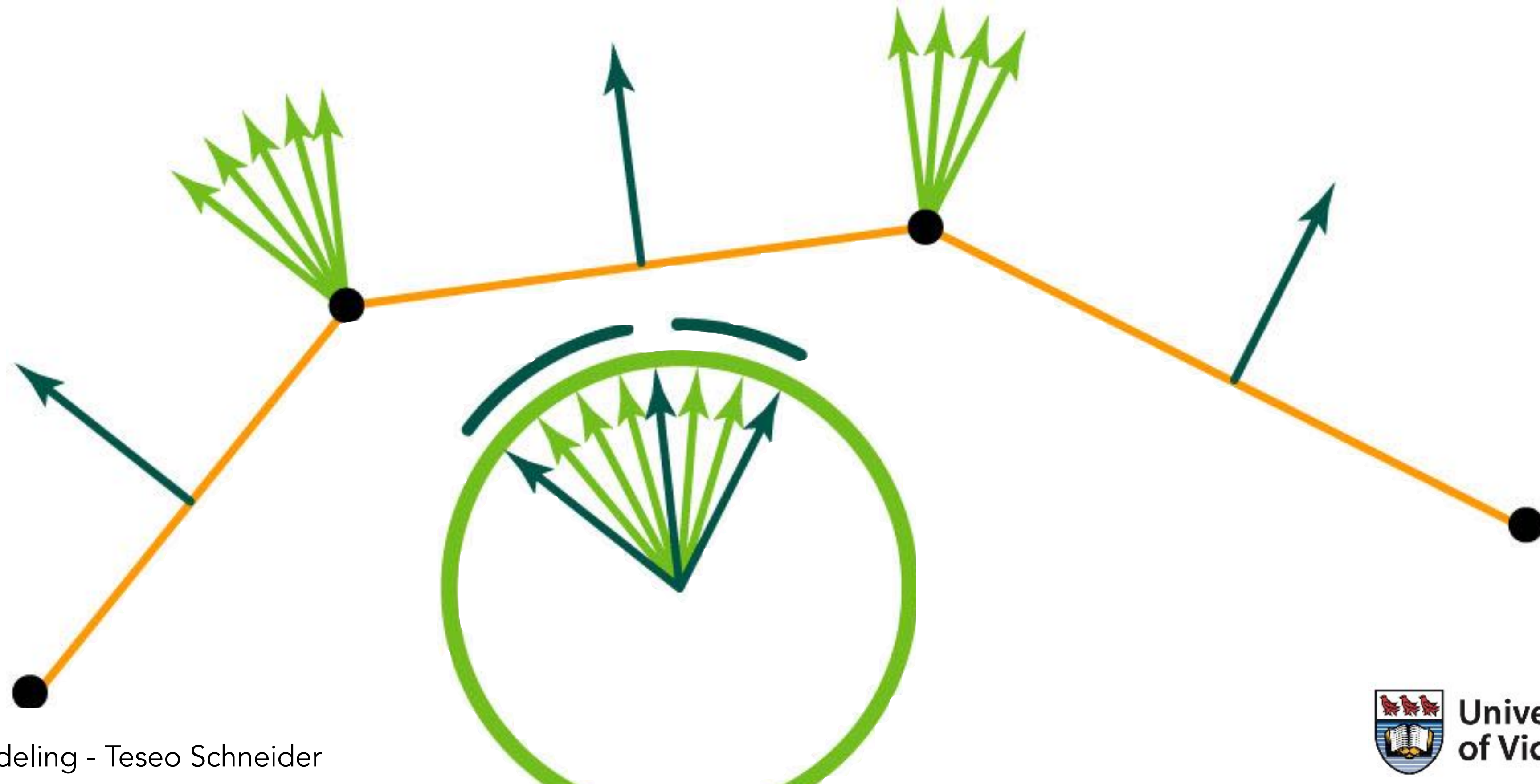
$$\text{tsc}(p) = \sum_{i=1}^n \alpha_i$$

- Sum of turning angles



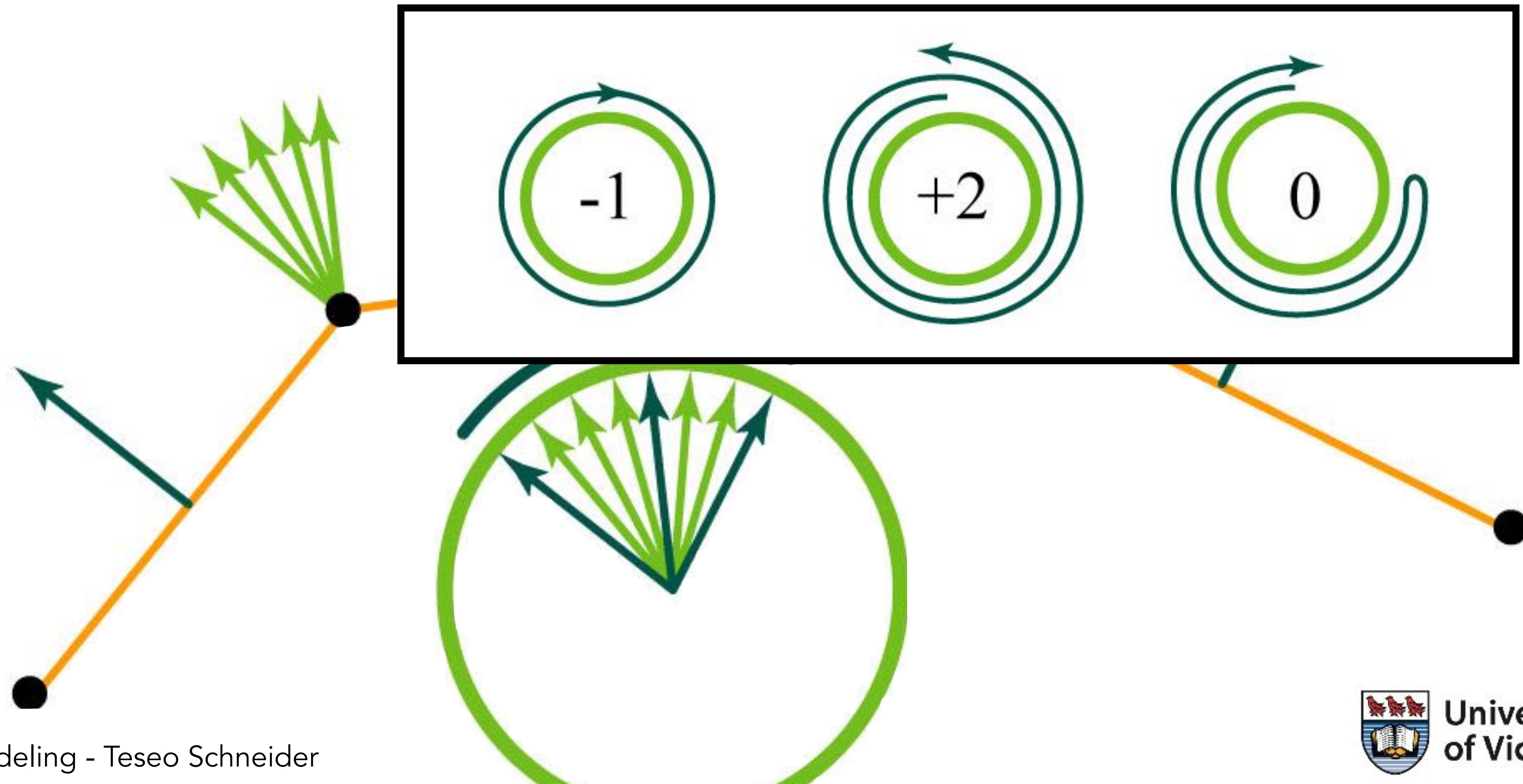
Discrete Gauss Map

- Edges map to points, vertices map to arcs.



Discrete Gauss Map

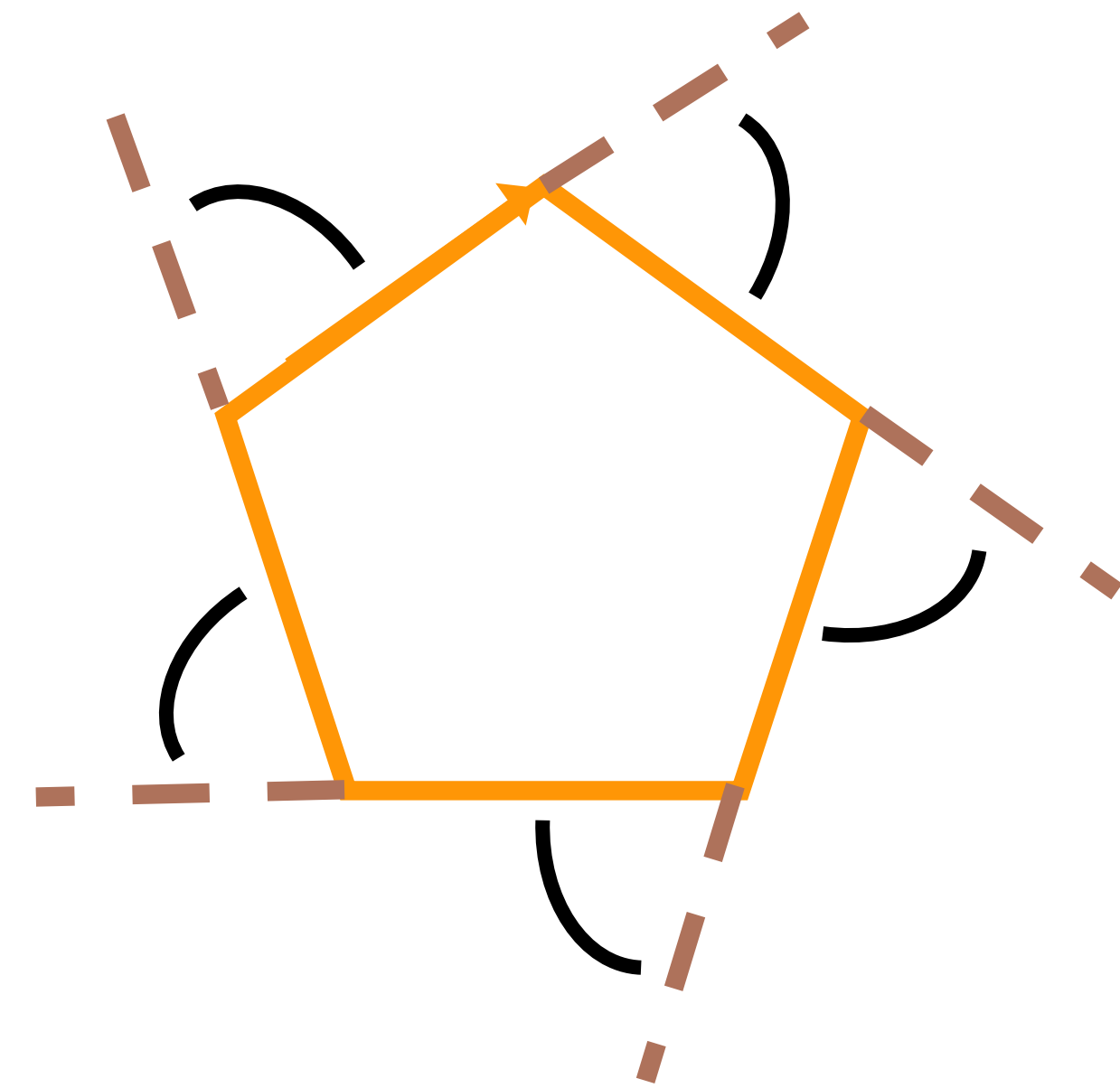
- Turning number well-defined for discrete curves.



Discrete Turning Number Theorem

$$\text{tsc}(p) = \sum_{i=1}^n \alpha_i = 2\pi k$$

- For a closed curve, the total signed curvature is an integer multiple of 2π .
- proof: sum of exterior angles



Turning Number Theorem

Continuous world

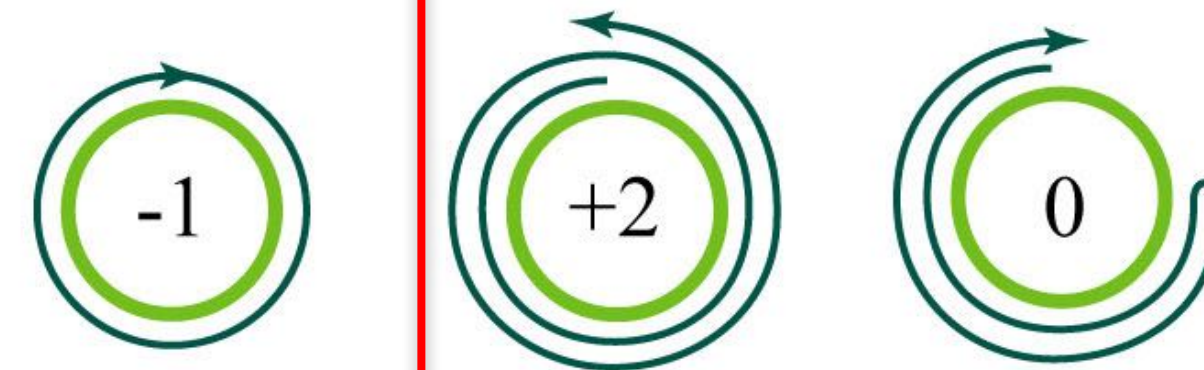
$$\int_{\gamma} \kappa dt = 2\pi k$$

k :



Discrete world

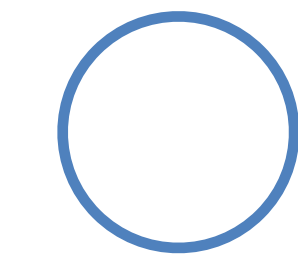
$$\sum_{i=1}^n \alpha_i = 2\pi k$$



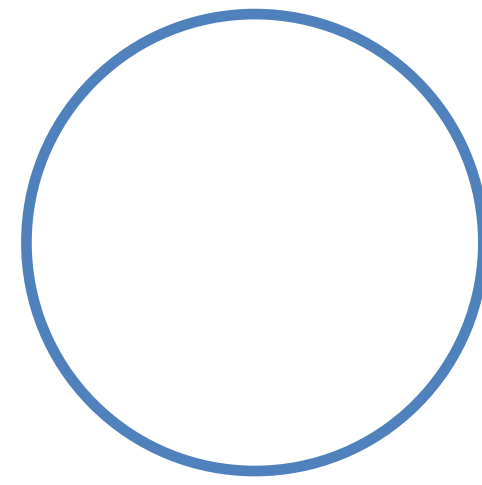
$$\kappa = \alpha_i \quad ??$$

Curvature is scale dependent

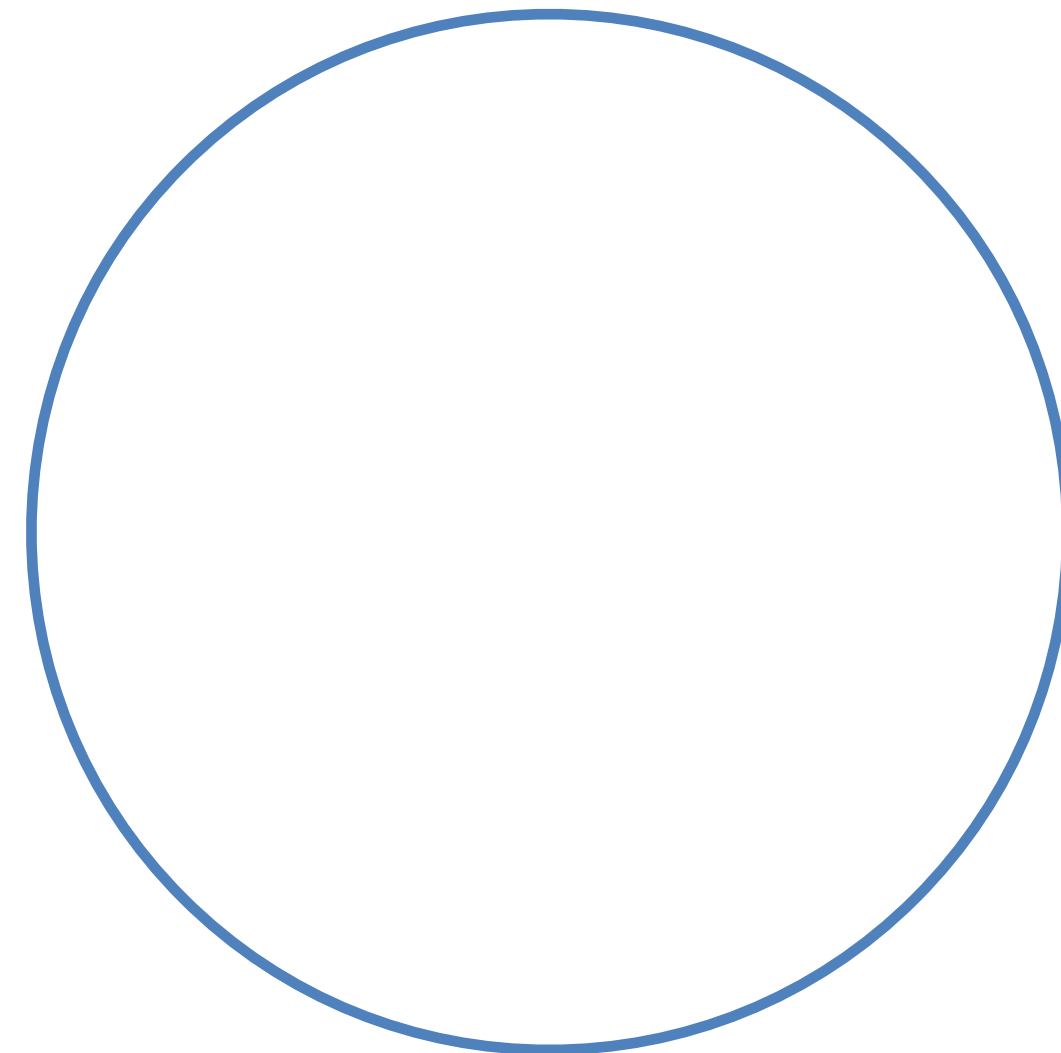
$$\kappa = \frac{1}{r}$$



κ



κ

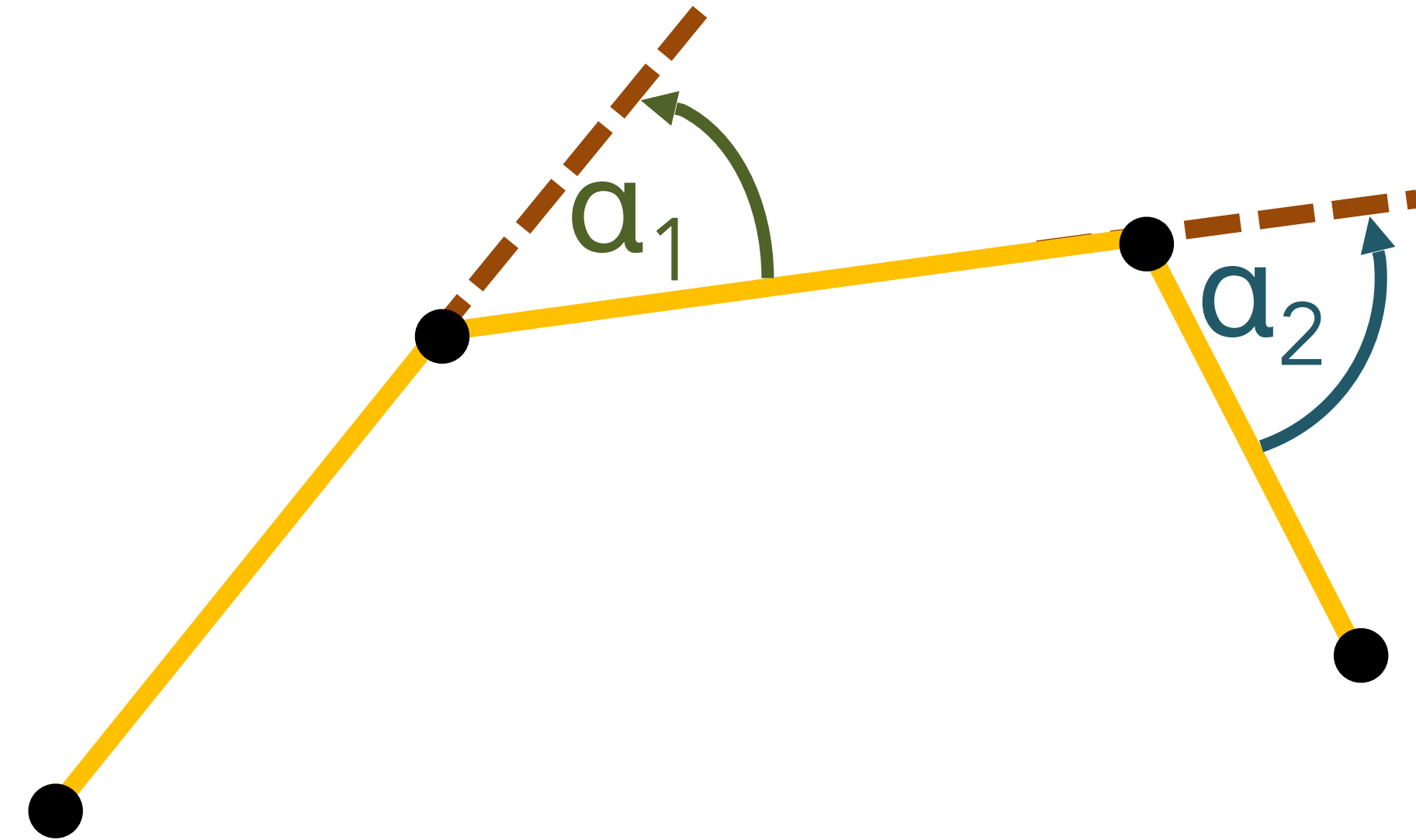


κ

α_i is scale-independent

Discrete Curvature – Integrated Quantity!

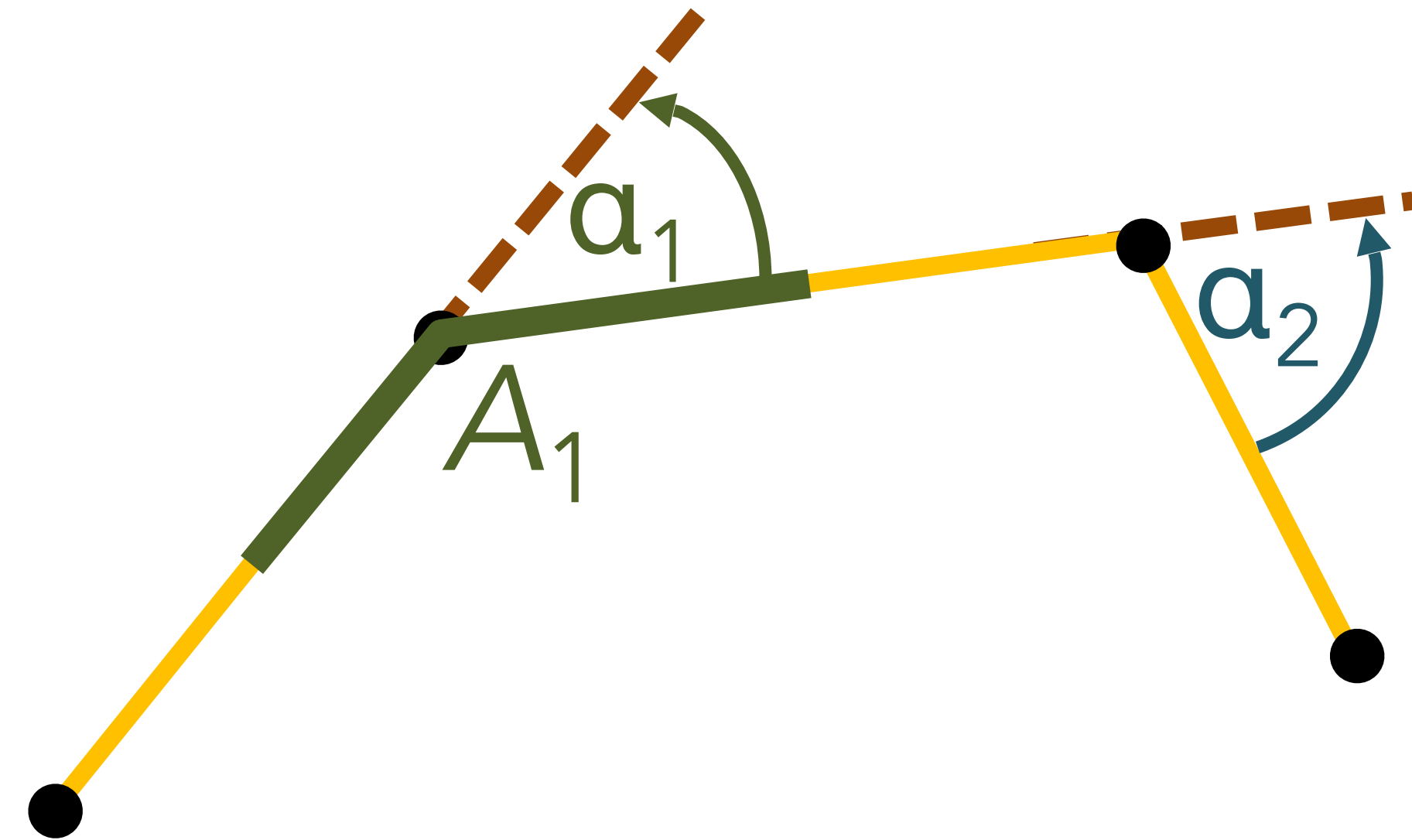
- Cannot view α_i as pointwise curvature
- It is *integrated curvature* over a local area associated with vertex i



Discrete Curvature – Integrated Quantity!

- Integrated over a local area associated with vertex i

$$\alpha_1 = A_1 \cdot \kappa_1$$

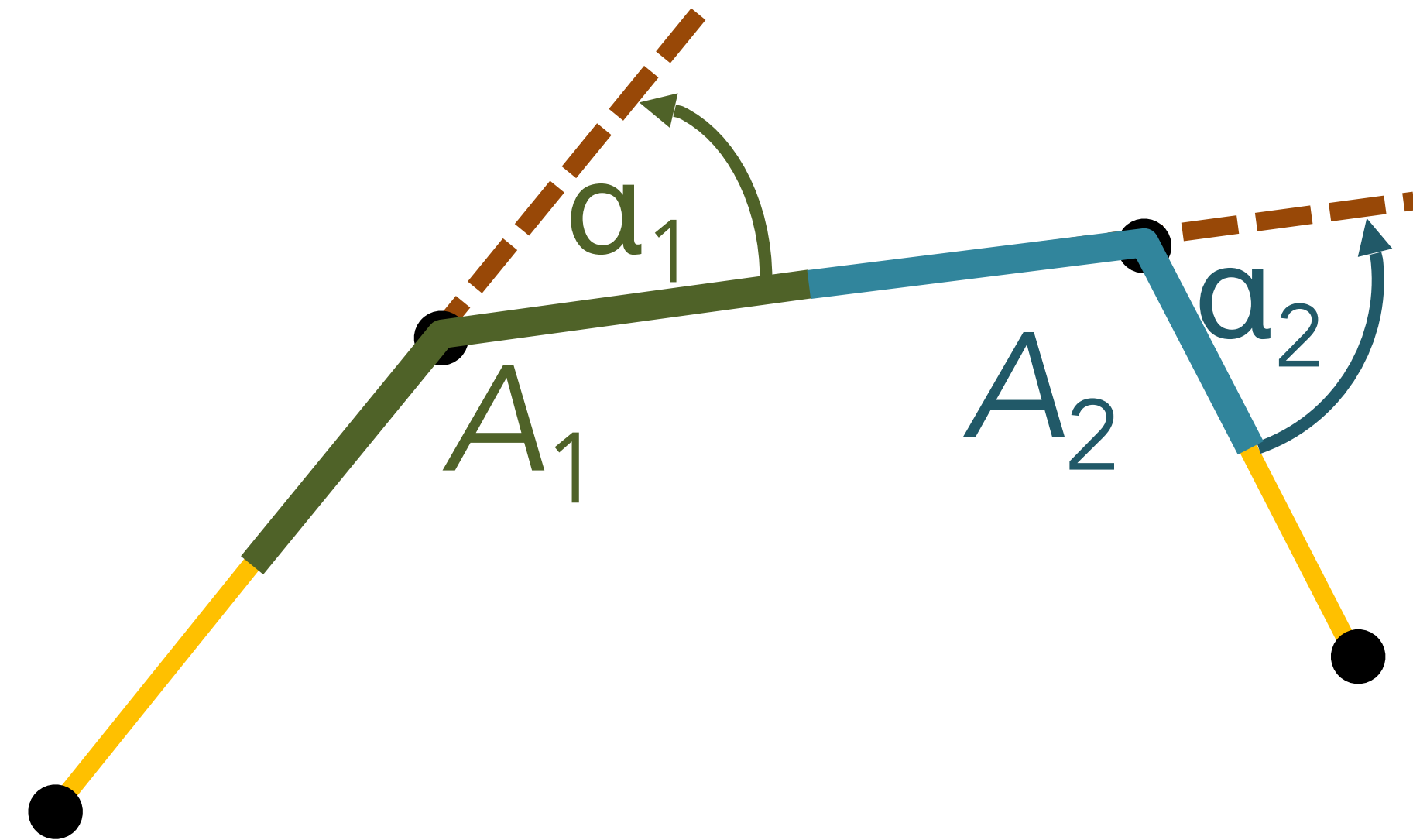


Discrete Curvature – Integrated Quantity!

- Integrated over a local area associated with vertex i

$$\alpha_1 = A_1 \cdot \kappa_1$$

$$\alpha_2 = A_2 \cdot \kappa_2$$



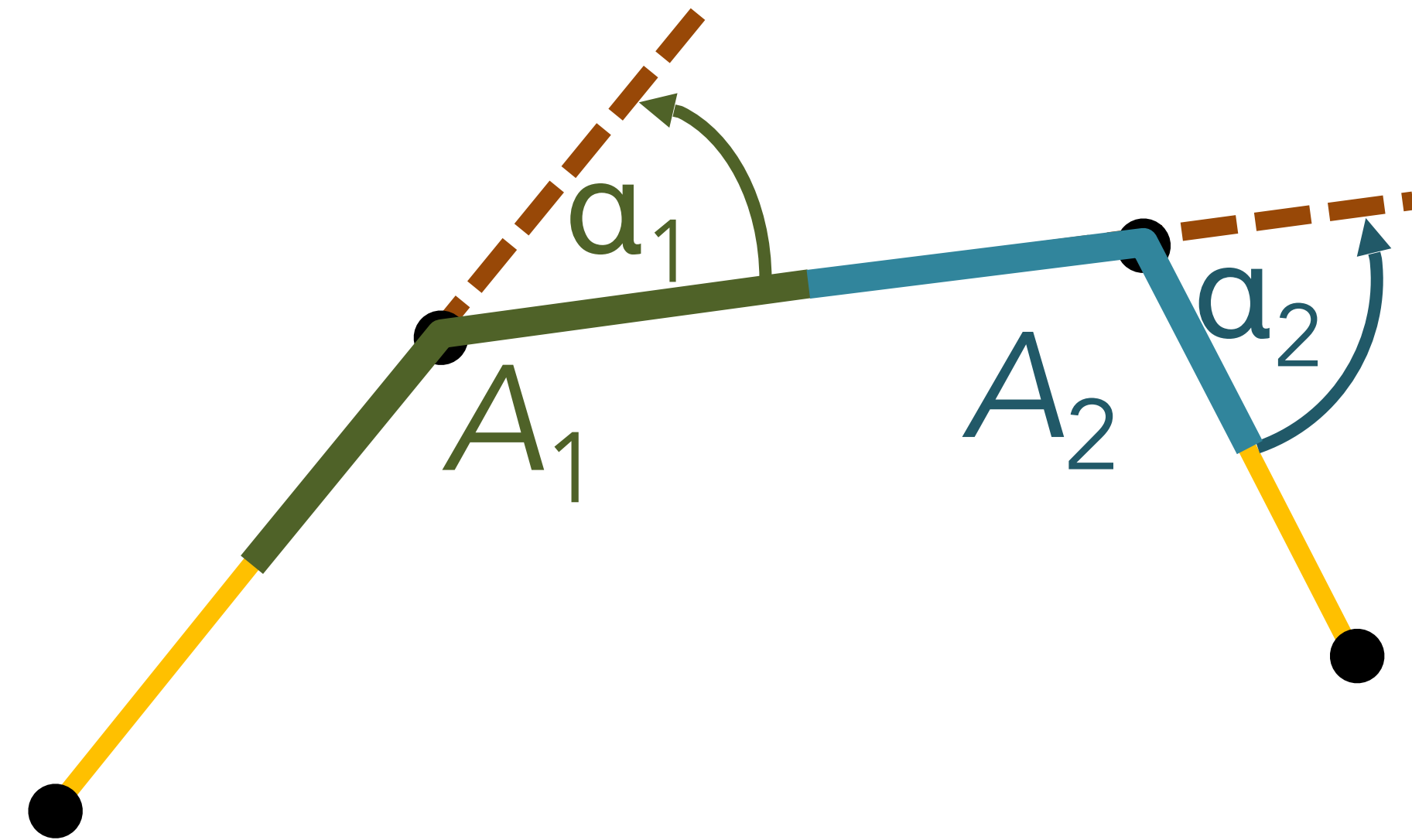
Discrete Curvature – Integrated Quantity!

- Integrated over a local area associated with vertex i

$$\alpha_1 = A_1 \cdot \kappa_1$$

$$\alpha_2 = A_2 \cdot \kappa_2$$

$$\sum A_i = \text{len}(p)$$



The vertex areas A_i form a covering of the curve.

They are pairwise disjoint (except endpoints).

Structure Preservation

- Arbitrary discrete curve
 - total signed curvature obeys
discrete turning number theorem
 - even coarse mesh (curve)
 - which continuous theorems to preserve?
 - that depends on the application...

*discrete analogue
of continuous theorem*

Convergence

- Consider refinement sequence
 - length of inscribed polygon approaches length of smooth curve
 - in general, discrete measure approaches continuous analogue
 - which refinement sequence?
 - depends on discrete operator
 - pathological sequences may exist
 - in what sense does the operator converge?
(pointwise, L_2 ; linear, quadratic)

Recap

Structure- preservation

For an arbitrary (even coarse) discrete curve, the discrete measure of curvature **obeys** the discrete turning number theorem.

Convergence

In the limit of a refinement sequence, discrete measures of length and curvature **agree** with continuous measures.

References

- Polygon Mesh Processing, Chapter 3
- <http://www.cs.cmu.edu/~kmcrane/Projects/DDG/paper.pdf>