PSet8 Report

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1 First Order Differential Equation

The equation we are to solve is

$$R\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q}{C} = V \tag{1}$$

for the values $V = 10volts$, $R = 3000\Omega$, $C = 1\mu F$ and the simulation time from zero to $t = 0.5\,ms$. Since the numbers are a little overwhelming, we make parameter changes to come up with better numbers for our simulation.

The change of variables is as follows:

$$\begin{aligned} &x \equiv \frac{Q}{RC} - \frac{V}{R}, \quad \tau \equiv \frac{t}{RC} \\ \Rightarrow &\frac{\mathrm{d}x}{\mathrm{d}\tau} + x = 0 \end{aligned} \tag{2}$$

Using this change in variables, the time limit becomes: $\tau = \frac{0.5ms}{10^{-6}F\times 3000\Omega} = \frac{1}{6}s$ Also, if we want the capacitor to start from being empty, the initial value for x will be: $x_0 = \frac{1}{300}\frac{volts}{\Omega}$

The analytical solution is obtained by integrating the equation $\frac{\mathrm{d}x}{x} = \mathrm{d}\tau$ which gives:

$$x = x_0 \exp(-\tau) \Rightarrow Q = CV(1 - e^{-\frac{t}{RC}}) \tag{3}$$

I integrated numerically for x and then used a reverse version of our change of variables, to find Q. Then, I plotted the analytical solution for Q and the integration in the same graph. (Fig.1)

For the second part, I found the difference in numeric and analytic method, δ, at $t = 0.5ms$ for various values of the time step. The data is available in table 1, and the plot, in Fig.1

h	δ
0.001	0.00282761
0.005	0.00284087
0.01	0.002868396
0.015	0.00286730
0.02	0.002895286
0.03	0.002952511
0.04	0.002950655
0.05	0.003009868

Table 1: The values of δ for different values of time step h.

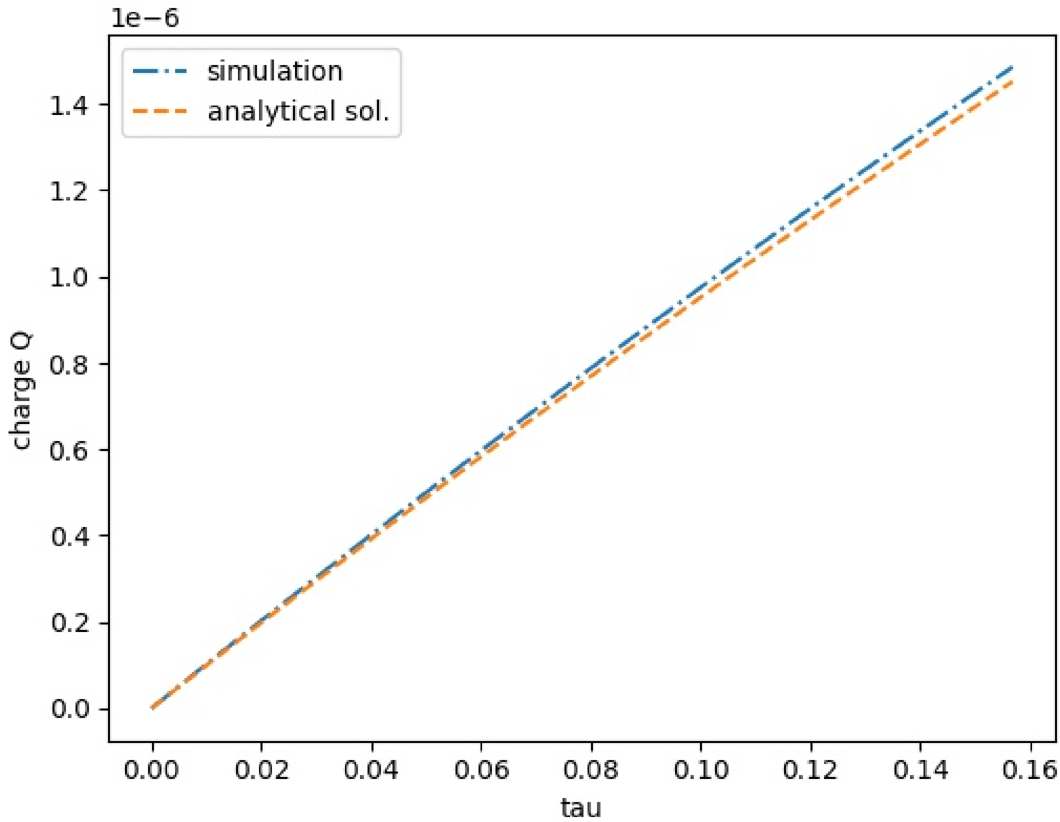

Figure 1: Analytical and numerical solution for the charging capacitor equation. The time step for the numerical solution here is 0.01

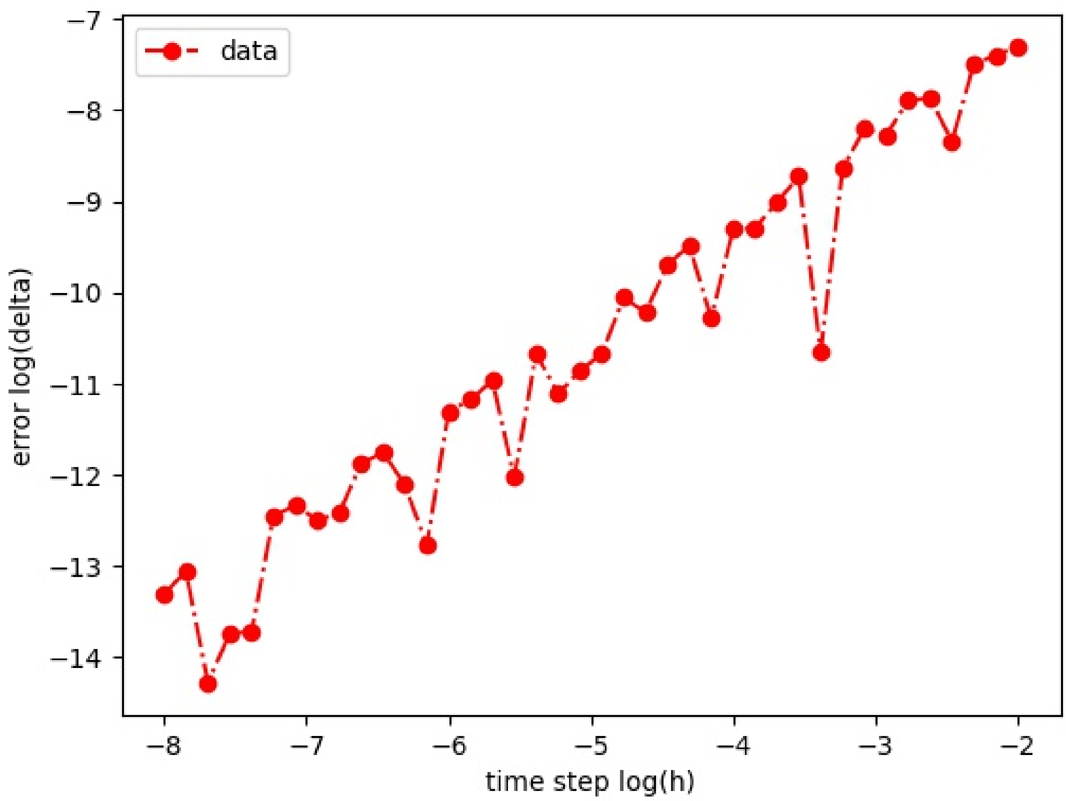

Figure 2: Plot for $\log(\delta)$ vs. $\log(h)$. We can see that for a value of h there's a minimum where the error is optimal and then the relation is linear.

2 2^{nd} Order ODE

I made all the functions to take variables `x_init, acc, step, time`. `x_init` is the initial position of the mass from the origin. `acc` is the function that returns the *acceleration* as a function of position `x`. The initial conditions are as follows:

$$\begin{aligned} \dot{x} &= 0 \\ x &= 1 \end{aligned} \tag{4}$$

For some methods like *verlat* I had to define extra initial conditions due to the nature of the algorithms.

I made a dictionary `data` that stores the results for different methods and makes the job of plotting and other stuff easier. I integrated for 60 time units and time step of `step = 0.01`. Then I plotted them in one graph. (Fig 2) Then, I plotted $\dot{x}$ as a function of x for each method separately.

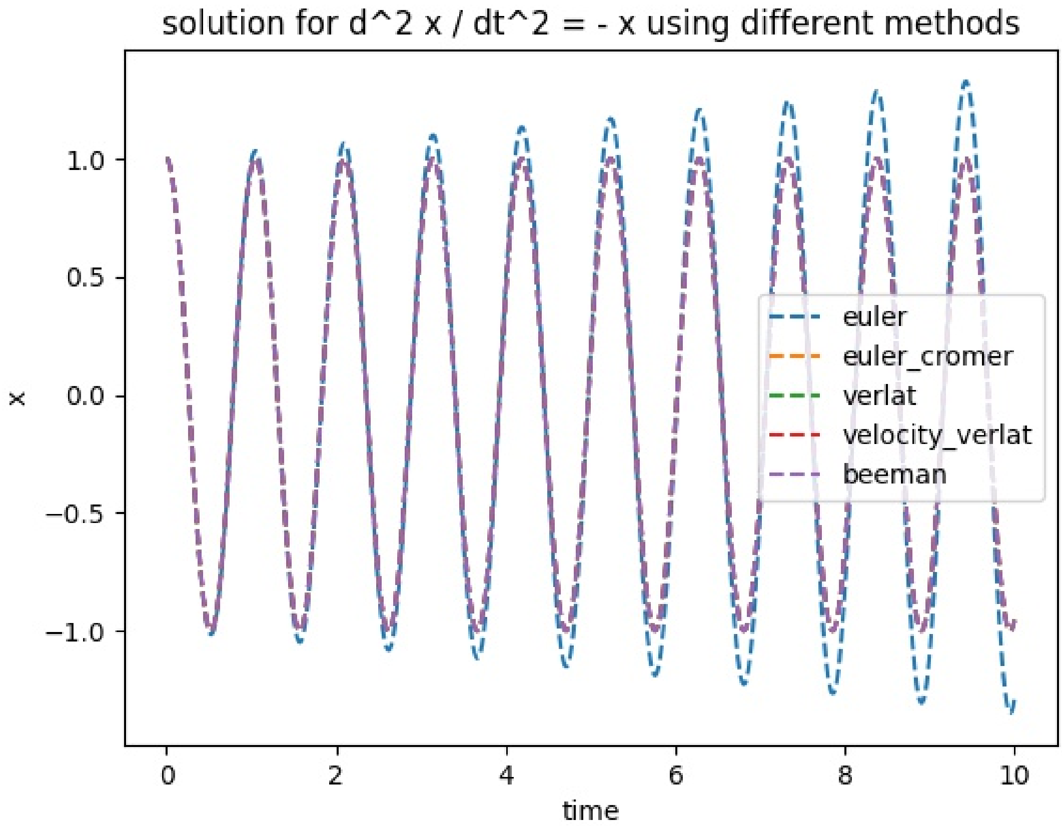

Figure 3: plot of the solutions using different integration methods from 0 to $t = 60$ with time step of 0.01

(Fig2)

As can be seen from Fig2, the *euler-cromer*, *verlat*, *velocity verlat*, and *beeman* methods have stability for this problem.

3 Instability in algorithms

Here we use the same change in variables that we did in equation2 to simulate the charging capacitor. The algorithm we are going to use here is as follows:

$$y_{n+1} = y_{n-1} + 2\dot{y}_n h \tag{5}$$

where h is the time step. Since this is a two step algorithm, we need a second initial value that we get using the Euler method. From that point forward, we use the algorithm in equation5. Then, we plot both the analytical solution (eq.3) and the numerical solution in one graph to compare them.

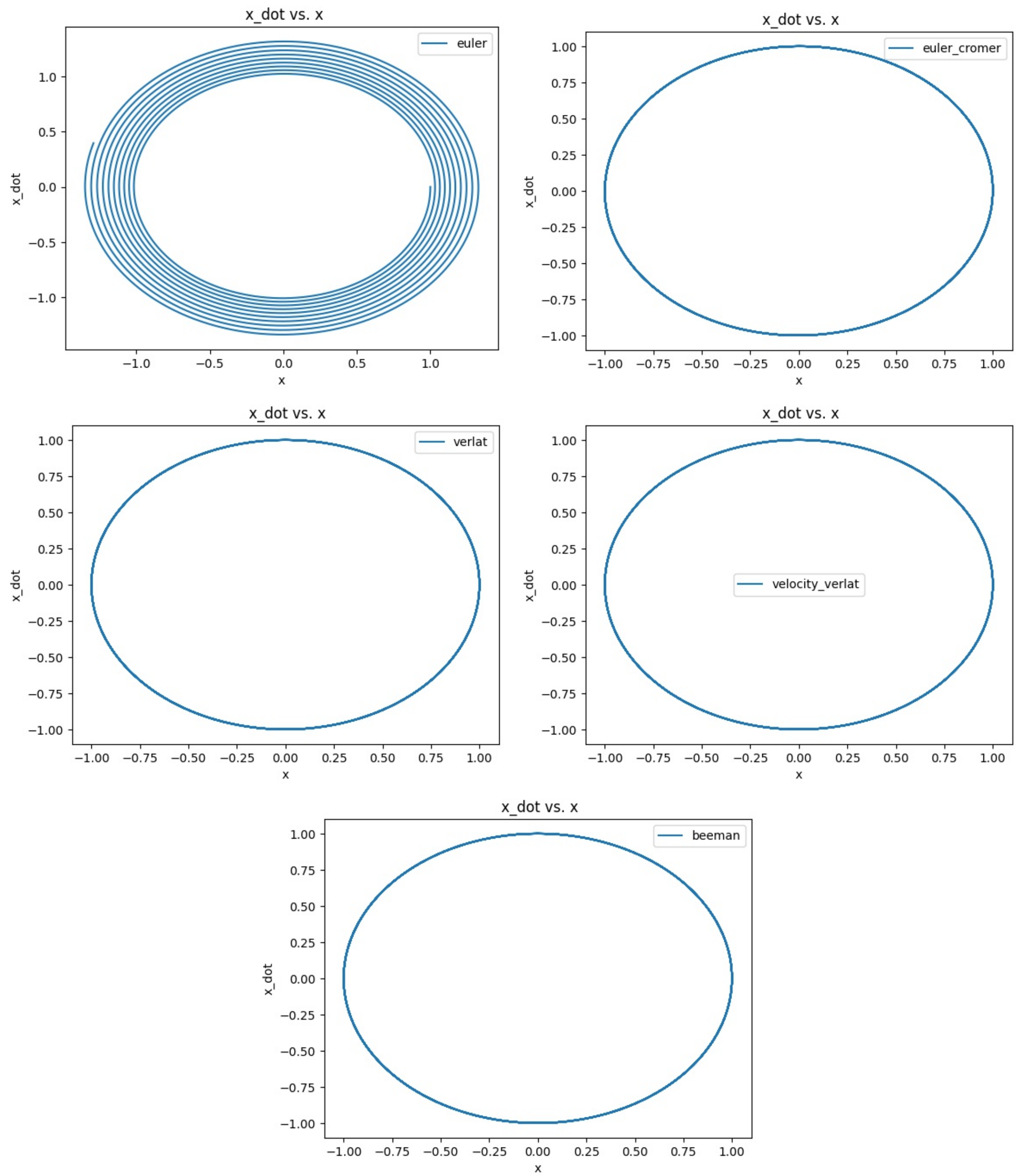

Figure 4: $\dot{x}$ vs. x for different methods.

In small values for terminating the integration, the numerical solution seems to be stable, but if we wait long enough, we can see that from about $\tau = 6.0$, the numerical integration shows signs of periodic motions and oscillation around the analytical solution. (Fig.3)

It can also be seen that for higher values of the time step, the instability manifests much sooner and explodes more rapidly.

4 Chaos

Here I made a function `stable_point(r)` which applies the function `f(r, x)` 1500 times and returns the last 200 values. This way we can cover the bifurcations to 128-periodic. Here $f(r, x) = x_{n+1} = 4rx_n(1 - x_n)$. The bifurcation plot is available in FIg4

I failed to find the values for δ, α. :-(

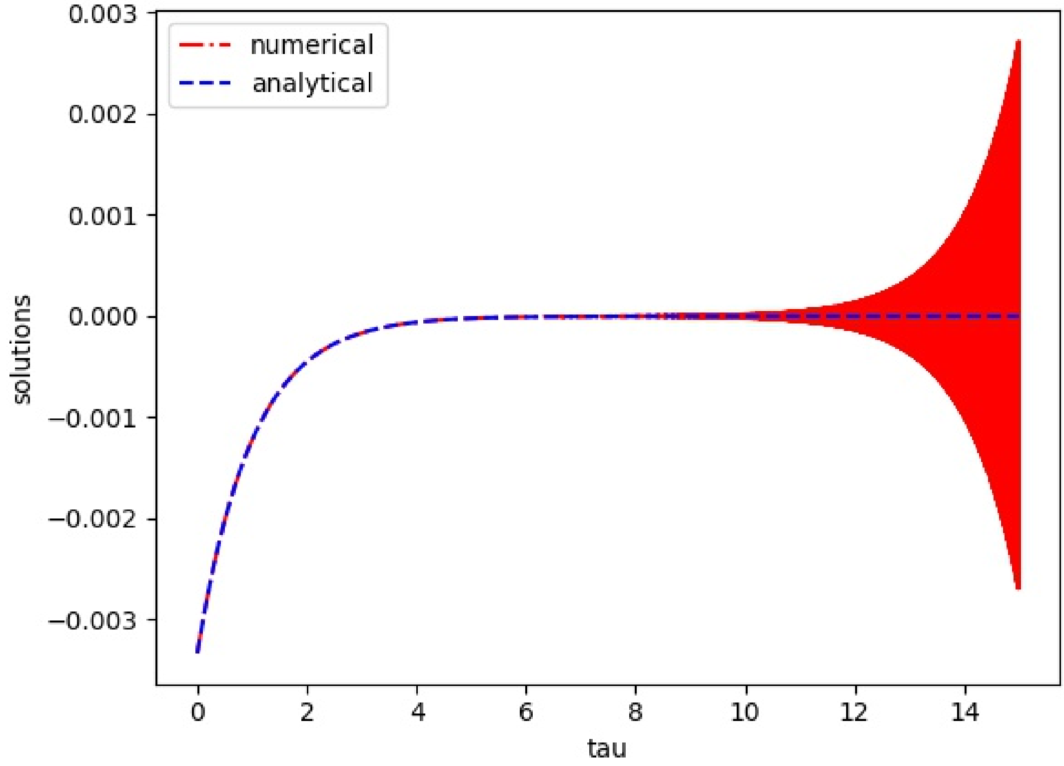

Figure 5: numerical solution using the algorithm from eq5, shows oscillation about the analytical solution for large enough τ, which represents instability for this algorithm in this perticular problem.

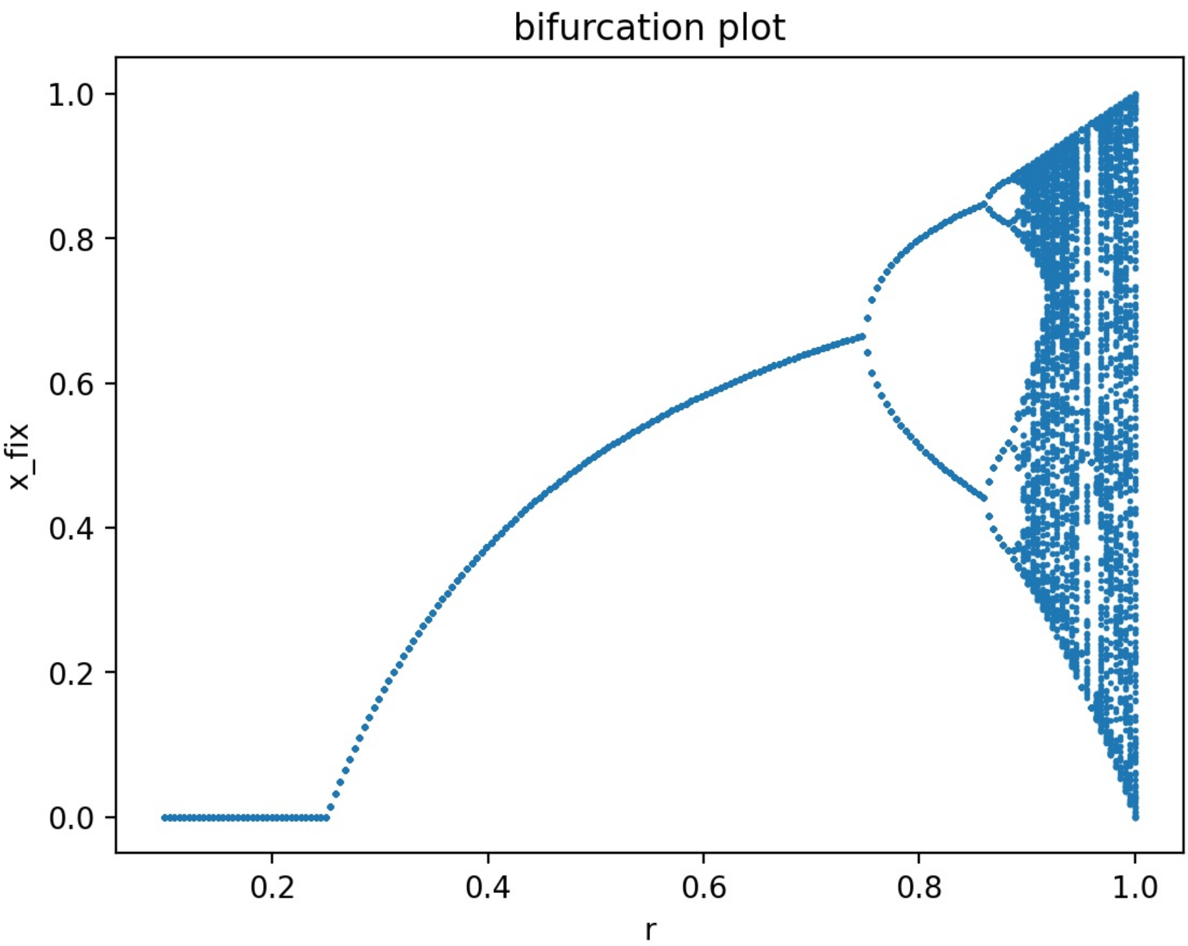

Figure 6: bifurcation plot for $f(r,x)$