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$$x^2_{(t)} = \left(x_{(t-\tau)} + a\ell\right)^2 = x^2_{(t-\tau)} + a^2\ell^2 + 2a\ell x_{(t-\tau)}$$

$$\Rightarrow \langle x^2_{(t)}\rangle = \langle x^2_{(t-\tau)}\rangle + \ell^2\underbrace{\langle a^2\rangle}_{1} + 2\langle a\rangle \ell \langle x_{(t-\tau)}\rangle$$

$$= \langle x^2_{(t-\tau)}\rangle + \ell^2 + 2\ell(p-q)\frac{\ell}{\tau}(p-q)(t-\tau)$$

$$= \langle x^2_{(t-\tau)}\rangle + \ell^2 + \frac{2\ell^2}{\tau}(p-q)^2(t-\tau)$$

$$= \langle x^2_{(t-2\tau)}\rangle + 2\ell^2 + \frac{2\ell^2}{\tau}(p-q)^2\left[(t-\tau) + (t-2\tau)\right]$$

$$= \ell^2\frac{t}{\tau} + \frac{2\ell^2}{\tau}(p-q)^2\sum_{i=1}^{\frac{t}{\tau}-1}(t-i\tau)$$

$$= \frac{\ell^2}{\tau}t + \frac{2\ell^2}{\tau}(p-q)^2\sum_{i=1}^{\frac{t}{\tau}-1} i\tau = \frac{\ell^2}{\tau}t + \frac{2\ell^2}{\tau}(p-q)^2\tau\,\frac{\frac{t}{\tau}\left(\frac{t}{\tau}-1\right)}{2}$$

$$\Rightarrow \sigma^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{\ell^2}{\tau}t^2 + \frac{\cancel{2}\ell^2}{\tau}(p-q)^2 t\left(\frac{t}{\tau}-1\right) - \frac{\ell^2}{\tau^2}(p-q)^2t^2$$

$$= \frac{\ell^2}{\tau^2}t^2 - \frac{\ell^2}{\tau}(p-q)^2 t$$

$$= \frac{\ell^2}{\tau^2}t\left(1-(p-q)^2\right) = \frac{\ell^2}{\tau}t\left[(p+q)^2-(p-q)^2\right]$$

$$\Rightarrow \boxed{\sigma^2 = \frac{4\ell^2}{\tau}pq\,t}$$