Signals and Systems

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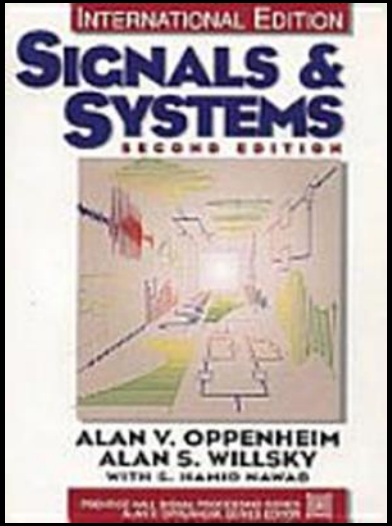
1399-1400_II

Course organization

- **Textbook**
  - Signals and Systems
    - Alan V. Oppenheim
    - Alan S. Willsky
- **References**
  - Fundamentals of Signals and Systems: Using Matlab
    - Edward W. Kamen and bonnie S. Heck
    - Prentice Hall
  - Signals and Systems : Continuous and Discrete
    - Ziemer, Trainer, Fannin
    - Macmillon



Course organization

- **Grading**
  - **Homework (20%)**
    - Computer Projects
      - Matlab or Python
      - DO NOT COPY other students' codes
    - Written Problems
  - **Tests (80%) Quizes**
    - Midterm 1(Closed book)
    - Final (Closed Book)

- **Submit all your works electronically**

- نمره تمرینات برای دانشجویان که کمتر از 50 % نمره امتحانی را اخذ کنند به طور کامل لحاظ نخواهد گردید.

Course organization

- Policies
  - On-line presentation
  - Attendance
  - Missing Exams / HW/ Quiz
  - Neatness
  - INDIVIDUALLY of your works

Overall Outline

- Introduction
- 1. Signals
- 2. Systems
- 3. Continuous-time Fourier Transform
- 4. Communication Systems
- 5. Discrete-Time Fourier Transform
- 6. Sampling
- 7. Applications of Discrete-Time Signal Processing
- 8. Fourier Series Representation
- 9. Laplace Transform
- 11. Z-Transform

LECTURE SUBJECT

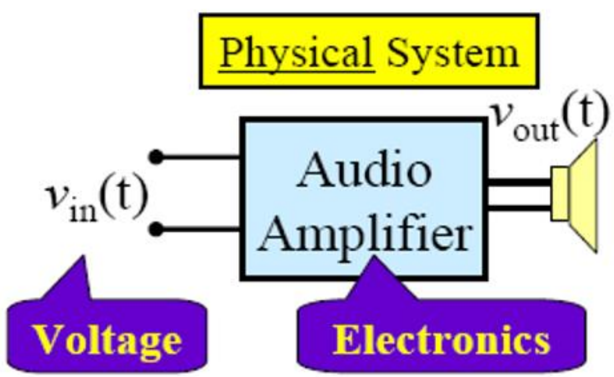
- Introduction to Signals and Their Properties
- Periodic and Exponential Signals, Discrete and Continuous Exponentials
- Discrete and Continuous Impulse Signals
- Systems and Their Properties, LTI Systems
- The convolution Theorem
- LTI Systems Described by Differential and Difference Equations
- Step Response of LTI Systems
- Fourier Series and their porperties
- Discrete Fourier Series and Their properties
- Frequency Shaping Filters
- CT Fourier Transform
- Discrete Time Fourier Transform
- Linear Filters in Prtacticei, First and Second Order Systems
- LTI Systems Described by Differential and Difference Equations
- Sampling, Nyquist Rate and Shannon Theorem
- Sampling with Hold, Discrete Time Implementation of Continuous Time Systems
- Discrete Time Sampling, Interpolation and Decimation
- Communications Systems
- Modulation, Demodulation, AM, variations of AM, PM, FM, PAM and PCM, Discrete Time Modulation
- The Laplace Transform and / or The z-Transform

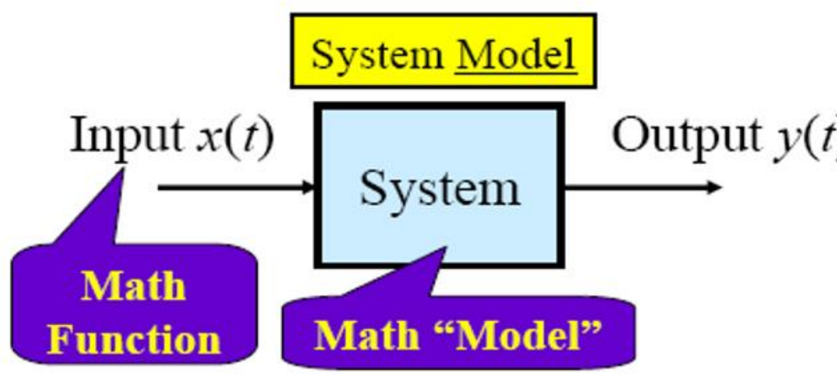
Definition

- Broadly speaking, a signal is a means to convey information
  - A page of book is considered a signal
  - A sound is a signal
  - sequence of bits stored in a flash memory
  - An image is a signal
- system is a process that generates signals or transforms signals
  - electronic book reader for the blind
  - A program that analyzes images, recognizing objects, faces, animals, is system

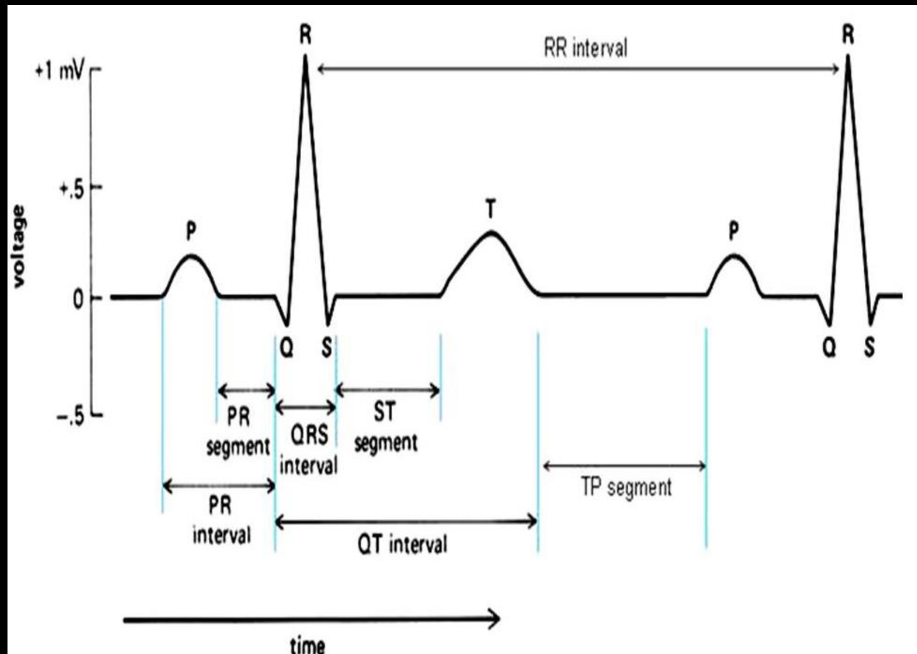
What *is "Signals & Systems" all about???*

- "**Signal**" = a time-varying voltage (or other quantity) that generally carries some information
- The job of the "**System**" is often to extract, modify, transform, or manipulate that carried information
- So… a big part of "**Signals & Systems**" is using **math models** to see what a system "does" to a signal





EKG (a.k.a ECG) Example

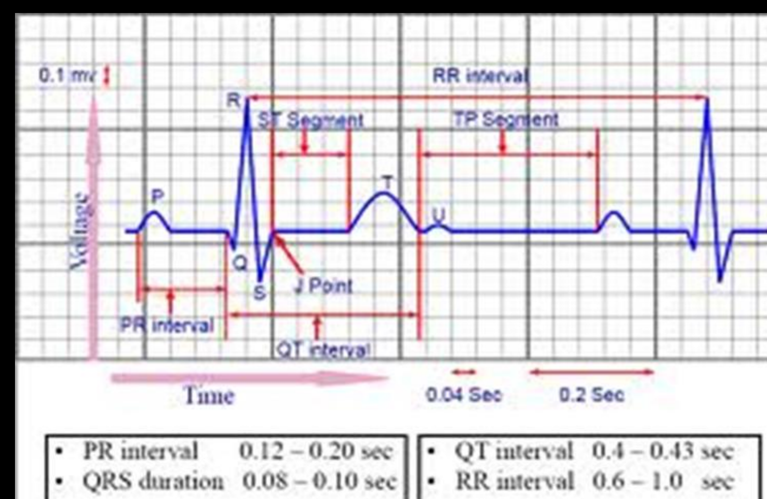


graph of voltage versus time of the electrical activity of the heart using electrodes placed on the skin

Conveys a large amount of information about the structure of the heart and the function of its electrical conduction system. Among other things, an ECG can be used to measure the rate and rhythm of heartbeats, the size and position of the [heart chambers]

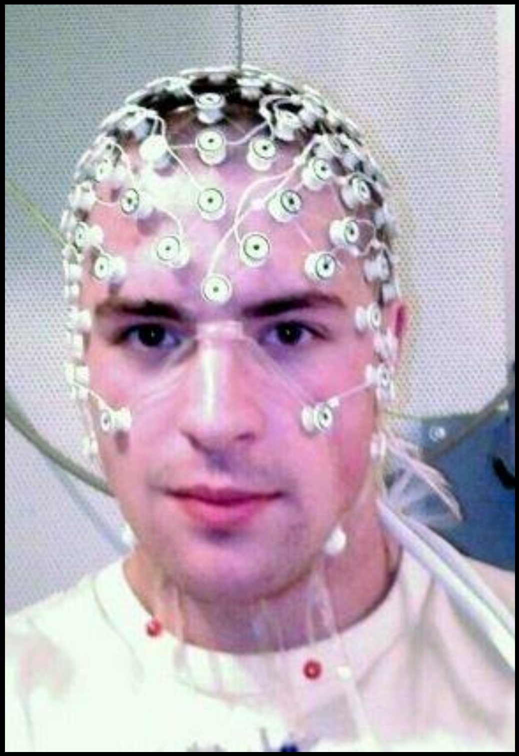
The presence of any damage to the heart's muscle cells or conduction system,

The effects of heart drugs, and the function of implanted [pacemakers].



Electroencephalography (EEG)

- **Electroencephalography** (**EEG**) is an electrophysiological monitoring method to record electrical activity on the scalp that has been shown to represent the macroscopic activity of the surface layer of the brain underneath



SIGNALS

- Signals are functions of independent variables that carry information. For example:
- Electrical signals ---voltages and currents in a circuit
- Acoustic signals ---audio or speech signals (analog or digital)
- Video signals ---intensity variations in an image (e.g. a CAT scan)
- Biological signals ---sequence of bases in a gene
- Data --- Databases, Log files, Internet searches, ..
- .

Some Application Areas

Application Area	Specific Uses of Signals & Systems
Telecommunications	Answering machines, modems, fax machines, cell phones, speaker phones
Speech and Audio	Voice mail, speaker verification, synthetic speech, audio compression (e.g., mp3)
Automotive	Engine control, antilock braking systems, active suspension, airbag control, system diagnosis
Medical	Hearing aids, remote monitoring, ultrasound imaging, magnetic resonance imaging (MRI)
Image Processing	3D animation, image enhancement, image compression (JPEG), video compression (MPEG), high-definition TV
Control Systems	Head positioning in disk drives, laser control (e.g., printers, CD/DVD drives), engine & motor control, robots
Military & Aerospace	Radar & sonar, navigation systems (e.g., GPS), secure communications, missile guidance, battlefield sensors

In each of these areas you can't build the electronics until your math models tell you what you need to build

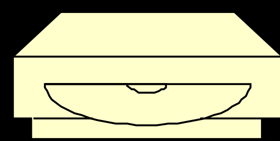
SIGNALS

Information expressed in different forms

Stock Price

$1.00, $1.20, $1.30, $1.30, …

Data File



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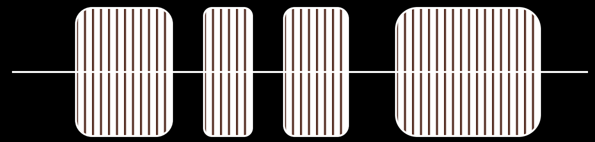
Transmit Waveform



$x(t)$

Primary interest of Electronic Engineers

SIGNALS PROCESSING AND ANALYSIS

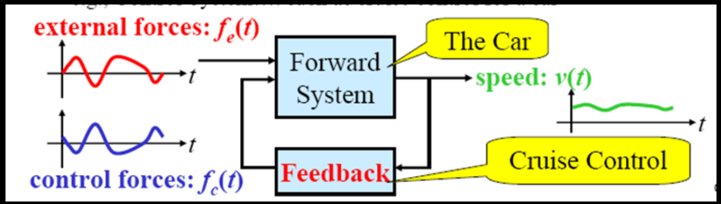
Processing: Methods and system that modify signals

<u>Analysis</u>:

- **What information is contained in the input signal *x*(*t*)?**
- **What changes do the System imposed on the input?**
- **What is the output signal *y*(*t*)?**

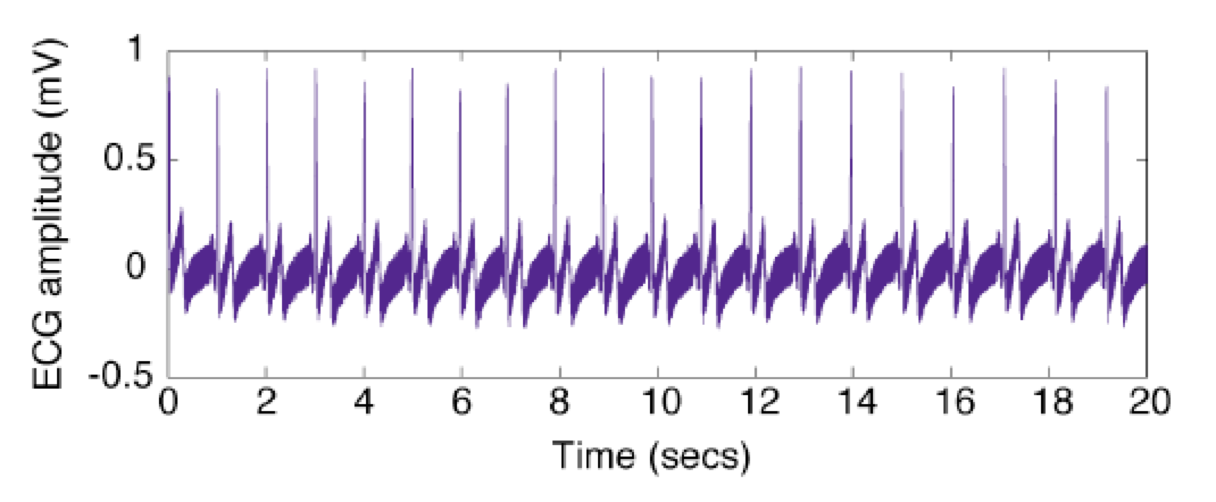
Common Signals & Systems Scenarios:

- Given a system, how it will respond to various inputs.
  - A circuit: to quantify its response to different voltage and current sources
  - Aircraft: characterize it to pilot commands and to wind gusts.
- Design system, to process signals in particular ways
  - design to enhance or restore signals that have been degraded in some way. (in aircraft communication with air traffic control tower often background noise in the cockpit has to be removed)
  - Underwater communication
  - Image restoration (Remote sensing)
- Design system, to **extract specific pieces of information** from signals
  - Estimate the heart rate from an electrocardiogram signals
  - Estimate the condition of heart health   Economic forcasting
- Design of signals with particular properties
  - Long distance communication through the atmosphere: requires the use of signals with frequencies in a particular part of the electromagnetic spectrum
- Modify or control the characteristics of a given system
  - design of aircraft autopilots and computer control systems. Crouse control of cars.



Signal Classification : Type of Independent Variable

- Time is often the independent variable. ***Example:*** the electrical activity of the heart recorded with chest electrodes — the electrocardiogram (ECG or EKG).



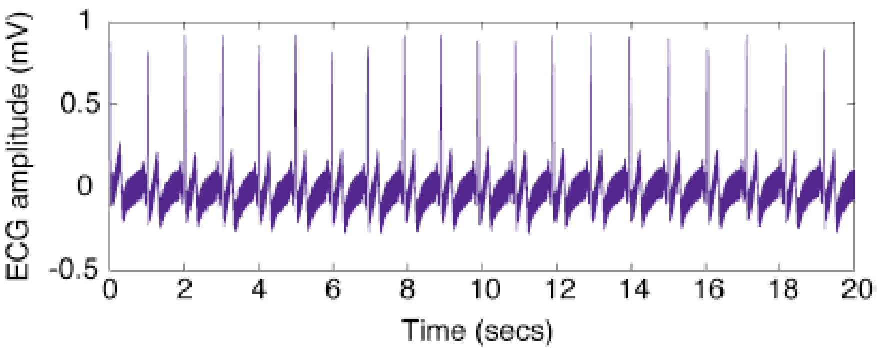
The variables can also be spatial

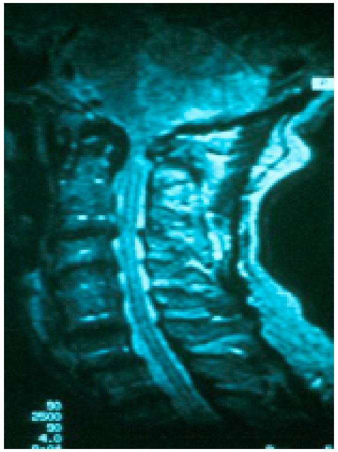
Eg. Cervical MRI

In this example, the signal is the intensity as a function of the spatial variables x and y.

Independent Variable Dimensionality

An independent variable can be 1-D (t in the EKG) or 2-D (x, y in an image).

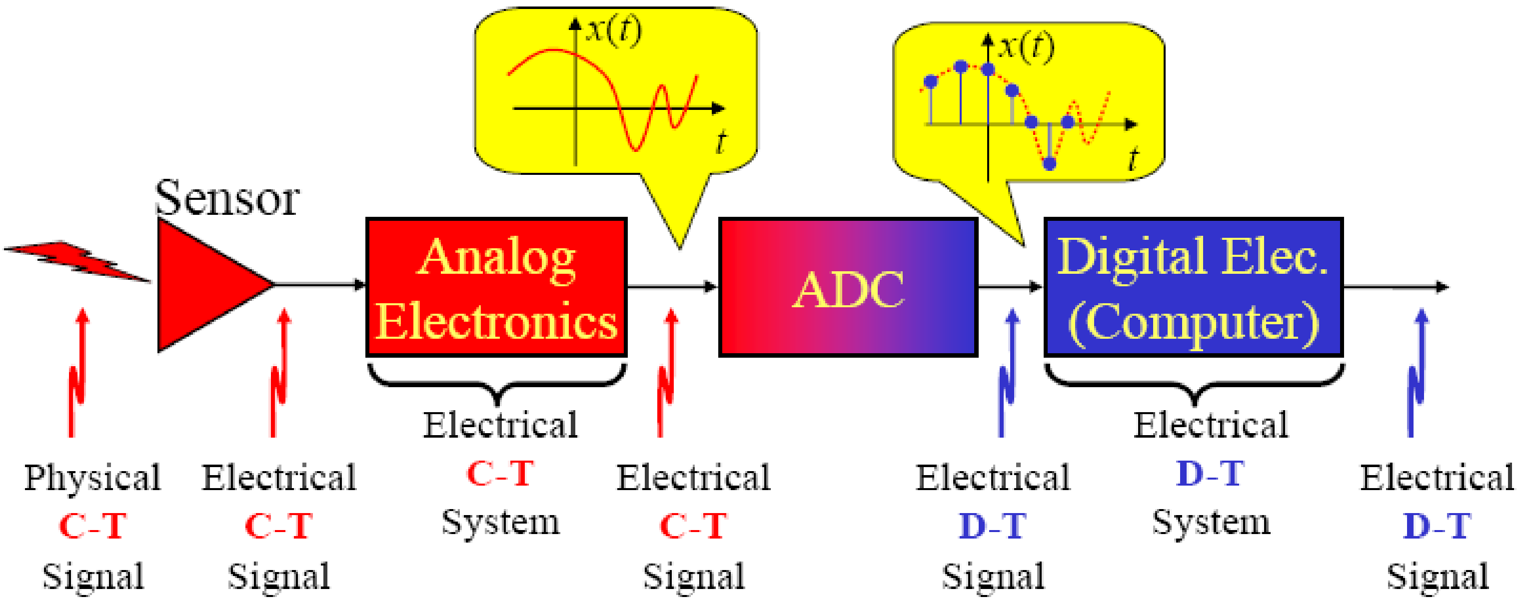




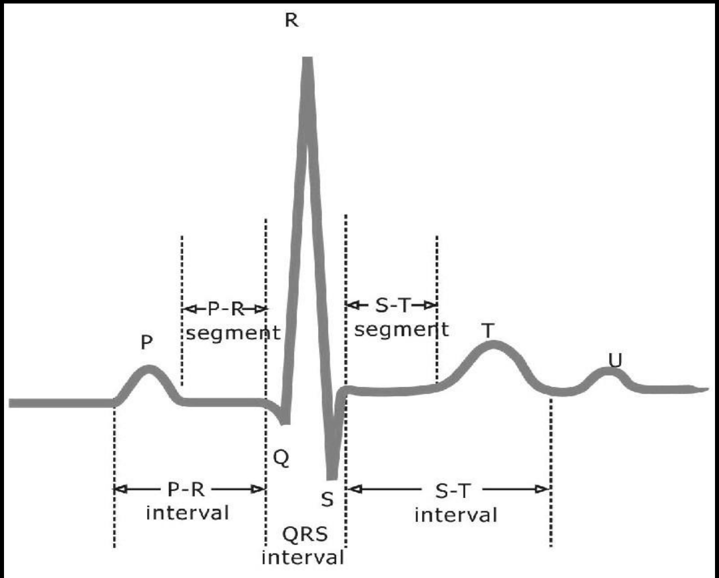
In EECE233, focus on 1-D for mathematical simplicity but the results can be extended to 2-D or even higher dimensions. Also, we will use a generic time t for the independent variable,whether it is time or space.

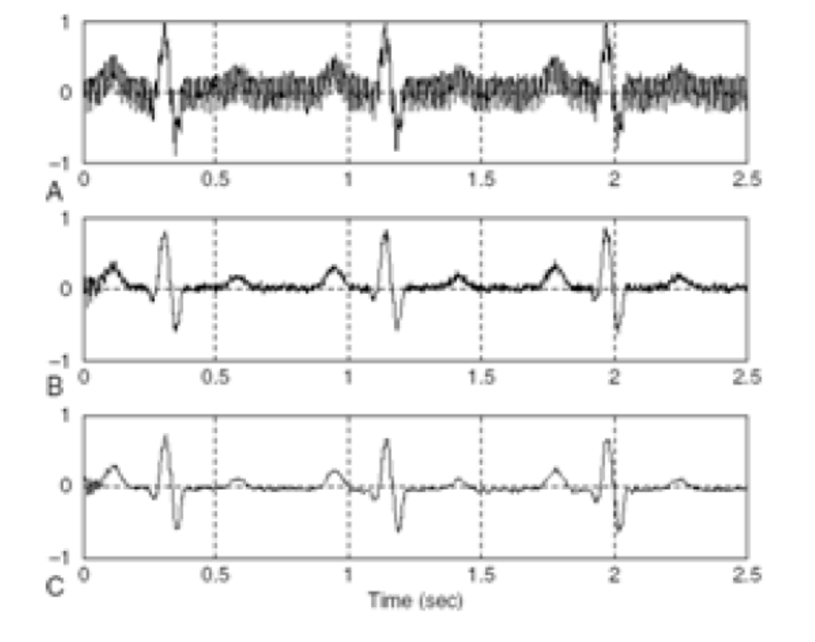
Continuous-Time & Discrete-Time

- Modern systems generally…
  - get a continuous-time signal from a sensor
  - a cont.-time system modifies the signal
  - an "analog-to-digital converter" (ADC or A-to-D) sample the signal to create a discrete-time signal … a "stream of numbers"
  - A discrete-time system to do the processing
  - and then (if desired) convert back to analog (not shown here)



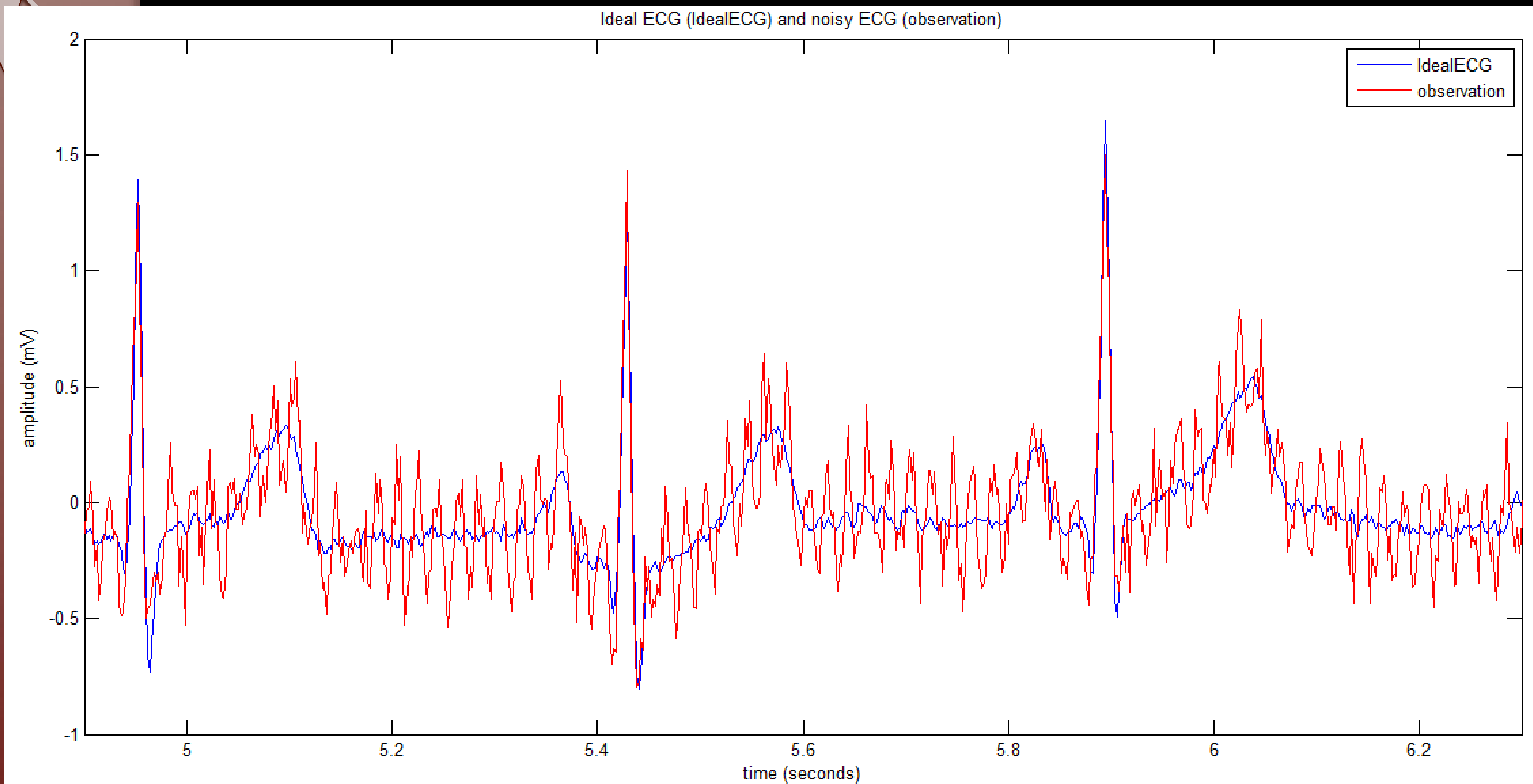
EKG Example





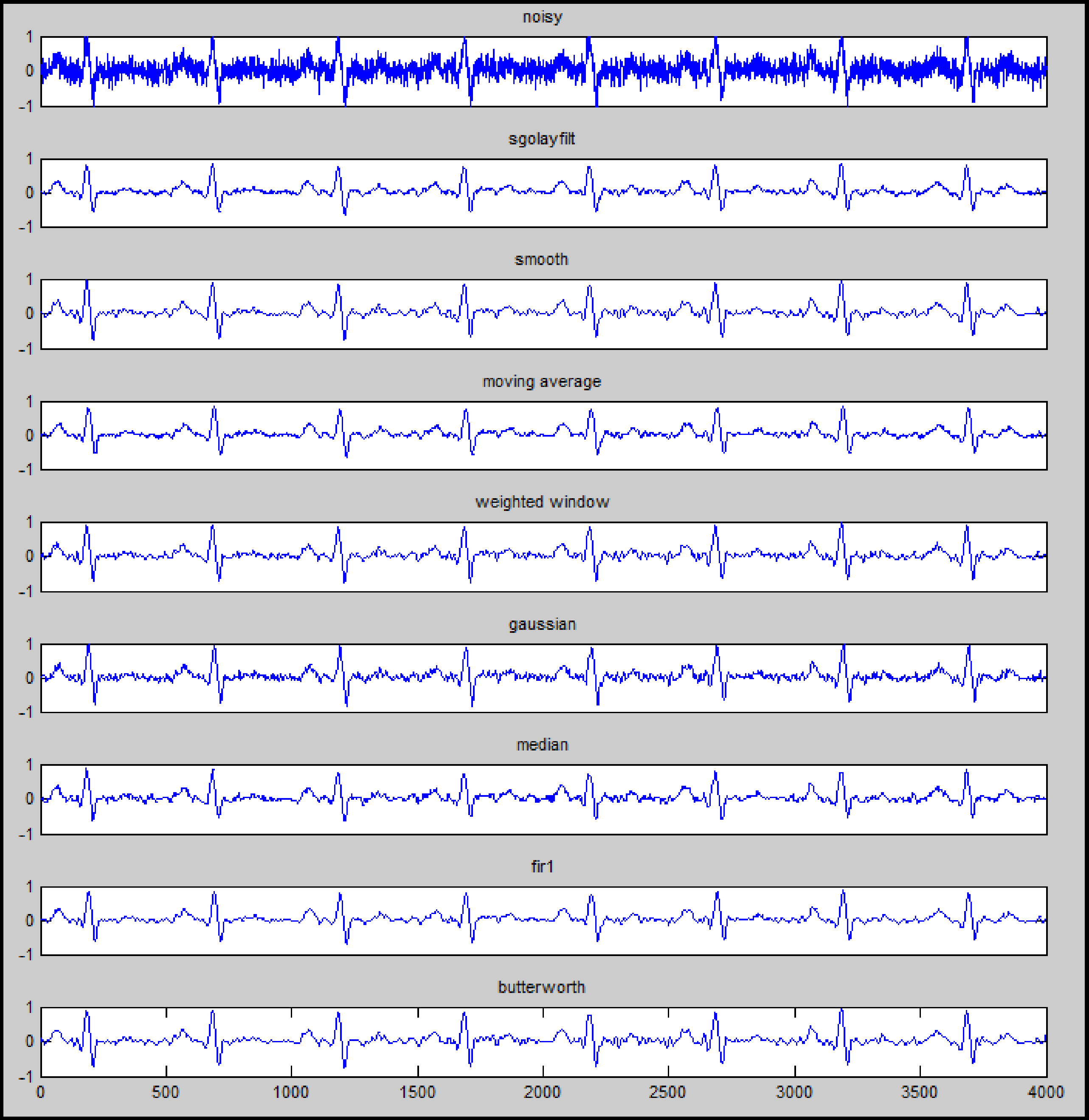
Results of ECG signal processing prior to implementation of algorithm.

Noisy EKG



Ideal ECG (IdealECG) and noisy ECG (observation)
IdealECG
observation
amplitude (mV)
time (seconds)
2
1.5
1
0.5
0
-0.5
-1
5
5.2
5.4
5.6
5.8
6
6.2

Noise and different Filtering Results



Our Studies

- Because there are many similarities between C-T and D-T signals and systems…
  - We will present many ideas "side-by-side"
  - You'll need to recognize the differences/similarities

Ch. 2
Differential Equations
Difference Equations

Ch. 2
C-T Convolution
D-T Convolution

Ch. 3 & 5
C-T Fourier Analysis & Freq. Response

Ch. 4 & 5
D-T Fourier Analysis & Freq. Response

Ch. 6 & 8
C-T Laplace Transform & Transfer Function

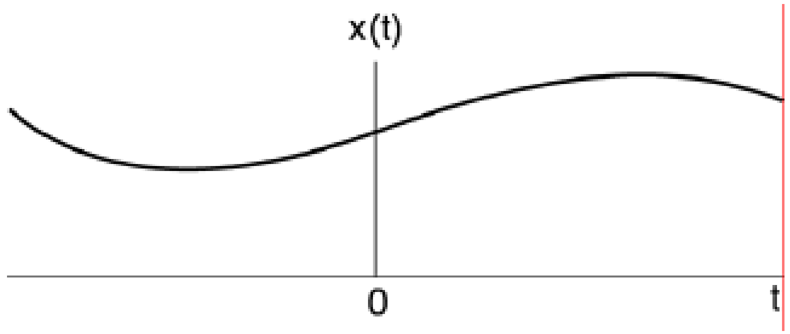
Ch. 7
D-T Z-Transform & Transfer Function

THE INDEPENDENT VARIABLES

- Can be continuous
  - — Trajectory of a space shuttle
  - — Mass density in a cross-section of a brain
- Can be discrete
  - — DNA base sequence
  - — Digital image pixels
- Can be 1-D, 2-D, ••• N-D
- For this course: Focus on a single (1-D) independent variable which we call "time".

  Continuous-Time (CT) signals: $x(t)$, t — continuous values
  Discrete-Time (DT) signals: $x[n]$, n — integer values only

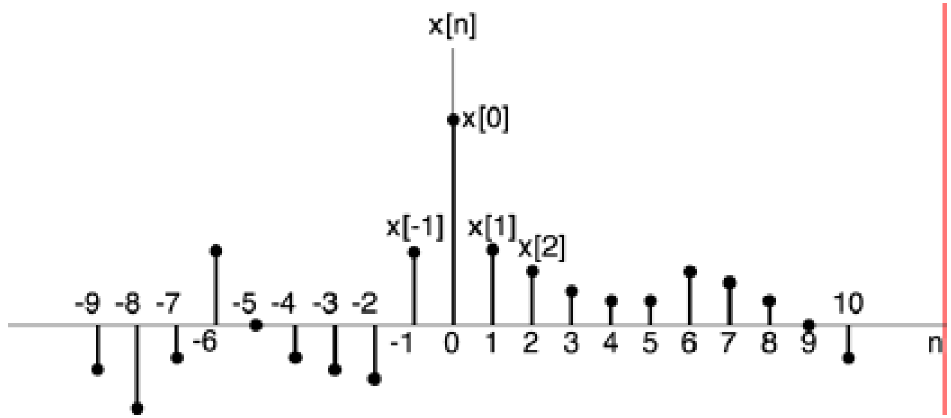
CT Signals



- Most of the signals in the physical world are CT signals—E.g. voltage & current, pressure, temperature, velocity, etc.

DT Signals

- $x[n]$, n — integer, time varies discretely

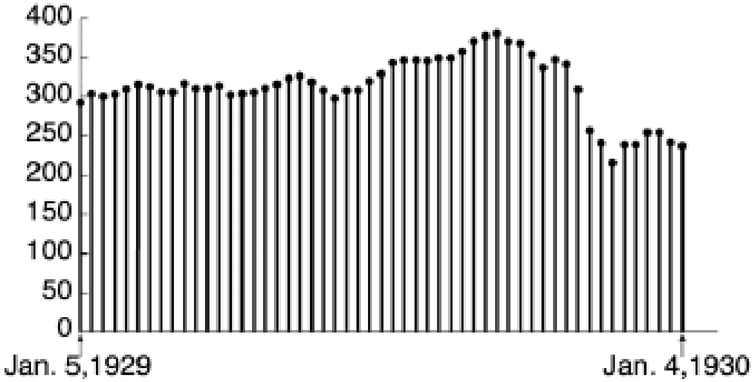


- Examples of DT signals in nature:
  — DNA base sequence
  — Population of the nth generation of certain species
  :

Our Notation : $x(t)$ — CT, $x[n]$ — DT

Many human-made DT Signals

Ex.#1 Weekly Dow-Jones industrial average

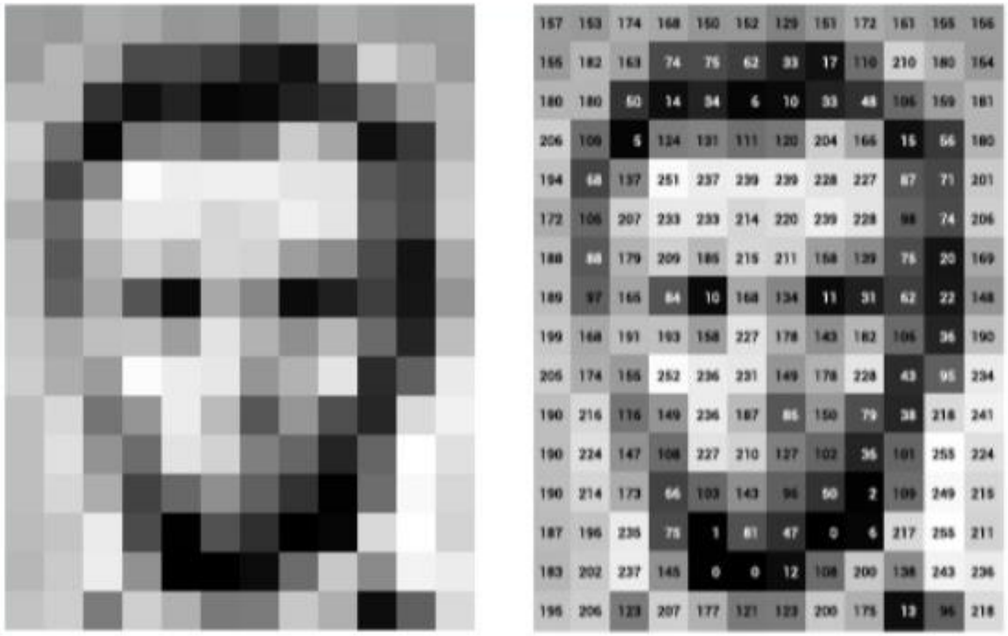


Ex.#2 digital image

Courtesy of Jason Oppenheim.
Used with permission.

Why DT? —Can be processed by modern digital computers and digital signal processors (DSPs).

Digital Image



157	153	174	168	150	152	129	151	172	161	155	156
155	182	163	74	75	62	33	17	110	210	180	154
180	180	50	14	34	6	10	33	48	106	159	181
206	109	5	124	131	111	120	204	166	15	56	180
194	68	137	251	237	239	239	228	227	87	71	201
172	105	207	233	233	214	220	239	228	98	74	206
188	88	179	209	185	215	211	158	139	75	20	169
189	97	165	84	10	168	134	11	31	62	22	148
199	168	191	193	158	227	178	143	182	106	36	190
205	174	155	252	236	231	149	178	228	43	95	234
190	216	116	149	236	187	86	150	79	38	218	241
190	224	147	108	227	210	127	102	36	101	255	224
190	214	173	66	103	143	96	50	2	109	249	215
187	196	235	75	1	81	47	0	6	217	255	211
183	202	237	145	0	0	12	108	200	138	243	236
195	206	123	207	177	121	123	200	175	13	96	218

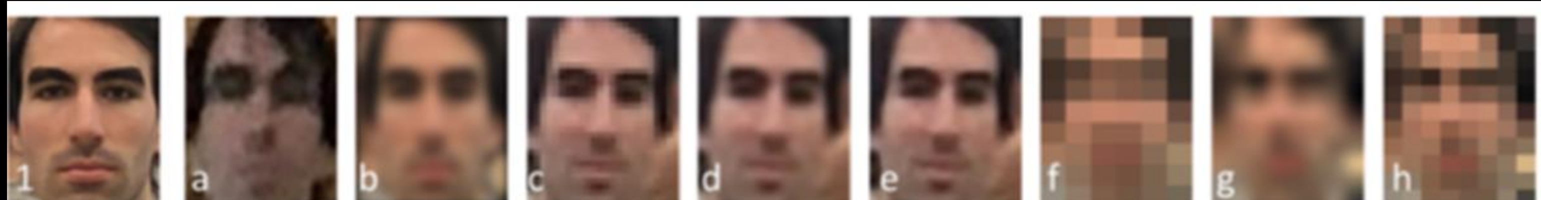
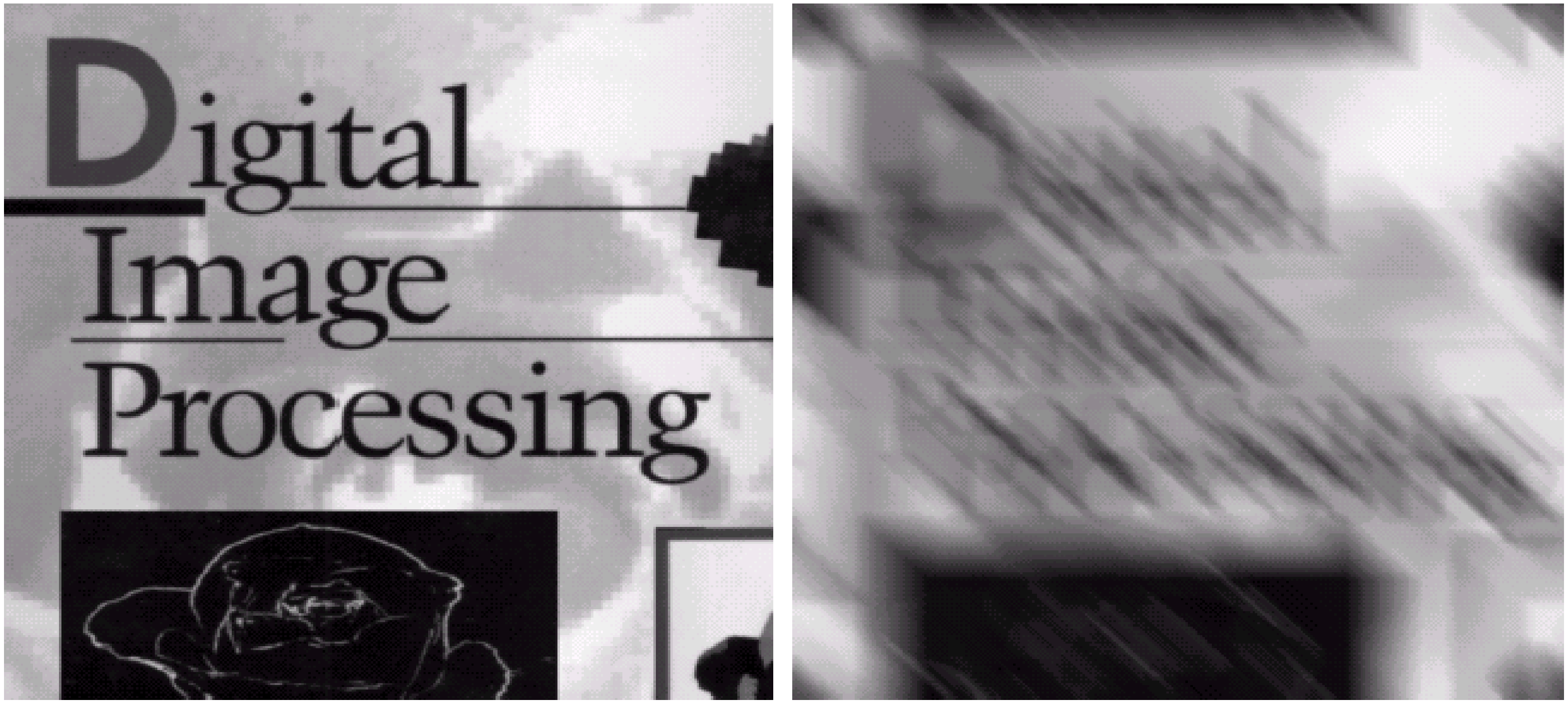


Fig. 1. Comparison of a high-quality face image (1) with low-quality face images (a–h).

Image De-blurring

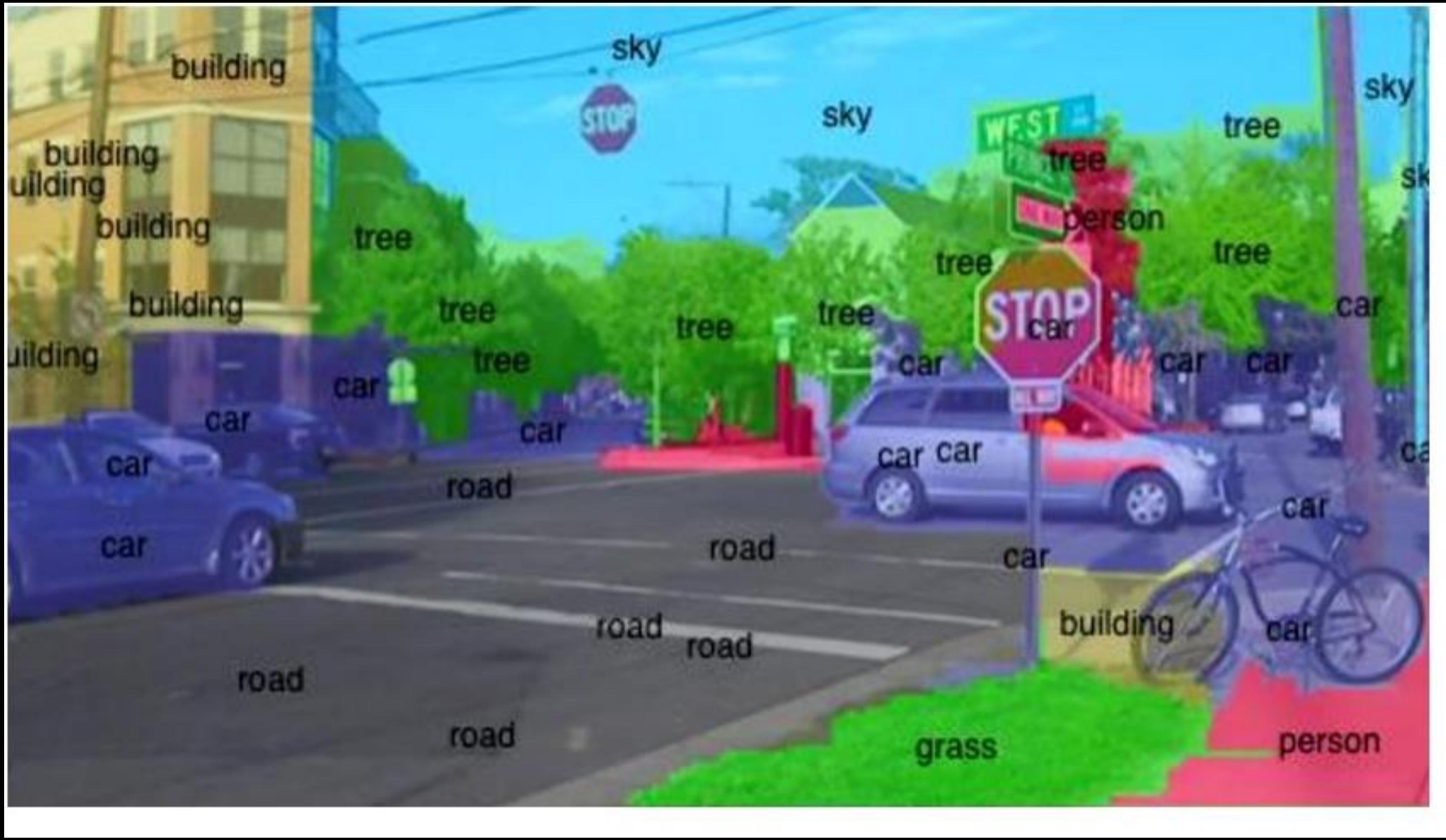
Estimation of degradation functions (model bases)



a b

FIGURE 5.26 (a) Original image. (b) Result of blurring using the function in Eq. (5.6-11) with $a = b = 0.1$ and $T = 1$.

Recognition



SYSTEMS

- For the most part, our view of systems will be from an input-output perspective:
- A system responds to applied input signals, and its response is described in terms of one or more output signals

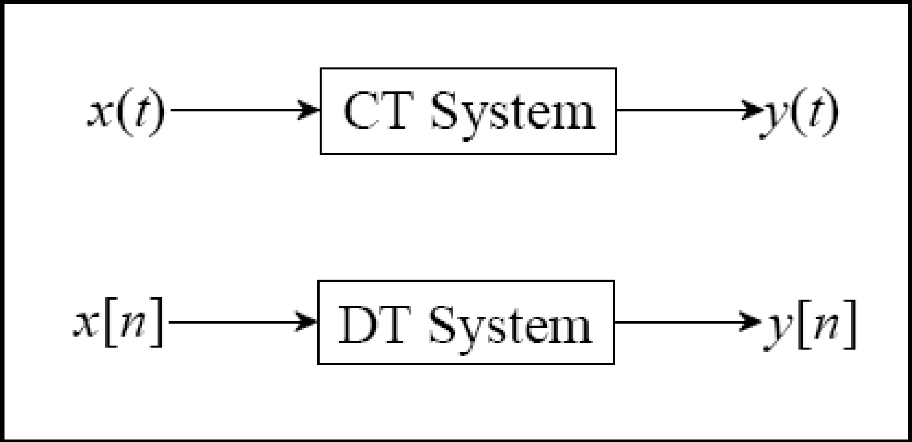
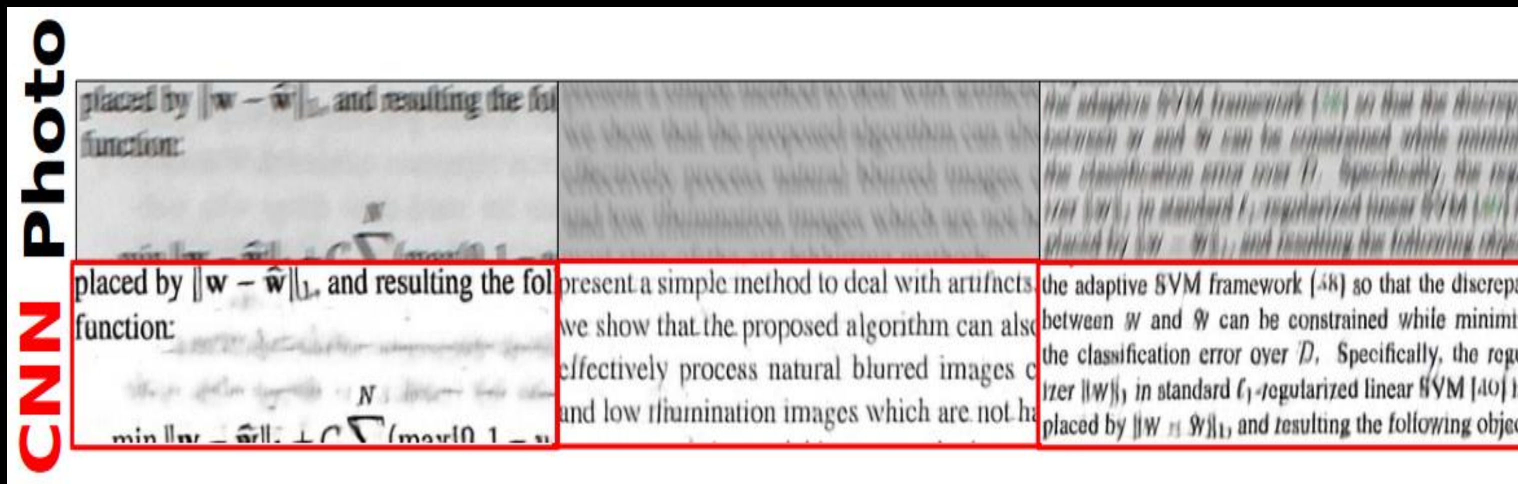
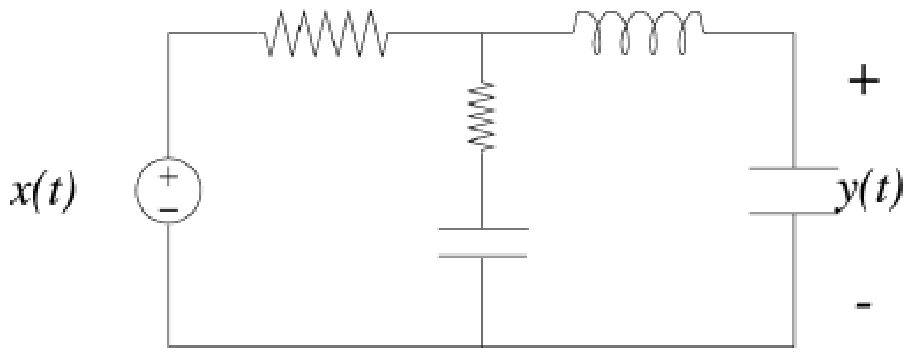


Image Restoration



EXAMPLES OF SYSTEMS

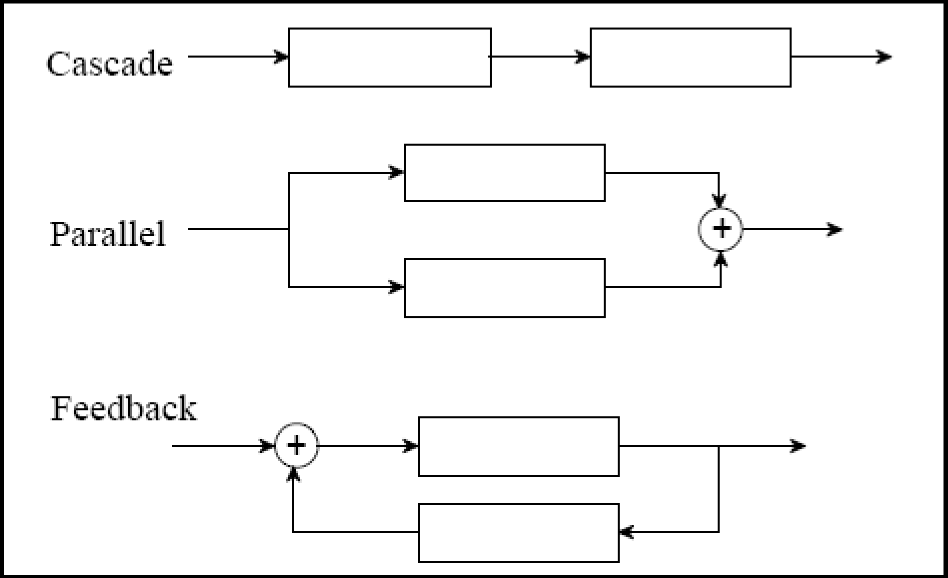
- An RLC circuit



- Dynamics of an aircraft or space vehicle
- An algorithm for analyzing financial and economic factors to predict bond prices
- An algorithm for post-flight analysis of a space launch
- An edge detection algorithm for medical images

SYSTEM INTERCONNECTIOINS

- An important concept is that of interconnecting systems
  — To build more complex systems by interconnecting simpler subsystems
  — To modify response of a system
- Signal flow (Block) diagram



Real and Complex Signals

A very important class of signals is:

- CT signals of the form $x(t) = e^{st}$
- DT signals of the form $x[n] = z^n$

where z and s are complex numbers. For both *exponential* CT and DT signals, x is a complex quantity and has:

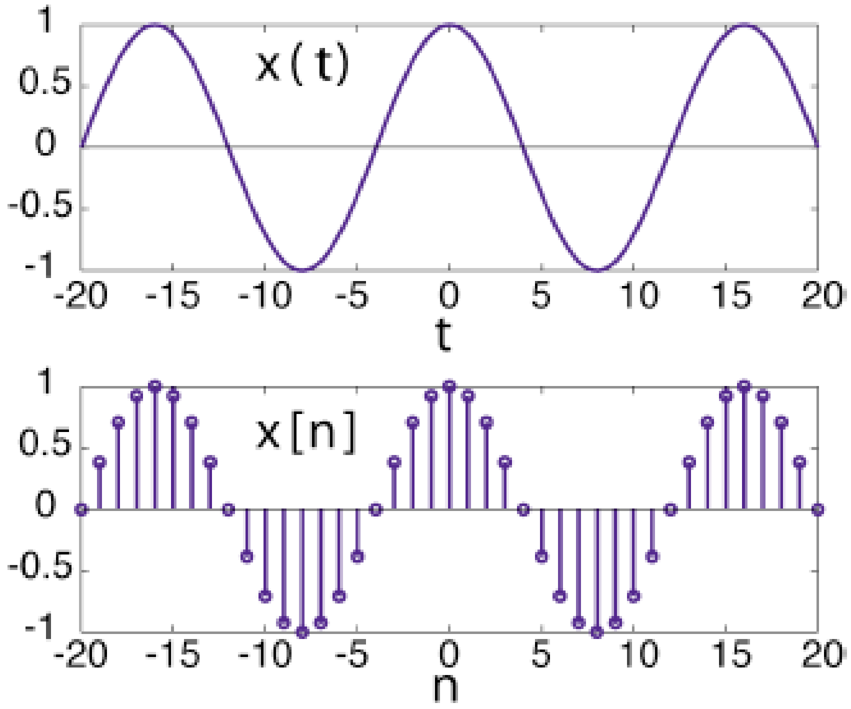
- a real and imaginary part, or equivalently
- a magnitude and a phase angle.

We will use whichever form that is convenient.

For example, suppose $s = j\pi/8$ and $z = e^{j\pi/8}$, then the real parts are

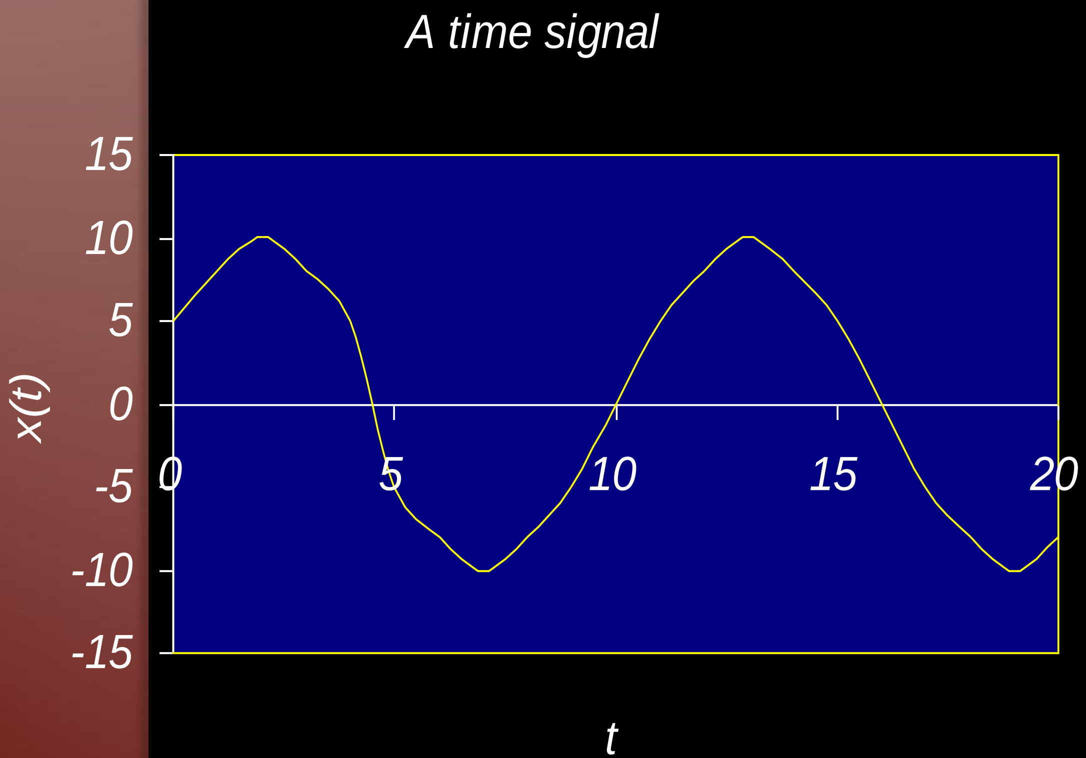
$$\mathcal{R}\{x(t) = e^{st}\} = \mathcal{R}\{e^{j\pi t/8}\} = cos(\pi t/8),$$

$$\mathcal{R}\{x[n] = z^{n}\} = \mathcal{R}\{e^{j\pi n/8}\} = cos[\pi n/8].$$



SIGNALS DESCRIPTION

To analyze signals, we must know how to describe or represent them in the first place.



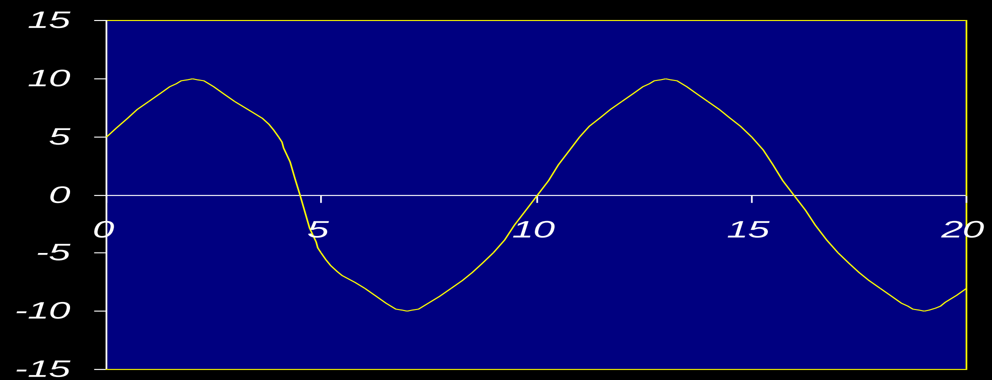
t	x(t)
0	0
1	5
2	8
3	10
4	8
5	5
⋮	⋮

Detail but not informative

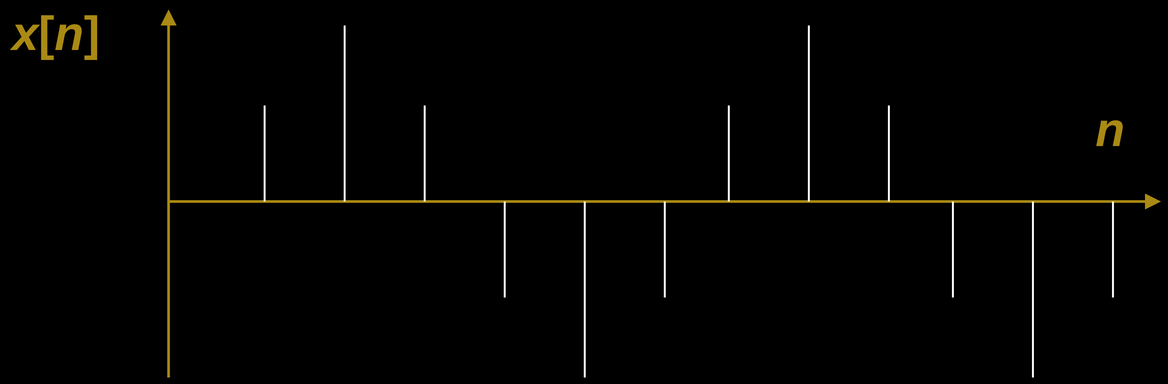
TIME SIGNALS DESCRIPTION

1. **Mathematical expression:** $x(t) = A\sin(\omega t + \phi)$

2. **Continuous (Analogue)**



3. **Discrete (Digital)**

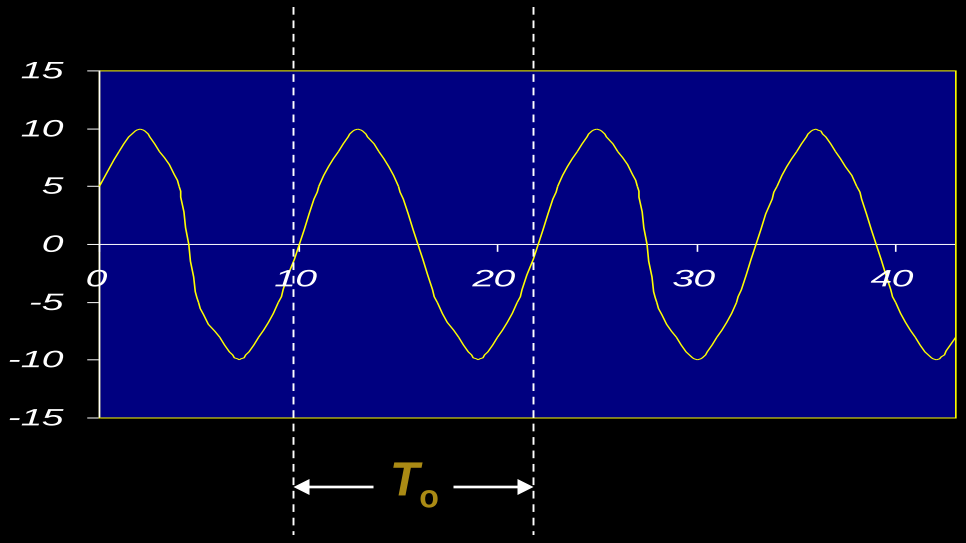


TIME SIGNALS DESCRIPTION

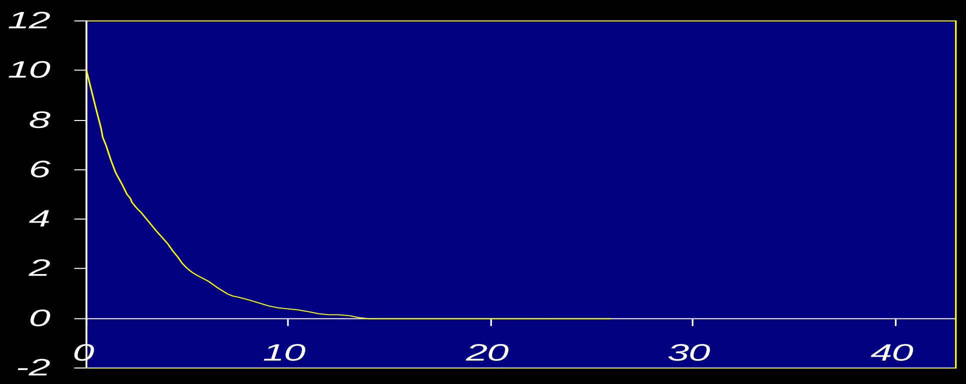
4. Periodic

$$x(t) = x(t+T_o)$$

$$\text{Period} = T_o$$

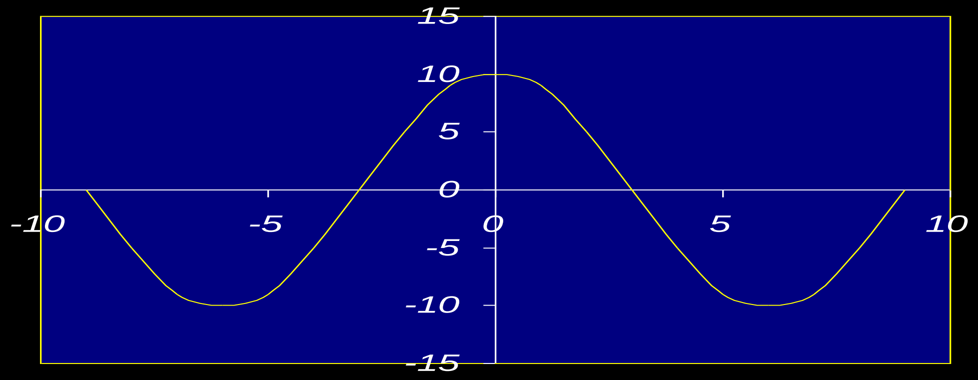


5. Aperiodic

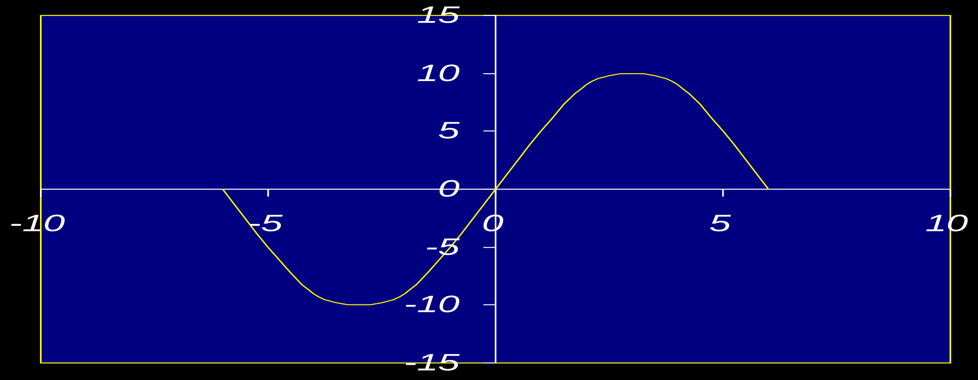


TIME SIGNALS DESCRIPTION

6. **Even signal** $x(t) = x(-t)$



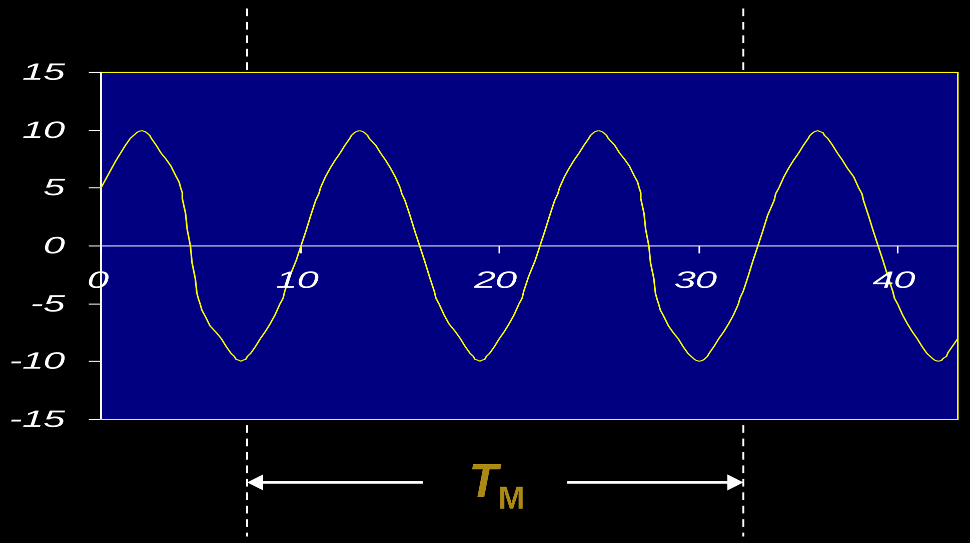
7. **Odd signal** $x(t) = -x(-t)$



TIME SIGNALS DESCRIPTION

8. Average/Mean/*DC* value

$$x_{DC} = \frac{1}{T_M} \int_{t_1}^{t_1+T_M} x(t)\,dt$$



9. *AC* value

$$x_{AC}(t) = x(t) - x_{DC}$$

DC: Direct Component
AC: Alternating Component

TIME SIGNALS DESCRIPTION

10. Energy

$$E = \int_{-\infty}^{\infty} \left| x(t) \right|^2 dt$$

11. Instantaneous Power

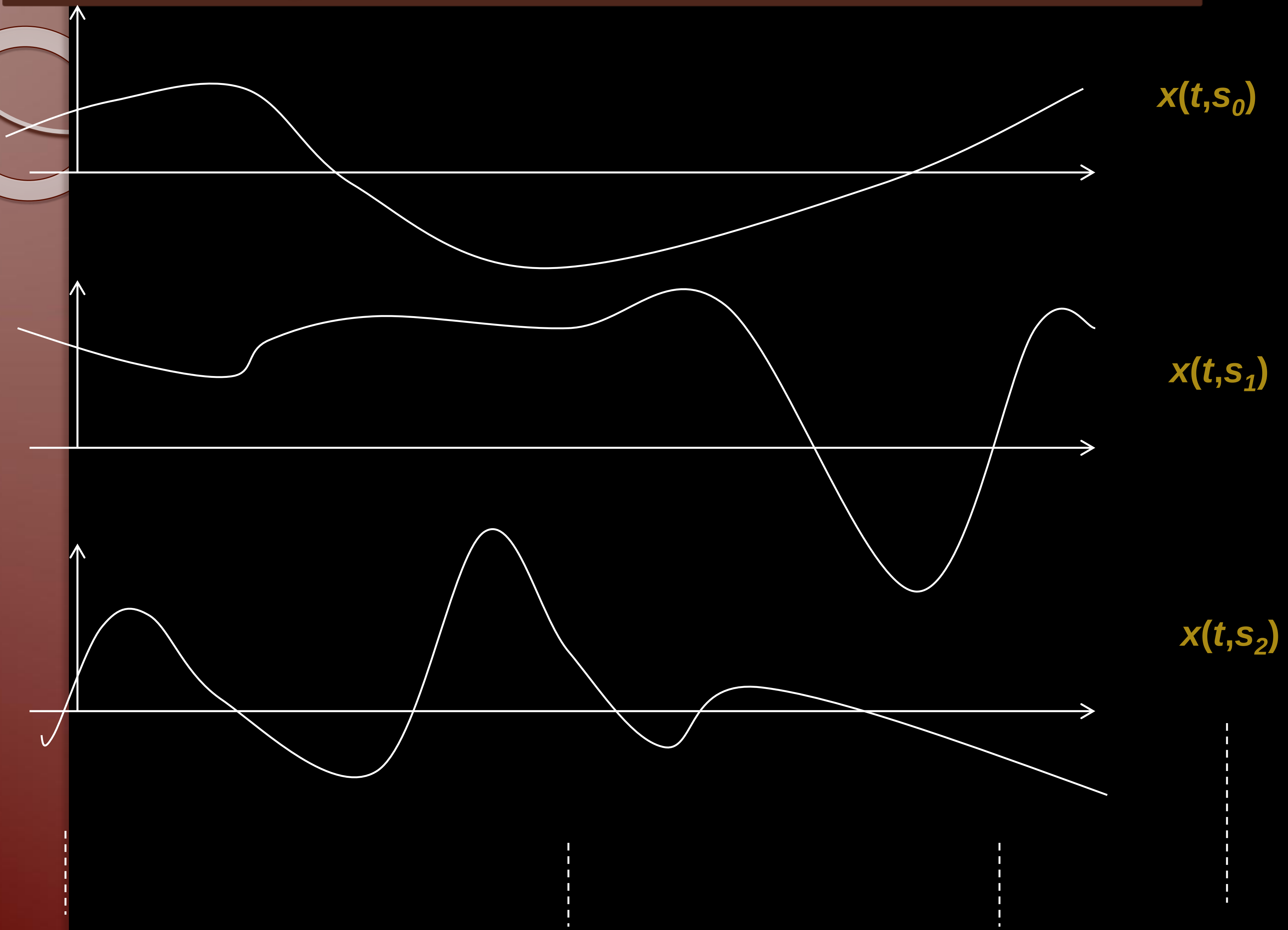
$$P(t) = \frac{\left| x(t) \right|^2}{R} \, watts$$

12. Average Power

$$P_{av} = \frac{1}{T_M} \int_{t_1}^{t_1 + T_M} P(t) dt$$

Note: For periodic signal, T_M is generally taken as T_o

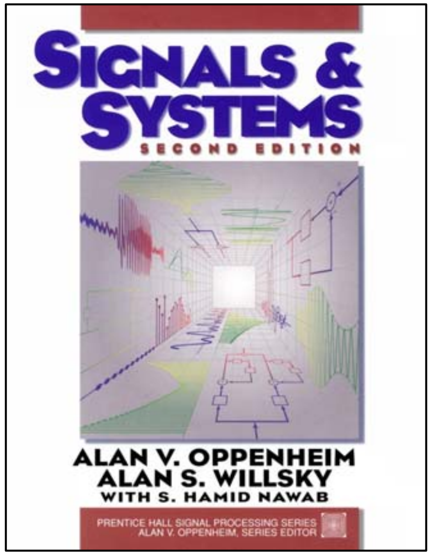
Random Signals

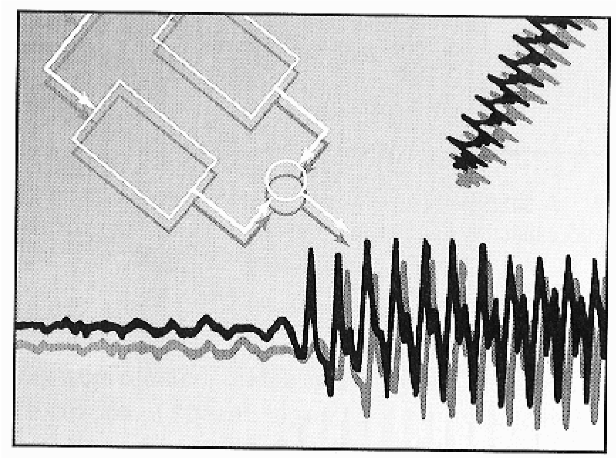


Fall 2020

Signals and Systems

Chapter SS-1
Signals and Systems



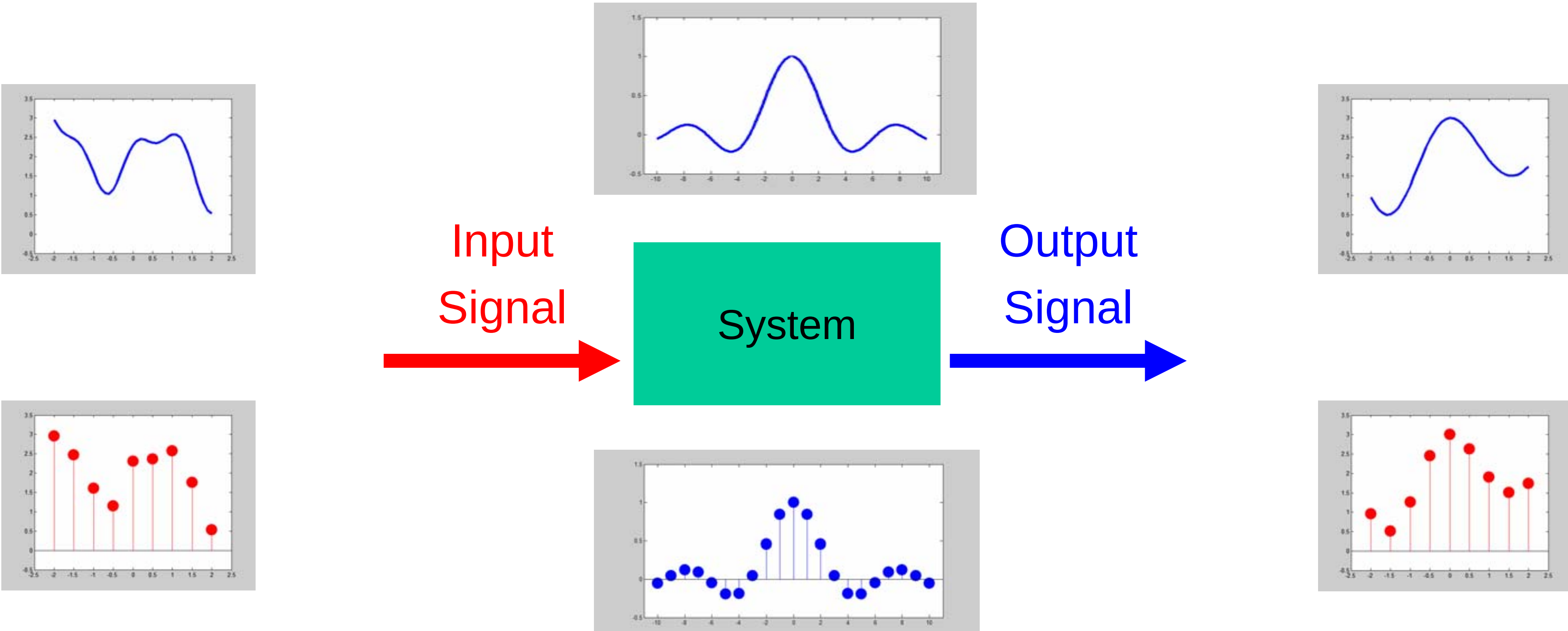


Figures and images used in these lecture notes are adopted from
"Signals & Systems" by Alan V. Oppenheim and Alan S. Willsky, 1997

- Introduction
- Continuous-Time & Discrete-Time Signals
- Transformations of the Independent Variable
- Exponential & Sinusoidal Signals
- The Unit Impulse & Unit Step Functions
- Continuous-Time & Discrete-Time Systems
- Basic System Properties

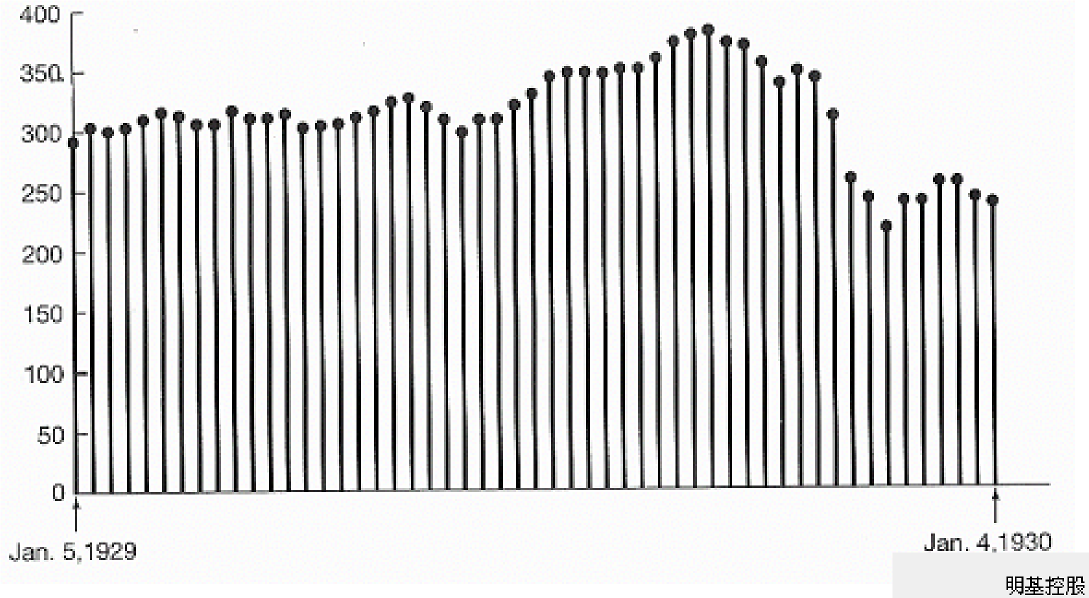
Signals & Systems:

- Is about using mathematical techniques to help describe and analyze systems which process signals
  - Signals are variables that carry information
  - Systems process input signals to produce output signals



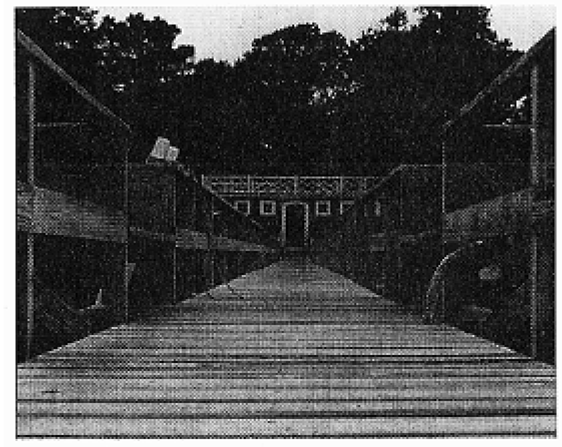
▪ Discrete-Time Signals:

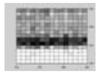
▪ The weekly Dow-Jones stock market index

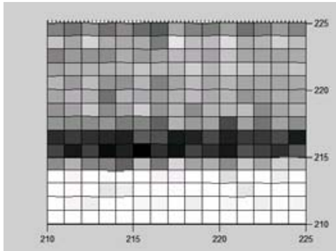


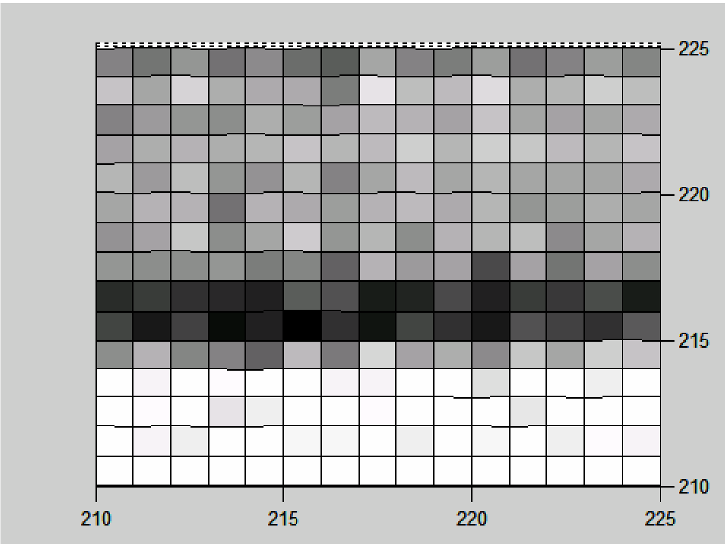
http://big5.jrj.com.cn/

- Discrete-Time Signals:
  - A monochromatic picture



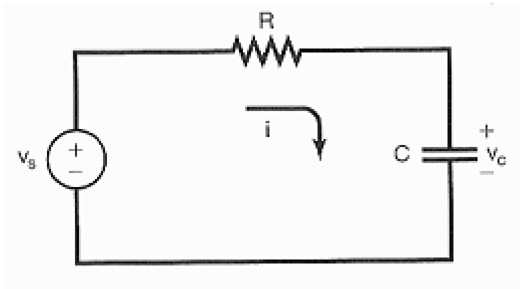




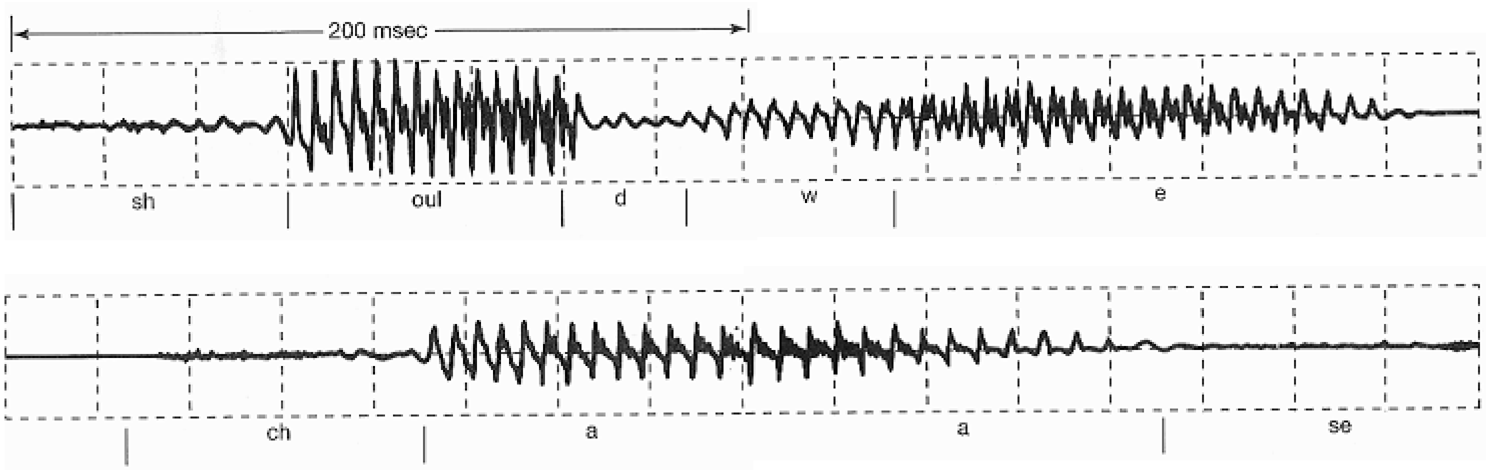


117	151	112	136	110	95	166	131	125	159	114	130	159	133
164	209	173	171	168	126	225	189	184	219	172	180	206	188
154	151	143	174	156	160	186	179	163	193	167	161	167	168
173	179	172	182	195	181	187	205	183	205	196	186	180	193
155	188	149	145	181	131	166	185	166	181	164	164	166	171
176	176	113	178	169	159	177	184	171	182	148	159	174	164
162	198	141	164	202	149	181	142	176	182	189	136	165	176
143	143	148	127	132	97	177	152	160	74	163	119	162	143
62	51	40	32	95	82	28	39	75	35	60	58	77	28
26	67	14	34	3	49	22	69	48	26	81	67	49	91
176	134	129	97	185	120	212	160	173	139	197	166	207	192
243	255	250	255	254	242	242	255	254	222	255	253	239	252
248	255	226	238	255	255	249	255	255	255	230	252	255	255
243	238	255	253	245	247	255	238	255	244	255	237	249	241

- Continuous-Time Signals:
  - Source voltage & capacity voltage in a simple RC circuit

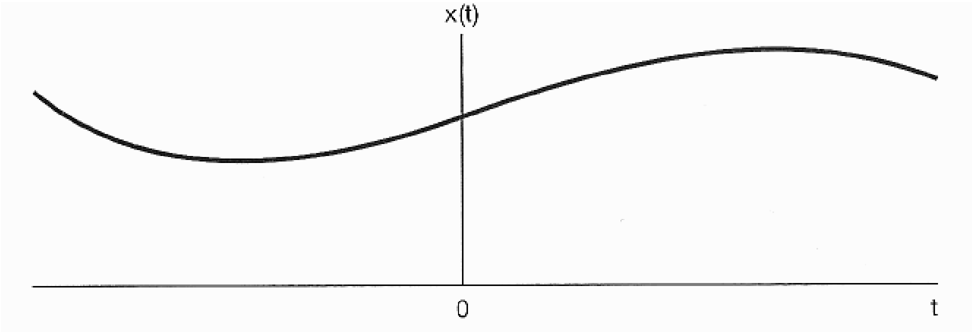


  - Recording of a speech signal

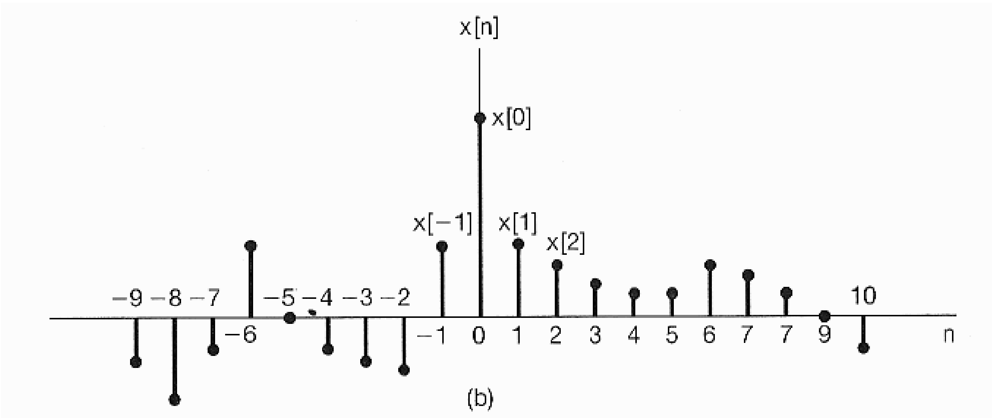


Graphical Representations of Signals:

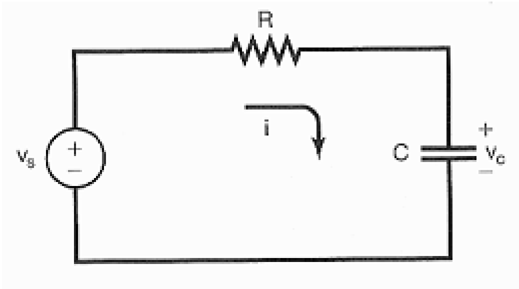
- Continuous-time signals $x(t)$ or $x_c(t)$



- Discrete-time signals $x[n]$ or $x_d[n]$



- Energy & Power of a resistor:



  - Instantaneous power

$$p(t) = v(t)i(t) = \frac{1}{R}v^2(t)$$

  - Total energy over a finite time interval

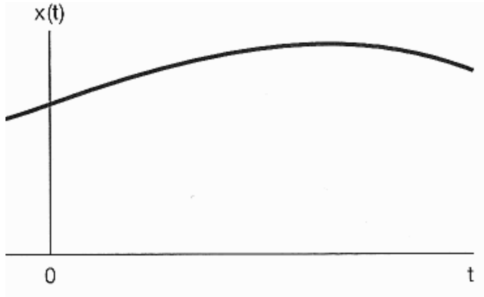
$$\int_{t_1}^{t_2} p(t)dt = \int_{t_1}^{t_2} \frac{1}{R}v^2(t)dt$$

  - Average power over a finite time interval

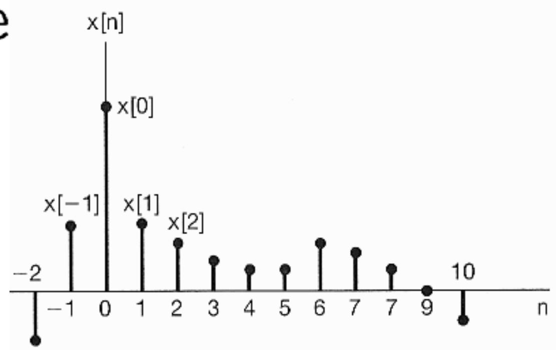
$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} p(t)dt = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \frac{1}{R}v^2(t)dt$$

Signal Energy & Power:

- Total energy over a finite time interval



$$E \triangleq \int_{t_1}^{t_2} |x(t)|^2 dt \qquad \text{continuous-time}$$



$$E \triangleq \sum_{n=n_1}^{n_2} |x[n]|^2 \qquad \text{discrete-time}$$

- Time-averaged power over a finite time interval

$$P \triangleq \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt \qquad \text{continuous-time}$$

$$P \triangleq \frac{1}{n_2 - n_1 + 1} \sum_{n=n_1}^{n_2} |x[n]|^2 \qquad \text{discrete-time}$$

Signal Energy & Power:

- Total energy over an infinite time interval

$$E_\infty \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |x(t)|^2 dt$$

$$E_\infty \triangleq \lim_{N\to\infty} \sum_{n=-N}^{+N} |x[n]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2$$

- Time-averaged power over an infinite time interval

$$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[t]|^2$$

$$E_\infty = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt$$
$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

Three Classes of Signals:

- Finite total energy & zero average power

$$0 \le E_\infty < \infty \quad \Rightarrow \quad P_\infty = \lim_{T\to\infty} \frac{E_\infty}{2T} = 0$$

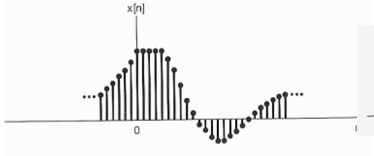
- Finite average power & infinite total energy

$$0 \le P_\infty < \infty \quad \Rightarrow \quad E_\infty = \infty \ \ (\text{if } P_\infty > 0)$$

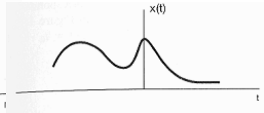
- Infinite average power & infinite total energy

$$P_\infty = \infty \quad \& \quad E_\infty = \infty$$

$x[n]$



$x(t)$

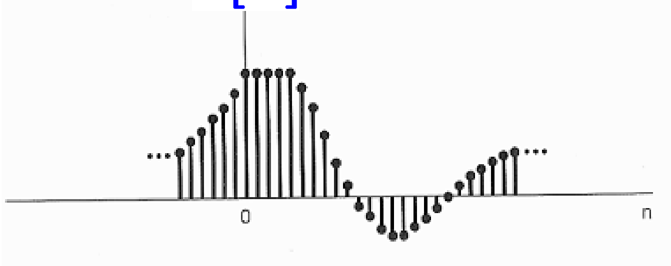


- Introduction
- Continuous-Time & Discrete-Time Signals
- Transformations of the Independent Variable
  - Time Shift
  - Time Reversal
  - Time Scaling
  - Periodic Signals
  - Even & Odd Signals
- Exponential & Sinusoidal Signals
- The Unit Impulse & Unit Step Functions
- Continuous-Time & Discrete-Time Systems
- Basic System Properties

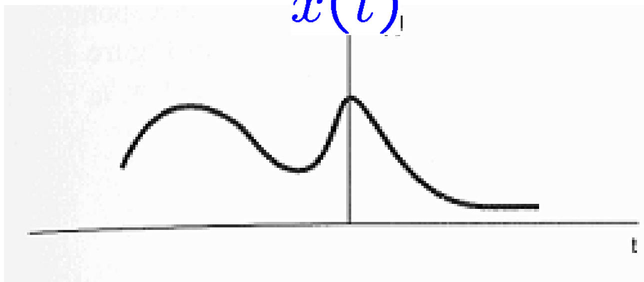
- Time Shift:

$$\begin{cases} n_0,\ t_0 > 0: & \text{delay} \\ n_0,\ t_0 < 0: & \text{advance} \end{cases}$$

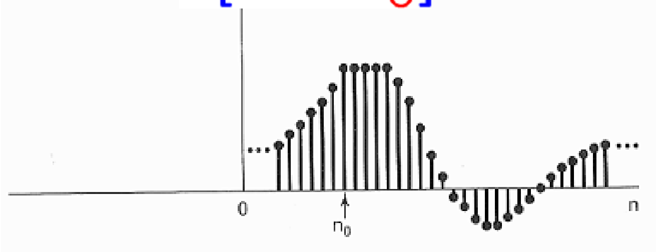
$x[n]$



$x(t)$



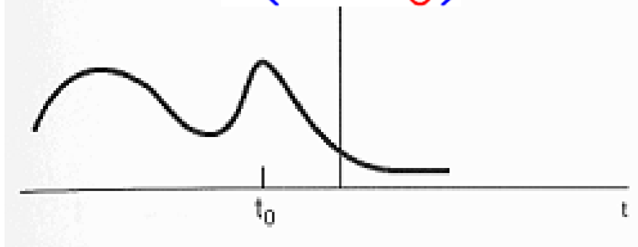
$x[n - n_0]$



$n_0 > 0$

$x[n - 8]$

$x(t - t_0)$

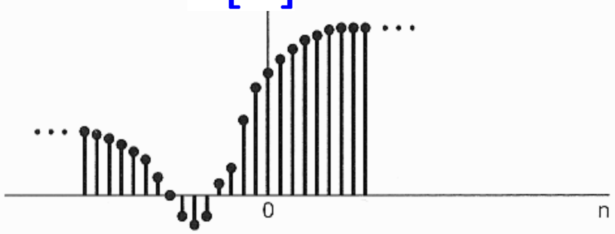


$t_0 < 0$

$x(t + 5)$

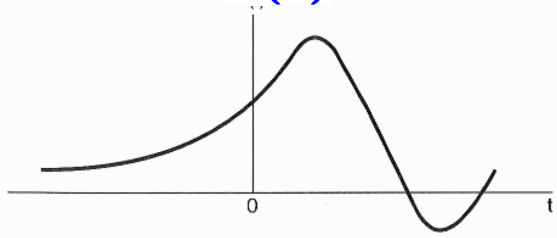
- Time Reversal:

$x[n]$



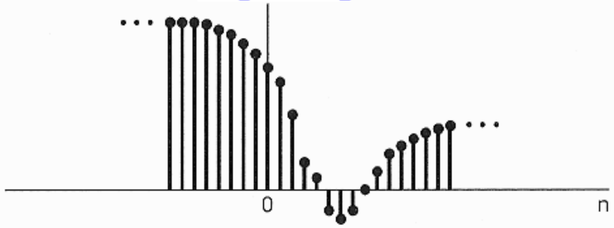
(a)

$x(t)$



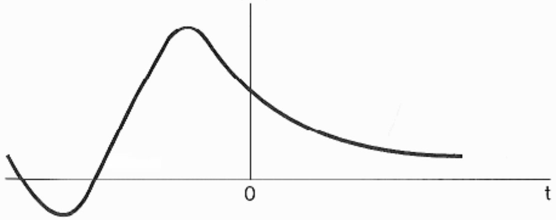
(a)

$x[-n]$



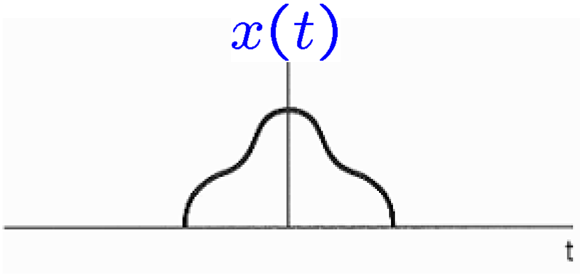
(b)

$x(-t)$

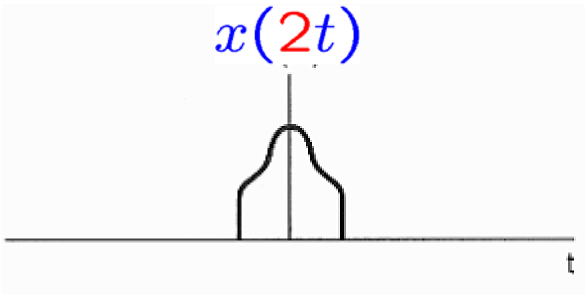


(b)

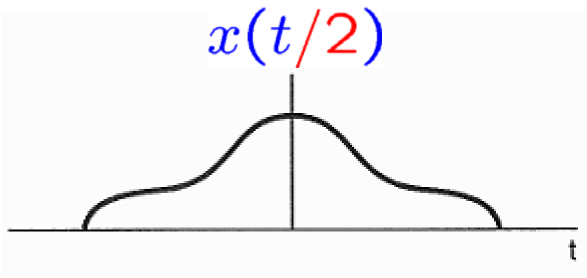
- Time Scaling:



$t \rightarrow 2t$

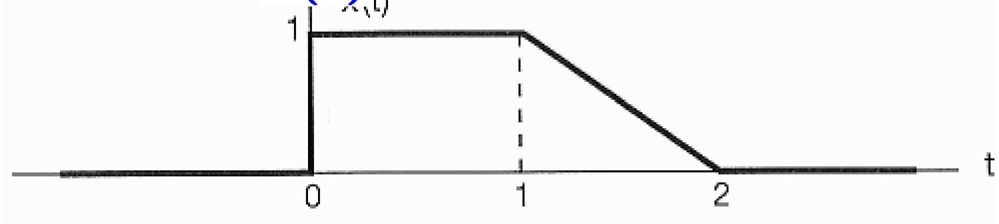


$t \rightarrow t/2$



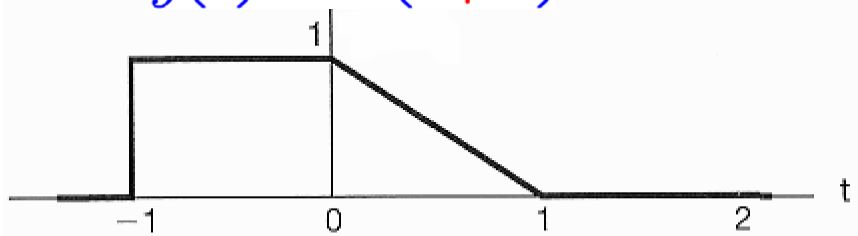
- $x(t) \rightarrow x(-t+1)$

$x(t)$



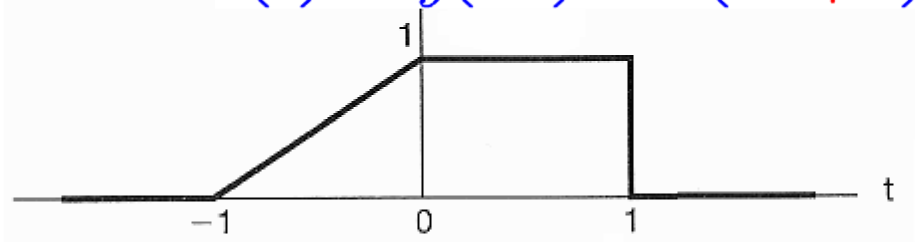
$t \rightarrow t+1$

$y(t) = x(t+1)$



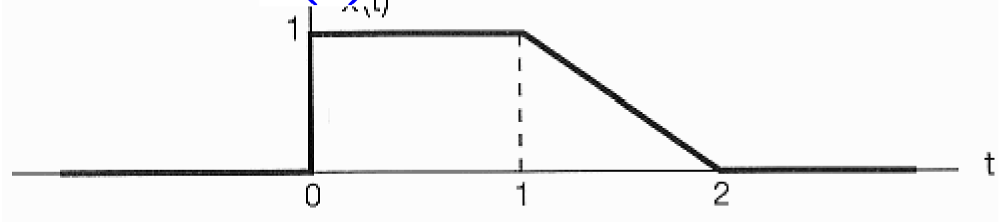
$t \rightarrow -t$

$z(t) = y(-t) = x(-t+1)$



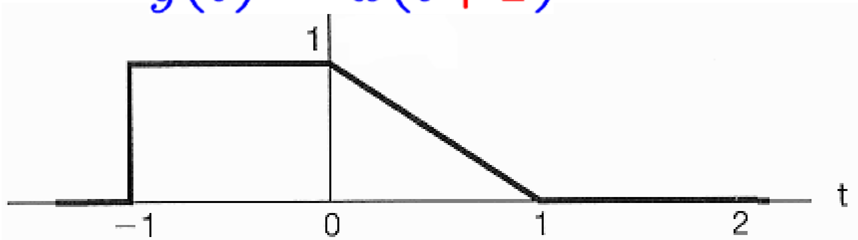
- $x(t) \rightarrow x(-t+1)$

$x(t)$



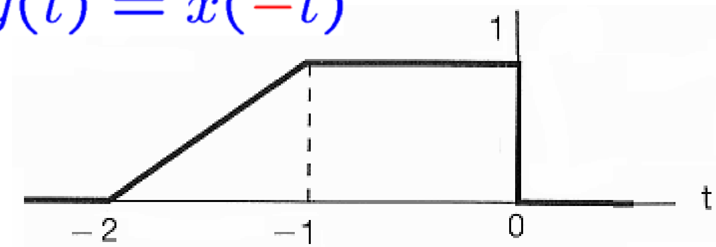
$t \rightarrow t+1$

$y(t) = x(t+1)$



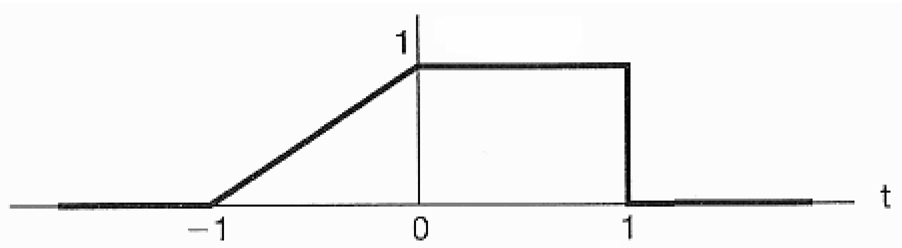
$t \rightarrow -t$

$y(t) = x(-t)$



$t \rightarrow -t$

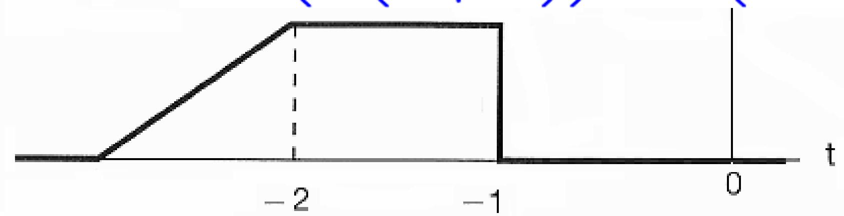
$z(t) = y(-t) = x(-t+1)$



$t \rightarrow t+1$

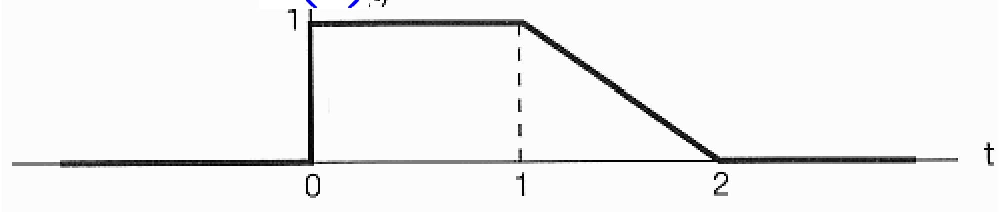
$z(t) = y(t+1)$

$= x(-(t+1)) = x(-t-1)$



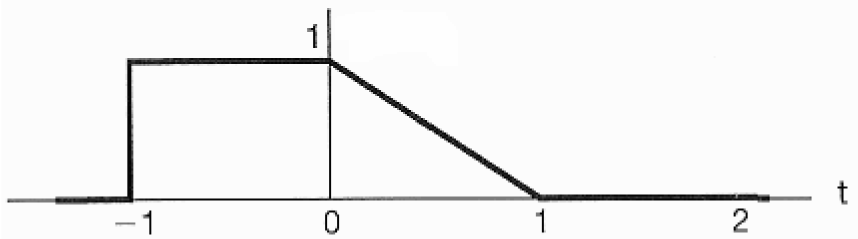
- $x(t) \rightarrow x(3/2\,t + 1)$

$x(t)$



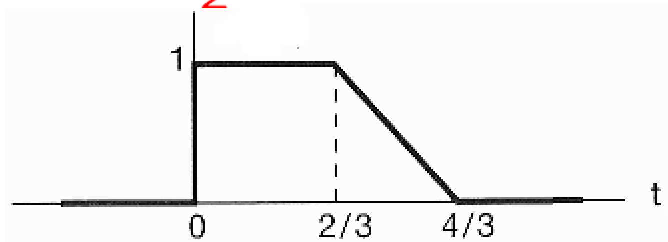
$t \rightarrow t+1$

$y(t) = x(t+1)$



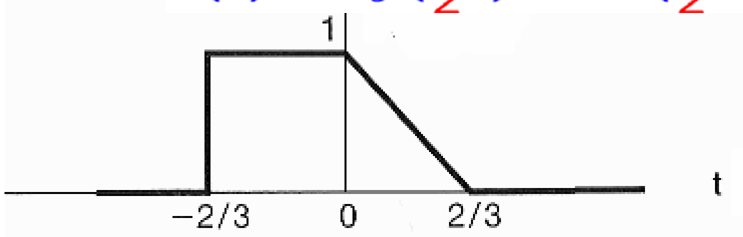
$t \rightarrow \frac{3}{2}t$

$y(t) = x(\frac{3}{2}t)$



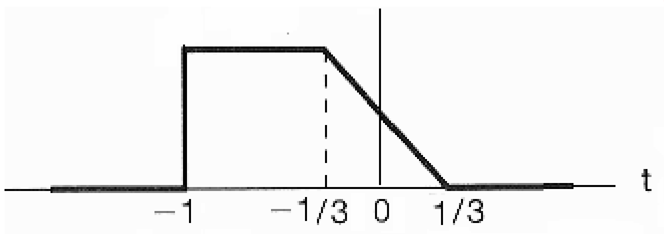
$t \rightarrow \frac{3}{2}t$

$z(t) = y(\frac{3}{2}t) = x(\frac{3}{2}t+1)$



$t \rightarrow t+1$

$z(t) = y(t+1) = x(\frac{3}{2}t+\frac{3}{2})$

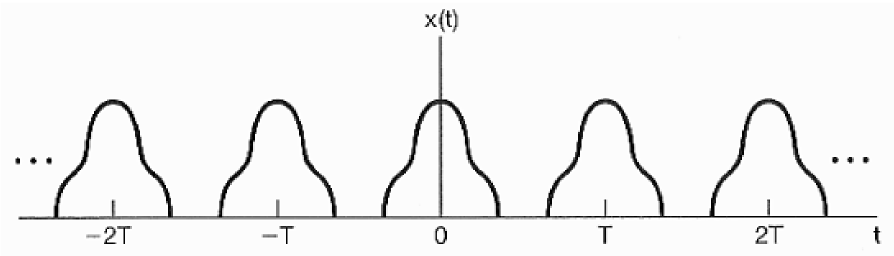


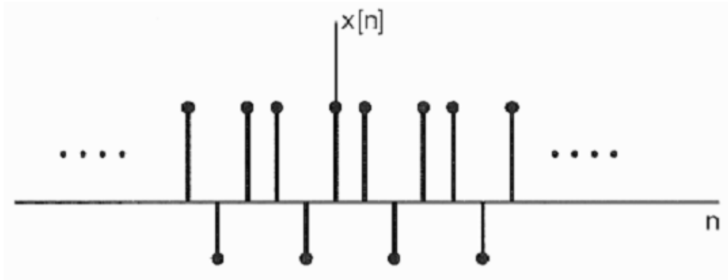
- $x(t) \rightarrow x(at - b)$

  - $|a| < 1$: linearly stretched
  - $|a| > 1$: linearly compressed
  - $a < 0$: time reversal
  - $b > 0$: delayed time shift
  - $b < 0$: advanced time shift

- Problems:
  - P1.21 for CT
  - P1.22 for DT

CT & DT Periodic Signals:





$N = 3$

$x(t) = x(t + T)$ for $T > 0$ and all values of t

$x[n] = x[n + N]$ for $N > 0$ and all values of n

Periodic Signals:

$$x(t) = x(t+T) \quad \text{for } T > 0 \text{ and all values of } t$$

$$x[n] = x[n+N] \quad \text{for } N > 0 \text{ and all values of } n$$

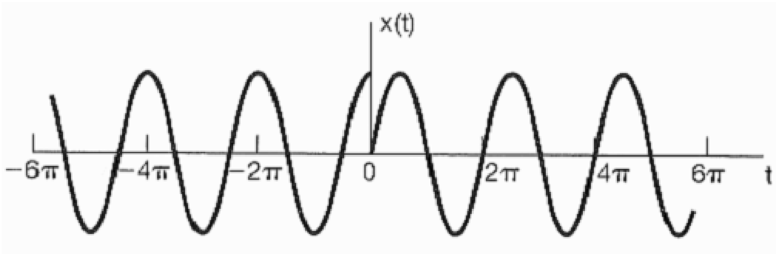
- A periodic signal is unchanged by a time shift of T or N
- They are also periodic with period
  - 2T, 3T, 4T, …
  - 2N, 3N, 4N, …
- T or N is called the fundamental period

  denoted as T_0 or N_0

- Periodic signal ?

$$x(t) = x(t+T) \quad \forall t, \quad T > 0$$

$$x(t) = \begin{cases} \cos(t), & \text{if } t < 0 \\ \sin(t), & \text{if } t \geq 0 \end{cases}$$

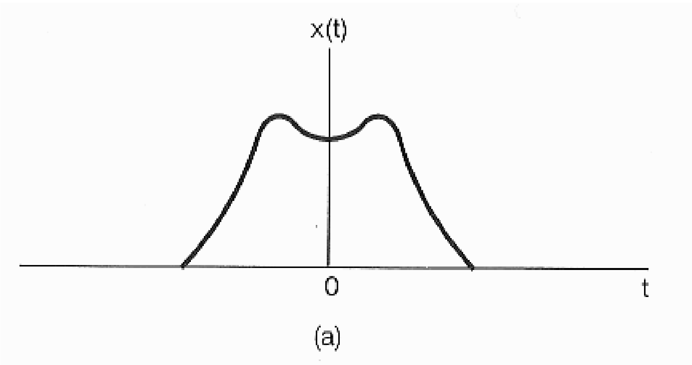


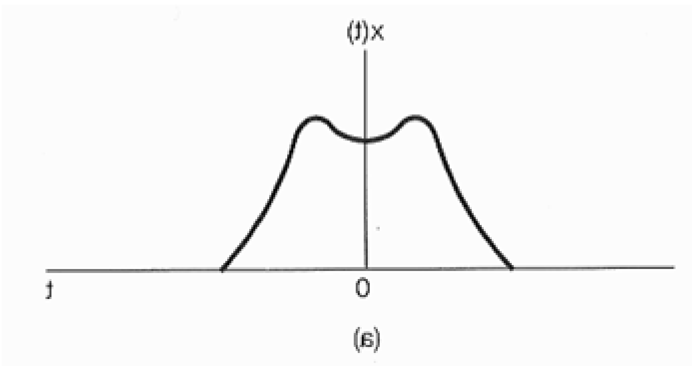
- Problems:
  - P1.25 for CT
  - P1.26 for DT

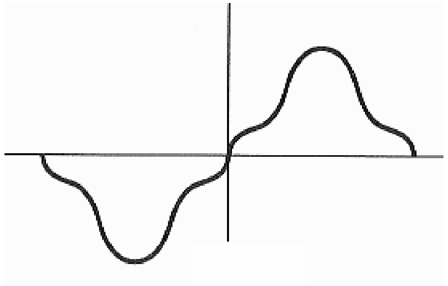
Even & odd signals:

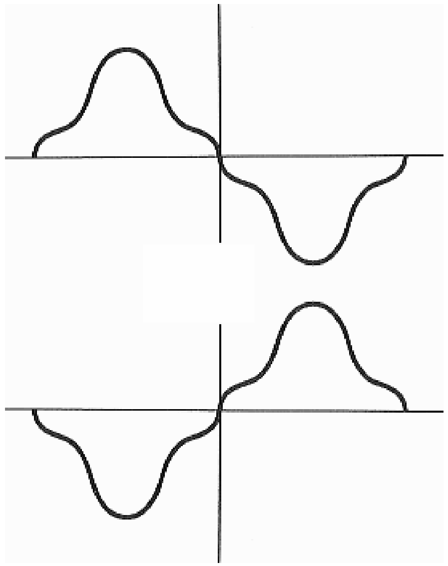
A signal is even if $x(-t) = x(t)$ or $x[-n] = x[n]$

A signal is odd if $x(-t) = -x(t)$ or $x[-n] = -x[n]$









- Even-odd decomposition of a signal:
  - Any signal can be broken into a sum of one even signal and one odd signal

$$\mathcal{E}v\left\{x(t)\right\} = \frac{1}{2}\left[\, x(t) + x(-t) \,\right] = \frac{1}{2}\left[\, x(-t) + x(t) \,\right]$$

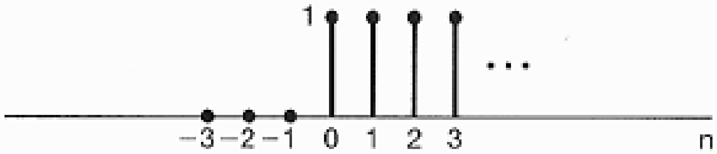
$$\mathcal{O}d\left\{x(t)\right\} = \frac{1}{2}\left[\, x(t) - x(-t) \,\right] = -\frac{1}{2}\left[\, x(-t) - x(t) \,\right]$$

$$\Rightarrow x(t) = \mathcal{E}v\left\{x(t)\right\} + \mathcal{O}d\left\{x(t)\right\}$$

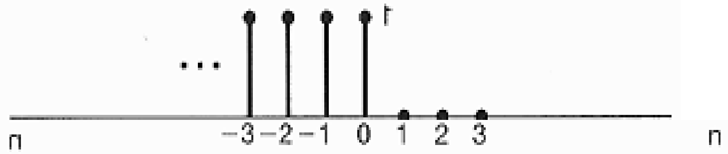
Even-odd decomposition of a DT signal:

$x[n]$

$$x[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

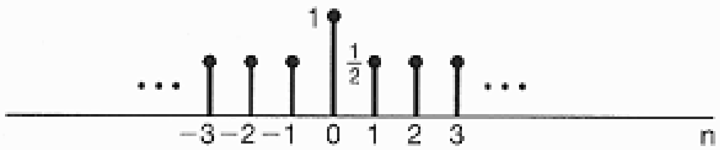


$x[-n]$



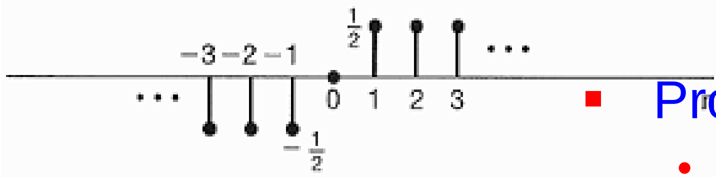
$$\mathcal{E}v\left\{x[n]\right\} = \frac{1}{2}\left[x[n] + x[-n]\right]$$

$$\mathcal{E}v\left\{x[n]\right\} = \begin{cases} \frac{1}{2}, & n < 0 \\ 1, & n = 0 \\ \frac{1}{2}, & n > 0 \end{cases}$$



$$\mathcal{O}d\left\{x[n]\right\} = \frac{1}{2}\left[x[n] - x[-n]\right]$$

$$\mathcal{O}d\left\{x[n]\right\} = \begin{cases} -\frac{1}{2}, & n < 0 \\ 0, & n = 0 \\ \frac{1}{2}, & n > 0 \end{cases}$$



- Problems:
  - P1.23 for CT
  - P1.24 for DT

Uniqueness of even-odd decomposition:

$$\text{Assume that } x(t) = \mathcal{E}v_1(t) + \mathcal{O}d_1(t)$$

$$\text{and } x(t) = \mathcal{E}v_2(t) + \mathcal{O}d_2(t)$$

$$\text{So, } \mathcal{E}v_1(t) + \mathcal{O}d_1(t) = \mathcal{E}v_2(t) + \mathcal{O}d_2(t)$$

$$\text{and } \mathcal{E}v_1(-t) + \mathcal{O}d_1(-t) = \mathcal{E}v_2(-t) + \mathcal{O}d_2(-t)$$

$$\text{Because } \begin{cases} \mathcal{E}v_1(-t) = \mathcal{E}v_1(t) \\ \mathcal{E}v_2(-t) = \mathcal{E}v_2(t) \end{cases} \text{ and } \begin{cases} \mathcal{O}d_1(-t) = -\mathcal{O}d_1(t) \\ \mathcal{O}d_2(-t) = -\mathcal{O}d_2(t) \end{cases}$$

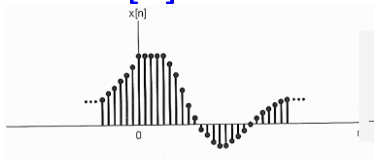
$$\text{Then, } \mathcal{E}v_1(t) - \mathcal{O}d_1(t) = \mathcal{E}v_2(t) - \mathcal{O}d_2(t)$$

$$\Rightarrow 2\mathcal{E}v_1(t) = 2\mathcal{E}v_2(t) \quad \text{or, } \mathcal{E}v_1(t) = \mathcal{E}v_2(t)$$

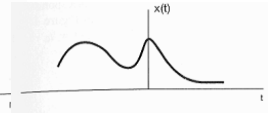
$$\Rightarrow 2\mathcal{O}d_1(t) = 2\mathcal{O}d_2(t) \quad \text{or, } \mathcal{O}d_1(t) = \mathcal{O}d_2(t)$$

Outline

$x[n]$



$x(t)$



- Introduction
- Continuous-Time & Discrete-Time Signals
- Transformations of the Independent Variable
  - Time Shift — $x[n-n_0]$ — $x(t-t_0)$
  - Time Reversal — $x[-n]$ — $x(-t)$
  - Time Scaling — $x[an]$ — $x(at)$
  - Periodic Signals — $x(t) = x(t+T)$, $x[n] = x[n+N]$
  - Even & Odd Signals

$$x(-t) = x(t), x[-n] = x[n]$$

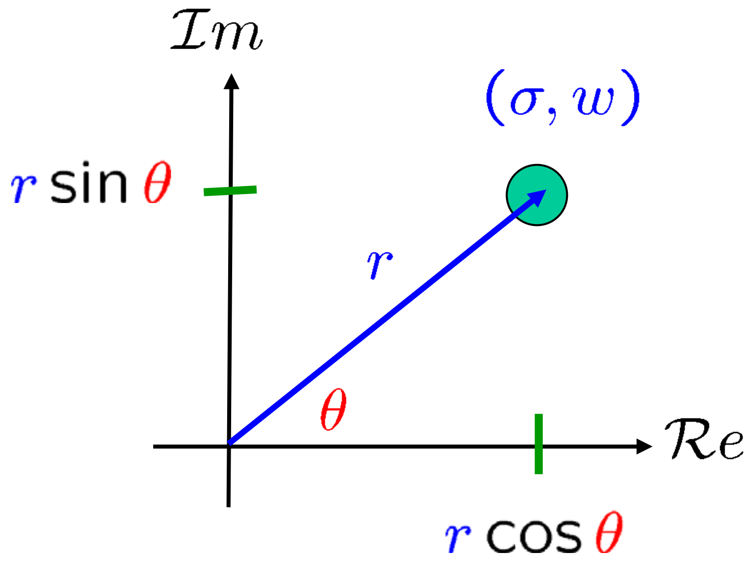
$$x(-t) = -x(t), x[-n] = -x[n]$$

$$\mathcal{E}v\left\{x[n]\right\} = \frac{1}{2}\left[x[n] + x[-n]\right]$$

$$\mathcal{O}d\left\{x[n]\right\} = \frac{1}{2}\left[x[n] - x[-n]\right]$$

- Exponential & Sinusoidal Signals
- The Unit Impulse & Unit Step Functions
- Continuous-Time & Discrete-Time Systems
- Basic System Properties

- Magnitude & Phase Representation:



$$j = \sqrt{-1}$$

$$\sigma + jw \quad \Rightarrow \quad \begin{cases} r = \sqrt{\sigma^2 + w^2} \\ \tan(\theta) = \frac{w}{\sigma} \end{cases}$$

$$\Rightarrow \quad \sigma + jw \;=\; r\, e^{j\theta}$$

- Euler's relation:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\Rightarrow \quad \sigma + jw = r\left(\cos\theta + j\sin\theta\right)$$

$$= (r\cos\theta) + j(r\sin\theta)$$

CT Complex Exponential Signals:

$$x(t) = Ce^{at}$$

- where C & a are, in general, complex numbers

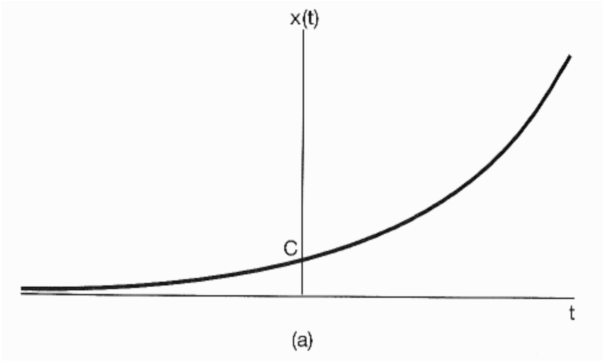
$$a = \sigma + jw$$

$$C = |C|\, e^{j\theta}$$

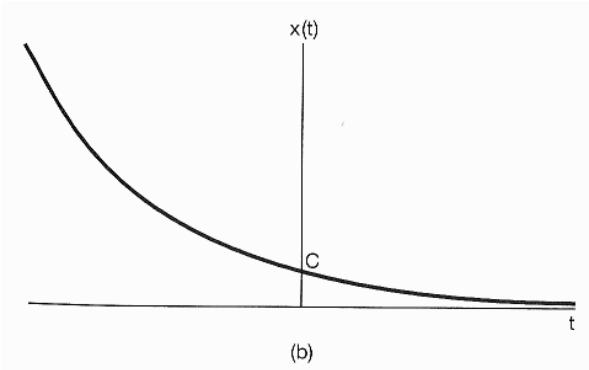
▪ Real exponential signals:

- If C & a are real

$$x(t) = Ce^{at}$$



$a > 0$



$a < 0$

- Periodic complex exponential signals: $e^{j\theta} = \cos\theta + j\sin\theta$
  - If a is purely imaginary $\quad a = \sigma + jw$

$$x(t) = e^{jw_0 t}$$

  - It is periodic
    - Because let $\quad T_0 = \frac{2\pi}{|w_0|}$
    - Then

$$e^{jw_0 T_0} = e^{jw_0 \frac{2\pi}{w_0}} = \cos(\quad) + j\sin(\quad) =$$

    - Hence

$$x(t + \quad) = x(t) \qquad e^{jw_0(t+T_0)} = e^{jw_0 t} e^{jw_0 T_0} = e^{jw_0 t}$$

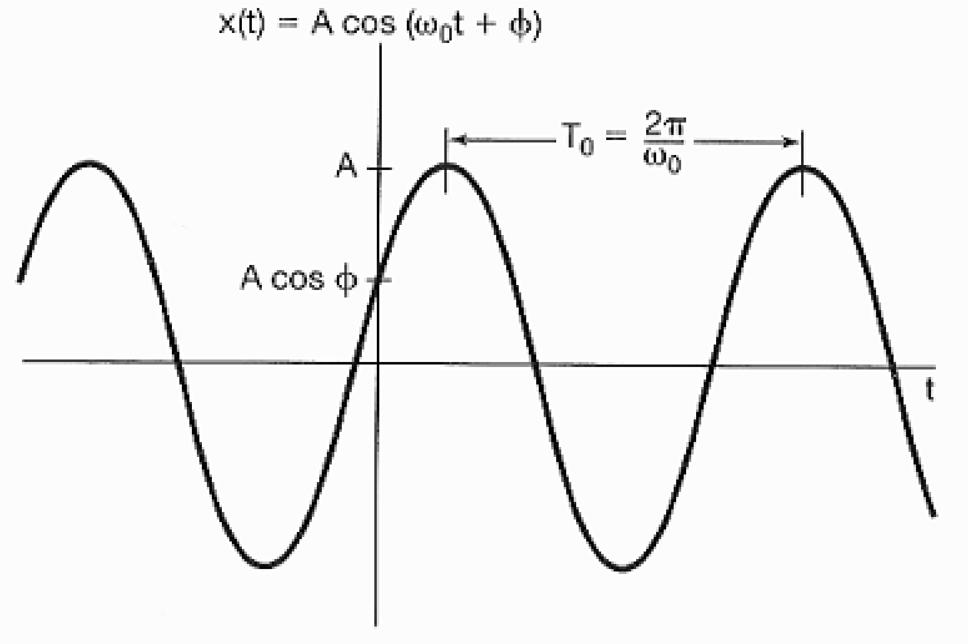
Periodic sinusoidal signals:

$$w_0 = 2\pi f_0$$

$$x(t) = A \cos(w_0 t + \phi)$$

$$T_0 = \frac{2\pi}{w_0}$$

$$T_0 = \frac{1}{f_0}$$



$T_0 : (sec)$

$w_0 : (rad/sec)$

$f_0 : (1/sec = Hz)$

Period & Frequency:

$$T_0 = \frac{2\pi}{w_0}$$

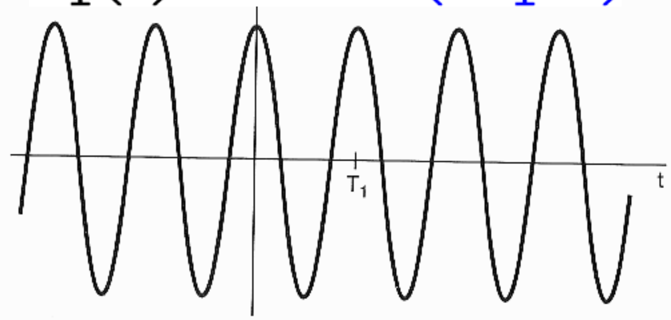
$$w_0 = 2\pi f_0$$

$$T_0 = \frac{1}{f_0}$$

w_1 w_2 w_3

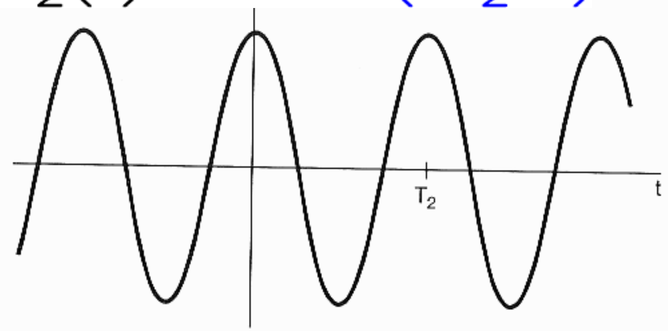
T_1 T_2 T_3

$x_1(t) = \cos(w_1 t)$



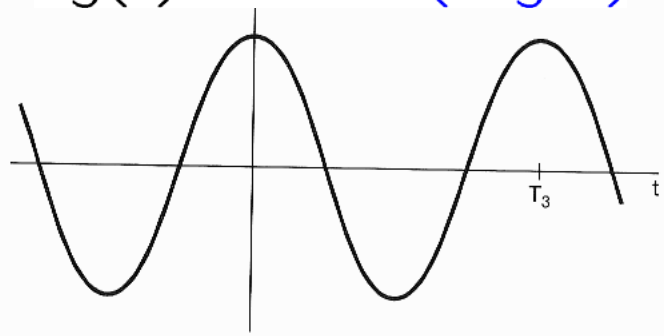
(a)

$x_2(t) = \cos(w_2 t)$



(b)

$x_3(t) = \cos(w_3 t)$



(c)

- Euler's relation:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\cos(\theta) = \mathcal{R}e\left\{e^{(j\theta)}\right\}$$

$$\sin(\theta) = \mathcal{I}m\left\{e^{(j\theta)}\right\}$$

$$e^{j(-\theta)} = \cos(-\theta) + j\sin(-\theta)$$

$$= \cos(\theta) - j\sin(\theta)$$

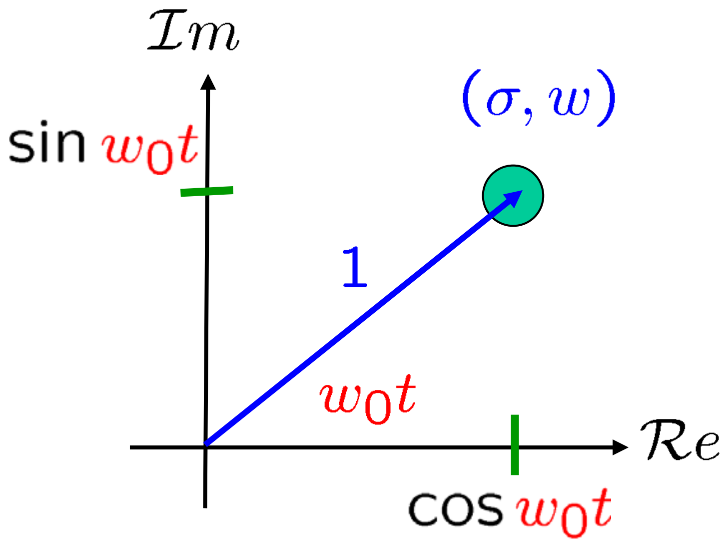
$$\Rightarrow \cos(\theta) = \frac{e^{(j\theta)} + e^{-(j\theta)}}{2}$$

$$\Rightarrow \sin(\theta) = \frac{e^{(j\theta)} - e^{-(j\theta)}}{2j}$$

$$\Rightarrow A\cos(w_0 t + \phi) = \frac{A}{2}\, e^{j(\phi + w_0 t)} + \frac{A}{2}\, e^{-j(\phi + w_0 t)}$$

$$= \frac{A}{2}\, e^{j\phi}\, e^{jw_0 t} + \frac{A}{2}\, e^{-j\phi}\, e^{-jw_0 t}$$

- Total energy & average power:



$$E_{\text{period}} = \int_0^{T_0} \left| e^{jw_0 t} \right|^2 dt$$

$$= \int_0^{T_0} 1 \cdot dt = T_0$$

$$P_{\text{period}} = \frac{1}{T_0} E_{\text{period}} = 1$$

$$E_\infty = \infty$$

$$P_\infty = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left| e^{jw_0 t} \right|^2 dt = 1$$

- Problem:
  - P1.3

Harmonically related periodic exponentials

$$e^{j0w_0t}, \quad e^{j1w_0t}, \quad e^{j2w_0t}, \quad e^{j3w_0t}, \quad \dots,$$

$$e^{j(-1)w_0t}, \quad e^{j(-2)w_0t}, \quad e^{j(-3)w_0t}, \quad \dots$$

$$\phi_k(t) = e^{\,j\,k\,w_0\,t}, \qquad k = 0, \pm 1, \pm 2, ...$$

- For $k = 0$, $\phi_k(t)$ is constant
- For $k \neq 0$, $\phi_k(t)$ is periodic with

  fundamental frequency $|k|w_0$ and

  fundamental period $\frac{T_0}{|k|}$

▪ General complex exponential signals:

$$C e^{at} = (|C| e^{j\theta})(e^{(r+jw_0)t})$$

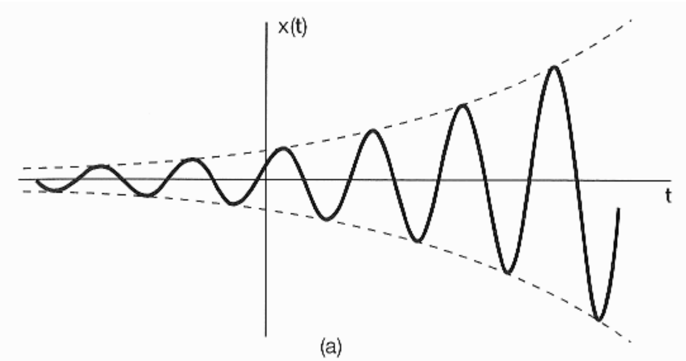
$$\sigma + jw = r\, e^{j\theta}$$

$$= (\ |C|\ e^{j\theta}\ )\ (\ e^{rt}\ e^{jw_0 t}\ )$$

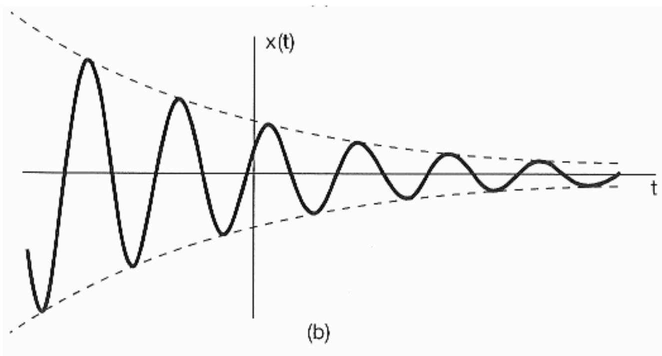
$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$= |C|\ e^{rt}\ e^{j(w_0 t+\theta)}$$

$$= \quad |C| e^{rt} \ \cos(w_0 t+\theta) \ + \ j |C| e^{rt} \ \sin(w_0 t+\theta)$$



(a)



(b)

▪ DT complex exponential signal or sequence:

$$x(t) = Ce^{at}$$

$$x[n] = Ce^{bn}$$

$$= C(e^{b})^{n} \quad \text{with } a = e^{b}$$

$$x[n] = Ca^{n}$$

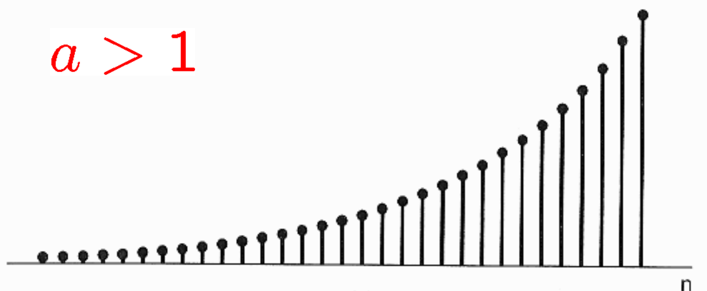
- where C & a are, in general, complex numbers

- Real exponential signals:

  - If C & a are real

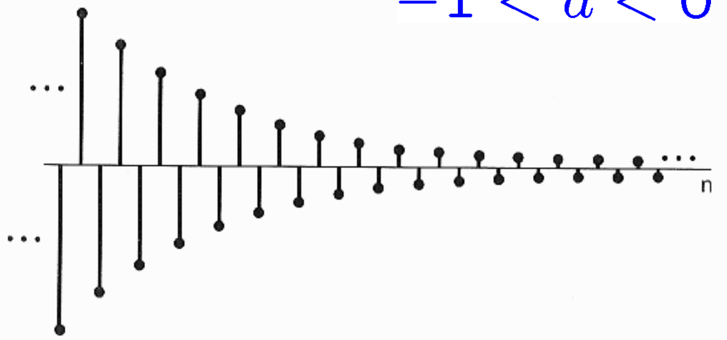
$$x[n] = Ca^n$$

$a > 1$



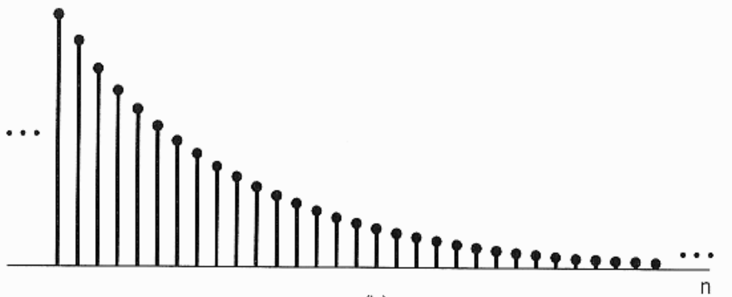
(a)

$-1 < a < 0$



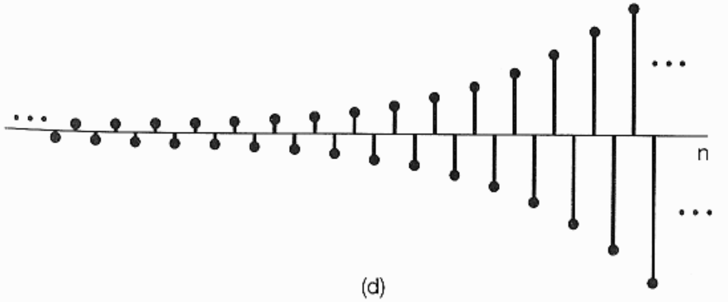
(c)

$0 < a < 1$



(b)

$a < -1$



(d)

DT Complex Exponential & Sinusoidal Signals

- If b is purely imaginary (or |a| = 1)

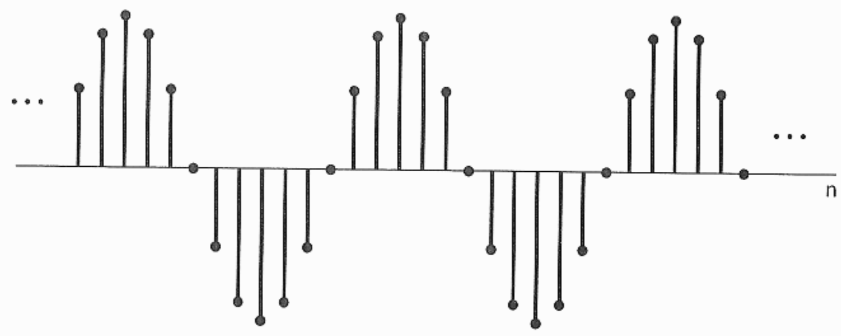
$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$x[n] = e^{jw_0 n}$$

$$= \cos(w_0 n) + j\sin(w_0 n)$$

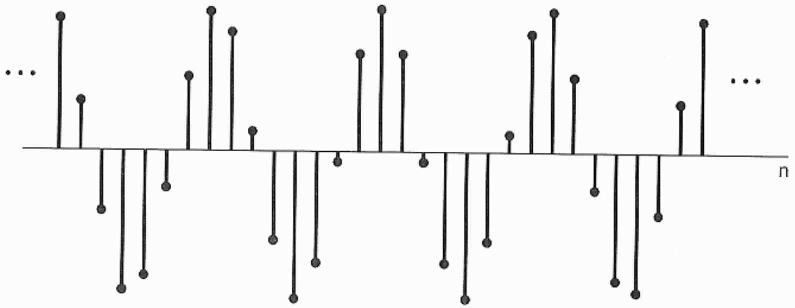
$$x[n] = A\cos(w_0 n + \phi)$$

$$x[n] = \cos\left(\frac{2\pi}{12} n\right)$$



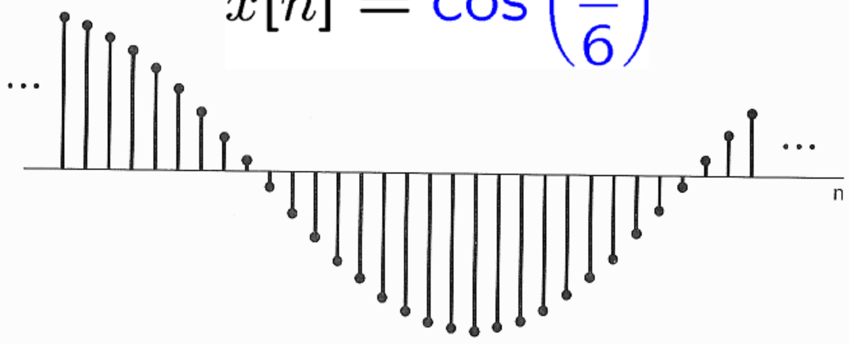
(a)

$$x[n] = \cos\left(\frac{4 \cdot 2\pi}{31} n\right)$$



(b)

$$x[n] = \cos\left(\frac{n}{6}\right)$$



(c)

- Euler's relation:

$$e^{jw_0 n} = \cos w_0 n + j \sin w_0 n$$

  - And,

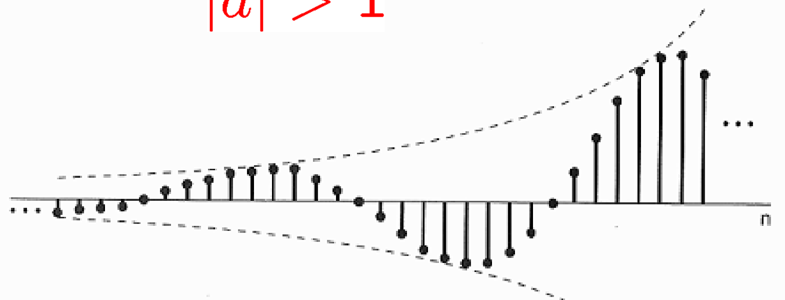
$$A\cos(w_0 n + \phi) = \frac{A}{2}\, e^{j\phi}\, e^{jw_0 n} + \frac{A}{2}\, e^{-j\phi}\, e^{-jw_0 n}$$

- General complex exponential signals:

$$Ca^n = (|C|e^{j\theta})((|a|e^{jw_0})^n)$$

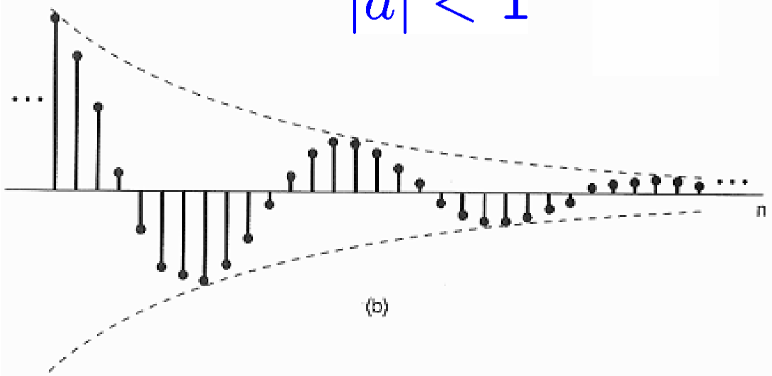
$$= |C||a|^n \ \cos(w_0 n + \theta) \ + \ j|C||a|^n \ \sin(w_0 n + \theta)$$

$|a| > 1$



(a)

$|a| < 1$



(b)

- Periodicity properties of DT complex exponentials:

$$e^{jw_0 n}$$

$$e^{j2\pi n} = \cos 2\pi n + j \sin 2\pi n$$

$$e^{j(w_0 + 2\pi)n} = e^{j2\pi n} e^{jw_0 n} = e^{jw_0 n}$$

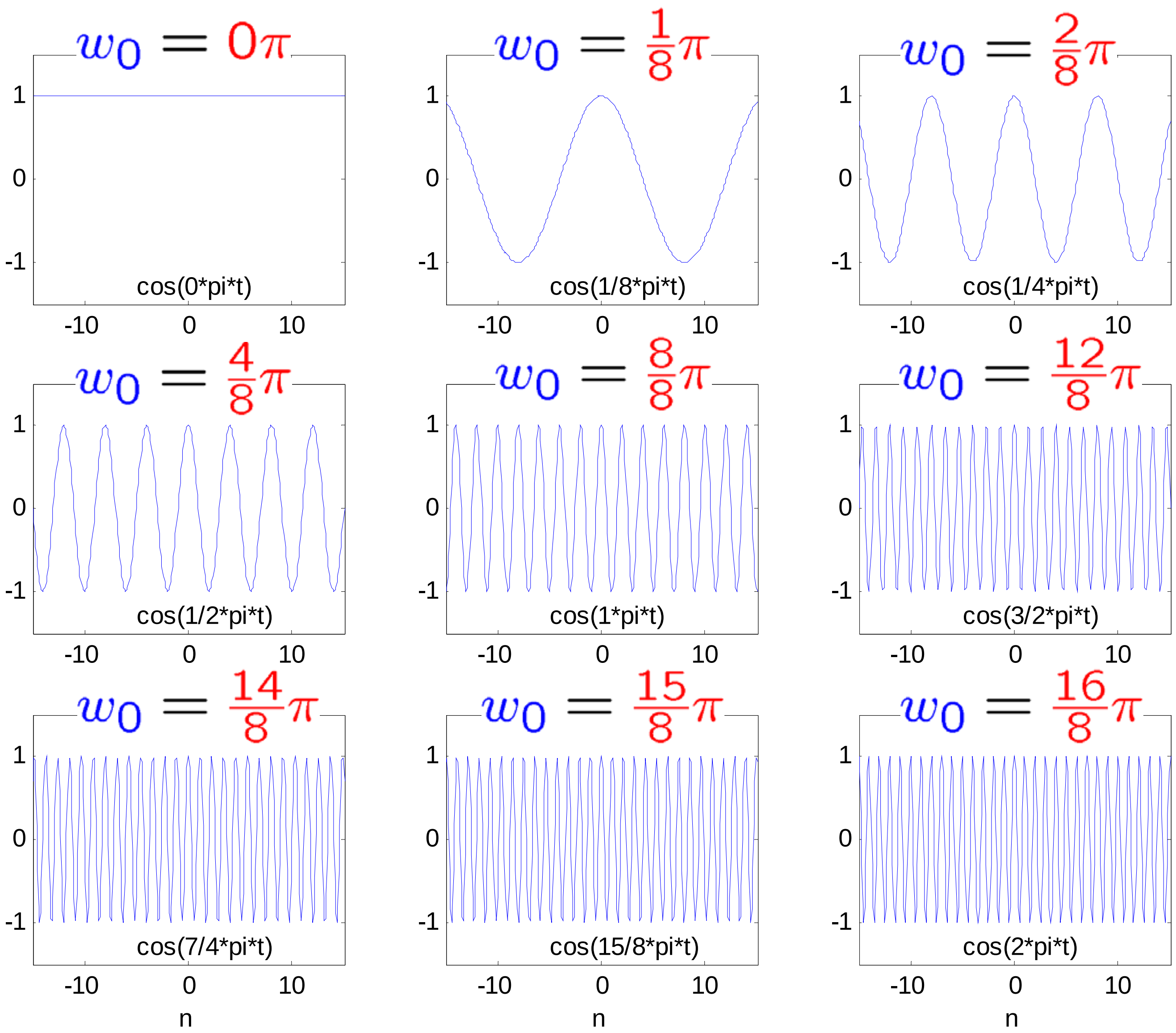
- The signal with frequency ω_0 is identical to the signals with frequencies $w_0 \pm 2\pi, \ w_0 \pm 4\pi, \ w_0 \pm 6\pi, \ \cdots$

- Only need to consider a frequency interval of length 2π
  - Usually use $0 \leq w_0 < 2\pi$ or $-\pi \leq w_0 < \pi,$

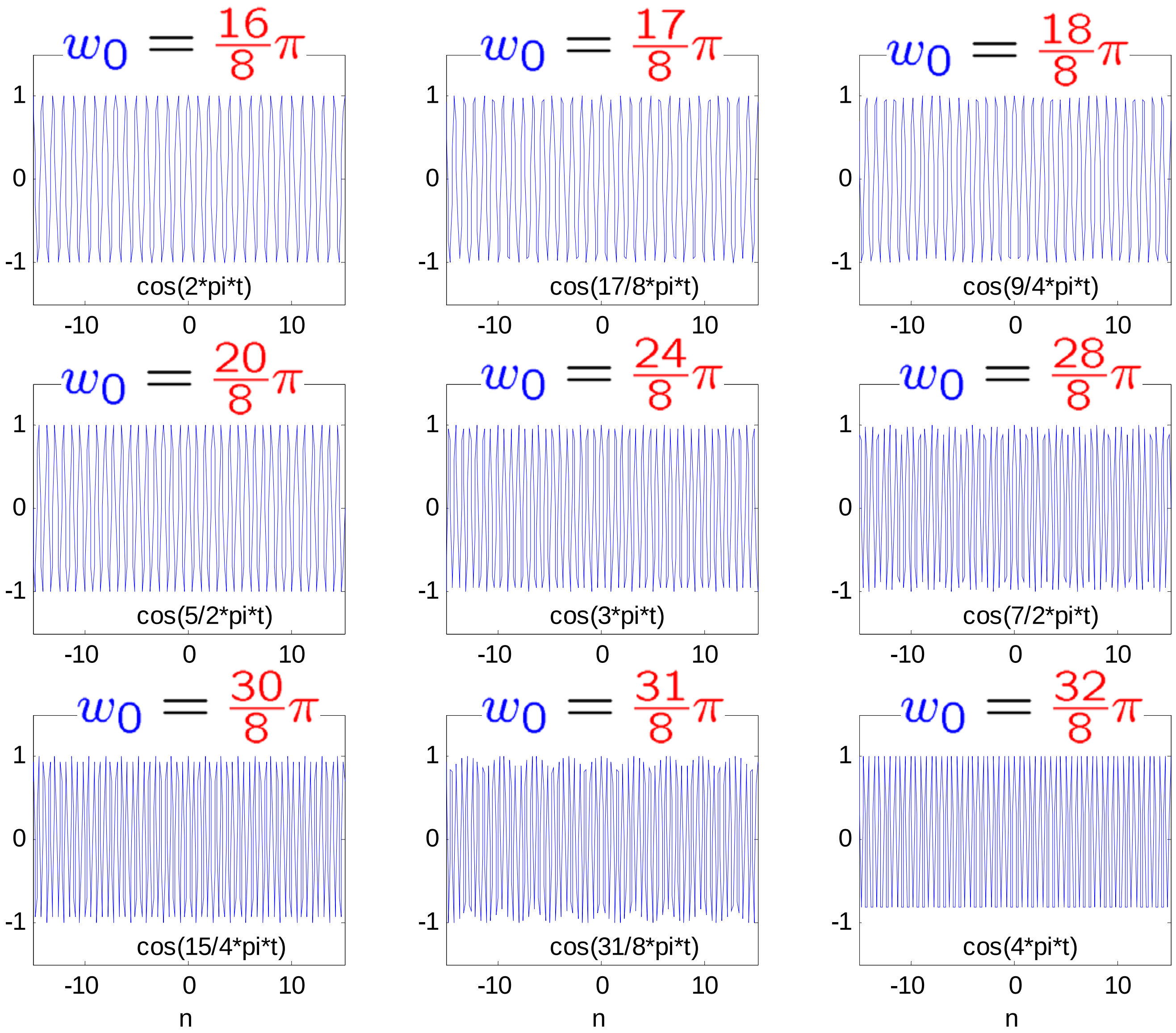
- The low frequencies are located at $w_0 = 0, \ \pm 2\pi, \ \cdots$
  The high frequencies are located at $w_0 = \pm\pi, \pm 3\pi, \ \cdots$

$$e^{j(0)n} = 1 \quad \text{and} \quad e^{j(\pi)n} = (e^{j(\pi)})^n = (-1)^n$$

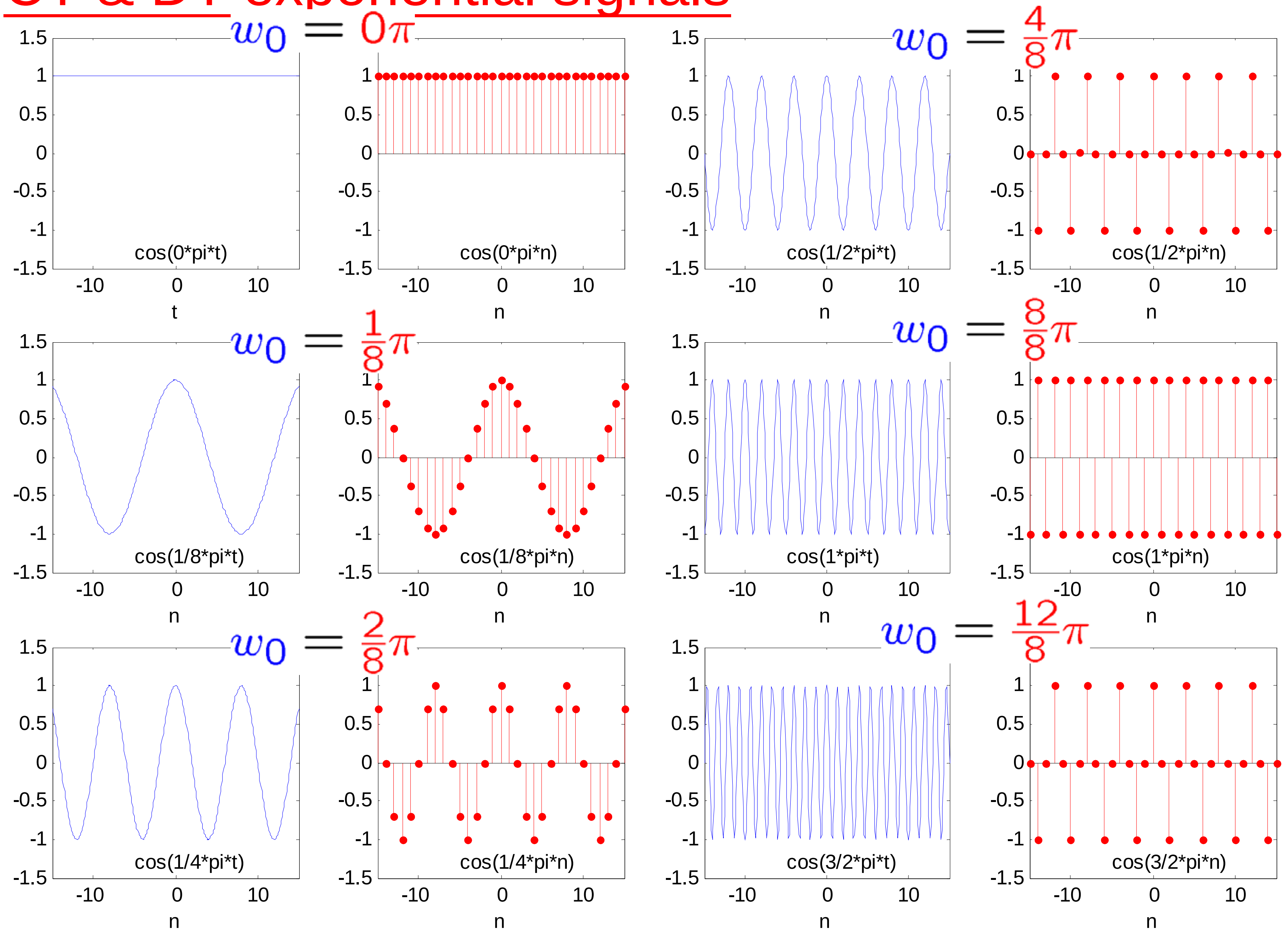
- CT exponential signals



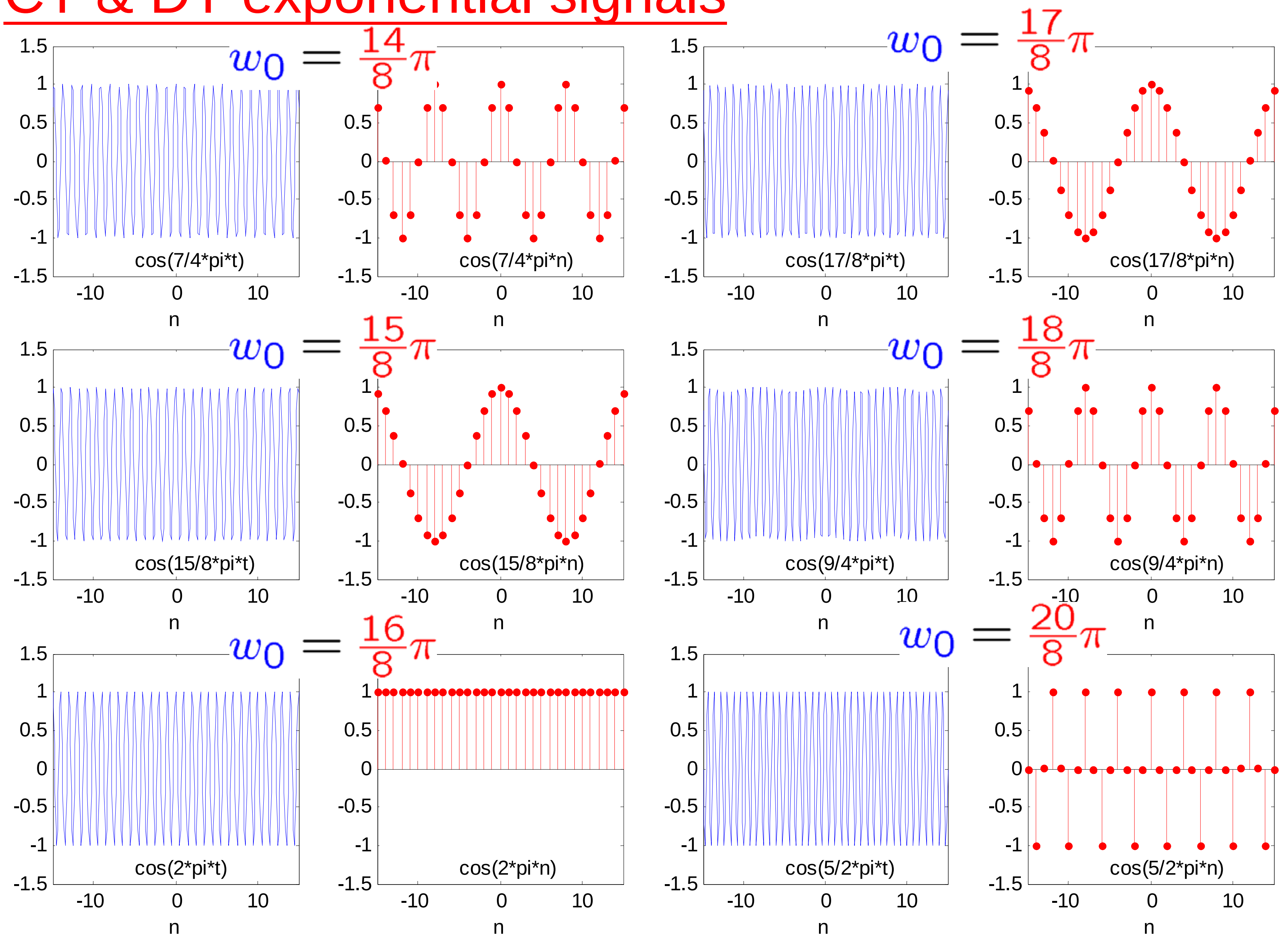
- CT exponential signals



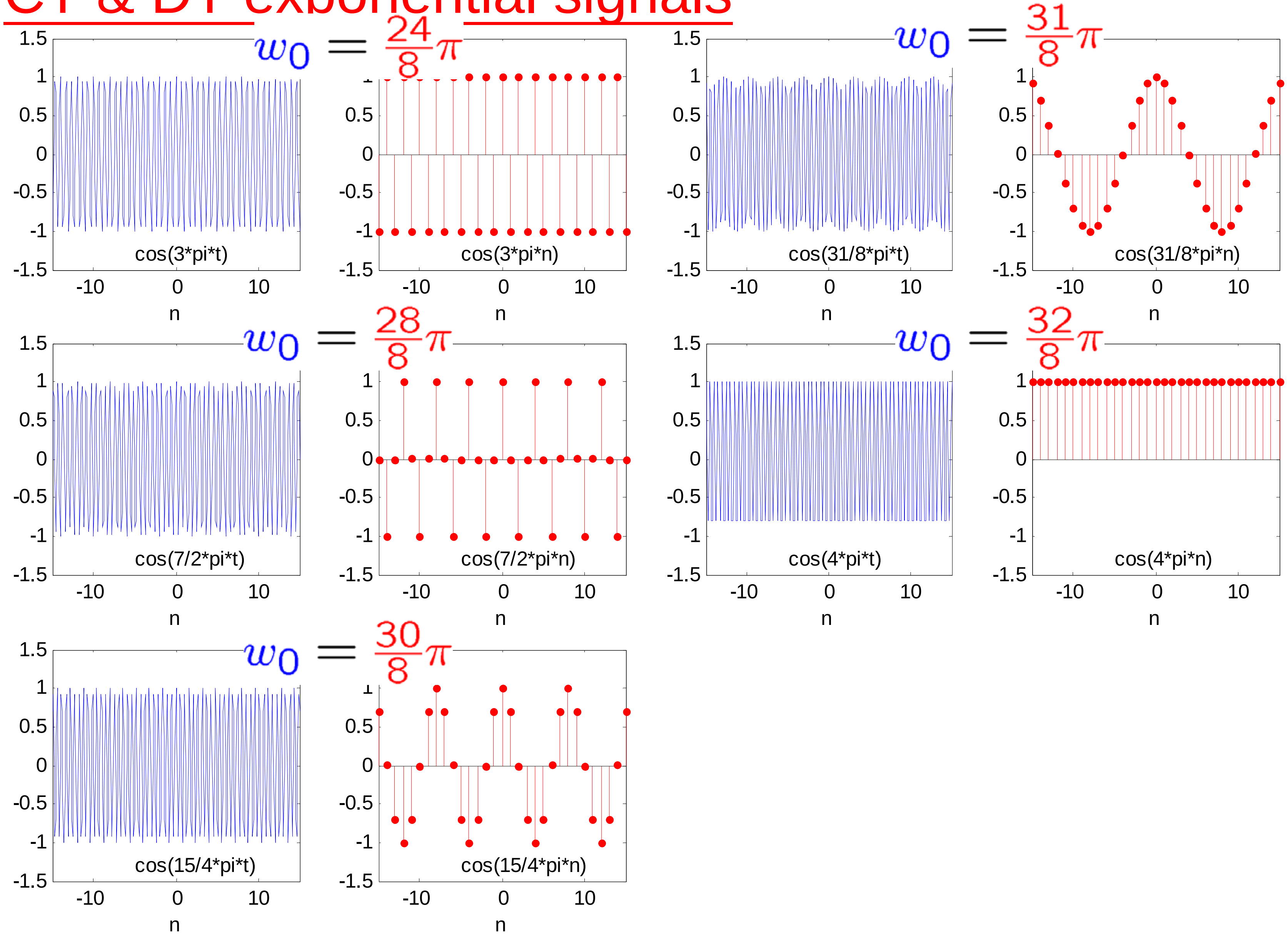
- CT & DT exponential signals



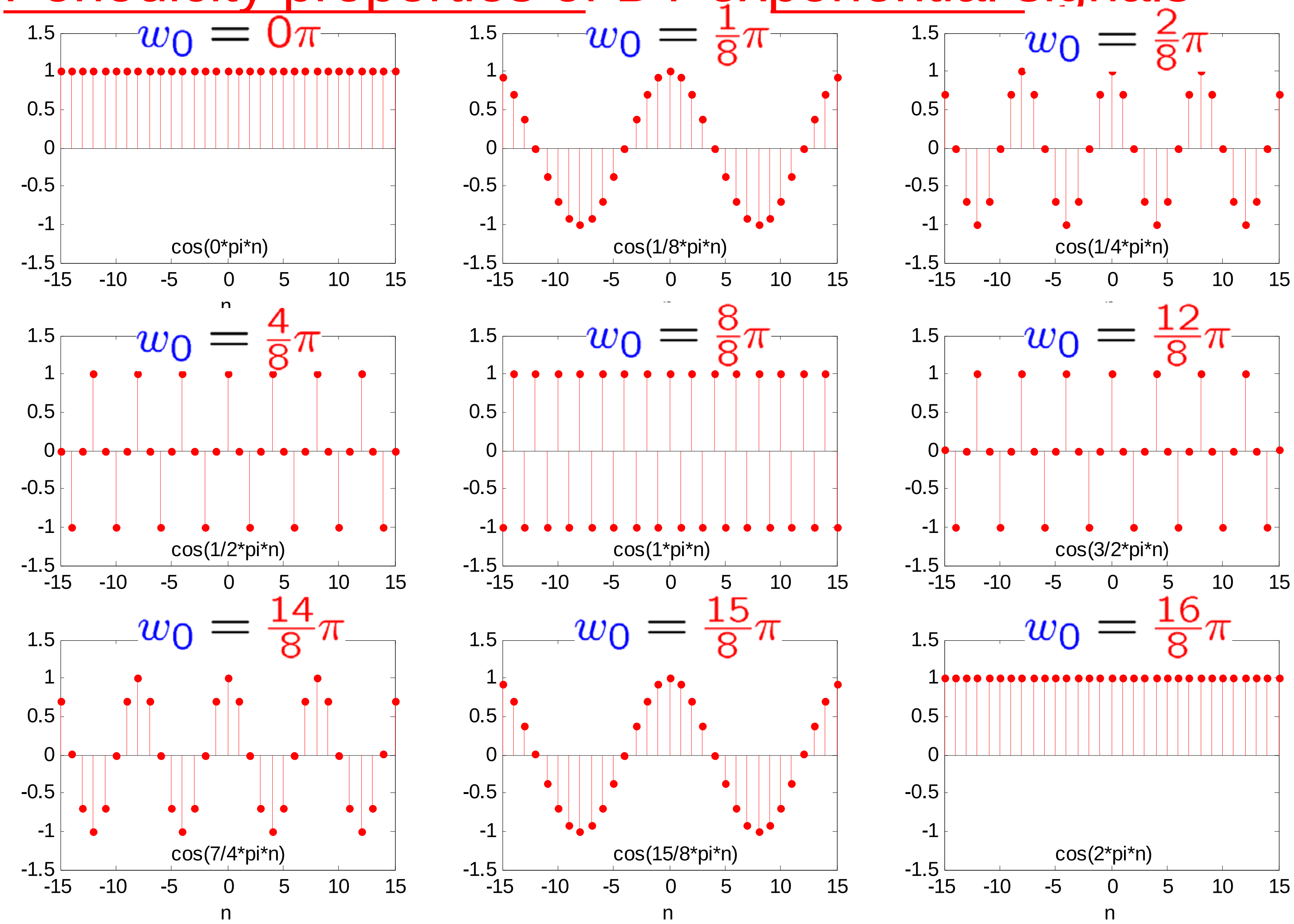
CT & DT exponential signals



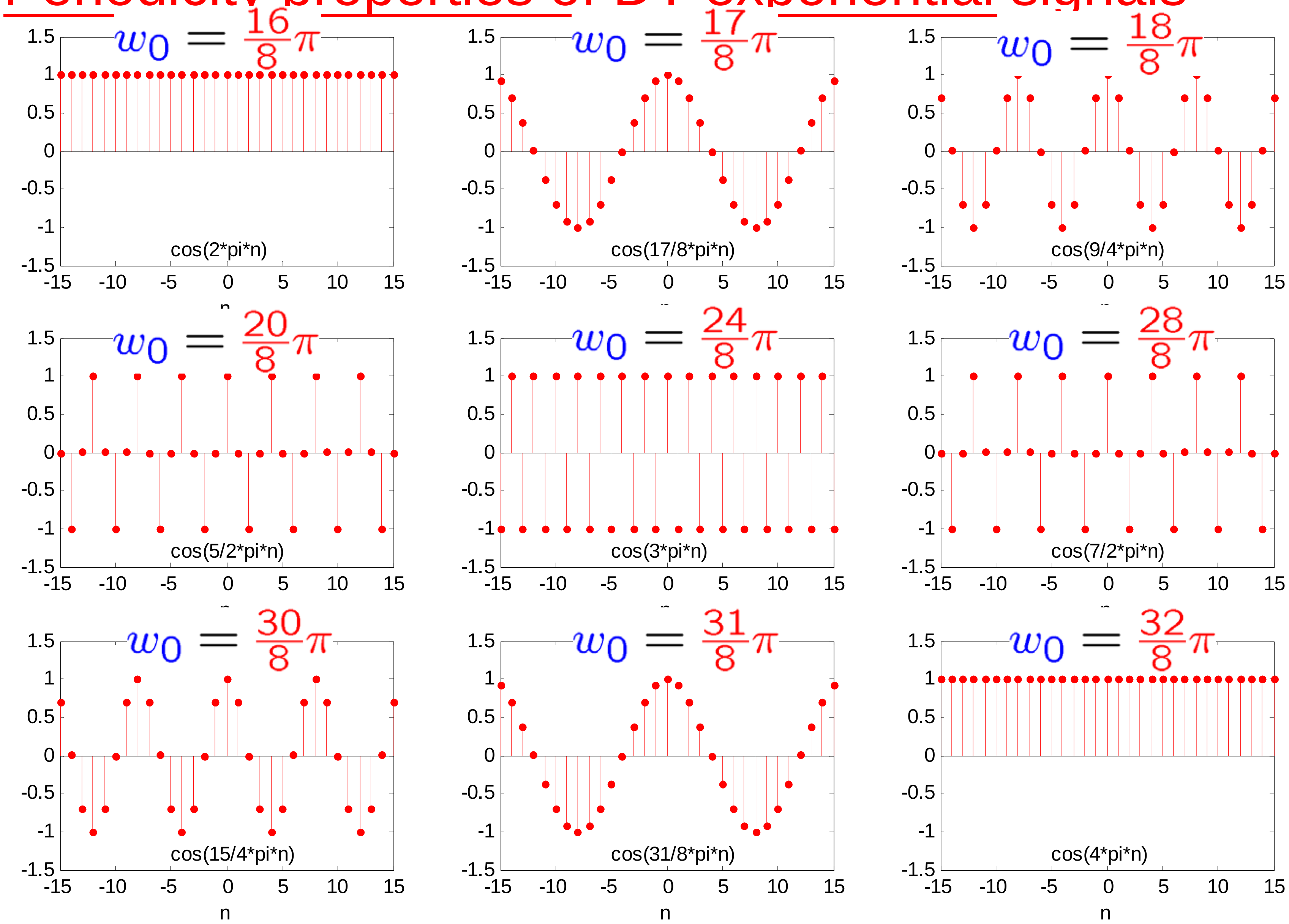
- CT & DT exponential signals



- Periodicity properties of DT exponential signals



- Periodicity properties of DT exponential signals



Periodicity properties of DT exponential signals

- Periodicity of N > 0

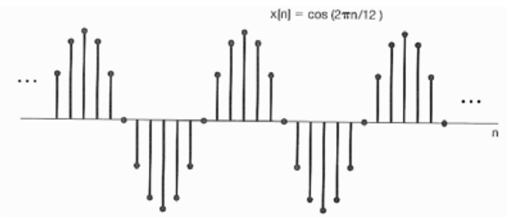
- Problem:
  - P1.35

$$e^{jw_0 n} \qquad e^{jw_0(n+N)} = e^{jw_0 n} \; e^{jw_0 N} = e^{jw_0 n} \quad \text{or} \quad e^{jw_0 N} = 1$$

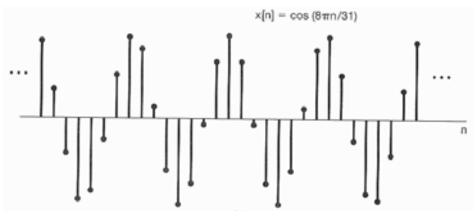
- That is, $w_0 N = 2\pi m$ or $\dfrac{w_0}{2\pi} = \dfrac{m}{N}$

- Hence, $e^{jw_0 n}$ is periodic if $\frac{w_0}{2\pi}$ is a rational number

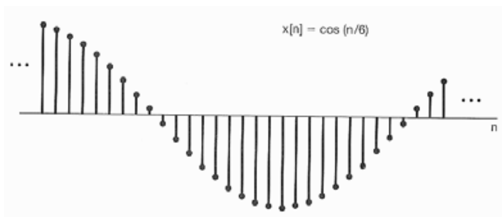
$$x[n] = \cos\left(\frac{2\pi}{12} n\right) \qquad x[n] = \cos\left(\frac{4 \cdot 2\pi}{31} n\right) \qquad x[n] = \cos\left(\frac{n}{6}\right)$$



(a)

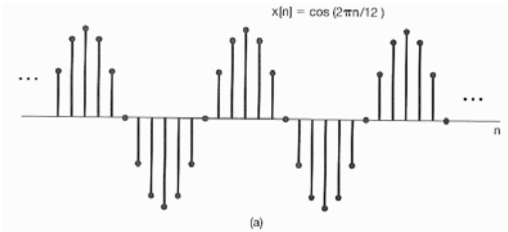


(b)



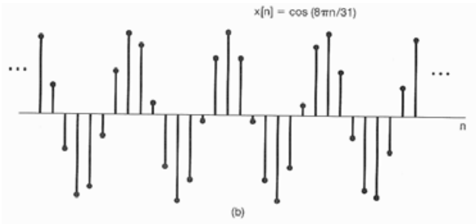
(c)

▪ Periodicity properties of DT exponential signals



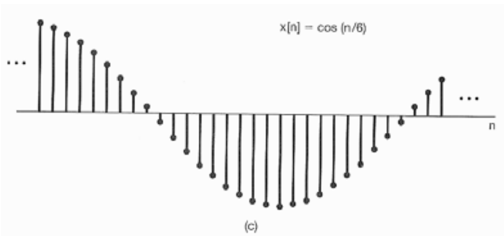
$$x(t) = \cos\left(\frac{2\pi}{12}t\right) \qquad T = 12?$$

$$x[n] = \cos\left(\frac{2\pi}{12}n\right) \qquad N = 12?$$



$$x(t) = \cos\left(\frac{4 \cdot 2\pi}{31}t\right) \qquad T = \frac{31}{4}?$$

$$x[n] = \cos\left(\frac{4 \cdot 2\pi}{31}n\right) \qquad N = \frac{31}{4}?$$



$$x(t) = \cos\left(\frac{1}{6}t\right) \qquad T = 12\pi?$$

$$x[n] = \cos\left(\frac{1}{6}n\right) \qquad N = 12\pi?$$

Harmonically related periodic exponentials

$$\phi_k[n] = e^{jk(w_0)n}, \quad = e^{jk(\frac{2\pi}{N})n}, \quad k = 0, \pm 1, \pm 2, \ldots$$

$$\phi_{k+N}[n] = e^{j(k+N)(\frac{2\pi}{N})n}$$

$$= e^{jk(\frac{2\pi}{N})n} \, e^{jN(\frac{2\pi}{N})n} = \phi_k[n]$$

- Only N distinct periodic exponentials in the set

$$\phi_0[n] = 1, \quad \phi_1[n] = e^{j(\frac{2\pi}{N}n)}, \quad \phi_2[n] = e^{j(2\frac{2\pi}{N}n)},$$

$$\ldots, \quad \phi_{N-1}[n] = e^{j(N-1)\frac{2\pi}{N}n}$$

$$\phi_N[n] = e^{j(N)\frac{2\pi}{N}n} = e^{j2\pi n} = 1 = \phi_0[n], \; ; \quad \phi_{N+1}[n] = \phi_1[n], \ldots$$

Comparison of CT & DT signals:

TABLE 1.1 Comparison of the signals $e^{j\omega_0 t}$ and $e^{j\omega_0 n}$.

CT e^{jw_0t}	DT e^{jw_0n}
Distinct signals for distinct values of ω_0	Identical signals for values of ω_0 separated by multiples of 2π
Periodic for any choice of ω_0	Periodic only if $\omega_0 = 2\pi m/N$ for some integers $N > 0$ and m.
Fundamental frequency ω_0	Fundamental frequency* ω_0/m
Fundamental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $\frac{2\pi}{\omega_0}$	Fundamental period* $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $m\left(\frac{2\pi}{\omega_0}\right)$

*Assumes that m and N do not have any factors in common.

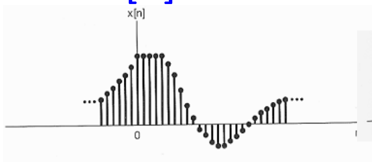
DT $\quad 1, \quad e^{j(\frac{2\pi}{N}n)}, \quad e^{j(2\frac{2\pi}{N}n)}, \quad \ldots, \quad e^{j(N-1)\frac{2\pi}{N}n}$

CT $\quad 1, \quad e^{j1w_0t}, \quad e^{j2w_0t}, \quad e^{j3w_0t}, \quad \cdots,$

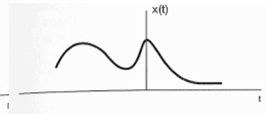
$e^{j(-1)w_0t}, \quad e^{j(-2)w_0t}, \quad e^{j(-3)w_0t}, \quad \cdots$

- Problem:
  - P1.36

$x[n]$



$x(t)$



- Introduction
- Continuous-Time & Discrete-Time Signals
- Transformations of the Independent Variable
  - Time Shift — $x[n-n_0]$ — $x(t-t_0)$
  - Time Reversal — $x[-n]$ — $x(-t)$
  - Time Scaling — $x[an]$ — $x(at)$
  - Periodic Signals — $x(t) = x(t+T)$ — $x[n] = x[n+N]$
  - Even & Odd Signals

$x(-t) = \quad x(t), x[-n] = \quad x[n]$

$x(-t) = -x(t), x[-n] = -x[n]$

$\mathcal{E}v\left\{x[n]\right\} = \frac{1}{2}\left[x[n] + x[-n]\right]$

$\mathcal{O}d\left\{x[n]\right\} = \frac{1}{2}\left[x[n] - x[-n]\right]$

$\phi_k(t) = e^{jkw_0 t}, k = 0, \pm 1, ...$

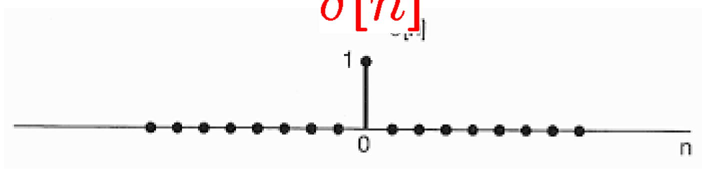
$\phi_k[n] = e^{jkw_0 n}, k = 0, ..., N-1$

- Exponential & Sinusoidal Signals
- The Unit Impulse & Unit Step Functions
- Continuous-Time & Discrete-Time Systems
- Basic System Properties

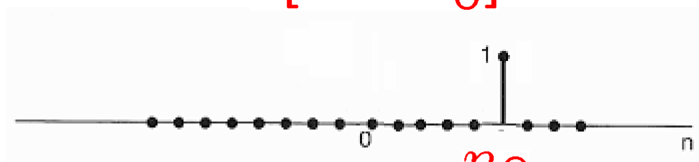
DT Unit Impulse & Unit Step Sequences

Unit impulse (or unit sample)

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

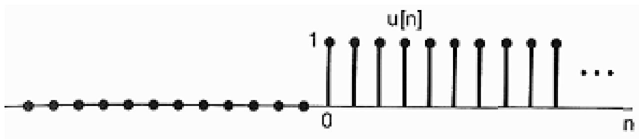


$$\delta[n- \quad] = \begin{cases} 1, & n = \\ 0, & n \neq \end{cases}$$

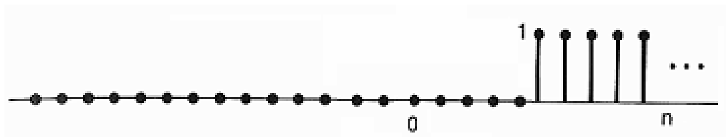


Unit step

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

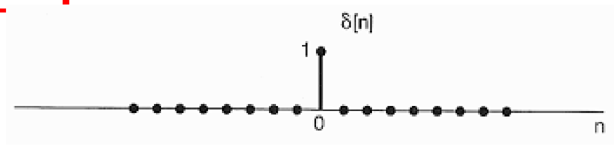


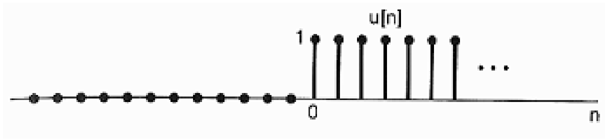
$$u[n- \quad] = \begin{cases} 0, & n < \\ 1, & n \geq \end{cases}$$



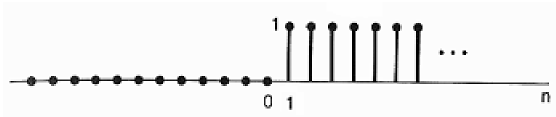
Relationship Between Impulse & Step

- First difference



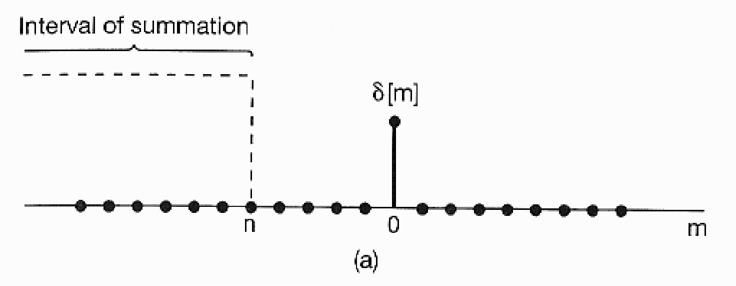


$$\delta[n] = u[n] - u[n-1]$$



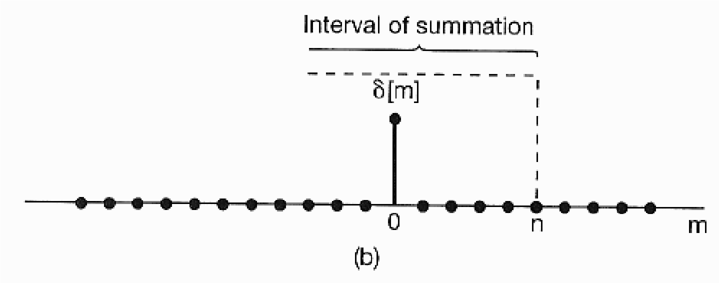
- Running sum

$$u[n] = \sum_{m=-\infty}^{n} \delta[m] \qquad = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$



(a)

$n < 0$



(b)

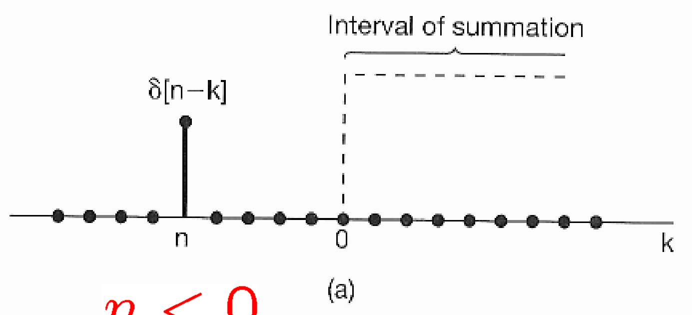
$n > 0$

Relationship Between Impulse & Step

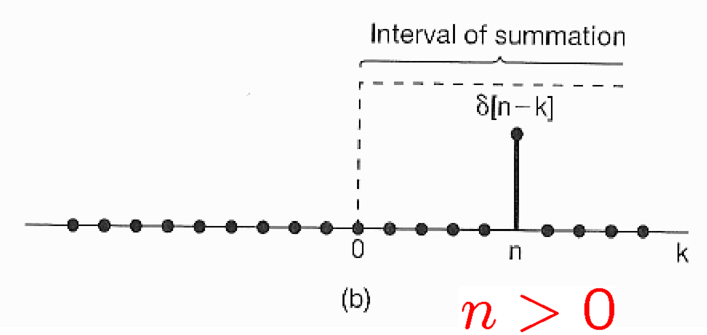
- Alternatively,

$$u[n] = \sum_{k=\infty}^{0} \delta[n-k], \quad \text{with } m = n - k$$

or,
$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$



(a)
$n < 0$

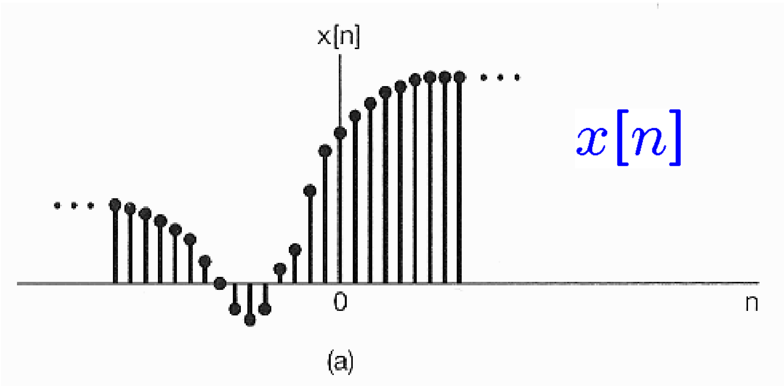


(b)
$n > 0$

Sample by Unit Impulse

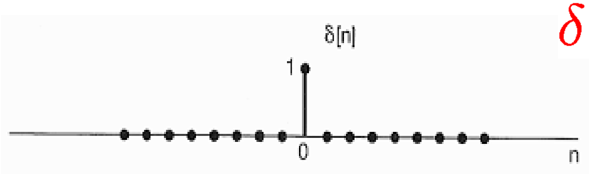
- For x[n]

$$x[n]\delta[n] = x[0]\delta[n]$$

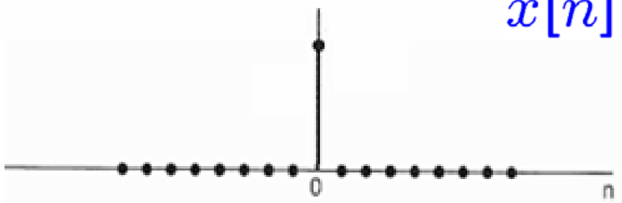


$x[n]$

$\delta[n]$

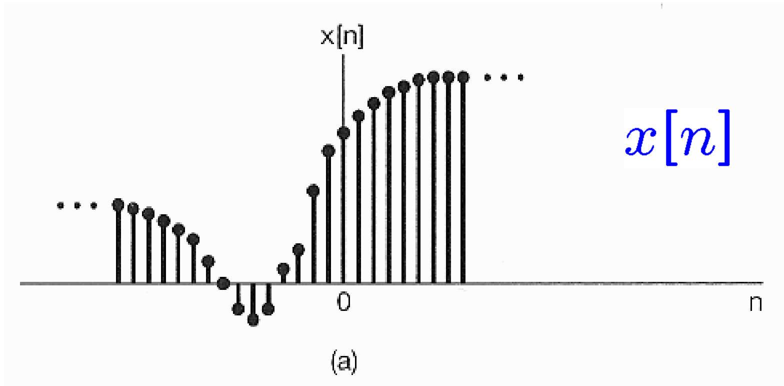


$x[n]\delta[n]$



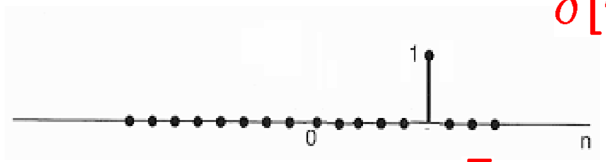
- More generally,

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$



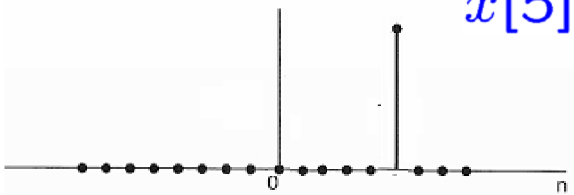
$x[n]$

$\delta[n-5]$



5

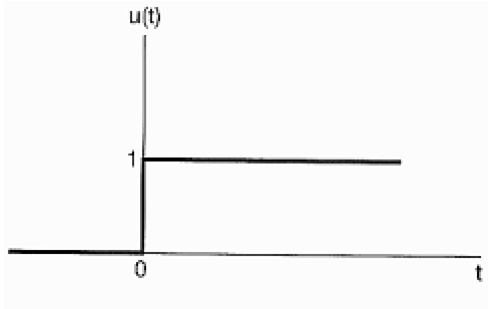
$x[5]\delta[n-5]$



CT Unit Impulse & Unit Step Functions

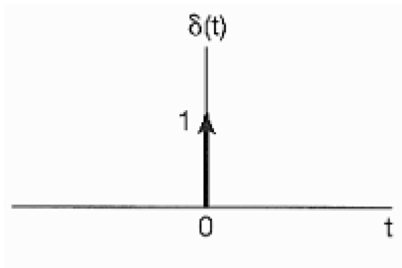
- Unit step function

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



- Unit impulse function

$$\delta(t)$$



Relationship Between Impulse & Step

- Running integral

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau \qquad = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

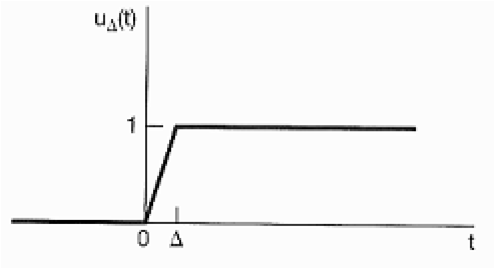
- First derivative

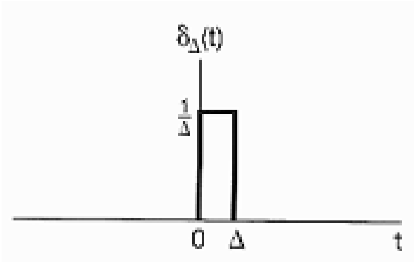
$$\delta(t) = \frac{du(t)}{dt}$$

  - But, u(t) is discontinuous at t = 0, hence, not differentiable
  - Use approximation

Relationship Between Impulse & Step

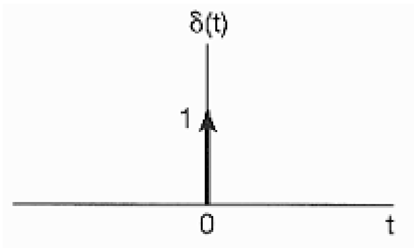
- Use approximation

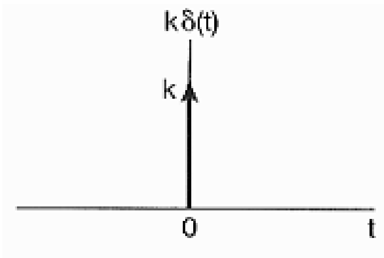




$$\delta_\Delta(t) = \frac{du_\Delta(t)}{dt}$$

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$





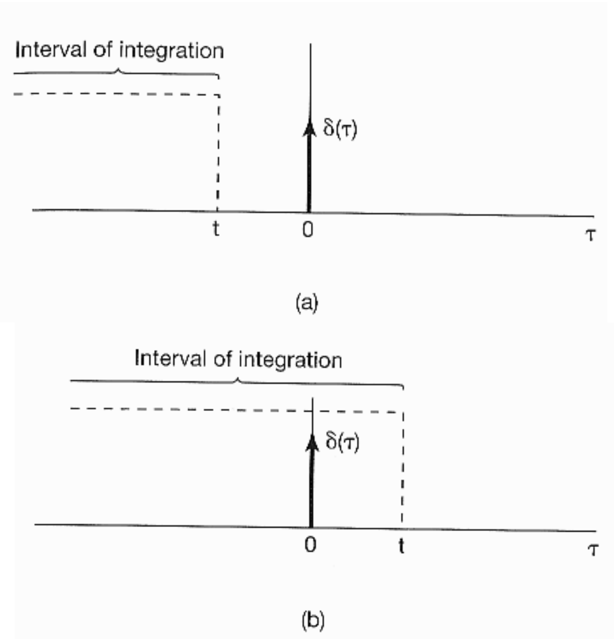
▪ Relationship Between Impulse & Step

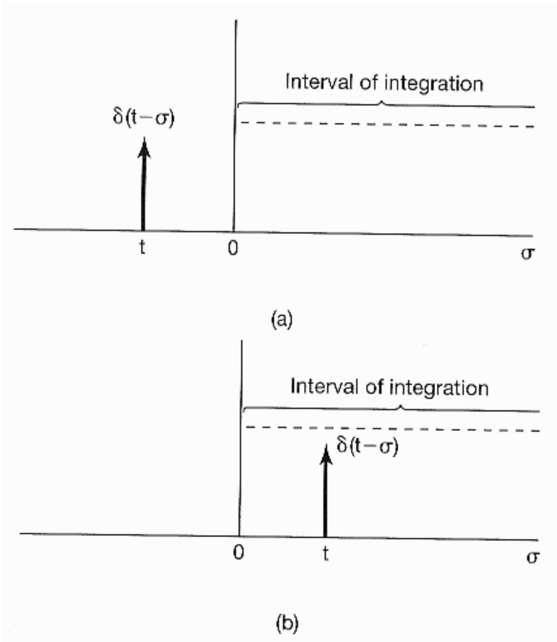
$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau \quad = \int_{\infty}^{0} \delta(t-\sigma)(-d\sigma) \quad = \int_{0}^{\infty} \delta(t-\sigma)(d\sigma)$$

$$\tau = t - \sigma$$

$$d\tau = -d\sigma$$

$$= \int_{0}^{\infty} \delta(t-\tau)(d\tau)$$





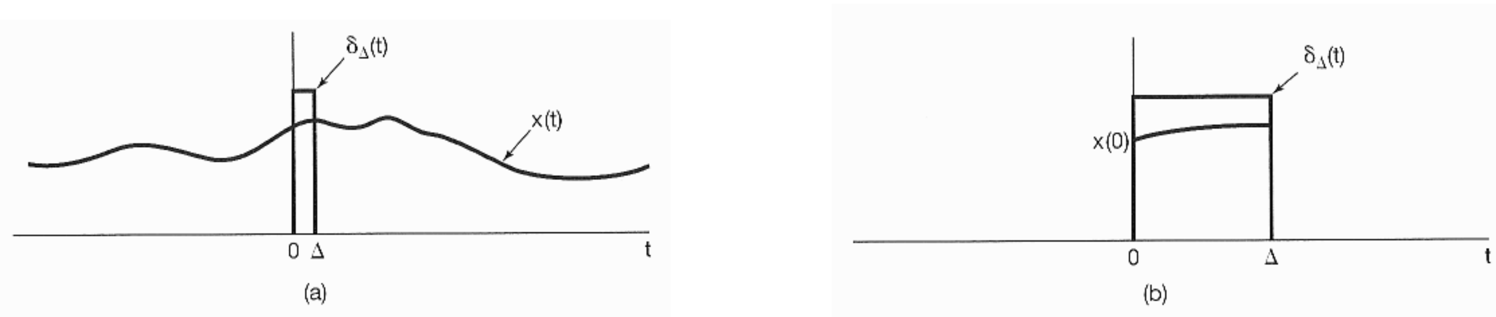
Sample by Unit Impulse Function

- For x(t)

$$x(t)\delta(t) = x(0)\delta(t)$$

- More generally,

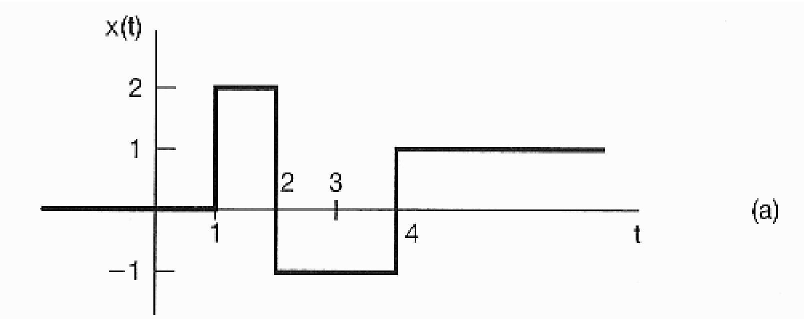
$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$



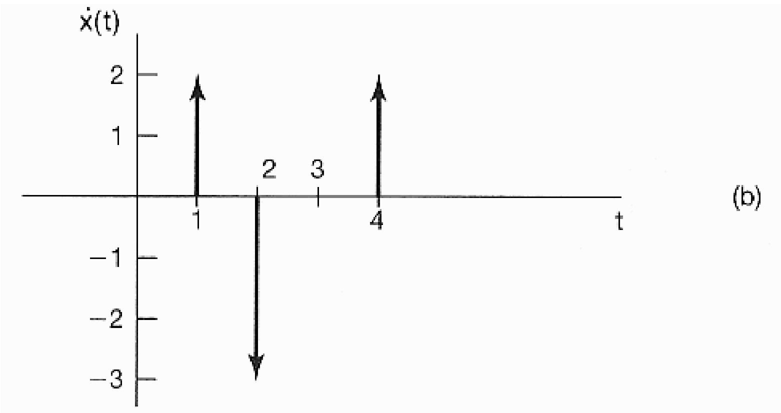
- Example 1.7:

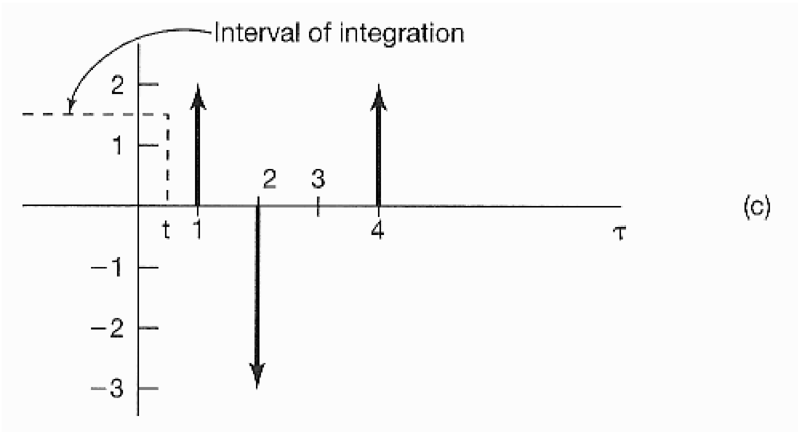
$$\delta(t) = \frac{du(t)}{dt}$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$



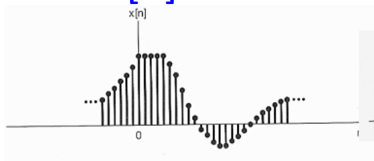
$$x(t) = \int_{0}^{t} \dot{x}(\tau) d\tau$$



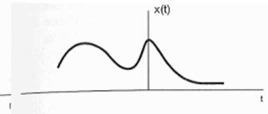


Outline

$x[n]$



$x(t)$



- Introduction
- Continuous-Time & Discrete-Time Signals
- Transformations of the Independent Variable
  - Time Shift $\quad x[n-n_0] \quad x(t-t_0)$
  - Time Reversal $\quad x[-n] \quad x(-t)$
  - Time Scaling $\quad x[an] \quad x(at)$
  - Periodic Signals $\quad x(t) = x(t+T) \quad x[n] = x[n+N]$
  - Even & Odd Signals

$$x(-t) = x(t), x[-n] = x[n]$$
$$x(-t) = -x(t), x[-n] = -x[n]$$
$$\mathcal{E}v\left\{x[n]\right\} = \frac{1}{2}\left[x[n] + x[-n]\right]$$
$$\mathcal{O}d\left\{x[n]\right\} = \frac{1}{2}\left[x[n] - x[-n]\right]$$

- Exponential & Sinusoidal Signals

$$\phi_k(t) = e^{jkw_0t}, k = 0, \pm 1, ...$$
$$\phi_k[n] = e^{jkw_0n}, k = 0, ..., N-1$$

- The Unit Impulse & Unit Step Functions

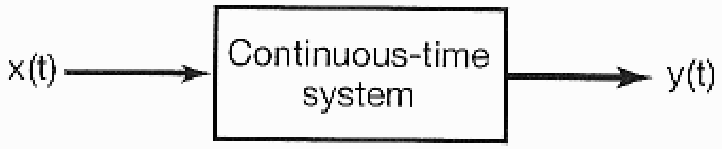
$$\delta[n], u[n]$$
$$\delta(t), u(t)$$

- Continuous-Time & Discrete-Time Systems
- Basic System Properties

Physical Systems & Mathematical Descriptions

- Examples of physical systems are signal processing, communications, electromechanical motors, automotive vehicles, chemical-processing plants

- A system can be viewed as a process in which input signals are transformed by the system or cause the system to respond in some way, resulting in other signals or outputs



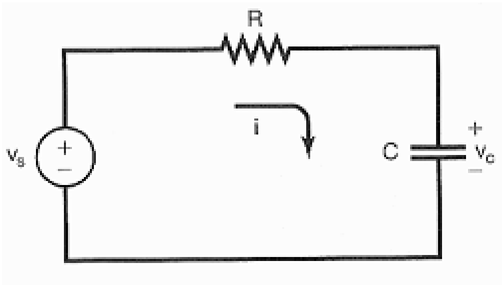
(a)

(b)

Simple examples of CT systems

- RC circuit

Input signal: $v_s(t)$

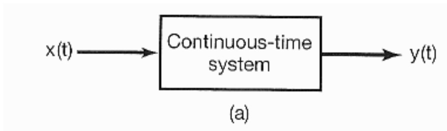


Output signal: $v_c(t)$

$$i(t) = \frac{v_s(t) - v_c(t)}{R} \qquad i(t) = C\frac{dv_c(t)}{dt}$$

$$\Rightarrow \frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t)$$

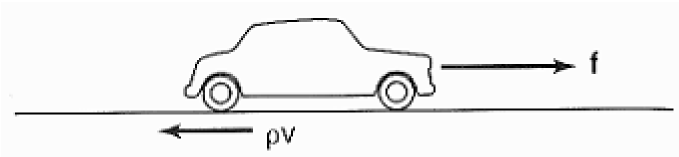
$$\Rightarrow \frac{dy(t)}{dt} + ay(t) = bx(t)$$



$x(t) \rightarrow y(t)$

Simple examples of CT systems

Automobile



Input signal: $f(t)$

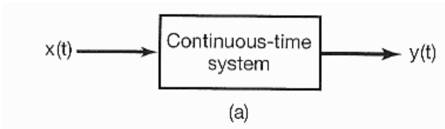
Output signal: $v(t)$

$$f(t) - \rho v(t) = m\frac{dv(t)}{dt}$$

$$\frac{dv(t)}{dt} = \frac{1}{m}[f(t) - \rho v(t)]$$

$$\Rightarrow \frac{dv(t)}{dt} + \frac{\rho}{m}v(t) = \frac{1}{m}f(t)$$

$$\Rightarrow \frac{dy(t)}{dt} + ay(t) = bx(t)$$



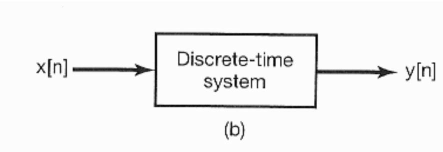
$$x(t) \rightarrow y(t)$$

- Simple examples of DT systems
  - Balance in a bank account

$$y[n] = 1.01y[n-1] + x[n]$$

or, $y[n] - 1.01y[n-1] = x[n]$

$$\Rightarrow \quad y[n] + ay[n-1] = bx[n]$$



$$x[n] \rightarrow y[n]$$

Simple examples of DT systems

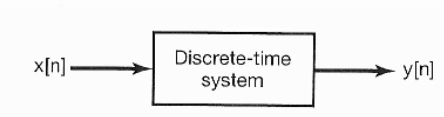
Digital simulation of differential equation

$$\frac{dv(t)}{dt} \approx \frac{v(n\Delta) - v((n-1)\Delta)}{\Delta} = \frac{v[n] - v[n-1]}{\Delta},$$

$$t = n\Delta$$

$$\frac{dv(t)}{dt} + \frac{\rho}{m}v(t) = \frac{1}{m}f(t)$$

$$\Rightarrow \; v[n] - \frac{m}{m+\rho\Delta}v[n-1] = \frac{\Delta}{m+\rho\Delta}f[n]$$



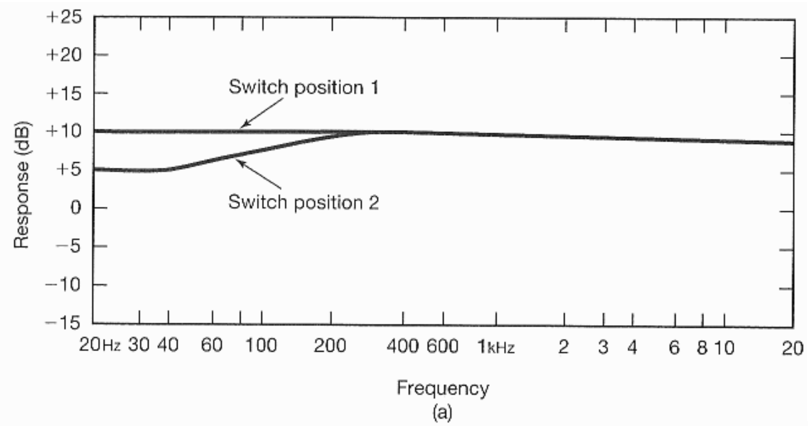
(b)

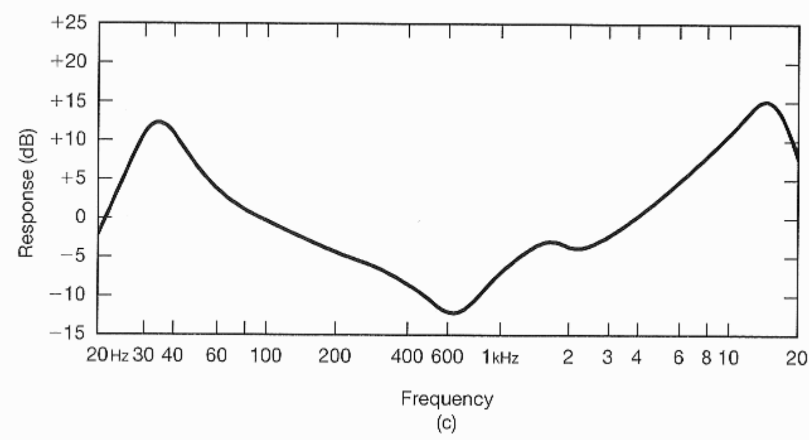
$$\Rightarrow \; y[n] + ay[n-1] = bx[n]$$

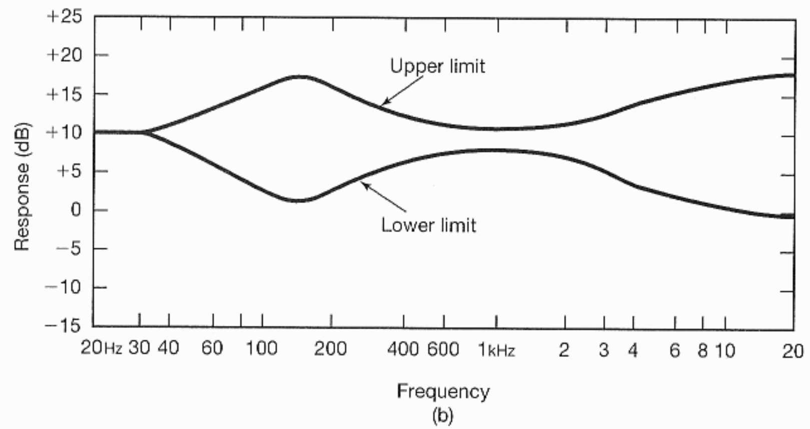
$$x[n] \rightarrow y[n]$$

Interconnections of Systems:

- Audio System:

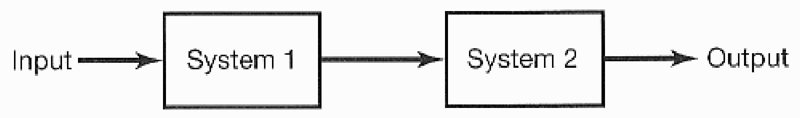






Interconnections of Systems

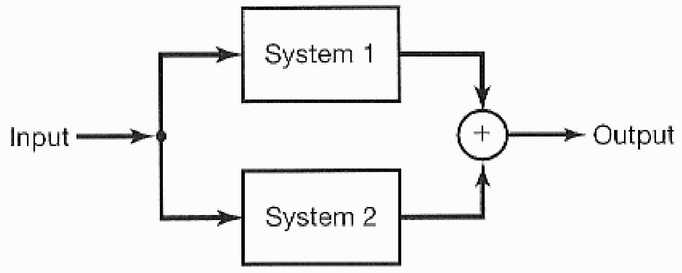
- Series or cascade interconnection of 2 systems



(a)

> e.x. radio receiver + amplifier

- Parallel interconnection of 2 systems

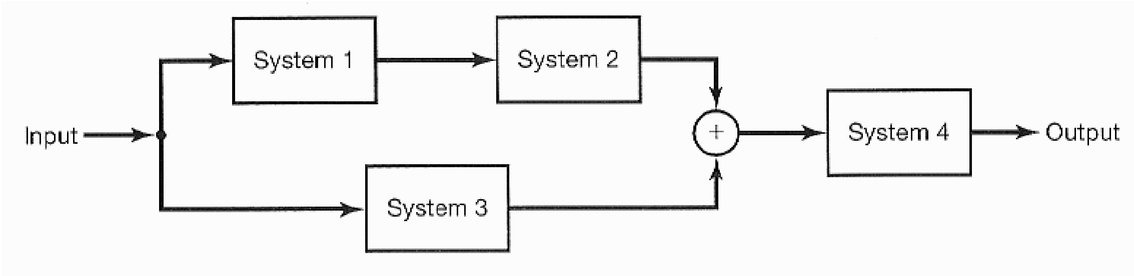


(b)

> e.x. audio system with several microphones or speakers

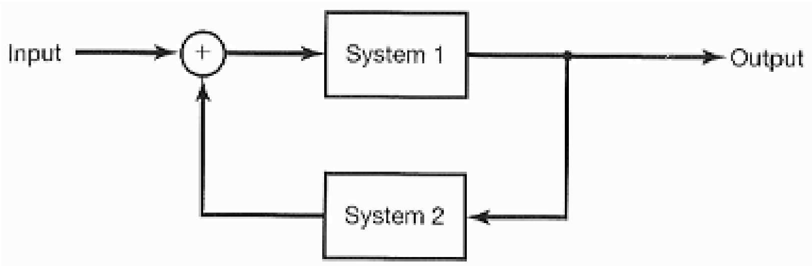
Interconnections of Systems

- Series-parallel interconnection



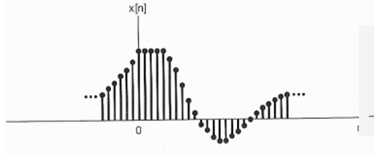
(c)

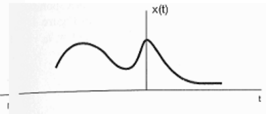
- Feedback interconnection



> e.x. cruise control, electrical circuit

- Introduction
- Continuous-Time & Discrete-Time Signals

$x[n]$ 

$x(t)$ 

- Transformations of the Independent Variable
  - Time Shift $\quad x[n-n_0] \quad x(t-t_0)$
  - Time Reversal $\quad x[-n] \quad x(-t)$
  - Time Scaling $\quad x[an] \quad x(at)$
  - Periodic Signals $\quad x(t) = x(t+T)$
  - Even & Odd Signals $\quad x[n] = x[n+N]$

$$x(-t) = x(t), x[-n] = x[n]$$
$$x(-t) = -x(t), x[-n] = -x[n]$$
$$\mathcal{E}v\left\{x[n]\right\} = \frac{1}{2}\left[x[n] + x[-n]\right]$$
$$\mathcal{O}d\left\{x[n]\right\} = \frac{1}{2}\left[x[n] - x[-n]\right]$$

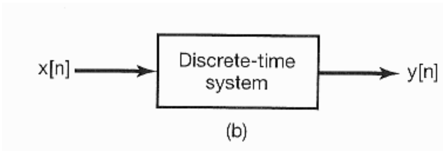
- Exponential & Sinusoidal Signals

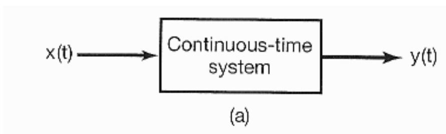
$$\phi_k(t) = e^{jkw_0 t}, k = 0, \pm 1, \ldots$$
$$\phi_k[n] = e^{jkw_0 n}, k = 0, \ldots, N-1$$

- The Unit Impulse & Unit Step Functions

$$\delta[n], u[n]$$
$$\delta(t), u(t)$$

- Continuous-Time & Discrete-Time Systems
- Basic System Properties
  - Systems with or without memory
  - Invertibility & Inverse Systems
  - Causality
  - Stability
  - Time Invariance
  - Linearity





$$x[n] \rightarrow y[n] \qquad x(t) \rightarrow y(t)$$

Systems with or without memory

Memoryless systems

- Output depends only on the input at that same time

$$y[n] = (2x[n] - x[n]^2)^2$$

$$y(t) = Rx(t)$$ (resistor)

Systems with memory

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$ (accumulator)

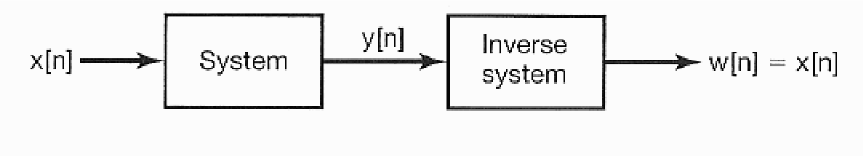
$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau$$

$$y[n] = x[n-1]$$ (delay)

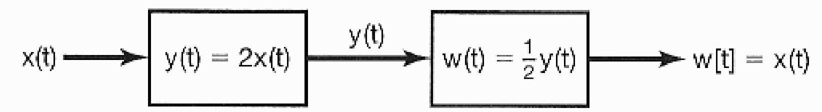
Invertibility & Inverse Systems

Invertible systems

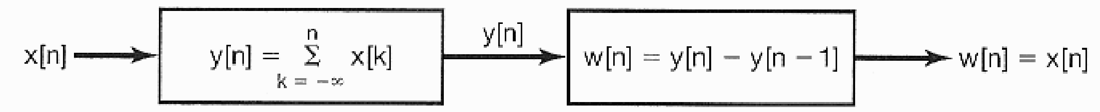
- Distinct inputs lead to distinct outputs



(a)



(b)



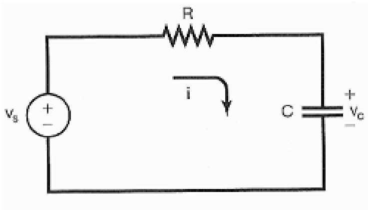
(c)

$y(t) = x(t)^2$ is not invertible

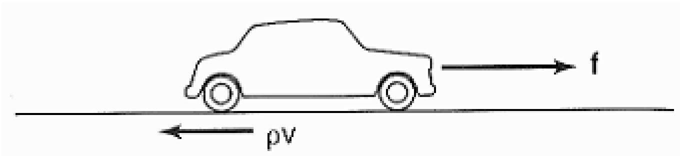
Causality

Causal systems

- Output depends only on input at present time & in the past



$$\Rightarrow \quad \frac{dv_c(t)}{dt} + \frac{1}{RC} v_c(t) = \frac{1}{RC} v_s(t)$$



$$\Rightarrow \quad \frac{dv(t)}{dt} + \frac{\rho}{m} v(t) = \frac{1}{m} f(t)$$

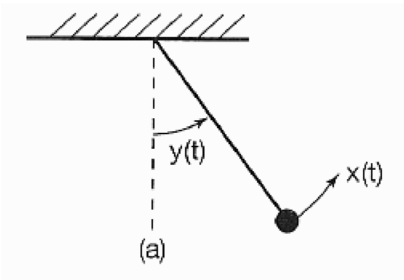
- Non-causal systems

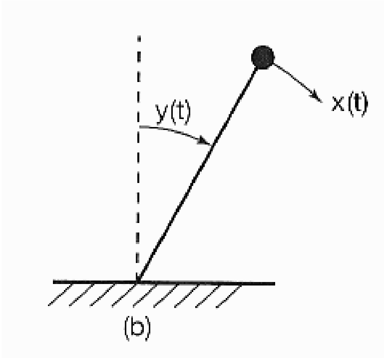
$$y[n] = x[n] - x[n+1]$$

$$y(t) = x(t+1)$$

$$y(t) = x(t)\cos(t+1) \quad ???$$

- Stability
  - Stable systems
    - Small inputs lead to responses that do not diverge
    - Every bounded input excites a bounded output
      - Bounded-input bounded-output stable (BIBO stable)
      - For all $|x(t)| < a$, then $|y(t)| < b$, for all t





- Balance in a bank account?

$$y[n] = 1.01y[n-1] + x[n]$$

- Example 1.13: Stability

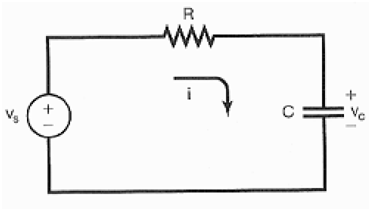
$$S_1 : \quad y(t) = t\, x(t)$$

$$S_2 : \quad y(t) = e^{x(t)}$$

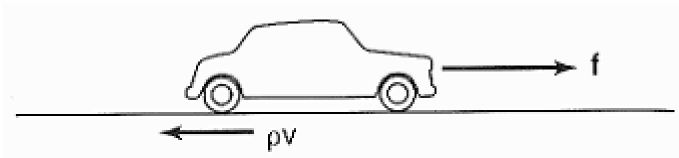
Time Invariance

Time-invariant systems

- Behavior & characteristics of system are fixed over time



$$\Rightarrow \frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t)$$



$$\Rightarrow \frac{dv(t)}{dt} + \frac{\rho}{m}v(t) = \frac{1}{m}f(t)$$

- A time shift in the input signal results in an identical time shift in the output signal

$$x[n] \to y[n] \quad \Longleftrightarrow \quad x[n-n_0] \to y[n-n_0]$$

Time Invariance

- Example of time-invariant system (Example 1.14)

$$y(t) = \sin[x(t)]$$

$$x_1(t) \qquad\qquad y_1(t) = \sin[x_1(t)]$$

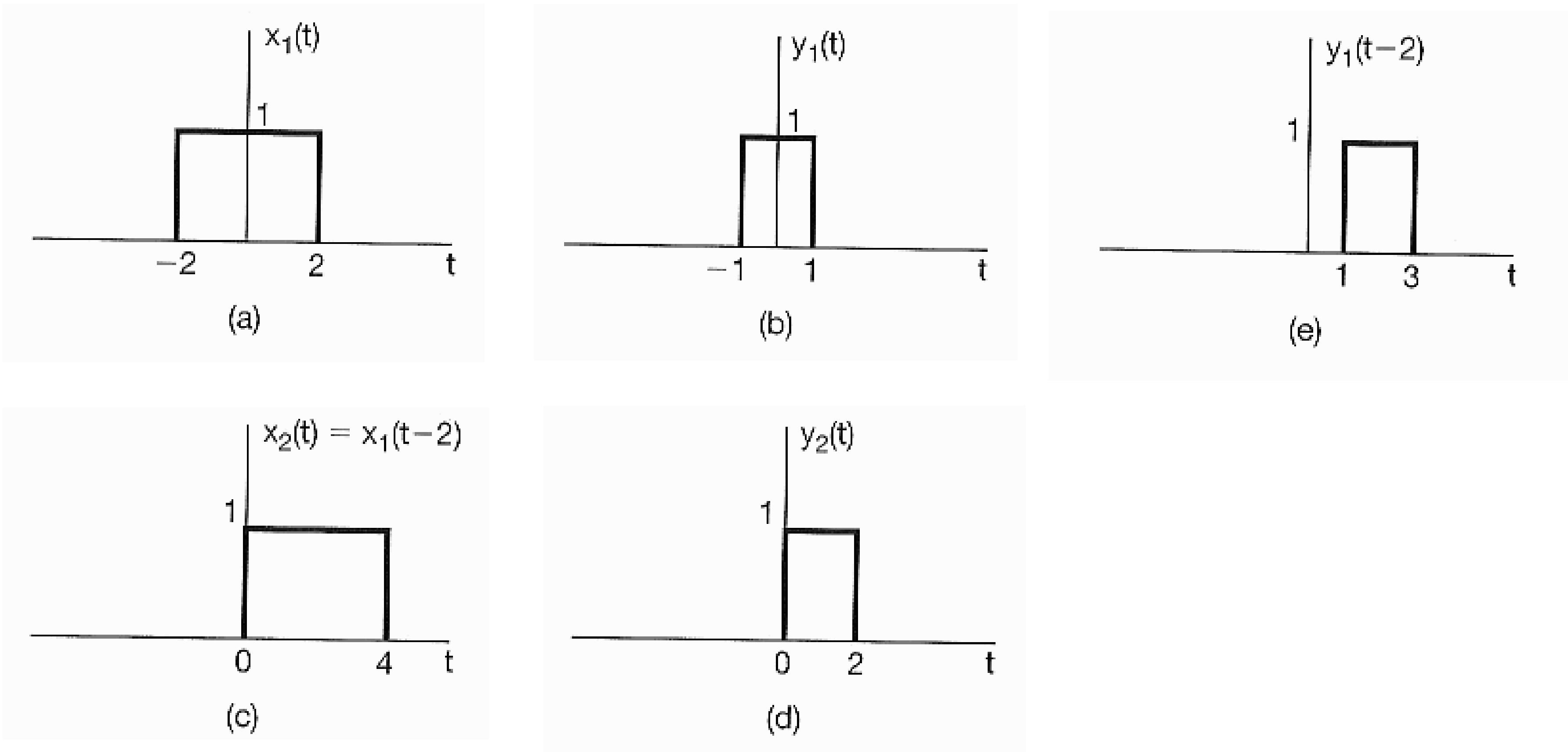
$$x_2(t) = x_1(t - t_0) \qquad\qquad y_2(t) = \sin[x_2(t)] = \sin[x_1(t - t_0)]$$

$$y_1(t - t_0) = \sin[x_1(t - t_0)]$$

$$y_2(t) = y_1(t - t_0)$$

- Time Invariance
  - Example of time-varying system (Example 1.16)

$$y(t) = x(2t)$$



Linearity

Linear systems

- If an input consists of the weighted sum of several signals, then the output is the superposition of the responses of the system to each of those signals

$$x_1[n] \to y_1[n]$$

$$x_2[n] \to y_2[n]$$

IF (1) $x_1[n] + x_2[n] \ \to \ y_1[n] + y_2[n]$ (additivity)

(2) $a\, x_1[n] \ \to \ a\, y_1[n]$ (scaling or homogeneity)

a: any complex constant

THEN, the system is linear

- Linearity
  - Linear systems
    - In general,

a, b: any complex constants

$$a x_1[n] + b x_2[n] \rightarrow a y_1[n] + b y_2[n] \qquad \text{for DT}$$

$$a x_1(t) + b x_2(t) \rightarrow a y_1(t) + b y_2(t) \qquad \text{for CT}$$

    - OR,

$$x[n] = \sum_k a_k x_k[n] = a_1 x_1[n] + a_2 x_2[n] + \ldots$$

$$\longrightarrow \; y[n] = \sum_k a_k y_k[n] = a_1 y_1[n] + a_2 y_2[n] + \ldots$$

This is known as the superposition property

- Linearity
  - Example 1.17: $S: \; y(t) = tx(t)$

$$x_1(t) \rightarrow y_1(t) = tx_1(t)$$

$$x_2(t) \rightarrow y_2(t) = tx_2(t)$$

$$x_3(t) = ax_1(t) + bx_2(t)$$

$$\rightarrow \quad y_3(t) = tx_3(t)$$

$$= t(ax_1(t) + bx_2(t)) = atx_1(t) + btx_2(t)$$

$$= ay_1(t) + by_2(t)$$

- Linearity
  - Example 1.18: $S: \; y(t) = (x(t))^2$

$$x_1(t) \quad \rightarrow \quad y_1(t) = (x_1(t))^2$$

$$x_2(t) \quad \rightarrow \quad y_2(t) = (x_2(t))^2$$

$$x_3(t) = a x_1(t) + b x_2(t)$$

$$\rightarrow \quad y_3(t) = (x_3(t))^2 \qquad = (a x_1(t) + b x_2(t))^2$$

$$= a^2 (x_1(t))^2 + b^2 (x_2(t))^2 + 2ab x_1(t) x_2(t)$$

$$= a^2 y_1(t) + b^2 y_2(t) + 2ab x_1(t) x_2(t)$$

- Linearity
  - Example 1.20: $S: \; y[n] = 2x[n] + 3$

$$x_1[n] \quad \longrightarrow \quad y_1[n] = 2x_1[n] + 3$$

$$x_2[n] \quad \longrightarrow \quad y_2[n] = 2x_2[n] + 3$$

$$x_3[n] = ax_1[n] + bx_2[n]$$

$$\begin{aligned} \longrightarrow \quad y_3[n] &= 2x_3[n] + 3 \\ &= 2(ax_1[n] + bx_2[n]) + 3 \\ &= a(2x_1[n] + 3) + b(2x_2[n] + 3) + 3 - 3a - 3b \\ &= ay_1[n] + by_2[n] + 3(1 - a - b) \end{aligned}$$

- Linearity
  - Example 1.20: $S: \ y[n] = 2x[n] + 3$

$$x_1[n] \quad \rightarrow \quad y_1[n] = 2x_1[n] + 3$$

$$x_2[n] \quad \rightarrow \quad y_2[n] = 2x_2[n] + 3$$

  - However,

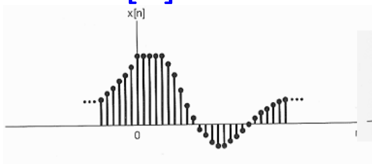
$$y_1[n] - y_2[n] = \Big(2x_1[n] + 3\Big) - \Big(2x_2[n] + 3\Big)$$

$$= 2\Big[x_1[n] - x_2[n]\Big]$$

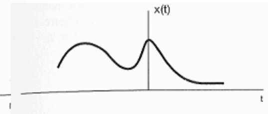
It is a incrementally linear system

- Introduction
- Continuous-Time & Discrete-Time Signals

$x[n]$



$x(t)$



- Transformations of the Independent Variable
  - Time Shift $x[n-n_0]$ $x(t-t_0)$
  - Time Reversal $x[-n]$ $x(-t)$
  - Time Scaling $x[an]$ $x(at)$
  - Periodic Signals $x(t)=x(t+T)$
  - Even & Odd Signals $x[n]=x[n+N]$

$$x(-t) = x(t), x[-n] = x[n]$$

$$x(-t) = -x(t), x[-n] = -x[n]$$

$$\mathcal{E}v\left\{x[n]\right\} = \frac{1}{2}\left[x[n] + x[-n]\right]$$

$$\mathcal{O}d\left\{x[n]\right\} = \frac{1}{2}\left[x[n] - x[-n]\right]$$

- Exponential & Sinusoidal Signals

$$\phi_k(t) = e^{jkw_0 t}, k = 0, \pm 1, \ldots$$

$$\phi_k[n] = e^{jkw_0 n}, k = 0, \ldots, N-1$$

- The Unit Impulse & Unit Step Functions

$$\delta[n], u[n]$$

$$\delta(t), u(t)$$

- Continuous-Time & Discrete-Time Systems
- Basic System Properties
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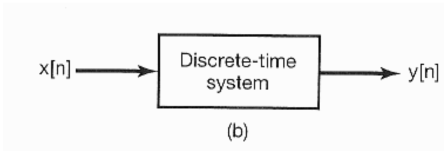
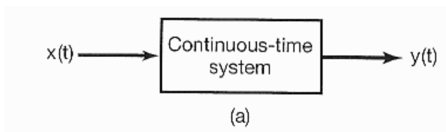
  - Invertibility & Inverse Systems
  - Causality
  - Stability
  - Time Invariance
  - Linearity





$$x[n] \rightarrow y[n]$$

$$x(t) \rightarrow y(t)$$

Flowchart

Signals & Systems (Chap 1)

LTI & Convolution (Chap 2)

Bounded/Convergent

Periodic

FS – CT – DT (Chap 3)

Aperiodic

FT
– CT (Chap 4)
– DT (Chap 5)

Unbounded/Non-convergent

LT – CT (Chap 9)

zT – DT (Chap 10)

Time-Frequency (Chap 6)

Communication (Chap 8)

CT-DT (Chap 7)

Control (Chap 11)

- Problem 1.26 (Page 61)

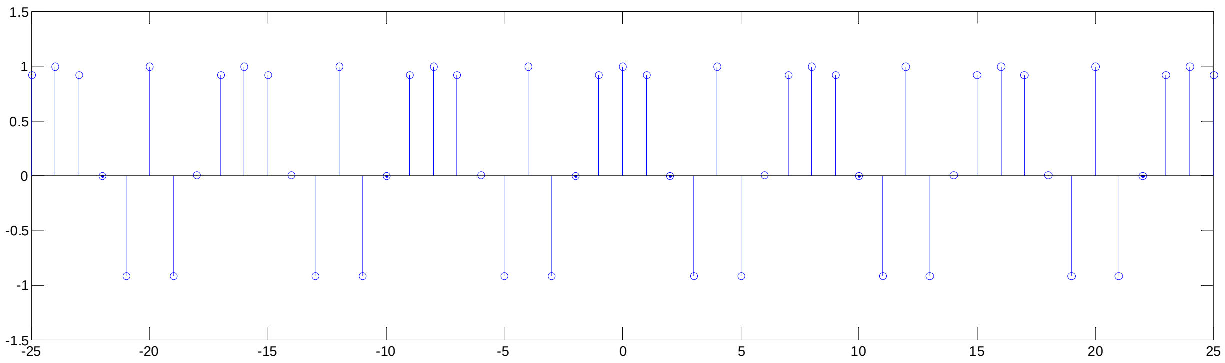
$$x[n] = \cos\left(\frac{\pi}{8}\,n^2\right)$$

```
L = 25;
n = -L:L;

x = cos( pi/8 * (n.^2) );

figure(1)
stem( n, x, 'o' ); hold on;

axis([-L L -1.5 1.5])
```



- Problem 1.27 (Page 62)

$(a)\ y(t) = x(t-2) + x(2-t)$ Time-Invariant?

$$x_1(t) \rightarrow y_1(t) \qquad y_1(t) = x_1(t-2) + x_1(2-t)$$

$$x_2(t) \rightarrow y_2(t) \qquad y_2(t) = x_2(t-2) + x_2(2-t)$$

$$x_2(t) = x_1(t - t_0)$$

$$\Rightarrow y_2(t) = x_2(t-2) + x_2(2-t)$$

$$= x_1((t-t_0)-2) + x_1(2-(t-t_0))$$

$$= x_1(t - t_0 - 2) + x_1(2 - t + t_0))$$

$$\Rightarrow y_2(t) = x_1(t - 2 - t_0) + x_1(2 - t - t_0)$$

$$= x_1(t - t_0 - 2) + x_1(2 - t - t_0)$$

- Problem 1.27 (Page 62)

$(a)\ y(t) = x(t-2) + x(2-t)$ Time-Invariant?

$x_1(t) = \delta(t)$

$y_1(t) = \delta(t-2) + \delta(2-t)$

$x_2(t) = \delta(t-3)$

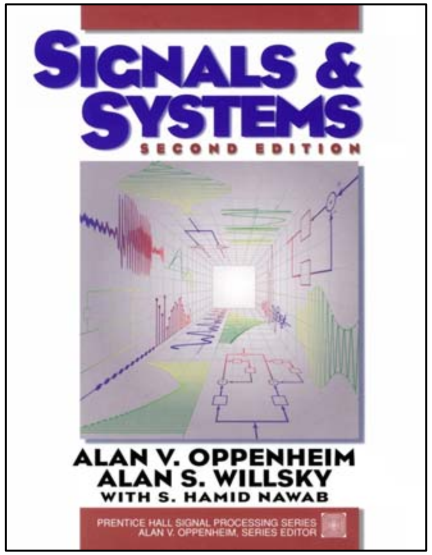
$$y_2(t) = \delta(t-2-3) + \delta(2-t-3)$$
$$= \delta(t-5) + \delta(-1-t)$$

$$\Rightarrow y_1(t-3) = \delta(t-3-2) + \delta(2-(t-3))$$
$$= \delta(t-5) + \delta(5-t)$$

Spring 2010

信號與系統
Signals and Systems

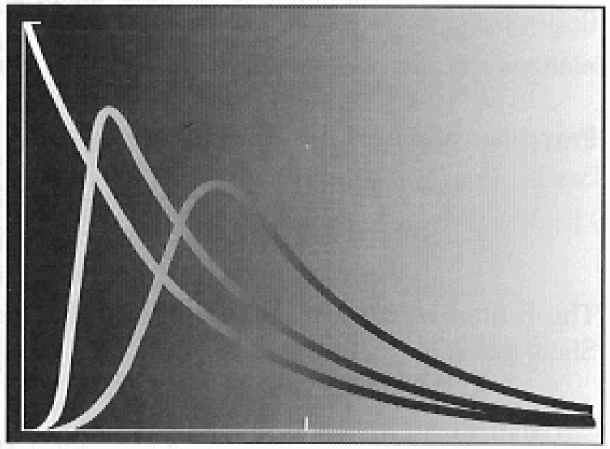
Chapter SS-2
Linear Time-Invariant Systems



Feng-Li Lian

NTU-EE

Feb10 – Jun10



Figures and images used in these lecture notes are adopted from
"Signals & Systems" by Alan V. Oppenheim and Alan S. Willsky, 1997

- Discrete-Time Linear Time-Invariant Systems
  - The convolution sum
- Continuous-Time Linear Time-Invariant Systems
  - The convolution integral
- Properties of Linear Time-Invariant Systems
- Causal Linear Time-Invariant Systems Described by Differential & Difference Equations
- Singularity Functions

$x[n] \rightarrow \boxed{\text{h[n]}} \rightarrow y[n]$ $\qquad$ $x(t) \rightarrow \boxed{\text{h(t)}} \rightarrow y(t)$

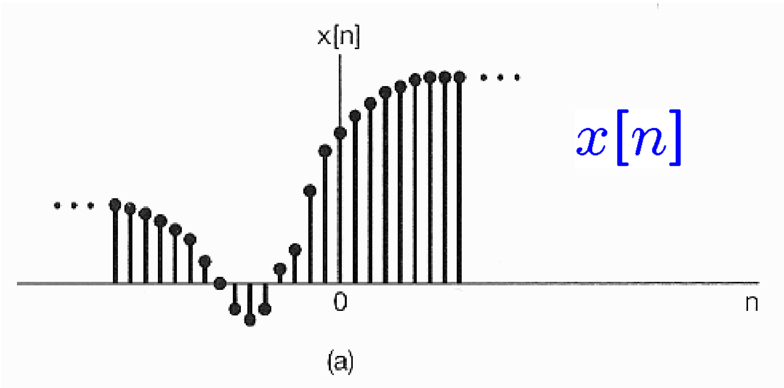
$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] \qquad x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Signals

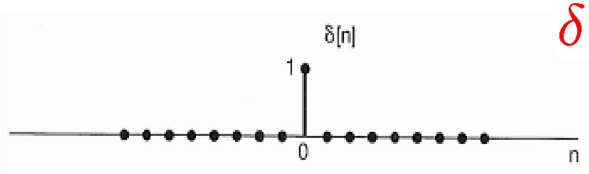
Systems

- Sample by Unit Impulse
- For x[n]

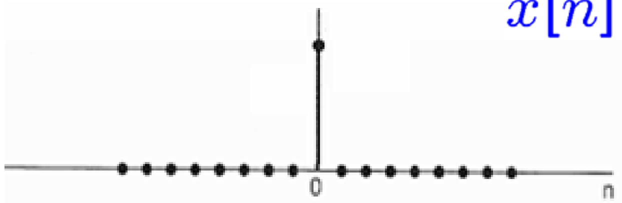
$$x[n]\delta[n] = x[0]\delta[n]$$



$x[n]$



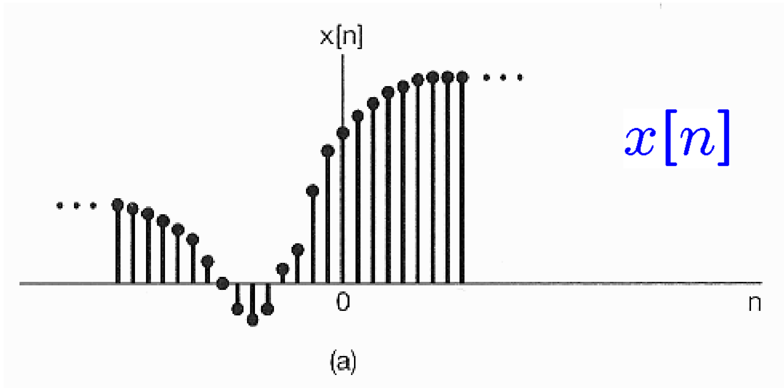
$\delta[n]$



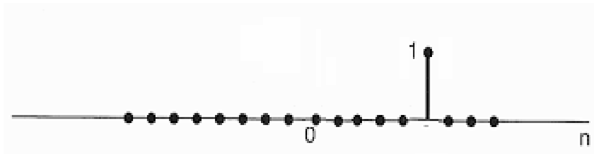
$x[n]\delta[n]$

- More generally,

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$

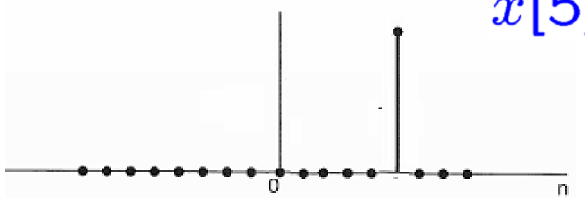


$x[n]$



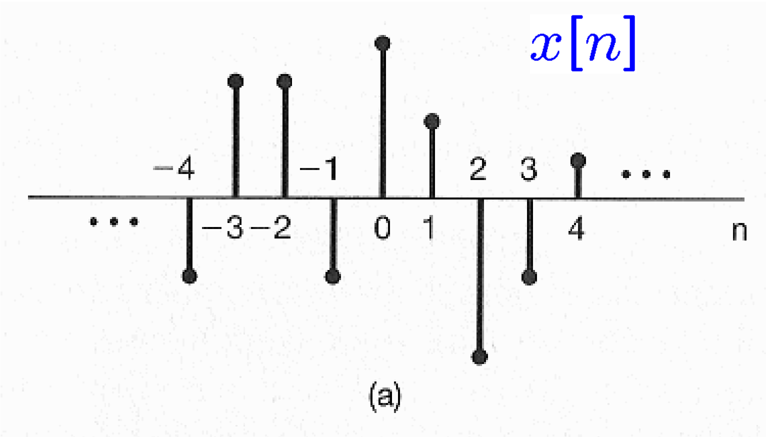
$\delta[n-5]$

5

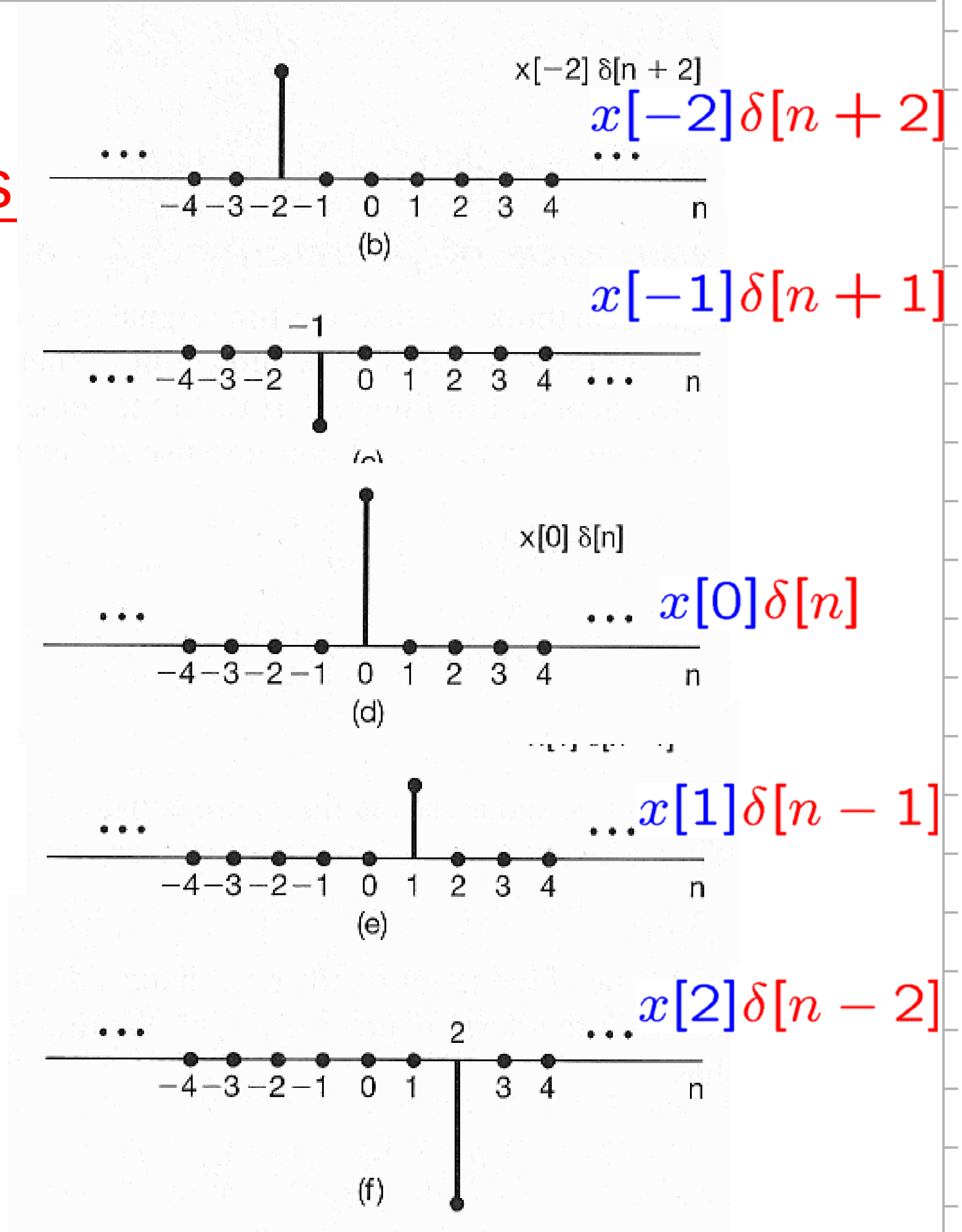


$x[5]\delta[n-5]$

- Representation of DT Signals by Impulses



(a)



- Representation of DT Signals by Impulses:
  - More generally,

$$x[n] = \cdots + x[-3]\delta[n+3] + x[-2]\delta[n+2]$$

$$+ x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1]$$

$$+ x[2]\delta[n-2] + x[3]\delta[n-3] + \cdots$$

$$= \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

- The sifting property of the DT unit impulse
- x[n] = a superposition of scaled versions of shifted unit impulses δ[n-k]

- DT Unit Impulse Response & Convolution Sum:

$$\text{input} \longrightarrow \boxed{\text{Linear System}} \longrightarrow \text{output}$$

$$\delta[n] \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_0[n]$$

$$\delta[n-1] \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_1[n]$$

$$\delta[n-2] \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_2[n]$$

$$\vdots$$

$$\delta[n-k] \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_k[n]$$

▪ DT Unit Impulse Response & Convolution Sum:

$$x[n] \longrightarrow \boxed{\text{Linear System}} \longrightarrow y[n]$$

$$x[0] \cdot \delta[n] \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_0[n] \cdot x[0]$$

$$x[1] \cdot \delta[n-1] \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_1[n] \cdot x[1]$$

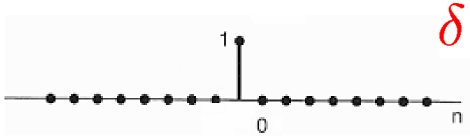
$$x[2] \cdot \delta[n-2] \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_2[n] \cdot x[2]$$

$$\vdots$$

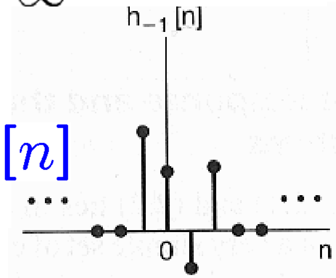
$$x[k] \cdot \delta[n-k] \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_k[n] \cdot x[k]$$

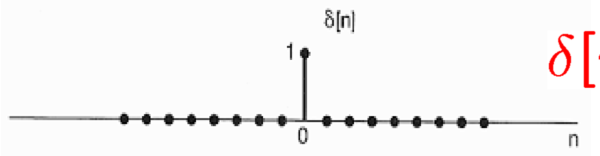
$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] \quad \Longrightarrow \quad y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n]$$

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] \quad \Longrightarrow \quad y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n]$$

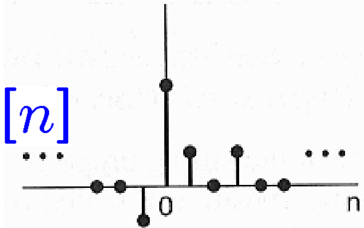
$x[-1]$


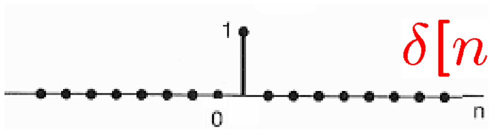
$\delta[n+1]$

$h_{-1}[n]$


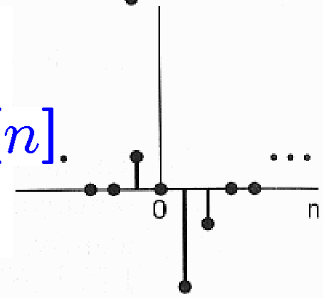
$x[0]$


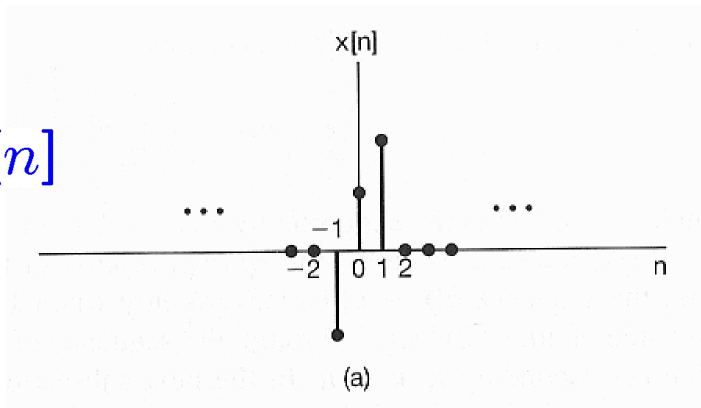
$\delta[n]$

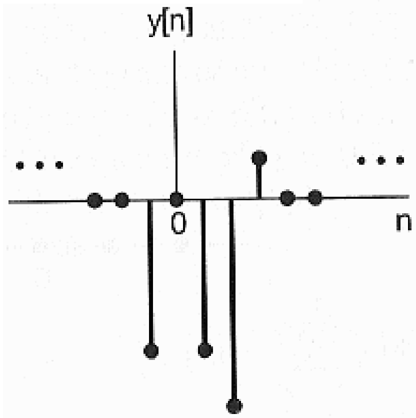
$h_0[n]$


$x[1]$


$\delta[n-1]$

$h_1[n]$


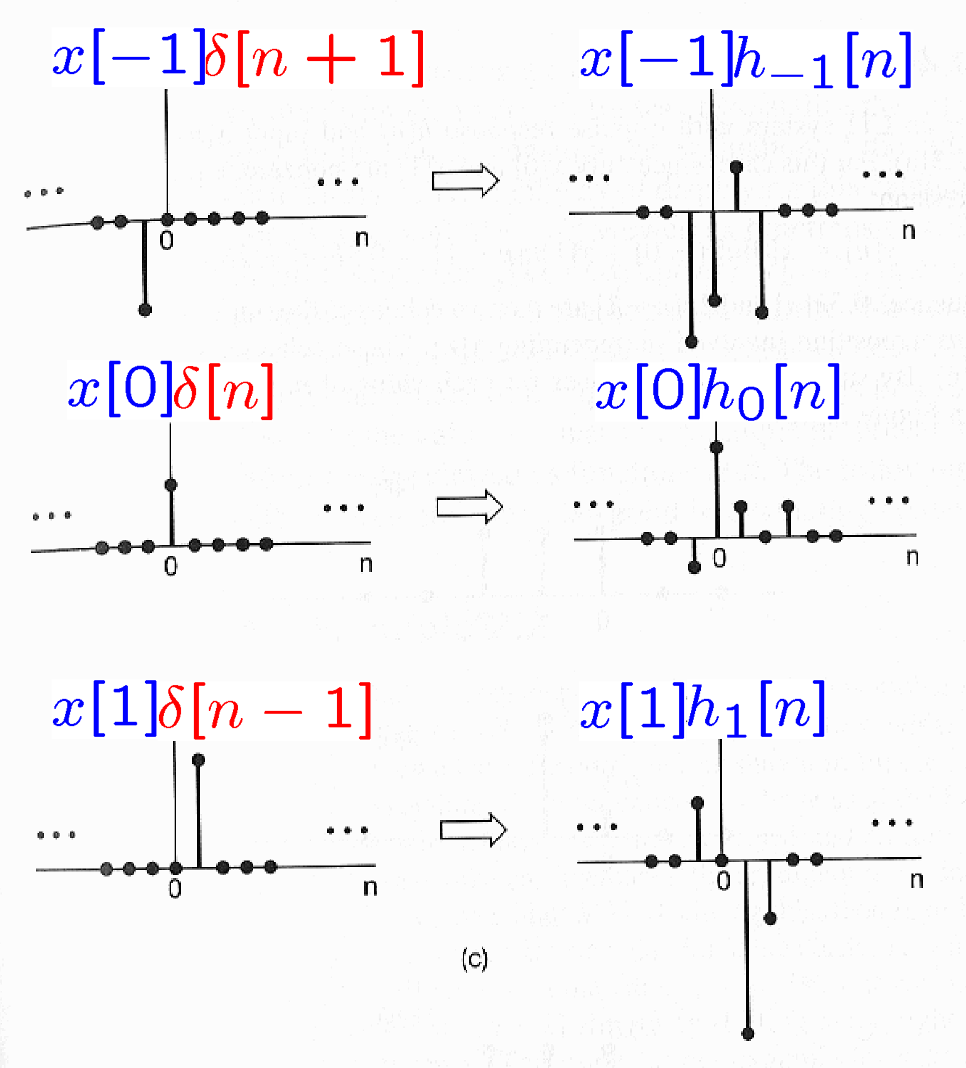
$x[n]$




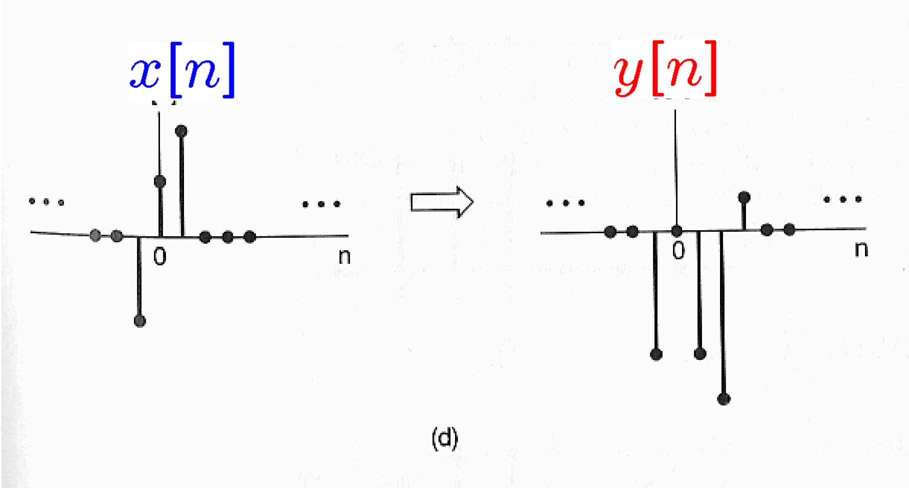
$y[n]$

DT LTI Systems: Convolution Sum

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] \quad \Longrightarrow \quad y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n]$$



(c)



(d)

$x[n] \longrightarrow$ Linear System $\longrightarrow y[n]$

- If the linear system (L) is also time-invariant (TI)
  - Then,

$$h_k[n] = h_0[n-k] = h[n-k]$$

- Hence, for an LTI system,

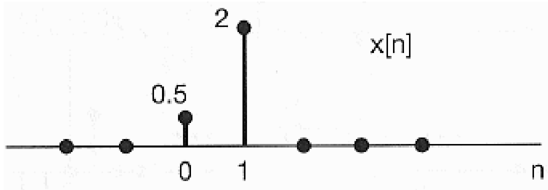
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=-\infty}^{+\infty} x[n-k]h[k]$$

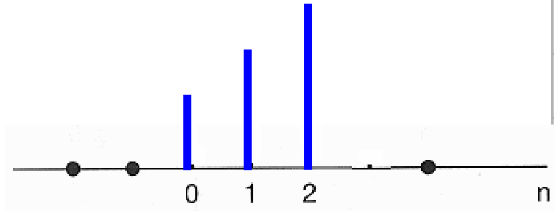
  - Known as the convolution of x[n] & h[n]
  - Referred as the convolution sum or superposition sum

- Symbolically, $y[n] = x[n] * h[n] = h[n] * x[n]$

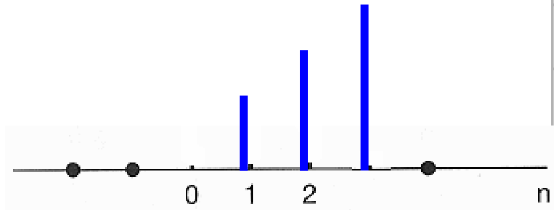
- Example 2.1:

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

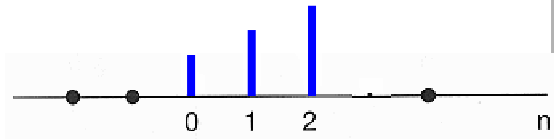




$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

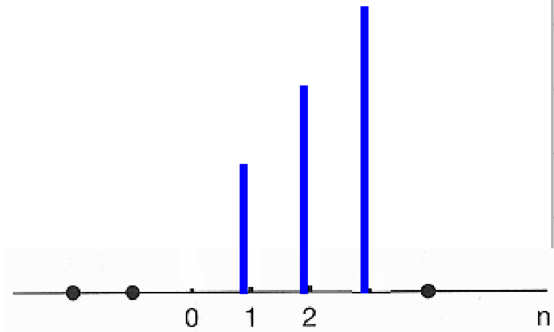


$$= \cdots + x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + \cdots$$



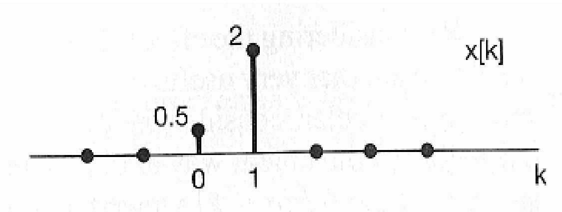
$$y[n] = x[0]h[n-0] + x[1]h[n-1]$$

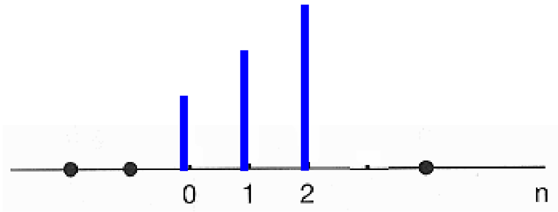
$$= 0.5h[n] + 2h[n-1]$$



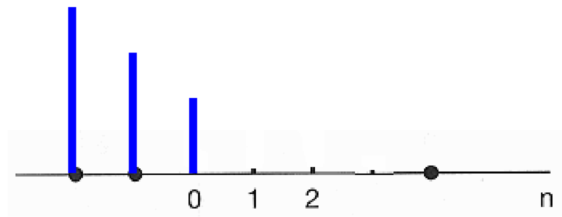
- **Example 2.2:** $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$



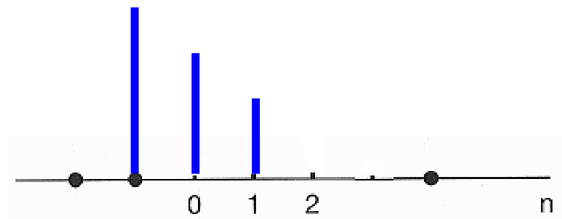


$$y[0] = \sum_{k=-\infty}^{+\infty} x[k]h[0-k]$$

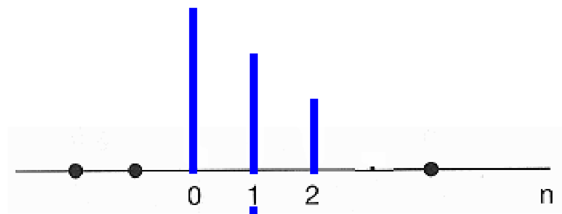


$$= \cdots + x[-1]h[1] + x[0]h[0] + x[1]h[-1] + x[2]h[-2] + \cdots = 0.5$$

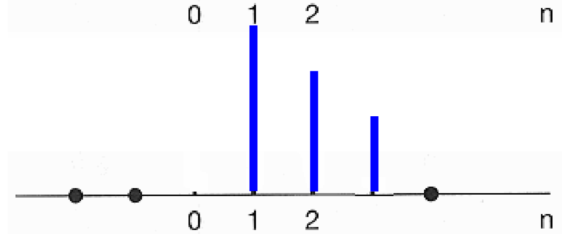
$$y[1] = \sum_{k=-\infty}^{+\infty} x[k]h[1-k]$$



$$= \cdots + x[-1]h[2] + x[0]h[1] + x[1]h[0] + x[2]h[-1] + \cdots = 4$$



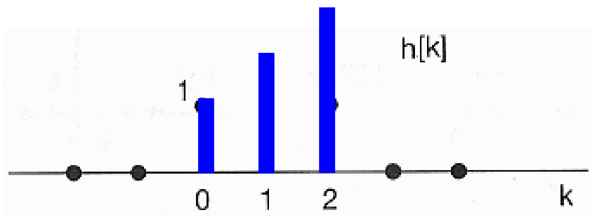
$$y[2] = \sum_{k=-\infty}^{+\infty} x[k]h[2-k] = 5.5$$

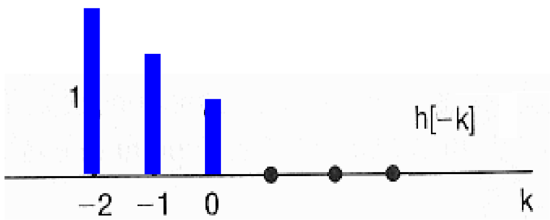


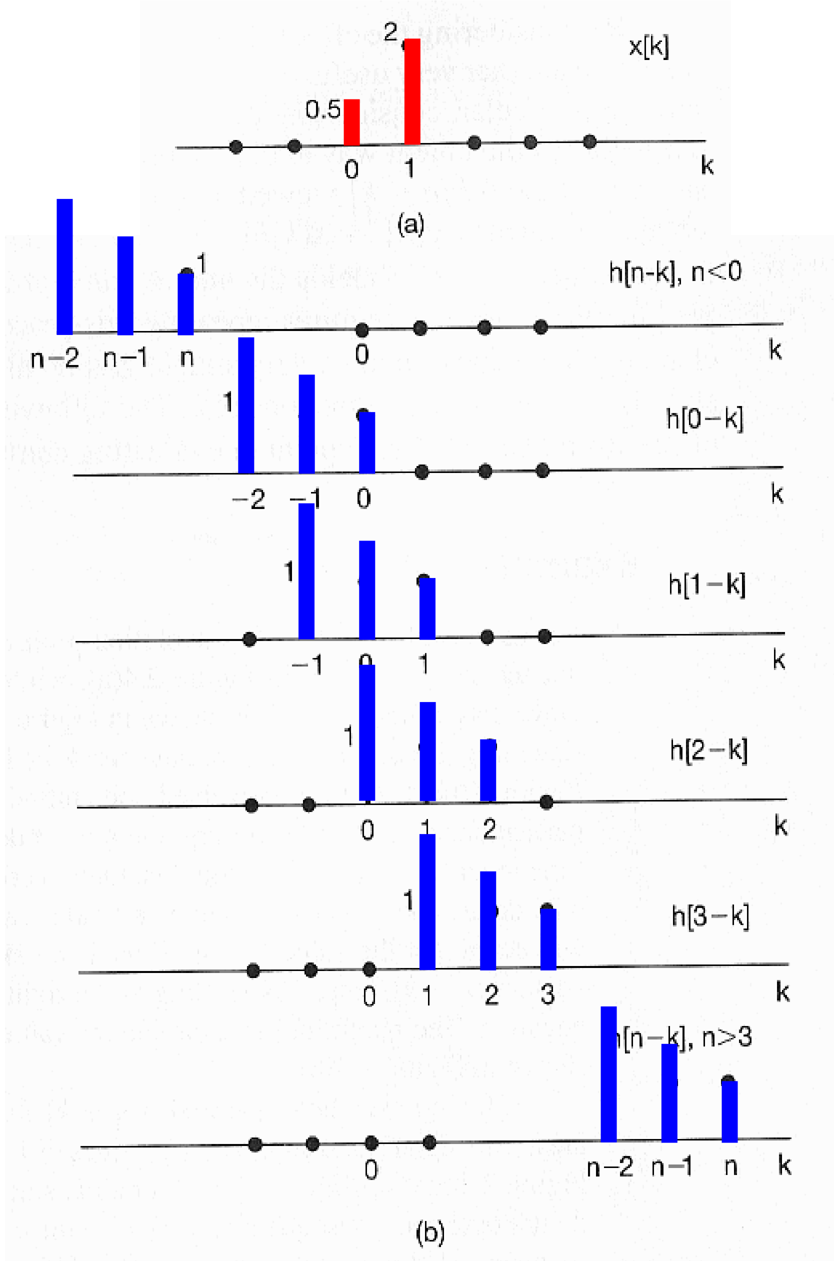
$$y[3] = \sum_{k=-\infty}^{+\infty} x[k]h[3-k] = 4.0$$

▪ Example 2.2:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

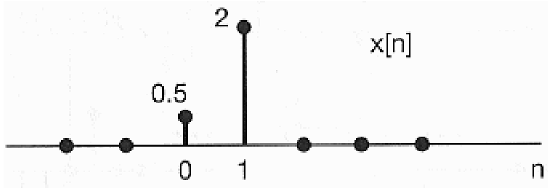


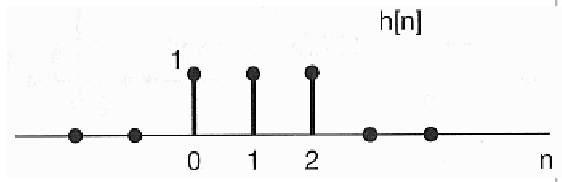




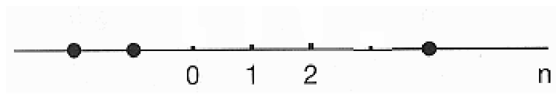
- Example 2.1:

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

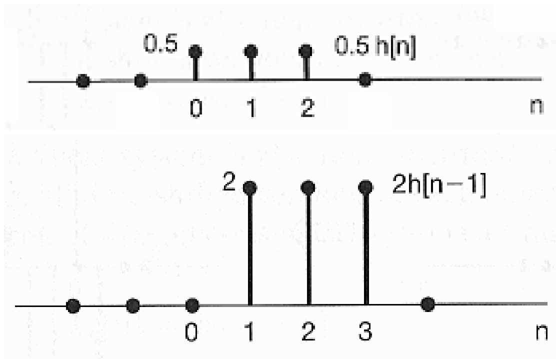




$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$



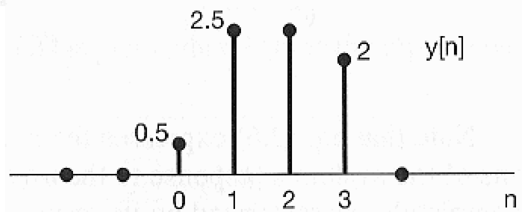
$$= \cdots + x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + \cdots$$



(b)

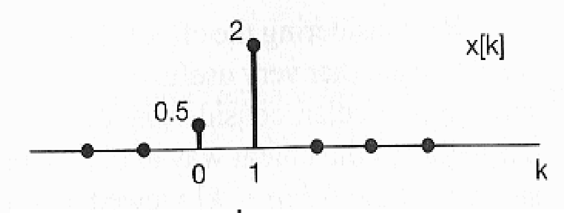
$$y[n] = x[0]h[n-0] + x[1]h[n-1]$$

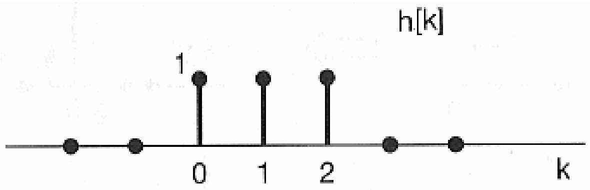
$$= 0.5h[n] + 2h[n-1]$$



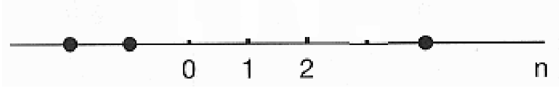
- Example 2.2: $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$

$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$





$$y[0] = \sum_{k=-\infty}^{+\infty} x[k]h[0-k]$$



$$= \cdots + x[-1]h[1] + x[0]h[0] + x[1]h[-1] + x[2]h[-2] + \cdots = 0.5$$

$$y[1] = \sum_{k=-\infty}^{+\infty} x[k]h[1-k] = 2.5$$

$$= \cdots + x[-1]h[2] + x[0]h[1] + x[1]h[0] + x[2]h[-1] + \cdots = 2.5$$

$$y[2] = \sum_{k=-\infty}^{+\infty} x[k]h[2-k] = 2.5$$

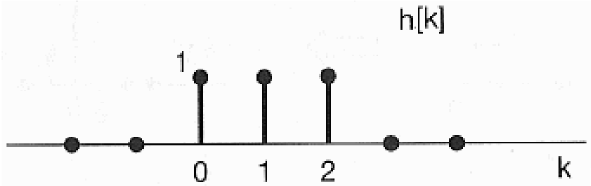
$$y[n] = 0 \text{ for } n < 0$$

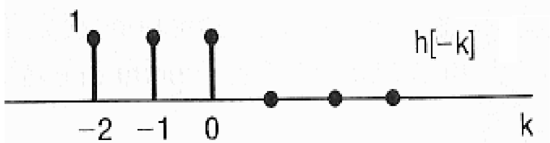
$$y[3] = \sum_{k=-\infty}^{+\infty} x[k]h[3-k] = 2.0$$

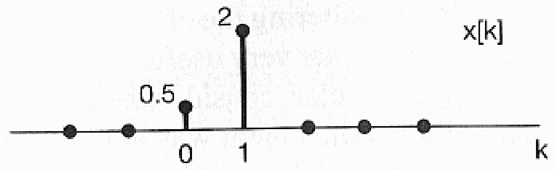
$$y[n] = 0 \text{ for } n > 3$$

- Example 2.2:

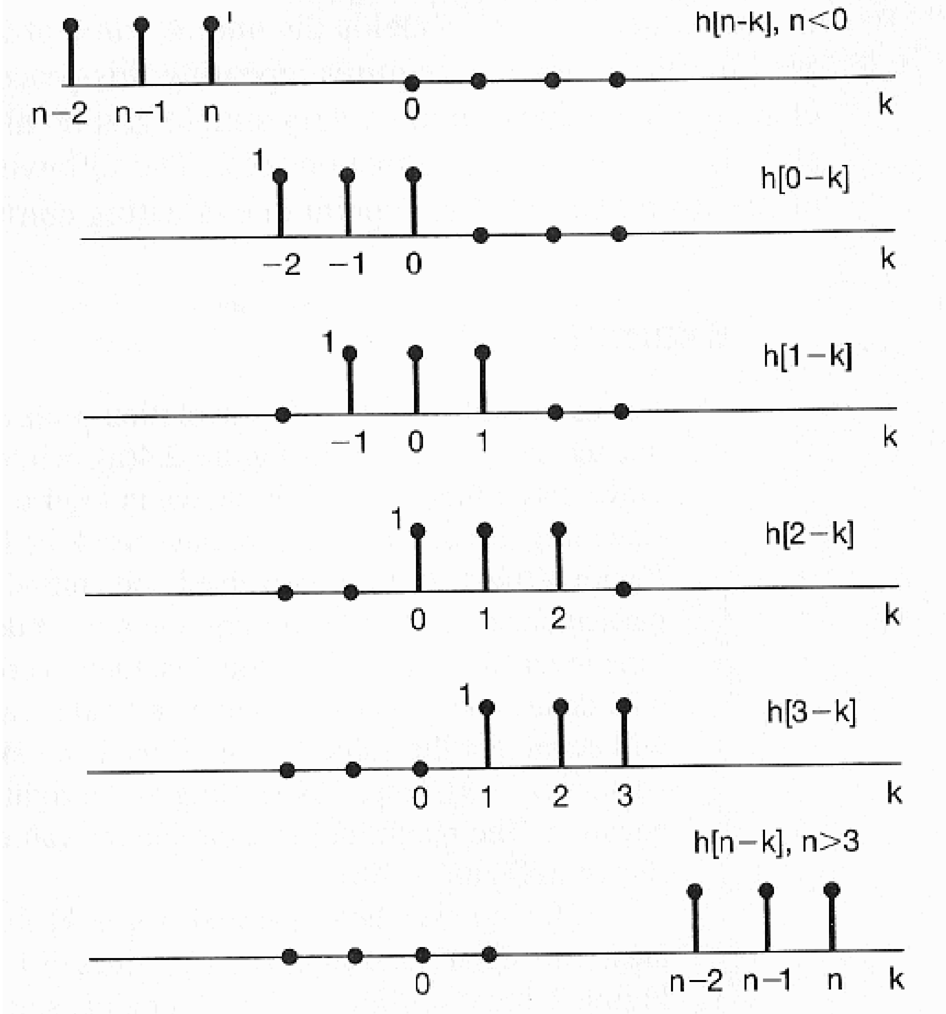
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$







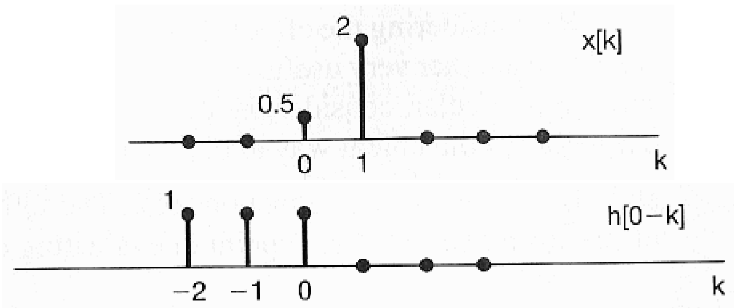
(a)

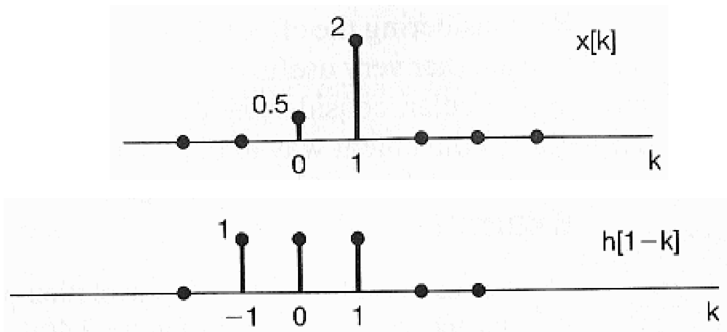


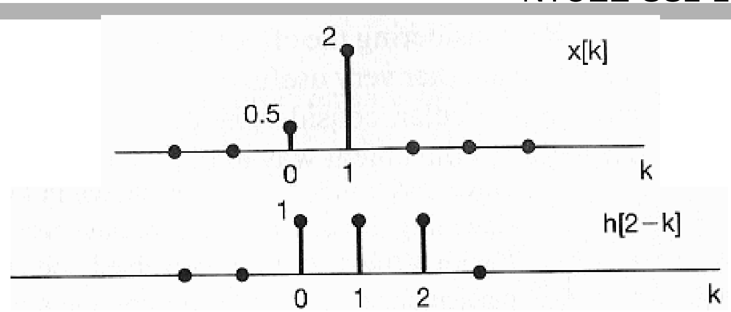
(b)

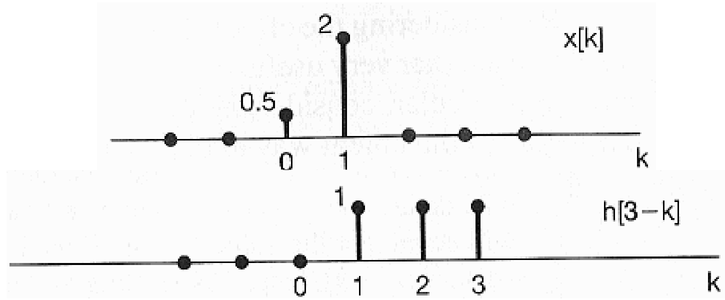
- Example 2.2:

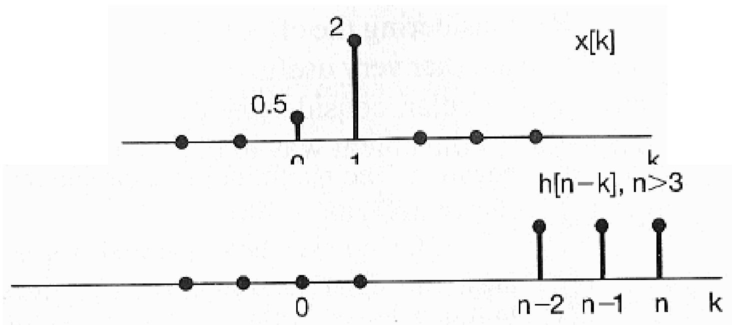
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

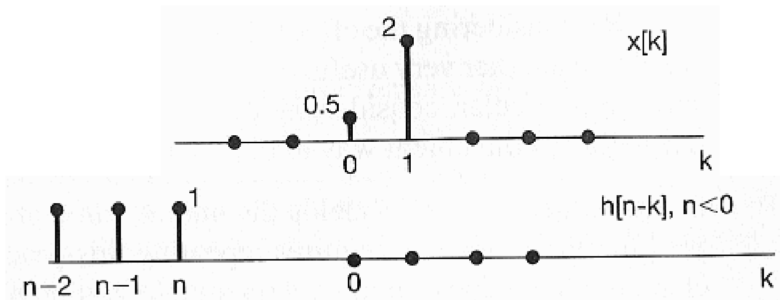




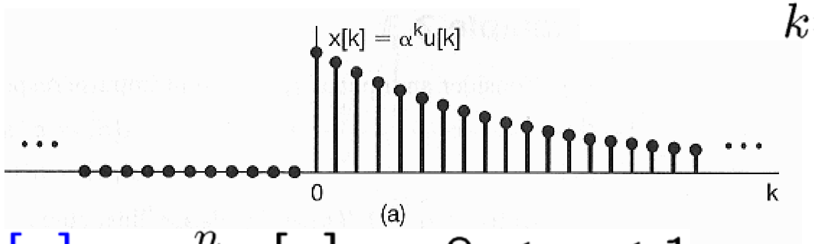




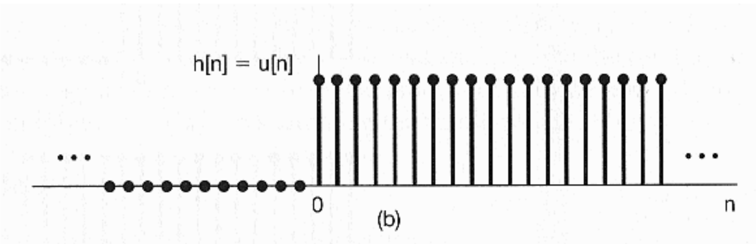




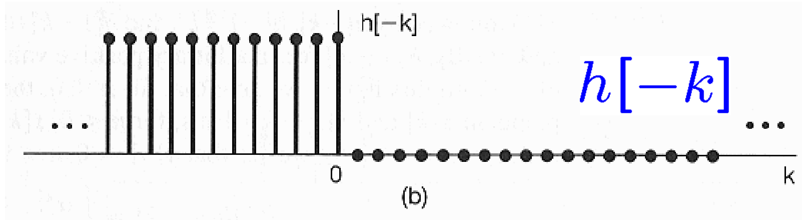
- Example 2.3: $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$



$x[n] = \alpha^n \, u[n], \quad 0 < \alpha < 1$

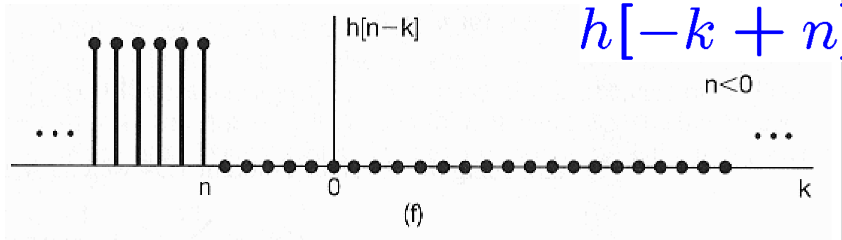


$h[n] = u[n]$



$h[-k]$

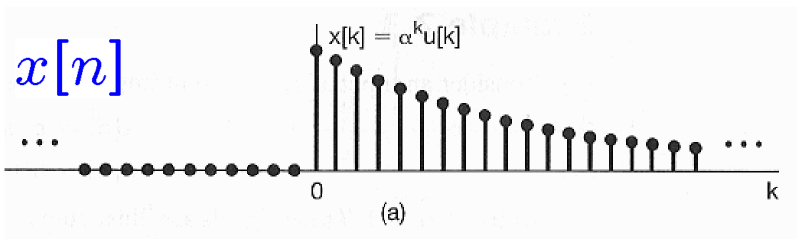
$n < 0$



$h[-k+n]$

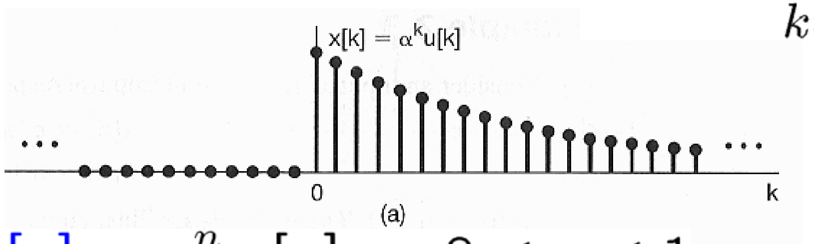
for $n < 0, \quad x[k] \; h[n-k] = 0$

$\Rightarrow \quad y[n] = 0$

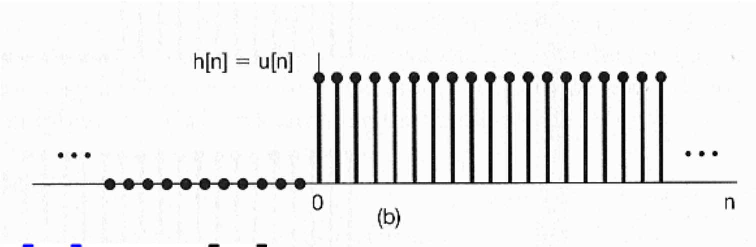


$x[n]$

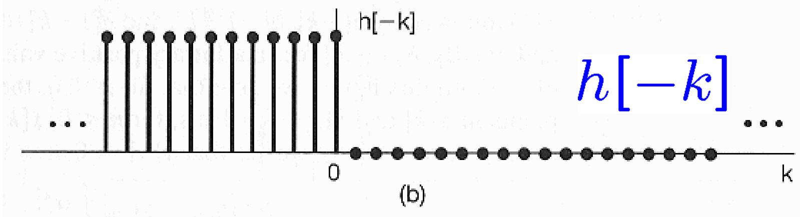
- Example 2.3: $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$



$x[n] = \alpha^n \, u[n], \quad 0 < \alpha < 1$



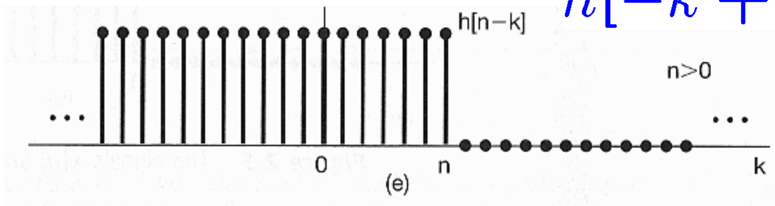
$h[n] = u[n]$



$h[-k]$

$n > 0$

$h[-k+n]$

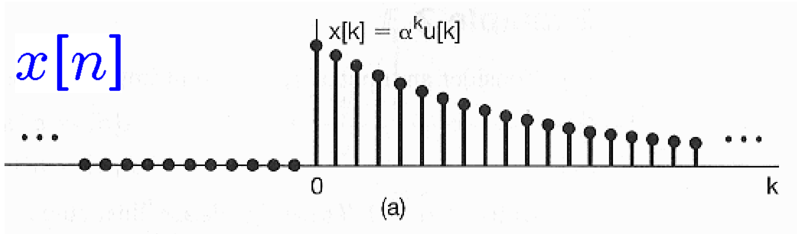


for $n \geq 0$,

$$x[k]\ h[n-k] = \begin{cases} \alpha^k, & 0 \leq k \leq n \\ 0, & \text{otherwise} \end{cases}$$

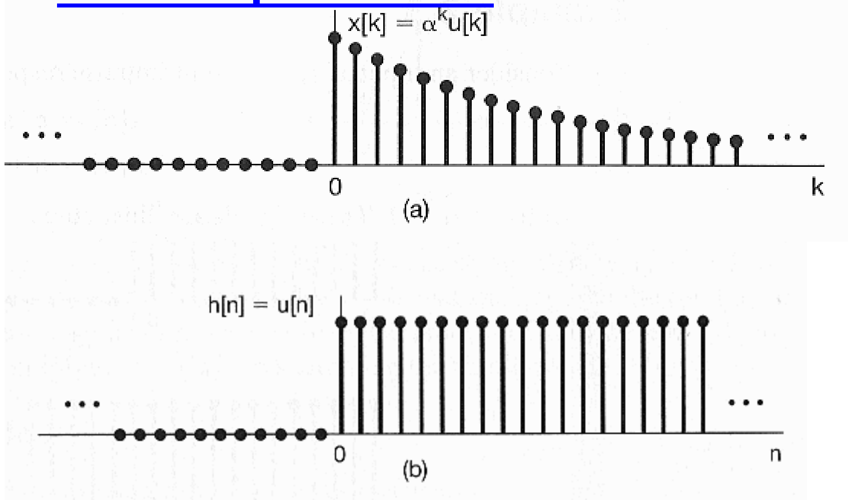
$$\Rightarrow \ y[n] = \sum_{k=0}^{n} \alpha^k \quad = \frac{1-\alpha^{n+1}}{1-\alpha}$$

$x[n]$



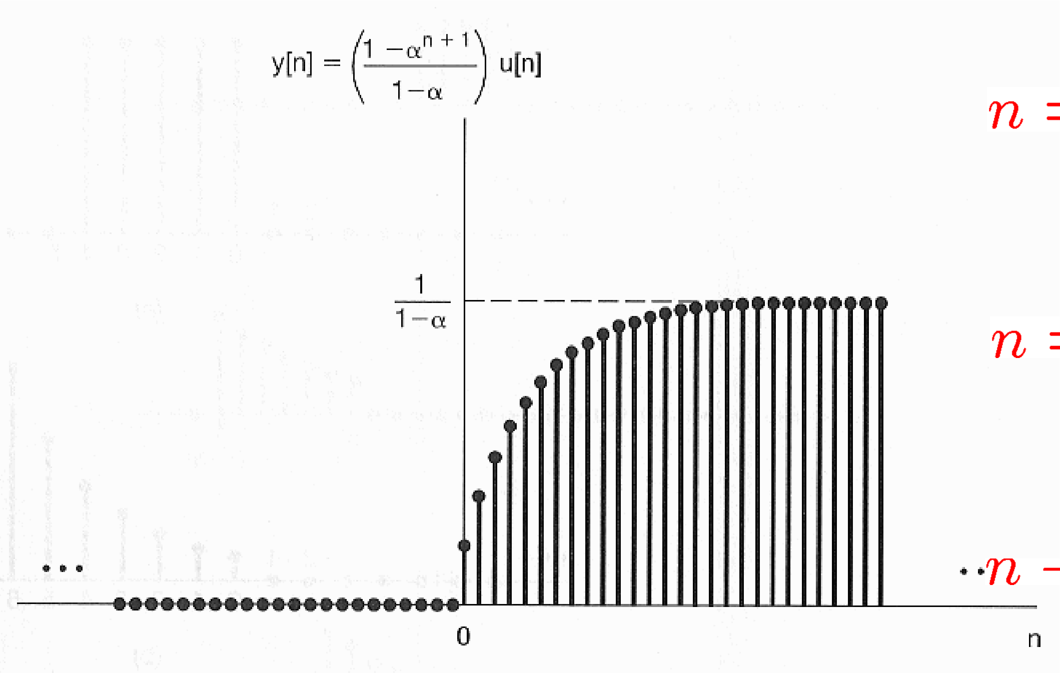
- Example 2.3:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$



for all n, $y[n] = \left(\dfrac{1-\alpha^{n+1}}{1-\alpha}\right) u[n]$

$$\alpha = \frac{7}{8}$$

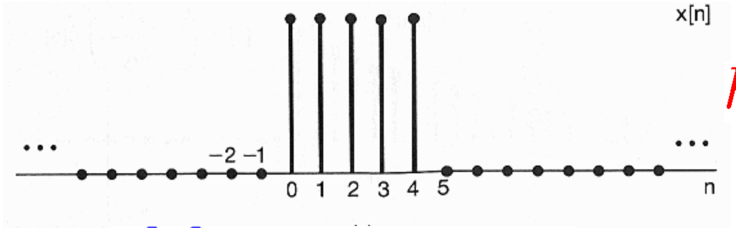


$n = 0$ $\quad y[0] = \dfrac{1-\frac{7}{8}}{1-\frac{7}{8}} = 1$

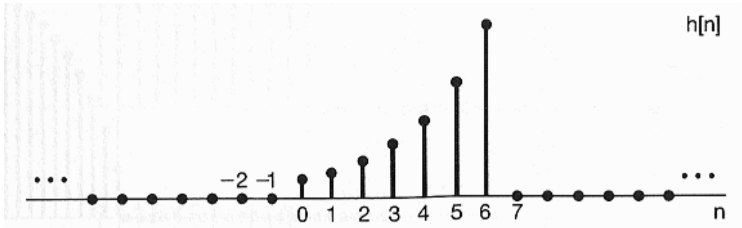
$n = 1$ $\quad y[1] = \dfrac{1-(\frac{7}{8})^2}{1-\frac{7}{8}} = \dfrac{15}{8}$

$n \to \infty$ $\quad y[n] = \dfrac{1-0}{1-\frac{7}{8}} = 8$

- Example 2.4:

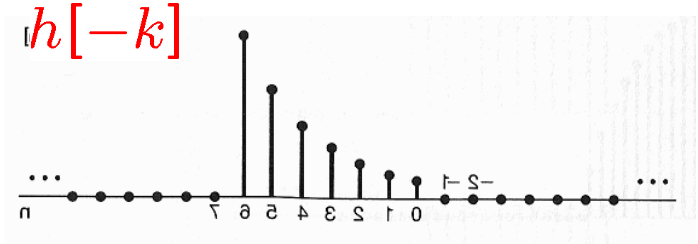


$x[n] = 1, \ 0 \leq n \leq 4$



$h[n] = \alpha^n, \ 0 \leq n \leq 6$

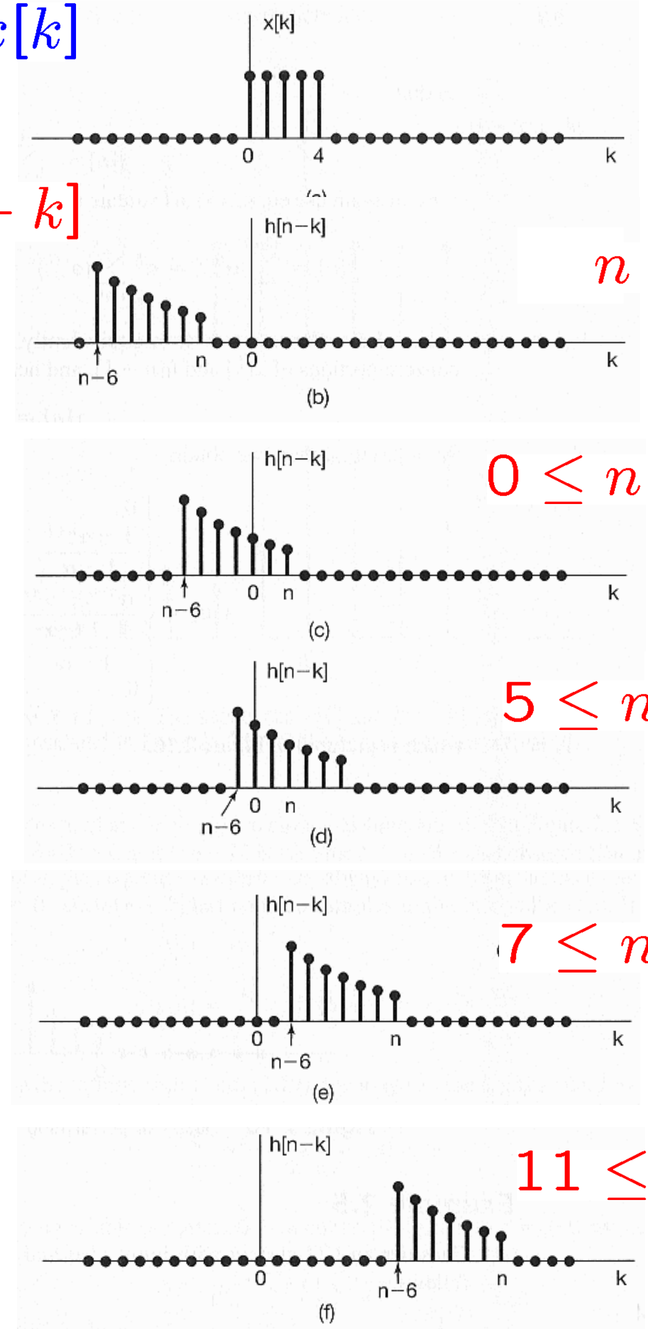
$h[-k]$



(d)

$x[k]$

$h[n-k]$



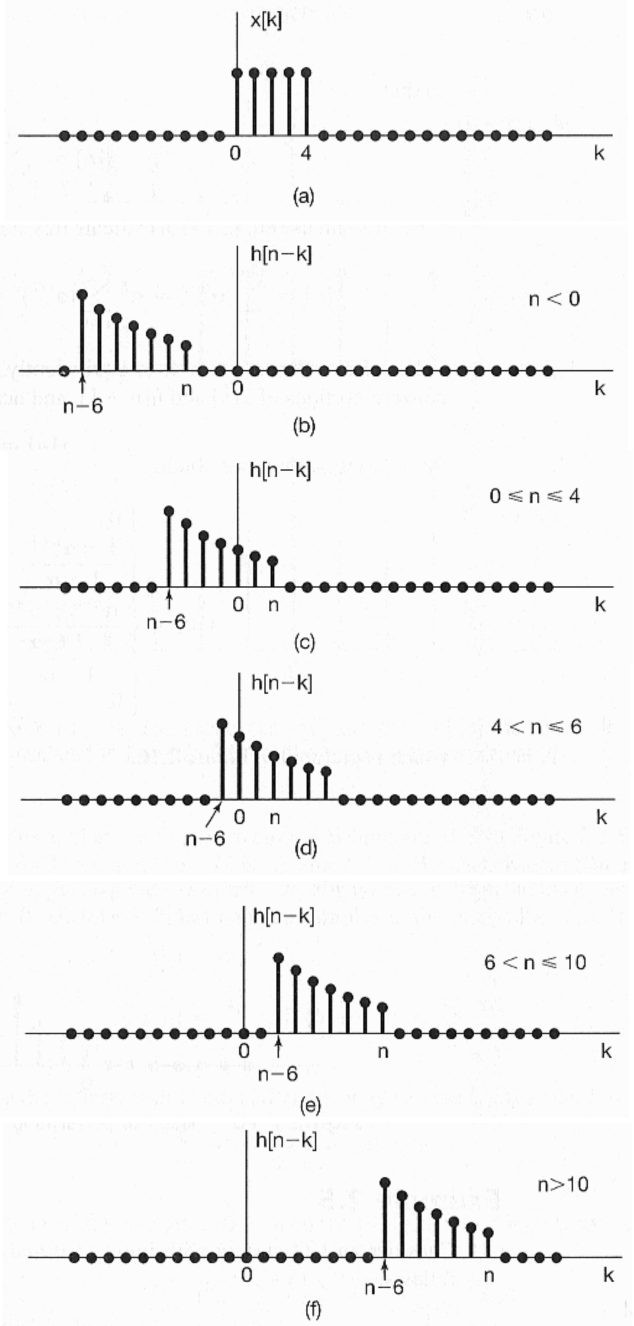
$n \leq -1$

$0 \leq n \leq 4$

$5 \leq n \leq 6$

$7 \leq n \leq 10$

$11 \leq n$



for $n < 0$, $\quad x[k]\, h[n-k] = 0 \quad \Rightarrow \quad y[n] = 0$

for $0 \le n \le 4$, $\quad x[k]\, h[n-k] = \begin{cases} \alpha^{n-k}, & 0 \le k \le n \\ 0, & \text{otherwise} \end{cases}$

$$\Rightarrow y[n] = \sum_{k=0}^{n} \alpha^{n-k} \qquad = \frac{1-\alpha^{n+1}}{1-\alpha}$$

for $4 < n \le 6$, $\quad x[k]\, h[n-k] = \begin{cases} \alpha^{n-k}, & 0 \le k \le 4 \\ 0, & \text{otherwise} \end{cases}$

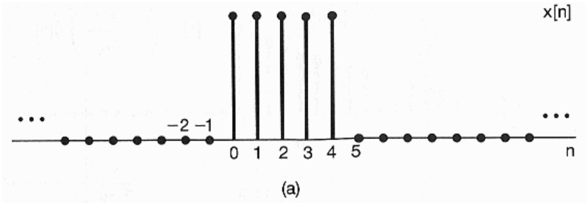
$$\Rightarrow y[n] = \sum_{k=0}^{4} \alpha^{n-k} \qquad = \frac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha}$$

for $6 < n \le 10$, $\quad x[k]\, h[n-k] = \begin{cases} \alpha^{n-k}, & (n-6) \le k \le 4 \\ 0, & \text{otherwise} \end{cases}$

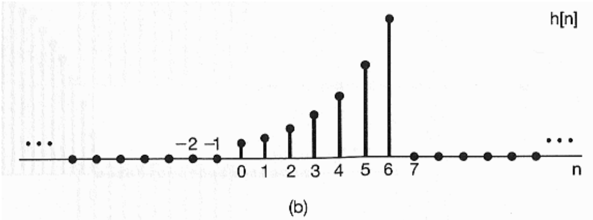
$$\Rightarrow y[n] = \sum_{k=n-6}^{4} \alpha^{n-k} \qquad = \frac{\alpha^{n-4}-\alpha^{7}}{1-\alpha}$$

for $n > 10$, $\quad y[n] = 0$

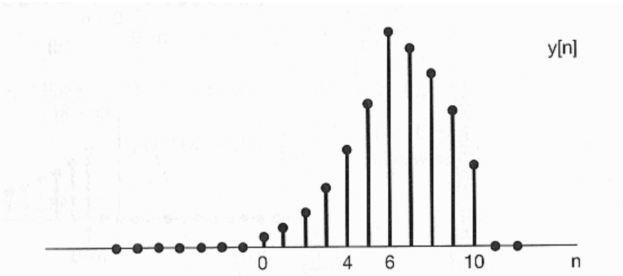
$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$



$$x[n] = 1, \ 0 \le n \le 4$$



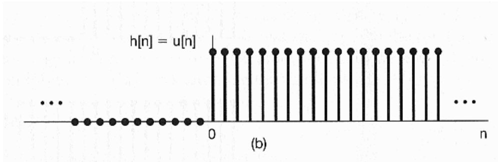
$$h[n] = \alpha^n, \ 0 \le n \le 6$$



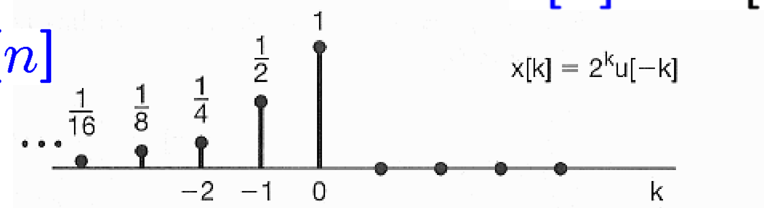
$$y[n] = \begin{cases} 0, & n < 0 \\ \frac{1-\alpha^{n+1}}{1-\alpha}, & 0 \le n \le 4 \\ \frac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha}, & 4 < n \le 6 \\ \frac{\alpha^{n-4}-\alpha^{7}}{1-\alpha}, & 6 < n \le 10 \\ 0, & 10 < n \end{cases}$$

- **Example 2.5:** $x[n] = 2^n u[-n]$ $\qquad x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$

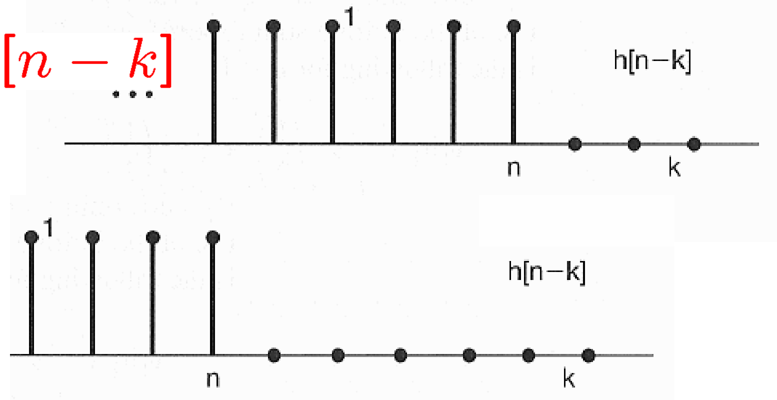
$h[n] = u[n]$



$x[n]$



$h[n-k]$



for $n \geq 0$, $\quad y[n] = \sum_{k=-\infty}^{0} x[k]\, h[n-k] = \sum_{k=-\infty}^{0} 2^k$

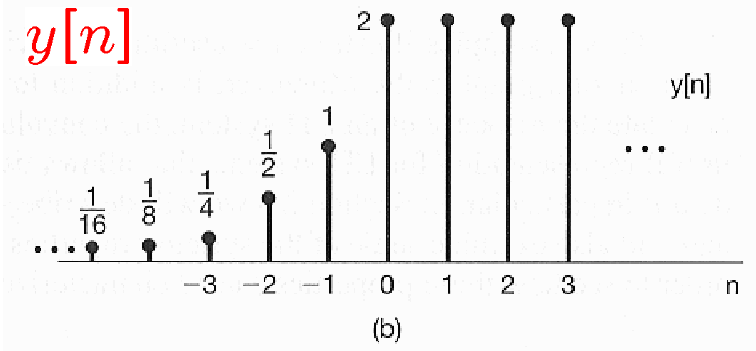
$$= \sum_{r=0}^{\infty} \left(\frac{1}{2}\right)^r = \frac{1}{1-(1/2)} = 2$$

for $n < 0$, $\quad y[n] = \sum_{k=-\infty}^{n} x[k]\, h[n-k] = \sum_{k=-\infty}^{n} 2^k$

$$= \sum_{l=-n}^{\infty} \left(\frac{1}{2}\right)^l = \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{m-n}$$

$$= \left(\frac{1}{2}\right)^{-n} \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m = 2^n \cdot 2 = 2^{n+1}$$

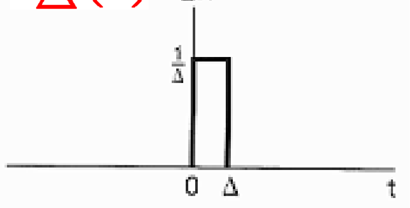
$y[n]$



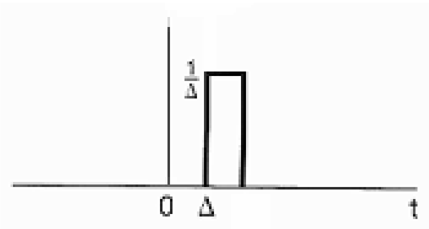
- Discrete-Time Linear Time-Invariant Systems
  - The convolution sum $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$ $\quad y[n] = x[n] * h[n]$
- Continuous-Time Linear Time-Invariant Systems
  - The convolution integral
- Properties of Linear Time-Invariant Systems
- Causal Linear Time-Invariant Systems Described by Differential & Difference Equations
- Singularity Functions

▪ Representation of CT Signals by Impulses:

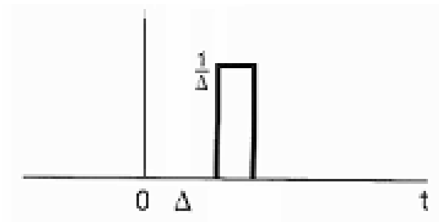
$\delta_\Delta(t)$



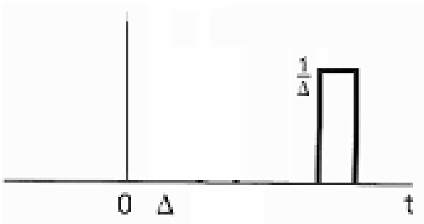
$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \leq t < \Delta \\ 0, & \text{otherwise} \end{cases}$$



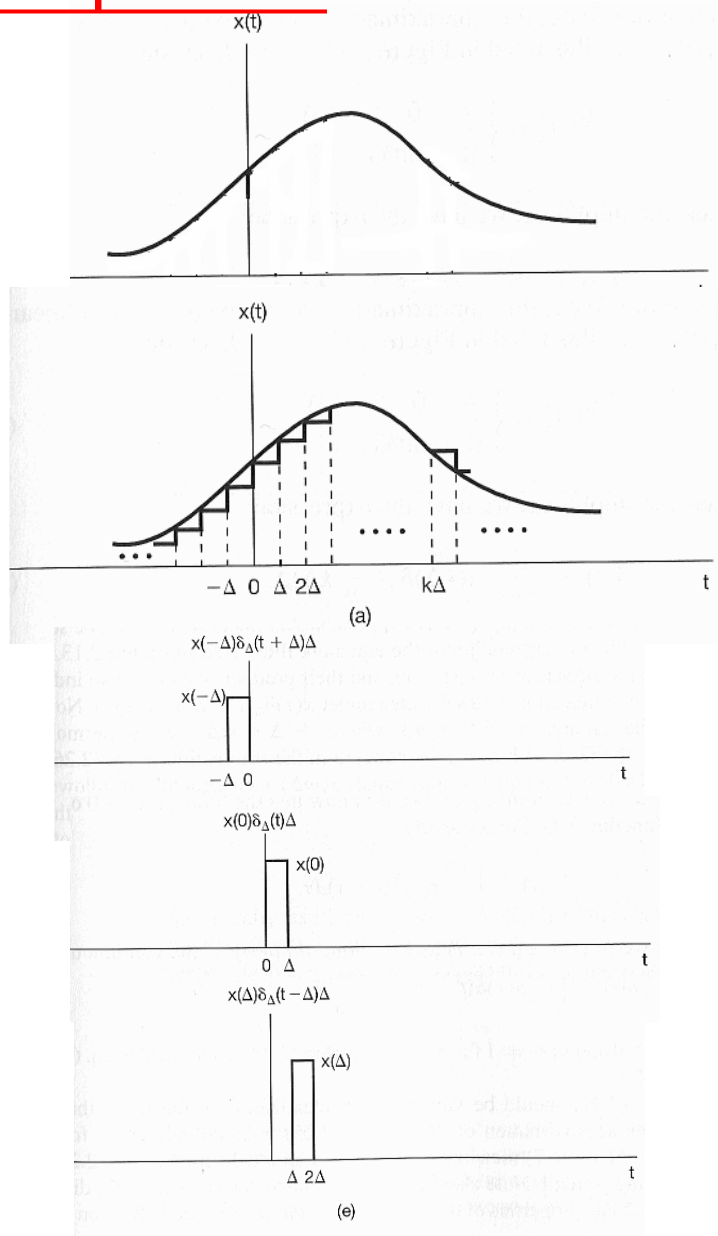
$\delta_\Delta(t-\Delta)$



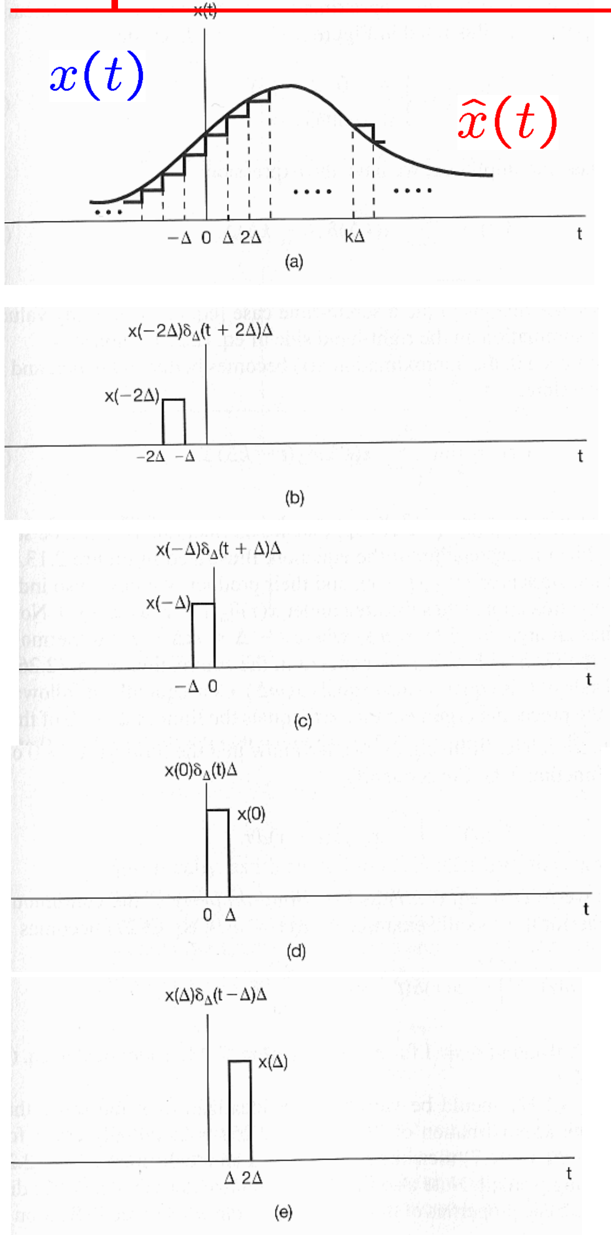
$\delta_\Delta(t-2\Delta)$



$\delta_\Delta(t-k\Delta)$



Representation of CT Signals by Impulses:



$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta$$

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta$$

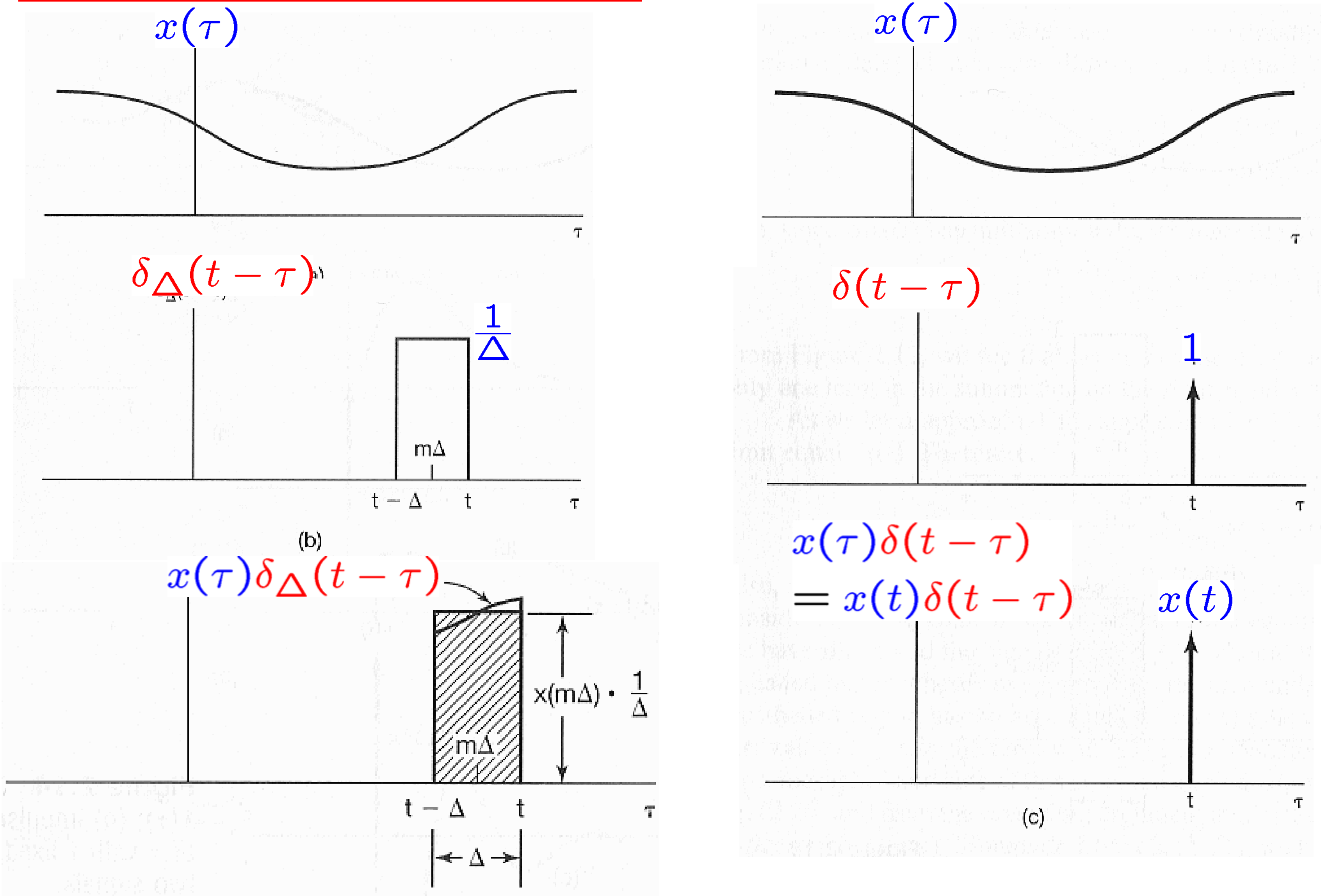
$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t - \tau)d\tau$$

the sifting property of CT impulse

$x(t) =$ an integral of weighted,

shifted impulses

- Graphical interpretation:



▪ CT Impulse Response & Convolution Integral:

$$\text{input} \longrightarrow \boxed{\text{Linear System}} \longrightarrow \text{output}$$

$$\delta_\Delta(t) \longrightarrow \boxed{\text{Linear System}} \longrightarrow \hat{h}_{0\Delta}(t)$$

$$\delta_\Delta(t-1\Delta) \longrightarrow \boxed{\text{Linear System}} \longrightarrow \hat{h}_{1\Delta}(t)$$

$$\delta_\Delta(t-2\Delta) \longrightarrow \boxed{\text{Linear System}} \longrightarrow \hat{h}_{2\Delta}(t)$$

$$\vdots$$

$$\delta_\Delta(t-k\Delta) \longrightarrow \boxed{\text{Linear System}} \longrightarrow \hat{h}_{k\Delta}(t)$$

▪ CT Impulse Response & Convolution Integral:

$$\text{input} \longrightarrow \boxed{\text{Linear System}} \longrightarrow \text{output}$$

$$x(0\Delta) \qquad \delta_{\Delta}(t) \longrightarrow \boxed{\text{Linear System}} \longrightarrow \hat{h}_{0\Delta}(t) \qquad x(0\Delta)$$

$$x(1\Delta) \qquad \delta_{\Delta}(t-1\Delta) \longrightarrow \boxed{\text{Linear System}} \longrightarrow \hat{h}_{1\Delta}(t) \qquad x(1\Delta)$$

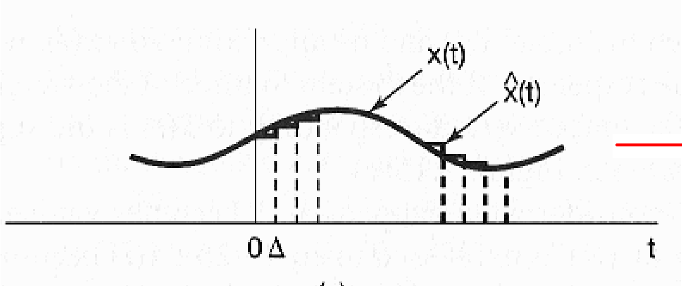
$$x(2\Delta) \qquad \delta_{\Delta}(t-2\Delta) \longrightarrow \boxed{\text{Linear System}} \longrightarrow \hat{h}_{2\Delta}(t) \qquad x(2\Delta)$$

$$\vdots$$

$$x(k\Delta) \qquad \delta_{\Delta}(t-k\Delta) \longrightarrow \boxed{\text{Linear System}} \longrightarrow \hat{h}_{k\Delta}(t) \qquad x(k\Delta)$$

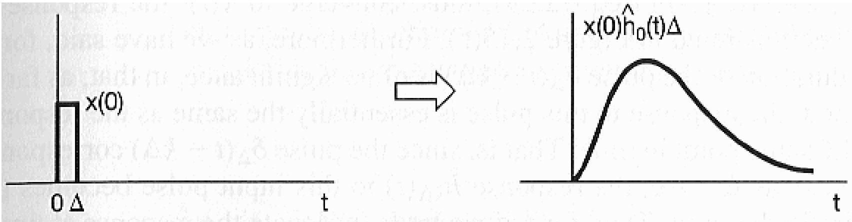
$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_{\Delta}(t-k\Delta)\Delta \implies \hat{y}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\,\hat{h}_{k\Delta}(t)\,\Delta$$

CT LTI Systems: Convolution Integral

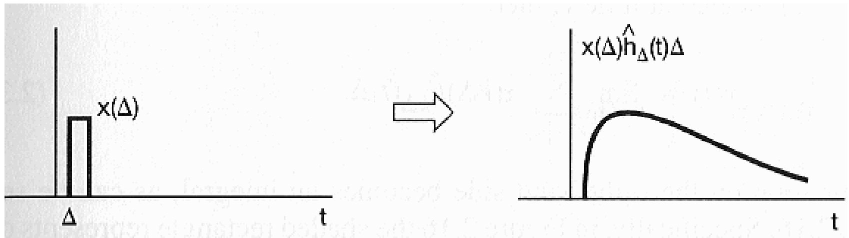


(a)

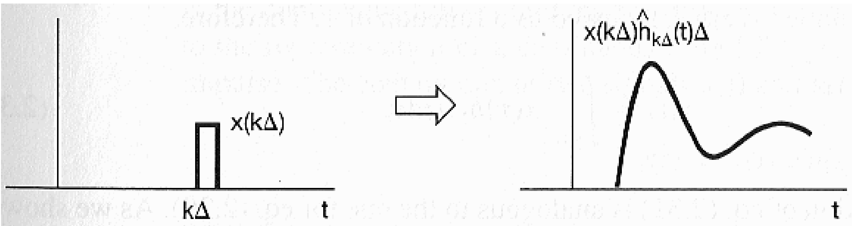
Linear System



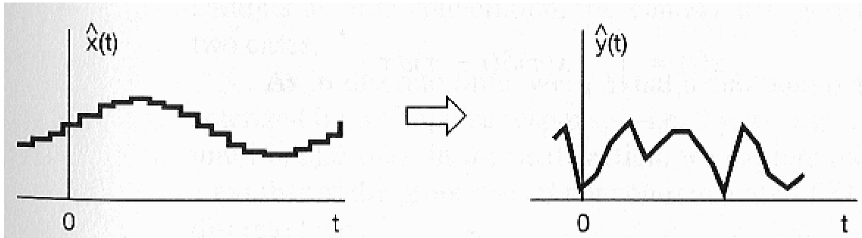
(b)



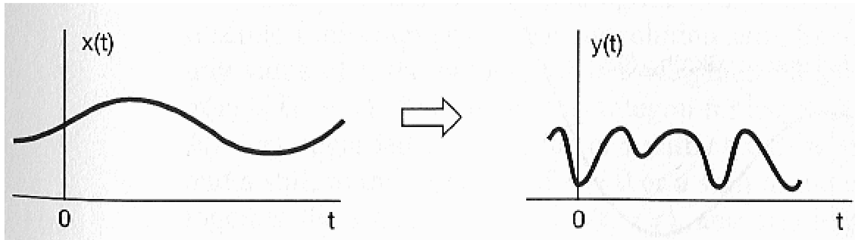
(c)



(d)



(e)



(f)

▪ CT Unit Impulse Response & Convolution Integral:

$$\hat{y}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\, \hat{h}_{k\Delta}(t)\, \Delta$$

$$y(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\, \hat{h}_{k\Delta}(t)\, \Delta$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\;\; h_\tau(t)\;\; d\tau$$

$$\delta(t-\tau) \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_\tau(t)$$

$$x(t) \longrightarrow \boxed{\text{Linear System}} \longrightarrow y(t)$$

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau \quad \Longrightarrow \quad y(t) = \int_{-\infty}^{+\infty} x(\tau)h_\tau(t)d\tau$$

- If the linear system (L) is also time-invariant (TI)
  - Then,

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$h_\tau(t) = h_0(t-\tau) = h(t-\tau)$$

- Hence, for an LTI system,

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \qquad = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$$

  - Known as the convolution of x(t) & h(t)
  - Referred as the convolution integral or the superposition integral
- Symbolically,

$$y(t) = x(t) * h(t) \qquad = h(t) * x(t)$$

- Example 2.6: $y(t) = \int_{-\infty}^{+\infty} x(\tau) h(t-\tau) d\tau$

$h(t) = u(t)$

$h(\tau)$

$x(t) = e^{-at}\, u(t)$

$a > 0$

$h(t-\tau)$

$t < 0$

$t > 0$

for $t < 0$, $x(\tau)\, h(t-\tau) = 0$

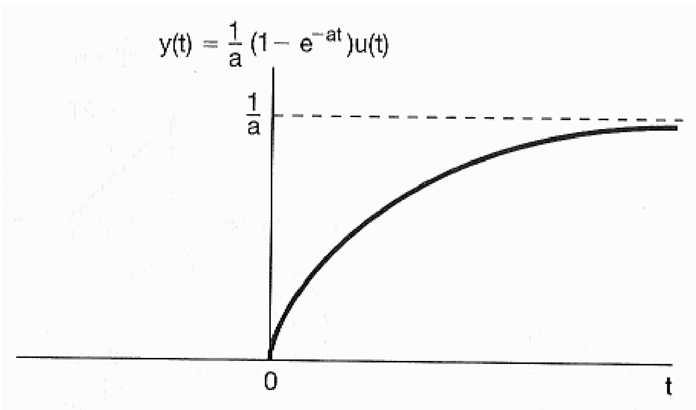
$$\Rightarrow \quad y(t) = \int_{-\infty}^{t} 0 \, d\tau = 0$$

for $t \geq 0$, $x(\tau)\, h(t-\tau) = \begin{cases} e^{-a\tau}, & 0 \leq \tau \leq t \\ 0, & \text{otherwise} \end{cases}$

$$\Rightarrow \quad y(t) = \int_{0}^{t} e^{-a\tau} d\tau$$

$$= -\frac{1}{a} e^{-a\tau} \Big|_{0}^{t}$$

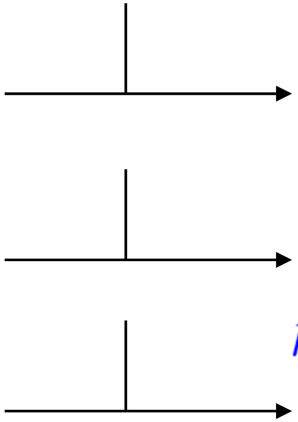
$$= \frac{1}{a}(1 - e^{-at})$$



- **Example 2.7:** $y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$

$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{otherwise} \end{cases}$$

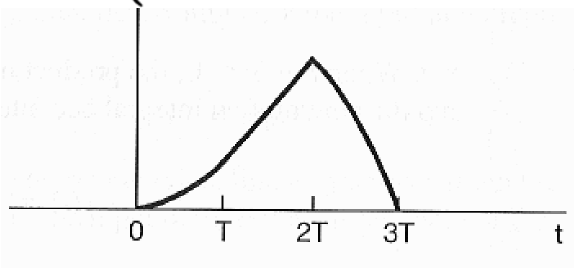
$$h(t) = \begin{cases} t, & 0 < t < 2T \\ 0, & \text{otherwise} \end{cases}$$



$h(\tau)$

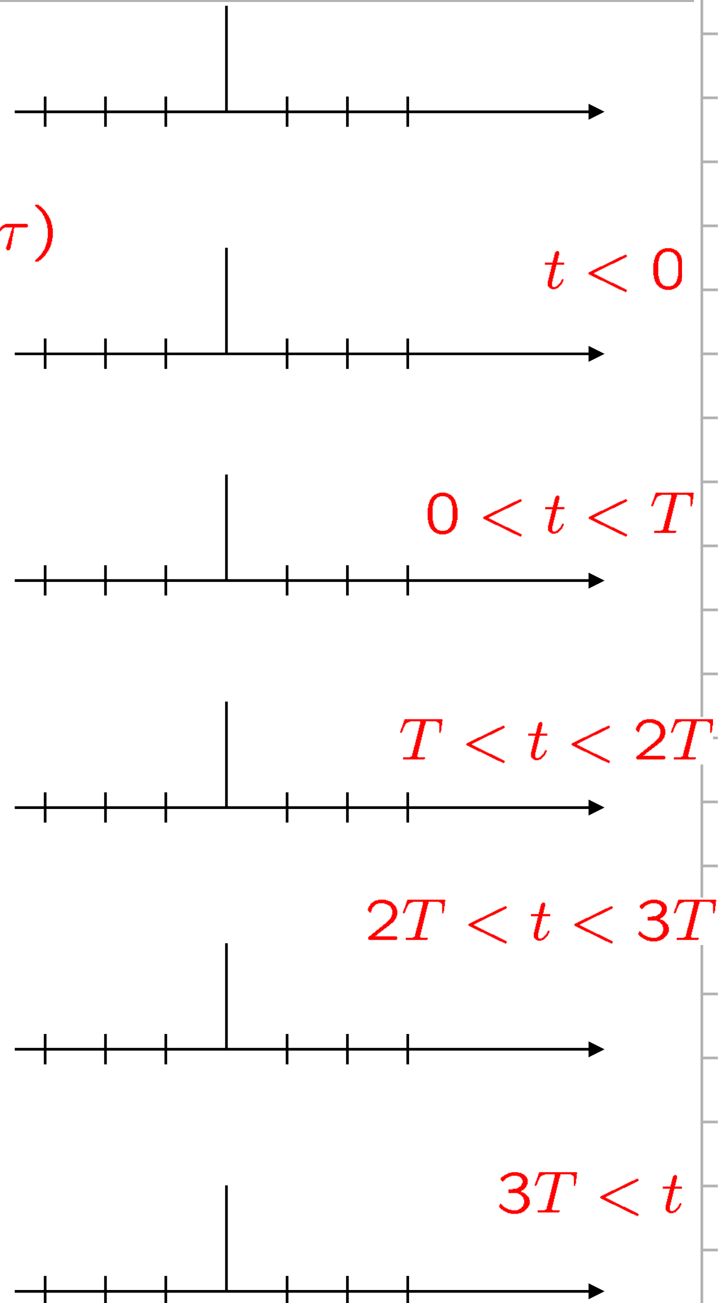
$h(-\tau)$

$$y(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2}t^2, & 0 < t < T \\ Tt - \frac{1}{2}T^2, & T < t < 2T \\ -\frac{1}{2}t^2 + Tt + \frac{3}{2}T^2, & 2T < t < 3T \\ 0, & 3T < t \end{cases}$$



$x(\tau)$

$h(t-\tau)$

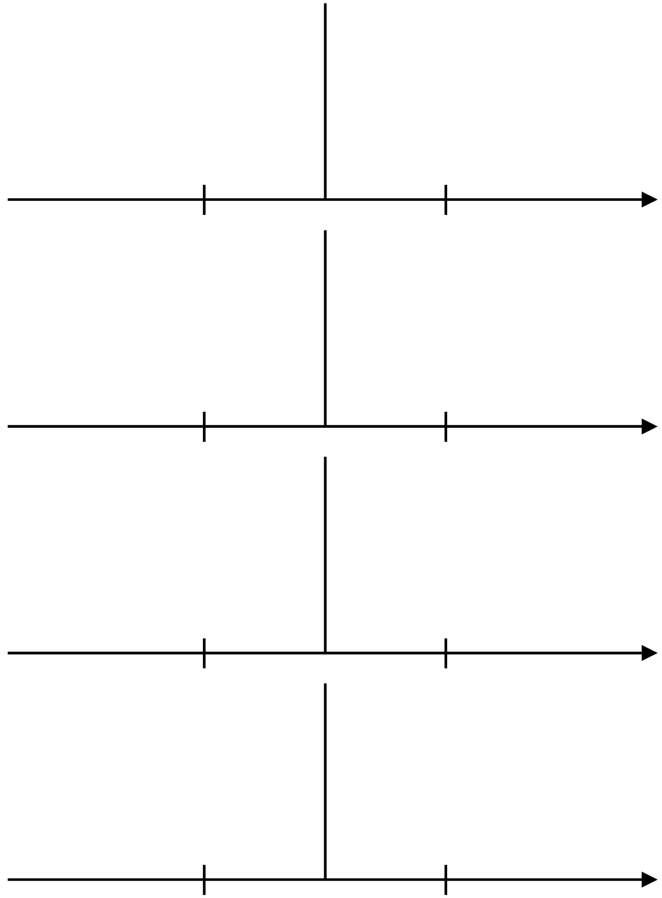


- **Example 2.8:** $x(t) = e^{2t}u(-t)$

$$h(t) = u(t-3)$$

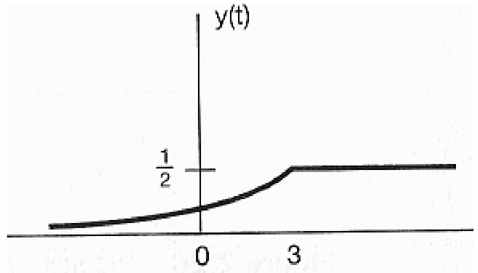
$$h(-\tau)$$

$$h(t-\tau)$$



for $t-3 \leq 0$, $\quad y(t) = \int_{-\infty}^{t-3} e^{2\tau} d\tau \qquad = \frac{1}{2}e^{2(t-3)}$

for $t-3 \geq 0$, $\quad y(t) = \int_{-\infty}^{0} e^{2\tau} d\tau \qquad = \frac{1}{2}$



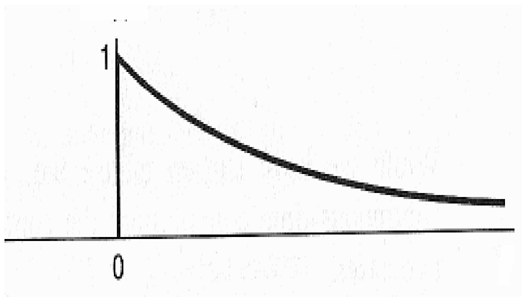
▪ Signal and System..

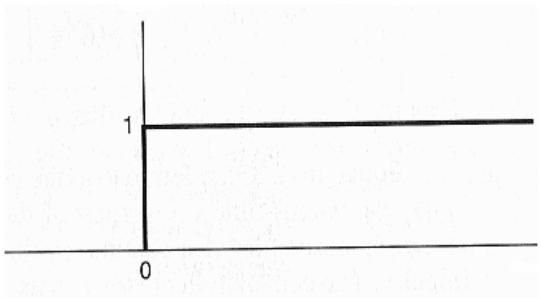
$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

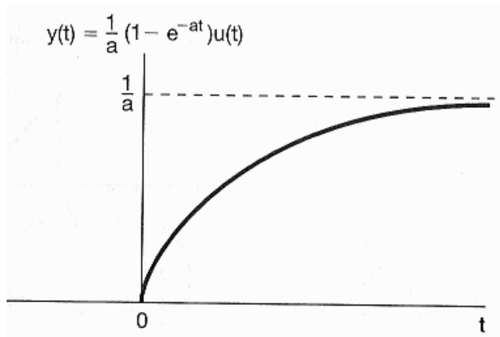
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=-\infty}^{+\infty} x[n-k]h[k] = x(t) * h(t)$$

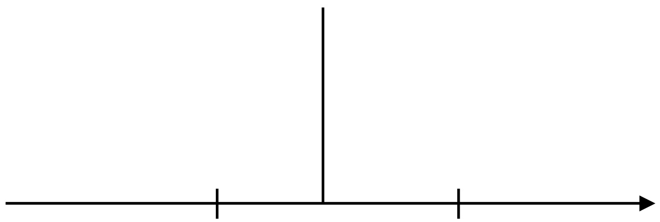
$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau = x[n] * h[n]$$

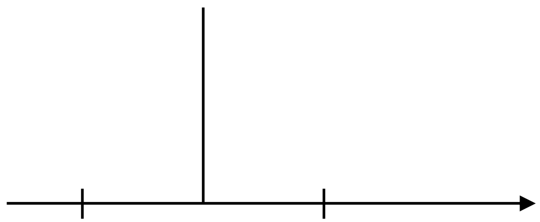
$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$











- Discrete-Time Linear Time-Invariant Systems
  - The convolution sum $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$ $\quad y[n] = x[n] * h[n]$
- Continuous-Time Linear Time-Invariant Systems
  - The convolution integral $y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$ $\quad y(t) = x(t) * h(t)$
- Properties of Linear Time-Invariant Systems
- Causal Linear Time-Invariant Systems Described by Differential & Difference Equations
- Singularity Functions

- Convolution Sum & Integral of LTI Systems:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = x[k] * h[n]$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$$

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$

$$y[n] = x[k] * h[n]$$
$$y(t) = x(t) * h(t)$$

- Properties of LTI Systems

  1. Commutative property
  2. Distributive property
  3. Associative property
  4. With or without memory
  5. Invertibility
  6. Causality
  7. Stability
  8. Unit step response

$$a \times b = b \times a$$
$$a + b = b + a$$
$$a \times (b + c) = a \times b + a \times c$$
$$a \times (b \times c) = (a \times b) \times c$$
$$= \cdots = a \times b \times c$$

$$x[n] \rightarrow \boxed{\text{h[n]}} \rightarrow y[n]$$
$$x(t) \rightarrow \boxed{\text{h(t)}} \rightarrow y(t)$$

$$\forall x[n] \quad \rightarrow \quad \forall y[n] \qquad\qquad h[n] = ?$$

- Commutative Property: $n - k = r$

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \qquad = \sum_{r=+\infty}^{-\infty} x[n-r]h[r]$$

$$= \sum_{r=-\infty}^{+\infty} h[r]x[n-r] \qquad = h[n] * x[n]$$

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \qquad \begin{aligned} t - \tau &= \sigma \\ -d\tau &= d\sigma \end{aligned}$$

$$= \int_{+\infty}^{-\infty} x(t-\sigma)h(\sigma)(-d\sigma) \qquad = \int_{-\infty}^{+\infty} x(t-\sigma)h(\sigma)d\sigma$$

$$= \int_{-\infty}^{+\infty} h(\sigma)x(t-\sigma)d\sigma \qquad = h(t) * x(t)$$

 $\boxed{a \times (b + c) \;=\; a \times b + a \times c}$

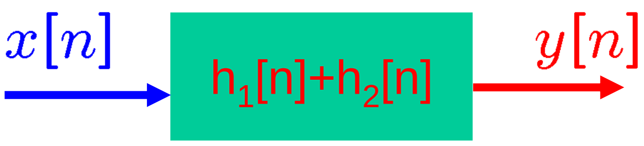
- Distributive Property:

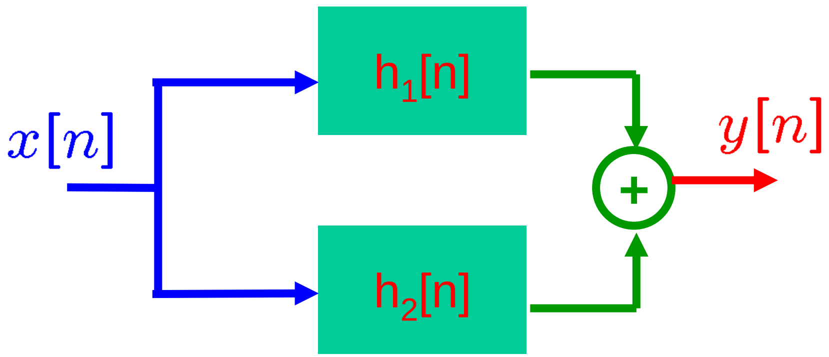
$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$x[n] * \Big(h_1[n] + h_2[n]\Big) = x[n] * h_1[n] + x[n] * h_2[n]$$

$$x(t) * \Big(h_1(t) + h_2(t)\Big) = x(t) * h_1(t) + x(t) * h_2(t)$$





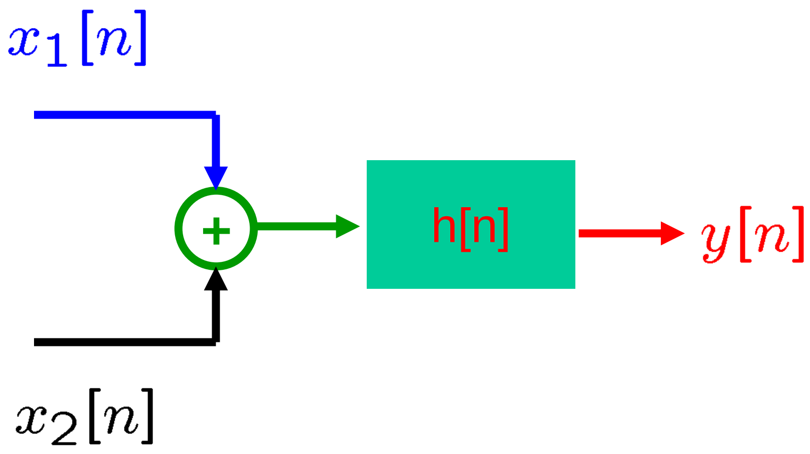
- Distributive Property:

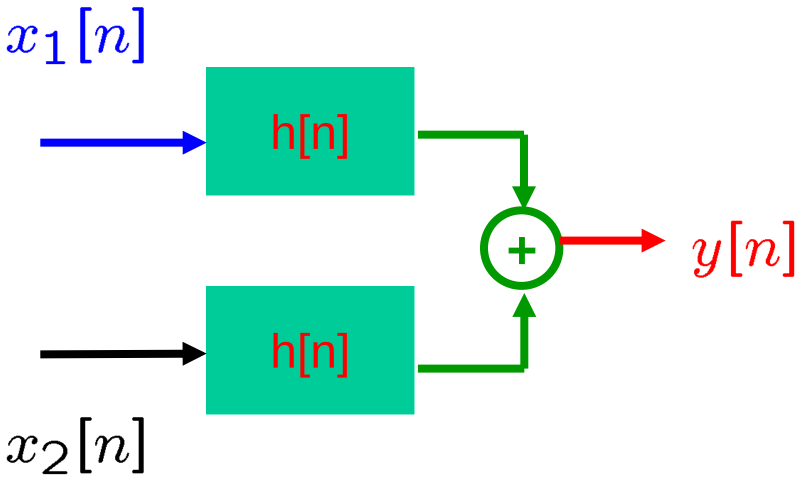
$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$\Big(x_1[n] + x_2[n]\Big) * h[n] = x_1[n] * h[n] + x_2[n] * h[n]$$

$$\Big(x_1(t) + x_2(t)\Big) * h(t) = x_1(t) * h(t) + x_2(t) * h(t)$$

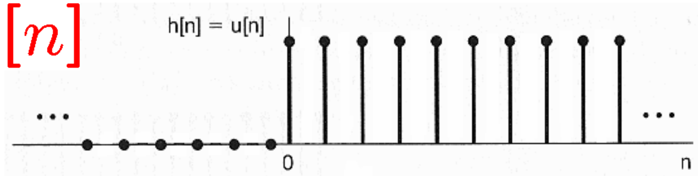


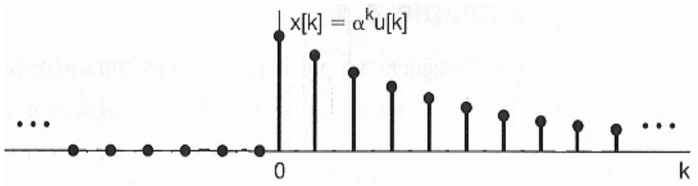


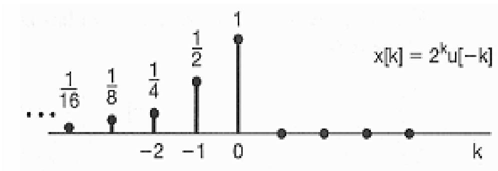
- Example 2.10

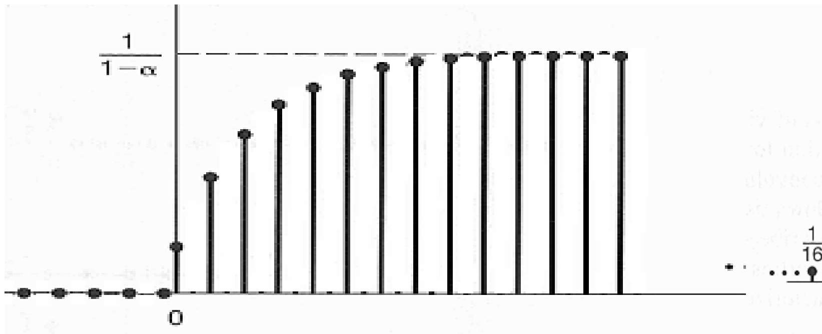
$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n]$$

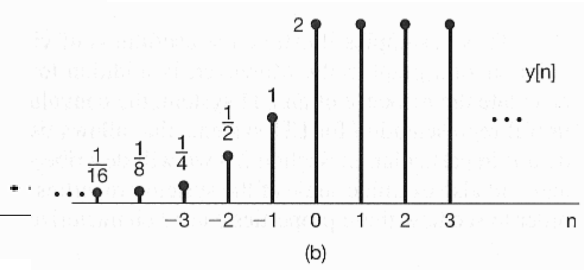
$h[n]$







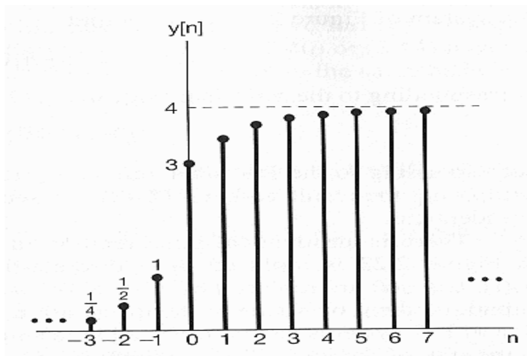




$$y[n] = x[n] * h[n]$$

$$= \left(x_1[n] + x_2[n]\right) * h[n]$$

$$= x_1[n] * h[n] + x_2[n] * h[n]$$



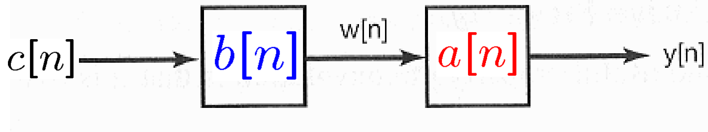
- **Associative Property:**

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

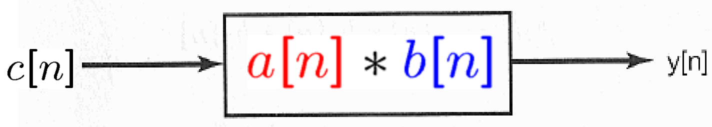
$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$a[n] * \Big(b[n] * c[n]\Big) = \Big(a[n] * b[n]\Big) * c[n]$$

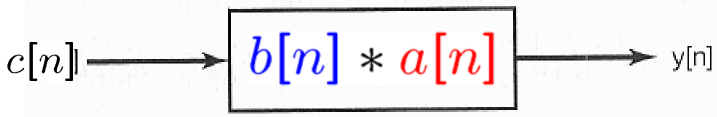
$$a(t) * \Big(b(t) * c(t)\Big) = \Big(a(t) * b(t)\Big) * c(t)$$



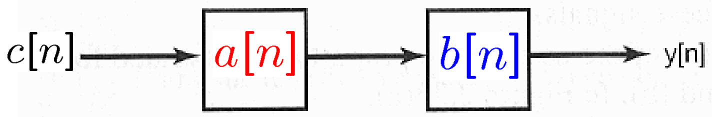
(a)



(c)



(b)



(d)

- Systems with or without memory
  - Memoryless systems
    - Output depends only on the input at that same time

$$y[n] = (2x[n] - x[n]^2)^2$$

$$y(t) = Rx(t)$$ (resistor)

  - Systems with memory

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$ (accumulator)

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau$$

$$y[n] = x[n-1]$$ (delay)

Memoryless:

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

- A DT LTI system is memoryless if $h[n] = 0$ for $n \neq 0$

- The impulse response: $h[n] = K\delta[n], \quad K = h[0]$

- The convolution sum: $y[n] = x[n] * h[n]$

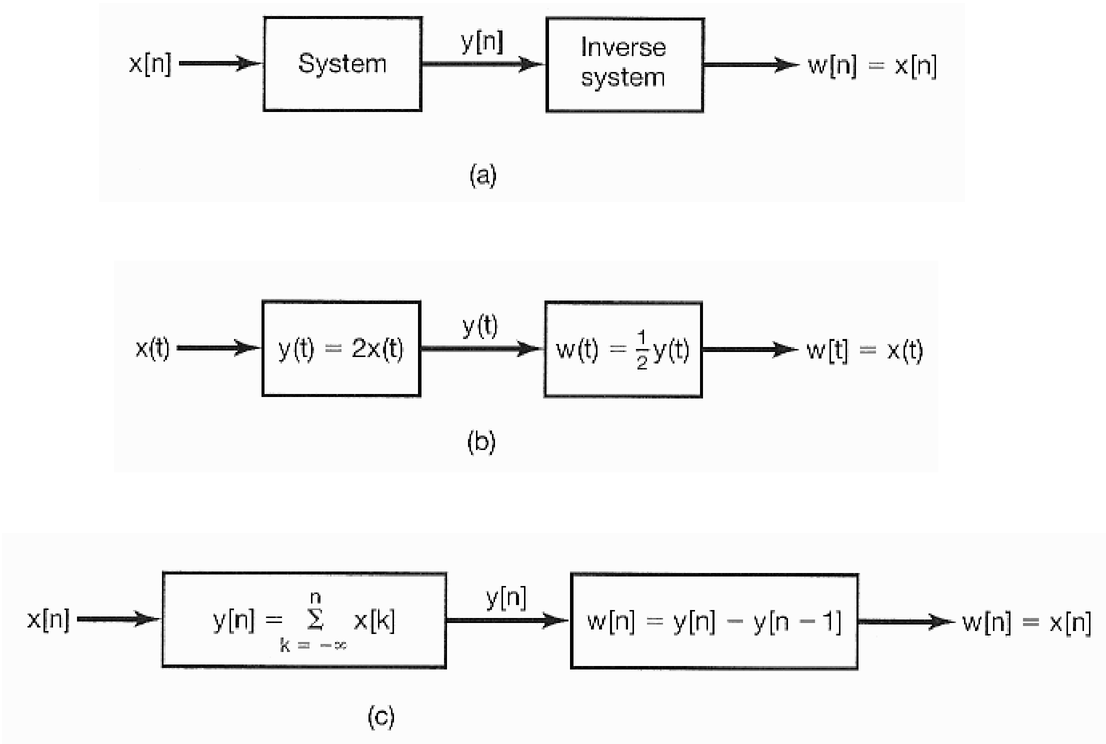
$$= Kx[n]$$

- Similarly, for CT LTI system: $y(t) = x(t) * h(t) = Kx(t)$

Invertibility & Inverse Systems

Invertible systems

- Distinct inputs lead to distinct outputs



$y(t) = x(t)^2$ is not invertible

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau) h(t-\tau) d\tau$$

- **Invertibility:**

$$x(t) \rightarrow \boxed{\text{h1(t)}} \rightarrow y(t) \rightarrow \boxed{\text{h2(t)}} \rightarrow w(t)$$

$$y(t) = x(t) * h_1(t) \qquad w(t) = y(t) * h_2(t)$$

$$\Rightarrow w(t) = x(t) * h_1(t) * h_2(t)$$

$$x(t) \rightarrow \boxed{\text{Identity System } \delta\text{(t)}} \rightarrow x(t)$$

$$x(t) = x(t) * \delta(t)$$

$$\Longrightarrow h_2(t) * h_1(t) = \delta(t)$$

Example 2.11: Pure time shift

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$x(t) \rightarrow \boxed{\text{h1(t)}} \rightarrow y(t) \rightarrow \boxed{\text{h2(t)}} \rightarrow w(t)$$

- $y(t) = x(t - t_0)$
  - delay if $t_0 > 0$
  - advance if $t_0 < 0$

$$\Rightarrow \quad h_1(t) = \delta(t - t_0) \quad \Rightarrow \quad x(t) * \delta(t - t_0) = x(t - t_0)$$

- $w(t) = x(t) = y(t + t_0)$

$$\Rightarrow \quad h_2(t) = \delta(t + t_0) \quad \Rightarrow \quad y(t) * \delta(t + t_0) = y(t + t_0)$$

$$\Rightarrow \quad h_1(t) * h_2(t) = \delta(t - t_0) * \delta(t + t_0) = \delta(t)$$

Example 2.12

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$x[n] \rightarrow \boxed{\text{h1[n]}} \rightarrow y[n] \rightarrow \boxed{\text{h2[n]}} \rightarrow w[n]$$

$$h_1[n] = u[n]$$

$$\Rightarrow \; y[n] = \sum_{k=-\infty}^{\infty} x[k]\, u[n-k] \; = \sum_{k=-\infty}^{n} x[k]$$

$\Rightarrow$ a running-sum operation

- Its inverse is a first difference operation:

$$w[n] = y[n] - y[n-1] \;\Rightarrow\; h_2[n] = \delta[n] - \delta[n-1]$$

$$\Rightarrow \; h_1[n] * h_2[n] = u[n] - u[n-1] = \delta[n]$$

- Causality:

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

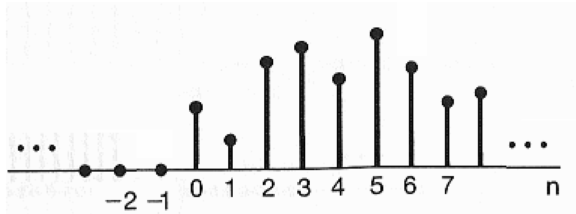
  - The output of a causal system depends only on the present and past values of the input to the system
  - Specifically, y[n] must not depend on x[k], for k > n

$$h[n-k] = 0, \qquad \text{for } k > n$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$h[n] = 0, \qquad \text{for } n < 0$$

  - It implies that the system is initially rest



  - A CT LTI system is causal if $h(t) = 0, \qquad \text{for } t < 0$

Convolution Sum & Integral

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\ h[n-k]$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\ h(t-\tau)\ d\tau$$

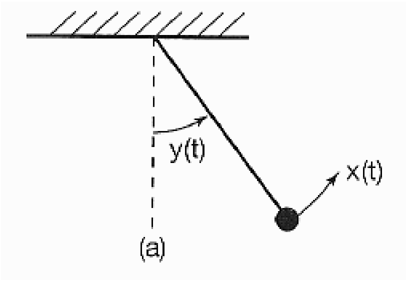
$$= \sum_{k=-\infty}^{n} x[k]\ h[n-k]$$

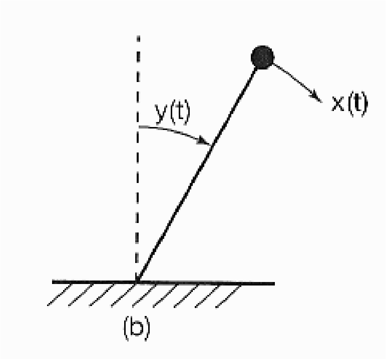
$$= \int_{-\infty}^{t} x(\tau)\ h(t-\tau)\ d\tau$$

$$= \sum_{k=0}^{\infty} h[k]\ x[n-k]$$

$$= \int_{0}^{\infty} h(\tau)\ x(t-\tau)\ d\tau$$

- Stability
  - Stable systems
    - Small inputs lead to responses that do not diverge
    - Every bounded input excites a bounded output
      - Bounded-input bounded-output stable (BIBO stable)
      - For all |x(t)| < a, then |y(t)| < b, for all t





- Balance in a bank account? $y[n] = 1.01y[n-1] + x[n]$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

- **Stability:**
  - A system is stable
    if every bounded input produces a bounded output

$$x[n] \rightarrow \boxed{\text{Stable LTI}} \rightarrow y[n]$$

$$\left|x[n]\right| < B \quad \text{for all } n \qquad\qquad \left|y[n]\right| = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$$

$$\Rightarrow y[n] \le \sum_{k=-\infty}^{+\infty} h[k] \quad x[n-k]$$

$$\Rightarrow y[n] \le \left(\sum_{k=-\infty}^{+\infty} h[k] \right)$$

if $\displaystyle\sum_{k=-\infty}^{+\infty} \left|h[k]\right| < \infty$ then, $y[n]$ is bounded

absolutely summable

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

- **Stability:**
  - For CT LTI stable system:

$$x(t) \rightarrow \boxed{\text{Stable LTI}} \rightarrow y(t)$$

$\left|x(t)\right| < B$ for all t

$$\left|y(t)\right| = \left| \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau \right|$$

$$\Rightarrow \left|y(t)\right| \leq \int_{-\infty}^{+\infty} \left|h(\tau)\right|\left|x(t-\tau)\right|d\tau$$

$$\Rightarrow \left|y(t)\right| \leq \quad \left(\int_{-\infty}^{+\infty} \left|h[(\tau)\right|d\tau \right)$$

if $\int_{-\infty}^{+\infty} \left|h(\tau)\right|d\tau < \infty$ then, $y(t)$ is bounded

absolutely integrable

Example 2.13: Pure time shift

- $y[n] = x[n - n_0]$ & $h[n] = \delta[n - n_0]$
- $y(t) = x(t - t_0)$ & $h(t) = \delta(t - t_0)$

$$\Rightarrow \sum_{n=-\infty}^{+\infty} \left| h[n] \right| = \sum_{n=-\infty}^{+\infty} \left| \delta[n - n_0] \right| = 1$$ absolutely summable

$$\Rightarrow \int_{-\infty}^{+\infty} \left| h(\tau) \right| = \int_{-\infty}^{+\infty} \left| \delta(\tau - t_0) \right| d\tau = 1$$ absolutely integrable

$\Rightarrow$ A (CT or DT) pure time shift is stable

Example 2.13: Accumulator

- $y[n] = \sum_{k=-\infty}^{n} x[k] \quad \& \quad h[n] = u[n]$

- $y(t) = \int_{-\infty}^{t} x(\tau) d\tau \quad \& \quad h(t) = u(t)$

$$\Rightarrow \quad \sum_{n=-\infty}^{+\infty} \left| h[n] \right| = \sum_{n=0}^{+\infty} \left| u[n] \right| = \infty$$

NOT absolutely summable

$$\Rightarrow \quad \int_{-\infty}^{+\infty} \left| h(\tau) \right| = \int_{0}^{\infty} \left| u(\tau) \right| d\tau = \infty$$

NOT absolutely integrable

$\Rightarrow$ A accumulator or integrator is NOT stable

$$h[n] = \delta[n] * h[n]$$

▪ Unit Step Response:

- For an LTI system, its impulse response is:

$$\delta[n] \rightarrow \boxed{\text{DT LTI}} \rightarrow h[n] \qquad \delta(t) \rightarrow \boxed{\text{CT LTI}} \rightarrow h(t)$$

- Its unit step response is:

$$u[n] \rightarrow \boxed{\text{DT LTI}} \rightarrow s[n] \qquad u(t) \rightarrow \boxed{\text{CT LTI}} \rightarrow s(t)$$

$$\Rightarrow \; s[n] = u[n] * h[n]$$

$$= \sum_{k=-\infty}^{+\infty} u[n-k]h[k]$$

$$= \sum_{k=-\infty}^{n} h[n]$$

$$\Rightarrow \; h[n] = s[n] - s[n-1]$$

$$\Rightarrow \; s(t) = u(t) * h(t)$$

$$= \int_{-\infty}^{+\infty} n(t-\tau)h(\tau)d\tau$$

$$= \int_{-\infty}^{t} h(\tau)d\tau$$

$$\Rightarrow \; h(t) = \frac{ds(t)}{dt}$$

- Discrete-Time Linear Time-Invariant Systems
  - The convolution sum $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$ $y[n] = x[n] * h[n]$
- Continuous-Time Linear Time-Invariant Systems
  - The convolution integral $y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$ $y(t) = x(t) * h(t)$
- Properties of Linear Time-Invariant Systems
  1. Commutative property $x(t) * h(t) = h(t) * x(t)$
  2. Distributive property $x(t) * \left(h_1(t) + h_2(t)\right) = x(t) * h_1(t) + x(t) * h_2(t)$
  3. Associative property $a(t) * \left(b(t) * c(t)\right) = \left(a(t) * b(t)\right) * c(t)$
  4. With or without memory $h[n] = 0$ for $n \neq 0$ $h(t) = 0,$ for $t < 0$
  5. Invertibility $h_2(t) * h_1(t) = \delta(t)$
  6. Causality
  7. Stability if $\int_{-\infty}^{+\infty} |h(\tau)| d\tau < \infty$
  8. Unit step response
- Causal Linear Time-Invariant Systems Described by Differential & Difference Equations
- Singularity Functions $h[n] = \left(\frac{1}{2}\right)^n u[n]$

Singularity Functions

- CT unit impulse function is one of singularity functions

$$\delta(t) = u_0(t)$$

$$u(t) = u_{-1}(t)$$

$$\int_{-\infty}^{\infty} \delta(\tau) d\tau = u(t)$$

$$\frac{d}{dt}\delta(t) = u_1(t)$$

$$\int_{-\infty}^{t} u(\tau) d\tau = u_{-2}(t)$$

$$\frac{d^2}{dt^2}\delta(t) = u_2(t)$$

$$\int_{-\infty}^{t} \left(\int_{-\infty}^{\tau} u(\sigma) d\sigma \right) d\tau = u_{-3}(t)$$

$$\frac{d^k}{dt^k}\delta(t) = u_k(t)$$

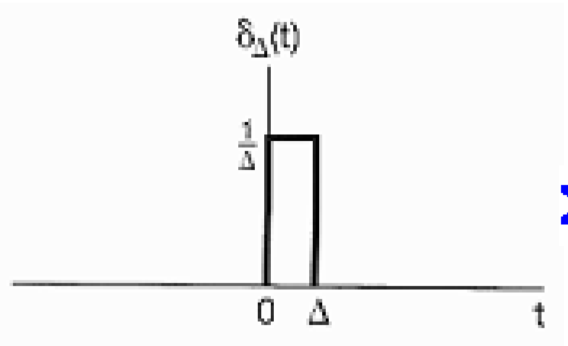
$$\int_{-\infty}^{t} \cdots \left(\int_{-\infty}^{\tau} u(\sigma) d\sigma \right) \cdots d\tau = u_{-k}(t)$$

- Singularity Functions

$$x(t) = x(t) * \delta(t)$$

$$\delta(t) = \delta(t) * \delta(t)$$

$$r_{\Delta}(t) = \delta_{\Delta}(t) * \delta_{\Delta}(t)$$



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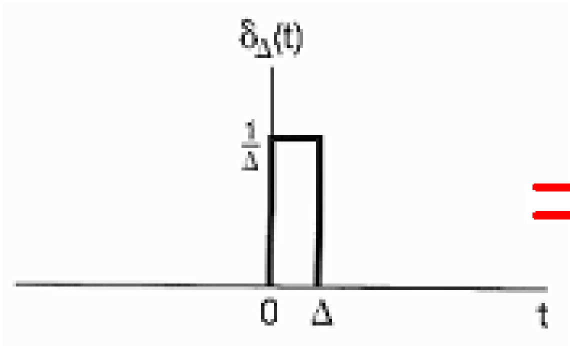

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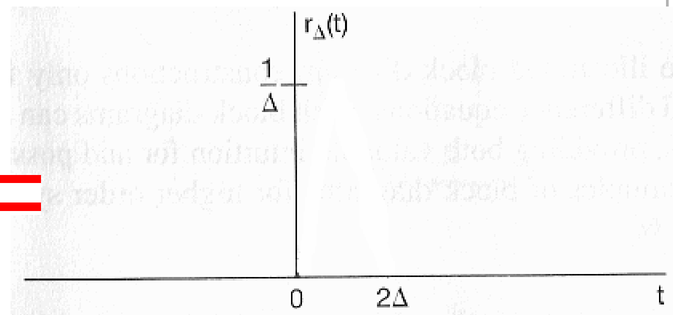


$$\lim_{\Delta \to 0} \delta_{\Delta}(t) = \delta(t) \qquad \Rightarrow \quad \lim_{\Delta \to 0} r_{\Delta}(t) = \delta(t)$$

- **Example 2.16** $\dfrac{d}{dt} y(t) + 2\, y(t) = x(t)$

with initial-rest condition

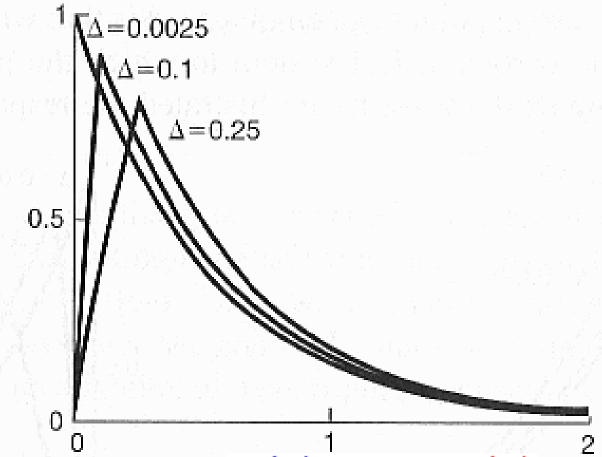
$x(t) = \delta_\triangle(t)$

$x(t) = r_\triangle(t)$

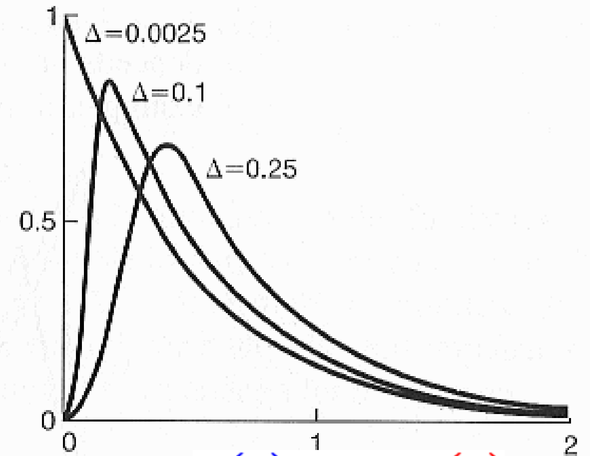
$x(t) = \delta_\triangle(t) * r_\triangle(t)$

$x(t) = r_\triangle(t) * r_\triangle(t)$

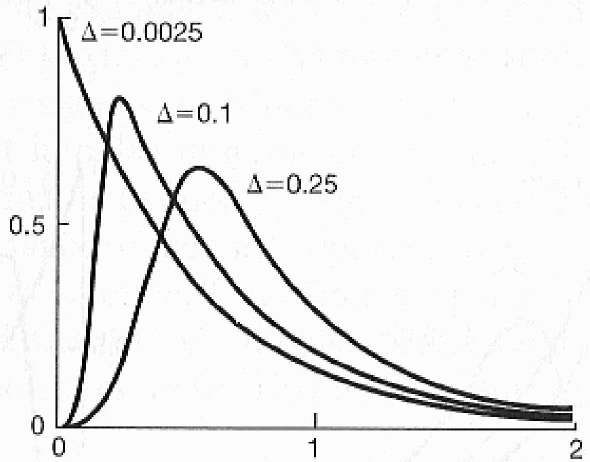
$x(t) = \delta(t)$



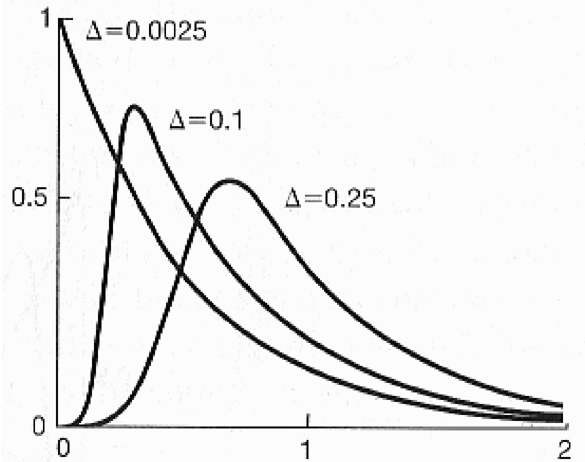
$x(t) = \delta_\triangle(t)$



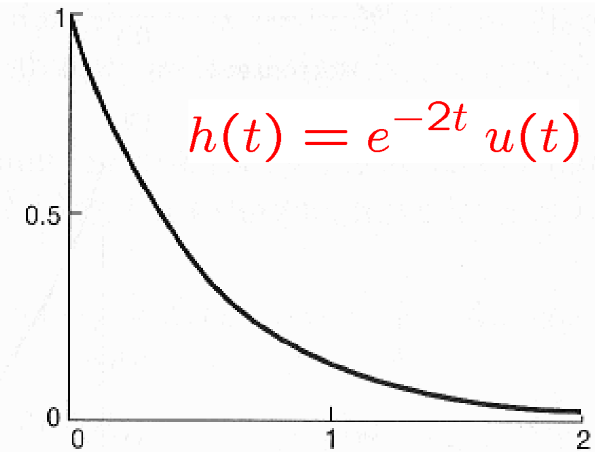
$x(t) = r_\triangle(t)$



$x(t) = \delta_\triangle(t) * r_\triangle(t)$



$x(t) = r_\triangle(t) * r_\triangle(t)$



$h(t) = e^{-2t}\, u(t)$

$x(t) = \delta(t)$

(e)

- Example 2.16 $\frac{d}{dt}y(t) + 20\, y(t) = x(t)$

with initial-rest condition

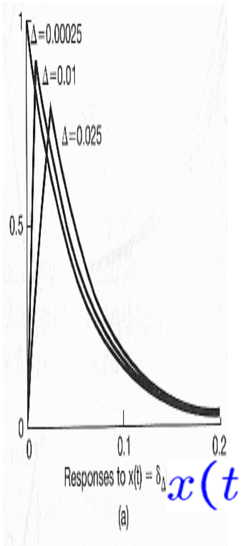
$x(t) = \delta_{\triangle}(t)$

$x(t) = r_{\triangle}(t)$

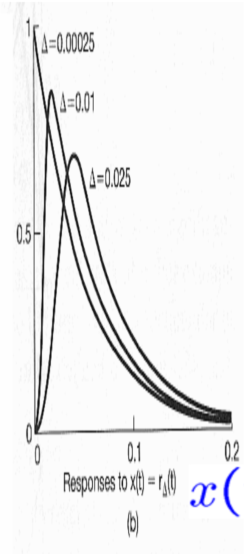
$x(t) = \delta_{\triangle}(t) * r_{\triangle}(t)$

$x(t) = r_{\triangle}(t) * r_{\triangle}(t)$

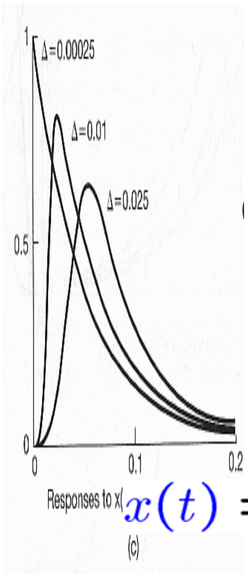
$x(t) = \delta(t)$



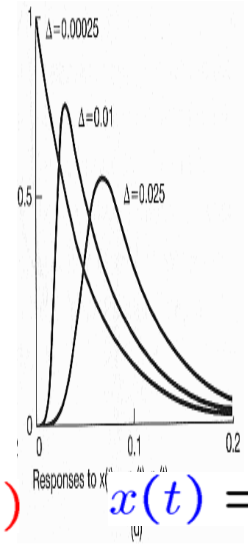
$x(t) = \delta_{\triangle}(t)$



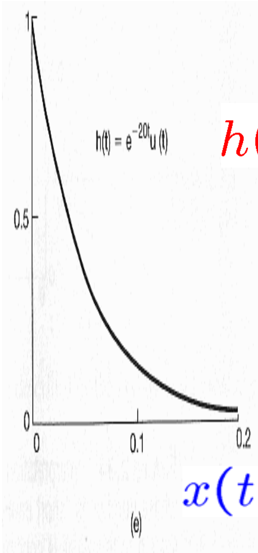
$x(t) = r_{\triangle}(t)$



$x(t) = \delta_{\triangle}(t) * r_{\triangle}(t)$



$x(t) = r_{\triangle}(t) * r_{\triangle}(t)$



$h(t) = e^{-20t}\, u(t)$

$x(t) = \delta(t)$

Defining the Unit Impulse through Convolution:

$$x(t) = x(t) * \delta(t)$$

- Let $x(t) = 1$,

$$1 = x(t) = x(t) * \delta(t) = \delta(t) * x(t)$$

$$= \int_{-\infty}^{\infty} \delta(\tau) x(t-\tau) d\tau = \int_{-\infty}^{\infty} \delta(\tau) d\tau$$

- So that the unit impulse has unit area

Defining Unit Impulse through Convolution:

- Alternatively, consdier an arbitrary signal $g(t)$,

$$g(-t) = g(-t) * \delta(t) = \int_{-\infty}^{\infty} g(\tau - t)\delta(\tau)d\tau$$

$$g(0) = \int_{-\infty}^{\infty} g(\tau)\delta(\tau)d\tau$$

- Define $x(t-\tau) = g(\tau)$

$$x(t) = g(0) = \int_{-\infty}^{\infty} g(\tau)\delta(\tau)d\tau$$

$$= \int_{-\infty}^{\infty} x(t-\tau)\delta(\tau)d\tau = x(t) * \delta(t)$$

Defining Unit Impulse through Convolution:

- Consider the signal $f(t)\delta(t)$

$$\int_{-\infty}^{\infty} g(\tau)f(\tau)\delta(\tau)d\tau = g(0)f(0)$$

- On the other hand, consider the signal $f(0)\delta(t)$

$$\int_{-\infty}^{\infty} g(\tau)f(0)\delta(\tau)d\tau = g(0)f(0)$$

- Therefore,

$$f(t)\delta(t) = f(0)\delta(t)$$

Unit Doublets of Derivative Operation:

- A system: Output is the derivative of input

$$y(t) = \frac{d}{dt}x(t)$$

$\Rightarrow$ The unit impulse response of the system

is the derivative of the unit impulse,

which is called the unit doublet $u_1(t)$

- That is, from $x(t) = x(t) * \delta(t)$, we have

$$\frac{d}{dt}x(t) = x(t) * u_1(t)$$

Unit Doublets of Derivative Operation:

- Similarly,

$$\frac{d^2}{dt^2}x(t) = x(t) * u_2(t)$$

- But,

$$\frac{d^2}{dt^2}x(t) = \frac{d}{dt}\left(\frac{d}{dt}x(t)\right) = \left(x(t) * u_1(t)\right) * u_1(t)$$

- Therefore,

$$u_2(t) = u_1(t) * u_1(t)$$

- In general,

$u_k(t)$, $k > 0$, the kth derivative of $\delta(t)$

$$u_k(t) = u_1(t) * \cdots * u_1(t)$$

Unit Doublets of Integration Operation:

- A system: Output is the integral of input

$$y(t) = \int_{-\infty}^{\infty} x(\tau) d\tau$$

- Therefore,

$$u(t) = \int_{-\infty}^{\infty} \delta(\tau) d\tau$$

- Hence, we have

$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau) d\tau$$

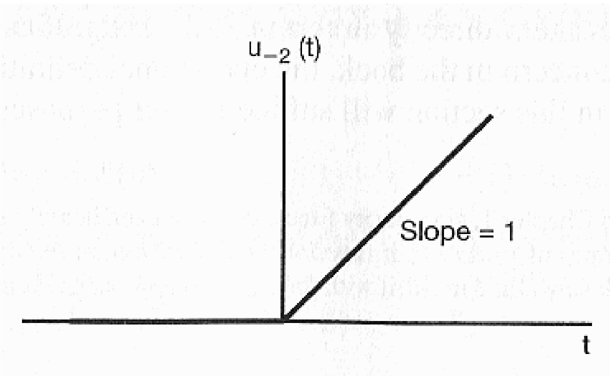
Unit Doublets of Integration Operation:

- Similarly,

$$u_{-2}(t) = u(t) * u(t) = \int_{-\infty}^{t} u(\tau)d\tau$$

- That is,

$$u_{-2}(t) = t\ u(t)$$ the unit ramp function



Unit Doublets of Integration Operation:

- Moreover,

$$x(t) * u_{-2}(t) = x(t) * u(t) * u(t)$$

$$= \left(\int_{-\infty}^{t} x(\sigma) d\sigma \right) * u(t)$$

$$= \int_{-\infty}^{t} \left(\int_{-\infty}^{\tau} x(\sigma) d\sigma \right) d\tau$$

- In general,

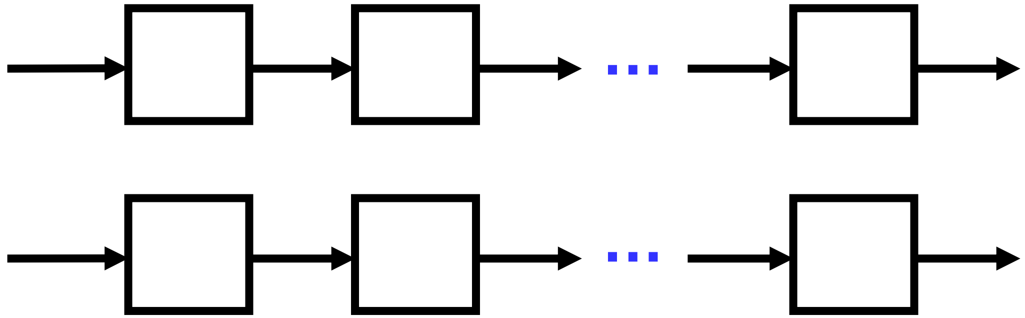
$$u_{-k}(t) = u(t) * \cdots * u(t) = \int_{-\infty}^{t} u_{-(k-1)}(\tau) d\tau$$

$$u_{-k}(t) = \frac{t^{k-1}}{(k-1)!} u(t)$$

In Summary

$$\delta(t) = u_0(t)$$

$$u(t) = u_{-1}(t)$$



$u_k(t)$

$k > 0,$

Impulse response of a cascade of k differentiators

$k < 0,$

Impulse response of a cascade of $|k|$ integrators

$$u(t) * u_1(t) = \delta(t) \quad \text{or,} \quad u_{-1}(t) * u_1(t) = u_0(t)$$

$$\Rightarrow \quad u_k(t) * u_r(t) = u_{k+r}(t)$$

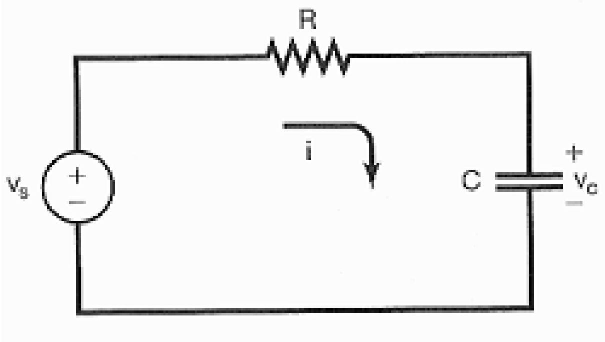
- Discrete-Time Linear Time-Invariant Systems
  - The convolution sum $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$ $\quad y[n] = x[n] * h[n]$
- Continuous-Time Linear Time-Invariant Systems
  - The convolution integral $y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$ $\quad y(t) = x(t) * h(t)$
- Properties of Linear Time-Invariant Systems
  1. Commutative property $x(t) * h(t) = h(t) * x(t)$
  2. Distributive property $x(t) * \left(h_1(t) + h_2(t)\right) = x(t) * h_1(t) + x(t) * h_2(t)$
  3. Associative property $a(t) * \left(b(t) * c(t)\right) = \left(a(t) * b(t)\right) * c(t)$
  4. With or without memory $h(t) = 0$ for $t \neq 0$
  5. Invertibility $h_2(t) * h_1(t) = \delta(t)$
  6. Causality $h(t) = 0,\quad$ for $t < 0$
  7. Stability if $\int_{-\infty}^{+\infty} \left|h(\tau)\right| d\tau < \infty$
  8. Unit step response
- Causal Linear Time-Invariant Systems Described by Differential & Difference Equations
- Singularity Functions

$$u_k(t) * u_r(t) = u_{k+r}(t)$$

Linear Constant-Coefficient Differential Equations

- e.x., RC circuit

Input signal: $v_s(t)$



Output signal: $v_c(t)$

$$\Rightarrow \frac{dv_c(t)}{dt} + \frac{1}{RC} v_c(t) = \frac{1}{RC} v_s(t)$$

$x(t) \rightarrow$ RC Circuit $\rightarrow y(t)$

$$\Rightarrow \frac{d}{dt} y(t) + a\, y(t) = b\, x(t)$$

- Provide an implicit specification of the system
- You have learned how to solve the equation in Diff Eqn

Linear Constant-Coefficient Differential Equations

- For a general CT LTI system, with N-th order,

$$x(t) \rightarrow \boxed{\text{CT LTI}} \rightarrow y(t)$$

$$a_N \frac{d^N}{dt^N} y(t) + a_{N-1} \frac{d^{N-1}}{dt^{N-1}} y(t) + \cdots + a_1 \frac{d}{dt} y(t) + a_0 y(t)$$

$$= b_M \frac{d^M}{dt^M} x(t) + b_{M-1} \frac{d^{M-1}}{dt^{M-1}} x(t) + \cdots + b_1 \frac{d}{dt} x(t) + b_0 x(t)$$

$$\Rightarrow \sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

$$\Rightarrow \ h(t) = ?$$

Linear Constant-Coefficient Difference Equations

- For a general DT LTI system, with N-th order,

$$x[n] \rightarrow \boxed{\text{DT LTI}} \rightarrow y[n]$$

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N]$$

$$= b_0 x[n] + b_1 x[n-1] + \cdots + b_{M-1} x[n-M+1] + b_M x[n-M]$$

$$\Rightarrow \sum_{k=0}^{N} a_k \, y[n-k] = \sum_{k=0}^{M} b_k \, x[n-k]$$

$$\Rightarrow \; h[n] = ?$$

- Recursive Equation:

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N]$$

$$= b_0 x[n] + b_1 x[n-1] + \cdots + b_{M-1} x[n-M+1] + b_M x[n-M]$$

$$\sum_{k=0}^{N} a_k \, y[n-k] = \sum_{k=0}^{M} b_k \, x[n-k]$$

$$\Rightarrow \; y[n] = \frac{1}{a_0} \left\{ \sum_{k=0}^{M} b_k \, x[n-k] - \sum_{k=1}^{N} a_k \, y[n-k] \right\}$$

Recursive Equation:

- For example, (Example 2.15)

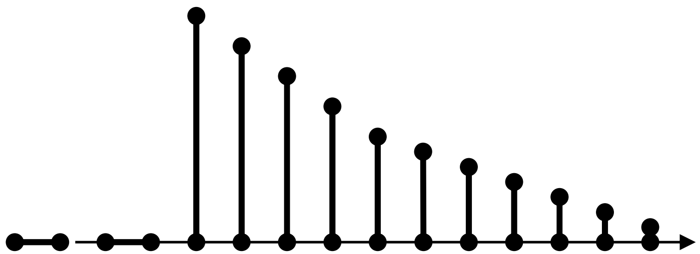
$$\delta[n] \rightarrow \boxed{\text{LTI}} \rightarrow h[n]$$
$$x[n] \rightarrow \boxed{\text{LTI}} \rightarrow y[n]$$

$$y[n] - \frac{1}{2}y[n-1] = x[n]$$

$$y[n] = 0, \quad \text{for } n \leq -1$$
$$x[n] = K\ \delta[n]$$

$$\Rightarrow \begin{cases} y[0] = x[0] + \frac{1}{2}y[-1] & = K \\ y[1] = x[1] + \frac{1}{2}y[0] & = K\ \frac{1}{2} \\ y[2] = x[2] + \frac{1}{2}y[1] & = K\ \left(\frac{1}{2}\right)^2 \\ \quad \vdots & \\ y[n] = x[n] + \frac{1}{2}y[n-1] & = K\ \left(\frac{1}{2}\right)^n \end{cases}$$

$$\Rightarrow h[n] = \left(\frac{1}{2}\right)^n u[n]$$



$\Rightarrow$ an Infinite Impulse Response (IIR) system

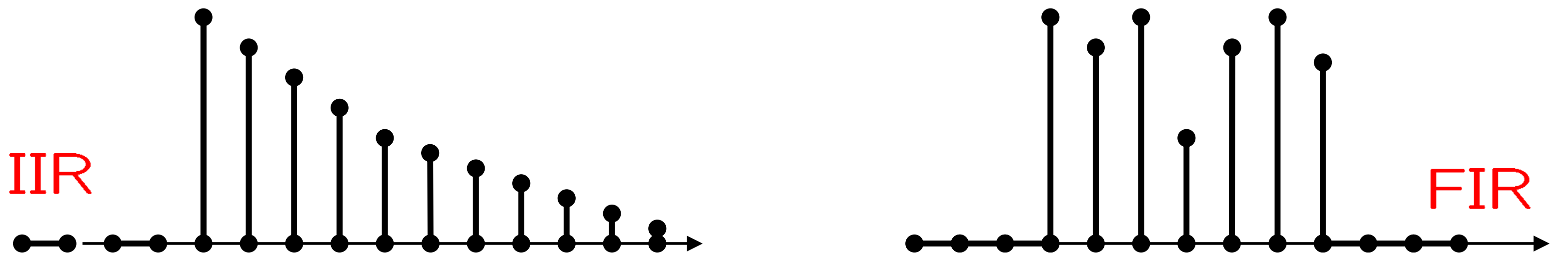
- Nonrecursive Equation: $\sum_{k=0}^{N} a_k \, y[n-k] = \sum_{k=0}^{M} b_k \, x[n-k]$
  - When N = 0,

$$\Rightarrow \; y[n] = \sum_{k=0}^{M} \left(\frac{b_k}{a_0} \right) x[n-k]$$

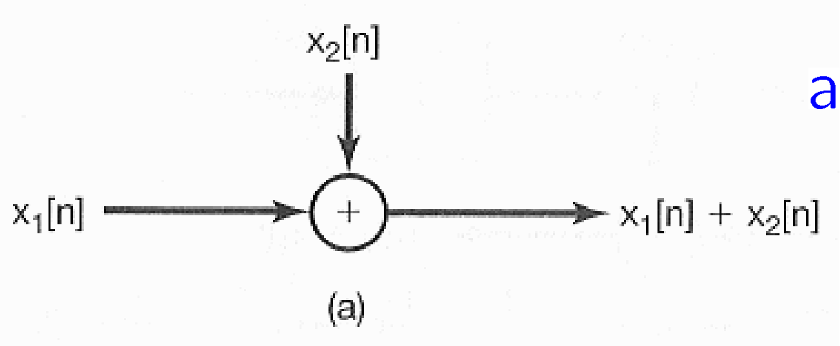
$$y[n] = \sum_{k=-\infty}^{+\infty} h[k] x[n-k]$$

$$\Rightarrow \; h[n] = \begin{cases} \frac{b_n}{a_0}, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

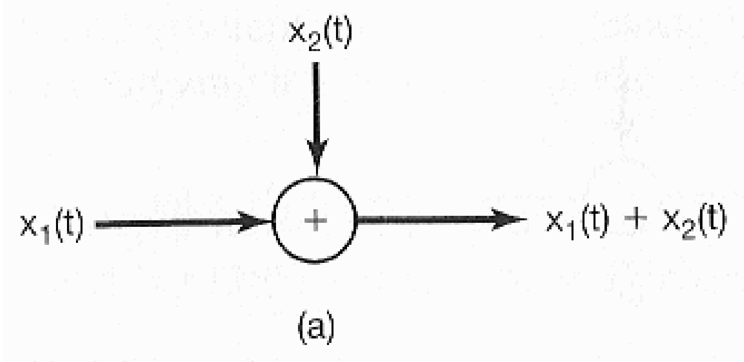
$\Rightarrow$ a Finite Impulse Response (FIR) system

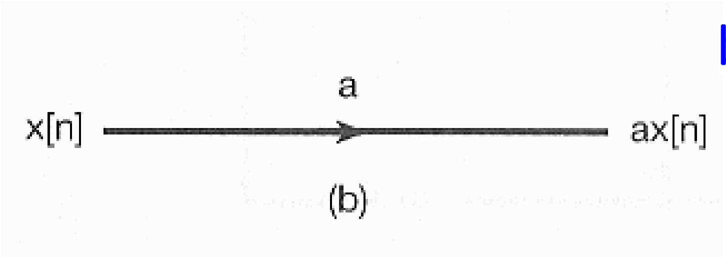


▪ Block Diagram Representations:

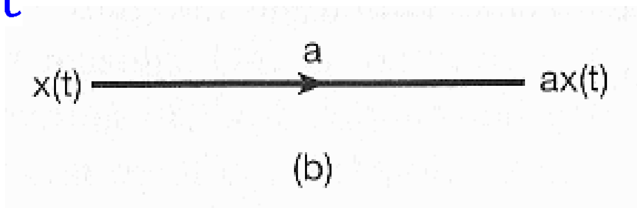


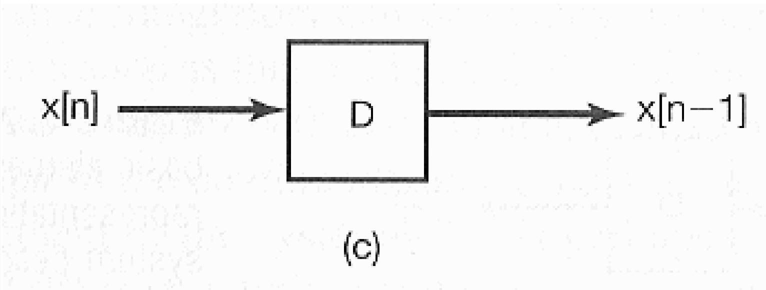
an adder



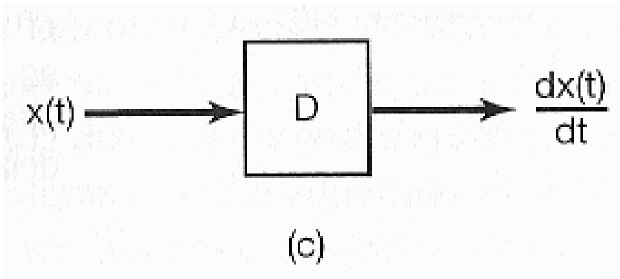


multiplication
by a coefficient





a unit delay/
differentiator



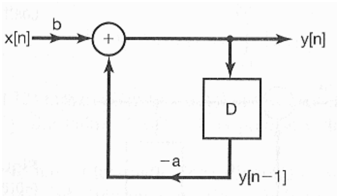
Block Diagram Representations:

$$y[n] + a y[n-1] = b x[n]$$

$$\frac{d}{dt} y(t) + a y(t) = b x(t)$$

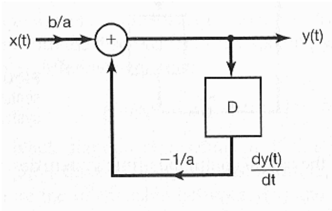
$$y[n] = -a y[n-1] + b x[n]$$

$$y(t) = -\frac{1}{a}\frac{d}{dt} y(t) + \frac{b}{a} x(t)$$



$$D \Longleftrightarrow z^{-1}$$

$$D \Longleftrightarrow s$$

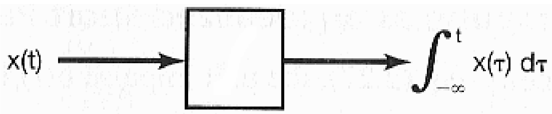


- Block Diagram Representations:

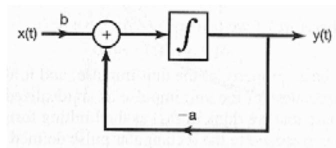
$$\frac{d}{dt}y(t) = bx(t) - ay(t)$$

$$\Rightarrow \; y(t) = \int_{-\infty}^{t} \left[bx(\tau) - ay(\tau)\right] d\tau$$

$$\Rightarrow \; y(t) = y(t_0) \; + \; \int_{t_0}^{t} \left[bx(\tau) - ay(\tau)\right] d\tau$$



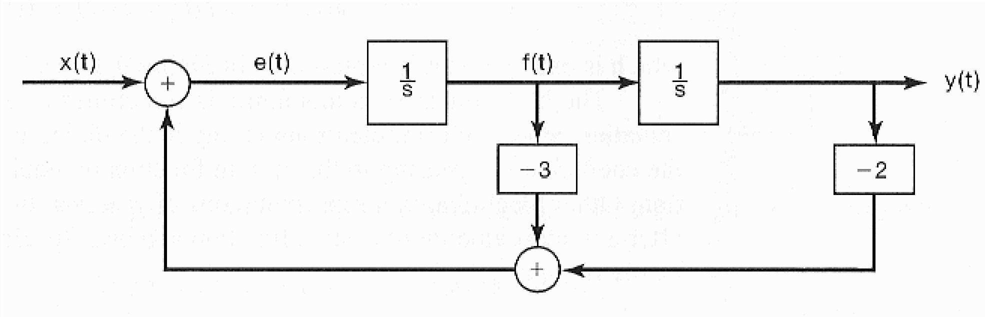
$$\int \Longleftrightarrow \frac{1}{s}$$



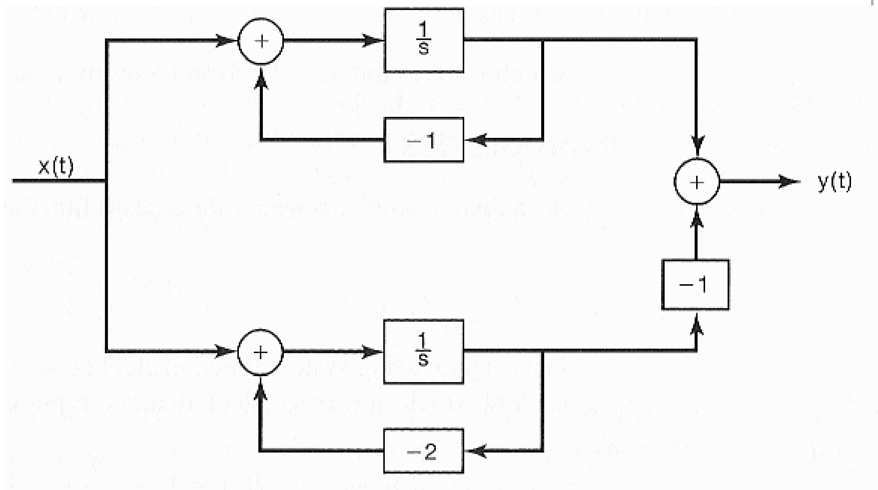
Block Diagram Representations:

$$\int \Longleftrightarrow \frac{1}{s}$$

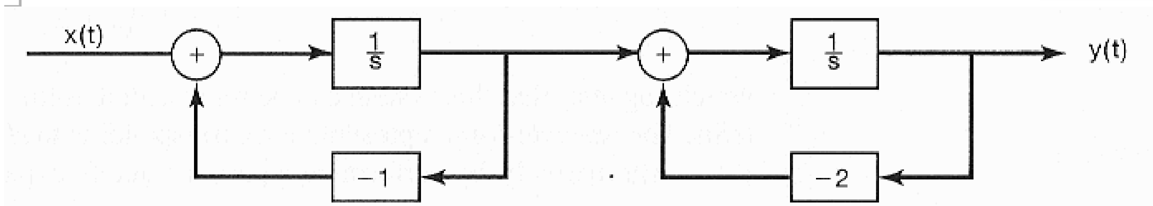
$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = x(t)$$



(a)



(c)



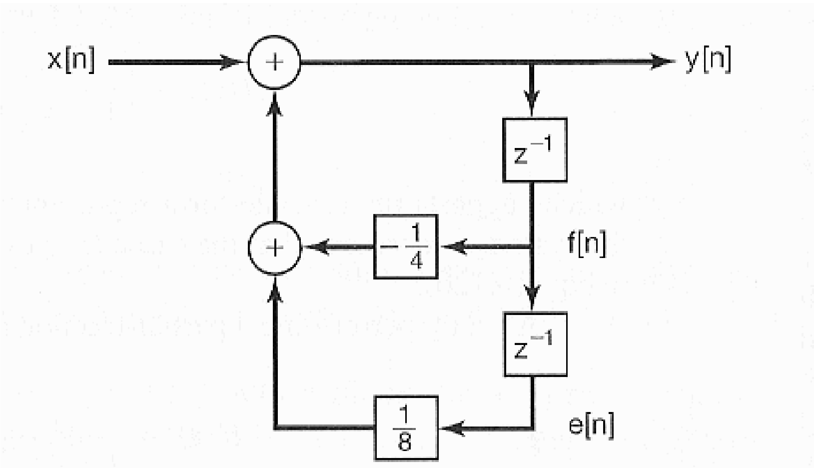
(b)

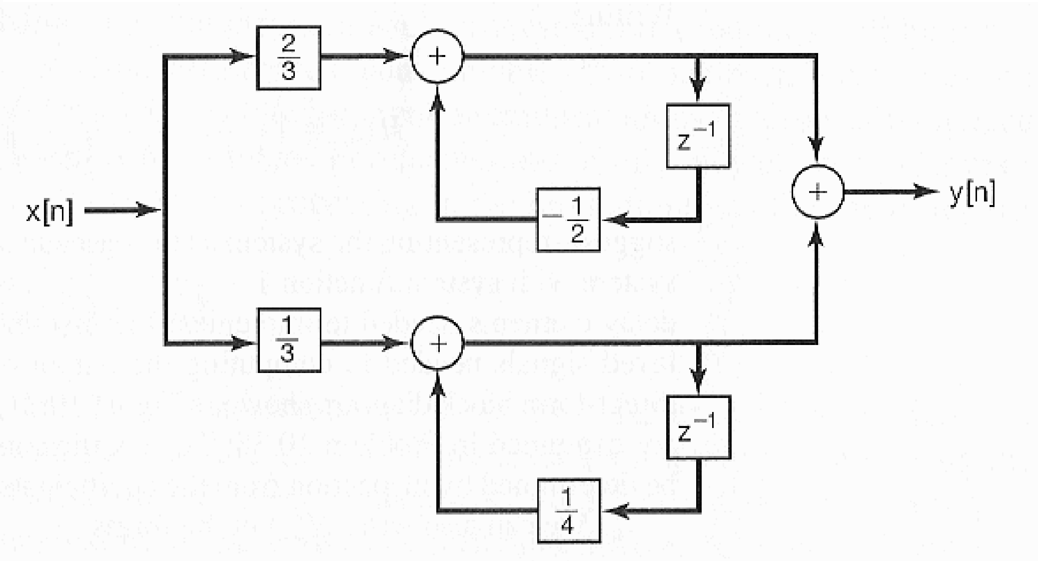
- Example 9.30 (pp.711)

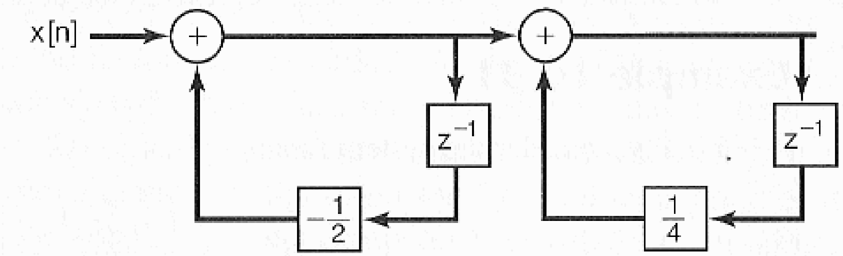
$$H(s) = \frac{1}{(s+1)(s+2)}$$

Block Diagram Representations:

$$y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n]$$







(b)

$$H(z) = \frac{1}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}$$

- Example 10.30 (pp.786)

- Discrete-Time Linear Time-Invariant Systems
  - The convolution sum

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$y[n] = x[n] * h[n]$$

- Continuous-Time Linear Time-Invariant Systems
  - The convolution integral

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$y(t) = x(t) * h(t)$$

- Properties of Linear Time-Invariant Systems
  1. Commutative property
  2. Distributive property
  3. Associative property
  4. With or without memory
  5. Invertibility
  6. Causality
  7. Stability
  8. Unit step response

$$x(t) * h(t) = h(t) * x(t)$$

$$x(t) * \left(h_1(t) + h_2(t)\right) = x(t) * h_1(t) + x(t) * h_2(t)$$

$$a(t) * \left(b(t) * c(t)\right) = \left(a(t) * b(t)\right) * c(t)$$

$h[n] = 0$ for $n \neq 0$ $\quad$ $h(t) = 0,$ for $t < 0$

$h_2(t) * h_1(t) = \delta(t)$ $\quad$ if $\int_{-\infty}^{+\infty} \left|h(\tau)\right| d\tau < \infty$

- Causal Linear Time-Invariant Systems Described by Differential & Difference Equations

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

- Singularity Functions

$$u_k(t) * u_r(t) = u_{k+r}(t)$$

Flowchart

Signals & Systems (Chap 1) LTI & Convolution (Chap 2)

Bounded/Convergent

Periodic

FS – CT
– DT
(Chap 3)

Aperiodic

FT – CT (Chap 4)
– DT (Chap 5)

Unbounded/Non-convergent

LT – CT (Chap 9)

zT – DT (Chap 10)

Time-Frequency (Chap 6) Communication (Chap 8)

CT-DT (Chap 7) Control (Chap 11)

Spring 2014

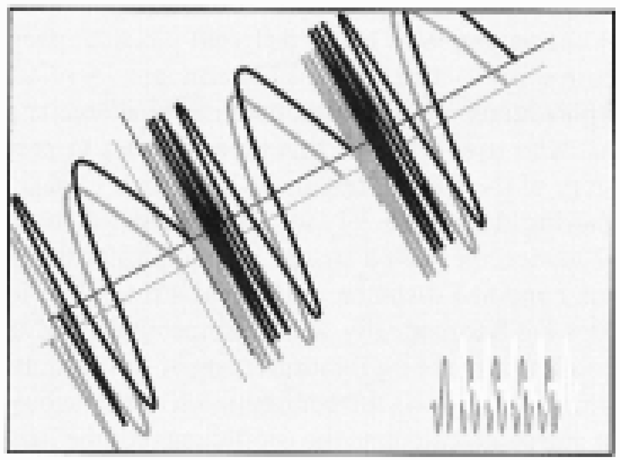
信號與系統
Signals and Systems

Chapter SS-3
Fourier Series Representation of Periodic Signals

Feng-Li Lian

NTU-EE

Feb14 – Jun14

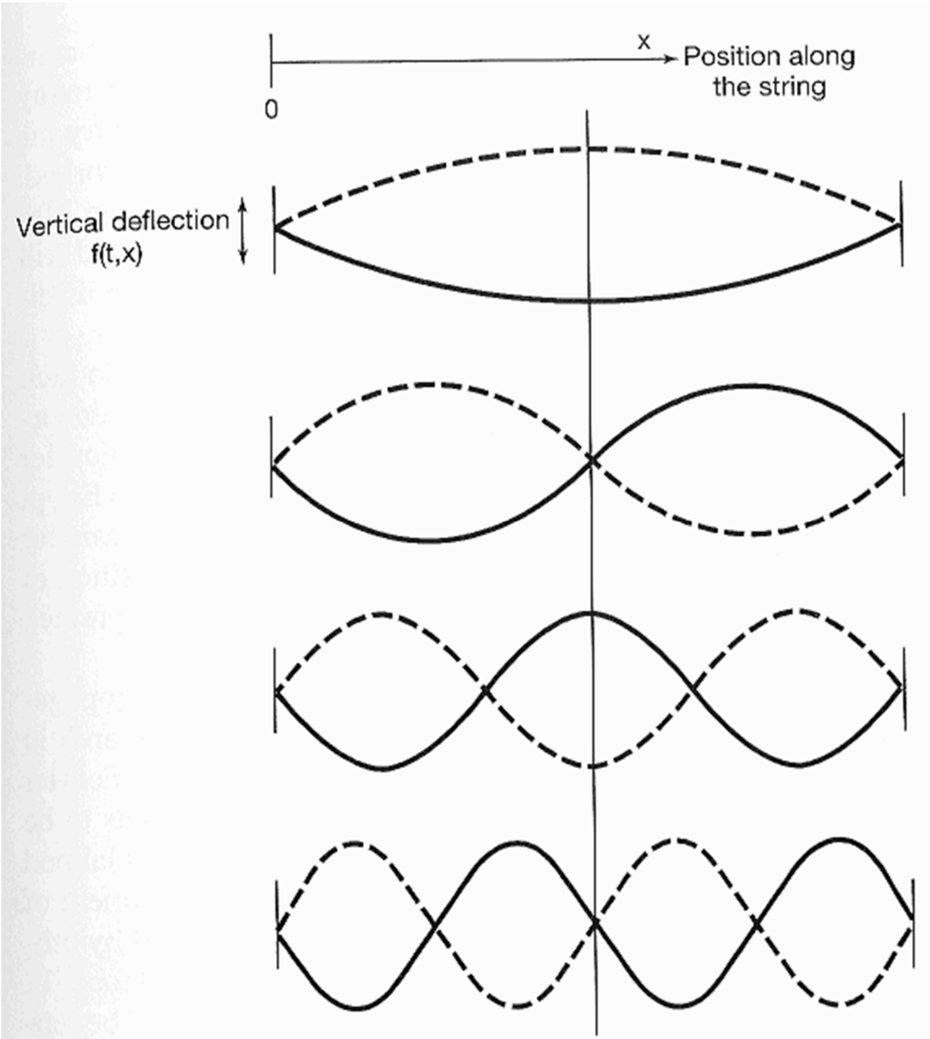


Figures and images used in these lecture notes are adopted from
"Signals & Systems" by Alan V. Oppenheim and Alan S. Willsky, 1997

Outline

- A Historical Perspective
- The Response of LTI Systems
  to Complex Exponentials
- Fourier Series Representation
  of Continuous-Time Periodic Signals
- Convergence of the Fourier Series
- Properties of Continuous-Time Fourier Series
- Fourier Series Representation
  of Discrete-Time Periodic Signals
- Properties of Discrete-Time Fourier Series
- Fourier Series & LTI Systems
- Filtering & Examples of CT & DT Filters

- L. Euler's study on the motion of a vibrating string in 1748

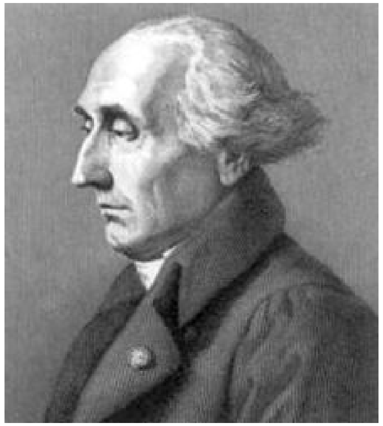


Leonhard Euler
1707-1783
Born in Switzerland
Photo from wikipedia

- L. Euler showed (in 1748)
  - The configuration of a vibrating string at some point in time is a linear combination of these normal modes

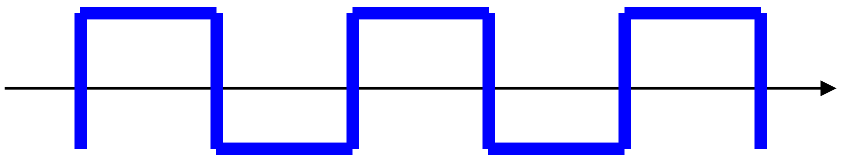
Daniel Bernoulli
1700-1782
Born in Dutch
Photo from wikipedia

- D. Bernoulli argued (in 1753)
  - All physical motions of a string could be represented by linear combinations of normal modes
  - But, he did not pursue this mathematically



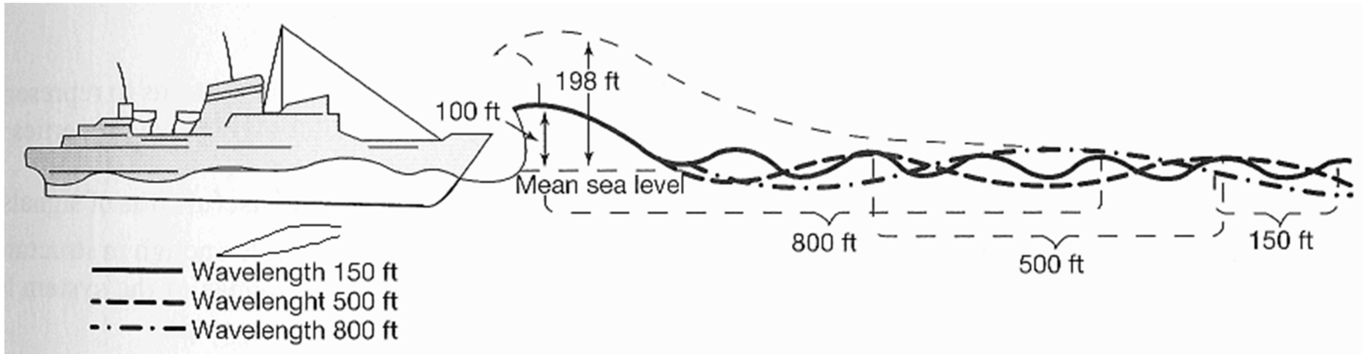
Joseph-Louis Lagrange
1736-1813
Born in Italy
Photo from wikipedia

- J.L. Lagrange strongly criticized (in 1759)
  - The use of trigonometric series in examination of vibrating strings
  - Impossible to represent signals with corners using trigonometric series



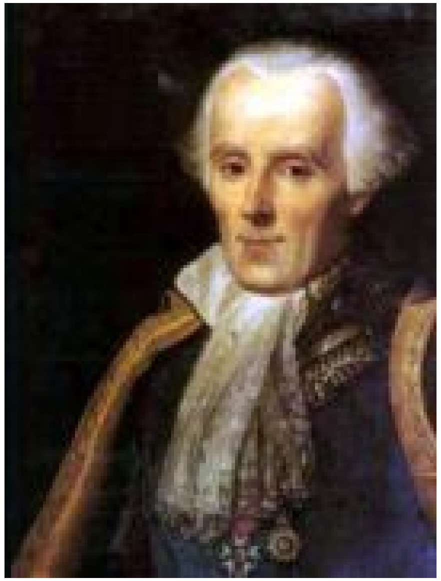
- In 1807, Jean Baptiste Joseph Fourier
  - Submitted a paper of using trigonometric series to represent "any" periodic signal
  - It is examined by S.F. Lacroix, G. Monge, P.S. de Laplace, and J.L. Lagrange,
  - But Lagrange rejected it!

- In 1822, Fourier published a book "Theorie analytique de la chaleur"
  - "The Analytical Theory of Heat"



Jean Baptiste Joseph Fourier
1768-1830
Born in France
Photo from wikipedia

Figure 1.2: A medallion by David d'Angers, the only known portrait of Lacroix, made two years prior to his death. [Académie des Sciences de l'Institut de France]



Sylvestre François de Lacroix
1765-1843
Born in France
Photo from
A short biography of Silvestre-François Lacroix
In Science Networks. Historical Studies, V35,
Lacroix and the Calculus, Birkhäuser Basel
2008, ISBN 978-3-7643-8638-2

Gaspard Monge, Comte de Péluse
1746-1818
Born in France
Photo from wikipedia

Pierre-Simon, Marquis de Laplace
1749-1827
Born in France
Photo from wikipedia

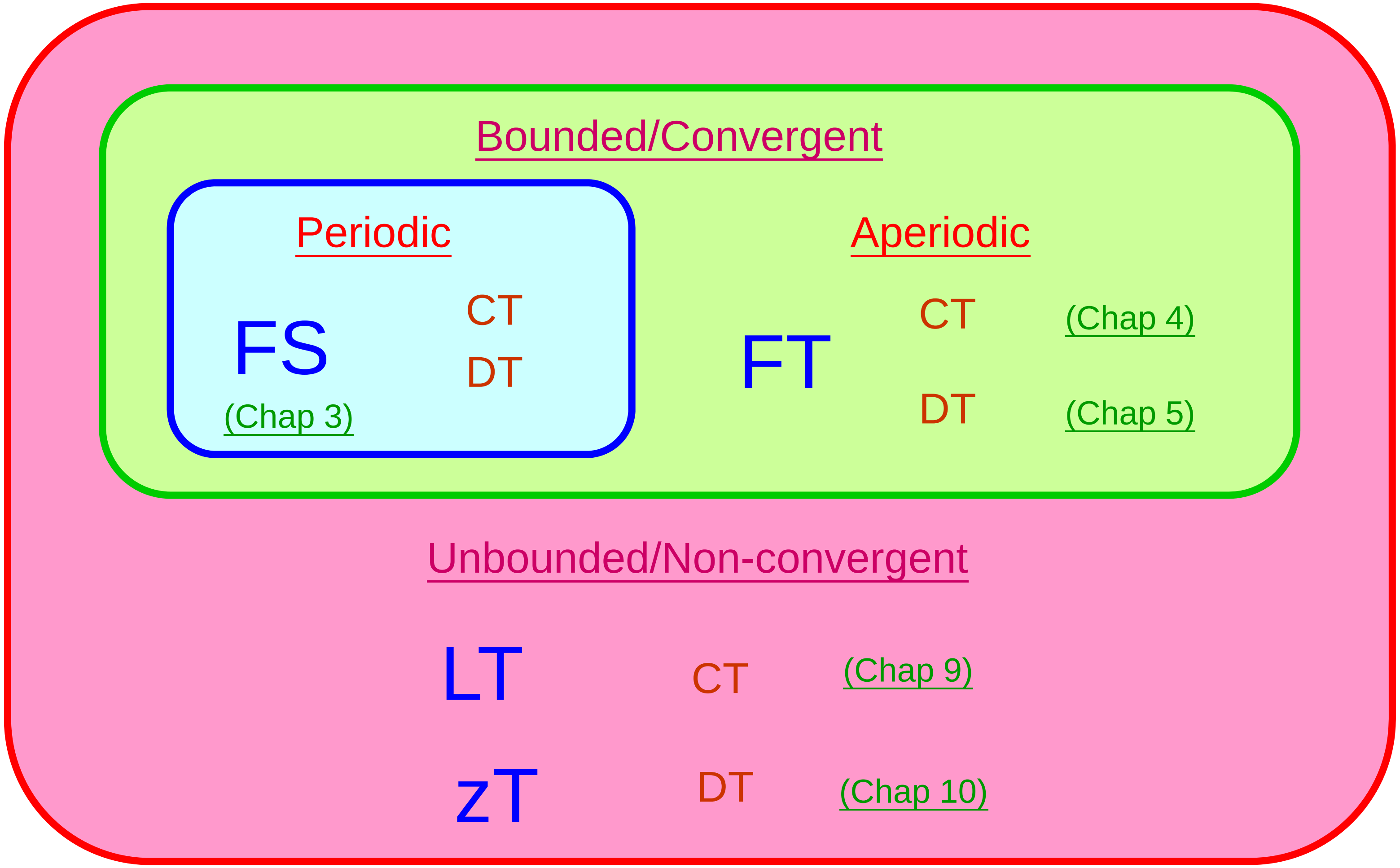
- Fourier's main contributions:
  - Studied vibration, heat diffusion, etc.
  - Found series of harmonically related sinusoids to be useful in representing the temperature distribution through a body
  - Claimed that "any" periodic signal could be represented by such a series (i.e., Fourier series discussed in Chap 3)
  - Obtained a representation for aperiodic signals (i.e., Fourier integral or transform discussed in Chap 4 & 5)
  - (Fourier did not actually contribute to the mathematical theory of Fourier series)

- Impact from Fourier's work:
  - Theory of integration, point-set topology, eigenfunction expansions, etc.
  - Motion of planets,
    periodic behavior of the earth's climate,
    wave in the ocean,
    radio & television stations
  - Harmonic time series in the 18th & 19th centuries
    - > Gauss etc. on discrete-time signals and systems
  - Faster Fourier transform (FFT) in the mid-1960s
    - > Cooley (IBM) & Tukey (Princeton) reinvented in 1965
    - > Can be found in Gauss's notebooks (in 1805)

Carl Friedrich Gauss (Gauß)
1777-1855
Born in Germany
Photo from wikipedia

James W. Cooley & John W. Tukey (1965):
"An algorithm for the machine calculation of complex Fourier series",
Math. Comput. 19, 297–301.

Signals & Systems (Chap 1) LTI & Convolution (Chap 2)



Time-Frequency (Chap 6) Communication (Chap 8)

CT-DT (Chap 7) Control (Chap 11)

Outline

- A Historical Perspective
- The Response of LTI Systems to Complex Exponentials
- Fourier Series Representation of Continuous-Time Periodic Signals
- Convergence of the Fourier Series
- Properties of Continuous-Time Fourier Series
- Fourier Series Representation of Discrete-Time Periodic Signals
- Properties of Discrete-Time Fourier Series
- Fourier Series & LTI Systems
- Filtering & Examples of CT & DT Filters

Basic Idea:

- To represent signals
  as linear combinations of basic signals

$\phi_i(t)$

Key Properties:

1. The set of basic signals can be used
   to construct a broad and useful class of signals

$x(t)$

2. The response of an LTI system to each signal
   should be simple enough
   in structure to provide us
   with a convenient representation
   for the response of the system
   to any signals constructed
   as linear combination of basic signals

$$x(t) \rightarrow \boxed{\text{LTI}}_{h(t)} \rightarrow y(t)$$

One of Choices:

- The set of complex exponential signals

$$\begin{cases} \text{signals of form } e^{st} \text{ in CT} \\ \text{signals of form } z^{n} \text{ in DT} \end{cases}$$

The Response of an LTI System:

input $\rightarrow$ LTI $\rightarrow$ output

$x(t)$ $h(t)$ $y(t)$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau) h(t-\tau) d\tau$$

$$\begin{cases} \text{CT:} \quad e^{st} \longrightarrow H(s) e^{st} \\ \text{DT:} \quad z^{n} \longrightarrow H(z) z^{n} \end{cases}$$

eigenfunction

eigenvalue

Response of LTI Systems to Complex Exponentials

Let $x(t) = e^{st}$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau) x(t-\tau) d\tau$$

$$= \int_{-\infty}^{+\infty} h(\tau) e^{s(t-\tau)} d\tau$$

$$= e^{st} \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau$$

$$\Rightarrow \; y(t) = H(s) x(t) = H(s) e^{st}$$

$$H(s) = \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau$$

Let $x[n] = z^n$

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k] x[n-k]$$

$$= \sum_{k=-\infty}^{+\infty} h[k] z^{n-k}$$

$$= z^n \sum_{k=-\infty}^{+\infty} h[k] z^{-k}$$

$$\Rightarrow \; y[n] = H(z) x[n] = H(z) z^n$$

$$H(z) = \sum_{k=-\infty}^{+\infty} h[k] z^{-k}$$

Eigenfunctions and Superposition Properties:

$$e^{s_k t} \longrightarrow \boxed{\text{LTI}} \longrightarrow H(s_k)\ e^{s_k t}$$

$$k = 1, 2, 3$$

$$e^{s_1 t} \longrightarrow H(s_1)\ e^{s_1 t}$$

$$e^{s_2 t} \longrightarrow H(s_2)\ e^{s_2 t}$$

$$e^{s_3 t} \longrightarrow H(s_3)\ e^{s_3 t}$$

$$x(t) = a_1\, e^{s_1 t} + a_2\, e^{s_2 t} + a_3\, e^{s_3 t}$$

$$y(t) = a_1\, H(s_1)\, e^{s_1 t} + a_2\, H(s_2)\, e^{s_2 t} + a_3\, H(s_3)\, e^{s_3 t}$$

$$\Rightarrow\ x(t) = \sum_k a_k e^{s_k t} \longrightarrow y(t) = \sum_k a_k H(s_k) e^{s_k t}$$

$$\Rightarrow\ x[n] = \sum_k a_k {z_k}^n \longrightarrow y[n] = \sum_k a_k H(z_k) {z_k}^n$$

Outline

- A Historical Perspective
- The Response of LTI Systems to Complex Exponentials
- Fourier Series Representation of Continuous-Time Periodic Signals
- Convergence of the Fourier Series
- Properties of Continuous-Time Fourier Series
- Fourier Series Representation of Discrete-Time Periodic Signals
- Properties of Discrete-Time Fourier Series
- Fourier Series & LTI Systems
- Filtering & Examples of CT & DT Filters

- Harmonically related complex exponentials

$$w_0 = \frac{2\pi}{T}$$

$$\phi_k(t) = e^{jkw_0 t} = e^{jk\left(\frac{2\pi}{T}\right)t}, \qquad k = 0, \pm 1, \pm 2, \ldots$$

- The Fourier Series Representation:

$$x(t) = \cdots a_{-2}\,\phi_{-2}(t) + a_{-1}\,\phi_{-1}(t) + a_0\,\phi_0(t) + a_1\,\phi_1(t) + a_2\,\phi_2(t) + \ldots$$

$$= \sum_{k=-\infty}^{+\infty} a_k\,\phi_k(t) = \sum_{k=-\infty}^{+\infty} a_k\,e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k\,e^{jk\left(\frac{2\pi}{T}\right)t}$$

$k = +1, -1$: the first harmonic components
or, the fundamental components

$k = +2, -2$: the second harmonic components

$\cdots$ etc.

- Example 3.2:

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk(2\pi)t}$$

$$a_0 = 1$$
$$a_1 = a_{-1} = \frac{1}{4}$$
$$a_2 = a_{-2} = \frac{1}{2}$$
$$a_3 = a_{-3} = \frac{1}{3}$$

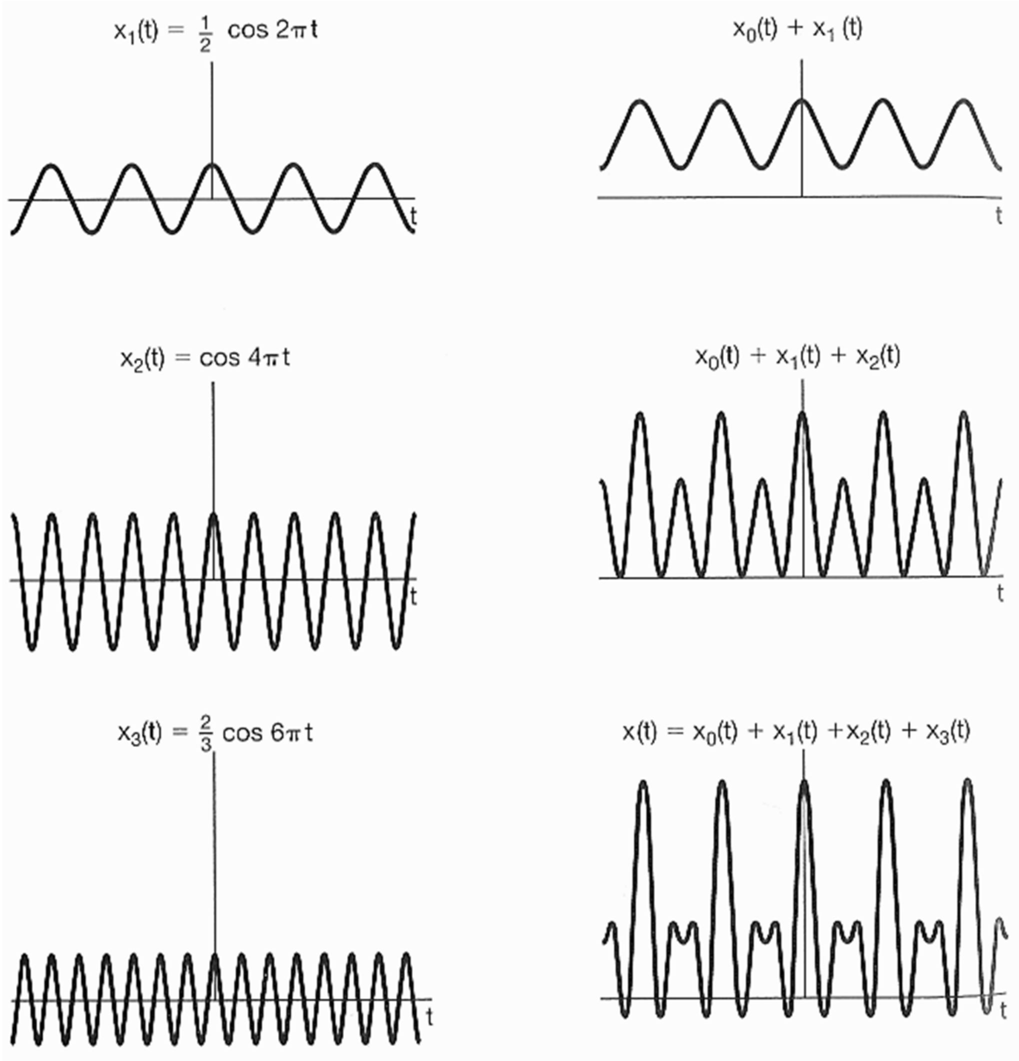
$$\Rightarrow \; x(t) = 1 + \frac{1}{4}(e^{j2\pi t} + e^{-j2\pi t}) + \frac{1}{2}(e^{j4\pi t} + e^{-j4\pi t})$$
$$+ \frac{1}{3}(e^{j6\pi t} + e^{-j6\pi t})$$

$$\Rightarrow \; x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t$$

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$
$$\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$$
$$\sin(\theta) = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$$

Fourier Series Representation of CT Periodic Signals

$$x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t$$



▪ Procedure of Determining the Coefficients:

$$w_0 = \frac{2\pi}{T}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

$$x(t)e^{-jnw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} e^{-jnw_0 t}$$

$$\int_0^T x(t)e^{-jnw_0 t}dt = \int_0^T \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} e^{-jnw_0 t} dt$$

$$= \sum_{k=-\infty}^{+\infty} a_k \left[\int_0^T e^{j(k-n)w_0 t} dt \right]$$

$$\int_0^T e^{j(k-n)w_0 t} dt = \int_0^T \cos\left((k-n)w_0 t\right)\ dt + j\int_0^T \sin\left((k-n)w_0 t\right)\ dt$$

Procedure of Determining the Coefficients:

$$\int_0^T e^{j(k-n)w_0 t} dt = \int_0^T \cos\left((k-n)w_0 t\right) \, dt + j \int_0^T \sin\left((k-n)w_0 t\right) \, dt$$

$$= \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases}$$

$$\Rightarrow \int_0^T x(t) e^{-jnw_0 t} dt = a_n T \qquad \Rightarrow a_n = \frac{1}{T} \int_0^T x(t) e^{-jnw_0 t} dt$$

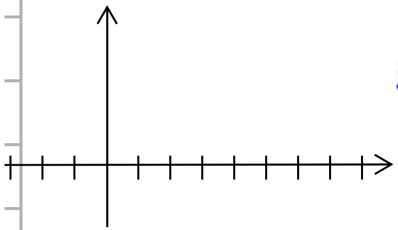
$$\Rightarrow a_k = \frac{1}{T} \int_0^T x(t) e^{-jkw_0 t} dt$$

- Furthermore,

$$\int_T e^{j(k-n)w_0 t} dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases} \qquad \Rightarrow a_k = \frac{1}{T} \int_T x(t) e^{-jkw_0 t} dt$$

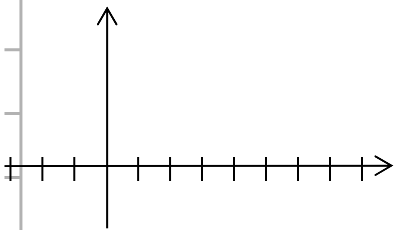
In Summary:

- The synthesis equation:



$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

- The analysis equation:



$$a_k = \frac{1}{T}\int_T x(t) e^{-jkw_0 t} dt = \frac{1}{T}\int_T x(t) e^{-jk(2\pi/T)t} dt$$

- $x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$: CT Fouries series pair

- $\{a_k\}$: the Fourier series coefficients
  or the spectral coefficients of $x(t)$

- $a_0 = \frac{1}{T}\int_T x(t) dt$, the dc or constant component of $x(t)$

- Example 3.4:

$$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$$

$$\sin(\theta) = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$$

$$x(t) = 1 + \sin w_0t + 2\cos w_0t + \cos\left(2w_0t + \frac{\pi}{4}\right)$$

$$\Rightarrow x(t) = 1 + \frac{1}{2j}\left[e^{jw_0t} - e^{-jw_0t}\right] + \left[e^{jw_0t} + e^{-jw_0t}\right]$$

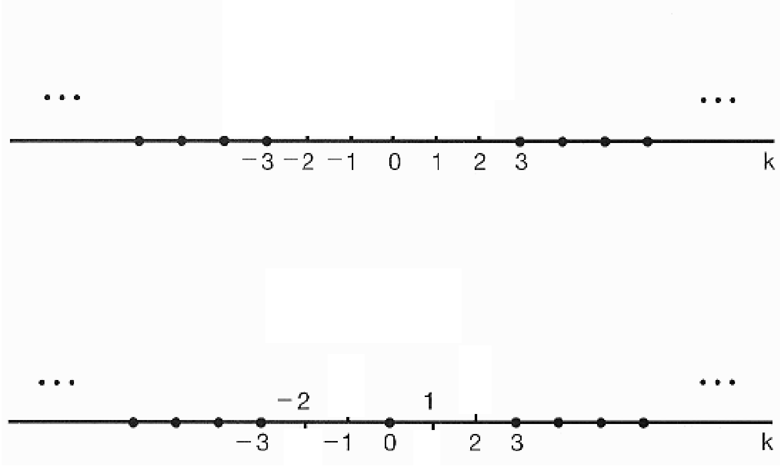
$$+\frac{1}{2}\left[e^{j(2w_0t+\pi/4)} + e^{-j(2w_0t+\pi/4)}\right]$$

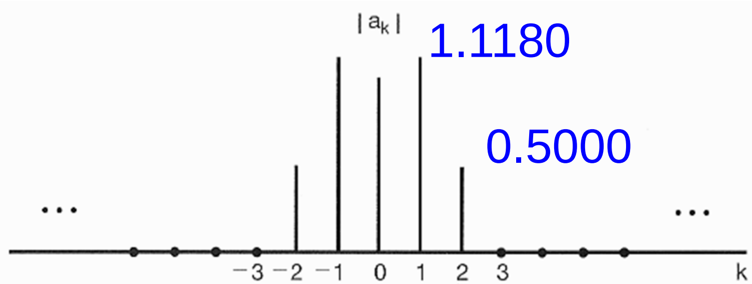
$$\Rightarrow x(t) = 1 + \left(1 + \frac{1}{2j}\right)e^{jw_0t} + \left(1 - \frac{1}{2j}\right)e^{-jw_0t}$$

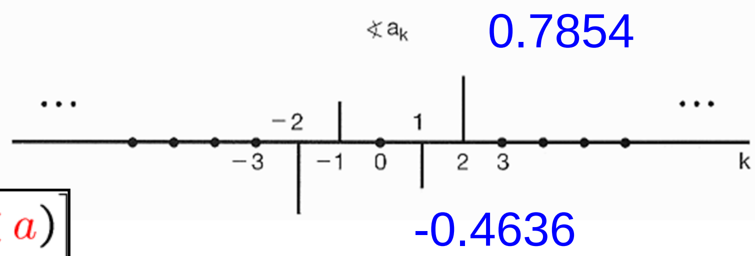
$$+\left(\frac{1}{2}e^{j(\pi/4)}\right)e^{j2w_0t} + \left(\frac{1}{2}e^{-j(\pi/4)}\right)e^{-j2w_0t}$$

- Example 3.4:

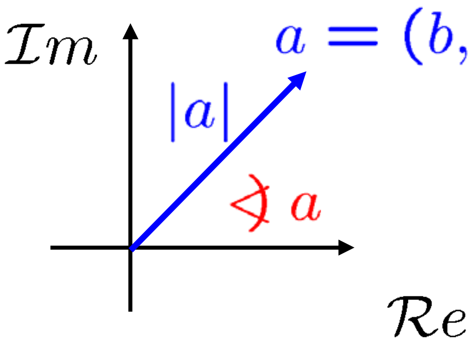
$$
\Rightarrow \begin{cases} a_0 = 1, \\ a_1 = \left(1 + \frac{1}{2j}\right) = 1 - \frac{1}{2}j, \\ a_{-1} = \left(1 - \frac{1}{2j}\right) = 1 + \frac{1}{2}j, \\ a_2 = \frac{1}{2}e^{j(\pi/4)} = \frac{\sqrt{2}}{4}(1+j), \\ a_{-2} = \frac{1}{2}e^{-j(\pi/4)} = \frac{\sqrt{2}}{4}(1-j), \\ a_k = 0, \qquad |k| > 2. \end{cases}
$$








```
>> a1 = 1-0.5j
>> abs(a1)
>> angle(a1)
```

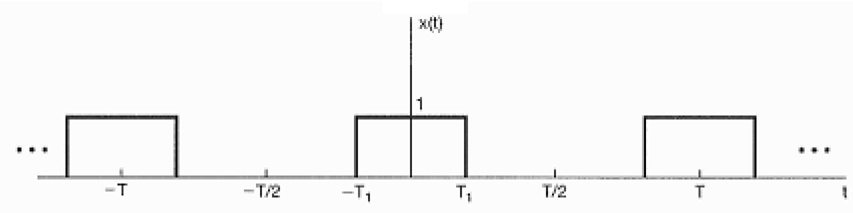


$$\boxed{a = |a| e^{j \measuredangle a}}$$

$$\boxed{a = |a| \left[\cos(\measuredangle a) + j \sin(\measuredangle a) \right]}$$

$$\boxed{a = b + jc = \sqrt{b^2+c^2} \left[\frac{b}{\sqrt{b^2+c^2}} + j \frac{c}{\sqrt{b^2+c^2}} \right]}$$

- Example 3.5: $a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$



$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

$k = 0$ $\quad a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}$

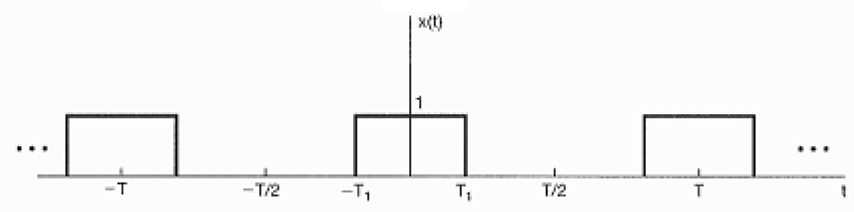
$k \neq 0$ $\quad a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jkw_0 t}dt = \frac{1}{T}\frac{1}{(-jkw_0)}e^{-jkw_0 t}\Big|_{-T_1}^{T_1}$

$$= \frac{1}{jkw_0 T}\left[e^{jkw_0 T_1} - e^{-jkw_0 T_1}\right] /$$

$$w_0 = \frac{2\pi}{T}$$

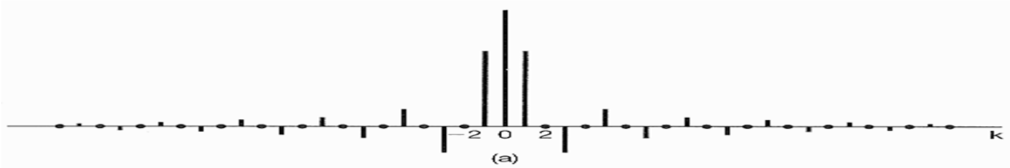
$$= \frac{2\sin(kw_0 T_1)}{kw_0 T} = \frac{\sin(kw_0 T_1)}{k\pi} = \frac{\sin(k(2\pi/T)T_1)}{k\pi}, \quad k \neq 0$$

- **Example 3.5:** $T = 4T_1$



$$a_k = \frac{\sin(k 2\pi \frac{T_1}{T})}{k\pi} = \frac{\sin(k\frac{\pi}{2})}{k\pi}$$

a_k

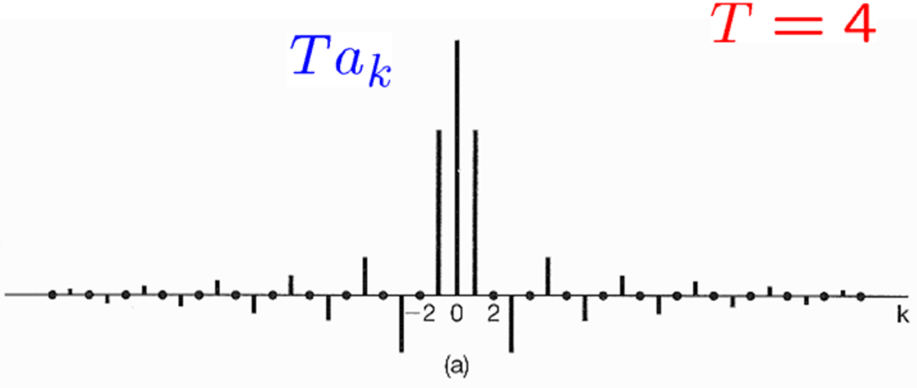


$$T \; a_k = T \frac{\sin(k 2\pi \frac{T_1}{T})}{k\pi} = T \frac{\sin(k\frac{\pi}{2})}{k\pi}$$

$T_1 = 1$

$T = 4$

$T a_k$



- Example 3.5:

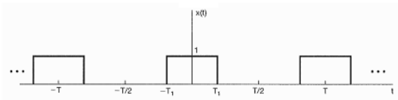
$$a_k = \frac{\sin(k 2\pi \frac{T_1}{T})}{k\pi}$$

$T_1 = 1$

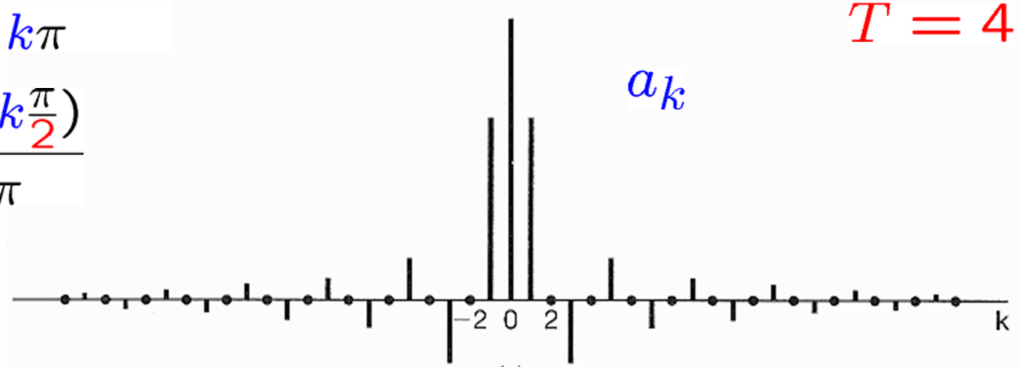
$T = 4$

$T = 4T_1$

$$a_k = \frac{\sin(k \frac{\pi}{2})}{k\pi}$$



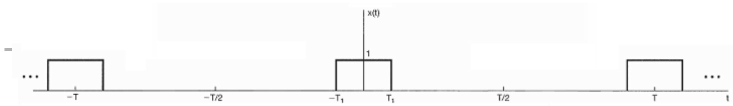
a_k



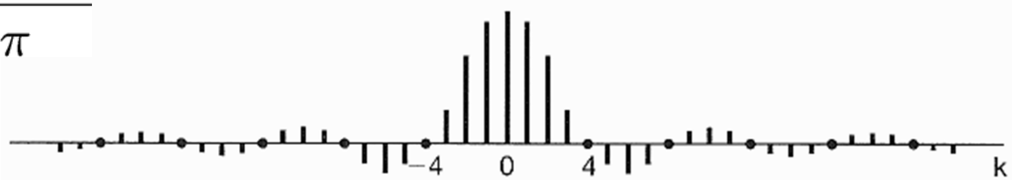
(a)

$T = 8T_1$

$$a_k = \frac{\sin(k \frac{\pi}{4})}{k\pi}$$



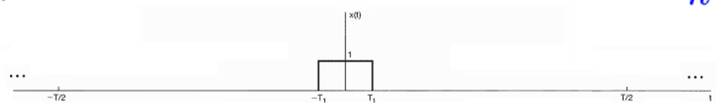
a_k



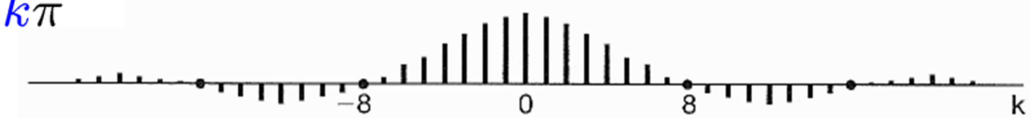
(b)

$T = 16T_1$

$$a_k = \frac{\sin(k \frac{\pi}{8})}{k\pi}$$



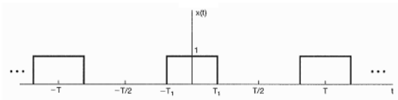
a_k



(c)

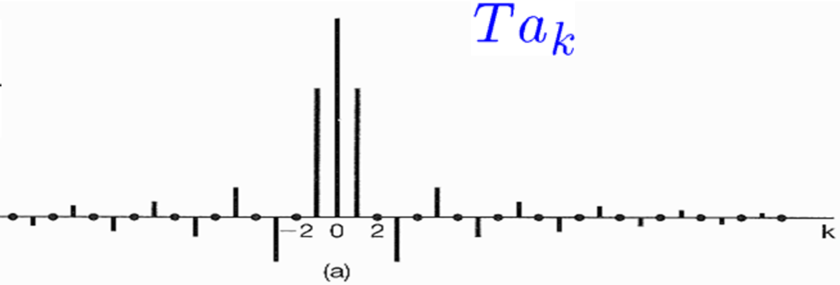
- Example 3.5: $T\ a_k = T\dfrac{\sin(k2\pi\frac{T_1}{T})}{k\pi}$

$T = 4T_1$



$Ta_k = T\dfrac{\sin(k\frac{\pi}{2})}{k\pi}$

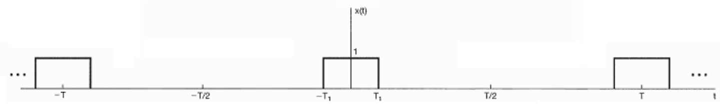
Ta_k

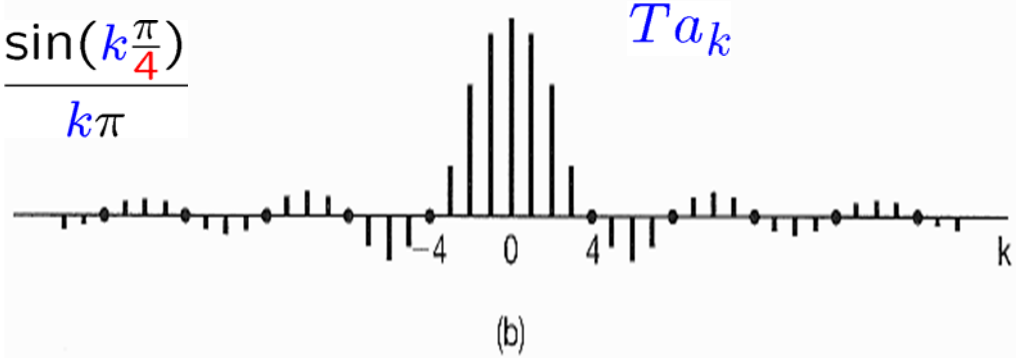


$T = 8T_1$

$Ta_k = T\dfrac{\sin(k\frac{\pi}{4})}{k\pi}$

Ta_k

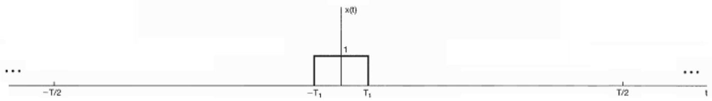


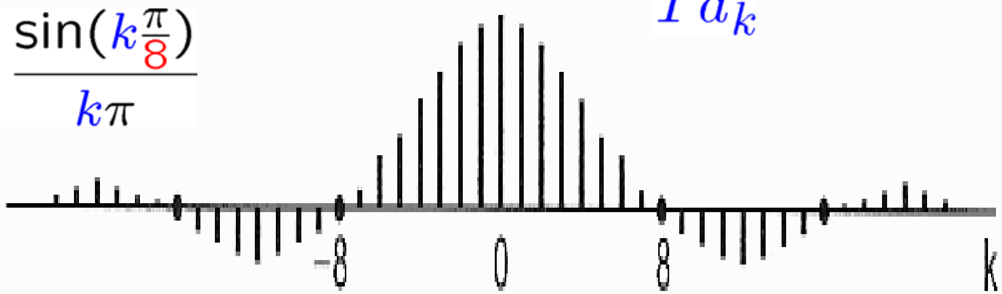


$T = 16T_1$

$Ta_k = T\dfrac{\sin(k\frac{\pi}{8})}{k\pi}$

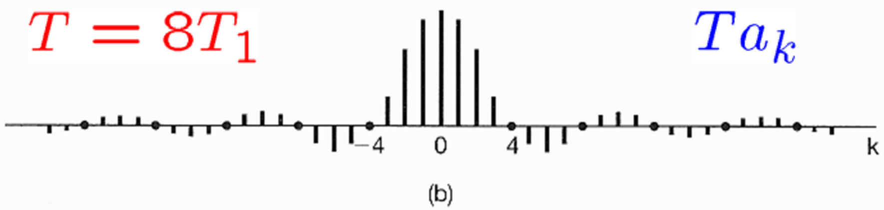
Ta_k

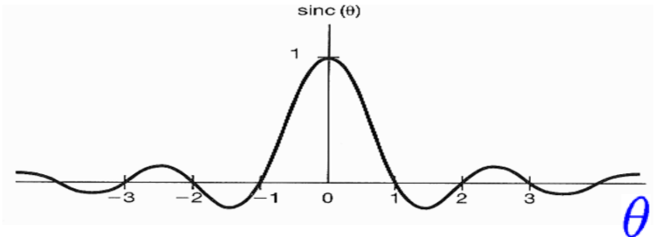




- Example 3.5:

$T = 8T_1$ $\qquad Ta_k$





θ

$$\text{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$$

$$Ta_k = T\,\frac{\sin(k\frac{\pi}{4})}{k\pi}$$

$$= \frac{1}{4}\;T\;\frac{\sin(\pi\frac{k}{4})}{\pi\frac{k}{4}}$$

$$= \frac{1}{4}\;T\;\text{sinc}(\frac{k}{4})$$

$$w_0 = \frac{2\pi}{T}$$

$$w = kw_0$$

$$Ta_k = \frac{2\sin(wT_1)}{w}$$

$$wT_1 = k\left(\frac{2\pi}{T}\right)\cdot T_1$$

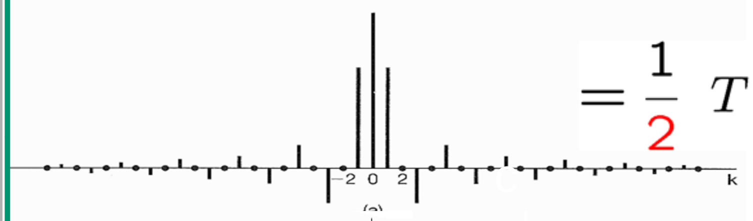
$$= \frac{2k\pi}{A}$$

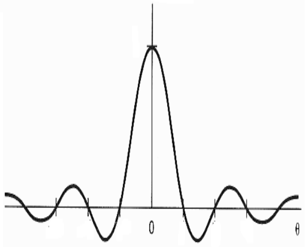
- Example 3.5:

$$T\ a_k = T\ \frac{\sin(k2\pi\frac{T_1}{T})}{k\pi}$$

$T = 4T_1$

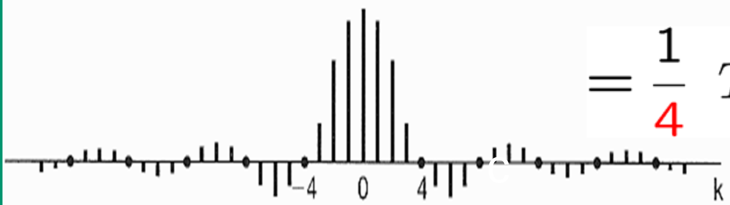
$$Ta_k = T\ \frac{\sin(k\frac{\pi}{2})}{k\pi} = \frac{1}{2}\ T\ \text{sinc}(\frac{k}{2})$$

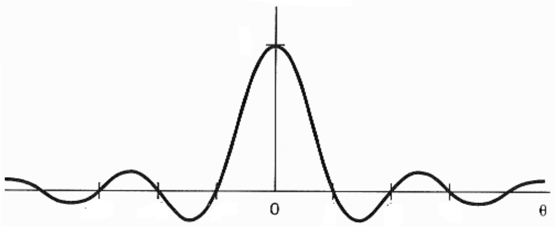




$T = 8T_1$

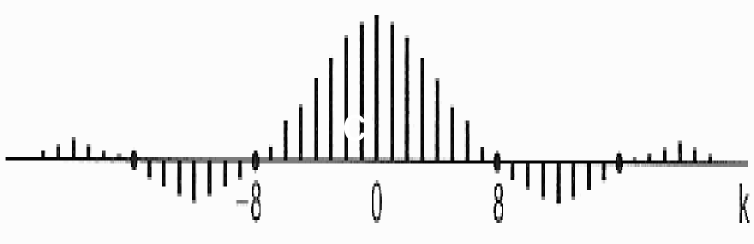
$$Ta_k = T\ \frac{\sin(k\frac{\pi}{4})}{k\pi} = \frac{1}{4}\ T\ \text{sinc}(\frac{k}{4})$$

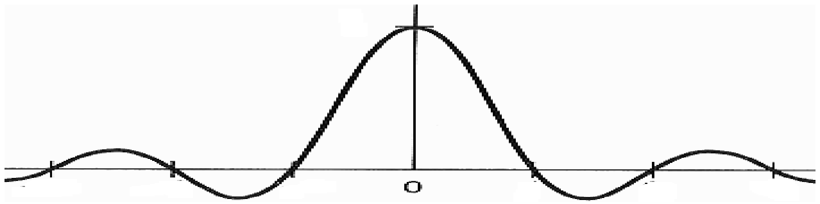




$T = 16T_1$

$$Ta_k = T\ \frac{\sin(k\frac{\pi}{8})}{k\pi} = \frac{1}{8}\ T\ \text{sinc}(\frac{k}{8})$$



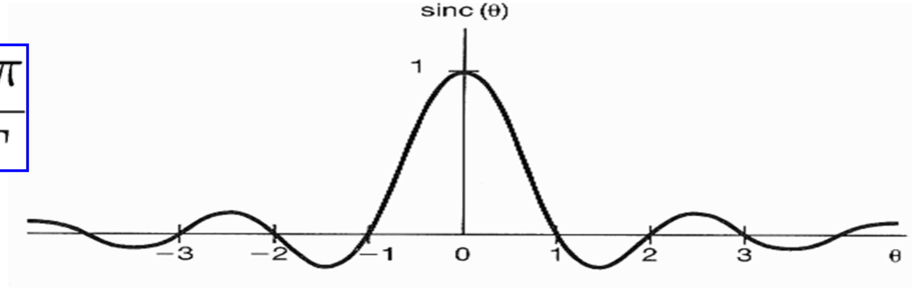


- Example 3.5:

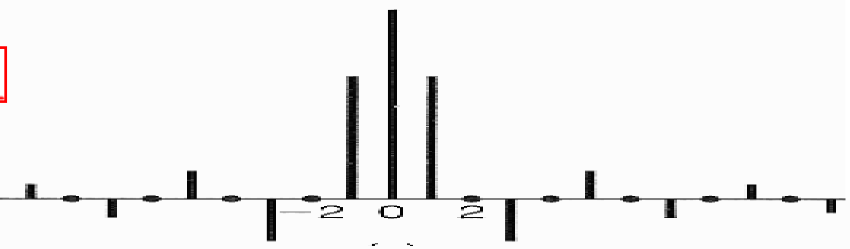
$$Ta_k = T \frac{2\sin(kw_0T_1)}{kw_0T}$$

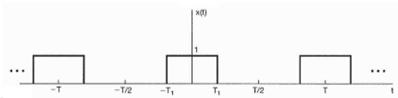
$$= T_1 \frac{2\sin(kw_0T_1)}{kw_0T_1}$$

$$\boxed{w_0 = \frac{2\pi}{T}}$$



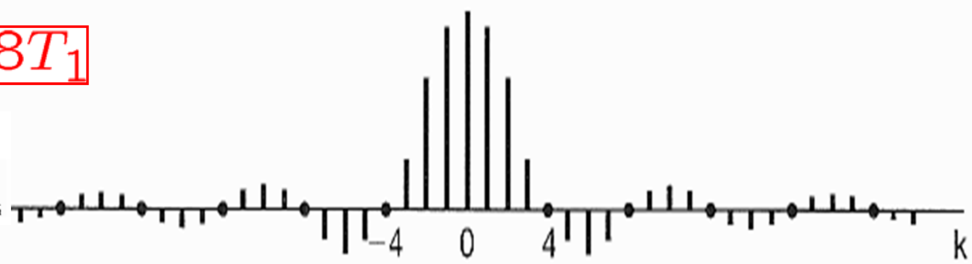
$\boxed{T = 4T_1}$

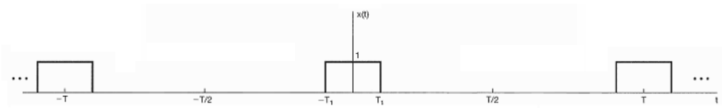




$$= \frac{1}{2}\ T\ \text{sinc}\left(\frac{k}{2}\right)$$

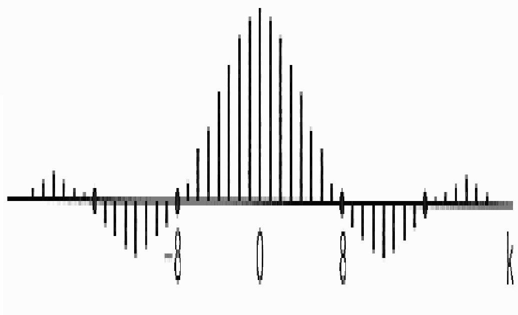
$\boxed{T = 8T_1}$

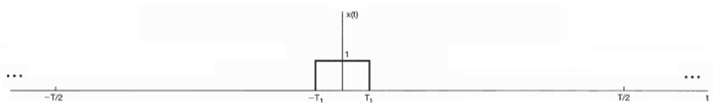




$$= \frac{1}{4}\ T\ \text{sinc}\left(\frac{k}{4}\right)$$

$\boxed{T = 16T_1}$





$$= \frac{1}{8}\ T\ \text{sinc}\left(\frac{k}{8}\right)$$

Fourier Series of Real Periodic Signals:

- If $x(t)$ is real, then $x^*(t) = x(t)$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0t}$$

$$(a+b)^* = (a^*+b^*)$$
$$(a\times b)^* = (a^*\times b^*)$$

$$\Rightarrow \; x(t) = x(t)^* = \left(\sum_{k=-\infty}^{+\infty} a_k e^{jkw_0t} \right)^*$$

$$= \sum_{k=-\infty}^{+\infty} a_k^* e^{-jkw_0t}$$

$$= \sum_{m=+\infty}^{-\infty} {a_{-m}}^* e^{jmw_0t} \qquad m = -k$$

$$= \sum_{k=-\infty}^{+\infty} a^*_{-k} e^{jkw_0t} \qquad k = m$$

$$\Rightarrow \; a^*_{-k} = a_k \quad \text{or,} \quad a_k^* = a_{-k}$$

Alternative Forms of the Fourier Series:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

$$\Rightarrow \; x(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jkw_0 t} + a_{-k} e^{-jkw_0 t} \right]$$

$$= a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jkw_0 t} + a_k^* e^{-jkw_0 t} \right]$$

$$a_k e^{jkw_0 t} + a_k^* e^{-jkw_0 t} = (R+jI)(C+jS)+(R-jI)(C-jS)$$

$$= (RC-IS)+j(RS+IC)+(RC-IS)-j(RS+IC)$$

$$= 2(RC-IS)$$

$$= a_0 + \sum_{k=1}^{\infty} 2\,\mathcal{R}e\left\{ a_k e^{jkw_0 t} \right\}$$

Alternative Forms of the Fourier Series:

- If $a_k = A_k \, e^{j\theta_k}$

$$\Rightarrow \; x(t) = a_0 + \sum_{k=1}^{\infty} 2 \, \mathcal{R}e \left\{ A_k \, e^{j\theta_k} \, e^{jkw_0 t} \right\}$$

$$= a_0 + \sum_{k=1}^{\infty} 2 \, \mathcal{R}e \left\{ A_k \, e^{j(kw_0 t + \theta_k)} \right\}$$

$$= a_0 + 2 \sum_{k=1}^{\infty} A_k \cos(kw_0 t + \theta_k)$$

$$\boxed{e^{j\theta} = \cos(\theta) + j\sin(\theta)}$$

- If $a_k = B_k + j \, C_k$

$$\boxed{(a+jb)(c+jd) = (ac-bd) + j(ad+bc)}$$

$$\boxed{C(a+b) = C(a)C(b) - S(a)S(b)}$$

$$\Rightarrow \; x(t) = a_0 + \sum_{k=1}^{\infty} 2 \, \mathcal{R}e \left\{ \left(B_k + j \, C_k \right) \, e^{jkw_0 t} \right\}$$

$$= a_0 + 2 \sum_{k=1}^{\infty} \left[B_k \cos(kw_0 t) - C_k \sin(kw_0 t) \right]$$

- A Historical Perspective
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- Fourier maintained that
  "any" periodic signal could be represented
  by a Fourier series

- The truth is that
  Fourier series can be used to represent
  an extremely large class of periodic signals

- The question is that
  when a periodic signal x(t) does in fact have a
  Fourier series representation?

$x(t)$

$$x_{FS}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t} \qquad a_k = \frac{1}{T}\int_T x(t) e^{-jk(2\pi/T)t} dt$$

$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jk(2\pi/T)t}$$

- One class of periodic signals:
  - Which have finite energy over a single period:

$$\int_T |x(t)|^2 \, dt < \infty \qquad \Rightarrow \qquad a_k = \frac{1}{T}\int_T x(t) e^{-jk w_0 t} dt < \infty$$

$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jk w_0 t}$$

$$e_N(t) = x(t) - x_N(t) \qquad\qquad e(t) = x(t) - \sum_{k=-\infty}^{+\infty} a_k e^{jk w_0 t}$$

$$E_N(t) = \int_T |e_N(t)|^2 \, dt \qquad\qquad E(t) = \int_T |e(t)|^2 \, dt \; = \; 0$$

$$\rightarrow \; 0 \quad \text{as } N \rightarrow \infty \qquad\qquad x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk w_0 t}, \quad \forall t \; ???$$

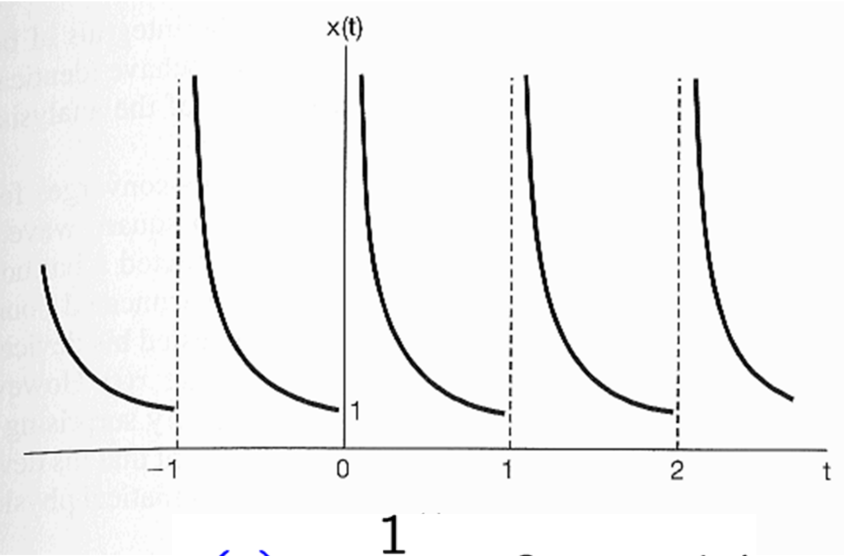
- The other class of periodic signals:
  - Which satisfy Dirichlet conditions:
  - Condition 1:
    - Over any period,
      x(t) must be absolutely integrable,
      i.e.,

Johann Peter Gustav Lejeune Dirichlet
1805-1859
Born in Germany
Photo from wikipedia

$$\int_T |x(t)|\, dt < \infty$$

$$\Rightarrow \quad |a_k| \;\leq\; \frac{1}{T}\int_T \left| x(t) e^{-jkw_0 t} \right| dt$$

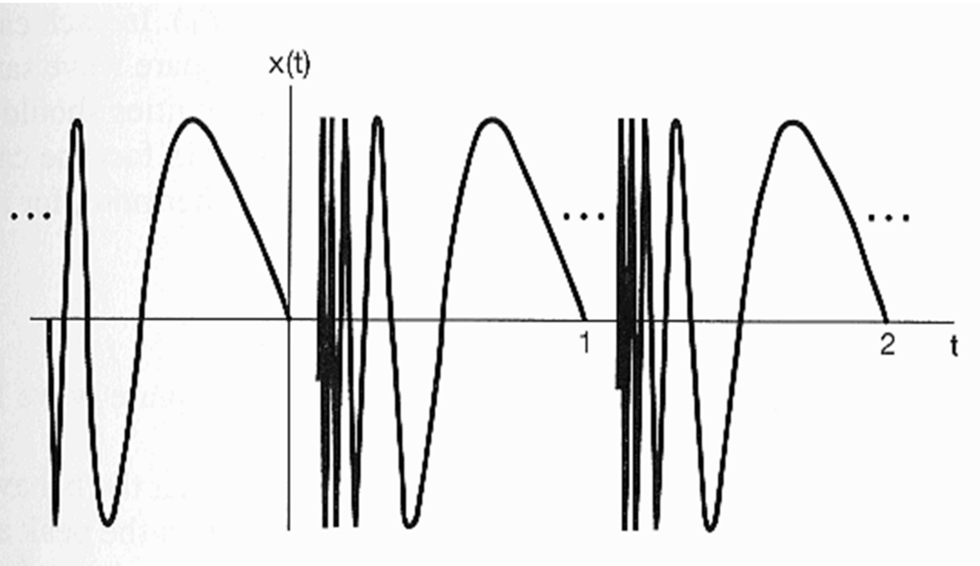
$$= \frac{1}{T}\int_T \left| x(t) \right| dt \;<\; \infty$$



$$x(t) = \frac{1}{t}, \qquad 0 < t \leq 1$$

The other class of periodic signals:

- Which satisfy Dirichlet conditions:
- Condition 2:
  - In any finite interval, x(t) is of bounded variation; i.e.,
  - There are no more than
    a finite number of maxima and minima
    during any single period of the signal

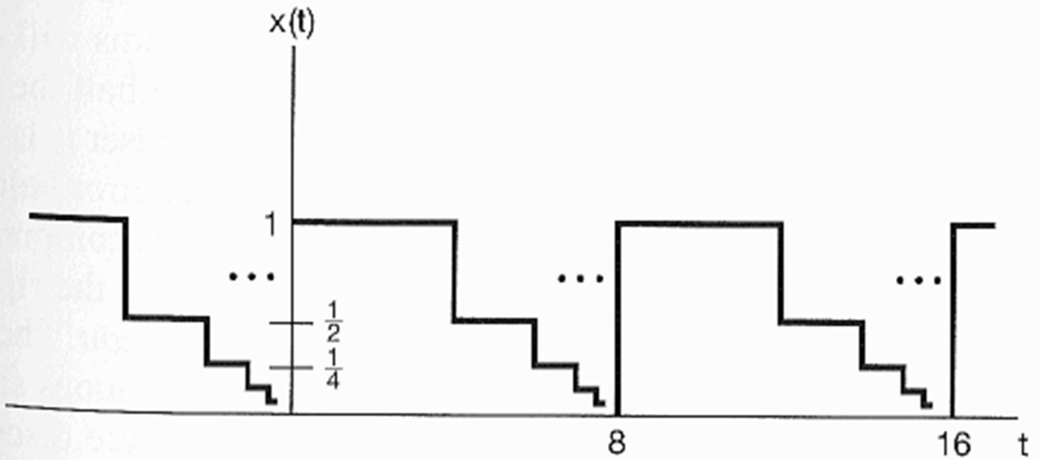


(b)

$$x(t) = \sin\left(\frac{2\pi}{t}\right), \quad 0 < t \leq 1$$

$$\int_0^1 |x(t)|\, dt < 1$$

- The other class of periodic signals:
  - Which satisfy Dirichlet conditions:
  - Condition 3:
    - In any finite interval,
      x(t) has only finite number of discontinuities.
    - Furthermore, each of these discontinuities is finite



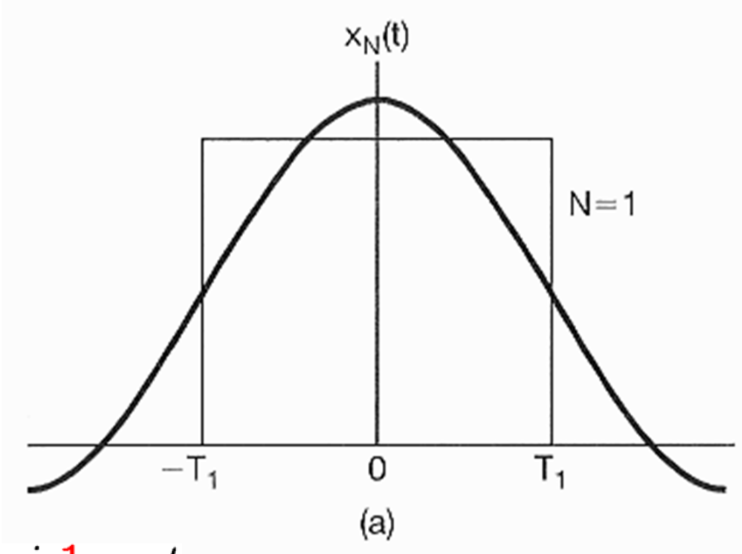
(c)

- How the Fourier series converges for a periodic signal with discontinuities

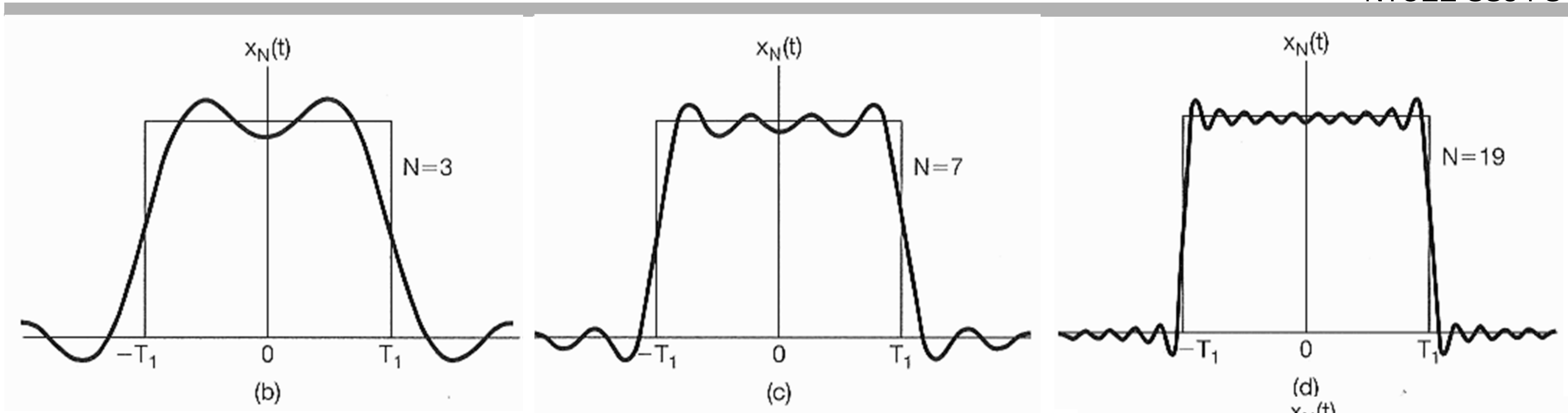
Albert Abraham Michelson
1852-1931
Polish-born German-American
Photo from wikipedia

  - In 1898,
    Albert Michelson (an American physicist)
    used his harmonic analyzer
    to compute
    the truncated Fourier series approximation
    for the square wave

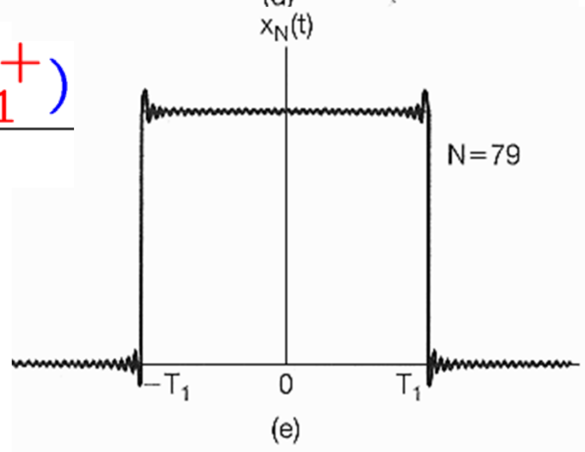
$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jkw_0 t}$$



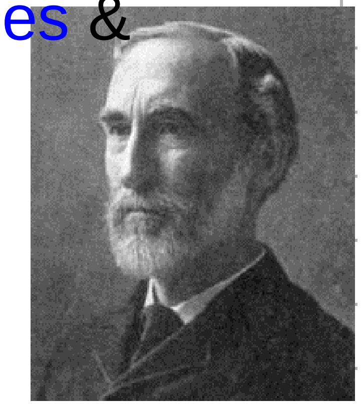
$$x_1(t) = a_{-1} e^{-j\cdot 1\cdot w_0 t} + a_0 + a_1 e^{j\cdot 1\cdot w_0 t}$$



$$x_N(T_1) = \frac{x(T_1^-) + x(T_1^+)}{2}$$



- Michelson wrote to Josiah Gibbs
- In 1899, Gibbs showed that
  - the partial sum near discontinuity exhibits ripples &
  - the peak amplitude remains constant with increasing N
- The Gibbs phenomenon



Josiah Willard Gibbs
1839-1903
Born in USA
Photo from wikipedia

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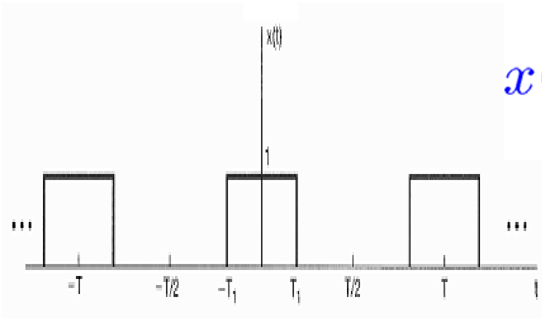
CT Fourier Series Representation:

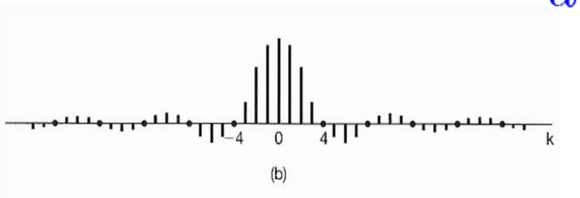
- The synthesis equation:



$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

- The analysis equation:

$$a_k = \frac{1}{T}\int_T x(t) e^{-jkw_0 t} dt = \frac{1}{T}\int_T x(t) e^{-jk(2\pi/T)t} dt$$



- $x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$: Fouries series pair

Outline

Section	Property
3.5.1	Linearity
3.5.2	Time Shifting
	Frequency Shifting
3.5.6	Conjugation
3.5.3	Time Reversal
3.5.4	Time Scaling
	Periodic Convolution
3.5.5	Multiplication
	Differentiation
	Integration
3.5.6	Conjugate Symmetry for Real Signals
3.5.6	Symmetry for Real and Even Signals
3.5.6	Symmetry for Real and Odd Signals
	Even-Odd Decomposition for Real Signals
3.5.7	Parseval's Relation for Periodic Signals

Linearity:

$$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0 t}dt$$

- $x(t), y(t)$: periodic signals with period T

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k \qquad x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

$$y(t) \xleftrightarrow{\mathcal{FS}} b_k \qquad y(t) = \sum_{m=-\infty}^{+\infty} b_m e^{jmw_0 t}$$

$$\Rightarrow \quad z(t) = Ax(t)+By(t) \xleftrightarrow{\mathcal{FS}} c_k = Aa_k+Bb_k$$

$$z(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jkw_0 t}$$

Add

Time Shifting:

- $x(t)$: periodic signal with period T

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$\Rightarrow \; x(t-t_0) \overset{\mathcal{FS}}{\longleftrightarrow} b_k = e^{-jkw_0 t_0} a_k = e^{-jk\left(\frac{2\pi}{T}\right)t_0} a_k$$

b/c $\quad b_k = \frac{1}{T}\int_T x(t-t_0) e^{-jkw_0 t} dt$

$$t - t_0 = \tau$$
$$t = \tau + t_0$$
$$dt = d\tau$$

$$= \frac{1}{T}\int_T x(\tau) e^{-jkw_0(\tau+t_0)} d\tau$$

$$= e^{-jkw_0 t_0} \; \frac{1}{T}\int_T x(\tau) e^{-jkw_0 \tau} d\tau$$

- Time Reversal:

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k \qquad x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}$$

$$\Rightarrow x(-t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_{-k} \qquad x(-t) = \sum_{k=-\infty}^{+\infty} a_k \, e^{-j\,k\left(\frac{2\pi}{T}\right)t}$$

$$= \sum_{m=-\infty}^{+\infty} a_{-m} \, e^{j\,m\left(\frac{2\pi}{T}\right)t}$$

$$-k = m$$

- If $x(t)$ is even, i.e., $x(-t) = x(t)$

  $\Rightarrow a_k$ is even, i.e., $a_{-k} = a_k$

- If $x(t)$ is odd, i.e., $x(-t) = -x(t)$

  $\Rightarrow a_k$ is odd, i.e., $a_{-k} = -a_k$

Time Scaling:

- $x(t)$: periodic signals with period T

  and fundamental frequency w_0

- $x(\alpha t)$: periodic signals with period $\frac{T}{\alpha}$

  and fundamental frequency αw_0

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}$$

$$x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k \, e^{j\,k\,w_0\,(\alpha\,t)} = \sum_{k=-\infty}^{+\infty} a_k \, e^{j\,k\,\alpha\,\left(\frac{2\pi}{T}\right)t}$$

$$= \sum_{k=-\infty}^{+\infty} a_k \, e^{j\,k\,(\alpha\,w_0)\,t} = \sum_{k=-\infty}^{+\infty} a_k \, e^{j\,k\,\left(\frac{2\pi}{(\frac{T}{\alpha})}\right)\,t}$$

▪ Multiplication:

$$z(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jkw_0 t}$$

- $x(t), y(t)$: periodic signals with period T

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k \qquad\qquad x(t) = \sum_{l=-\infty}^{+\infty} a_l e^{jlw_0 t}$$

$$y(t) \xleftrightarrow{\mathcal{FS}} b_k \qquad\qquad y(t) = \sum_{m=-\infty}^{+\infty} b_m e^{jmw_0 t}$$

$\Rightarrow x(t)y(t)$: also periodic with T

$$z(t) = x(t)y(t) \xleftrightarrow{\mathcal{FS}} c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

$$= \left(\sum_{l=-\infty}^{+\infty} e^{jlw_0 t}\right)\left(\sum_{m=-\infty}^{+\infty} e^{jmw_0 t}\right)$$

$$= \sum_{l=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} e^{j(l+m)w_0 t}$$

$$= \sum_{k=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} e^{j(\ k\ )w_0 t}$$

$$(a+b+c)(d+e+f)$$
$$= ad + ae + af$$
$$+bd + be + bf$$
$$+cd + ce + cf$$

Add

$$\boxed{x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}}$$

Differentiation:

- $x(t)$: periodic signals with period T

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$\frac{d}{dt} x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} jkw_0 \, a_k$$

$$\frac{d}{dt} x(t) = \sum_{k=-\infty}^{+\infty} a_k \frac{d}{dt}\left(e^{jkw_0 t} \right)$$

$$= \sum_{k=-\infty}^{+\infty} a_k \left(jkw_0 t \right) e^{jkw_0 t}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

Integration:

- $x(t)$: periodic signals with period T

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$\int_{-\infty}^{t} x(\tau)d\tau \stackrel{\mathcal{FS}}{\longleftrightarrow} \frac{1}{jkw_0} a_k$$

only if $a_0 = 0$,
it is finite velaued
and periodic

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

$$= \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

▪ Conjugation & Conjugate Symmetry:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \qquad x(t)^* \overset{\mathcal{FS}}{\longleftrightarrow} a_{-k}{}^*$$

$$x(t)^* = \left(\sum_{k=-\infty}^{+\infty} a_k \; e^{jkw_0 t} \right)^*$$

$$= \sum_{k=-\infty}^{+\infty} a_k^* \; e^{-jkw_0 t}$$

$$= \sum_{k=-\infty}^{+\infty} a_k^* \; e^{j(-k)w_0 t}$$

$$-k = m$$

$$= \sum_{m=+\infty}^{-\infty} a_{-m}^* \left(e^{j \; m \; w_0 t} \right)$$

$$m = k$$

$$x^*(t) = \sum_{k=-\infty}^{+\infty} a_{-k}^* \left(e^{j \; k \; w_0 t} \right)$$

Compare!

$$x(t) = \sum a_k e^{jkw_0 t}$$

Conjugation & Conjugate Symmetry:

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \qquad x(t)^* \overset{\mathcal{FS}}{\longleftrightarrow} a_{-k}{}^*$$

- $x(t) = x(t)^* \Rightarrow a_{-k} = a_k^*$

  $x(t)$ is real $\Rightarrow \{a_k\}$ are conjugate symmetric

- $x(t) = x(t)^*$ & $x(-t) = x(t) \Rightarrow a_{-k} = a_k^*$ & $a_{-k} = a_k$

  $\Rightarrow a_k = a_k^*$

  $x(t)$ is real & even $\Rightarrow \{a_k\}$ are real & even

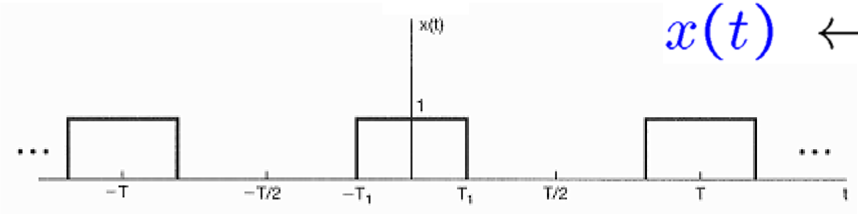
- $x(t)$ is real & odd $\Rightarrow \{a_k\}$ are purely imaginary & odd

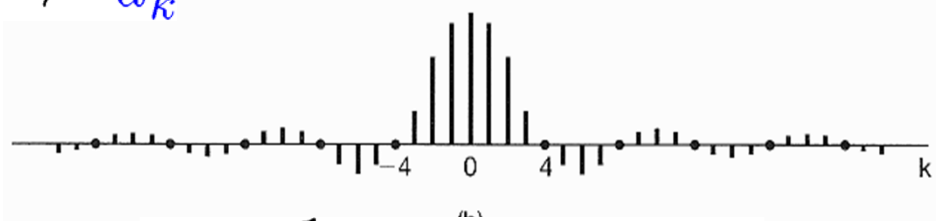
  $\Rightarrow a_k^* = -a_k$

Parseval's relation for CT periodic signals:

- As shown in Problem 3.46:

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$





$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} \qquad a_k = \frac{1}{T}\int_T x(t) e^{-jkw_0 t} dt$$

$$\frac{1}{T}\int_T \left|x(t)\right|^2 dt = \sum_{k=-\infty}^{+\infty} \left|a_k\right|^2$$

- Parseval's relation states that
  the total average power in a periodic signal
  equals
  the sum of the average powers
  in all of its harmonic components

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T} a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\left\{\begin{array}{l} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{array}\right.$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\left\{\begin{array}{ll} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{array}\right.$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

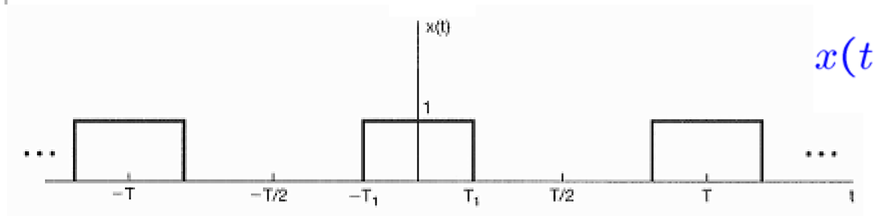
Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

- Example 3.6: $x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$

$$a_k = \frac{1}{T}\int_T x(t) e^{-jkw_0 t} dt$$

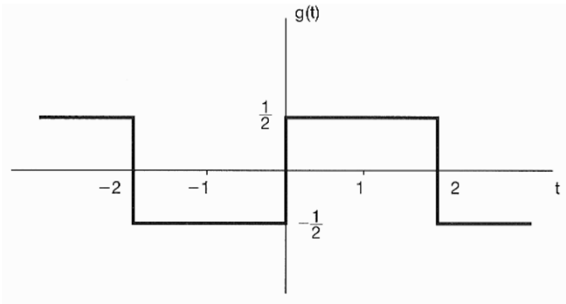
$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$



$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

$$a_0 = \frac{2T_1}{T}$$

$$a_k = \frac{\sin(k(2\pi/T)T_1)}{k\pi}, \quad k \neq 0$$



$$g(t) = x(t-1) - 1/2 \quad \text{with } T = 4, T_1 = 1$$

$$x(t-1) \overset{\mathcal{FS}}{\longleftrightarrow} b_k = a_k e^{-jk\pi/2}$$

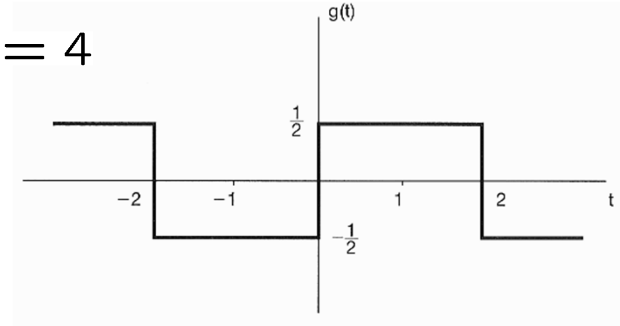
$$g(t) = x(t-1) - 1/2 \overset{\mathcal{FS}}{\longleftrightarrow} \begin{cases} a_k e^{-jk\pi/2}, & \text{for } k \neq 0 \\ a_0 - 1/2, & \text{for } k = 0 \end{cases}$$

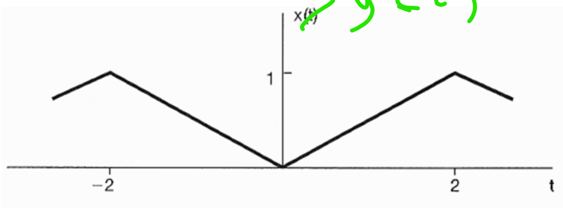
$$g(t) \overset{\mathcal{FS}}{\longleftrightarrow} \begin{cases} \frac{\sin(k\pi/2)}{k\pi} e^{-jk\pi/2}, & \text{for } k \neq 0 \\ 0, & \text{for } k = 0 \end{cases}$$

- Example 3.7:

$T = 4$

$\omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{2}$





$y(t)$

$$g(t) \overset{\mathcal{FS}}{\longleftrightarrow} d_k$$

$$y(t) \overset{\mathcal{FS}}{\longleftrightarrow} e_k$$

$$\frac{d}{dt}y(t) \overset{\mathcal{FS}}{\longleftrightarrow} jkw_0\, e_k$$

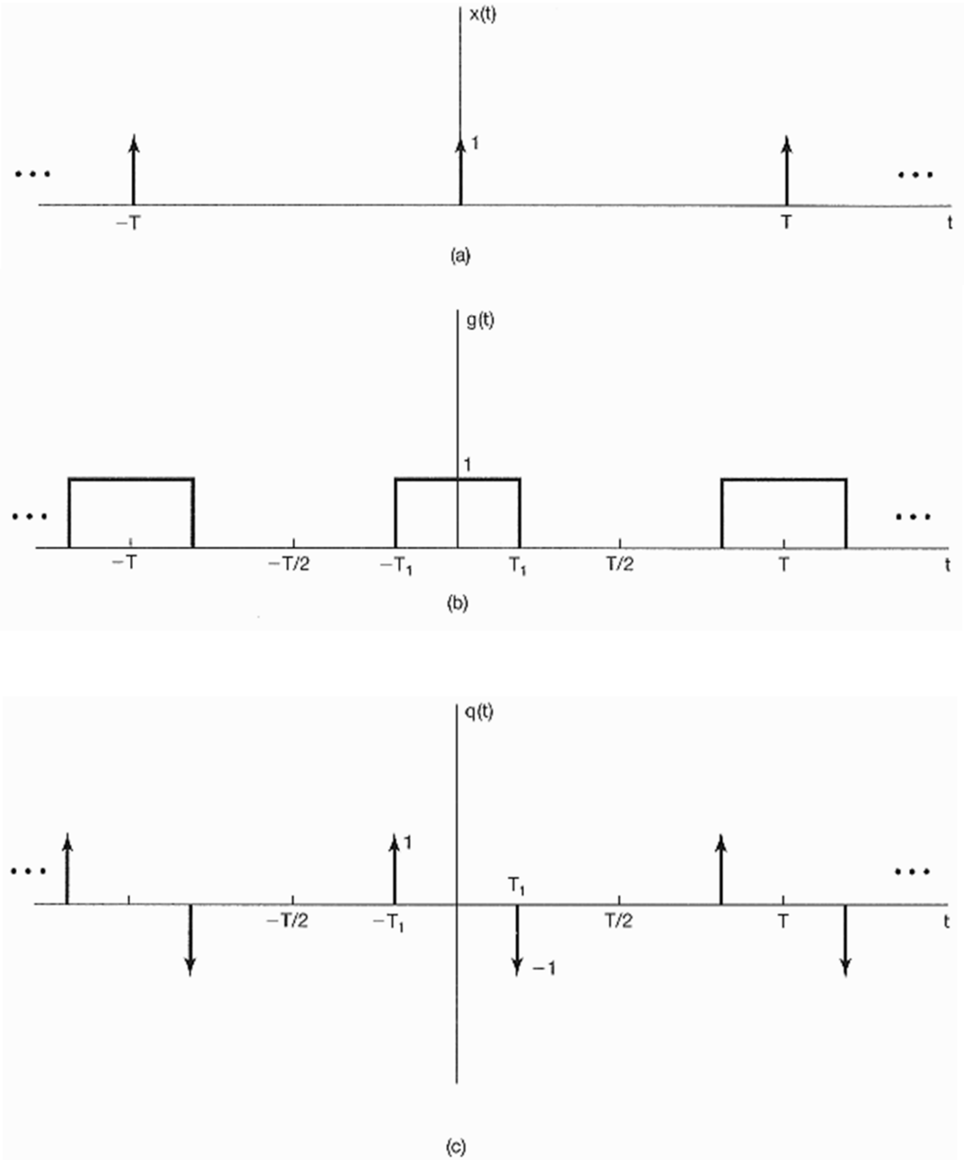
$d_k = jk\omega_0\, e_k$

$$g(t) = \frac{d}{dt}y(t) \iff d_k = jk(\pi/2)e_k$$

$$e_k = \begin{cases} \frac{2}{jk\pi}\, d_k = \frac{2\sin(\pi k/2)}{j(k\pi)^2}e^{-jk\pi/2}, & \text{for } k \neq 0 \\ \frac{1}{2}, & \text{for } k = 0 \end{cases}$$

- Example 3.8:

$$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0 t}dt$$



$$x(t) \xleftrightarrow{\mathcal{FS}} a_k = \frac{1}{T}$$

$$g(t) \xleftrightarrow{\mathcal{FS}} c_k$$

$$q(t) = \frac{d}{dt}g(t) \iff b_k = jkw_0 c_k$$

$$q(t) \xleftrightarrow{\mathcal{FS}} b_k$$

$$q(t) = x(t+T_1) - x(t-T_1)$$

$$\iff b_k = e^{jkw_0 T_1}a_k - e^{-jkw_0 T_1}a_k$$

- Example 3.8:

$$b_k = e^{jkw_0T_1}a_k - e^{-jkw_0T_1}a_k$$

$$= \frac{1}{T}\left[e^{jkw_0T_1} - e^{-jkw_0T_1}\right]$$

$$= \frac{2j\sin(kw_0T_1)}{T}$$

$$b_k = jkw_0c_k$$

$$k \neq 0 \qquad c_k = \frac{b_k}{jkw_0} = \frac{2j\sin(kw_0T_1)}{jkw_0T} = \frac{\sin(kw_0T_1)}{k\pi}$$

$$k = 0 \qquad c_0 = \frac{2T_1}{T}$$

- A Historical Perspective
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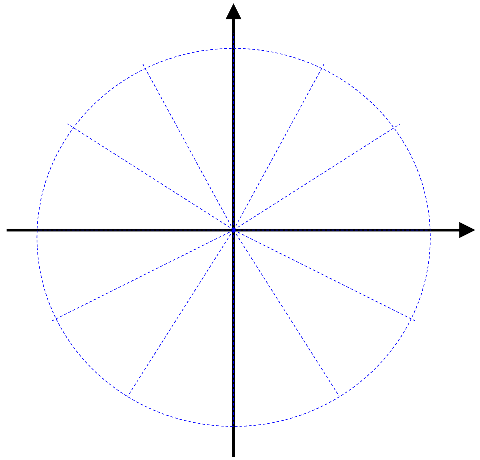
- Filtering & Examples of CT & DT Filters

- Harmonically related complex exponentials

$$\phi_k[n] = e^{jkw_0 n} = e^{jk\left(\frac{2\pi}{N}\right)n}, \qquad k = 0, \pm 1, \pm 2, \ldots$$

$$\phi_{k+N}[n] = e^{j(k+N)\left(\frac{2\pi}{N}\right)n} = e^{jk\left(\frac{2\pi}{N}\right)n} e^{jN\left(\frac{2\pi}{N}\right)n}$$

$$\Rightarrow \ \phi_k[n] = \phi_{k+N}[n] = \cdots = \phi_{k+rN}[n]$$



- The Fourier Series Representation:

$$x[n] = \sum_{k=<N>} a_k \phi_k[n] = \sum_{k=<N>} a_k e^{jkw_0 n} = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

▪ Procedure of Determining the Coefficients:

$$x[0] = \sum_{k=<N>} a_k$$

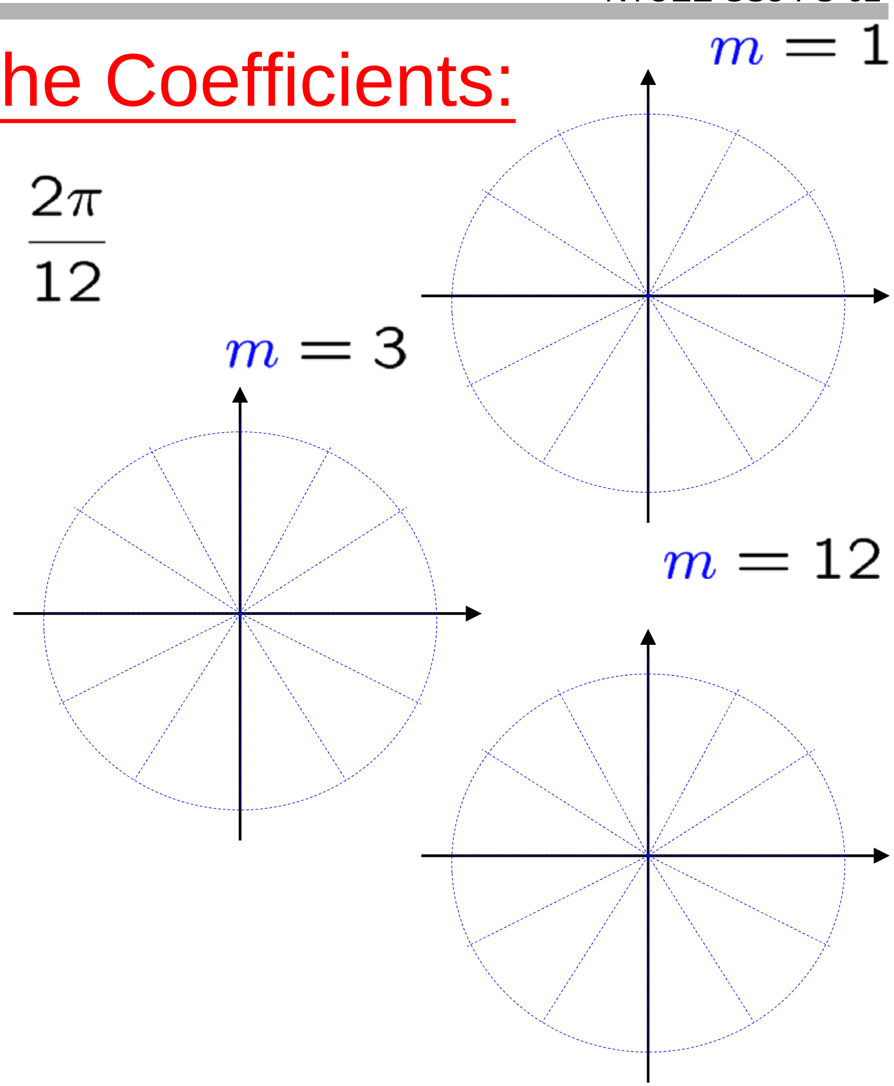
$$x[1] = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)}$$

$$x[2] = \sum_{k=<N>} a_k e^{jk2\left(\frac{2\pi}{N}\right)}$$

$$\vdots$$

$$x[N-1] = \sum_{k=<N>} a_k e^{jk(N-1)\left(\frac{2\pi}{N}\right)}$$

$$x[N] = \sum_{k=<N>} a_k e^{jk(N)\left(\frac{2\pi}{N}\right)}$$



and $$\sum_{n=<N>} e^{jm\left(\frac{2\pi}{N}\right)n} = \begin{cases} N, & m = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$$

▪ Procedure of Determining the Coefficients:

$$x[n] = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$\sum_{n=<N>} e^{-jr\left(\frac{2\pi}{N}\right)n} \qquad \sum_{n=<N>} e^{-jr\left(\frac{2\pi}{N}\right)n}$$

$$\sum_{n=<N>} x[n] e^{-jr\left(\frac{2\pi}{N}\right)n} = \sum_{n=<N>} \sum_{k=<N>} a_k e^{j(k-r)\left(\frac{2\pi}{N}\right)n}$$

$$\sum_{n=<N>} x[n] e^{-jr\left(\frac{2\pi}{N}\right)n} = \sum_{k=<N>} a_k \sum_{n=<N>} e^{j(k-r)\left(\frac{2\pi}{N}\right)n} = a_r N$$

$$\Rightarrow a_r = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jr\left(\frac{2\pi}{N}\right)n}$$

In Summary:

- The synthesis equation:

$$x[n] = \sum_{k=<N>} a_k e^{jkw_0 n} = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

- The analysis equation:

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jkw_0 n} = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$$a_k = a_{k+N}$$

- $x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$: DT Fouries series pair

- $\{a_k\}$: the Fourier series coefficients

  or the spectral coefficients of $x[n]$

- Example 3.11:

$$x[n] = 1 + \sin\left(\frac{2\pi}{N}\right)n + 3\cos\left(\frac{2\pi}{N}\right)n + \cos\left(\frac{4\pi}{N}n + \frac{\pi}{2}\right)$$

$$\Rightarrow x[n] = 1 + \frac{1}{2j}\left[e^{j\left(\frac{2\pi}{N}\right)n} - e^{-j\left(\frac{2\pi}{N}\right)n}\right] + \frac{3}{2}\left[e^{j\left(\frac{2\pi}{N}\right)n} + e^{-j\left(\frac{2\pi}{N}\right)n}\right]$$

$$+ \frac{1}{2}\left[e^{j\left(\frac{4\pi}{N}n+\frac{\pi}{2}\right)} + e^{-j\left(\frac{4\pi}{N}n+\frac{\pi}{2}\right)}\right]$$

$$\Rightarrow x[n] = 1 + \left(\frac{3}{2} + \frac{1}{2j}\right)e^{j\left(\frac{2\pi}{N}\right)n} + \left(\frac{3}{2} - \frac{1}{2j}\right)e^{-j\left(\frac{2\pi}{N}\right)n}$$

$$+ \frac{1}{2}e^{j\left(\frac{\pi}{2}\right)}\,e^{j2\left(\frac{2\pi}{N}\right)n} + \frac{1}{2}e^{-j\left(\frac{\pi}{2}\right)}\,e^{-j2\left(\frac{2\pi}{N}\right)n}$$

- Example 3.11:

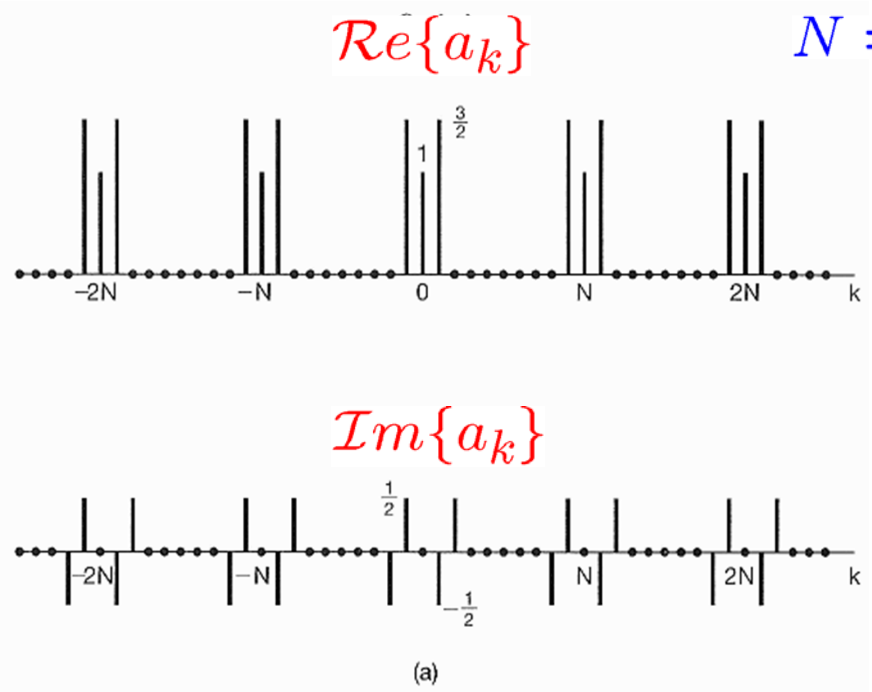
$$a = |a| e^{j \measuredangle a}$$

$$a = |a| \left[\cos(\measuredangle a) + j \sin(\measuredangle a) \right]$$

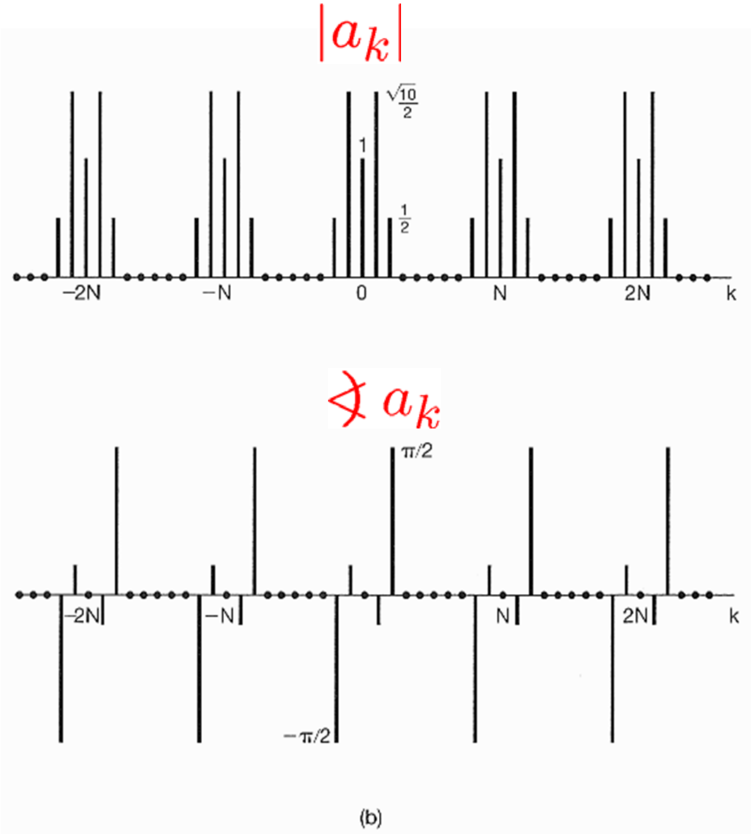
$$a = b + jc = \sqrt{b^2+c^2} \left[\frac{b}{\sqrt{b^2+c^2}} + j \frac{c}{\sqrt{b^2+c^2}} \right]$$

$$\Rightarrow \begin{cases} a_0 &= 1 \\ a_1 &= \left(\frac{3}{2} + \frac{1}{2j}\right) = \frac{3}{2} - \frac{1}{2}j \\ a_{-1} &= \left(\frac{3}{2} - \frac{1}{2j}\right) = \frac{3}{2} + \frac{1}{2}j \\ a_2 &= \frac{1}{2}j \\ a_{-2} &= -\frac{1}{2}j \\ a_k &= 0, \quad \text{others in} < N > \end{cases}$$

$N = 10$

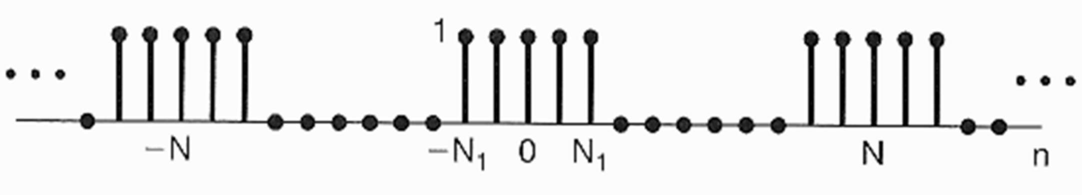


(a)



(b)

- **Example 3.12:** $a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$



$$x[n] = \begin{cases} 1, & -N_1 \le n \le N_1 \\ 0, & \text{others in } <N> \end{cases}$$

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} 1 \cdot e^{-jk\left(\frac{2\pi}{N}\right)n} = \frac{1}{N} \sum_{n=-N_1}^{N_1} \left(e^{-jk\left(\frac{2\pi}{N}\right)}\right)^n$$

$$= \frac{1}{N}\left[\left(\cdot\right)^{-N_1} + \left(\cdot\right)^{-N_1+1} + \cdots + \left(\cdot\right)^{N_1}\right]$$

$$= \frac{1}{N}\left(\cdot\right)^{-N_1}\left[\frac{1-\left(\cdot\right)^{(2N_1+1)}}{1-\left(\cdot\right)}\right] \qquad \left(\cdot\right) \ne 1$$

$$= \frac{1}{N}\left(\cdot\right)^{-N_1}\left[1 + \left(\cdot\right)^1 + \cdots + \left(\cdot\right)^{2N_1}\right]$$

- Let $m = n + N_1$ or $n = m - N_1$

$$a_k = \frac{1}{N} \sum_{m=0}^{2N_1} e^{-jk\left(\frac{2\pi}{N}\right)(m-N_1)} = \frac{1}{N} e^{jk\left(\frac{2\pi}{N}\right)N_1} \sum_{m=0}^{2N_1} e^{-jk\left(\frac{2\pi}{N}\right)m}$$

Example 3.12:

$$1 - e^{-j\theta}$$
$$= e^{-j\frac{\theta}{2}} e^{j\frac{\theta}{2}} - e^{-j\frac{\theta}{2}} e^{-j\frac{\theta}{2}}$$
$$= e^{-j\frac{\theta}{2}} \left(e^{j\frac{\theta}{2}} - e^{-j\frac{\theta}{2}} \right)$$

- $k = 0, \pm N, \pm 2N, \ldots$

$$a_k = \frac{2N_1 + 1}{N}$$

- $k \neq 0, \pm N, \pm 2N, \ldots$

$$(N_1 + \frac{1}{2})$$
$$(\frac{1}{2})$$

$$a_k = \frac{1}{N} e^{jk\left(\frac{2\pi}{N}\right)N_1} \left(\frac{1 - e^{-jk\left(\frac{2\pi}{N}\right)(2N_1+1)}}{1 - e^{-jk\left(\frac{2\pi}{N}\right)}} \right)$$

$$= \frac{1}{N} \frac{e^{-jk\left(\frac{2\pi}{2N}\right)} \left[e^{jk\left(\frac{2\pi}{2N}\right)(2N_1+1)} - e^{-jk\left(\frac{2\pi}{2N}\right)(2N_1+1)} \right]}{e^{-jk\left(\frac{2\pi}{2N}\right)} \left[e^{jk\left(\frac{2\pi}{2N}\right)} - e^{-jk\left(\frac{2\pi}{2N}\right)} \right]}$$

$$= \frac{1}{N} \frac{\sin\left[\left(\frac{2\pi}{N}\right) k (N_1 + \frac{1}{2})\right]}{\sin\left[\left(\frac{\pi}{N}\right) k\right]}$$

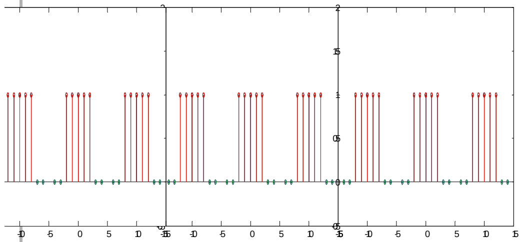
Example 3.12:

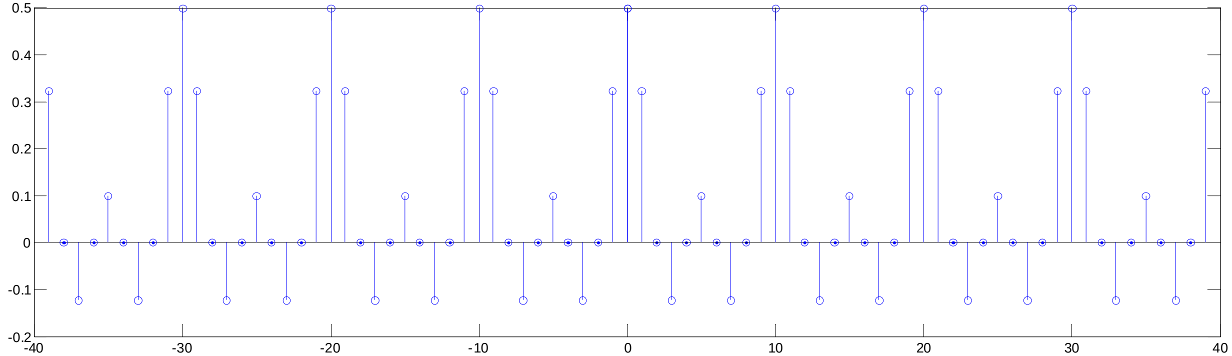
$$a_k = \frac{1}{N} \frac{\sin\left[\left(\frac{2\pi}{N}\right) k (N_1 + \frac{1}{2})\right]}{\sin\left[\left(\frac{\pi}{N}\right) k\right]}$$

- $2N_1 + 1 = 5$

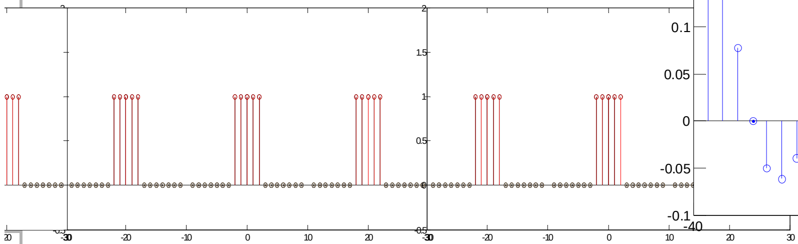
a_k

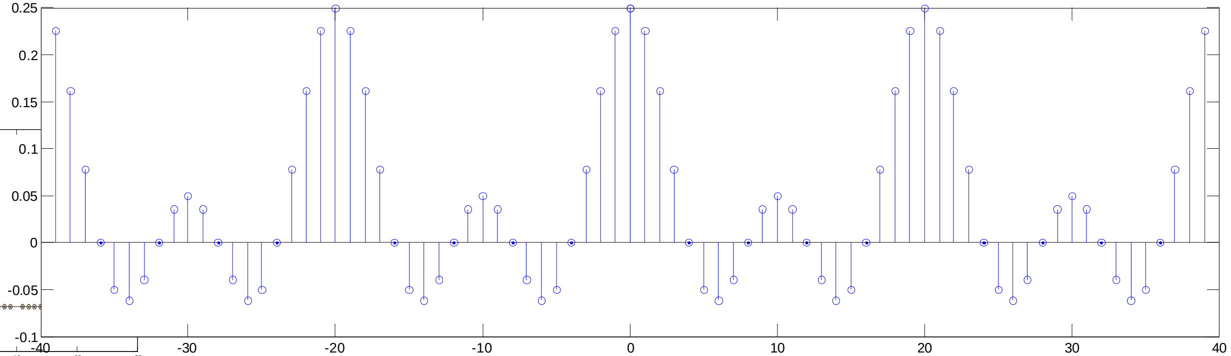
- $N = 10$



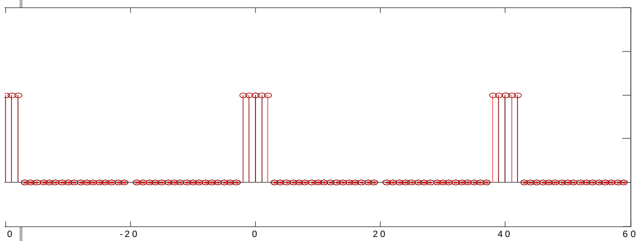


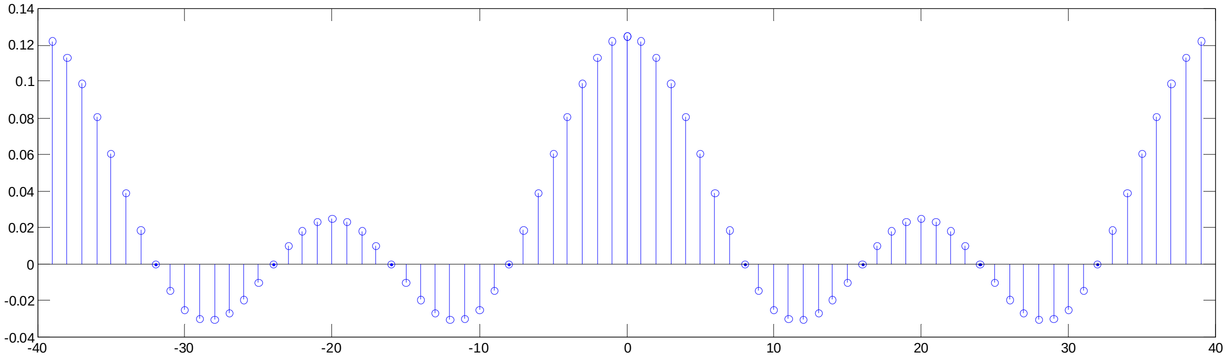
- $N = 20$





- $N = 40$





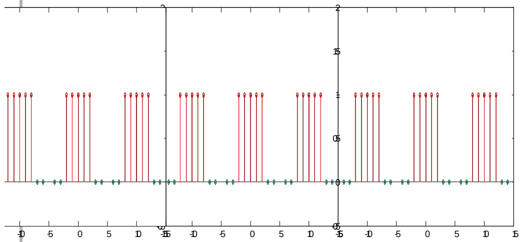
Example 3.12:

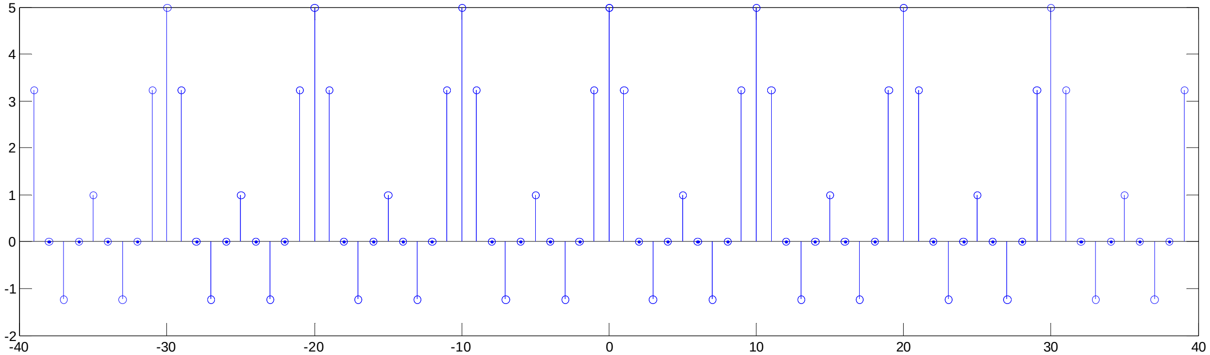
$$a_k = \frac{1}{N}\frac{\sin\left[\left(\frac{2\pi}{N}\right)k\left(N_1+\frac{1}{2}\right)\right]}{\sin\left[\left(\frac{\pi}{N}\right)k\right]}$$

- $2N_1 + 1 = 5$

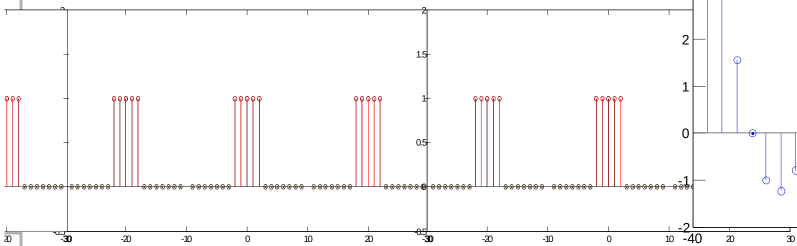
$N a_k$

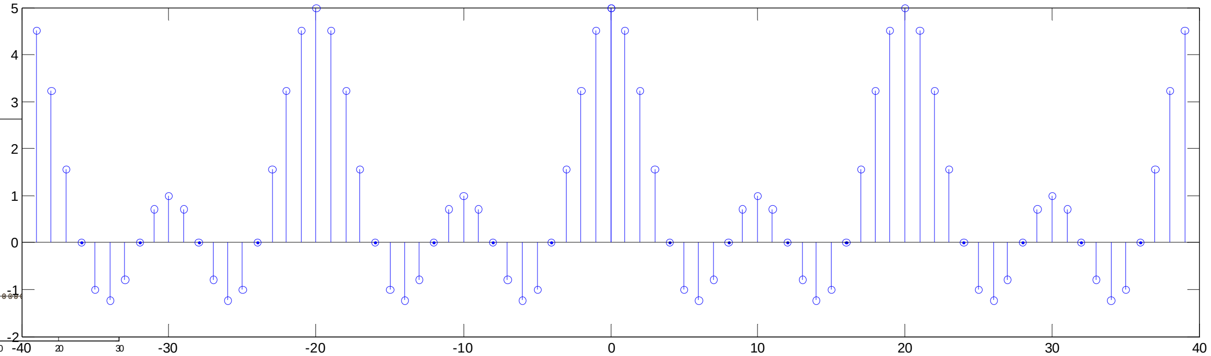
- $N = 10$



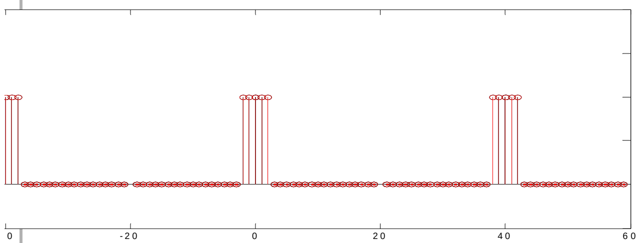


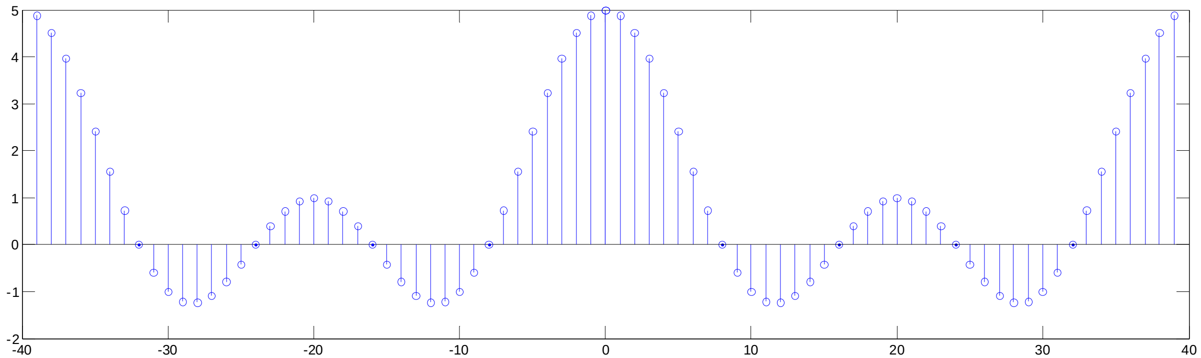
- $N = 20$



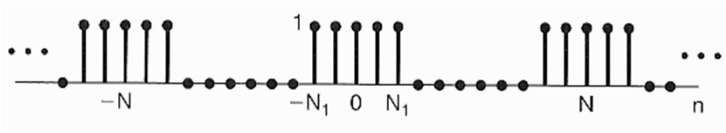


- $N = 40$



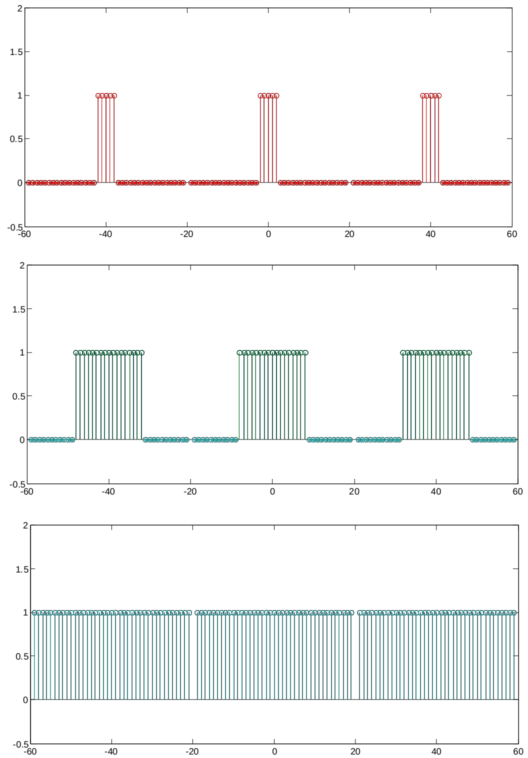


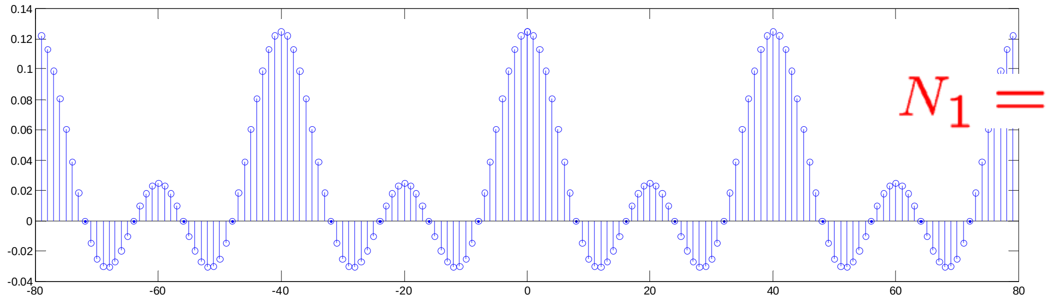
Example 3.12:



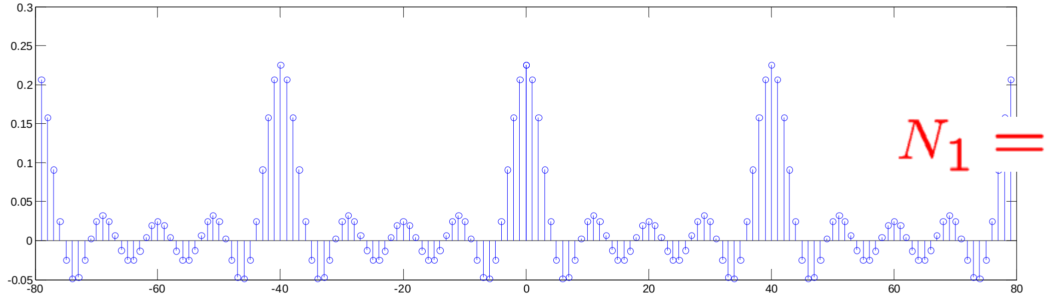
- $N = 40$

$$a_k = \frac{1}{N} \frac{\sin\left[\left(\frac{2\pi}{N}\right) k (N_1 + \frac{1}{2})\right]}{\sin\left[\left(\frac{\pi}{N}\right) k\right]}$$

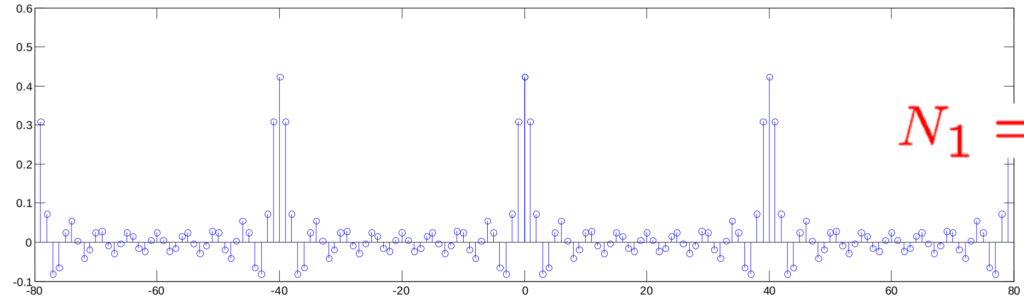




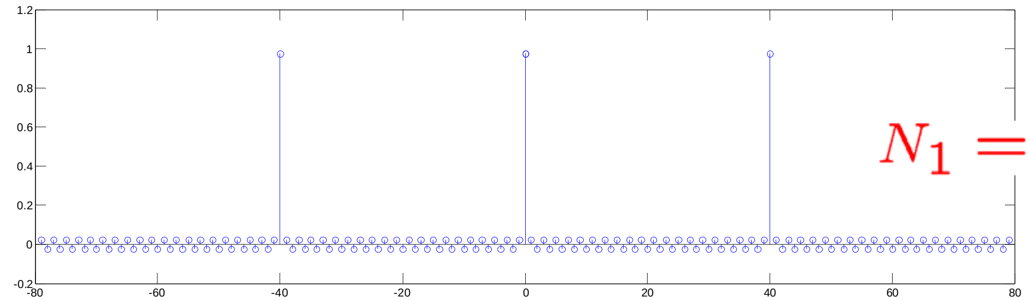
$N_1 = 2$



$N_1 = 4$

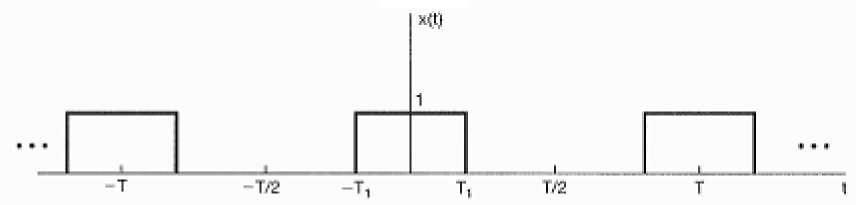


$N_1 = 8$



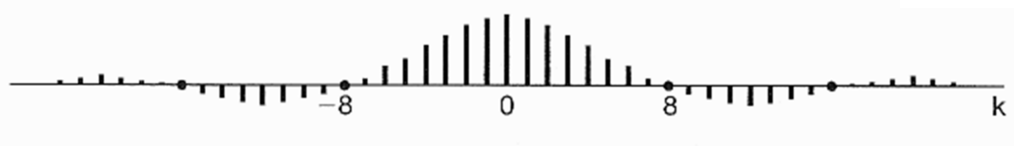
$N_1 = 19$

- Examples 3.5 (CT) & 3.12 (DT):



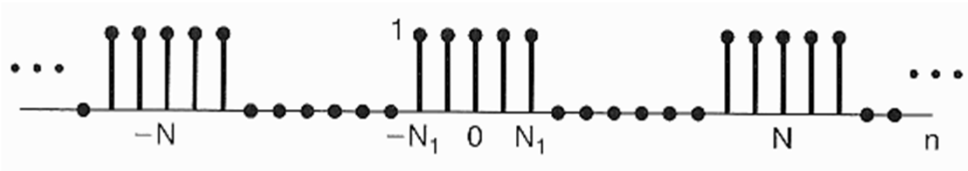
$$Ta_k = T\frac{\sin(k\frac{\pi}{8})}{k\pi}$$

$T = 16T_1$



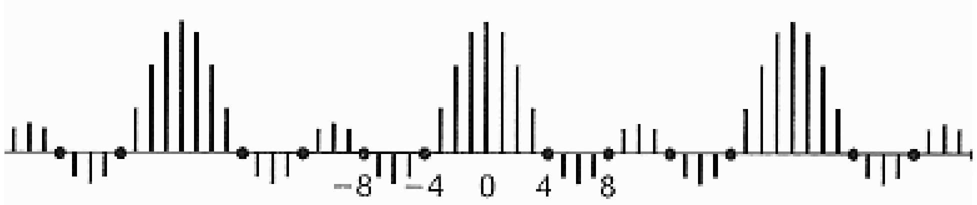
Ta_k

e^{jkw_0t}



- $N = 20$

$$a_k = \frac{1}{N}\frac{\sin\left[\left(\frac{2\pi}{N}\right)k(N_1+\frac{1}{2})\right]}{\sin\left[\left(\frac{\pi}{N}\right)k\right]}$$



$$a_k = \frac{2N_1+1}{N}$$

e^{jkw_0n}

▪ Partial Sum:

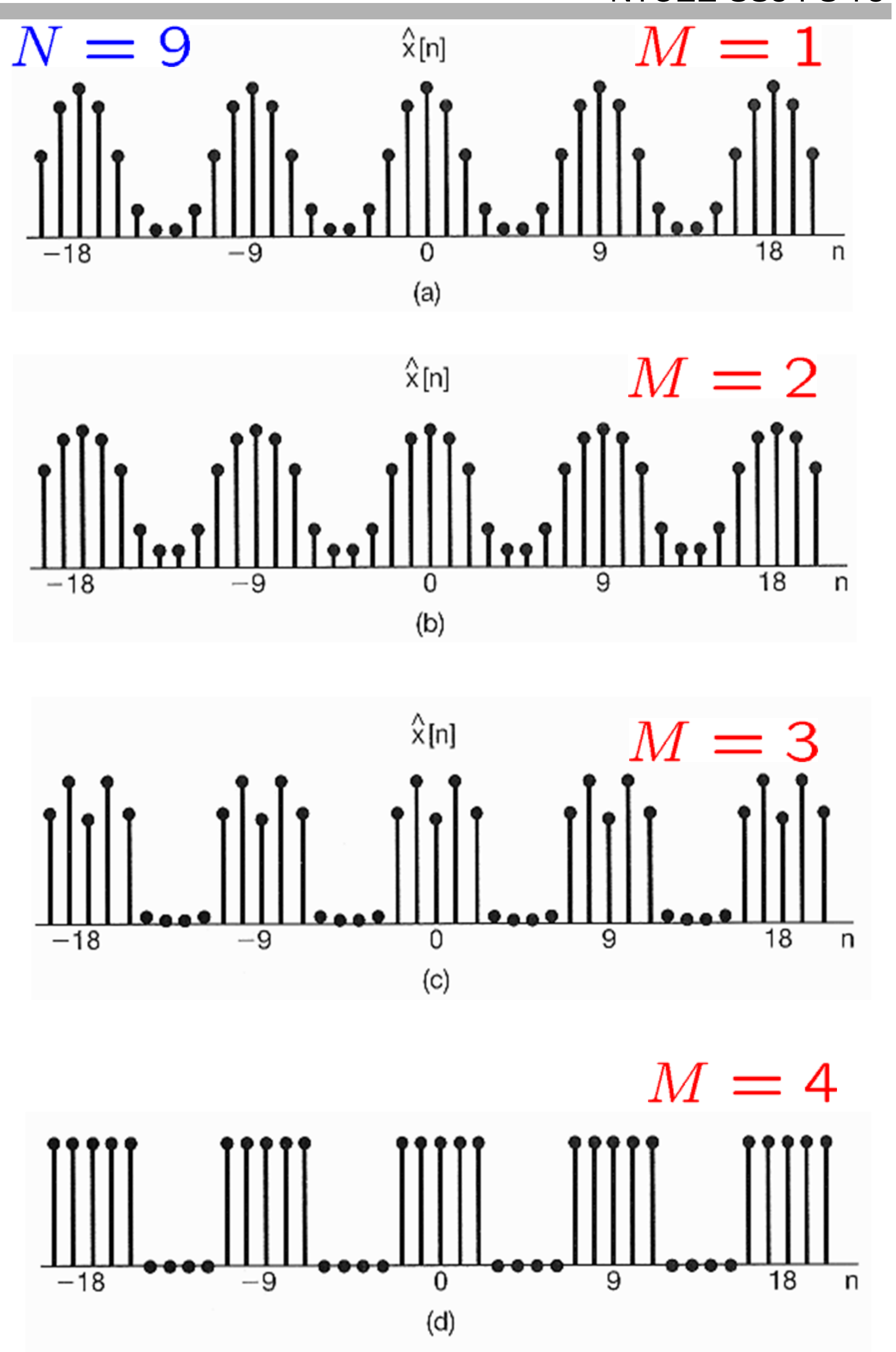
$$x[n] = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

- If N is odd

$$\hat{x}[n] = \sum_{k=-M}^{M} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

- If N is even

$$\hat{x}[n] = \sum_{k=-M+1}^{M} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$



- A Historical Perspective
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- Properties of Continuous-Time Fourier Series
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- Fourier Series & LTI Systems
- Filtering & Examples of CT & DT Filters

Outline

Section	Property
	Linearity
	Time Shifting
	Frequency Shifting
	Conjugation
	Time Reversal
	Time Scaling
	Periodic Convolution
3.7.1	Multiplication
3.7.2	First Difference
	Running Sum
	Conjugate Symmetry for Real Signals
	Symmetry for Real and Even Signals
	Symmetry for Real and Odd Signals
	Even-Odd Decomposition for Real Signals
3.7.3	Parseval's Relation for Periodic Signals

Properties of DT Fourier Series

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N \rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N \rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right) a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

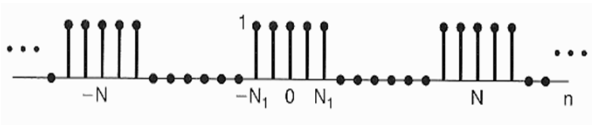
Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$

- In Summary:

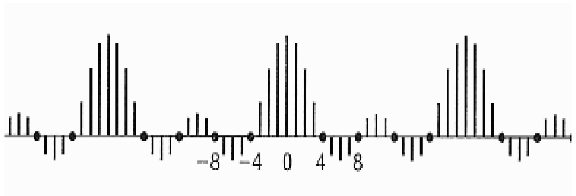
  - The synthesis equation:

$$x[n] = \sum_{k=<N>} a_k e^{jkw_0 n} \qquad = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$



  - The analysis equation:

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jkw_0 n} = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\left(\frac{2\pi}{N}\right)n}$$



$$a_k = a_{k+N}$$

  - $x[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$: DT Fouries series pair

$$x[n] = \sum_{k=<N>} a_k e^{jkw_0 n}$$

- Linearity:

  - $x[n], y[n]$: periodic signals with period N

$$x[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$y[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} b_k$$

$$\Rightarrow z[n] = Ax[n] + By[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} c_k = Aa_k + Bb_k$$

- Time Shifting:

$$x[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$\Rightarrow x[n-n_0] \stackrel{\mathcal{FS}}{\longleftrightarrow} e^{-jkw_0 n_0} a_k = e^{-jk\left(\frac{2\pi}{N}\right)n_0} a_k$$

Multiplication:

- $x[n], y[n]$: periodic signals with period N

$$x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k \qquad x[n] = \sum_{l=<N>} a_l e^{jlw_0 n}$$

$$y[n] \overset{\mathcal{FS}}{\longleftrightarrow} b_k \qquad y[n] = \sum_{m=<N>} b_m e^{jmw_0 n}$$

$\Rightarrow$ $x[n]y[n]$: also periodic with N

$$\boxed{c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}}$$

$$x[n]y[n] \overset{\mathcal{FS}}{\longleftrightarrow} d_k = \sum_{l=<N>} a_l \, b_{k-l}$$

$\Rightarrow$ a periodic convolution

Add

- First Difference:

$$x[n] = \sum_{l=<N>} a_k e^{jkw_0 n}$$

$$x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$\Rightarrow x[n-n_0] \overset{\mathcal{FS}}{\longleftrightarrow} e^{-jkw_0 n_0} a_k = e^{-jk\left(\frac{2\pi}{N}\right)n_0} a_k$$

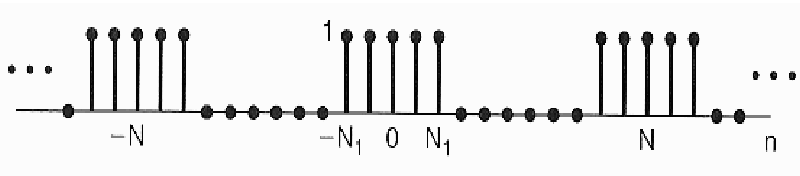
$$\Rightarrow x[n-1] \overset{\mathcal{FS}}{\longleftrightarrow} e^{-jkw_0} a_k = e^{-jk\left(\frac{2\pi}{N}\right)} a_k$$

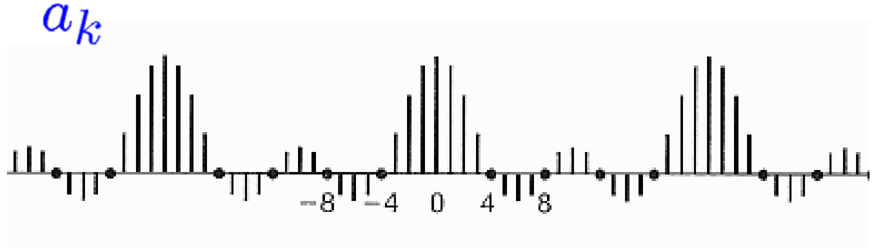
$$x[n] - x[n-1] \overset{\mathcal{FS}}{\longleftrightarrow} \left(1 - e^{-jk\left(\frac{2\pi}{N}\right)}\right) a_k$$

Parseval's relation for DT periodic signals:

- As shown in Problem 3.57:

$$x[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$$





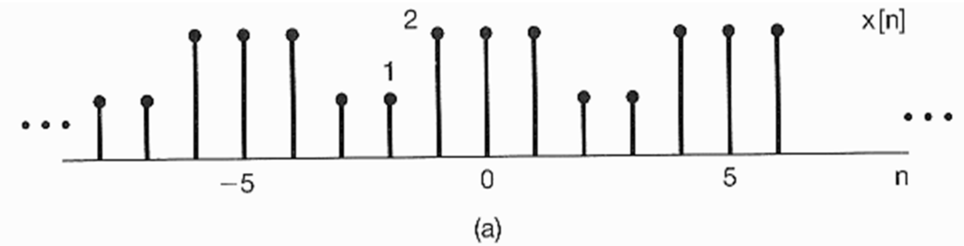
$$x[n] = \sum_{k=<N>} a_k e^{jkw_0 n} \qquad a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jkw_0 n}$$

$$\frac{1}{N} \sum_{k=<N>} \left| x[n] \right|^2 = \sum_{k=<N>} \left| a_k \right|^2$$

- Parseval's relation states that
  the total average power in a periodic signal
  equals
  the sum of the average powers
  in all of its harmonic components
  (only N distinct harmonic components in DT)

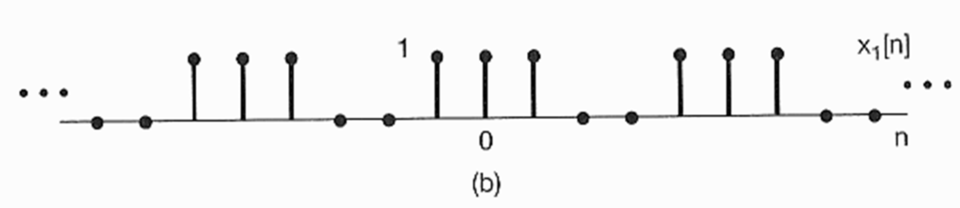
- Example 3.13:

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jkw_0 n}$$



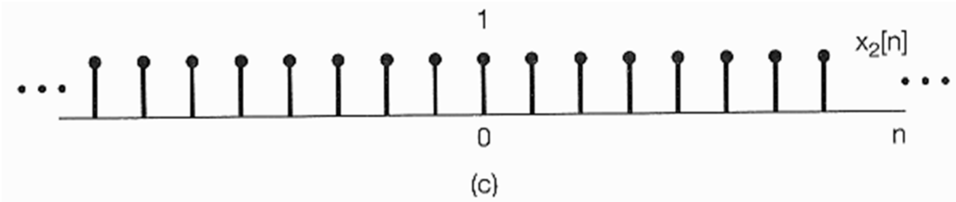
(a)

$$x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$



(b)

$$x_1[n] \overset{\mathcal{FS}}{\longleftrightarrow} b_k$$



(c)

$$x_2[n] \overset{\mathcal{FS}}{\longleftrightarrow} c_k$$

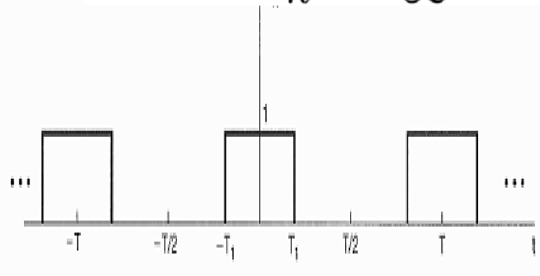
$$\Rightarrow b_k = \begin{cases} \frac{1}{5}\frac{\sin(3\pi k/5)}{\sin(\pi k/5)}, & \text{for } k \neq 0, \pm 5, \pm 10, \ldots \\ \frac{3}{5}, & \text{for } k = 0, \pm 5, \pm 10, \ldots \end{cases}$$

$$x[n] = x_1[n] + x_2[n]$$

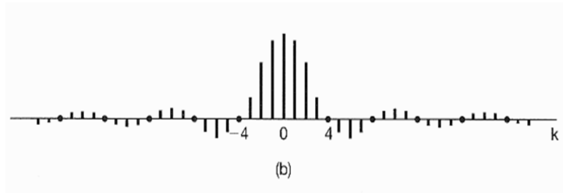
$$\Rightarrow c_k = \begin{cases} 0, & \text{for } k \neq 0, \pm 5, \pm 10, \ldots \\ 1, & \text{for } k = 0, \pm 5, \pm 10, \ldots \end{cases} \Rightarrow \quad a_k = b_k + c_k$$

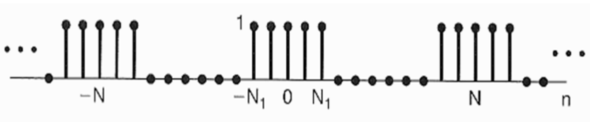
CT & DT Fourier Series Representation:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} \qquad\qquad a_k = \frac{1}{T}\int_T x(t) e^{-jkw_0 t} dt$$

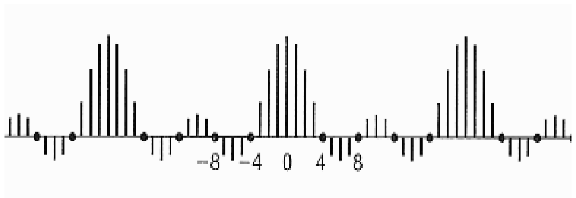


$$x(t) \xleftrightarrow{\mathcal{CTFS}} a_k$$





$$x[n] \xleftrightarrow{\mathcal{DTFS}} a_k$$



$$x[n] = \sum_{k=<N>} a_k e^{jkw_0 n} \qquad\qquad a_k = \frac{1}{N}\sum_{n=<N>} x[n] e^{-jkw_0 n}$$

- A Historical Perspective
- The Response of LTI Systems to Complex Exponentials
- Fourier Series Representation of Continuous-Time Periodic Signals
- Convergence of the Fourier Series
- Properties of Continuous-Time Fourier Series
- Fourier Series Representation of Discrete-Time Periodic Signals
- Properties of Discrete-Time Fourier Series
- Fourier Series & LTI Systems
- Filtering & Examples of CT & DT Filters

The Response of an LTI System:

On pages 12-14

$$in \rightarrow \boxed{\text{LTI}} \rightarrow out \qquad \begin{cases} \text{CT:} & e^{st} \longrightarrow H(s)e^{st} \\ \text{DT:} & z^{n} \longrightarrow H(z)z^{n} \end{cases}$$

$$H(s) = \int_{-\infty}^{+\infty} h(t)e^{-st}dt$$ $\Rightarrow$ the impulse response

$$H(z) = \sum_{k=-\infty}^{+\infty} h[k]z^{-k}$$ $\Rightarrow$ the system functions

- If $s = jw$ or $z = e^{jw}$:

$$H(jw) = \int_{-\infty}^{+\infty} h(t)e^{-jwt}dt$$

$\Rightarrow$ the frequency response

$$H(e^{jw}) = \sum_{n=-\infty}^{+\infty} h[n]e^{-jwn}$$

- **In Summary:**

$$\boxed{a = |a| e^{j \measuredangle a}}$$

$$\boxed{H = |H| e^{j \measuredangle H}}$$

$$in \rightarrow \boxed{\begin{array}{c}\text{LTI}\\ H(s/z/w)\end{array}} \rightarrow out$$

($s_i = jw_i$ or $z_i = e^{jw_i}$)

$$\begin{cases} \text{CT:} \quad e^{s_i t} \longrightarrow H(s_i) e^{s_i t} \\ \text{DT:} \quad z_i^n \longrightarrow H(z_i) z_i^n \end{cases}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k \, e^{jkw_0 t} \longrightarrow y(t) = \sum_{k=-\infty}^{+\infty} a_k \, H(jkw_0) \, e^{jkw_0 t}$$

$$x[n] = \sum_{k=<N>} a_k \, e^{jk\left(\frac{2\pi}{N}\right)n} \longrightarrow y[n] = \sum_{k=<N>} a_k \, H\left(e^{j\left(\frac{2\pi}{N}\right)k}\right) \, e^{jk\left(\frac{2\pi}{N}\right)n}$$

Examples 3.16 & 3.17

- A Historical Perspective
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- Fourier Series Representation of Continuous-Time Periodic Signals
- Convergence of the Fourier Series
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- Properties of Discrete-Time Fourier Series
- Fourier Series & LTI Systems
- Filtering & Examples of CT & DT Filters

Filtering:

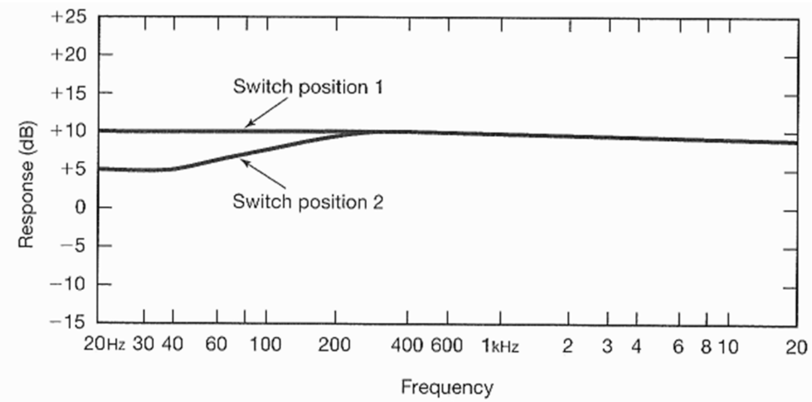
$in \longrightarrow \boxed{\text{filter}} \longrightarrow out$

- Change the relative amplitudes of the frequency components in a signal,
  - Frequency-shaping filters

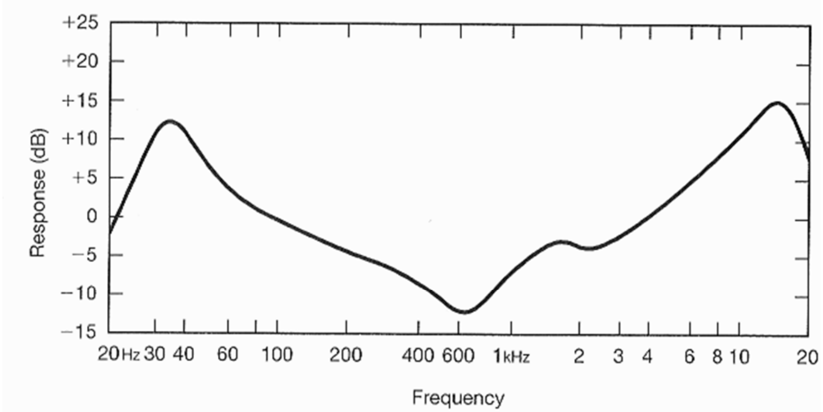
- OR, significantly attenuate or eliminate some frequency components entirely
  - Frequency-selective filters

Frequency-Shaping Filters:

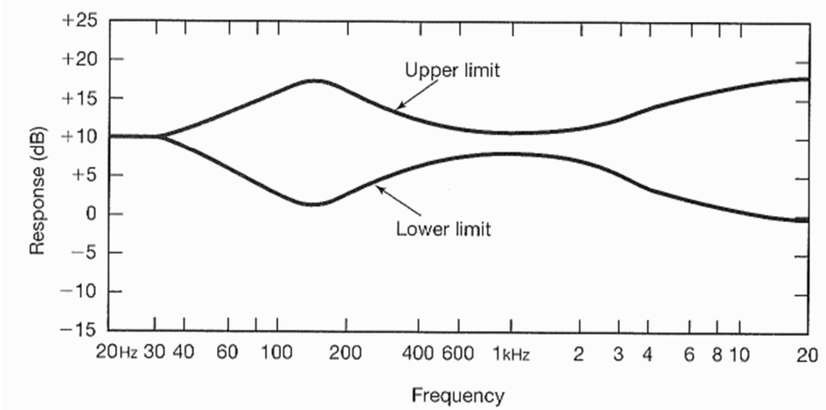
- Audio System:



(a)



(c)

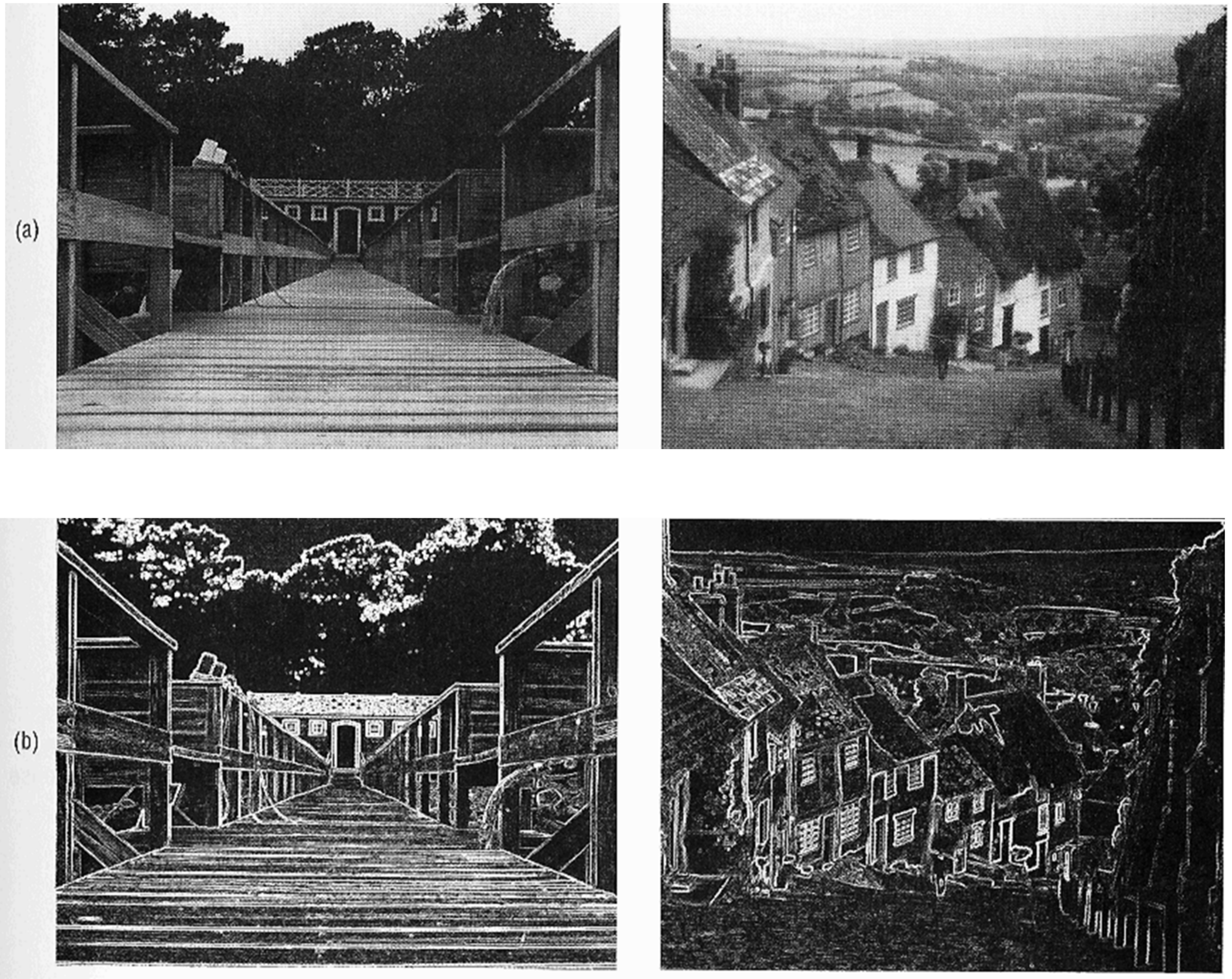


(b)

- **Frequency-Shaping Filters:**

$x(t) \longrightarrow$ Differentiating Filter $\longrightarrow y(t) = \frac{d}{dt}x(t)$

  - Differentiating filter on enhancing edges: $H(jw) = jw$



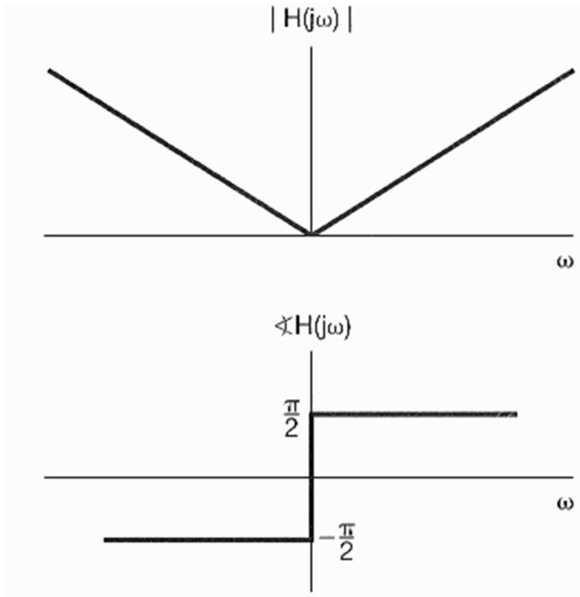
(a)

(b)

- Frequency-Shaping Filters:
  - Differentiating filter:

$x(t)$ → Differentiating Filter → $y(t) = \frac{d}{dt}x(t)$

$$H(jw) = jw$$



Frequency-Shaping Filters:

$$1 \pm e^{-j\theta} = e^{-j\frac{\theta}{2}} \left(e^{j\frac{\theta}{2}} \pm e^{-j\frac{\theta}{2}} \right)$$

- A simple DT filter: Two-point average

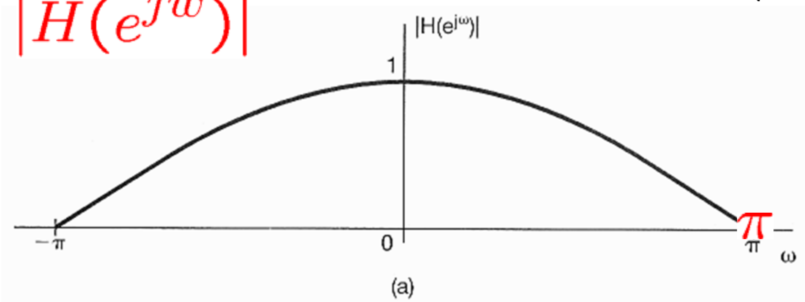
$$y[n] = \frac{1}{2}\left(x[n] + x[n-1]\right) = \qquad x[n] = H(e^{jw})\, x[n]$$

$$\Rightarrow \; h[n] = \frac{1}{2}\left(\delta[n] + \delta[n-1]\right)$$

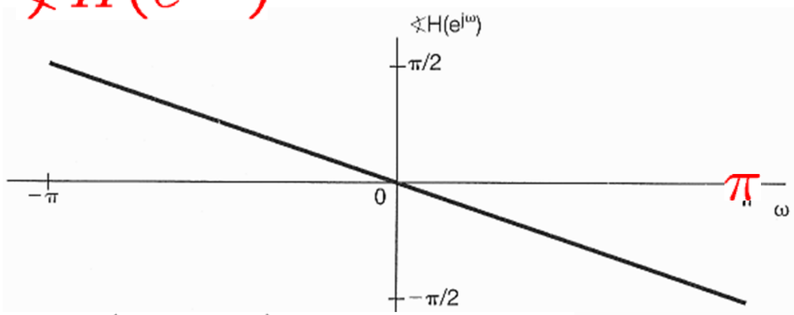
$$\Rightarrow \; H(e^{jw}) = \frac{1}{2}\left[1 + e^{-jw}\right] = \frac{1}{2} e^{-j\left(\frac{w}{2}\right)} \left[e^{j\left(\frac{w}{2}\right)} + e^{-j\left(\frac{w}{2}\right)} \right]$$

$$= e^{-j\left(\frac{w}{2}\right)} \cos\left(\frac{w}{2}\right)$$

$|H(e^{jw})|$



$\sphericalangle H(e^{jw})$



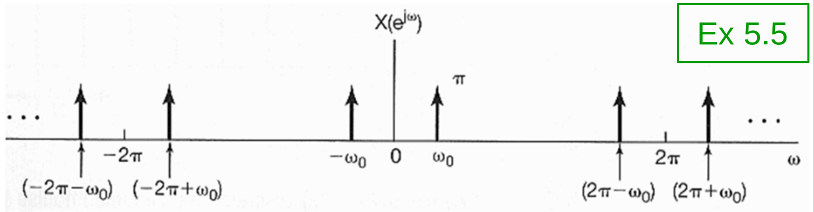
if $x[n] = K e^{j\left(\frac{\pi}{2}\right)\cdot n}$ then $y[n] = H\left(e^{j\left(\frac{\pi}{2}\right)}\right) K e^{j\left(\frac{\pi}{2}\right)\cdot n}$

Filtering $h[n] = \frac{1}{2}\left(\delta[n] + \delta[n-1]\right)$

Ex 5.5

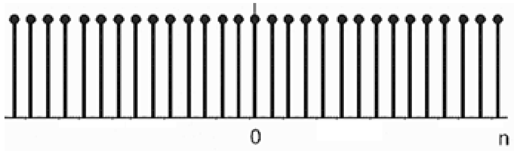
$$y[n] = H\left(e^{jw_0}\right) \; K e^{jw_0 n}$$

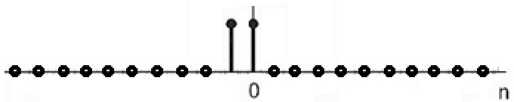
$$x[n] = \cos(w_0 n) \xleftrightarrow{\mathcal{DTFT}} X(e^{jw})$$

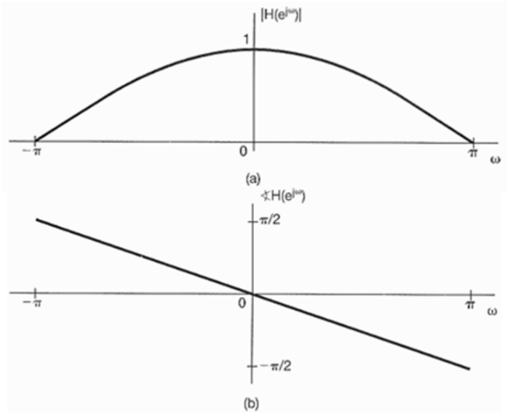


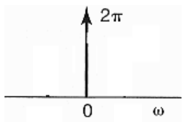
$$= \sum_{l=-\infty}^{+\infty} \pi\,\delta\left(w - w_0 - 2\pi l\right) + \sum_{l=-\infty}^{+\infty} \pi\,\delta\left(w + w_0 - 2\pi l\right)$$

$\boxed{w_0 = 0}$ $\qquad x[n] = 1$

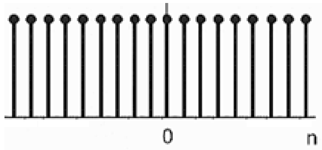


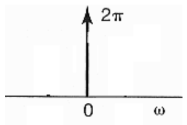


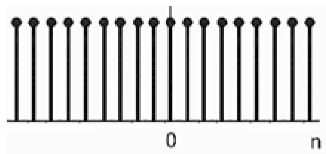




$h[n] * x[n]$





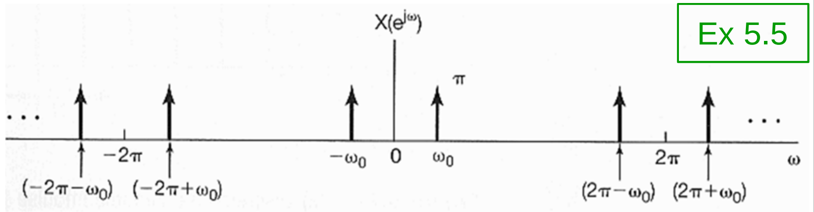


$$H\left(e^{jw_0}\right) \; X\left(e^{jw_0}\right)$$

Ex 5.5

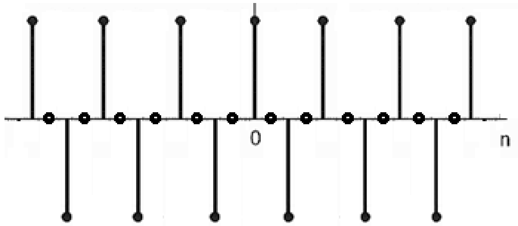
$$y[n] = H\left(e^{jw_0}\right) \; Ke^{jw_0 n}$$

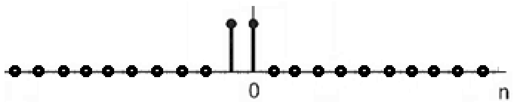
$$x[n] = \cos(w_0 n) \quad \xleftrightarrow{\mathcal{DTFT}} \quad X(e^{jw})$$

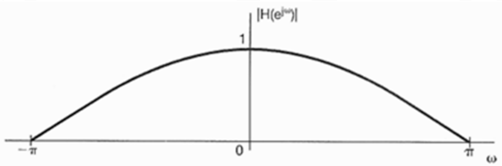


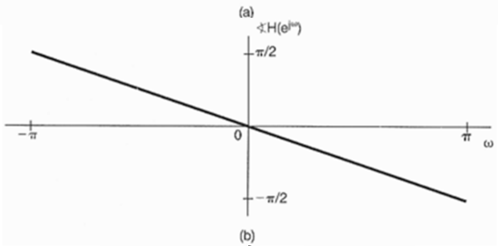
$$= \sum_{l=-\infty}^{+\infty} \pi\,\delta\left(w - w_0 - 2\pi l\right) + \sum_{l=-\infty}^{+\infty} \pi\,\delta\left(w + w_0 - 2\pi l\right)$$

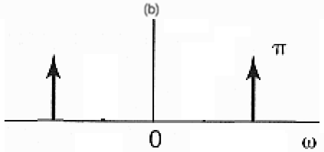
$$\boxed{w_0 = \frac{\pi}{2}} \qquad \boxed{x[n] = \cos(\frac{\pi}{2}n)}$$



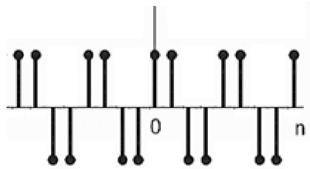


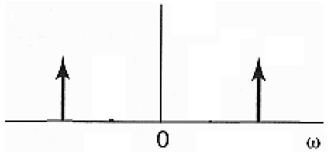


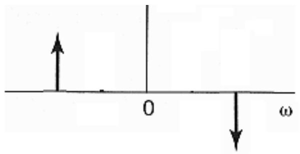


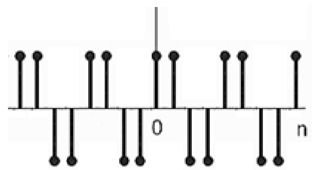


$$h[n] * x[n]$$









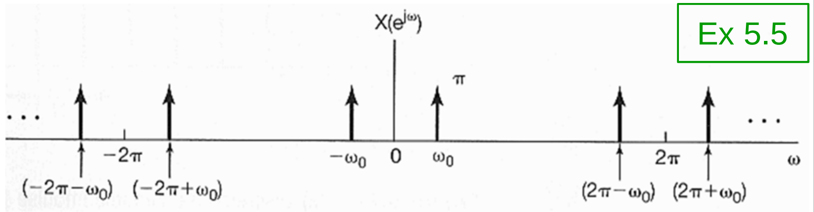
$$H\left(e^{jw_0}\right) \; X\left(e^{jw_0}\right)$$

Filtering $h[n] = \frac{1}{2}\left(\delta[n] + \delta[n-1]\right)$

Ex 5.5

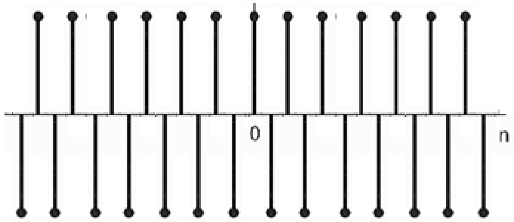
$$y[n] = H\left(e^{jw_0}\right) \; K e^{jw_0 n}$$

$$x[n] = \cos(w_0 n) \quad \xleftrightarrow{\mathcal{DTFT}} \quad X(e^{jw})$$

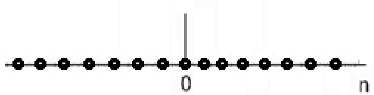


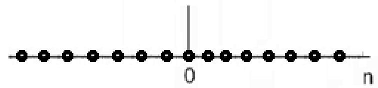
$$= \sum_{l=-\infty}^{+\infty} \pi\, \delta\left(w - w_0 - 2\pi l\right) + \sum_{l=-\infty}^{+\infty} \pi\, \delta\left(w + w_0 - 2\pi l\right)$$

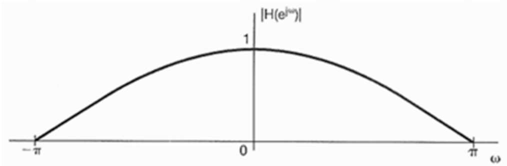
$\boxed{w_0 = \pi}$ $\boxed{x[n] = \cos(\pi n)}$

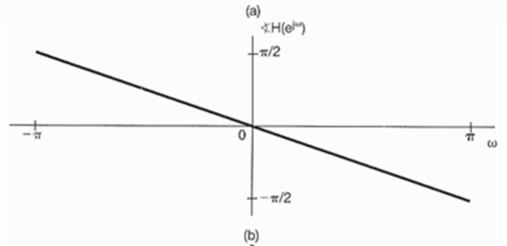


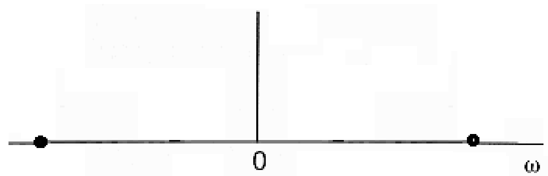
$$h[n] * x[n]$$

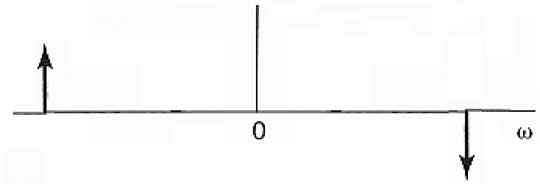


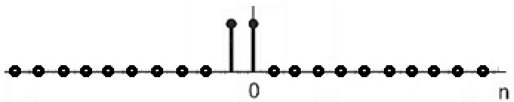


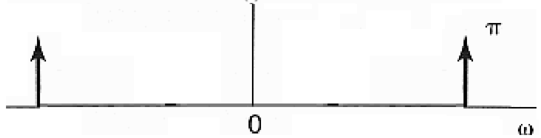








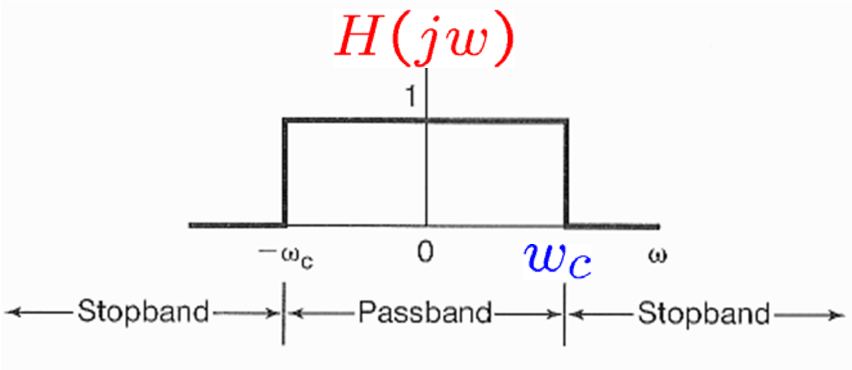




$$H\left(e^{jw_0}\right) \; X\left(e^{jw_0}\right)$$

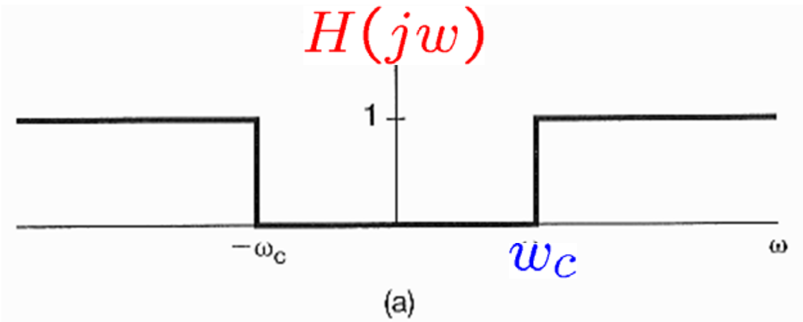
Frequency-Selective Filters:

- Select some bands of frequencies and reject others



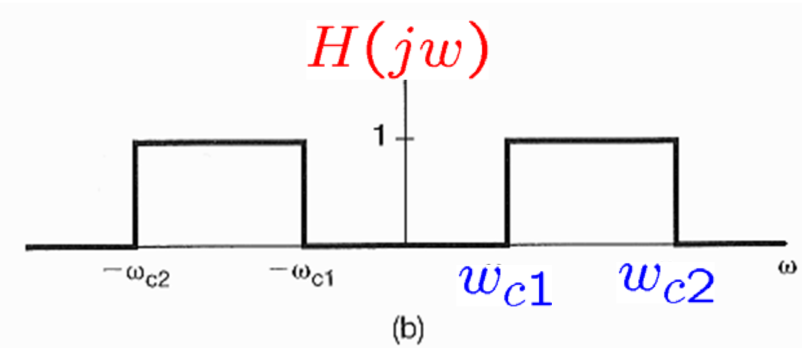
CT ideal lowpass filter

$$H(jw) = \begin{cases} 1, & |w| \leq w_c \\ 0, & |w| > w_c \end{cases}$$



CT ideal highpass filter

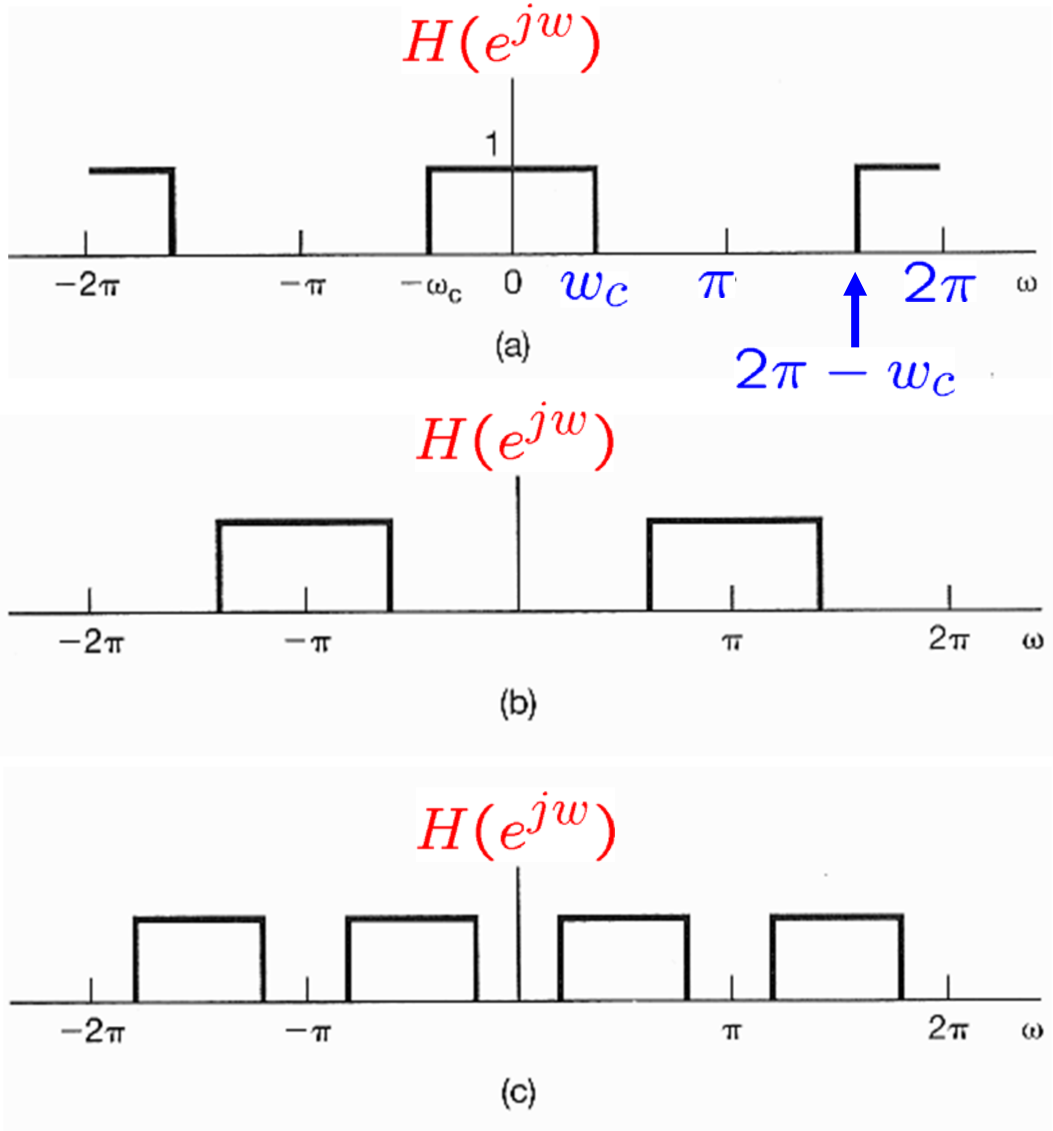
$$H(jw) = \begin{cases} 0, & |w| < w_c \\ 1, & |w| \geq w_c \end{cases}$$



CT ideal bandpass filter

$$H(jw) = \begin{cases} 1, & w_{c1} \leq |w| \leq w_{c2} \\ 0, & \text{otherwise} \end{cases}$$

- **Frequency-Selective Filters:**
  - Select some bands of frequencies and reject others



DT ideal lowpass filter

DT ideal highpass filter

DT ideal bandpass filter

- A Historical Perspective
- The Response of LTI Systems to Complex Exponentials
- Fourier Series Representation of Continuous-Time Periodic Signals
- Convergence of the Fourier Series
- Properties of Continuous-Time Fourier Series
- Fourier Series Representation of Discrete-Time Periodic Signals
- Properties of Discrete-Time Fourier Series
- Fourier Series & LTI Systems
- Filtering & Examples of CT & DT Filters

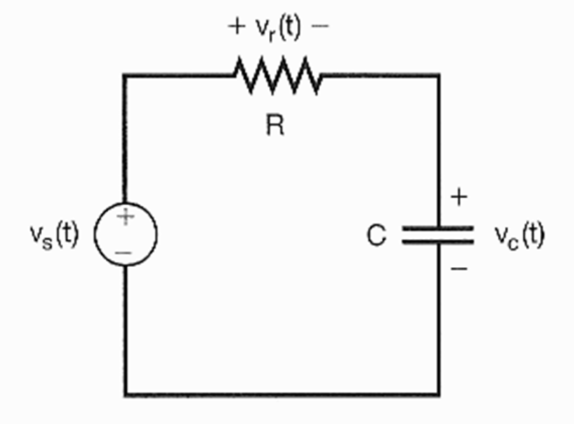
A Simple RC Lowpass Filter:

Input signal:

$v_s(t) = e^{jwt}$

$\delta(t)$

$u(t)$



Output signal:

$v_c(t) = H(jw)e^{jwt}$

$h(t)$

$s(t)$

$$\Rightarrow \ RC\,\frac{d}{dt}v_c(t) + v_c(t) = v_s(t)$$

$$\Rightarrow \ RC\,\frac{d}{dt}\left[H(jw)e^{jwt}\right] + H(jw)e^{jwt} = e^{jwt}$$

$$\Rightarrow \ RC\ jw\ H(jw)e^{jwt} + H(jw)e^{jwt} = e^{jwt}$$

$$\Rightarrow \ H(jw)e^{jwt} = \frac{1}{1+RCjw}e^{jwt}$$

- A Simple RC Lowpass Filter:

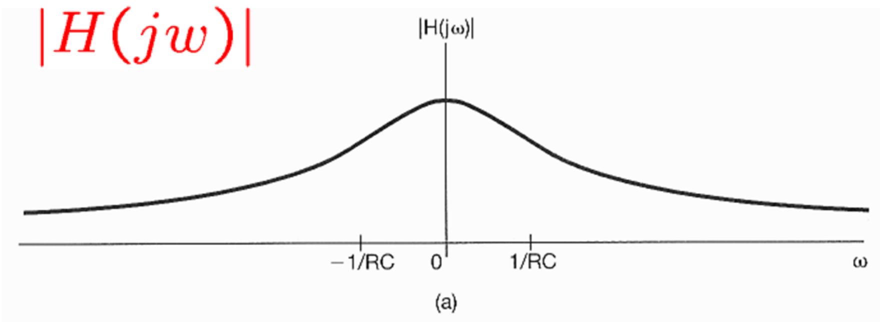
$$H(jw) = \int_{-\infty}^{+\infty} h(t) e^{-jwt} dt$$

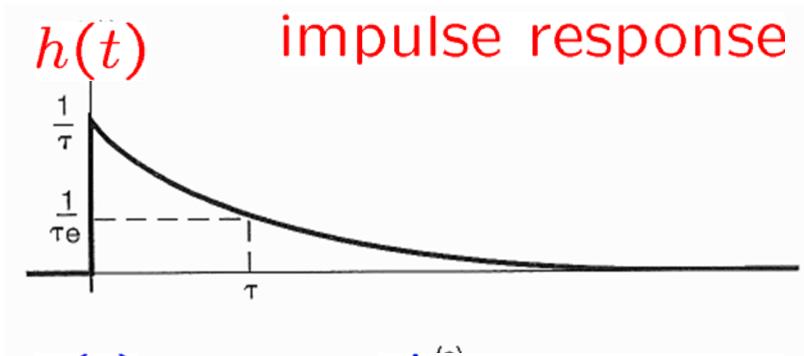
$$\Rightarrow \; H(jw) = \frac{1}{1 + RCjw}$$

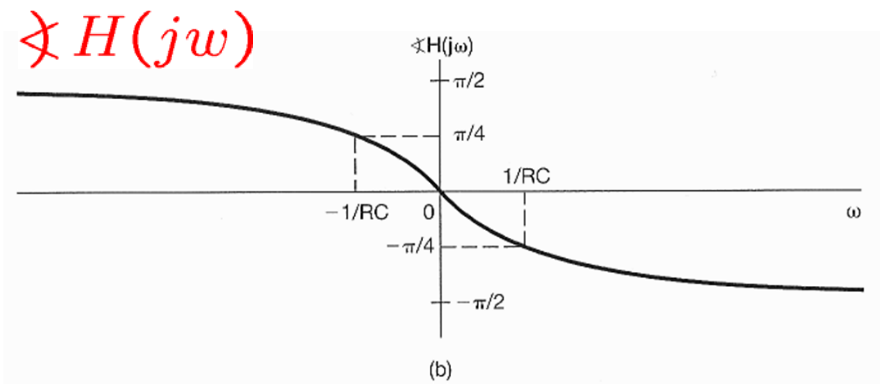
$$\Rightarrow \; h(t) = \frac{1}{RC} e^{-t/RC} \; u(t)$$

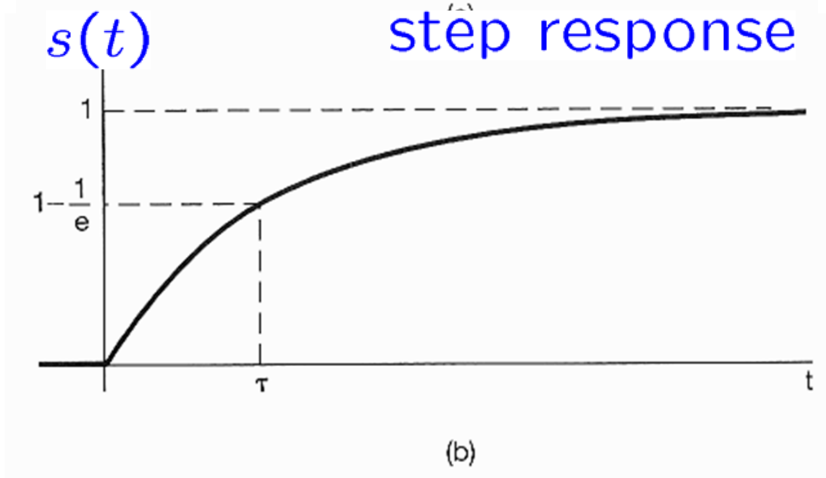
$$\Rightarrow \; s(t) = \left[1 - e^{-t/RC}\right] \; u(t)$$

$$\boxed{H = |H| e^{j \sphericalangle H}}$$









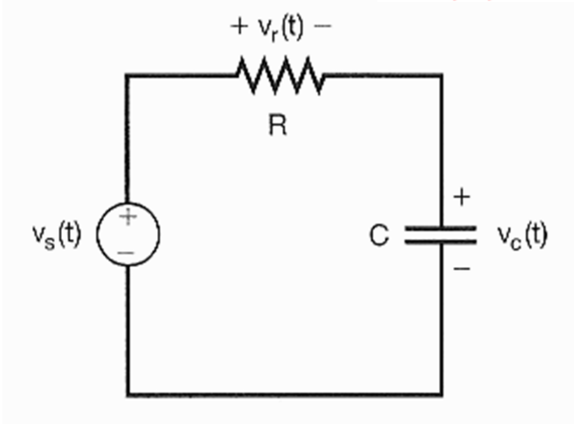
- A Simple RC Highpass Filter:

$h(t)$
$s(t)$

Output signal:
$v_r(t) = G(jw)e^{jwt}$

Input signal:
$v_s(t) = e^{jwt}$

$\delta(t)$
$u(t)$



$$\Rightarrow \; RC\,\frac{d}{dt}v_r(t) + v_r(t) = RC\,\frac{d}{dt}v_s(t)$$

$$\Rightarrow \; RC\,\frac{d}{dt}\left[G(jw)e^{jwt}\right] + G(jw)e^{jwt} = RC\,\frac{d}{dt}e^{jwt}$$

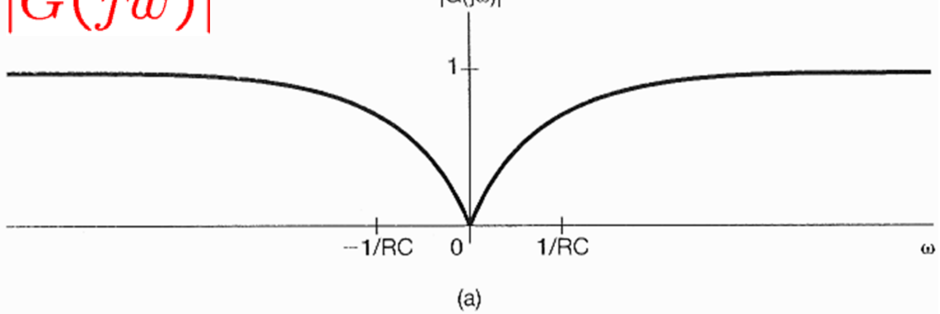
$$\Rightarrow \; RC\;jw\;G(jw)e^{jwt} + G(jw)e^{jwt} = RC\;jw\;e^{jwt}$$

$$\Rightarrow \; G(jw)e^{jwt} = \frac{jw\,RC}{1 + jw\,RC}e^{jwt}$$

▪ A Simple RC Highpass Filter:

$$\Rightarrow \; G(jw) = \frac{jw\,RC}{1 + jw\,RC}$$

$|G(jw)|$

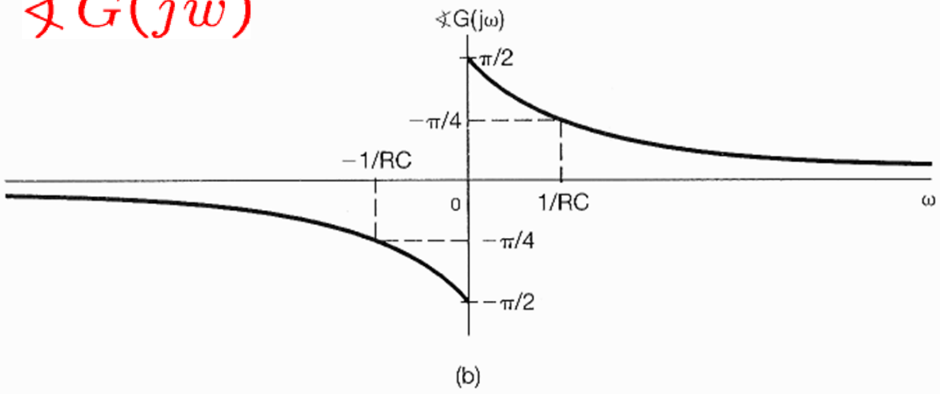


(a)

$$v_r(t) = v_s(t) - v_c(t)$$

$$\Rightarrow \; v_r(t) = e^{-t/RC}\, u(t)$$

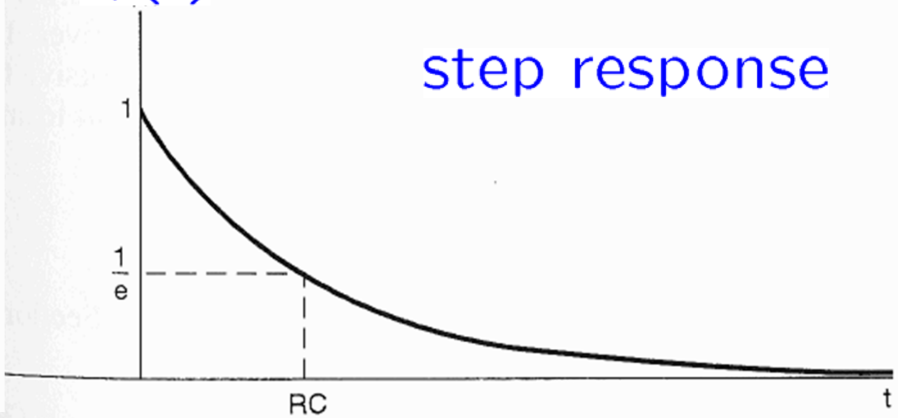
$\sphericalangle G(jw)$



(b)

$v_r(t)$

step response



First-Order Recursive DT Filters:

$$y[n] - ay[n-1] = x[n]$$

- If $x[n] = e^{jwn}$, then $y[n] = H(e^{jw})e^{jwn}$

where $H(e^{jw})$: the frequency response

$$\Rightarrow \; H(e^{jw}) \; e^{jwn} - a \; H(e^{jw}) \; e^{jw(n-1)} = e^{jwn}$$

$$\Rightarrow \; \left[1 - a \; e^{-jw}\right] H(e^{jw}) \; e^{jwn} = e^{jwn}$$

$$\Rightarrow \; H(e^{jw}) = \frac{1}{1 - a \; e^{-jw}}$$

First-Order Recursive DT Filters:

$$H(e^{jw}) = \frac{1}{1 - a\, e^{-jw}}$$

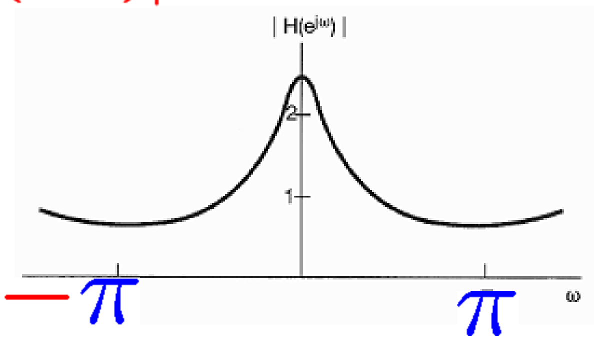
$$y[n] = ay[n-1] + x[n]$$

lowpass filter: $0 < a < 1$

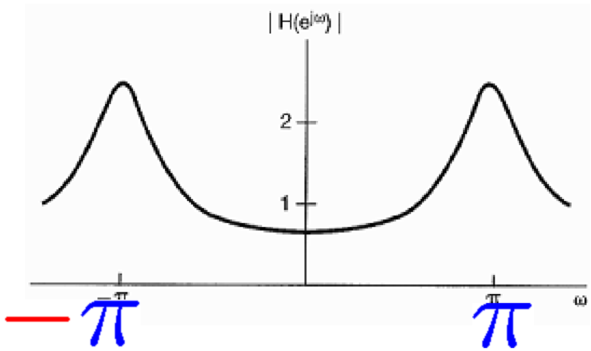
highpass filter: $-1 < a < 0$

$|H(e^{jw})|$

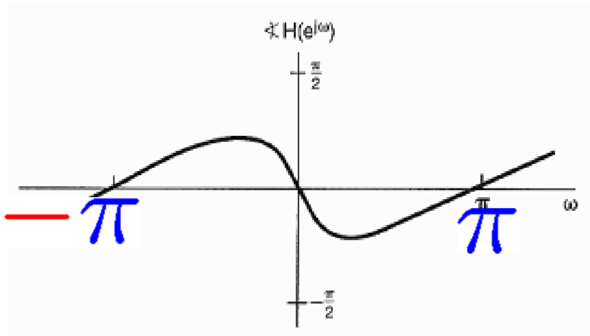
$a = 0.6$

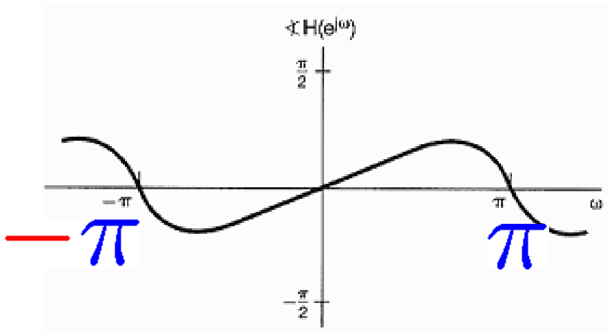


$a = -0.6$



$\sphericalangle H(e^{jw})$

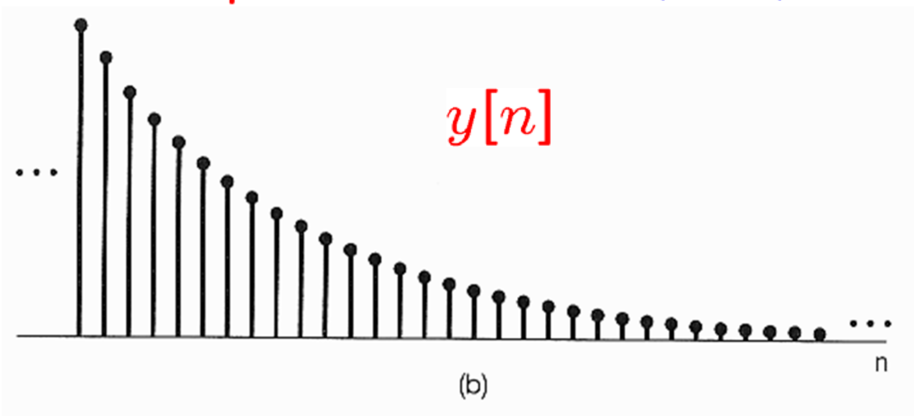




- **First-Order Recursive DT Filters:**

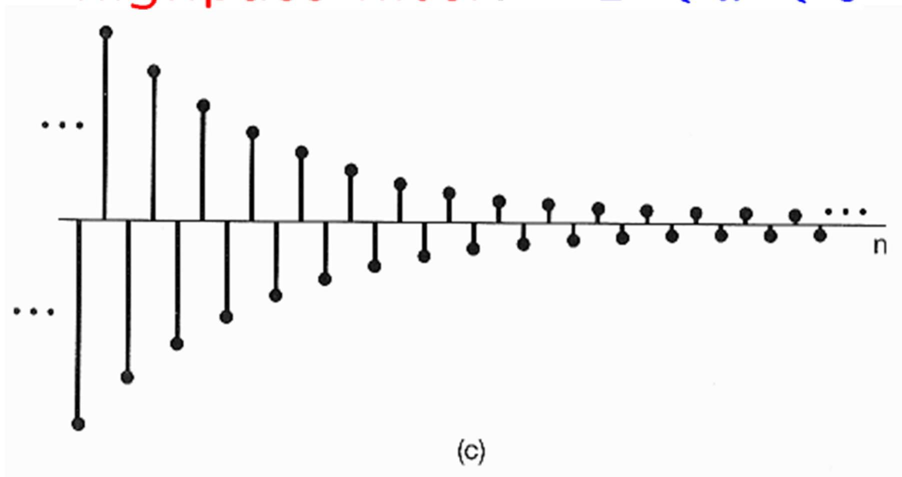
$$y[n] = ay[n-1] + x[n]$$

lowpass filter: $0 < a < 1$

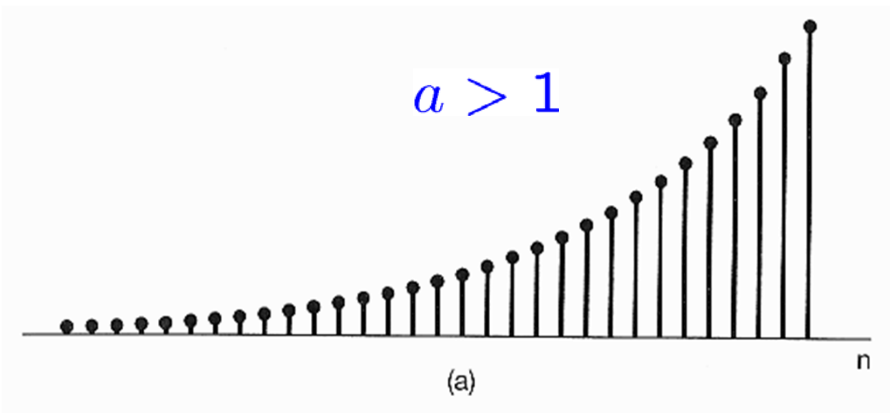


(b)

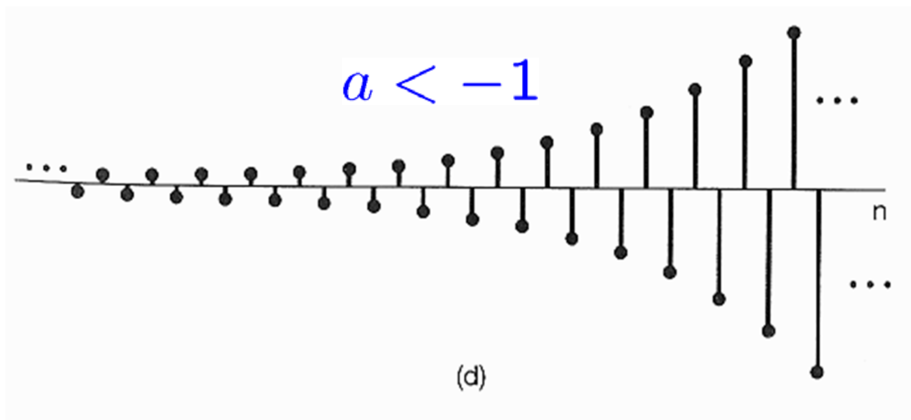
highpass filter: $-1 < a < 0$



(c)



(a)



(d)

Nonrecursive DT Filters:

- An FIR nonrecursive difference equation:

$$y[n] = \sum_{k=-N}^{M} b_k \, x[n-k]$$

$$= b_{-N} \, x[n+N] + b_{-N+1} \, x[n+N-1] + \cdots +$$

$$+ b_{-1} \, x[n+1] + b_0 \, x[n] + b_1 \, x[n-1] + \cdots + b_M \, x[n-M]$$

$$b_k =$$

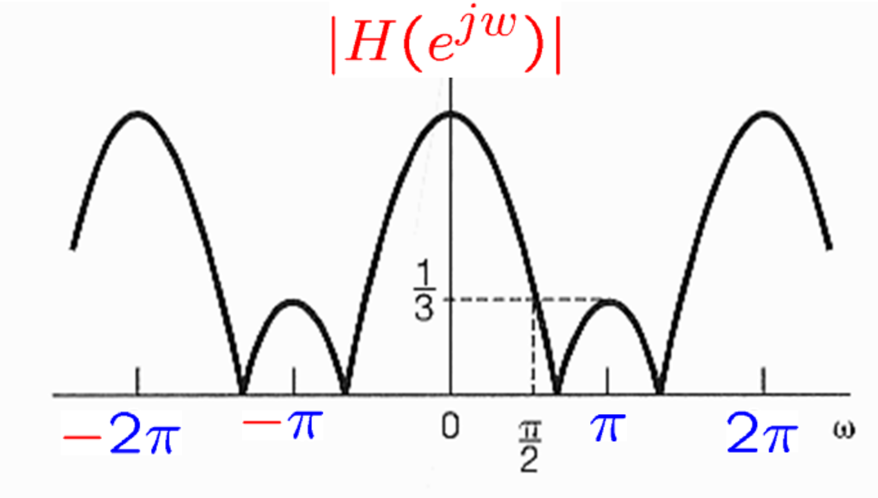
$$b_k =$$

- Nonrecursive DT Filters:
  - Three-point moving average (lowpass) filter:

$$y[n] = \frac{1}{3}\Big(x[n+1] + x[n] + x[n-1]\Big)$$

$$\Rightarrow\ h[n] = \frac{1}{3}\Big(\delta[n+1] + \delta[n] + \delta[n-1]\Big)$$

$$\Rightarrow\ H(e^{jw}) = \frac{1}{3}\Big(e^{jw} + 1 + e^{-jw}\Big) = \frac{1}{3}\Big(1 + 2\cos w\Big)$$



$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$$

$$\sin(\theta) = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$$

- Nonrecursive DT Filters:
  - N+M+1 moving average (lowpass) filter:

$$y[n] = \frac{1}{N+M+1} \sum_{k=-N}^{M} x[n-k]$$

$$\Rightarrow H(e^{jw}) = \frac{1}{N+M+1} \sum_{k=-N}^{M} e^{-jwk}$$

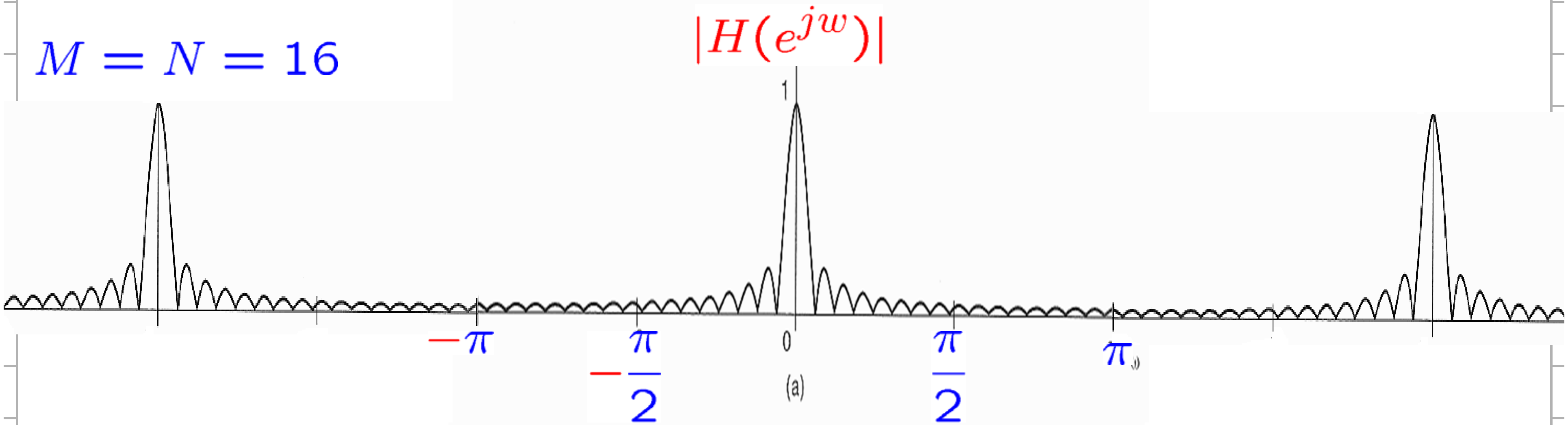
ss3-67

$$\Rightarrow H(e^{jw}) = \frac{1}{N+M+1} e^{jw\left(\frac{N-M}{2}\right)} \frac{\sin\left((M+N+1)\frac{w}{2}\right)}{\sin\left(\frac{w}{2}\right)}$$

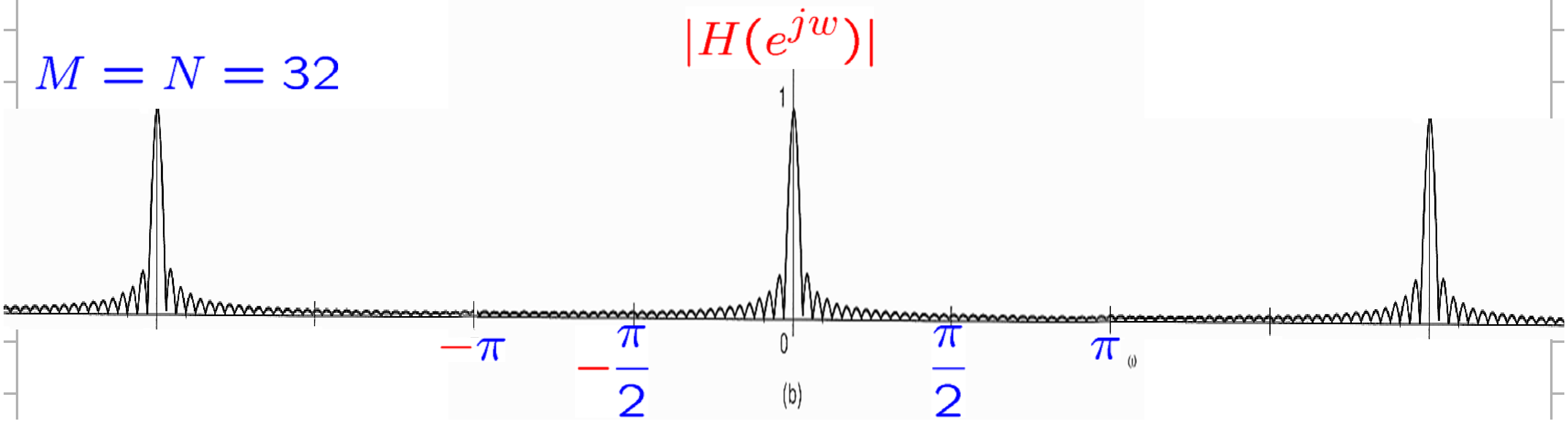
$$\frac{1-e^{-ja}}{1-e^{-jb}} = \frac{e^{-j\frac{a}{2}}\left(e^{j\frac{a}{2}} - e^{-j\frac{a}{2}}\right)}{e^{-j\frac{b}{2}}\left(e^{j\frac{b}{2}} - e^{-j\frac{b}{2}}\right)}$$

- Nonrecursive DT Filters:
  - N+M+1 moving average (lowpass) filter:

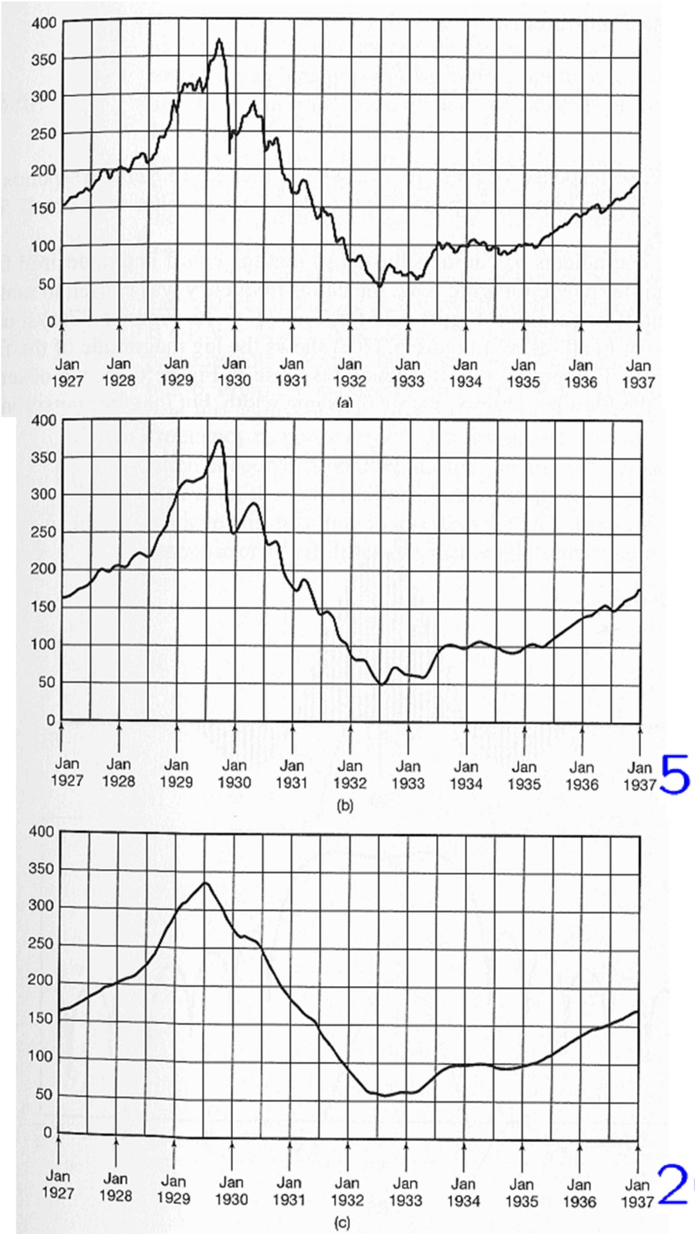
$M = N = 16$



$M = N = 32$



- Lowpass Filtering on Dow Jones Weekly Stock Market Index:



http://big5.jrj.com.cn/

51-day moving average

201-day moving average

$$\boxed{1 \pm e^{-j\theta} = e^{-j\frac{\theta}{2}}\left(e^{j\frac{\theta}{2}} \pm e^{-j\frac{\theta}{2}}\right)}$$

- **Nonrecursive DT Filters:**
  - Highpass filters:

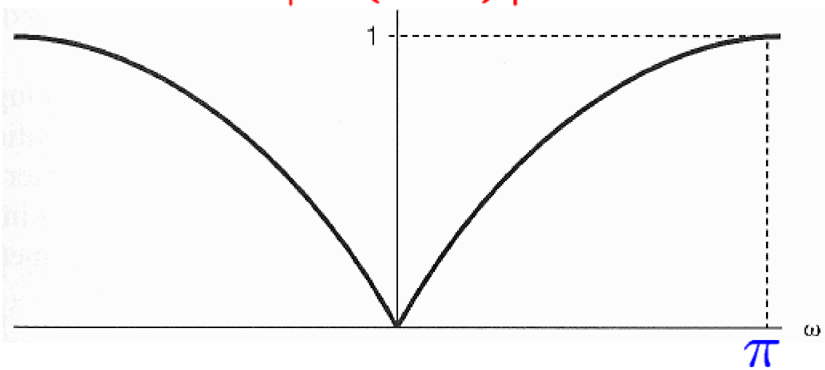
$$y[n] = \frac{x[n] - x[n-1]}{2}$$

$$\Rightarrow h[n] = \frac{1}{2}\left\{\delta[n] - \delta[n-1]\right\}$$

$$\Rightarrow H(e^{jw}) = \frac{1}{2}\left[1 - e^{-jw}\right] = \frac{1}{2}e^{-j\left(\frac{w}{2}\right)}\left[e^{j\left(\frac{w}{2}\right)} - e^{-j\left(\frac{w}{2}\right)}\right]$$

$$= j\, e^{-j\left(\frac{w}{2}\right)} \sin\left(\frac{w}{2}\right)$$

$|H(e^{jw})|$



Correction

- On page 235, Eq. 3.139

$$\boxed{1 \pm e^{-j\theta} = e^{-j\frac{\theta}{2}}\left(e^{j\frac{\theta}{2}} \pm e^{-j\frac{\theta}{2}}\right)}$$

$$\Rightarrow H(e^{jw}) = \frac{1}{2}\left[1 + e^{-jw}\right] = \frac{1}{2}e^{-j\left(\frac{w}{2}\right)}\left[e^{j\left(\frac{w}{2}\right)} + e^{-j\left(\frac{w}{2}\right)}\right]$$

$$= e^{-j\left(\frac{w}{2}\right)} \cos\left(\frac{w}{2}\right)$$

- On page 249, Eq. 3.164

$$\Rightarrow H(e^{jw}) = \frac{1}{2}\left[1 - e^{-jw}\right] = \frac{1}{2}e^{-j\left(\frac{w}{2}\right)}\left[e^{j\left(\frac{w}{2}\right)} - e^{-j\left(\frac{w}{2}\right)}\right]$$

$$= j\, e^{-j\left(\frac{w}{2}\right)} \sin\left(\frac{w}{2}\right)$$

- A Historical Perspective
- The Response of LTI Systems to Complex Exponentials
- FS Representation of CT Periodic Signals
- Convergence of the FS
- Properties of CT FS
  - Linearity, Time Shifting, Frequency Shifting, Conjugation
  - Time Reversal, Time Scaling, Periodic Convolution, Multiplication
  - Differentiation, Integration, Conjugate Symmetry for Real Signals
  - Symmetry for Real and Even Signals, Symmetry for Real and Odd Signals
  - Even-Odd Decomposition for Real Signals, Parseval's Relation for Periodic Signals
- FS Representation of DT Periodic Signals
- Properties of DT FS
  - Multiplication, First Difference, Running Sum
- FS & LTI Systems
- Filtering
  - Frequency-shaping filters & Frequency-selective filters
- Examples of CT & DT Filters

Signals & Systems (Chap 1)

LTI & Convolution (Chap 2)

Bounded/Convergent

Periodic

FS CT DT (Chap 3)

Aperiodic

FT CT (Chap 4) DT (Chap 5)

Unbounded/Non-convergent

LT CT (Chap 9)

zT DT (Chap 10)

Time-Frequency (Chap 6)

Communication (Chap 8)

CT-DT (Chap 7)

Control (Chap 11)

Spring 2011

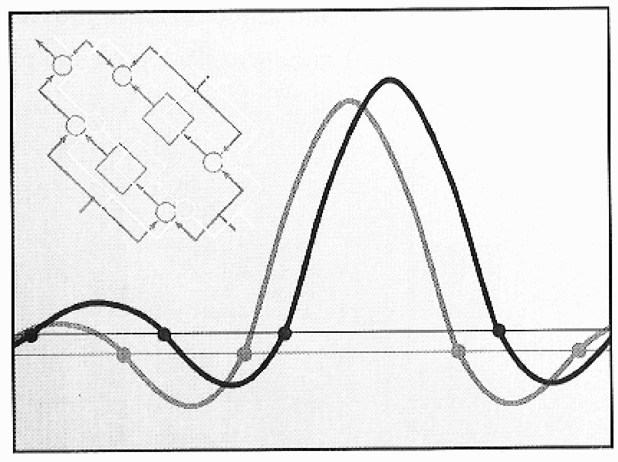
信號與系統
Signals and Systems

Chapter SS-4
The Continuous-Time Fourier Transform

Feng-Li Lian

NTU-EE

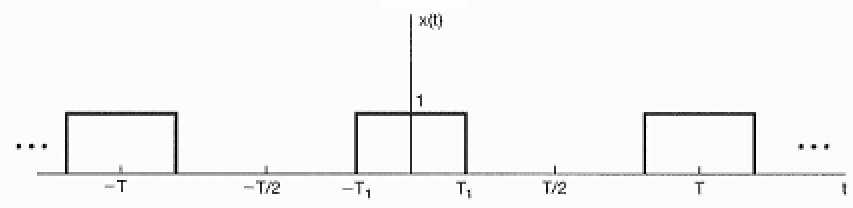
Feb11 – Jun11



Figures and images used in these lecture notes are adopted from
"Signals & Systems" by Alan V. Oppenheim and Alan S. Willsky, 1997

- Representation of Aperiodic Signals:
  the Continuous-Time Fourier Transform
- The Fourier Transform for Periodic Signals
- Properties
  of the Continuous-Time Fourier Transform
- The Convolution Property
- The Multiplication Property
- Systems Characterized by
  Linear Constant-Coefficient Differential Equations

- Example 3.5: $a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$



$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

$$k = 0 \qquad a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}$$

$$k \neq 0 \qquad a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jkw_0 t}dt = \frac{1}{T}\frac{1}{(-jkw_0)}e^{-jkw_0 t}\Big|_{-T_1}^{T_1}$$

$$= \frac{1}{jkw_0 T}\left[e^{jkw_0 T_1} - e^{-jkw_0 T_1}\right] /$$

$$w_0 = \frac{2\pi}{T}$$

$$= \frac{2\sin(kw_0 T_1)}{kw_0 T} = \frac{\sin(kw_0 T_1)}{k\pi} = \frac{\sin(k(2\pi/T)T_1)}{k\pi}, \quad k \neq 0$$

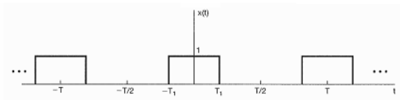
- Example 3.5:

$$T\ a_k = T\ \frac{\sin(k 2\pi \frac{T_1}{T})}{k\pi}$$

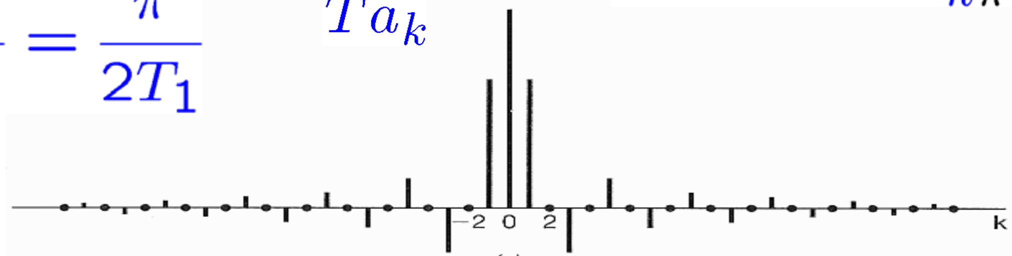
$T = 4T_1$

$$w_0 = \frac{2\pi}{4T_1} = \frac{\pi}{2T_1}$$

$$Ta_k = T\ \frac{\sin(k\frac{\pi}{2})}{k\pi}$$



Ta_k

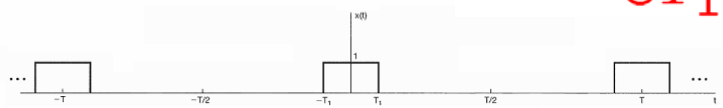


(a)

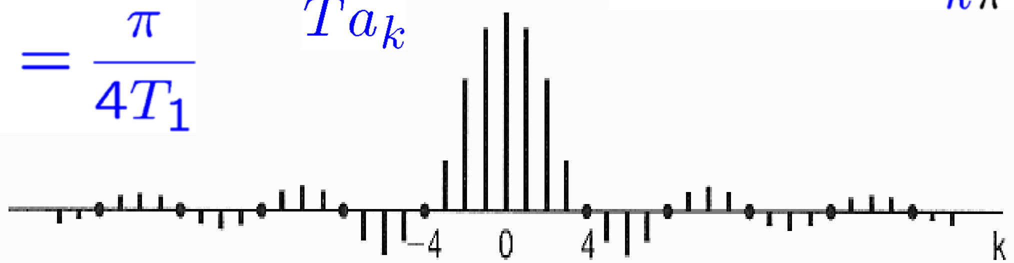
$T = 8T_1$

$$w_0 = \frac{2\pi}{8T_1} = \frac{\pi}{4T_1}$$

$$Ta_k = T\ \frac{\sin(k\frac{\pi}{4})}{k\pi}$$



Ta_k

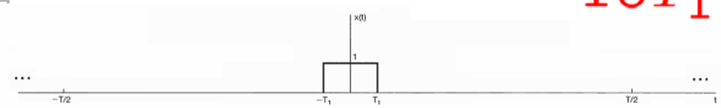


(b)

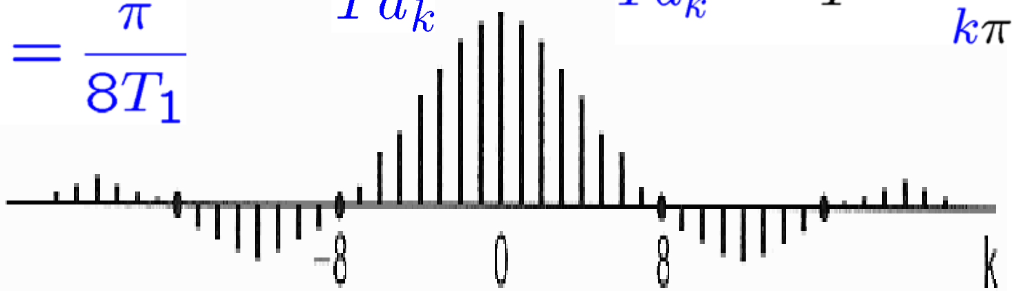
$T = 16T_1$

$$w_0 = \frac{2\pi}{16T_1} = \frac{\pi}{8T_1}$$

$$Ta_k = T\ \frac{\sin(k\frac{\pi}{8})}{k\pi}$$



Ta_k

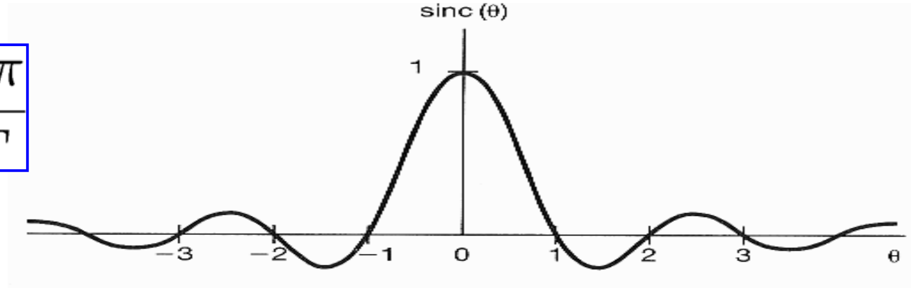


- Example 3.5:

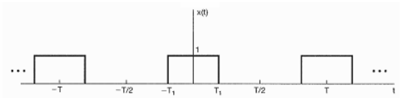
$$T a_k = T \, \frac{2 \sin(k w_0 T_1)}{k w_0 T}$$

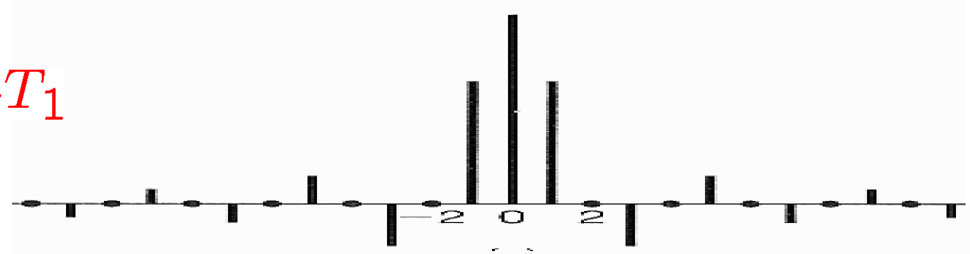
$$= T_1 \, \frac{2 \sin(k w_0 T_1)}{k w_0 T_1}$$

$$\boxed{w_0 = \frac{2\pi}{T}}$$

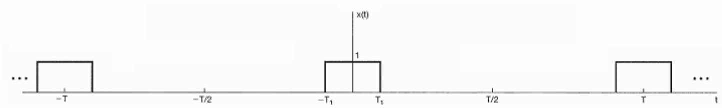


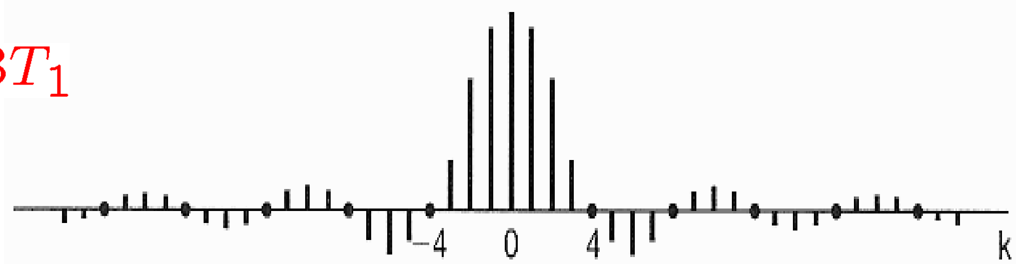
$T = 4T_1$



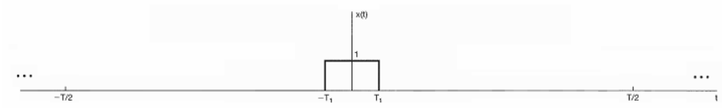


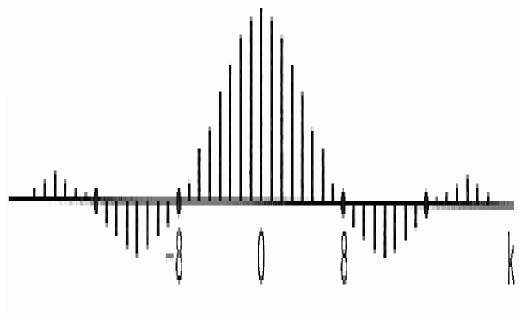
$T = 8T_1$





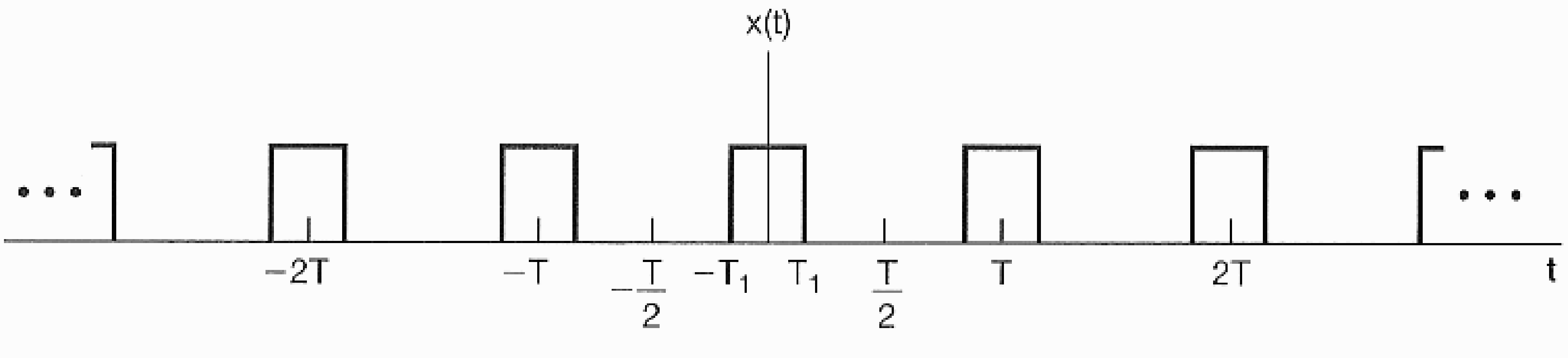
$T = 16T_1$





Page 193, Ex 3.5

CT Fourier Transform of an Aperiodic Signal:



$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

$$a_k = \frac{2\sin(kw_0 T_1)}{kw_0 T}$$

Fourier series coefficients

$$Ta_k = \left.\frac{2\sin(wT_1)}{w}\right|_{w=kw_0}$$

w as a continuous variable

$$Ta_k = \frac{2\sin(wT_1)}{w}$$

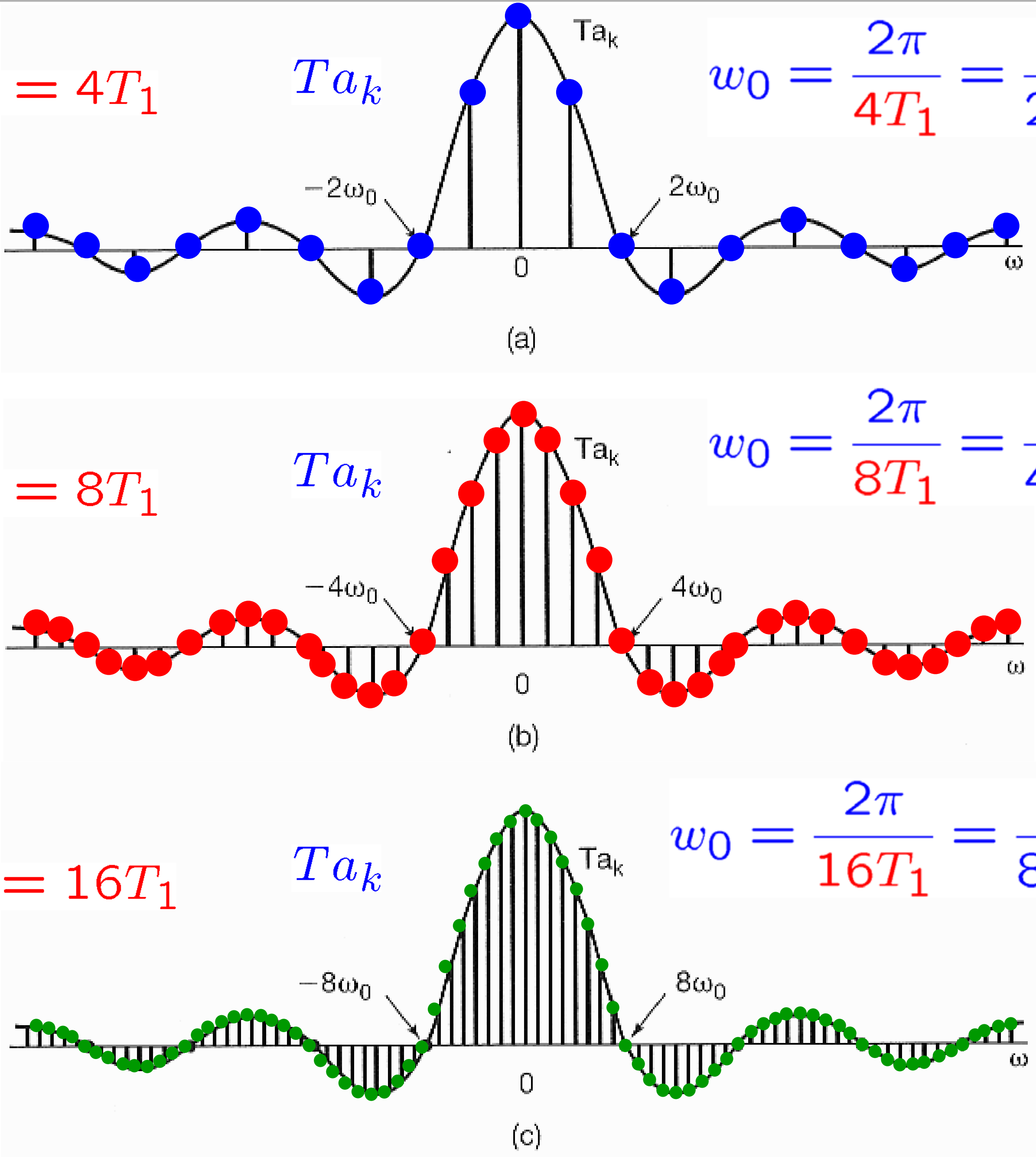
$$w = kw_0 = k\frac{2\pi}{T}$$

$$w_0 = \frac{2\pi}{T}$$

$T = 4T_1$ $\quad Ta_k$ $\quad w_0 = \frac{2\pi}{4T_1} = \frac{\pi}{2T_1}$

$T = 8T_1$ $\quad Ta_k$ $\quad w_0 = \frac{2\pi}{8T_1} = \frac{\pi}{4T_1}$

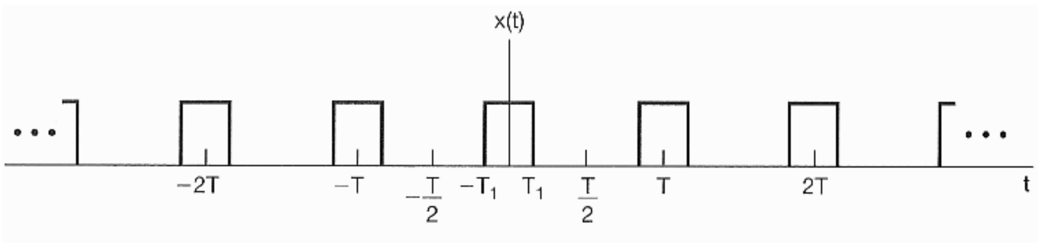
$T = 16T_1$ $\quad Ta_k$ $\quad w_0 = \frac{2\pi}{16T_1} = \frac{\pi}{8T_1}$

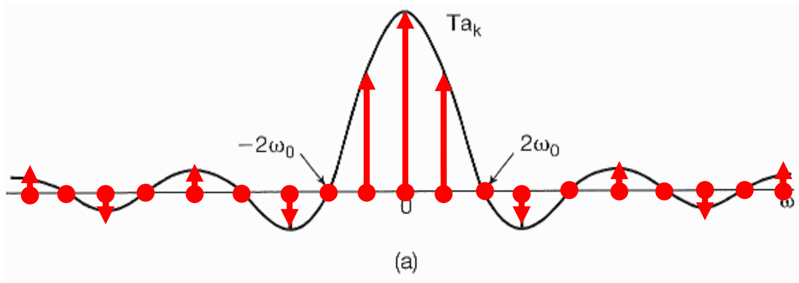


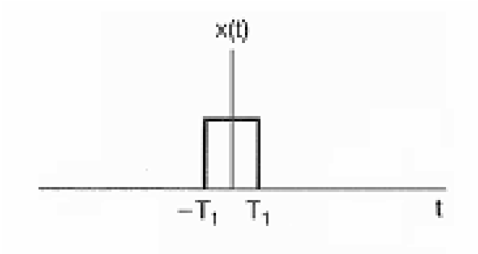
Representation of Aperiodic Signals: CT Fourier Transform

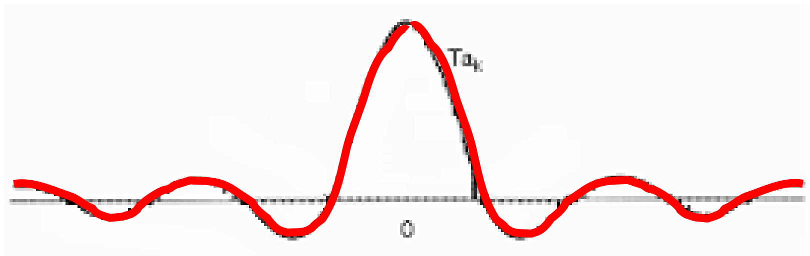
$$\boxed{w = kw_0 = k\frac{2\pi}{T}}$$

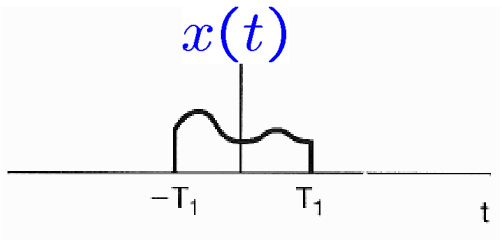
$$T \to \infty \;\Rightarrow\; \{Ta_k\} \to \left.\frac{2\sin(wT_1)}{w}\right|_{w=kw_0}$$





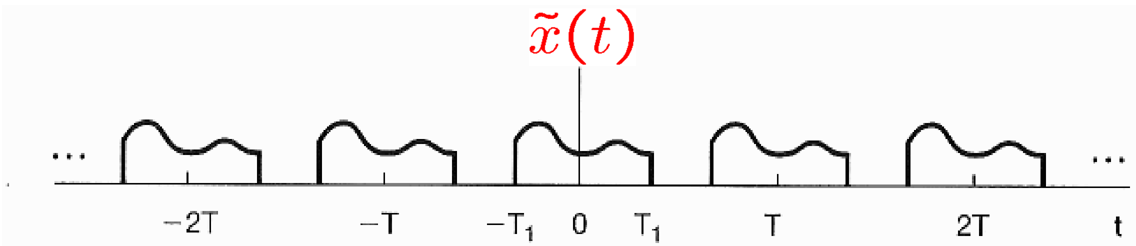






(a)

an aperiodic signal



(b)

a periodic signal

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jkw_0 t} dt$$

$$\Rightarrow a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jkw_0 t} dt = \frac{1}{T}\int_{-\infty}^{+\infty} x(t) e^{-jkw_0 t} dt$$

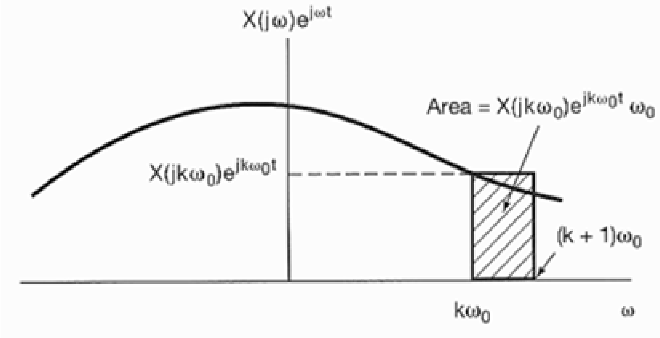
- Define the envelope $X(jw)$ of Ta_k as

$$Ta_k = \frac{2\sin(wT_1)}{w}$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t) e^{-jwt} dt$$

- Then,

$$a_k = \frac{1}{T} X(jkw_0)$$



- Hence,

$$w_0 = \frac{2\pi}{T}$$

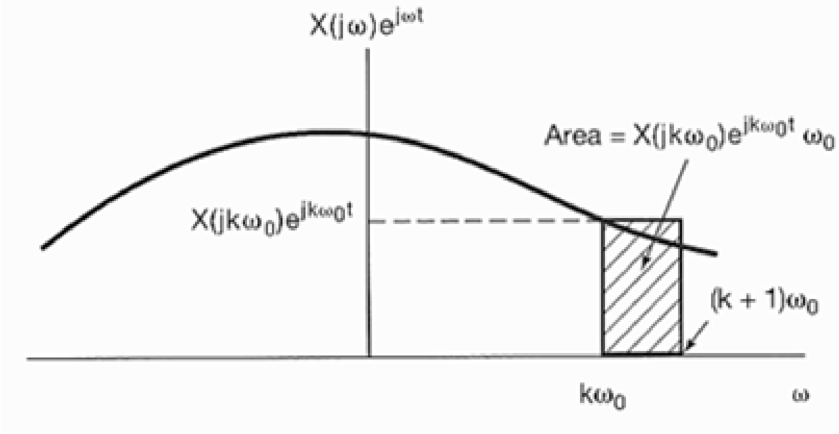
$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} X(jkw_0) e^{jkw_0 t}$$

$$\frac{1}{T} = \frac{1}{2\pi} w_0$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jkw_0) e^{jkw_0 t} w_0$$

- As $T \to \infty$, $\tilde{x}(t) \to x(t)$

also $w_0 \to 0$



$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

- inverse Fourier transform eqn

- synthesis eqn

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

- $X(jw)$: Fourier Transform of $x(t)$
  spectrum

- analysis eqn

$$a_k = \frac{1}{T}X(jw)\Big|_{w=kw_0}$$

Sufficient conditions for the convergence of FT

$$x(t) \xrightarrow{\mathcal{CTFT}} X(jw) \qquad X(jw) = \int_{-\infty}^{+\infty} x(t) e^{-jwt} dt$$

$$\hat{x}(t) \xleftarrow{\mathcal{CTIFT}} X(jw) \qquad \hat{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(jw) e^{jwt} dw$$

$$e(t) = \hat{x}(t) - x(t)$$

- If $x(t)$ has finite energy

  i.e., square integrable, $\int_{-\infty}^{+\infty} |x(t)|^2 dt < \infty$

  $\Rightarrow$ $X(jw)$ is finite

  $\Rightarrow$ $\int_{-\infty}^{+\infty} |e(t)|^2 dt = 0$

- Sufficient conditions for the convergence of FT
  - Dirichlet conditions:

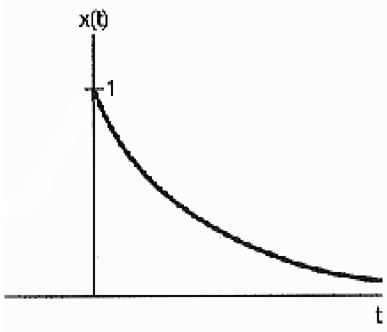
1. x(t) be absolutely integrable; that is, $\int_{-\infty}^{+\infty} |x(t)| dt < \infty$

2. x(t) have a finite number of maxima and minima within any finite interval

3. x(t) have a finite number of discontinuities within any finite interval
Furthermore, each of these discontinuities must be finite

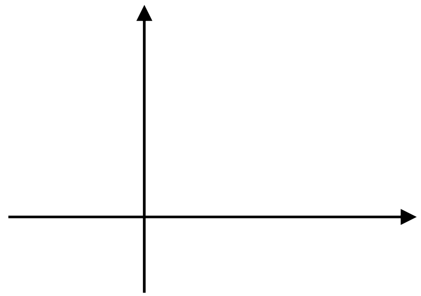
- Example 4.1:

$$x(t) = e^{-at}\,u(t), \quad a > 0$$



$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$



$$\Rightarrow X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

$$= \int_{-\infty}^{\infty} e^{-at}\,u(t)\,e^{-jwt}dt$$

$$= \int_{0}^{\infty} e^{-at}\,e^{-jwt}dt$$

$$= \int_{0}^{\infty} e^{-(a+jw)t}\,dt$$

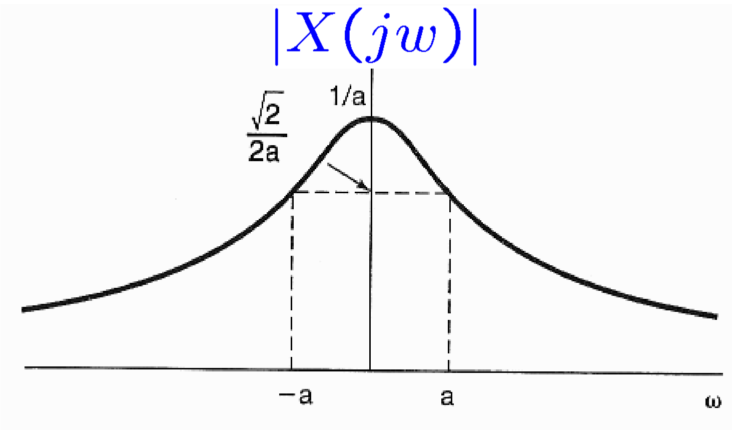
$$= -\frac{1}{a+jw}e^{-(a+jw)t}\Big|_{0}^{\infty}$$

$$= 0 - \left(-\frac{1}{a+jw}e^{-(a+jw)0}\right)$$

$$= \frac{1}{a+jw}, \quad a > 0$$

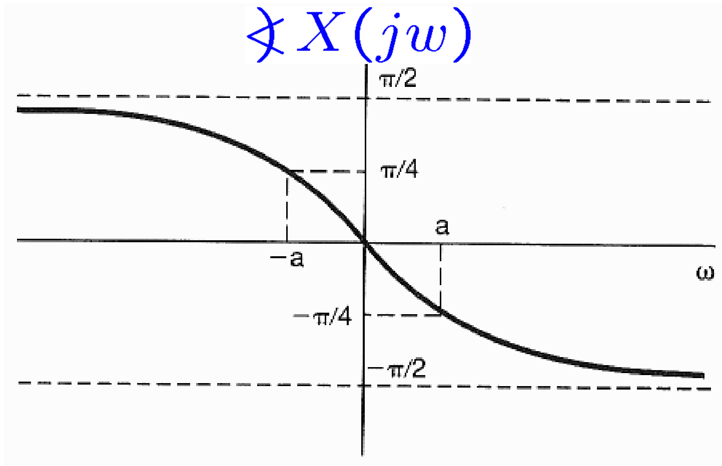
Example 4.1:

$$\Rightarrow \; X(jw) = \frac{1}{a + jw}, \quad a > 0$$



(a)

$$\Rightarrow |X(jw)| = \frac{1}{\sqrt{a^2 + w^2}}$$



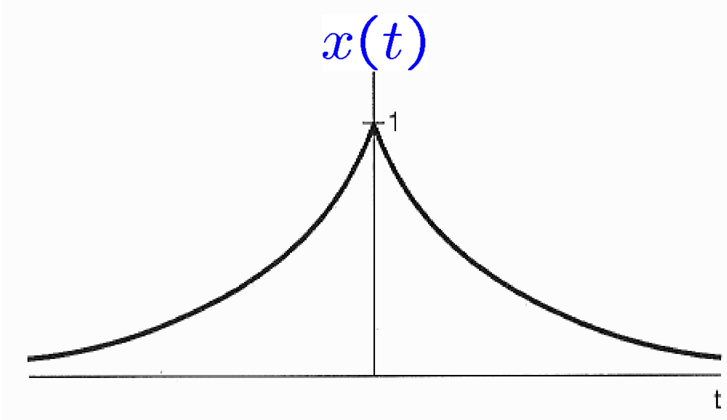
(b)

$$\Rightarrow \sphericalangle X(jw) = -\tan^{-1}\left(\frac{w}{a}\right)$$

$$\boxed{X(jw) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt}$$

- **Example 4.2:**

$$x(t) = e^{-a|t|}, \quad a > 0$$

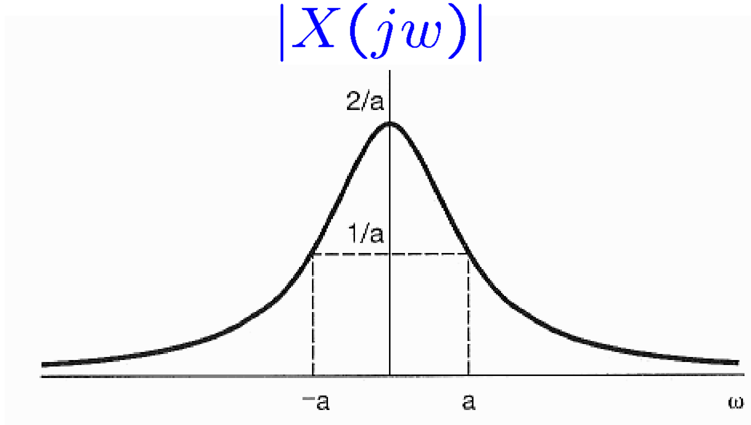


$$\Rightarrow X(jw) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-jwt} dt$$

$$= \int_{-\infty}^{0} e^{at} e^{-jwt} dt + \int_{0}^{\infty} e^{-at} e^{-jwt} dt$$

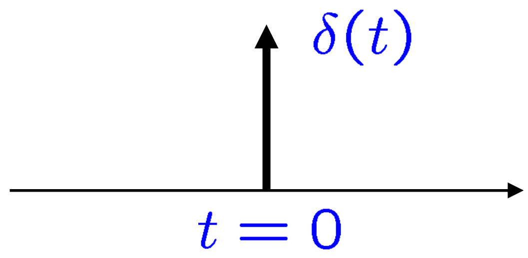
$$= \frac{1}{a - jw} + \frac{1}{a + jw}$$

$$= \frac{2a}{a^2 + w^2}$$



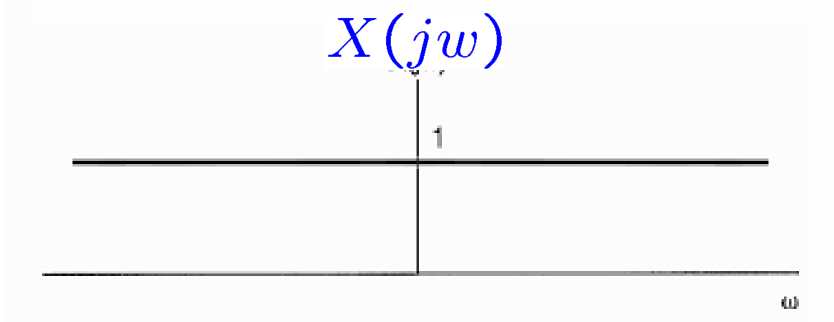
$$\boxed{X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt}$$

- Example 4.3:



$x(t) = \delta(t)$, i.e., unit impules

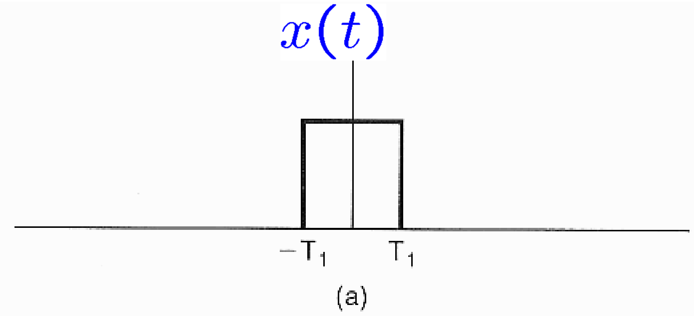
$$\Rightarrow X(jw) = \int_{-\infty}^{\infty} \delta(t)\, e^{-jwt} dt = 1$$



$$X(jw) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt$$

- Example 4.4:

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$

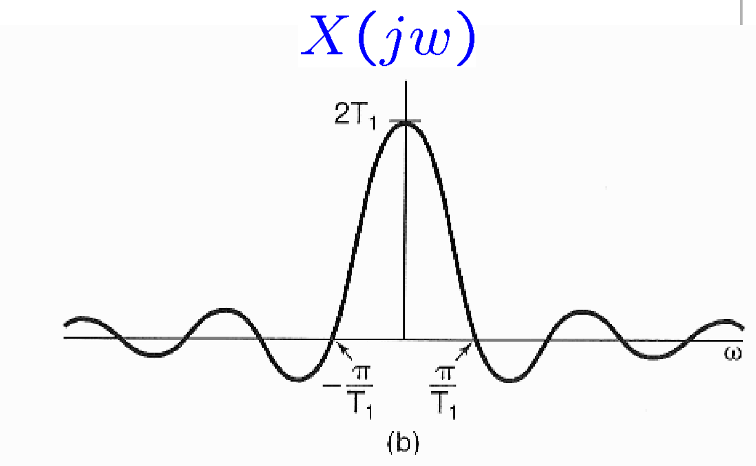


$$\Rightarrow X(jw) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt = \int_{-T_1}^{T_1} e^{-jwt} dt$$

$$= \frac{1}{-jw} e^{-jwt} \Big|_{-T_1}^{T_1}$$

$$= \frac{1}{-jw} \left(e^{-jwT_1} - e^{jwT_1} \right)$$

$$= \frac{1}{jw} \left(e^{jwT_1} - e^{-jwT_1} \right)$$



$$= 2 \frac{\sin(wT_1)}{w} = 2T_1 \frac{\sin(\pi \; wT_1 \; /\pi)}{\pi \; wT_1 \; /\pi)}$$

- Example 4.5:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw) e^{jwt} dw$$

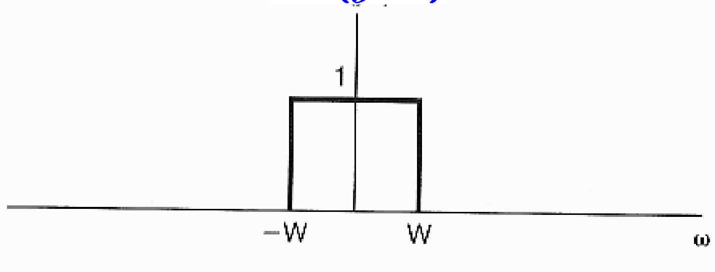
$$X(jw) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt$$

$$X(jw) = \begin{cases} 1, & |w| < W \\ 0, & |w| > W \end{cases}$$

$$\Rightarrow \; x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{jwt} dw$$

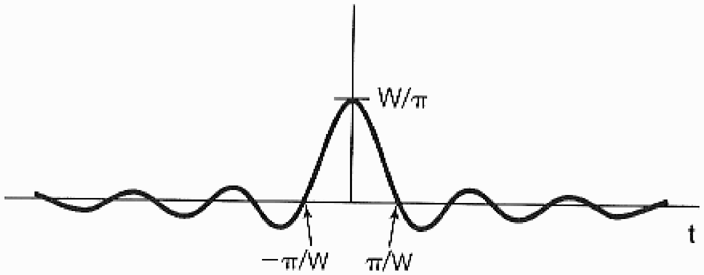
$$= \frac{\sin(Wt)}{\pi t}$$

$X(jw)$



(a)

$x(t)$

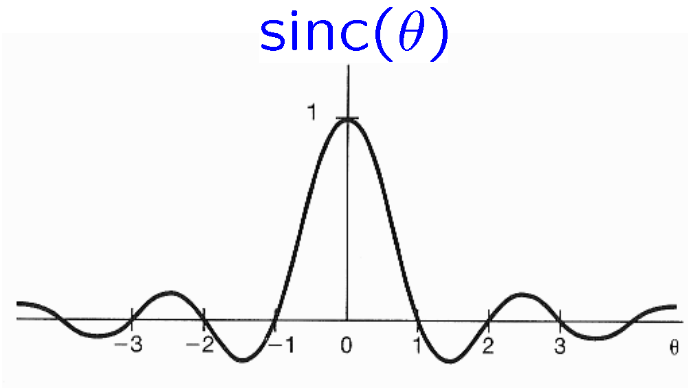


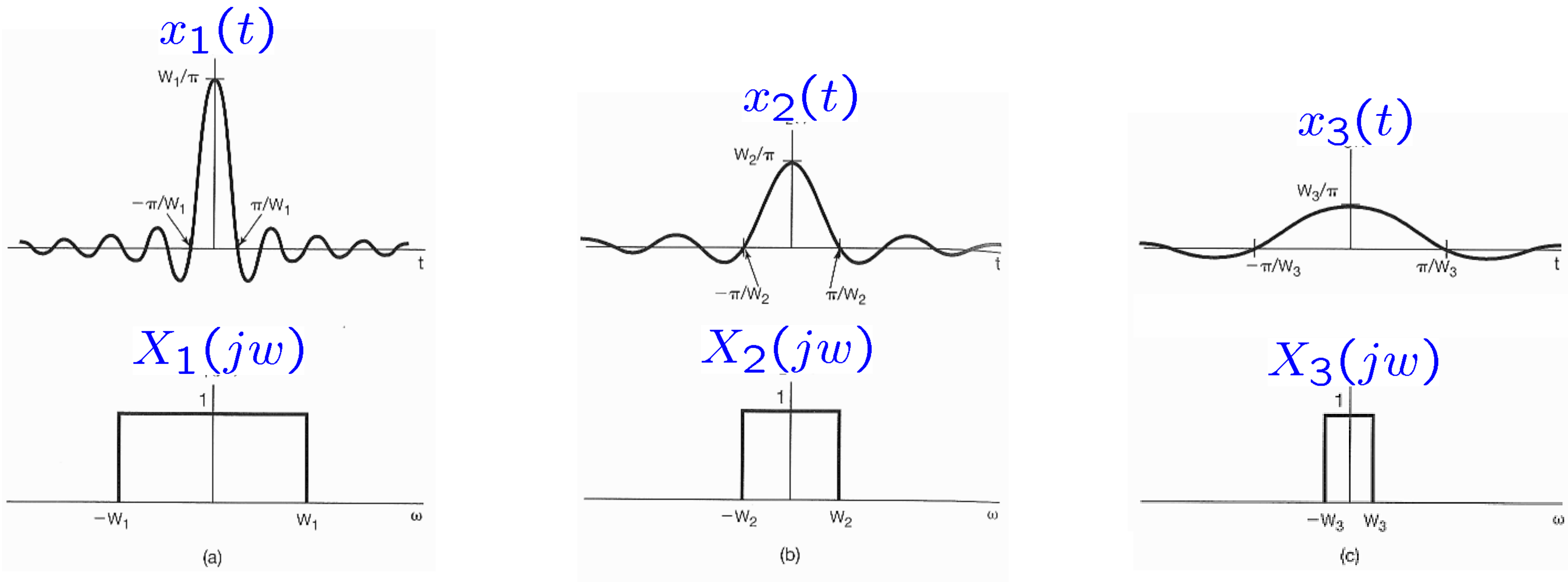
(b)

▪ sinc functions:

$$\text{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$$

$$\frac{\sin(Wt)}{\pi t} = \frac{W}{\pi}\text{sinc}\left(\frac{Wt}{\pi}\right)$$

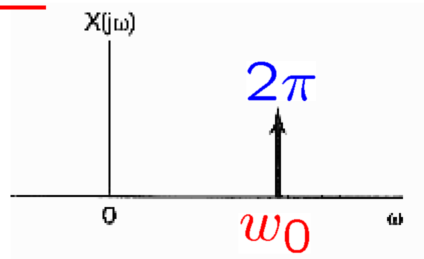




- Representation of Aperiodic Signals:
  the Continuous-Time Fourier Transform
- The Fourier Transform for Periodic Signals
- Properties
  of the Continuous-Time Fourier Transform
- The Convolution Property
- The Multiplication Property
- Systems Characterized by
  Linear Constant-Coefficient Differential Equations

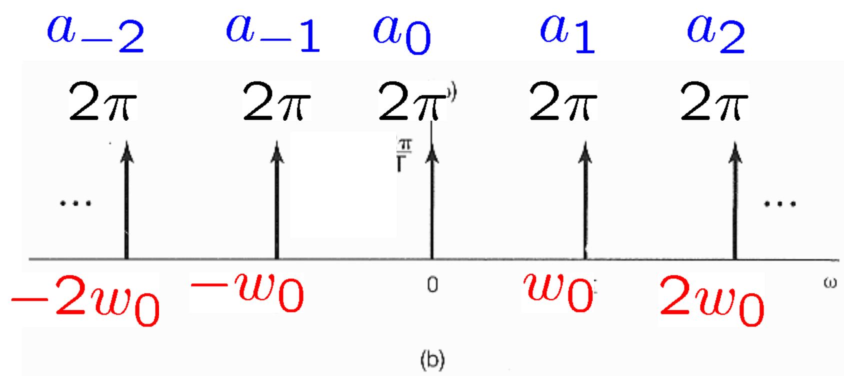
Fourier Transform from Fourier Series:

$$X(jw) = 2\pi \delta(w - w_0)$$



$$\Rightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi \delta(w - w_0) e^{jwt} dw$$

$$= e^{j w_0 t}$$



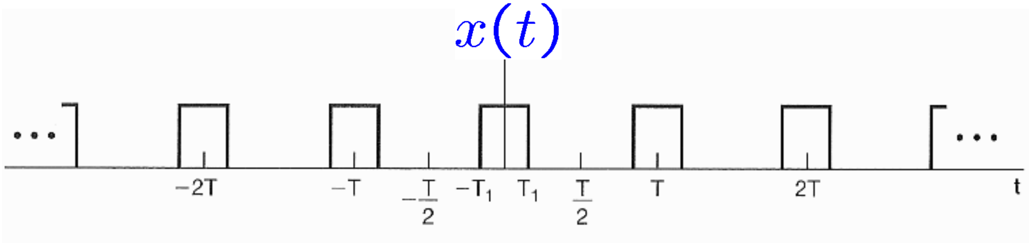
- more generally,

$$X(jw) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(w - k w_0)$$

$$\Rightarrow x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

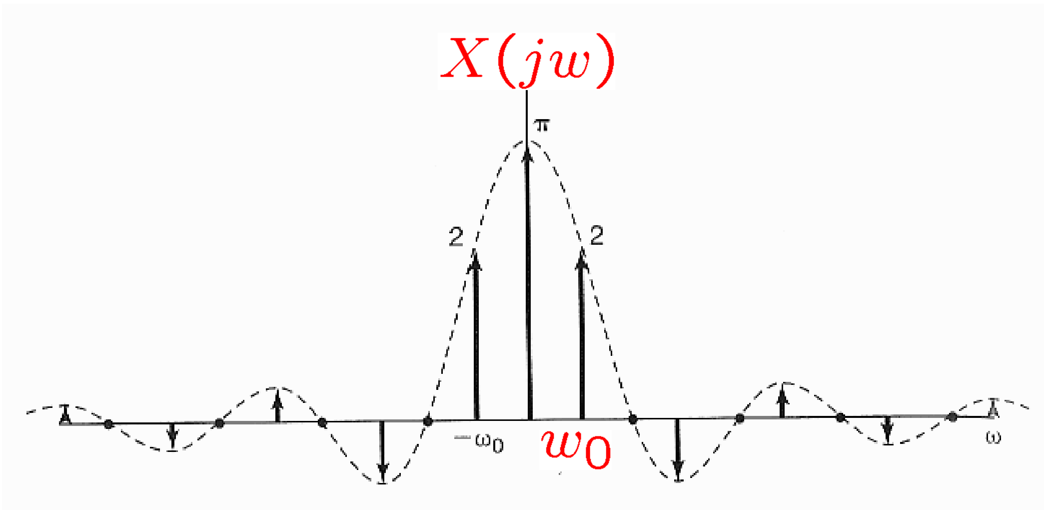
Fourier series represntation of a periodic signal

- Example 4.6:



$$\Rightarrow \quad a_k = \frac{\sin(k w_0 T_1)}{\pi k}$$

$$\Rightarrow \; X(jw) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(w - k w_0) = \sum_{k=-\infty}^{+\infty} \frac{2\sin(k w_0 T_1)}{k} \delta(w - k w_0)$$



- Example 4.7:

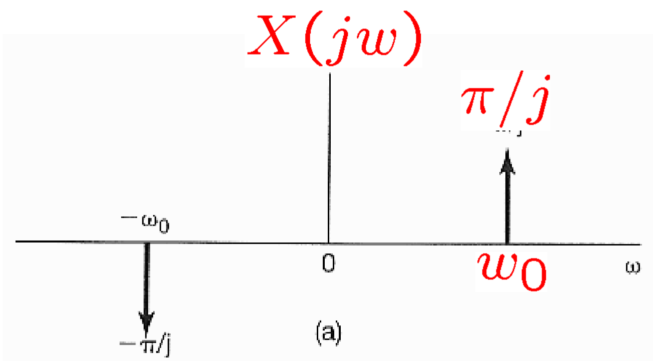
$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

$$x(t) = \sin(w_0 t) = \frac{e^{jw_0 t} - e^{-jw_0 t}}{2j}$$

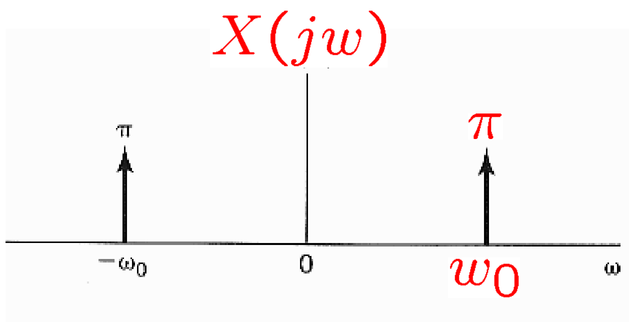
$$\Rightarrow \quad a_1 = \frac{1}{2j} \qquad a_{-1} = -\frac{1}{2j} \qquad a_k = 0, \quad k \neq 1, -1$$

$$x(t) = \cos(w_0 t) = \frac{e^{jw_0 t} + e^{-jw_0 t}}{2}$$

$$\Rightarrow \quad a_1 = \frac{1}{2} \qquad a_{-1} = \frac{1}{2} \qquad a_k = 0, \quad k \neq 1, -1$$



(a)



(b)

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{+T/2} \tilde{x}(t) e^{-jkw_0 t} dt$$

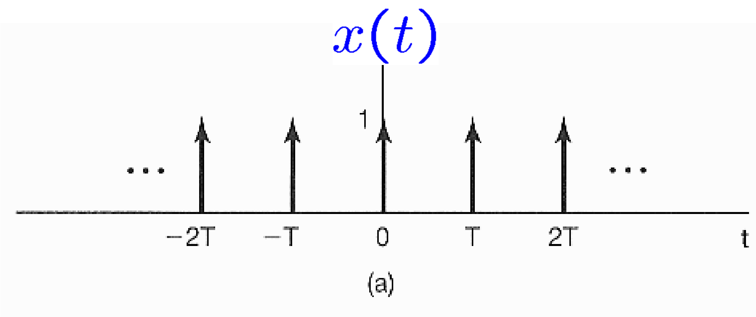
$$X(jw) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(w - kw_0)$$

- Example 4.8:

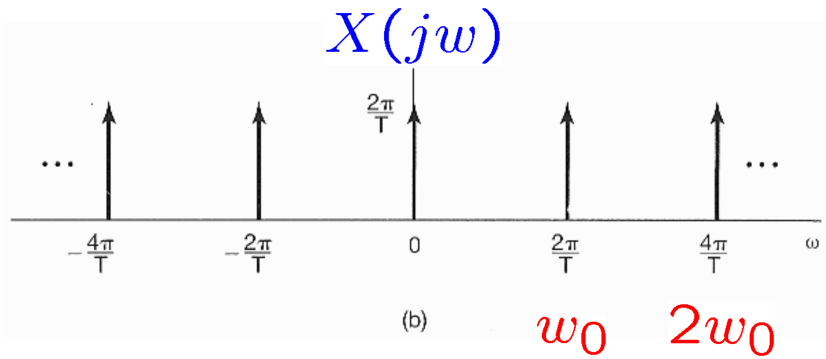
$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT)$$

$$\Rightarrow \ a_k = \frac{1}{T}\int_{-T/2}^{+T/2} \delta(t) e^{-jkw_0 t} dt = \frac{1}{T}$$

$$\Rightarrow \ X(jw) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(w - \frac{2\pi}{T} k)$$



(a)



(b)

- Representation of Aperiodic Signals:
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- The Multiplication Property
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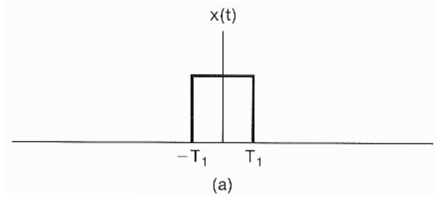
Outline

Section	Property
4.3.1	Linearity
4.3.2	Time Shifting
4.3.6	Frequency Shifting
4.3.3	Conjugation
4.3.5	Time Reversal
4.3.5	Time and Frequency Scaling
4.4	Convolution
4.5	Multiplication
4.3.4	Differentiation in Time
4.3.4	Integration
4.3.6	Differentiation in Frequency
4.3.3	Conjugate Symmetry for Real Signals
4.3.3	Symmetry for Real and Even Signals
4.3.3	Symmetry for Real and Odd Signals
4.3.3	Even-Odd Decomposition for Real Signals
4.3.7	Parseval's Relation for Aperiodic Signals

Outline

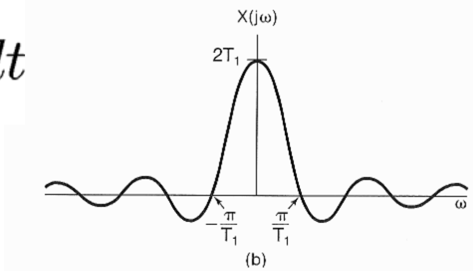
Property	CTFS	DTFS	CTFT	DTFT	LT	zT
Linearity	3.5.1		4.3.1	5.3.2	9.5.1	10.5.1
Time Shifting	3.5.2		4.3.2	5.3.3	9.5.2	10.5.2
Frequency Shifting (in s, z)			4.3.6	5.3.3	9.5.3	10.5.3
Conjugation	3.5.6		4.3.3	5.3.4	9.5.5	10.5.6
Time Reversal	3.5.3		4.3.5	5.3.6		10.5.4
Time & Frequency Scaling	3.5.4		4.3.5	5.3.7	9.5.4	10.5.5
(Periodic) Convolution			4.4	5.4	9.5.6	10.5.7
Multiplication	3.5.5	3.7.2	4.5	5.5		
Differentiation/First Difference		3.7.2	4.3.4, 4.3.6	5.3.5, 5.3.8	9.5.7, 9.5.8	10.5.7, 10.5.8
Integration/Running Sum (Accumulation)			4.3.4	5.3.5	9.5.9	10.5.7
Conjugate Symmetry for Real Signals	3.5.6		4.3.3	5.3.4		
Symmetry for Real and Even Signals	3.5.6		4.3.3	5.3.4		
Symmetry for Real and Odd Signals	3.5.6		4.3.3	5.3.4		
Even-Odd Decomposition for Real Signals			4.3.3	5.3.4		
Parseval's Relation for (A)Periodic Signals	3.5.7	3.7.3	4.3.7	5.3.9		
Initial- and Final-Value Theorems					9.5.10	10.5.9

Fourier Transform Pair:



- Synthesis equation: $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

- Analysis equation: $$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$



- Notations:

$$X(jw) = \mathcal{F}\{x(t)\}$$

$$\frac{1}{a+jw} = \mathcal{F}\{e^{-at}u(t)\}$$

$$x(t) = \mathcal{F}^{-1}\{X(jw)\}$$

$$e^{-at}u(t) = \mathcal{F}^{-1}\{\frac{1}{a+jw}\}$$

$$x(t) \xleftrightarrow{\mathcal{CTFT}} X(jw)$$

$$e^{-at}u(t) \xleftrightarrow{\mathcal{CTFT}} \frac{1}{a+jw}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(jw) e^{jwt} dw$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t) e^{-jwt} dt$$

- **Linearity:**

$$x(t) \xleftrightarrow{\mathcal{F}} X(jw)$$

$$y(t) \xleftrightarrow{\mathcal{F}} Y(jw)$$

$$\Rightarrow \quad a\, x(t) + b\, y(t) \xleftrightarrow{\mathcal{F}} a\, X(jw) + b\, Y(jw)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

- Time Shifting:

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(jw)$$

$$\Rightarrow x(t-t_0) \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-jwt_0}X(jw)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$x(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jw(t-t_0)}dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(e^{-jwt_0}X(jw)\right) e^{jwt}dw$$

$$Y(jw) = \int_{-\infty}^{+\infty} x(t-t_0)e^{-jwt}dt$$

$$= \int_{-\infty}^{+\infty} x(\tau)e^{-jw(\tau+t_0)}d\tau$$

$$= e^{-jwt_0}\int_{-\infty}^{+\infty} x(\tau)e^{-jw\tau}d\tau$$

- Time Shift $\rightarrow$ Phase Shift:

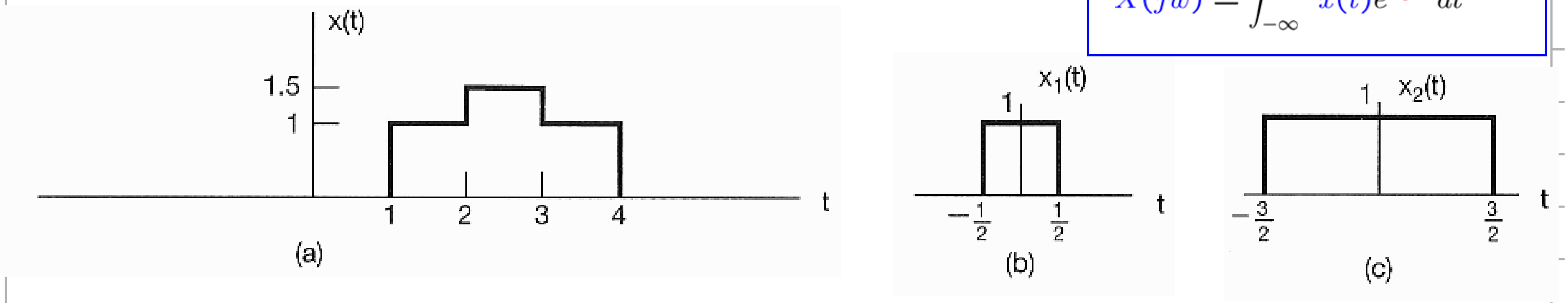
$$\mathcal{F}\{x(t)\} = X(jw) = |X(jw)|e^{j\sphericalangle X(jw)}$$

$$\mathcal{F}\{x(t-t_0)\} = e^{-jwt_0}X(jw) = |X(jw)|e^{j[\sphericalangle X(jw) - wt_0]}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

- Example 4.9:



$$x(t) = \frac{1}{2}x_1(t-2.5) + x_2(t-2.5)$$

$$X_1(jw) = \frac{2\sin(w/2)}{w}$$

$$X_2(jw) = \frac{2\sin(3w/2)}{w}$$

$$\Rightarrow \ X(jw) = e^{-j5w/2}\left\{\frac{\sin(w/2) + 2\sin(3w/2)}{w}\right\}$$

$X(jw) = \int_{-\infty}^{+\infty} x(t) e^{-jwt} dt$

- Conjugation & Conjugate Symmetry:

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(jw)$$

$$x(t)^* \overset{\mathcal{F}}{\longleftrightarrow} X^*(-jw)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(jw) e^{jwt} dw$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(jw) \quad e^{jwt} \quad dw$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(jw) \quad e \quad {}^{jwt} dw$$

$$= \frac{1}{2\pi} \int_{\infty}^{\infty} X(\quad j\bar{w}) e^{j\bar{w}t} \quad d\bar{w}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\quad j\bar{w}) e^{j\bar{w}t} d\bar{w}$$

Conjugation & Conjugate Symmetry:

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(jw)$$

$$x(t)^* \stackrel{\mathcal{F}}{\longleftrightarrow} X^*(-jw)$$

- $x(t) = x^*(t) \Rightarrow X(-jw) = X^*(jw)$

  $x(t)$ is real $\Rightarrow$ $X(jw)$ is conjugate symmetric

- $x(t) = x^*(t)$ & $x(-t) = x(t)$

  $\Rightarrow X(-jw) = X^*(jw)$ & $X(-jw) = X(jw)$

  $\Rightarrow X(jw) = X^*(jw)$

  $x(t)$ is real & even $\Rightarrow$ $X(jw)$ are real & even

- $x(t)$ is real & odd $\Rightarrow$ $X(jw)$ are purely imaginary & odd

▪ Conjugation & Conjugate Symmetry:

If $x(t)$ is a real function

$$x(t) = \mathcal{E}v\{x(t)\} + \mathcal{O}d\{x(t)\} = x_e(t) + x_o(t)$$

$$\Rightarrow \ \mathcal{F}\{x(t)\} = \mathcal{F}\{x_e(t)\} + \mathcal{F}\{x_o(t)\}$$

$\Rightarrow \ \mathcal{F}\{x_e(t)\}$: a real function

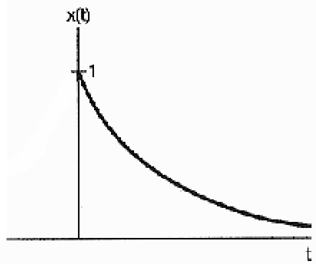
$\Rightarrow \ \mathcal{F}\{x_o(t)\}$: a purely imaginary function

$$x(t) \ \stackrel{\mathcal{F}}{\longleftrightarrow} \ X(jw)$$

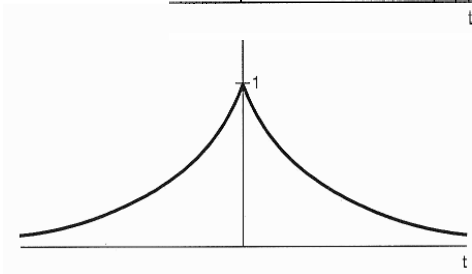
$$\mathcal{E}v\{x(t)\} \ \stackrel{\mathcal{F}}{\longleftrightarrow} \ \mathcal{R}e\{X(jw)\}$$

$$\mathcal{O}d\{x(t)\} \ \stackrel{\mathcal{F}}{\longleftrightarrow} \ j\,\mathcal{I}m\{X(jw)\}$$

- Example 4.10:



Ex 4.1
$$e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a+jw}$$



Ex 4.2
$$e^{-a|t|} \overset{\mathcal{F}}{\longleftrightarrow} ?$$

$$x(t) = e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t)$$

$$= 2\left[\frac{e^{-at}u(t) + e^{at}u(-t)}{2}\right] \qquad = 2\mathcal{E}v\left\{e^{-at}u(t)\right\}$$

$$\mathcal{E}v\left\{e^{-at}u(t)\right\} \overset{\mathcal{F}}{\longleftrightarrow} \mathcal{R}e\left\{\frac{1}{a+jw}\right\}$$

$$\mathcal{O}d\left\{e^{-at}u(t)\right\} \overset{\mathcal{F}}{\longleftrightarrow} j\,\mathcal{I}m\left\{\frac{1}{a+jw}\right\}$$

$$X(jw) = 2\mathcal{R}e\left\{\frac{1}{a+jw}\right\} = \frac{2a}{a^2+w^2}$$

- Differentiation & Integration:

$$\boxed{X(jw) = \int_{-\infty}^{+\infty} x(t) e^{-jwt} dt}$$

$$x(t) \xleftrightarrow{\mathcal{F}} X(jw)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(jw) e^{jwt} dw$$

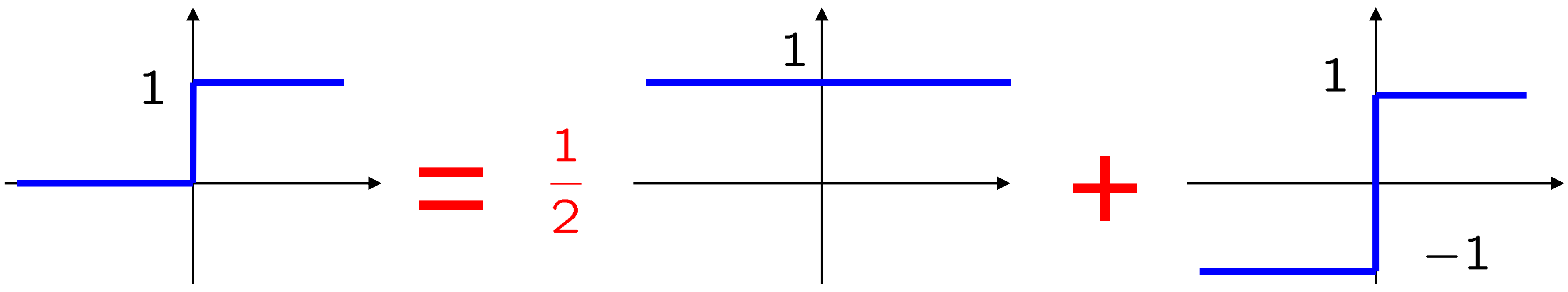
$$\frac{d}{dt} x(t) \xleftrightarrow{\mathcal{F}} jw X(jw)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(jw) \qquad e^{jwt} \quad dw$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(jw) \qquad e^{jwt} \quad dw$$

$$\int_{-\infty}^{t} x(\tau) d\tau \xleftrightarrow{\mathcal{F}} \frac{1}{jw} X(jw) + \pi X(0) \delta(w)$$

dc or average value



$u(t) \qquad \frac{1}{2} \qquad 1(t) \qquad \text{sgn}(t)$

$$1 \overset{\mathcal{FT}}{\longleftrightarrow} 2\pi\delta(jw) \qquad \text{sgn(t)} \overset{\mathcal{FT}}{\longleftrightarrow} S(jw)$$

$$\frac{d}{dt}\,\text{sgn(t)} \overset{\mathcal{FT}}{\longleftrightarrow} jw\; S(jw)$$

$$2\;\delta(t) \overset{\mathcal{FT}}{\longleftrightarrow} jw\; S(jw) \qquad \Rightarrow U(jw) =$$

$$\delta(t) \overset{\mathcal{FT}}{\longleftrightarrow} 1$$

$$\Rightarrow S(jw) =$$

- Example 4.11:

$$x(t) = u(t) \overset{\mathcal{F}}{\longleftrightarrow} X(jw) = ?$$

$$g(t) = \delta(t) \overset{\mathcal{F}}{\longleftrightarrow} G(jw) = 1$$

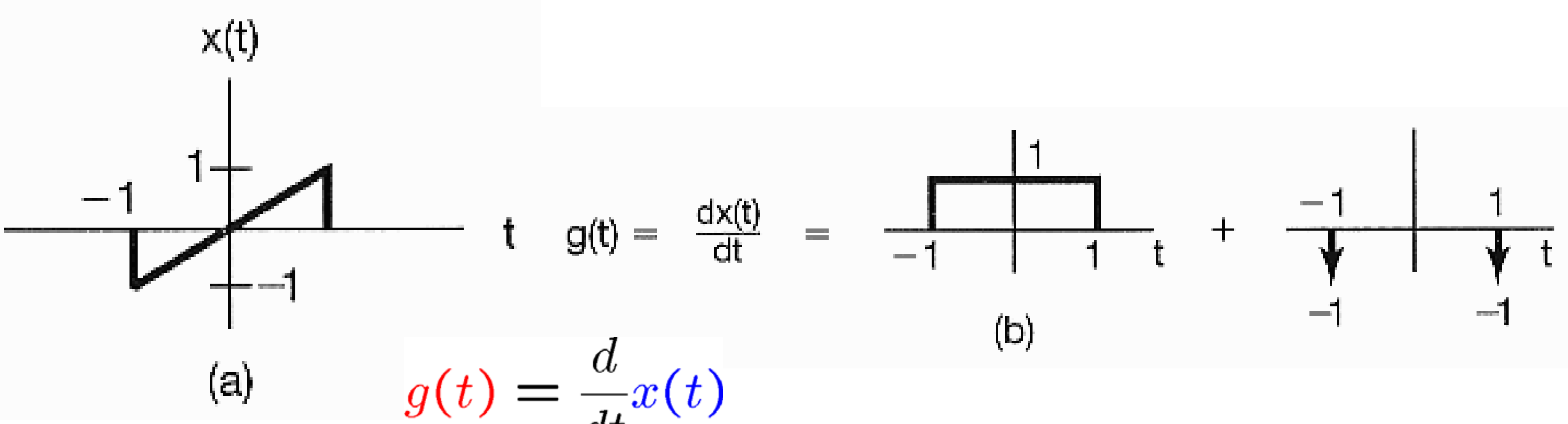
$$x(t) = \int_{-\infty}^{t} g(\tau) d\tau$$

$$X(jw) = \frac{1}{jw} G(jw) + \pi G(0) \delta(w)$$

$$= \frac{1}{jw} + \pi \delta(w)$$

$$\delta(t) = \frac{d}{dt} u(t) \overset{\mathcal{F}}{\longleftrightarrow} jw \left[\frac{1}{jw} + \pi \delta(w) \right] = 1$$

- Example 4.12:



$$g(t) = \frac{d}{dt}x(t)$$

$$G(jw) = \frac{2\sin(w)}{w} - e^{jw} - e^{-jw}$$

$$\Rightarrow \; X(jw) = \frac{G(jw)}{jw} + \pi G(0)\delta(w)$$

$$= \frac{2\sin(w)}{jw^2} - \frac{2\cos(w)}{jw}$$

- Time & Frequency Scaling:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(jw)$$

$$x(at) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|}X\left(\frac{jw}{a}\right)$$

$$\frac{1}{|b|}x\left(\frac{t}{b}\right) \stackrel{\mathcal{F}}{\longleftrightarrow} X(jbw)$$

$$x(-t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(-jw)$$

$$x(\quad t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jw\quad t}dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\quad \bar{w})e^{j\,\bar{w}\,t}\quad d\bar{w}$$

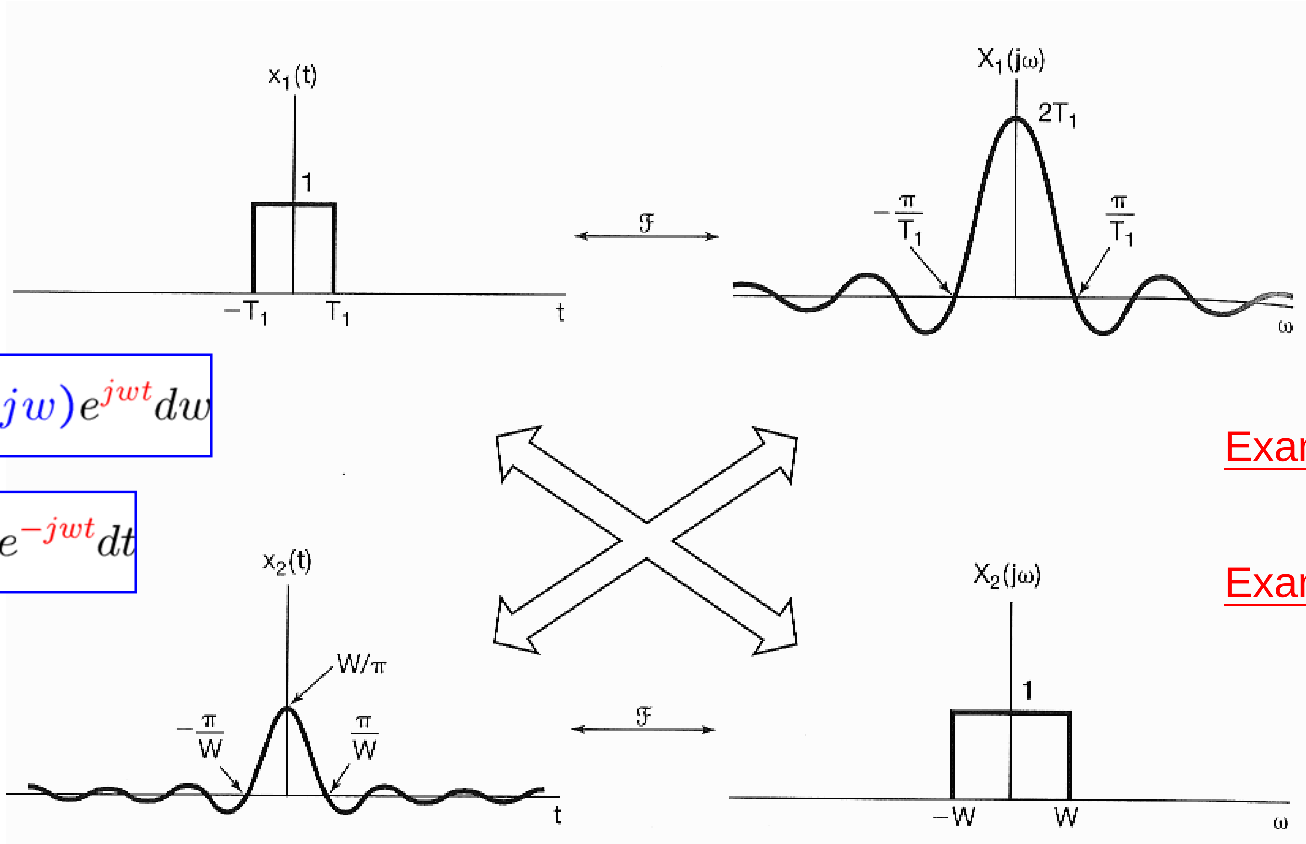
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \quad X(j\quad \bar{w})e^{j\,\bar{w}\,t}d\bar{w}$$

$$= \frac{1}{2\pi}\int_{+\infty}^{-\infty} X(j\quad \bar{w})e^{j\,\bar{w}\,t}\quad d\bar{w}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \quad X(j\quad \bar{w})e^{j\,\bar{w}\,t}d\bar{w}$$

- **Duality:**

$$x_1(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X_1(jw) = \frac{2\sin(wT_1)}{w}$$



$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

Example 4.4

Example 4.5

$$x_2(t) = \frac{\sin(Wt)}{\pi t} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X_2(jw) = \begin{cases} 1, & |w| < W \\ 0, & |w| > W \end{cases}$$

- Duality:

$$X(jw) = \int_{-\infty}^{+\infty} x(t) e^{-jwt} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(jw) e^{jwt} dw$$

$$B(s) = \int_{-\infty}^{+\infty} A(\tau) e^{-js\tau} d\tau$$

$$B(-s) = \int_{-\infty}^{+\infty} A(\tau) e^{js\tau} d\tau$$

$$A(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} B(s) e^{js\tau} ds$$

$$A(s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} B(\tau) e^{js\tau} d\tau$$

$$A(-s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} B(\tau) e^{-js\tau} d\tau$$

- Duality:

$$x(t - t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-jwt_0}X(jw)$$

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(jw)$$

$$\frac{d}{dt}x(t) \overset{\mathcal{F}}{\longleftrightarrow} jwX(jw)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{jw}X(jw) + \pi X(0)\delta(w)$$

f(t) <--> F(jw)F(t) <--> 2 * pi * f(-w)

$$-jtx(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{d}{dw}X(jw)$$

$$e^{jw_0t}x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j(w-w_0))$$

$$-\frac{1}{jt}x(t) + \pi x(0)\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{w} X(\eta)d\eta$$

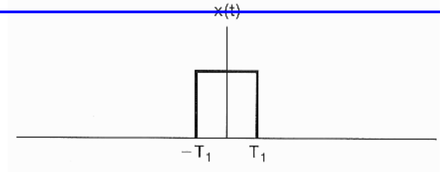
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

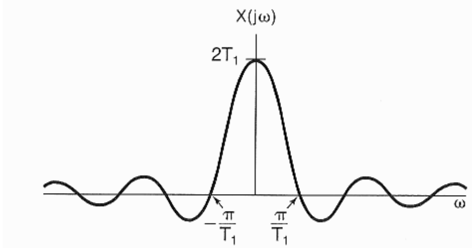
- Parseval's relation:

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(jw)$$

$$\Rightarrow \int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(jw)|^2 dw$$



(a)



(b)

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} x(t)x^*(t)dt$$

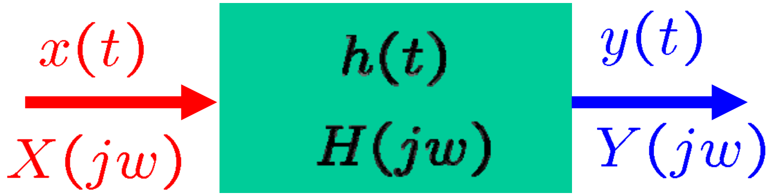
$$= \int_{-\infty}^{+\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(jw)e^{-jwt}dw\right]dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(jw)\left[\int_{-\infty}^{+\infty} x(t)e^{-jwt}dt\right]dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(jw)|^2 dw$$

- Representation of Aperiodic Signals:
  the Continuous-Time Fourier Transform
- The Fourier Transform for Periodic Signals
- Properties
  of the Continuous-Time Fourier Transform
- The Convolution Property
- The Multiplication Property
- Systems Characterized by
  Linear Constant-Coefficient Differential Equations

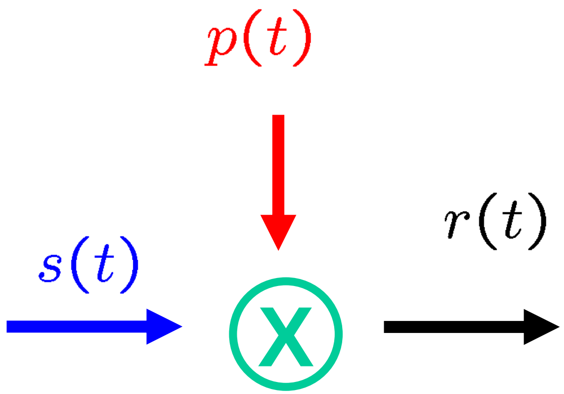
- Convolution Property:



$$y(t) = x(t) * h(t) \;\stackrel{\mathcal{F}}{\longleftrightarrow}\; Y(jw) = X(jw)H(jw)$$

$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- Multiplication Property:



$$r(t) = s(t)p(t) \;\stackrel{\mathcal{F}}{\longleftrightarrow}\; R(jw) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)P(j(w-\theta))d\theta$$

- **From Superposition (or Linearity):** $H(jkw_0) = \int_{-\infty}^{\infty} h(t) e^{-jkw_0 t} dt$

$h(t)$ $H(jw)$ Linear System

$$e^{jkw_0 t} \rightarrow H(jkw_0) e^{jkw_0 t}$$

$$X(jkw_0) e^{jkw_0 t} w_0 \rightarrow X(jkw_0) H(jkw_0) e^{jkw_0 t} w_0$$

$$\sum_{k=-\infty}^{+\infty} X(jkw_0) e^{jkw_0 t} w_0 \rightarrow \sum_{k=-\infty}^{+\infty} X(jkw_0) H(jkw_0) e^{jkw_0 t} w_0$$

$$x(t) \rightarrow y(t)$$

$$= \lim_{w_0 \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jkw_0) e^{jkw_0 t} w_0 \qquad = \lim_{w_0 \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jkw_0) H(jkw_0) e^{jkw_0 t} w_0$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(jw) e^{jwt} dw \qquad = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(jw) H(jw) e^{jwt} dw$$

- From Superposition (or Linearity):

$$\frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jkw_0)e^{jkw_0 t}w_0 \longrightarrow \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jkw_0)H(jkw_0)e^{jkw_0 t}w_0$$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)H(jw)e^{jwt}dw$$

Since $y(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Y(jw)e^{jwt}dw$

$$\Rightarrow \; Y(jw) = X(jw)H(jw)$$

$$y(t) = x(t) * h(t) \stackrel{\mathcal{F}}{\longleftrightarrow} Y(jw) = X(jw)H(jw)$$

- From Convolution Integral:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

$$x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-jwt_0}X(jw)$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$\Rightarrow\ Y(jw) = \mathcal{F}\{y(t)\} = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau\right]e^{-jwt}dt$$

$$= \int_{-\infty}^{+\infty} x(\tau)\left[\int_{-\infty}^{+\infty} h(t-\tau)e^{-jwt}dt\right]d\tau$$

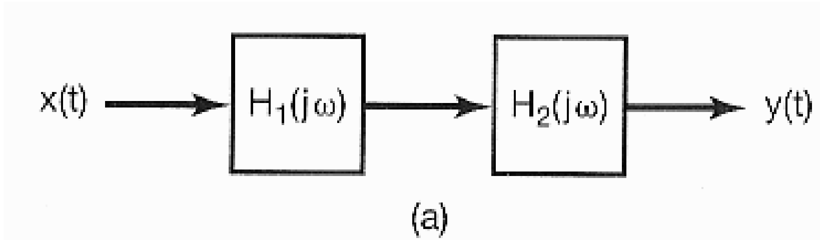
$$= \int_{-\infty}^{+\infty} x(\tau)\left[e^{-jw\tau}\int_{-\infty}^{+\infty} h(\sigma)e^{-jw\sigma}d\sigma\right]d\tau$$

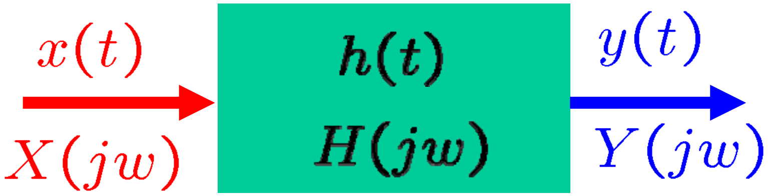
$$= \int_{-\infty}^{+\infty} x(\tau)\left[e^{-jw\tau}H(jw)\right]d\tau$$

$$= H(jw)\int_{-\infty}^{+\infty} x(\tau)e^{-jw\tau}d\tau$$

$$\Rightarrow\ Y(jw) = H(jw)X(jw)$$

- Equivalent LTI Systems:



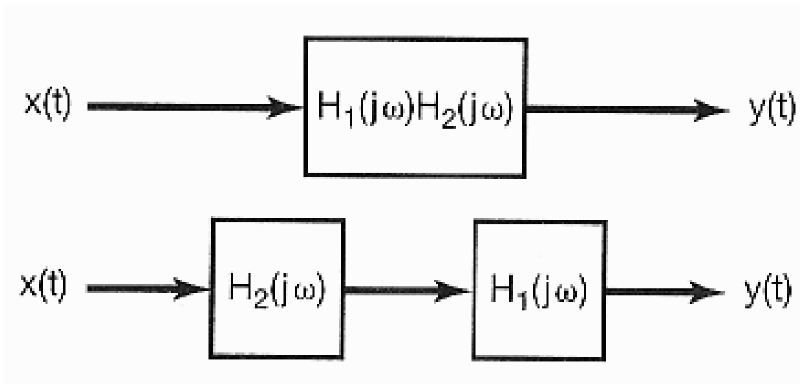


$$h(t) \stackrel{\mathcal{F}}{\longleftrightarrow} H(jw)$$

impulse response | frequency response

$$y(t) = x(t) * h(t)$$

$$Y(jw) = X(jw)H(jw)$$



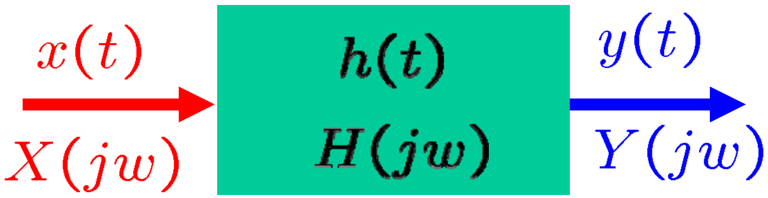
$$\Rightarrow Y(jw) = H_1(jw)H_2(jw)X(jw)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

$$x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-jwt_0}X(jw)$$

Example 4.15: Time Shift



$$h(t) = \delta(t-t_0)$$

$$\Rightarrow H(jw) = e^{-jwt_0}$$

$$\begin{aligned} Y(jw) &= H(jw)X(jw) \\ &= e^{-jwt_0}X(jw) \end{aligned}$$

$$\Rightarrow y(t) = x(t-t_0)$$

Examples 4.16 & 17: Differentiator & Integrator

$$y(t) = \frac{d}{dt}x(t) \quad \Rightarrow \quad Y(jw) = jwX(jw)$$

$$\Rightarrow \quad H(jw) = jw$$

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau \quad \Rightarrow \quad h(t) = u(t)$$

impulse response

$$\Rightarrow \quad H(jw) = \frac{1}{jw} + \pi\delta(w)$$

$$\Rightarrow \quad Y(jw) = H(jw)X(jw)$$

$$= \frac{1}{jw}X(jw) + \pi\delta(w)X(jw)$$

$$= \frac{1}{jw}X(jw) + \pi\delta(w)X(0)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

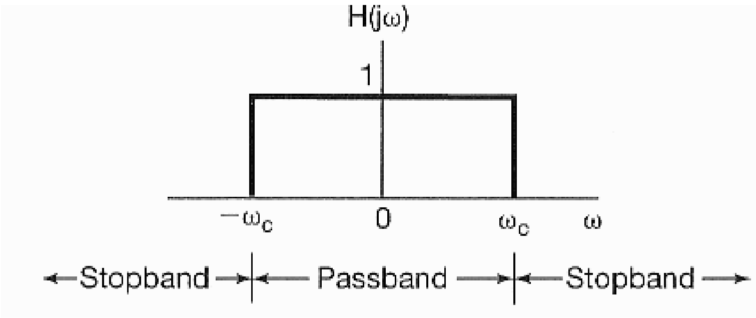
$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

- Example 4.18: Ideal Lowpass Filter

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

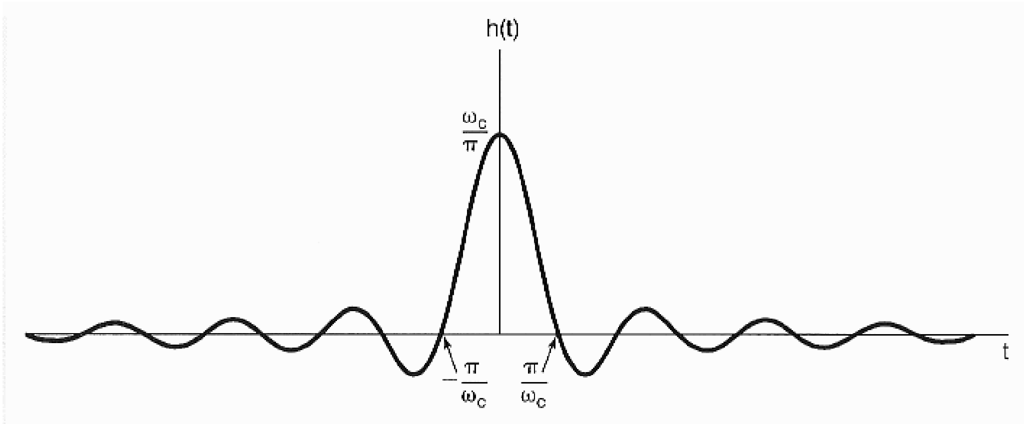
$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

$$H(jw) = \begin{cases} 1, & |w| < w_c \\ 0, & |w| > w_c \end{cases}$$



$$\Rightarrow\ h(t) = \frac{1}{2\pi}\int_{-w_c}^{+w_c} e^{jwt}dw$$

$$= \frac{\sin(w_c t)}{\pi t}$$



- Filter Design:

$$H(jw) = \int_{-\infty}^{\infty} h(t) e^{-jwt} dt$$

$\delta(t)$

$x(t)$

$h(t)$

$$y(t) = h(t) * x(t)$$
$$= \int_{-\infty}^{\infty} h(t-\tau) x(\tau) d\tau$$

$$H(jw) = \int_{-\infty}^{\infty} h(t) e^{-jwt} dt$$

$$X(jw) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt$$

$$\Rightarrow \; Y(jw) = H(jw) X(jw)$$

$$\Rightarrow \; y(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} Y(jw) e^{jwt} dw$$

▪ Filter Design:

$$H(jw) = \int_{-\infty}^{\infty} h(t)e^{-jwt}dt$$

$\delta(t)$ → Filter LTI System → $h(t)$

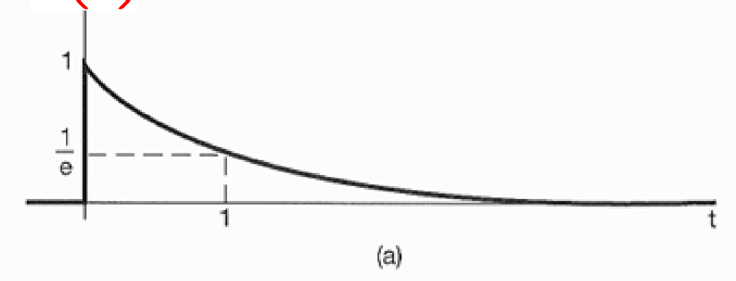
$x(t)$ → Filter LTI System → $y(t) = h(t) * x(t)$

$$Y(jw) = H(jw)X(jw)$$

RC circuit $\quad h(t) = e^{-t}u(t) \xleftrightarrow{\mathcal{F}} H(jw) = \dfrac{1}{jw+1}$

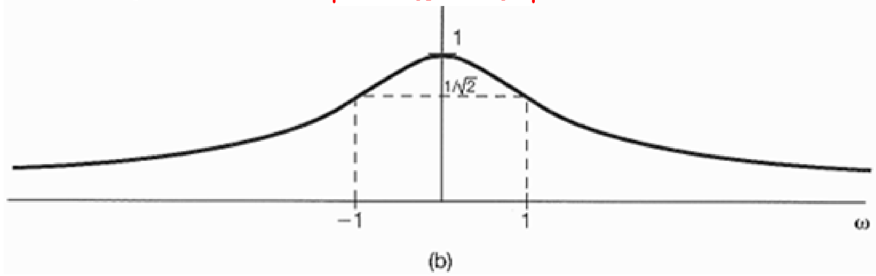
impulse response $\qquad$ frequency response

$h(t)$



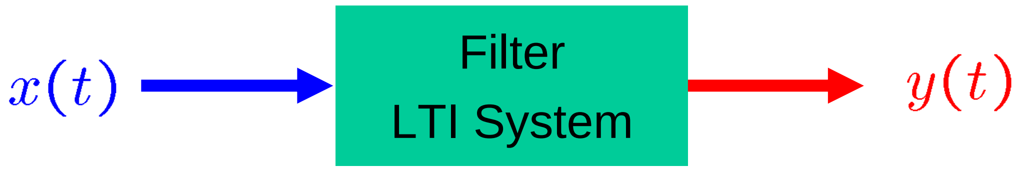
(a)

$|H(jw)|$



(b)

- Example 4.19:



$$h(t) = e^{-at}u(t), \quad a > 0 \qquad \Rightarrow \quad H(jw) = \frac{1}{a + jw}$$

$$x(t) = e^{-bt}u(t), \quad b > 0 \qquad \Rightarrow \quad X(jw) = \frac{1}{b + jw}$$

$$\Rightarrow \quad Y(jw) = H(jw)X(jw) \quad = \frac{1}{a + jw}\,\frac{1}{b + jw}$$

if $a \neq b$

$$= \frac{1}{b - a}\left[\frac{1}{a + jw} - \frac{1}{b + jw}\right]$$

- Example 4.19:

if $a \neq b$ $\quad Y(jw) = \dfrac{1}{b-a}\left[\dfrac{1}{a+jw} - \dfrac{1}{b+jw}\right]$

$$\Rightarrow \; y(t) = \frac{1}{b-a}\left[e^{-at}u(t) - e^{-bt}u(t)\right]$$

if $a = b$ $\quad Y(jw) = \dfrac{1}{(a+jw)^2}$

since $\quad e^{-at}u(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \dfrac{1}{a+jw}$

and $\quad t\, e^{-at}u(t) \stackrel{\mathcal{F}}{\longleftrightarrow} j\,\dfrac{d}{dw}\left[\dfrac{1}{a+jw}\right] = \dfrac{1}{(a+jw)^2}$

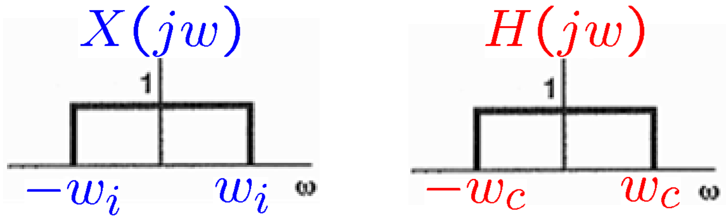
$$\Rightarrow \; y(t) = te^{-at}u(t)$$

- Example 4.20:

$$h(t) = \frac{\sin(w_c t)}{\pi t} \qquad H(jw) = \int_{-\infty}^{\infty} h(t) e^{-jwt} dt$$

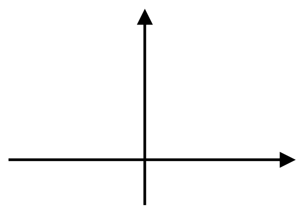
$$x(t) = \frac{\sin(w_i t)}{\pi t} \longrightarrow \boxed{\text{Filter LTI System}} \longrightarrow y(t) = ?$$

$$\Rightarrow \quad Y(jw) = H(jw) X(jw)$$



$$\Rightarrow \quad X(jw) = \begin{cases} 1, & |w| \le w_i \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow \quad H(jw) = \begin{cases} 1, & |w| \le w_c \\ 0, & \text{otherwise} \end{cases}$$



$$\Rightarrow \quad Y(jw) = \begin{cases} 1, & |w| \le w_0 \\ 0, & \text{otherwise} \end{cases}$$

$$w_0 = \min(w_c, w_i)$$

$$\Rightarrow \quad y(t) = \frac{\sin(w_0 t)}{\pi t}$$

$$\Rightarrow \quad y(t) = \begin{cases} \dfrac{\sin(w_c t)}{\pi t}, & w_c \le w_i \\ \dfrac{\sin(w_i t)}{\pi t}, & w_c \ge w_i \end{cases}$$

- Representation of Aperiodic Signals:
  the Continuous-Time Fourier Transform
- The Fourier Transform for Periodic Signals
- Properties of the Continuous-Time Fourier Transform
- The Convolution Property
- The Multiplication Property
- Systems Characterized by Linear Constant-Coefficient Differential Equations

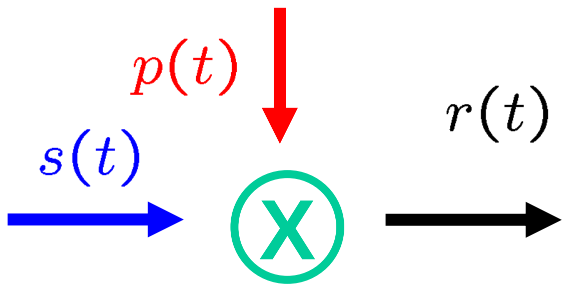
- Convolution & Multiplication:

$$y(t) = x(t) * h(t) \xleftrightarrow{\mathcal{F}} Y(jw) = X(jw)H(jw)$$

$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$r(t) = s(t)p(t) \xleftrightarrow{\mathcal{F}} R(jw) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)P(j(w-\theta))d\theta$$

- Multiplication of One Signal by Another:
  - Scale or modulate the amplitude of the other signal
  - Modulation



Multiplication Property

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

$$r(t) = s(t)p(t)$$

$$\Rightarrow \quad R(jw) = \int_{-\infty}^{\infty} r(t)e^{-jwt}dt$$

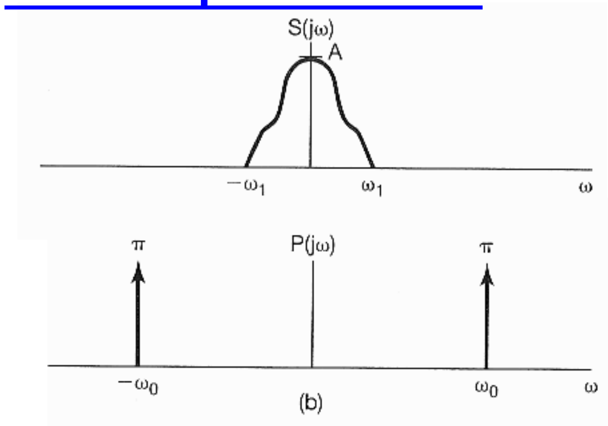
$$= \int_{-\infty}^{\infty} s(t)p(t)e^{-jwt}dt$$

$$= \int_{-\infty}^{\infty} s(t)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} P(j\theta)e^{j\theta t}d\theta\right\}e^{-jwt}dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} P(j\theta)\left[\int_{-\infty}^{\infty} s(t)e^{-j(w-\theta)t}dt\right]d\theta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} P(j\theta)S(j(w-\theta))d\theta \qquad = \frac{1}{2\pi}\int_{-\infty}^{\infty} P(j(w-\theta))S(j\theta)d\theta$$

- Example 4.21:



$$r(t) = s(t)p(t)$$

$$s(t) \stackrel{\mathcal{F}}{\longleftrightarrow} S(jw)$$

$$p(t) \stackrel{\mathcal{F}}{\longleftrightarrow} P(jw)$$

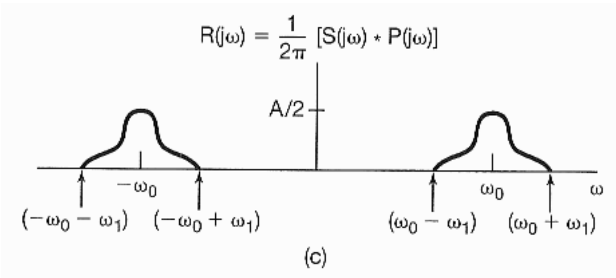
$$p(t) = \cos(w_0 t)$$

$$P(jw) = \pi\delta(w - w_0) + \pi\delta(w + w_0)$$

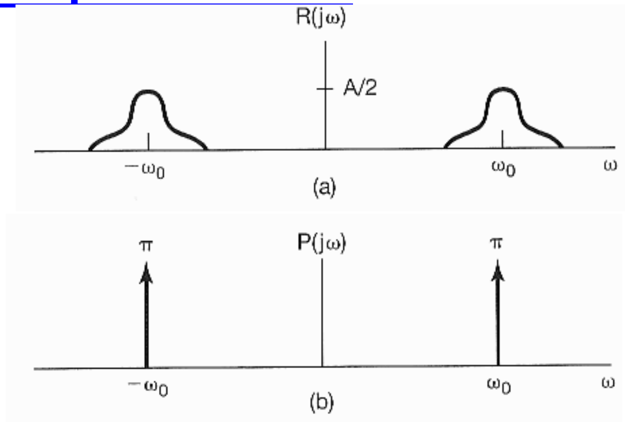
$$R(jw) = \frac{1}{2\pi}[S(jw) * P(jw)]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)P(j(w-\theta))d\theta$$

$$= \frac{1}{2}S\left(j(w-w_0)\right) + \frac{1}{2}S\left(j(w+w_0)\right)$$



- Example 4.22:



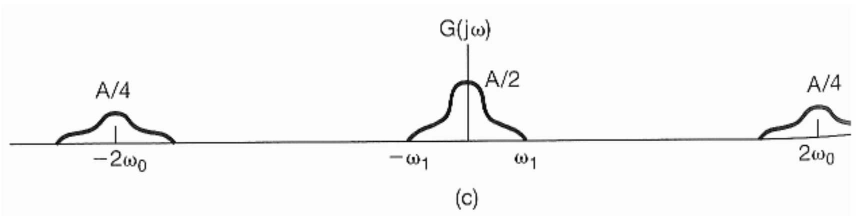
$$g(t) = r(t)p(t)$$

$$r(t) \overset{\mathcal{F}}{\longleftrightarrow} R(jw)$$

$$p(t) \overset{\mathcal{F}}{\longleftrightarrow} P(jw)$$

$$p(t) = \cos(w_0 t)$$

$$G(jw) = \frac{1}{2\pi}\left[R(jw) * P(jw)\right]$$



$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

- Example 4.23:

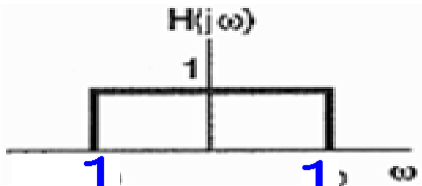
$$x(t) = \frac{\sin(t)\sin(t/2)}{\pi t^2}$$

$$X(jw) = \int_{-\infty}^{\infty} \frac{\sin(t)\sin(t/2)}{\pi t^2} e^{-jwt}dt$$

$$= \pi \left(\frac{\sin(t)}{\pi t}\right)\left(\frac{\sin(t/2)}{\pi t}\right)$$

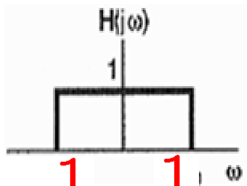
$$X(jw) = \frac{1}{2}\mathcal{F}\left\{\frac{\sin(t)}{\pi t}\right\} * \mathcal{F}\left\{\frac{\sin(t/2)}{\pi t}\right\}$$

$\frac{1}{2}$



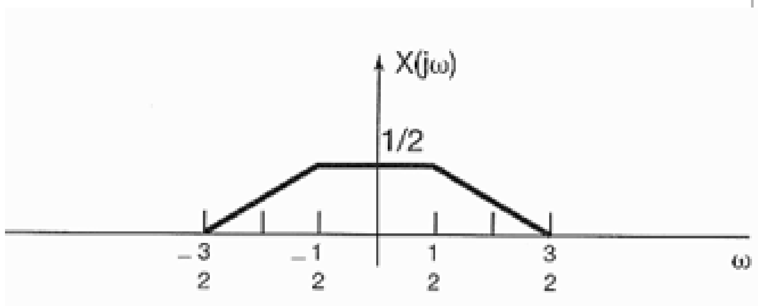
-1 $\quad$ 1

$*$

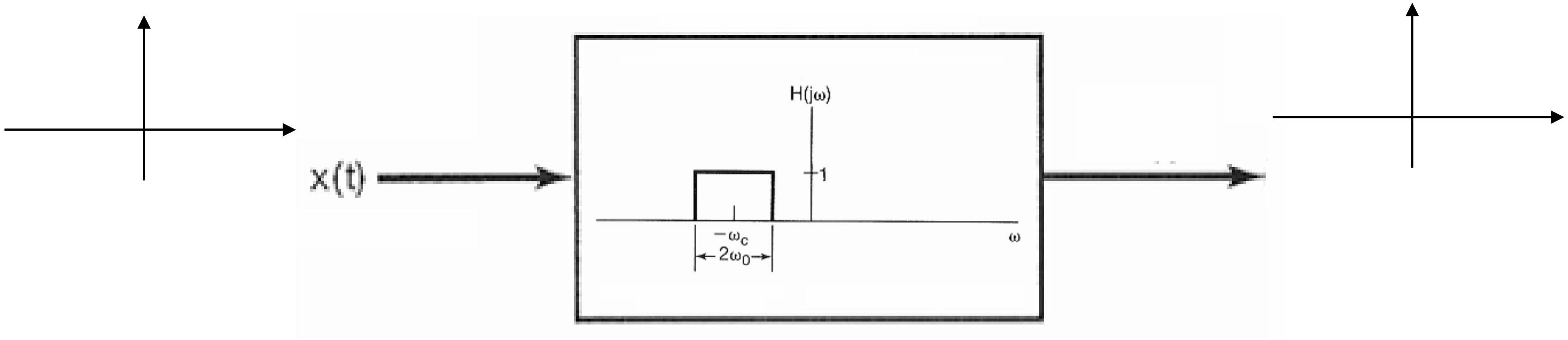


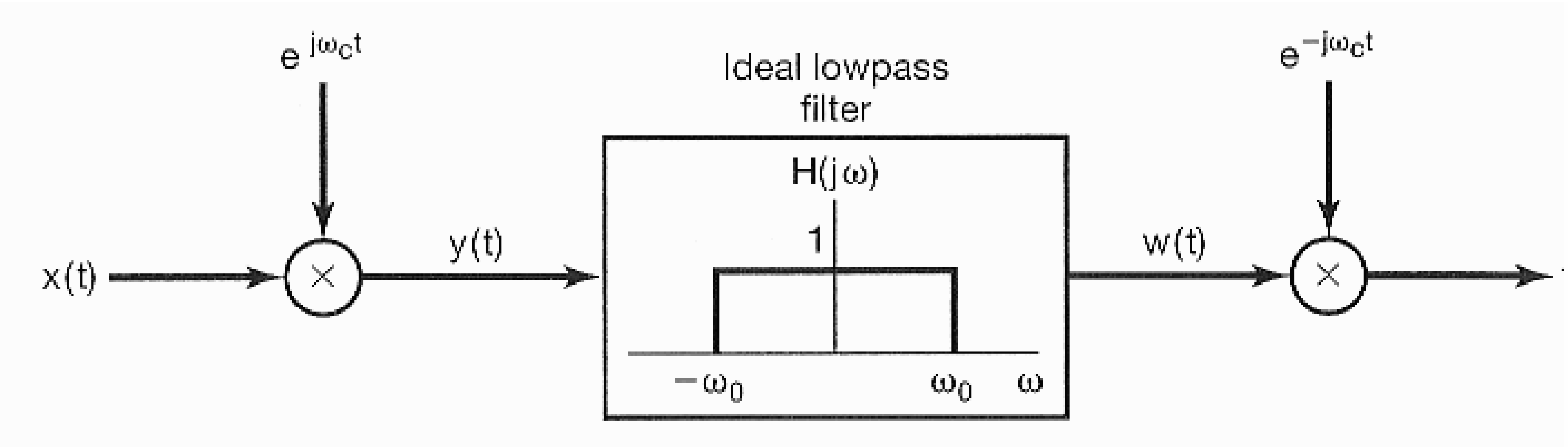
$-\frac{1}{2}$ $\quad$ $\frac{1}{2}$

$=$



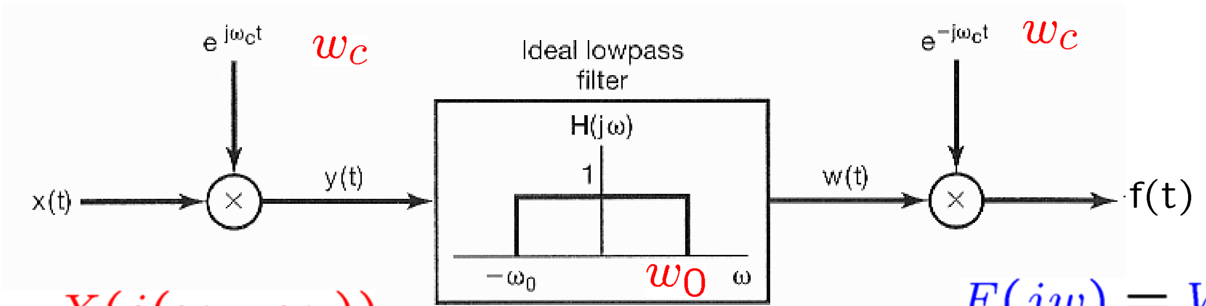
- Bandpass Filter Using Amplitude Modulation:





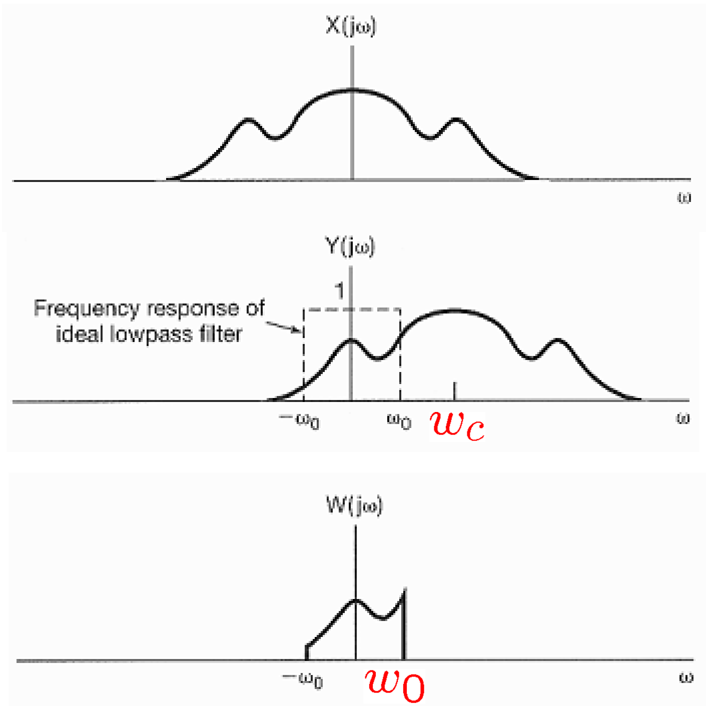
Bandpass Filter Using Amplitude Modulation:

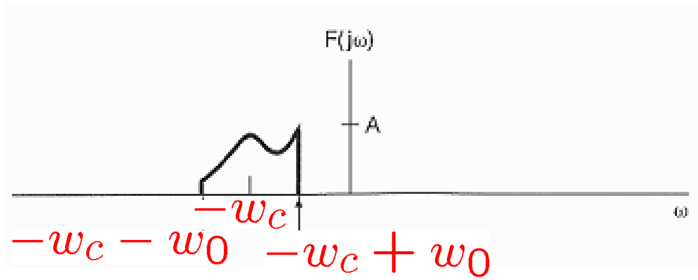
$$e^{jw_c t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(w - w_c)$$

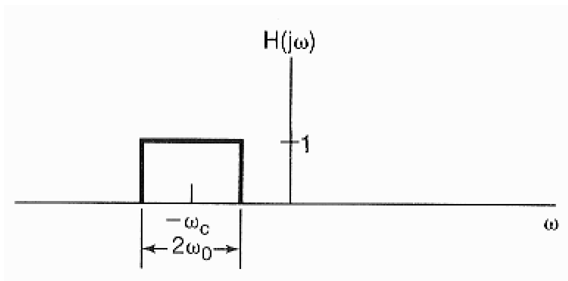


$$Y(jw) = X(j(w - w_c))$$

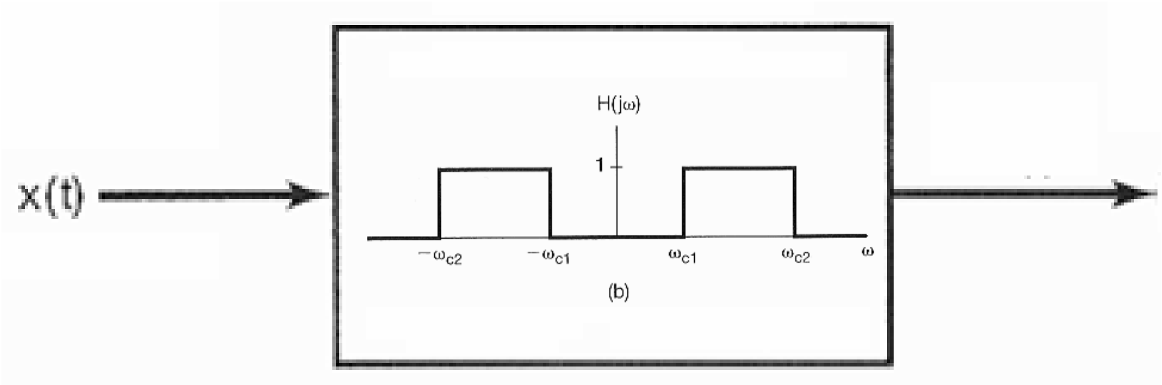
$$F(jw) = W(j(w + w_c))$$



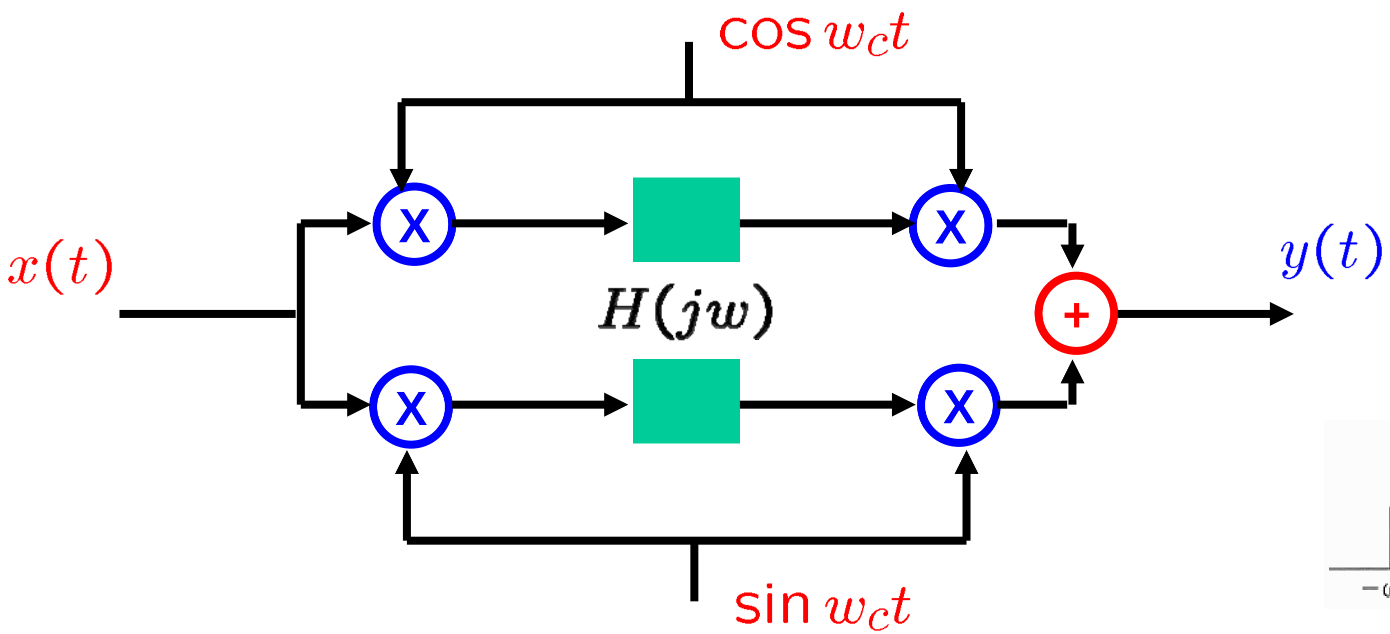




- Bandpass Filter Using Amplitude Modulation:



  - On Page 349-350, Problem 4.46



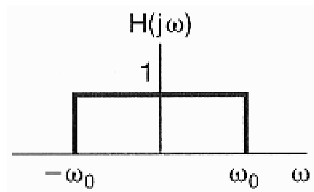


TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t)+by(t)$	$aX(j\omega)+bY(j\omega)$
4.3.2	Time Shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t}x(t)dt$	$\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega)=X^*(-j\omega) \\ \Re e\{X(j\omega)\}=\Re e\{X(-j\omega)\} \\ \Im m\{X(j\omega)\}=-\Im m\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert=\lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega)=-\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t)=\mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t)=\mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\Re e\{X(j\omega)\}$ $j\Im m\{X(j\omega)\}$

4.3.7 Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty}\lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\lvert X(j\omega)\rvert^2 d\omega$$

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\,\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \; k \neq 0$ $\left(\text{this is the Fourier series representation for any choice of } T > 0\right)$
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k

and $x(t+T)=x(t)$		
$\sum_{n=-\infty}^{+\infty} \delta(t-nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta\left(\omega-\frac{2\pi k}{T}\right)$	$a_k=\frac{1}{T}$ for all k
$x(t)\begin{cases}1, & \|t\|<T_1\\ 0, & \|t\|>T_1\end{cases}$	$\frac{2\sin\omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega)=\begin{cases}1, & \|\omega\|<W\\ 0, & \|\omega\|>W\end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega}+\pi\delta(\omega)$	—
$\delta(t-t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \mathcal{R}e\{a\}>0$	$\frac{1}{a+j\omega}$	—
$te^{-at}u(t), \mathcal{R}e\{a\}>0$	$\frac{1}{(a+j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$, $\mathcal{R}e\{a\}>0$	$\frac{1}{(a+j\omega)^n}$	—

- Representation of Aperiodic Signals:
  the Continuous-Time Fourier Transform
- The Fourier Transform for Periodic Signals
- Properties of the Continuous-Time Fourier Transform
- The Convolution Property
- The Multiplication Property
- Systems Characterized by Linear Constant-Coefficient Differential Equations

- A useful class of CT LTI systems:

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t)$$

$$= b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$Y(jw) = X(jw) H(jw) \qquad H(jw) = \frac{Y(jw)}{X(jw)}$$

Systems Characterized by Linear Constant-Coefficient Differential Equations

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$\sum_{k=0}^{N} a_k \qquad \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \qquad \frac{d^k x(t)}{dt^k}$$

$$\sum_{k=0}^{N} a_k (jw)^k \;\; Y(jw) = \sum_{k=0}^{M} b_k (jw)^k \;\; X(jw)$$

$$Y(jw) \left[\sum_{k=0}^{N} a_k (jw)^k \right] = X(jw) \left[\sum_{k=0}^{M} b_k (jw)^k \right]$$

$$\Rightarrow \quad H(jw) = \frac{Y(jw)}{X(jw)} = \frac{\sum_{k=0}^{M} b_k (jw)^k}{\sum_{k=0}^{N} a_k (jw)^k} = \frac{b_M (jw)^M + \cdots + b_1 (jw) + b_0}{a_N (jw)^N + \cdots + a_1 (jw) + a_0}$$

- Examples 4.24 & 4.25:

$$H = \frac{Y}{X}$$

$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

$$(jw)Y(jw) + aY(jw) = X(jw)$$

$$\Rightarrow \quad H(jw) = \frac{1}{jw + a}$$

$$\Rightarrow \quad h(t) = e^{-at}u(t)$$

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3\,y(t) = \frac{dx(t)}{dt} + 2\,x(t)$$

$$\Rightarrow \quad H(jw) = \frac{(jw) + 2}{(jw)^2 + 4(jw) + 3} = \frac{(jw + 2)}{(jw + 1)(jw + 3)}$$

$$= \frac{1/2}{jw + 1} + \frac{1/2}{jw + 3}$$

$$\Rightarrow \quad h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

▪ Example 4.26:

$$x(t) = e^{-t}u(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t) = ???$$

$$H(jw) = \frac{(jw+2)}{(jw+1)(jw+3)}$$

$$\Rightarrow \quad Y(jw) = X(jw)H(jw)$$

$$= \left[\frac{1}{jw+1}\right]\left[\frac{jw+2}{(jw+1)(jw+3)}\right]$$

$$= \frac{jw+2}{(jw+1)^2(jw+3)}$$

$$= \frac{\frac{1}{4}}{jw+1} + \frac{\frac{1}{2}}{(jw+1)^2} - \frac{\frac{1}{4}}{jw+3}$$

$$\Rightarrow \quad y(t) = \left[\frac{1}{4}e^{-t} + \frac{1}{2}t\,e^{-t} - \frac{1}{4}e^{-3t}\right]u(t)$$

- Representation of Aperiodic Signals: the CT FT
- The FT for Periodic Signals
- Properties of the CT FT
  - Linearity
  - Time Shifting
  - Frequency Shifting
  - Conjugation
  - Time Reversal
  - Time and Frequency Scaling
  - Convolution
  - Multiplication
  - Differentiation in Time
  - Integration
  - Differentiation in Frequency
  - Conjugate Symmetry for Real Signals
  - Symmetry for Real and Even Signals & for Real and Odd Signals
  - Even-Odd Decomposition for Real Signals
  - Parseval's Relation for Aperiodic Signals
- The Convolution Property
- The Multiplication Property
- Systems Characterized by Linear Constant-Coefficient Differential Equations

Questions for Chapter 4

- Why to study FT
  - In order to analyze or represent aperiodic signals
- How to develop FT
  - From FS and let T -> infinity
- Do periodic signals have FT
  - Yes, their FT is function of isolated impulses
- Why to know the properties of FT
  - Avoid using the fundamental formulas of FT to compute the FT
- What the duality of FT and why
  - FT and IFT have almost identical integration formulas
- Why to know the convolution property
  - To analyze system response and/or design proper circuits
  - To simplify computation
- Why to know the multiplication property
  - For signal modulation with different-frequency carriers
  - To simplify computation

X(jw) of Aperiodic Signals and a_k of Periodic Signals

$$a_k = \frac{1}{T} X(jw) \Big|_{w = kw_0}$$

$$X(jw) = \sum_{k=-\infty}^{+\infty} 2\pi \; a_k \; \delta(w - kw_0)$$

$$= \sum_{k=-\infty}^{+\infty} 2\pi \; \frac{1}{T} X(jkw_0) \; \delta(w - kw_0)$$

$$w = mw_0$$

$$X(jmw_0) = \sum_{k=-\infty}^{+\infty} 2\pi \; \frac{1}{T} X(jkw_0) \; \delta(mw_0 - kw_0)$$

$$= 2\pi \; \frac{1}{T} \; X(jmw_0)$$

$$\Rightarrow \; 2\pi = T$$

X(jw) of Aperiodic Signals and a_k of Periodic Signals

$$a_k = \frac{1}{T} X_a(jw) \Big|_{w = kw_0}$$

$$X_p(jw) = \sum_{k=-\infty}^{+\infty} 2\pi \; a_k \; \delta(w - kw_0)$$

$$= \sum_{k=-\infty}^{+\infty} 2\pi \; \frac{1}{T} X_a(jkw_0) \; \delta(w - kw_0)$$

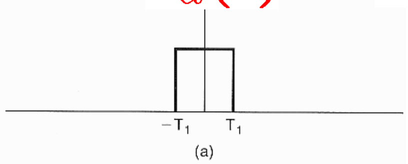
$$w = mw_0$$

$$X_p(jmw_0) = \sum_{k=-\infty}^{+\infty} 2\pi \; \frac{1}{T} X_a(jkw_0) \; \delta(mw_0 - kw_0)$$

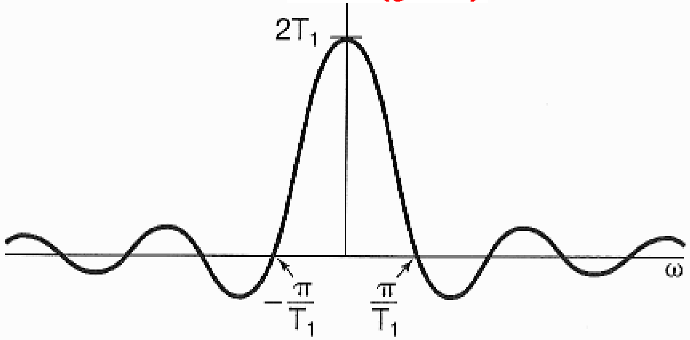
$$= 2\pi \; \frac{1}{T} \; X_a(jmw_0)$$

X(jw) of Aperiodic Signals and a_k of Periodic Signals

$x_a(t)$ $\quad T_1 = 1$

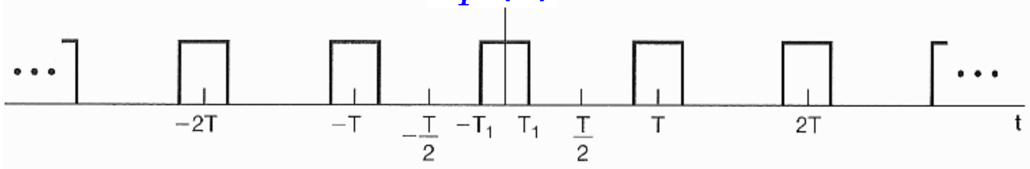


$X_a(jw)$



$T = 4 \qquad w_0 = 2\pi/4 = \pi/2 \qquad X_a(jw) = 2\,\dfrac{\sin(wT_1)}{w} = 2\,\dfrac{\sin(w)}{w}$

$x_p(t)$



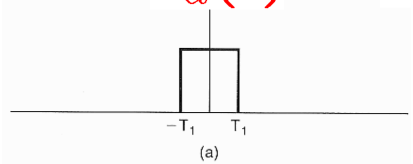
$$\Rightarrow \quad a_k = \frac{1}{T} X_a(jw)\Big|_{w=kw_0}$$

$$= \frac{\sin(k\pi/2)}{\pi k}$$

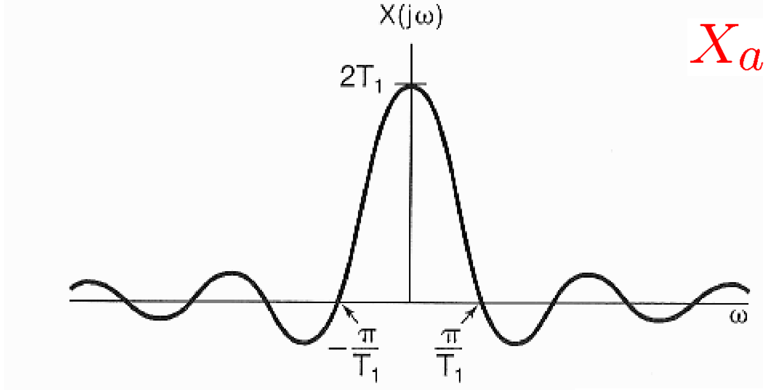
$$\Rightarrow \quad X_p(jw) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(w - kw_0) = \sum_{k=-\infty}^{+\infty} \frac{2\sin(k\pi/2)}{k} \delta(w - kw_0)$$

$$\Rightarrow \quad X_p(jmw_0) = \frac{2\sin(m\pi/2)}{m}$$

$x_a(t)$ $\quad T_1 = 1$



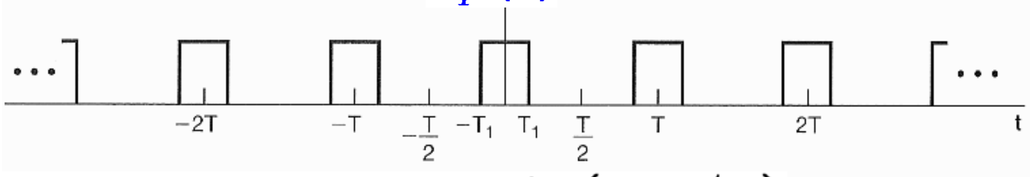
$X_a(jw)$



$T = 4 \qquad w_0 = 2\pi/4 = \pi/2$

$$X_a(jw) = 2\frac{\sin(wT_1)}{w} = 2\frac{\sin(w)}{w}$$

$x_p(t)$



$$\Rightarrow \quad a_k = \frac{1}{T} X_a(jw)\Big|_{w=kw_0}$$

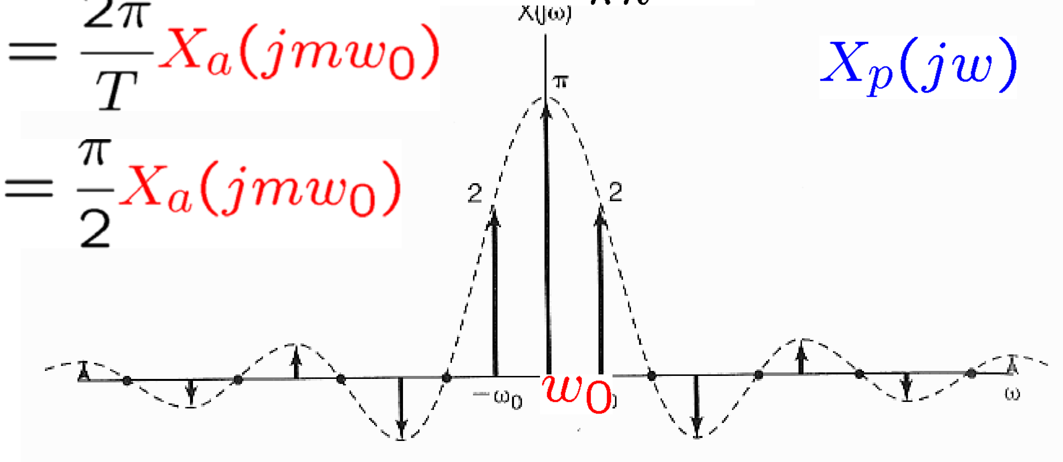
$$\Rightarrow \quad X_p(jmw_0) = \frac{2\sin(m\pi/2)}{m} = 2\pi a_m = \frac{\sin(k\pi/2)}{\pi k}$$

$$= \frac{2\pi}{T} X_a(jmw_0)$$

$$= \frac{\pi}{2} X_a(jmw_0)$$

m	0	1
a_m	$1/2$	$1/\pi$
$2\pi a_m$	π	2
$X_p(jmw_0)$	π	2
$X_a(jmw_0)$	2	$4/\pi$

$X_p(jw)$



Flowchart

Signals & Systems (Chap 1) LTI & Convolution (Chap 2)

Bounded/Convergent

Periodic

FS (Chap 3)
– CT
– DT

Aperiodic

FT
– CT (Chap 4)
– DT (Chap 5)

Unbounded/Non-convergent

LT – CT (Chap 9)

zT – DT (Chap 10)

Time-Frequency (Chap 6) Communication (Chap 8)

CT-DT (Chap 7) Control (Chap 11)

Spring 2011

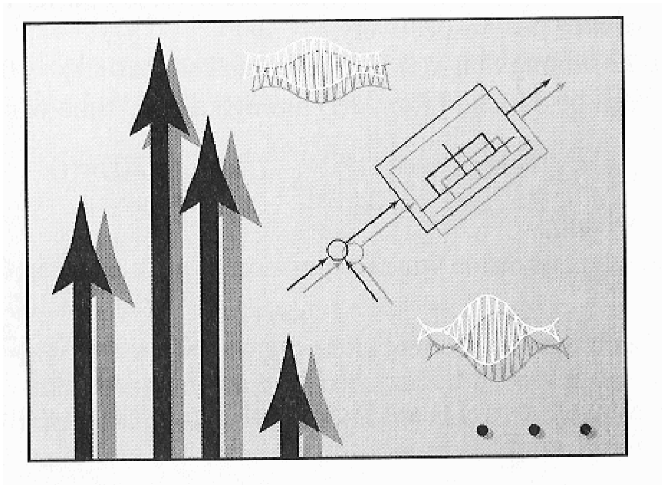
信號與系統
Signals and Systems

Chapter SS-8
Communication Systems

Feng-Li Lian

NTU-EE

Feb11 – Jun11



Figures and images used in these lecture notes are adopted from
"Signals & Systems" by Alan V. Oppenheim and Alan S. Willsky, 1997

Flowchart

Introduction (Chap 1)

LTI & Convolution (Chap 2)

Bounded/Convergent

Periodic

FS – CT
– DT
(Chap 3)

Aperiodic

FT – CT (Chap 4)
– DT (Chap 5)

Unbounded/Non-convergent

LT – CT (Chap 9)

zT – DT (Chap 10)

Time-Frequency (Chap 6)

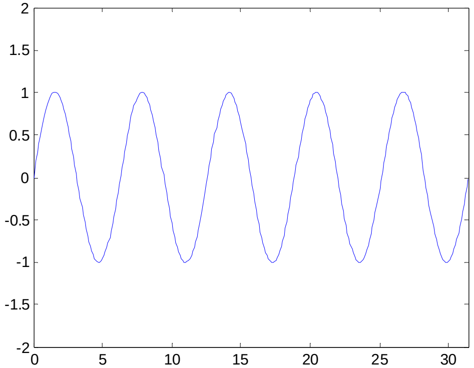
CT-DT (Chap 7)

Communication (Chap 8)

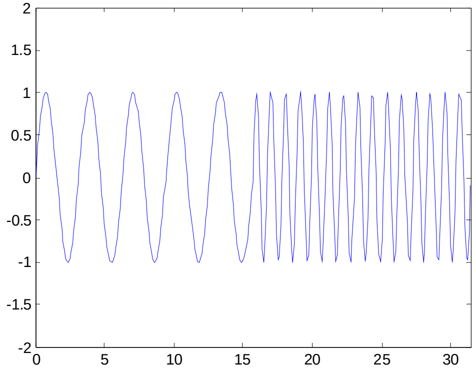
Control (Chap 11)

Digital Signal Processing (dsp-8)

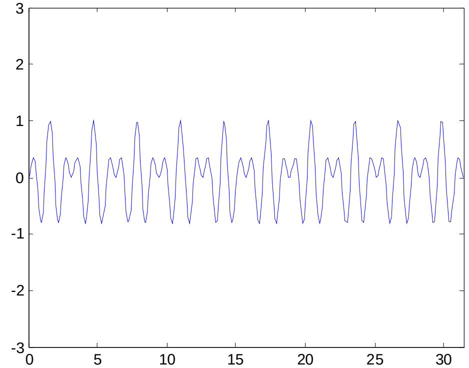
Introduction



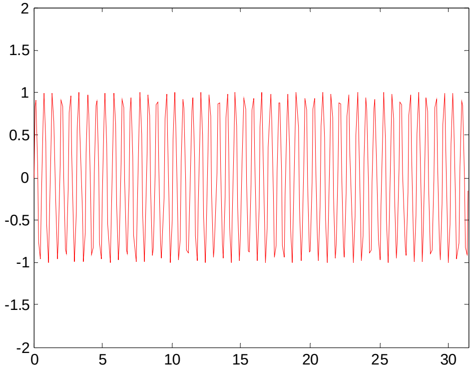
$x(t) = \sin(t)$



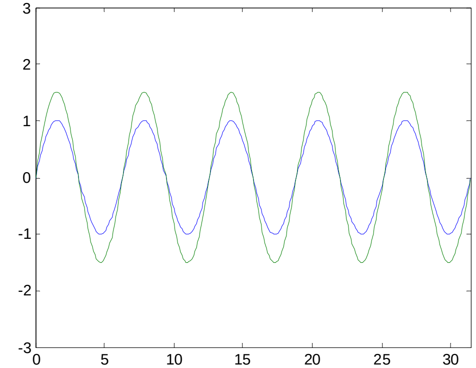
$\sin(2t)$ $\sin(6t)$



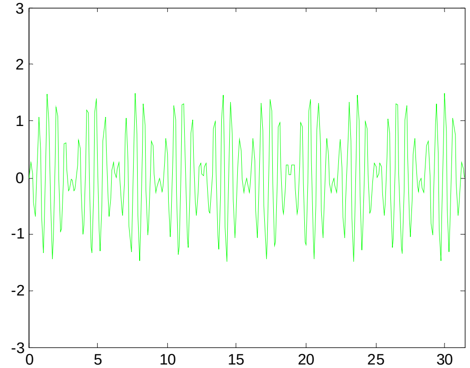
$z_1 = 1 \cdot \sin(t) \cdot \sin(5t)$



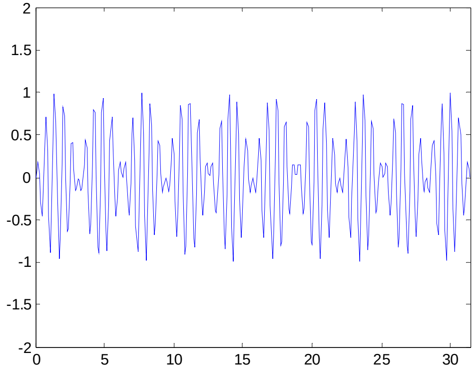
$c(t) = \sin(10t)$



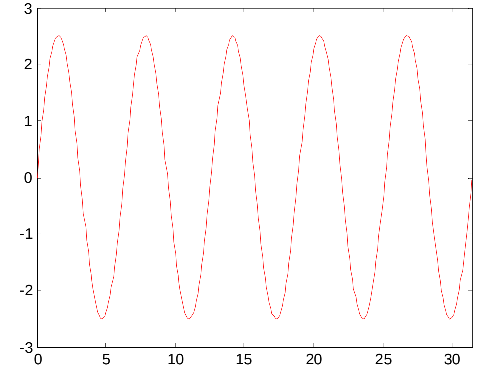
$\sin(t)$ $1.5 \sin(t)$



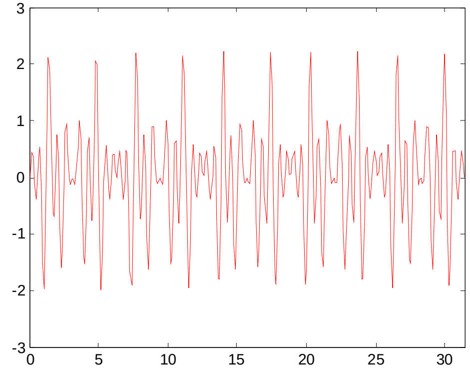
$z_2 = 1.5 \cdot \sin(t) \cdot \sin(10t)$



$\sin(10t) \cdot \sin(t)$

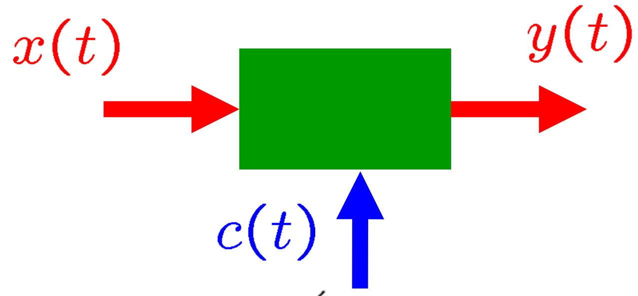


$\sin(t) + 1.5 \sin(t)$



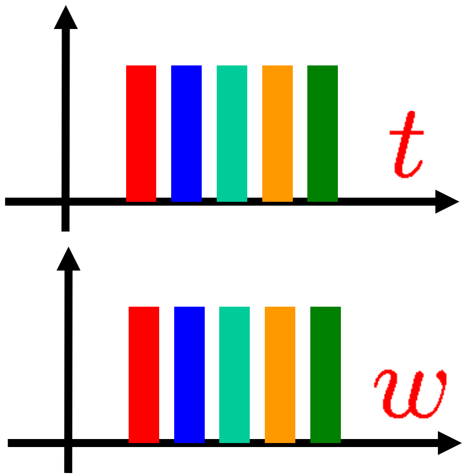
$z_1 + z_2$

- **Modulation & Demodulation:**
  - M: Embedding an information-bearing signal into a second signal
  - D: Extracting the information-bearing signal
  - Methods:
    - > Amplitude Modulation (AM)
    - > Frequency Modulation (FM)



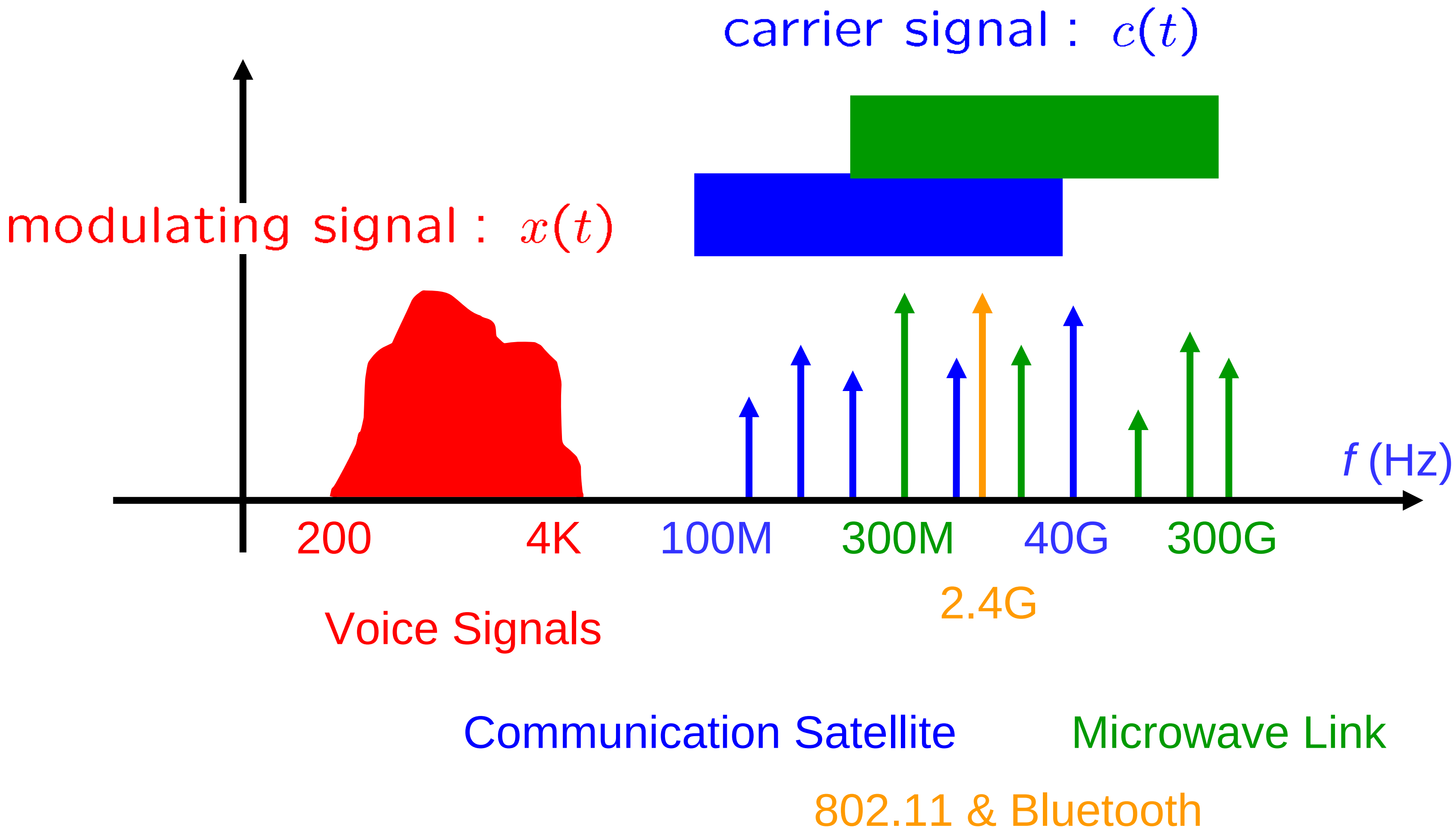
$$= A(t)\ \cos\Big(w(t)\cdot t + \theta(t)\Big)$$

- **Multiplexing & Demultiplexing:**
  - Simultaneous transmission of more than one signal with overlapping spectra over the same channel
  - Methods:
    - > Time-Division Multiplexing (TDM)
    - > Frequency-Division Multiplexing (FDM)



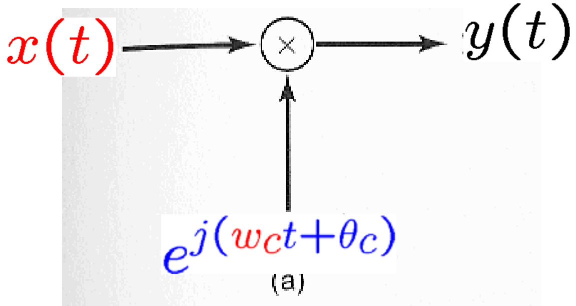
- Complex Exponential & Sinusoidal Amplitude Modulation & Demodulation
- Frequency-Division Multiplexing
- Single-Sideband Sinusoidal Amplitude Modulation
- Amplitude Modulation with a Pulse-Train Carrier
- Pulse-Amplitude Modulation
- Sinusoidal Frequency Modulation
- Discrete-Time Modulation

▪ Signal Frequency Characteristics:



modulated signal : $y(t) = x(t)\ c(t)$

AM with a Complex Exponential Carrier:

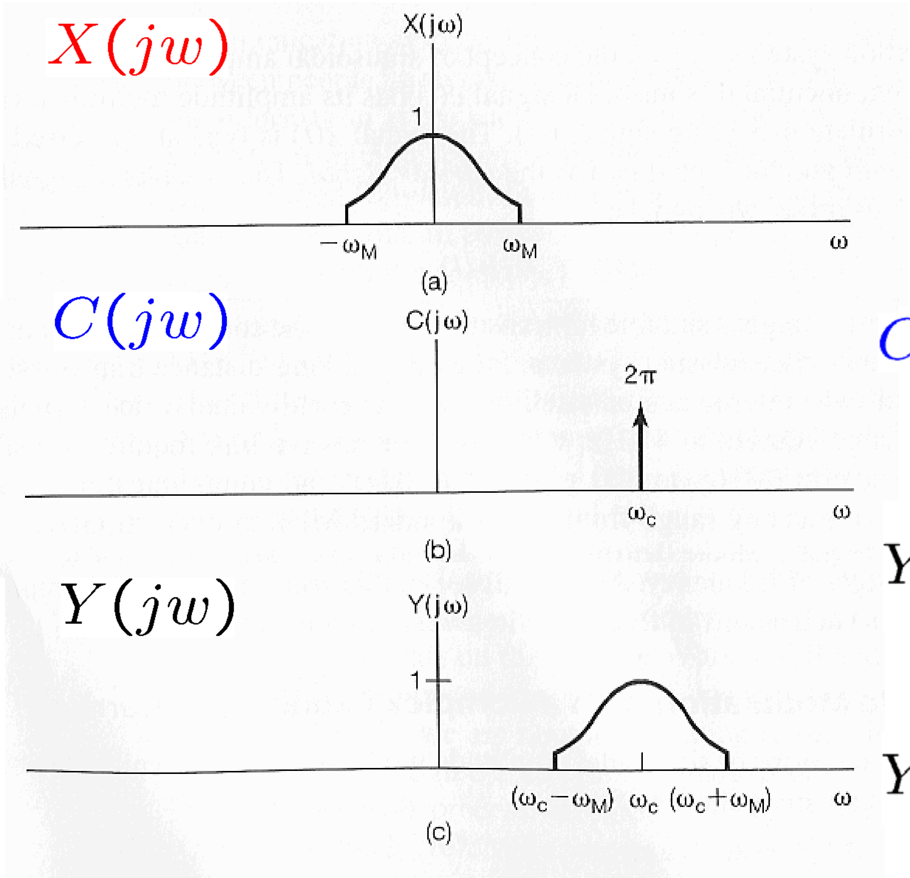


w_c : carrier frequency

$$c(t) = e^{j(w_c t+\theta_c)}$$

$$y(t) = x(t)\ c(t) \quad = x(t)\ e^{j w_c t}$$

$$\theta_c = 0$$

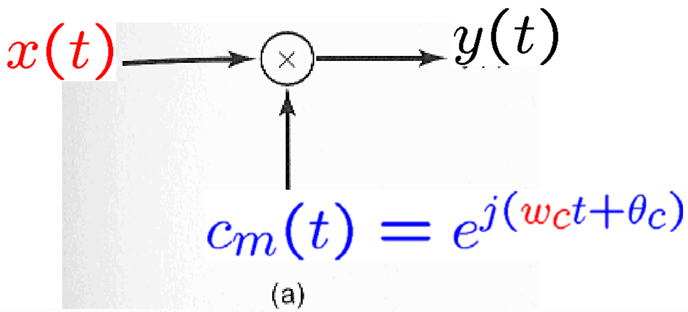


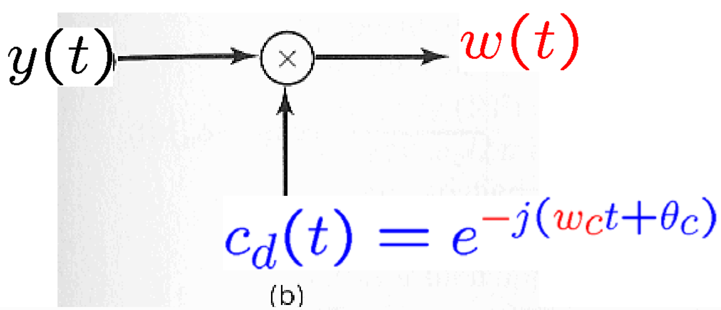
$$C(jw) = 2\pi\ \delta(w - w_c)$$

$$Y(jw) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)\ C(j(w-\theta))\, d\theta$$

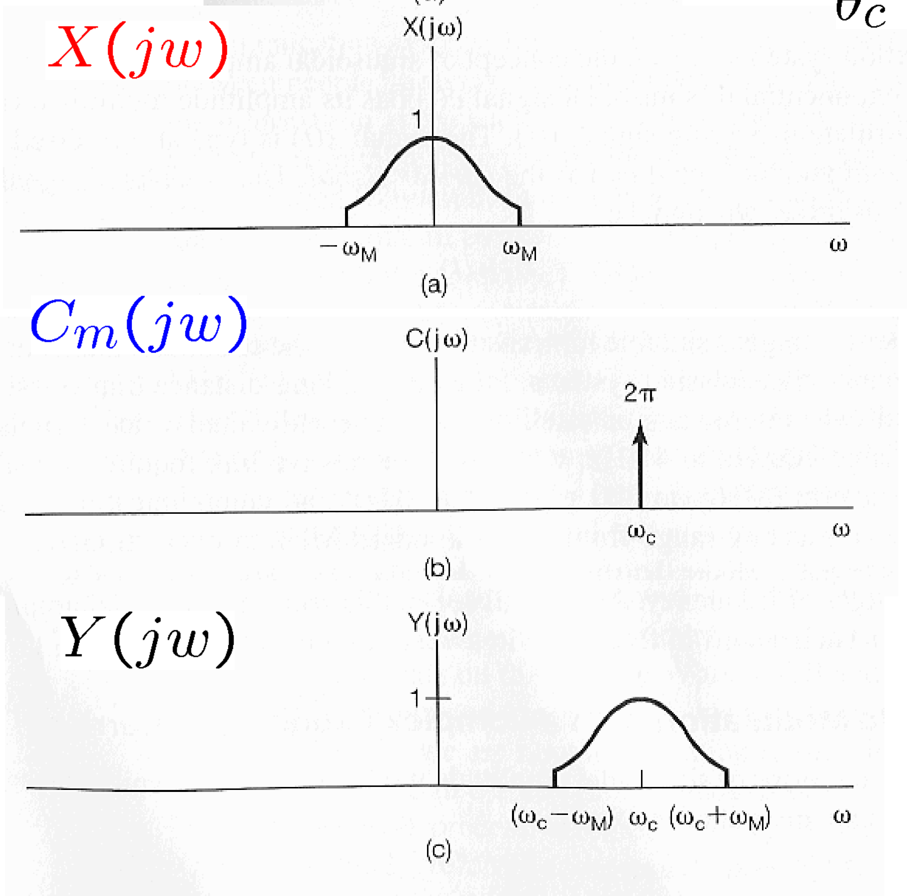
$$Y(jw) = X\left(j(w - w_c)\right)$$

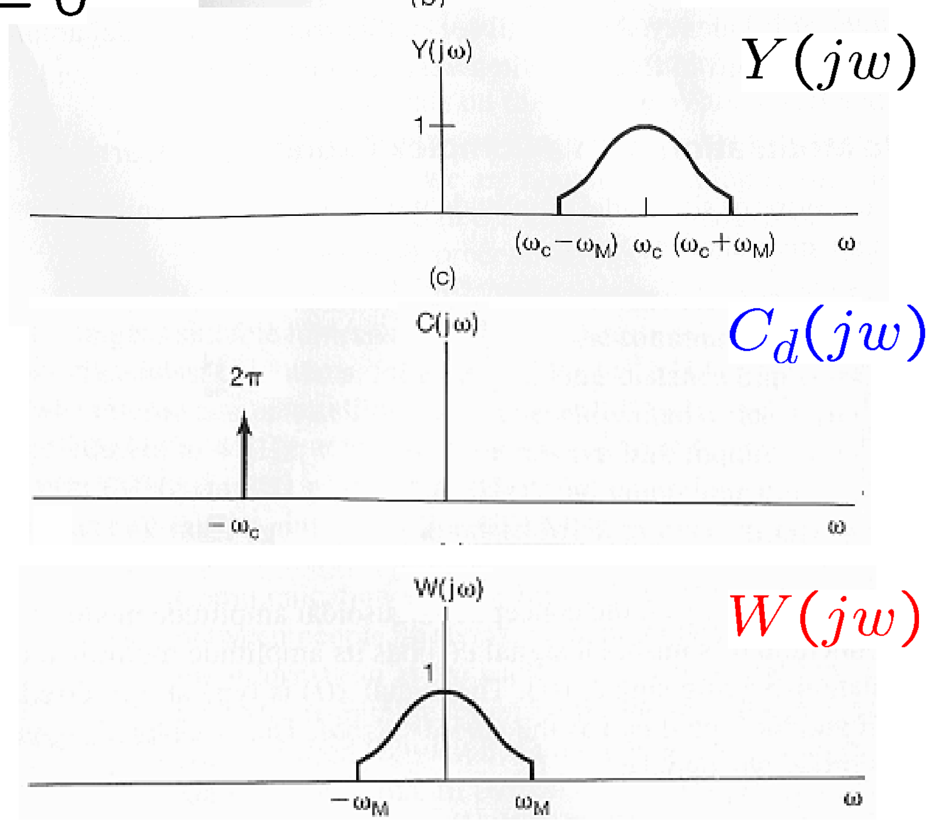
AM with a Complex Exponential Carrier:



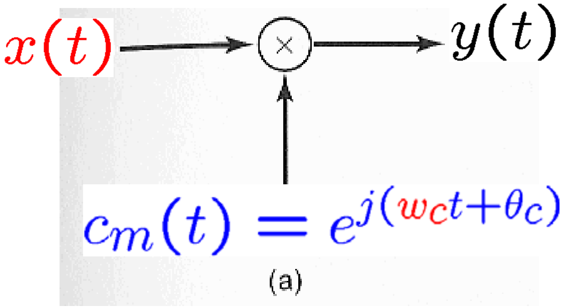


$\theta_c = 0$

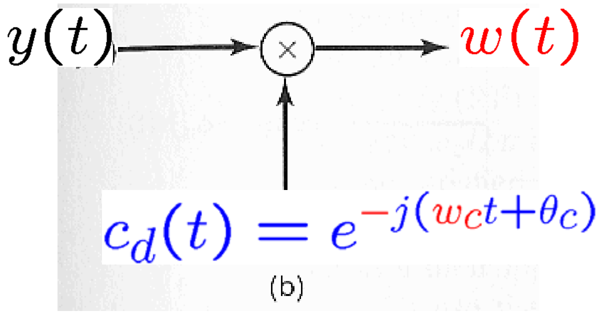




- **AM with a Complex Exponential Carrier:**



$$\theta_c = 0$$



$$y(t) = x(t)\ c_m(t)$$

$$= x(t)\ e^{jw_c t}$$

$$w(t) = y(t)\ c_d(t)$$

$$= y(t)\ e^{-jw_c t}$$

$$= x(t)\ e^{jw_c t}\ e^{-jw_c t}$$

$$\Rightarrow\ w(t) = x(t)$$

$$Y(jw) = X\left(j(w - w_c)\right)$$

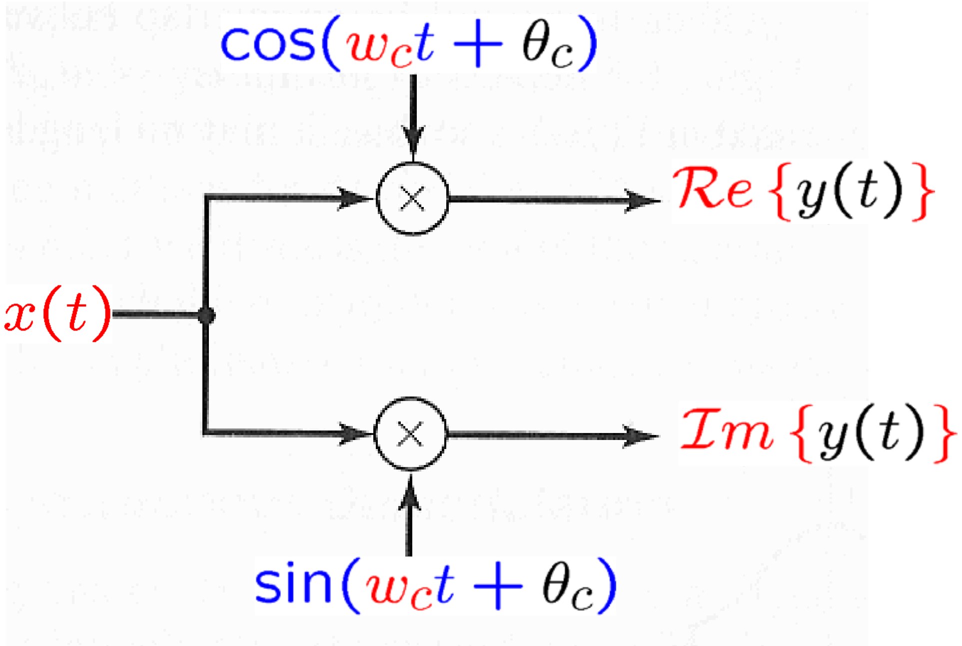
$$W(jw) = Y\left(j(w + w_c)\right)$$

$$\Rightarrow\ W(jw) = X(jw)$$

▪ AM with Sinusoidal Carriers:

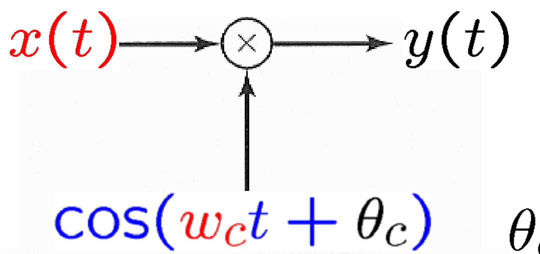
$$c(t) = e^{jw_c t} = \cos(w_c t) + j \sin(w_c t)$$

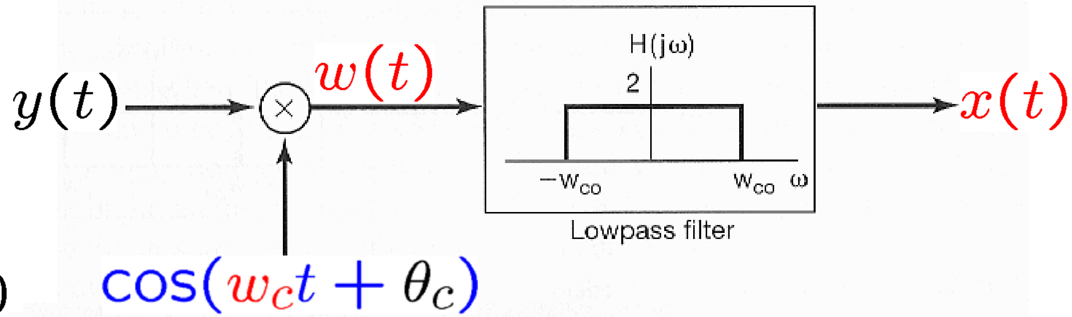
$$\Rightarrow y(t) = x(t) \cos(w_c t) + j\, x(t) \sin(w_c t)$$



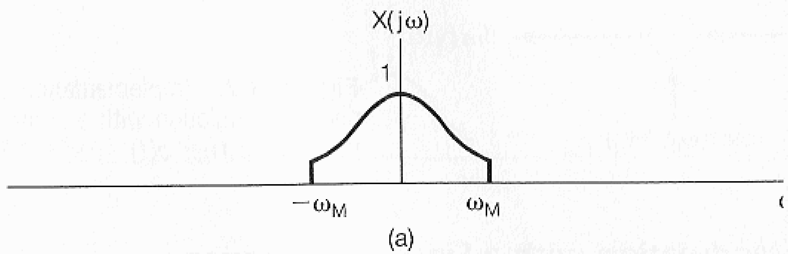
phase difference of $c_1(\cdot), c_2(\cdot)$?

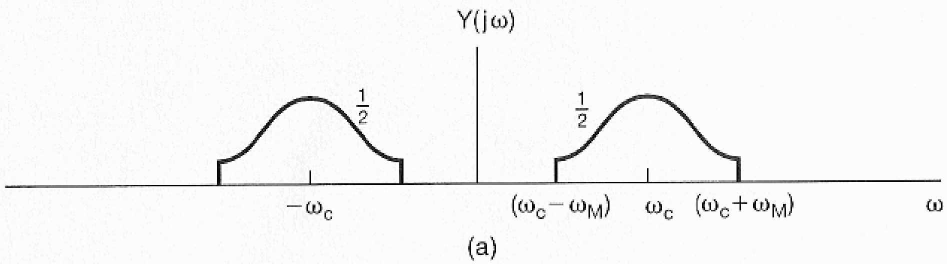
AM with a Sinusoidal Carrier:



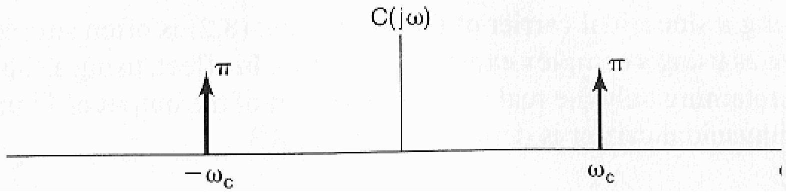


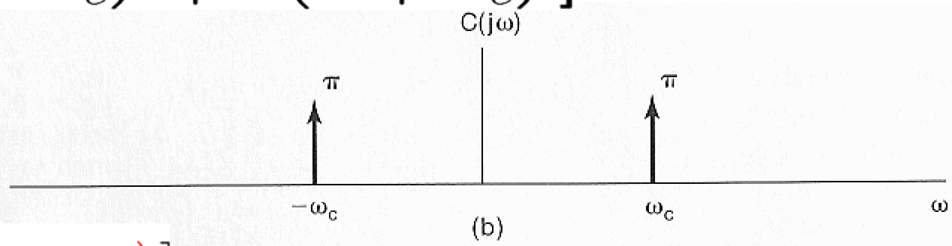
$\theta_c = 0$



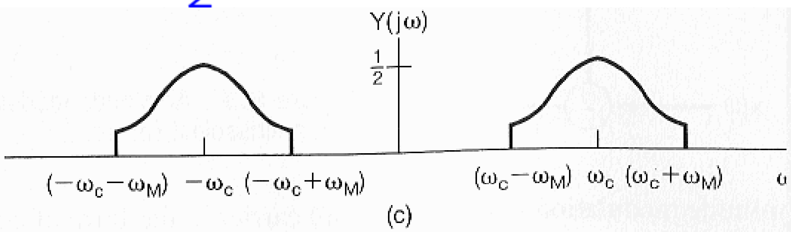


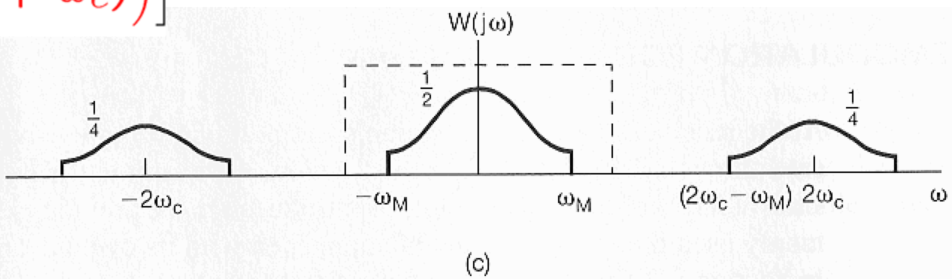
$$C(jw) = \pi[\ \delta(w - w_c)\ +\ \delta(w + w_c)\ ]$$



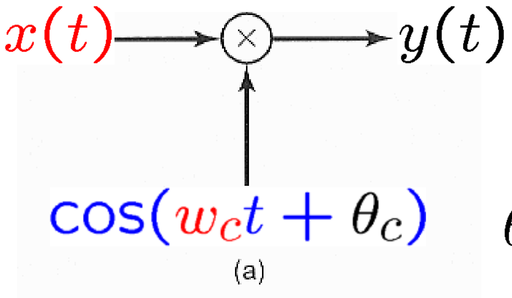


$$Y(jw) = \frac{1}{2}\left[X\left(j(w - w_c)\right) + X\left(j(w + w_c)\right)\right]$$



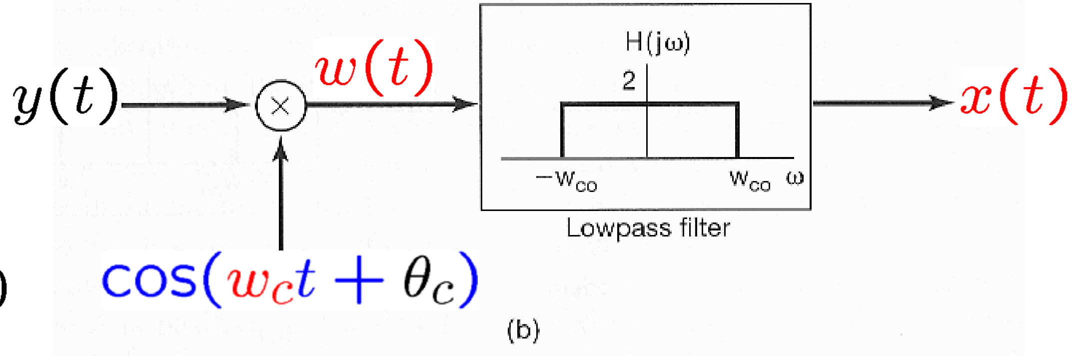


- AM with a Sinusoidal Carrier:



(a)

$\theta_c = 0$



(b)

$$y(t) = x(t)\cos(w_c t) \qquad w(t) = y(t)\cos(w_c t)$$

$$\Rightarrow w(t) = x(t)\cos^2(w_c t)$$

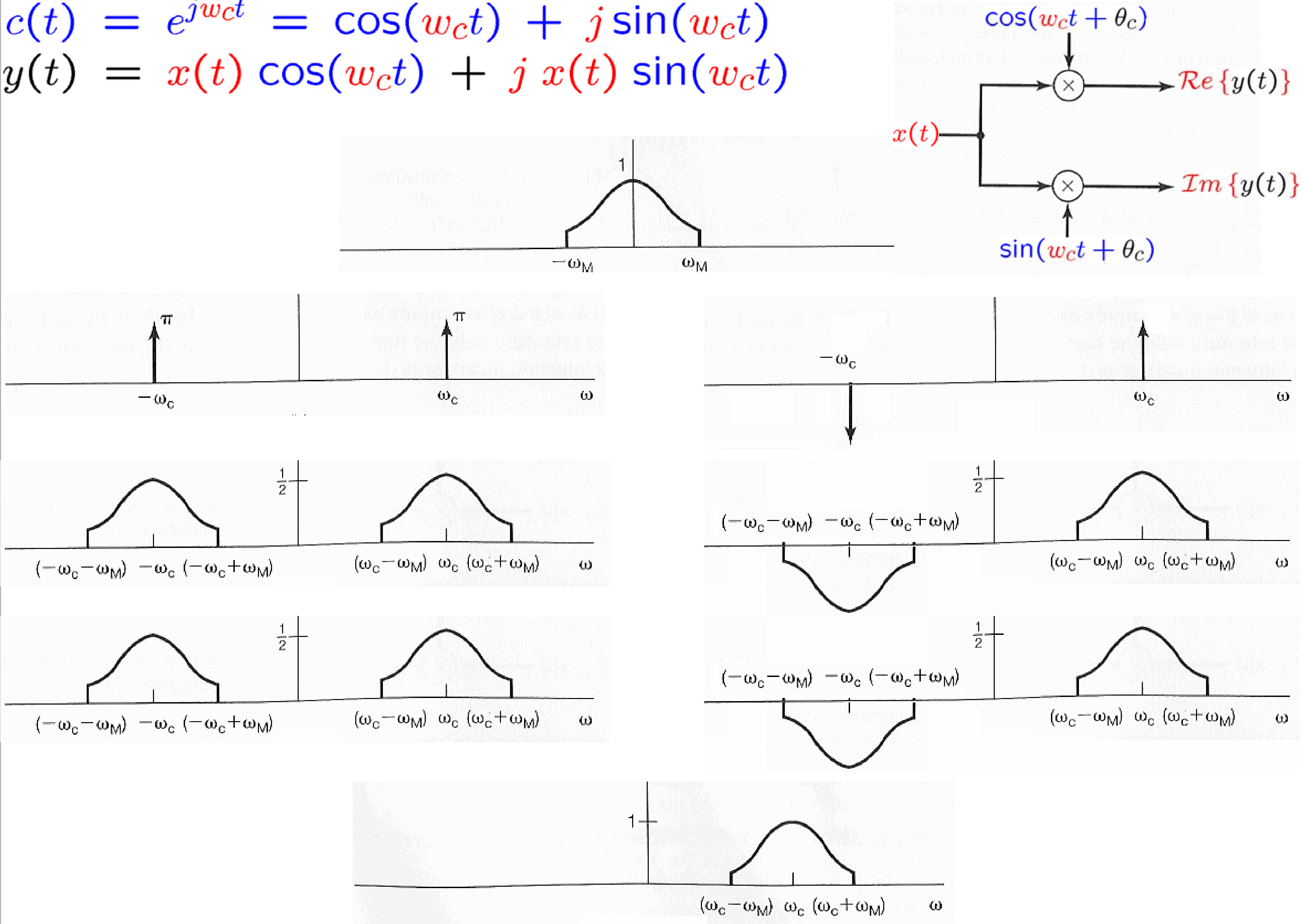
$$= x(t)\left[\frac{1}{2} + \frac{1}{2}\cos(2w_c t)\right]$$

$$= \frac{1}{2}x(t) + \frac{1}{2}x(t)\cos(2w_c t)$$

Complex Exponential & Sinusoidal Amplitude Modulation & Demodulation

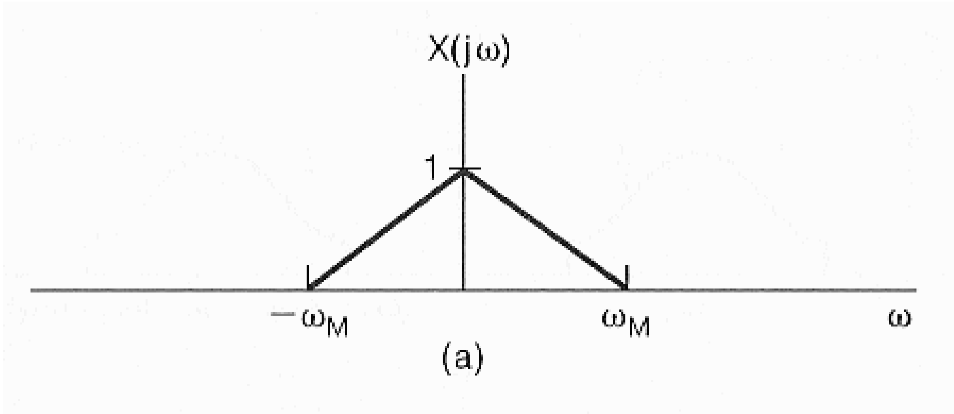
$$c(t) = e^{jw_ct} = \cos(w_ct) + j\sin(w_ct)$$

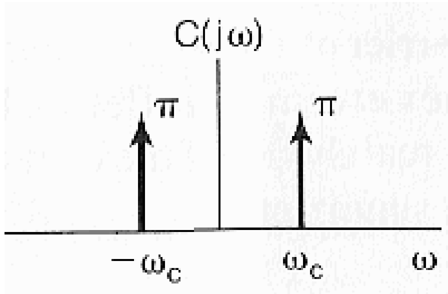
$$y(t) = x(t)\cos(w_ct) + j\,x(t)\sin(w_ct)$$

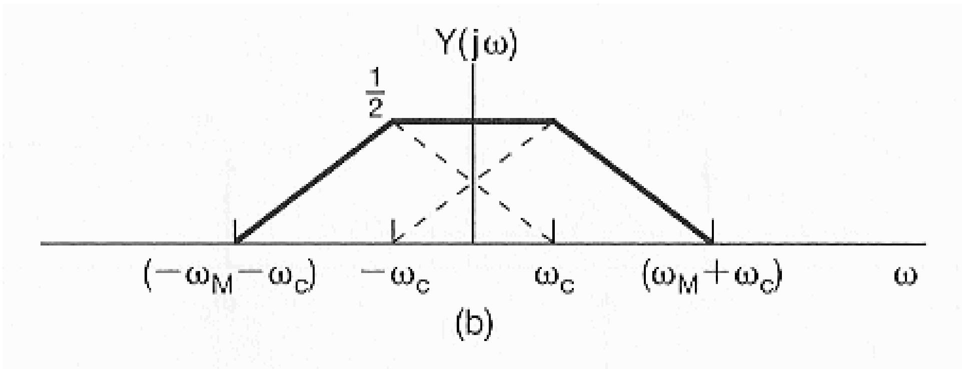


Overlapping of AM with a Sinusoidal Carrier:

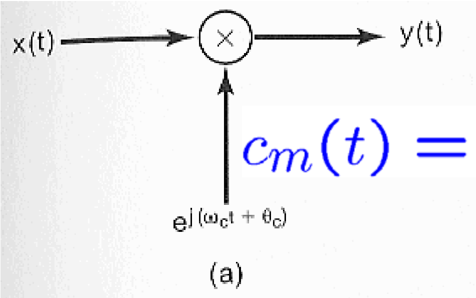
- If $w_c < w_M$,



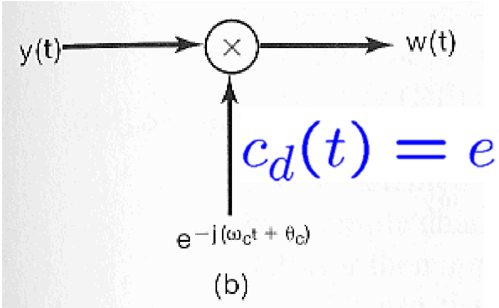




- Not Synchronized in Phase:



$$c_m(t) = e^{j(w_c t+\theta_c)}$$



$$c_d(t) = e^{-j(w_c t+\phi_c)}$$

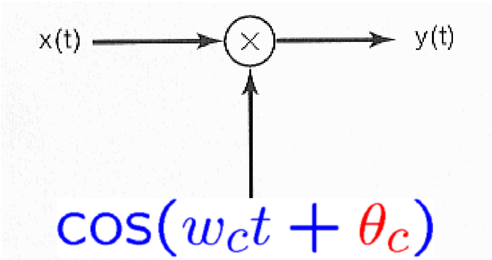
$$\theta_c \neq \phi_c$$

$$y(t) = x(t)\ c_m(t) = x(t)\ e^{j(w_c t+\theta_c)}$$

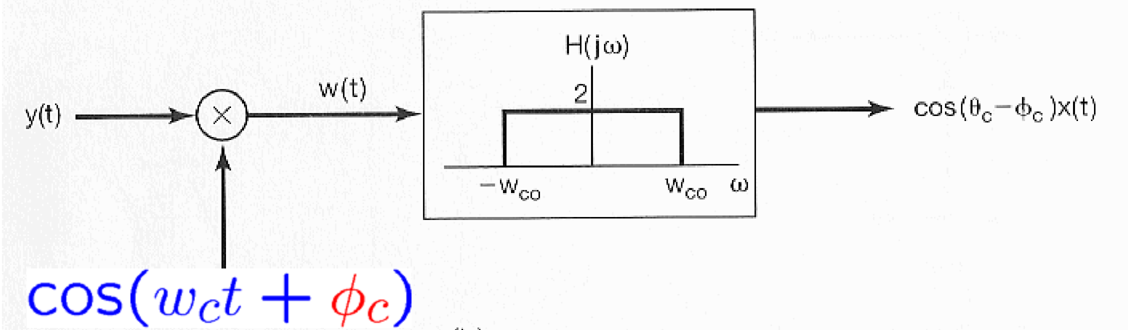
$$w(t) = y(t)\ c_d(t) = y(t)\ e^{-j(w_c t+\phi_c)} = x(t)\ e^{j(\theta_c-\phi_c)}$$

$\Rightarrow$ ONLY $\quad |x(t)| = |w(t)|$

Not Synchronized in Phase:



(a)



(b)

$$y(t) = x(t)\cos(w_c t + \theta_c) \qquad w(t) = y(t)\cos(w_c t + \phi_c)$$

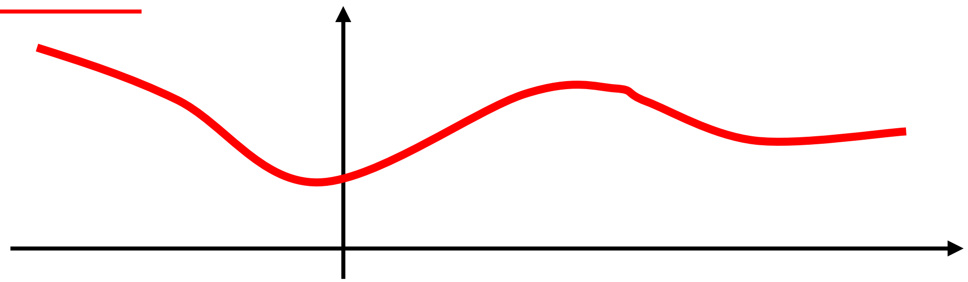
$$\Rightarrow w(t) = x(t)\cos(w_c t + \theta_c)\cos(w_c t + \phi_c)$$

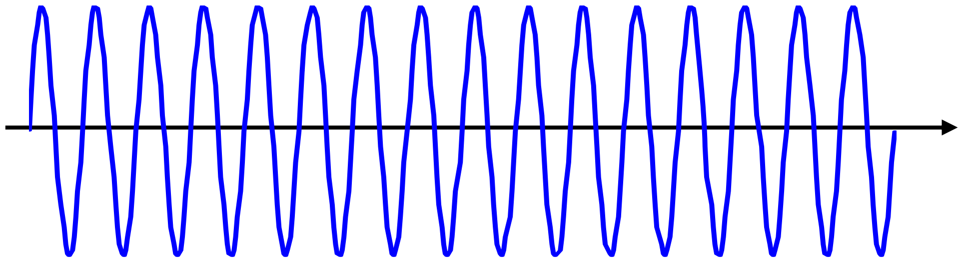
$$= x(t)\left[\frac{1}{2}\cos(\theta_c - \phi_c) + \frac{1}{2}\cos(2w_c t + \theta_c + \phi_c)\right]$$

$$= \frac{1}{2}\cos(\theta_c - \phi_c)\,x(t) + \frac{1}{2}\,x(t)\,\cos(2w_c t + \theta_c + \phi_c)$$

Asynchronous Demodulation:

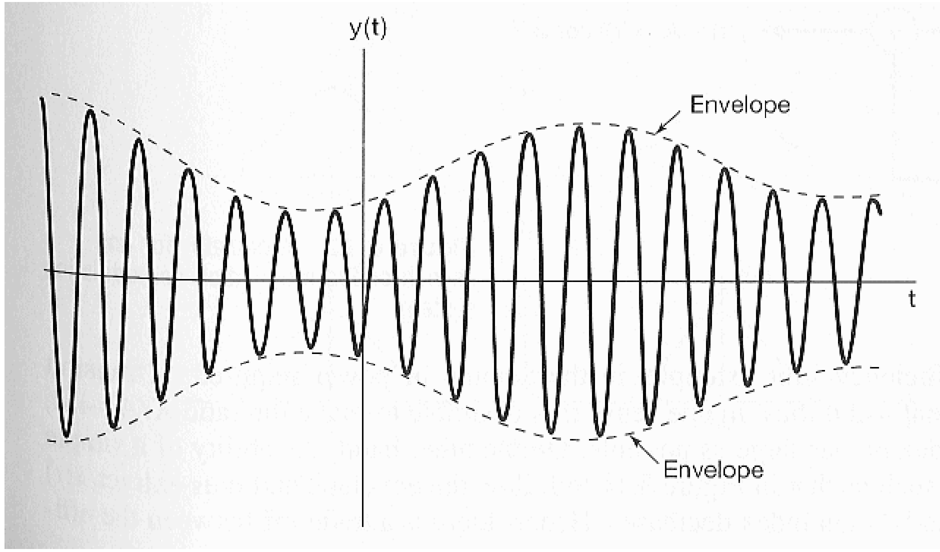
- $w_c >> w_M$
- $x(t) > 0, \quad \forall t$
  - In audio transmission over a RF channel
    - > w_M: 15 - 20 Hz
    - > $w_c/2\pi$: 500kHz – 2 MHz



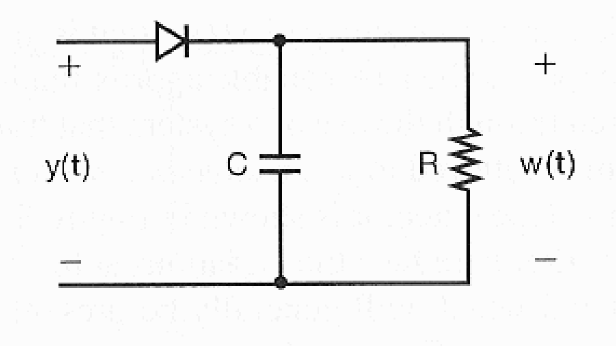


$$y(t) = x(t) \cos(w_c t + \theta_c)$$

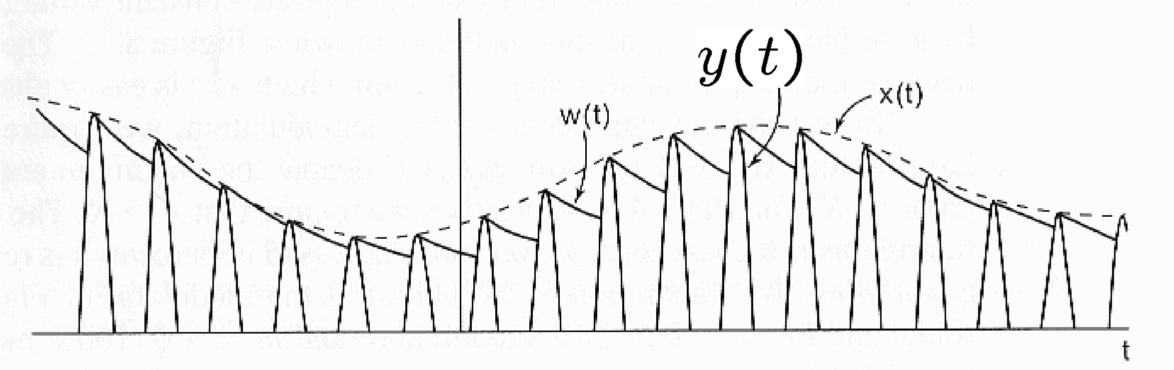
$$\approx x(t)$$



Envelope Detector:

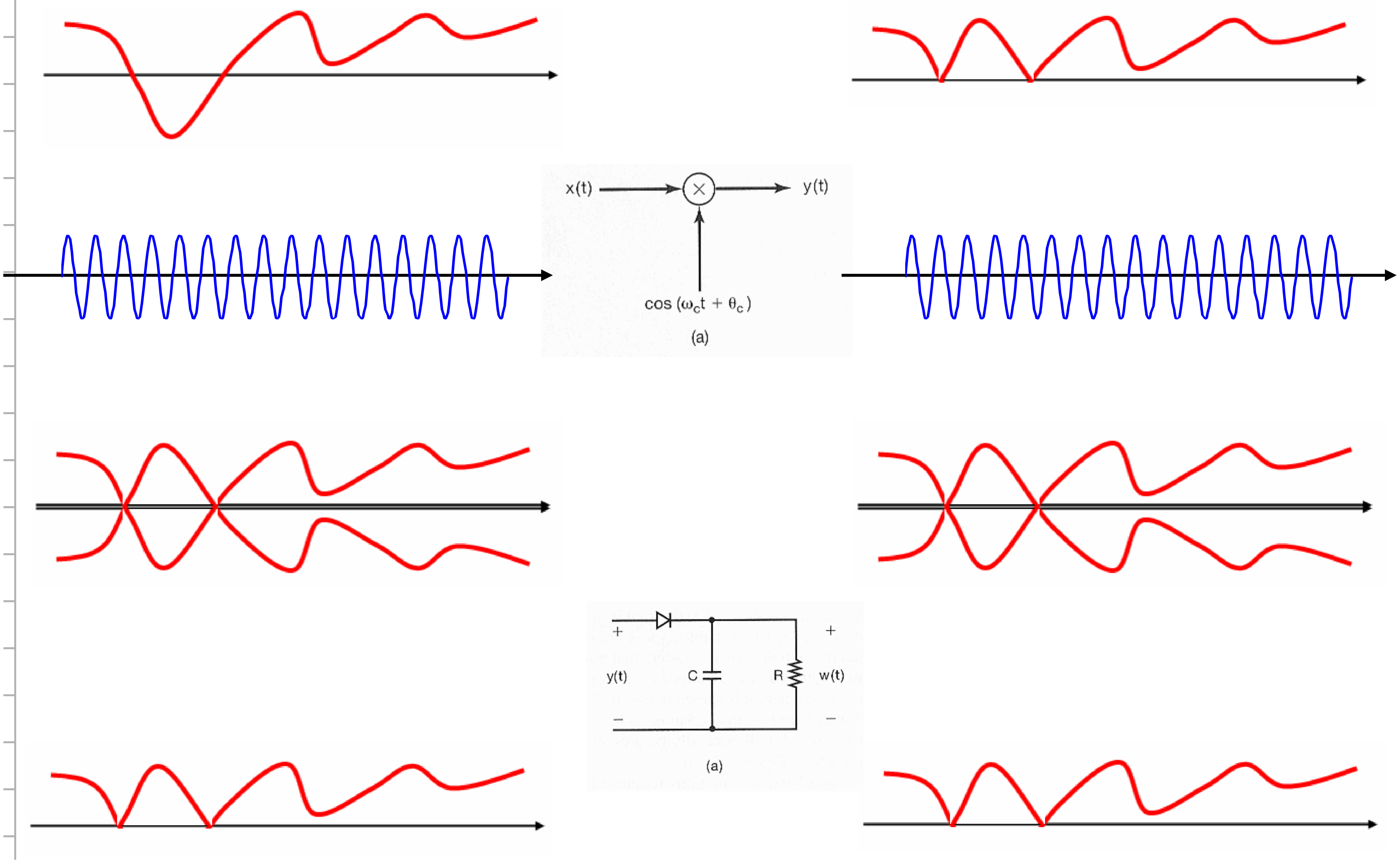


(a)



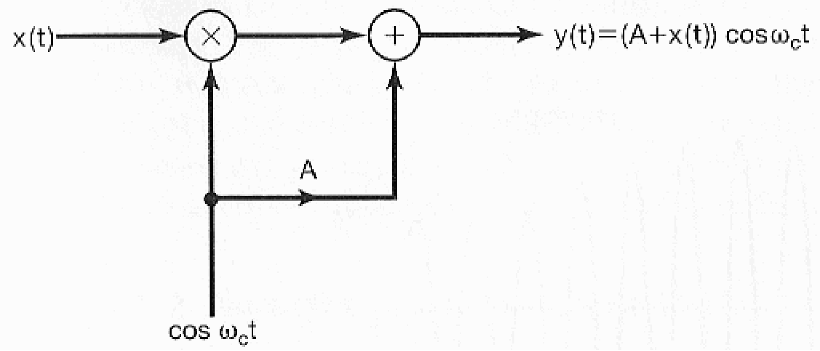
(b)

- Asynchronous Demodulation:



Asynchronous Demodulation:

- $w_c >> w_M$

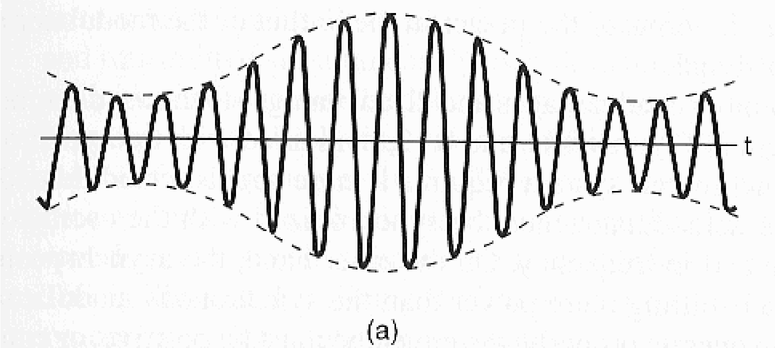


- $x(t) > 0, \quad \forall t$

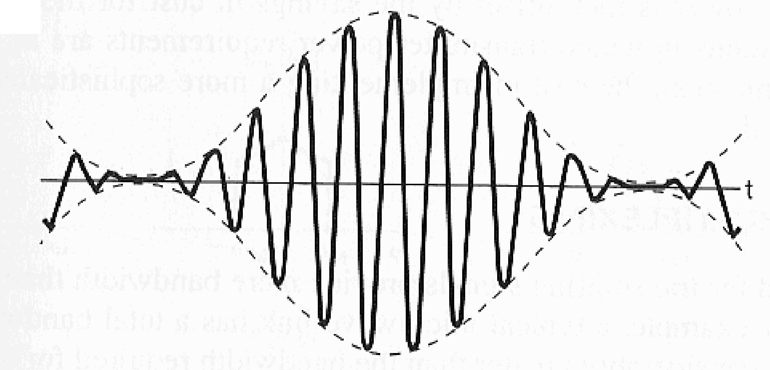
If not, $x(t) \rightarrow x(t) + A \;\; > \;\; 0$

$A \geq K, \quad |x(t)| \leq K$

- $\frac{K}{A}$: modulation index m, in %



$m = 0.5$

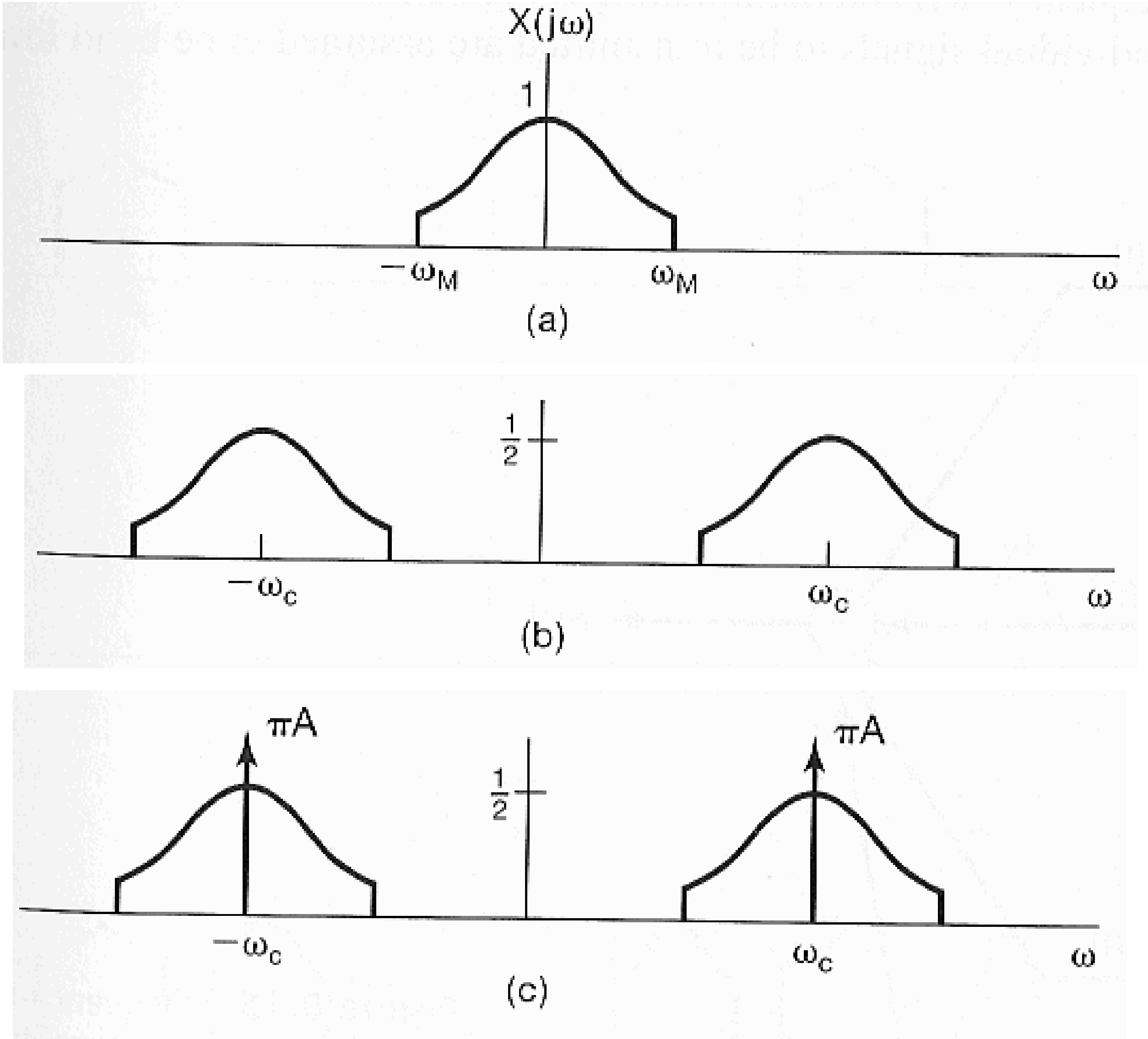


$m = 1.0$

- Synchronous & Asynchronous Demodulation:

$x(t)$ cos$(w_c t)$

$[x(t) + A]$ cos$(w_c t)$



Outline

- Complex Exponential & Sinusoidal Amplitude Modulation & Demodulation
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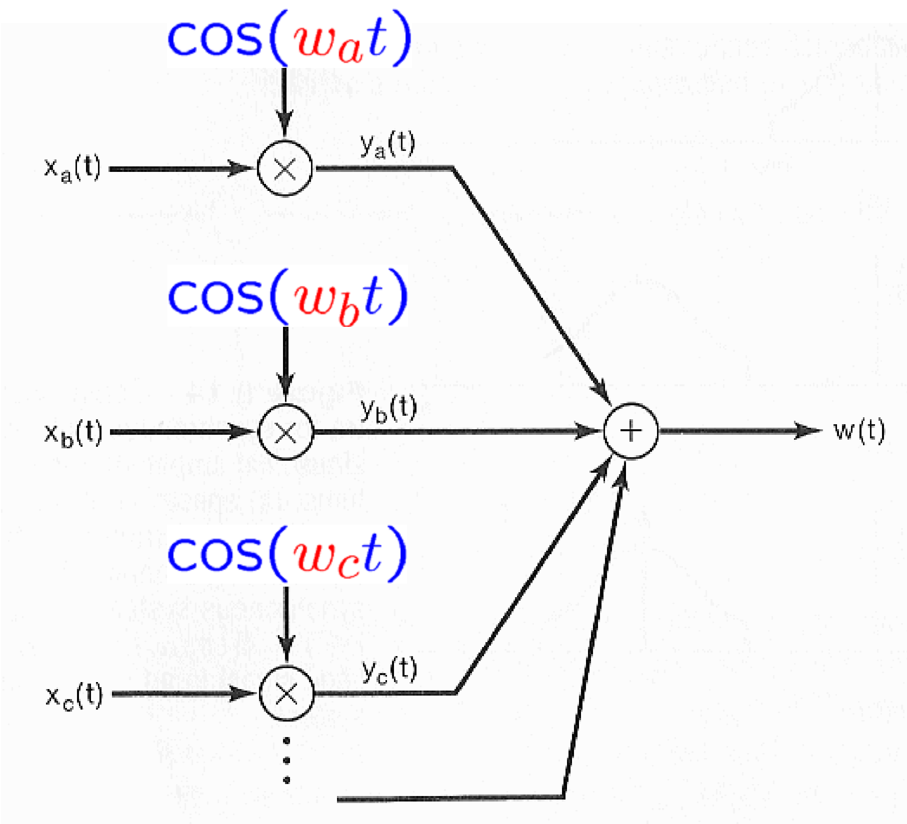
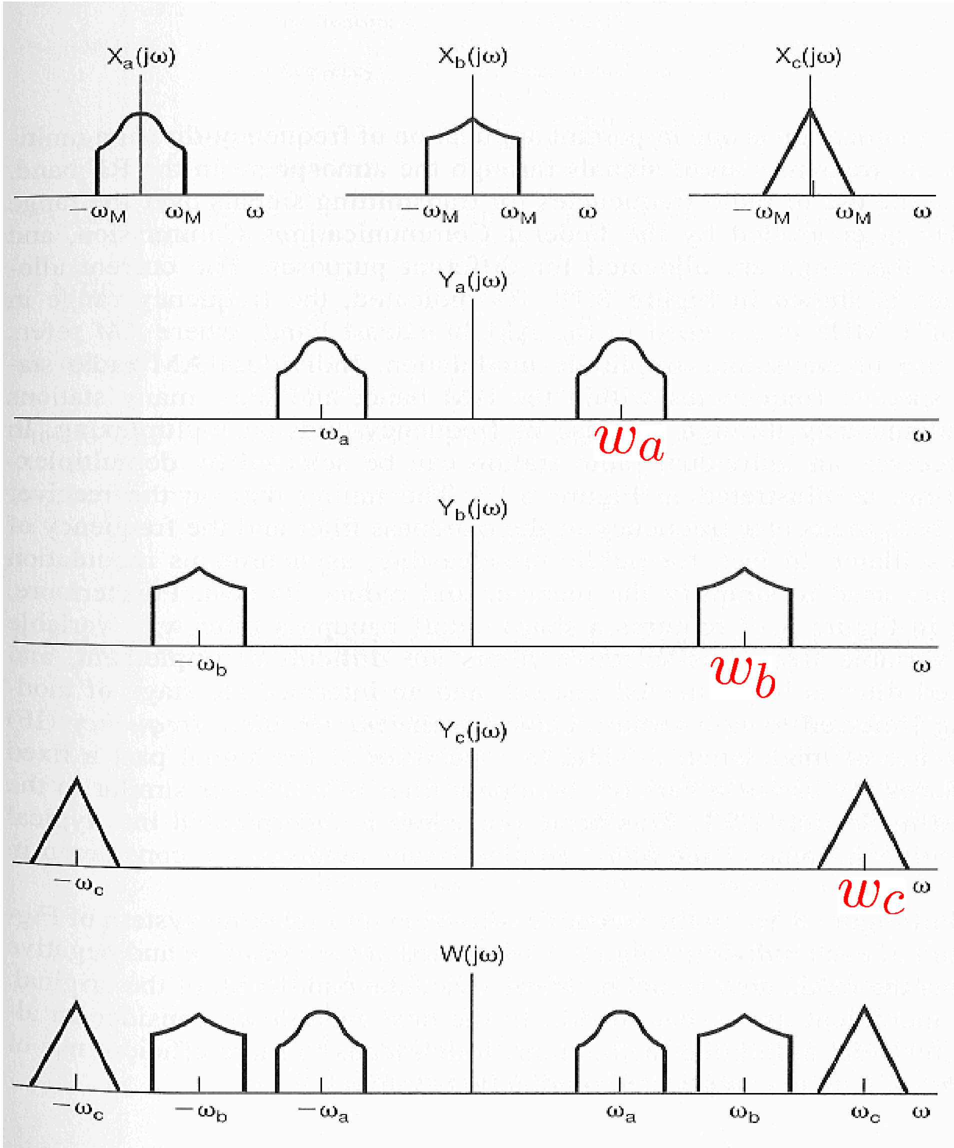
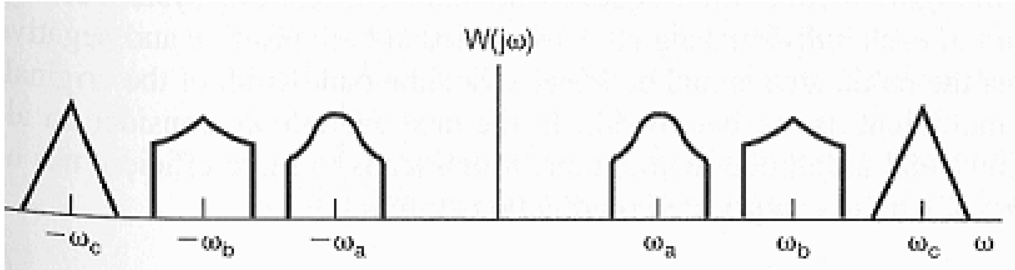
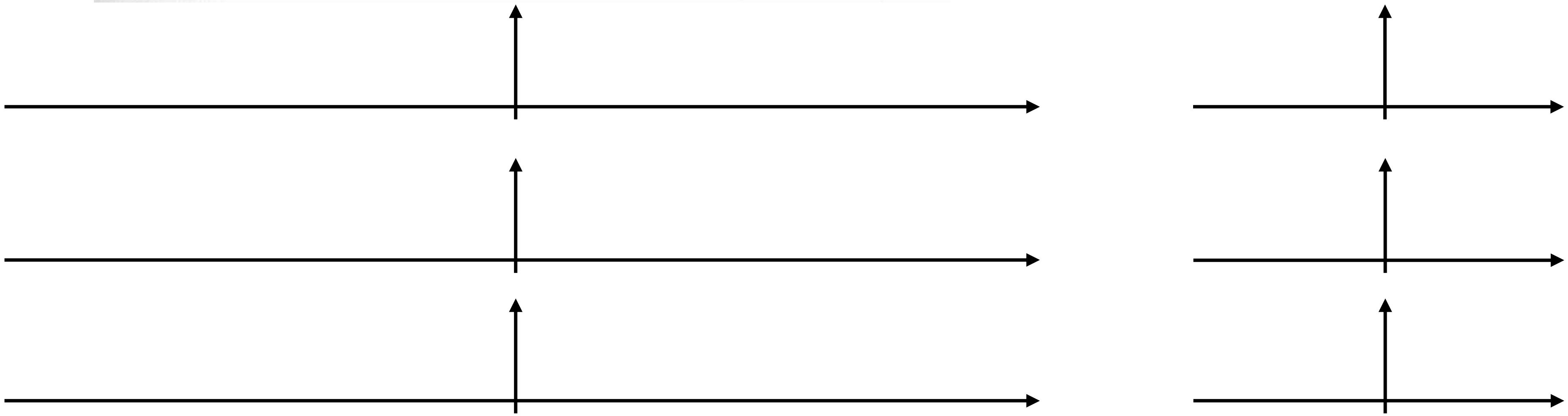
- FDM Using Sinusoidal AM:

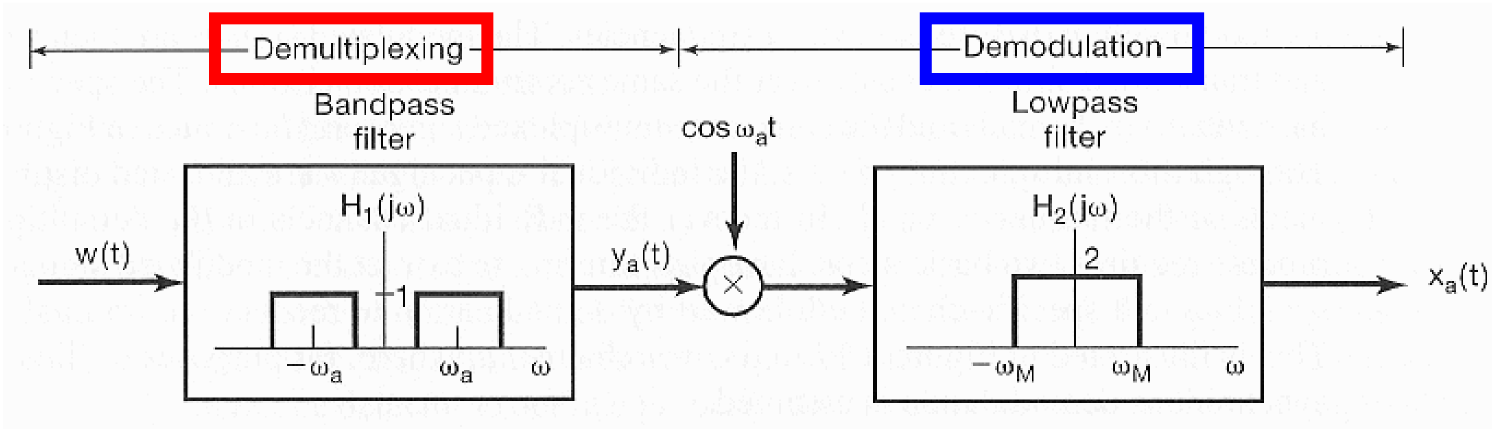




- Demultiplexing and Demodulation:







Allocation of Frequencies in the RF Spectrum

Frequency range	Designation	Typical uses	Propagation method	Channel features
30–300 Hz	ELF (extremely low frequency)	Macrowave, submarine communication	Megametric waves	Penetration of conducting earth and seawater
0.3–3 kHz	VF (voice frequency)	Data terminals, telephony	Copper wire	
3–30 kHz	VLF (very low frequency)	Navigation, telephone, telegraph, frequency and timing standards	Surface ducting (ground wave)	Low attenuation, little fading, extremely stable phase and frequency, large antennas
30–300 kHz	LF (low frequency)	Industrial (power line) communication, aeronautical and maritime long-range navigation, radio beacons	Mostly surface ducting	Slight fading, high atmospheric pulse
0.3–3 MHz	MF (medium frequency)	Mobile, AM broadcasting, amateur, public safety	Ducting and ionospheric reflection (sky wave)	Increased fading, but reliable
3–30 MHz	HF (high frequency)	Military communication, aeronautical mobile, international fixed, amateur and citizen's band, industrial	Ionospheric reflecting sky wave, 50–400 km layer altitudes	Intermittent and frequency-selective fading, multipath
30–300 MHz	VHF (very high frequency)	FM and TV broadcast, land transportation (taxis, buses, railroad)	Sky wave (ionospheric and tropospheric scatter)	Fading, scattering, and multipath
0.3–3 GHz	UHF (ultra high frequency)	UHF TV, space telemetry, radar, military	Transhorizon tropospheric scatter and line-of-sight relaying	
3–30 GHz	SHF (super high frequency)	Satellite and space communication, common carrier (CC), microwave	Line-of-sight ionosphere penetration	Ionospheric penetration, extraterrestrial noise, high directly
30–300 GHz	EHF (extremely high frequency)	Experimental, government, radio astronomy	Line of sight	Water vapor and oxygen absorption
10^3–10^7 GHz	Infrared, visible light, ultraviolet	Optical communications	Line of sight	

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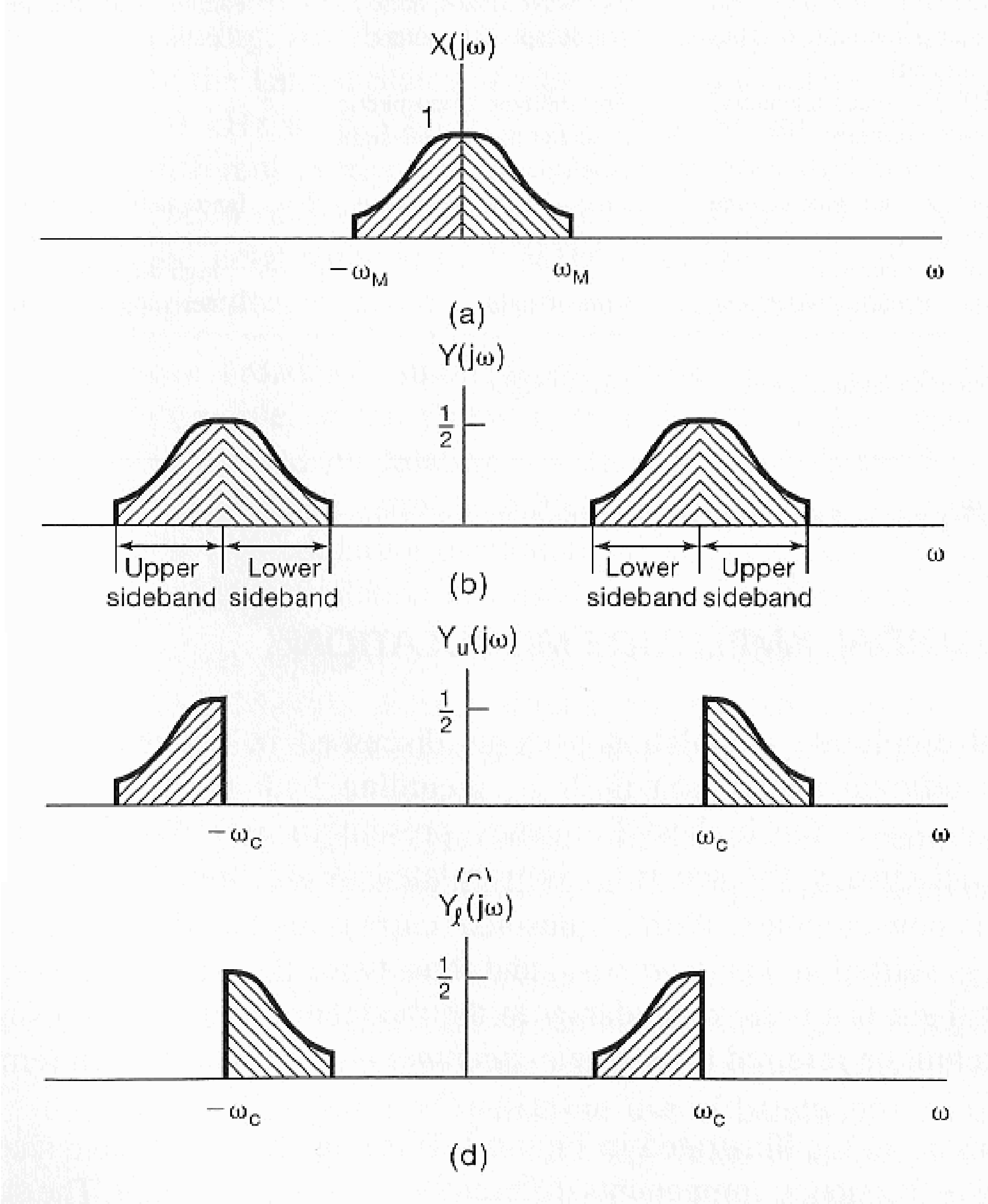
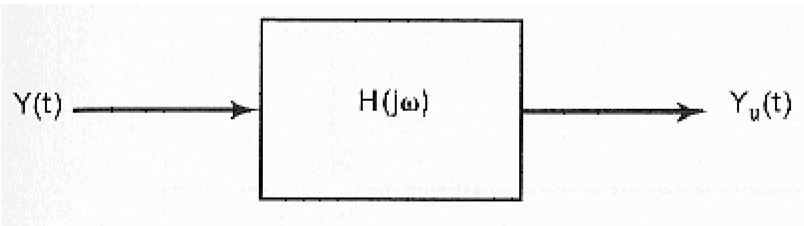
- SSB Modulation:

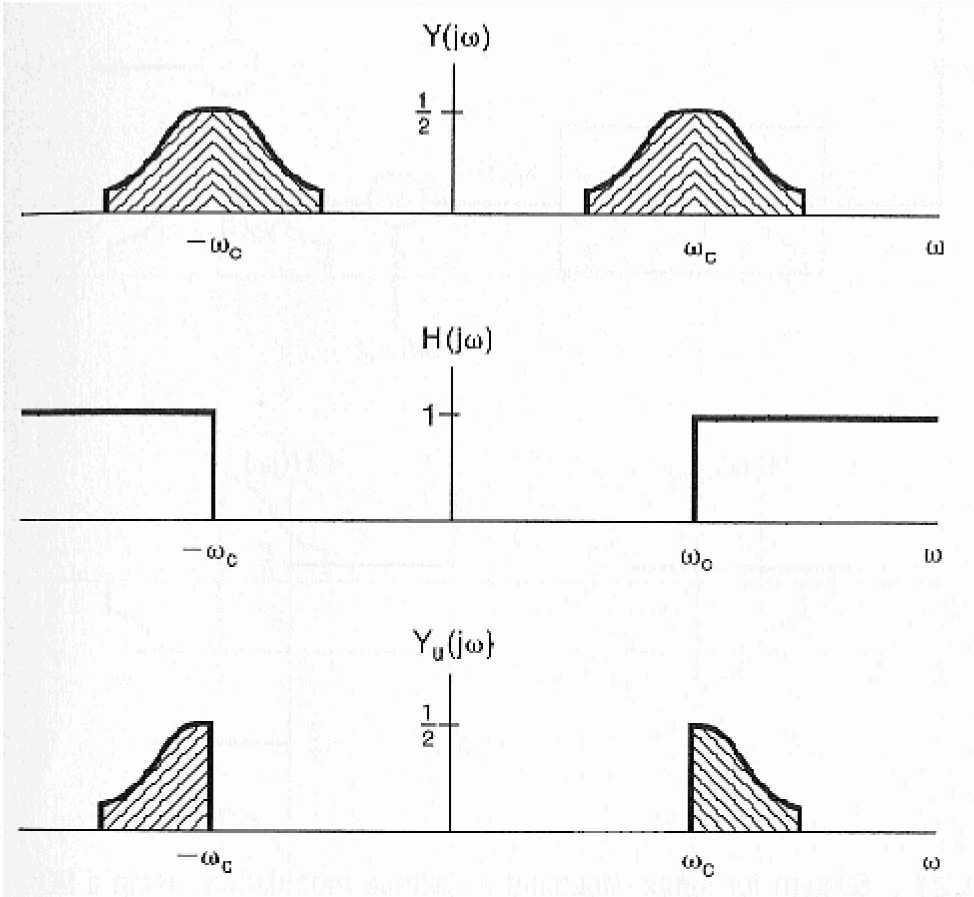
upper sidebands

lower sidebands

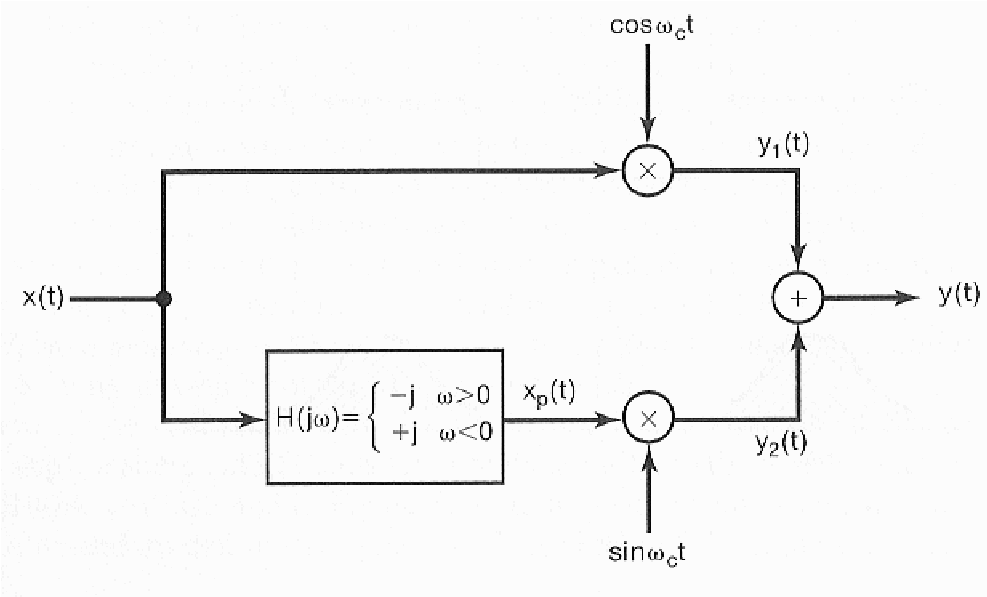


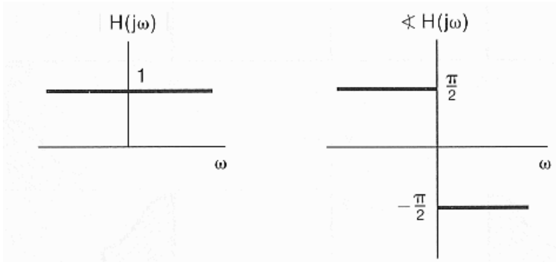
- Retain Upper Sidebands Using Ideal Highpass Filter





Retain Lower Sidebands Using Phase-Shift Network

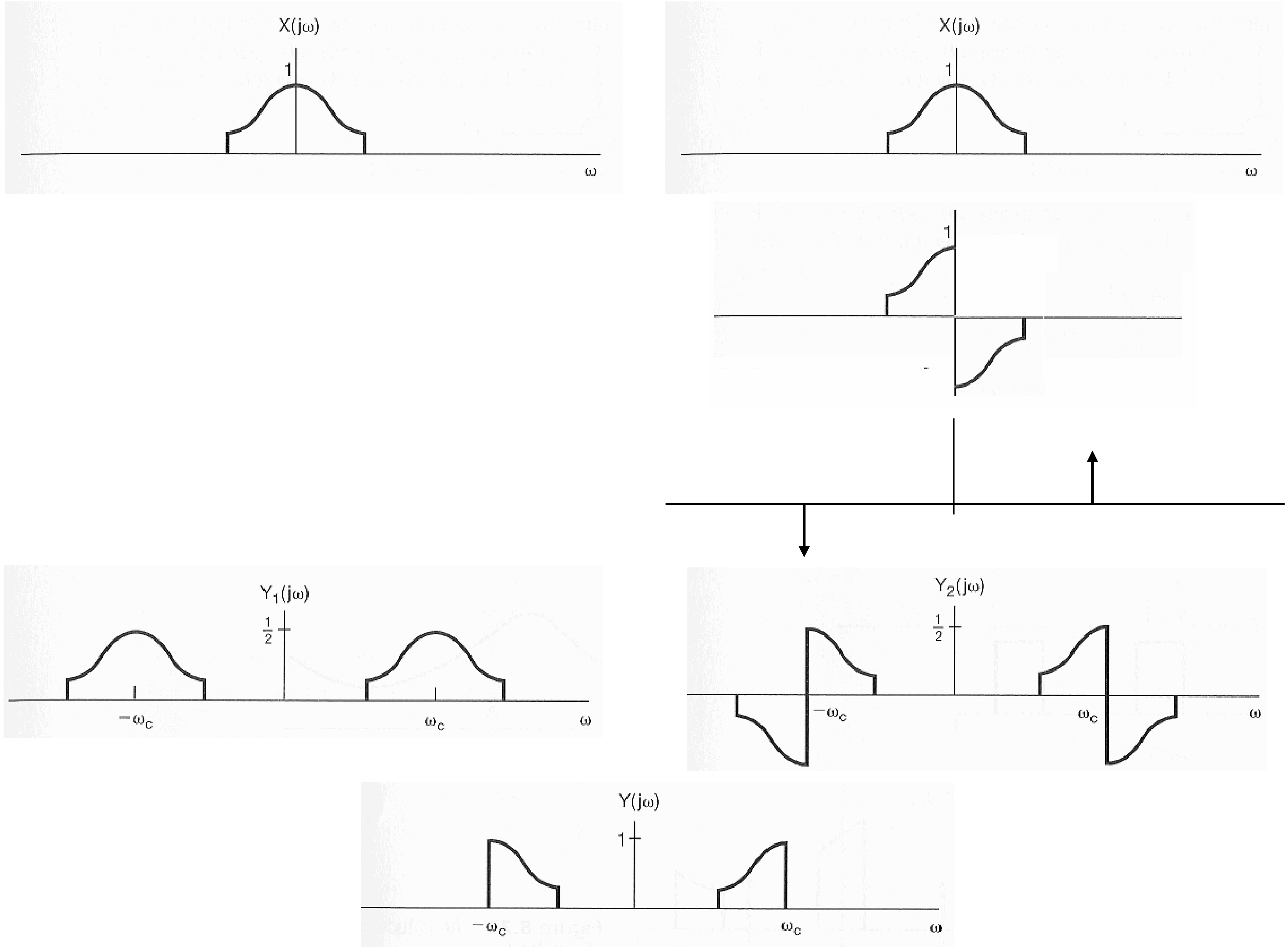




- Retain Lower Sidebands $H(jw) = \begin{cases} -j, & w > 0 \\ +j, & w < 0 \end{cases}$

- Retain Upper Sidebands $H(jw) = \begin{cases} +j, & w > 0 \\ -j, & w < 0 \end{cases}$

Single-Sideband Sinusoidal Amplitude Modulation



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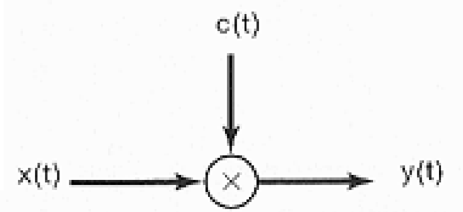
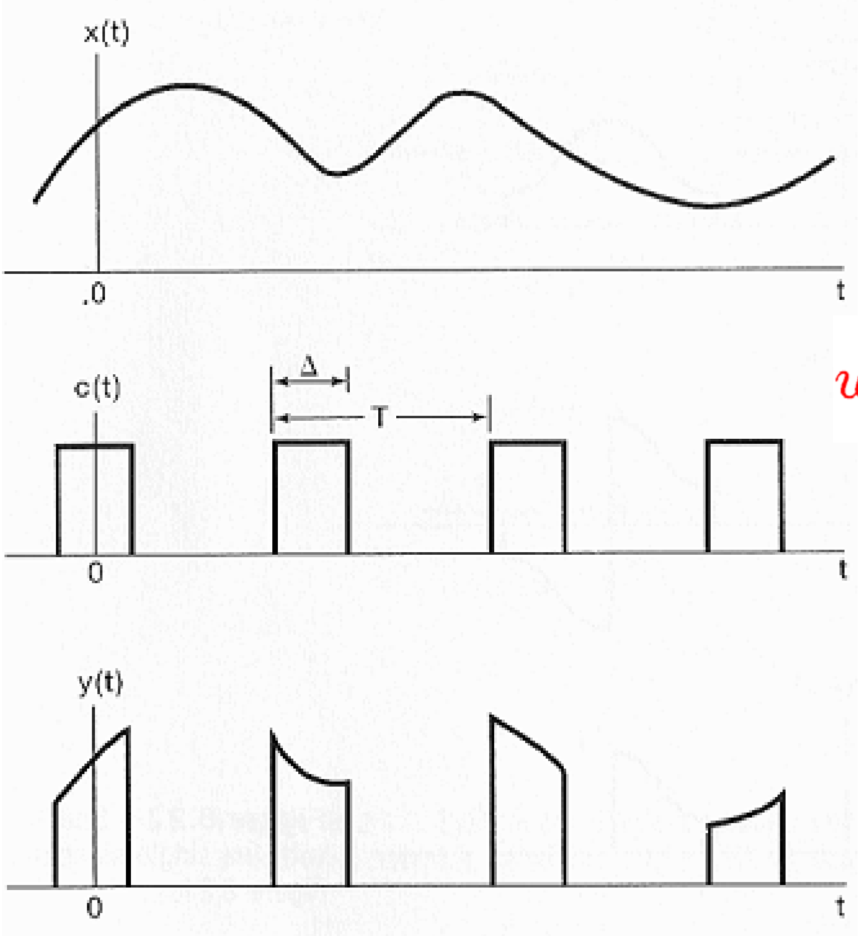
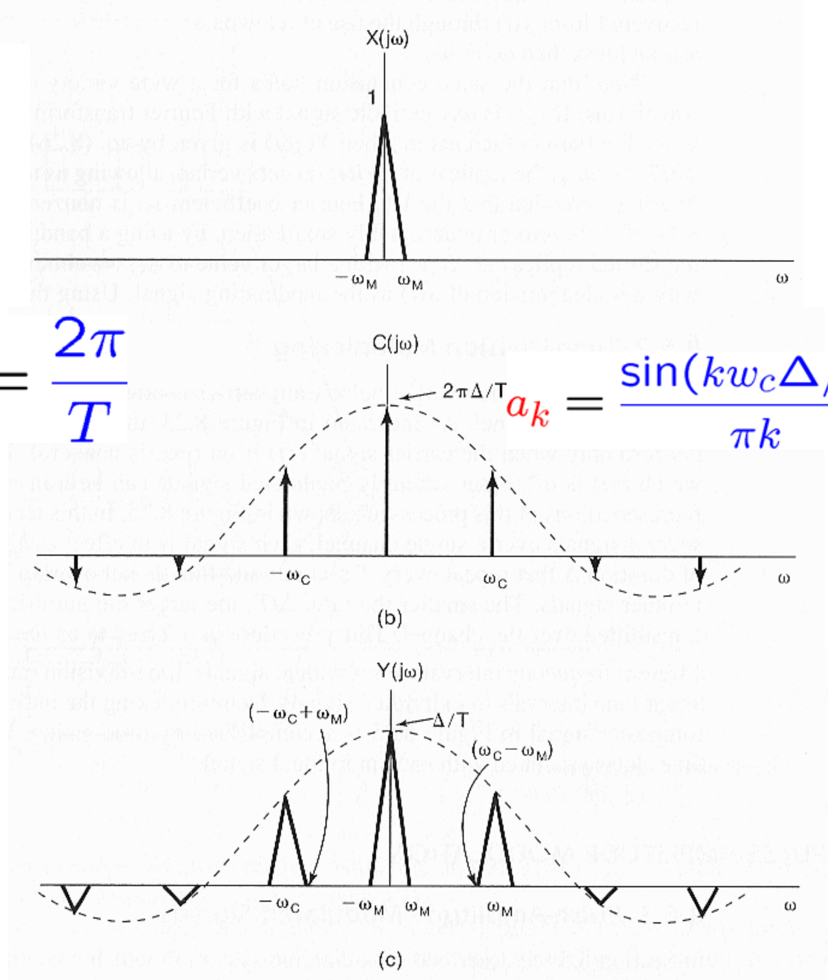
▪ Modulation of a Pulse-Train Carrier:



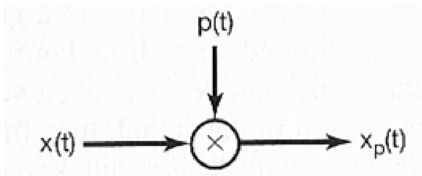


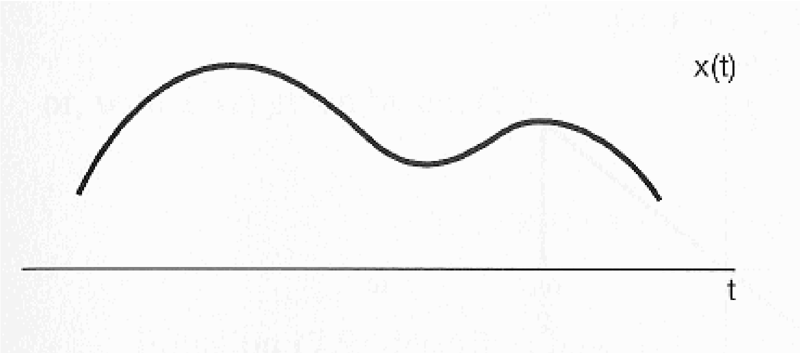
$$w_c = \frac{2\pi}{T}$$

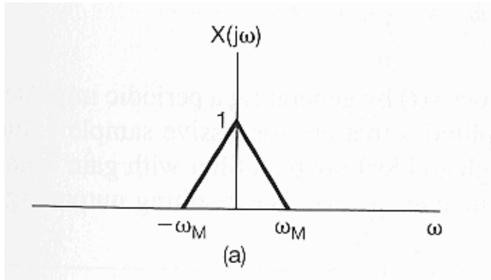
$$a_k = \frac{\sin(k w_c \Delta/2)}{\pi k}$$

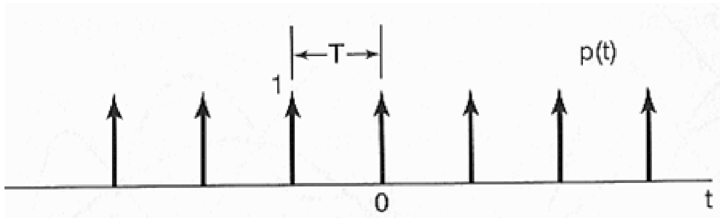


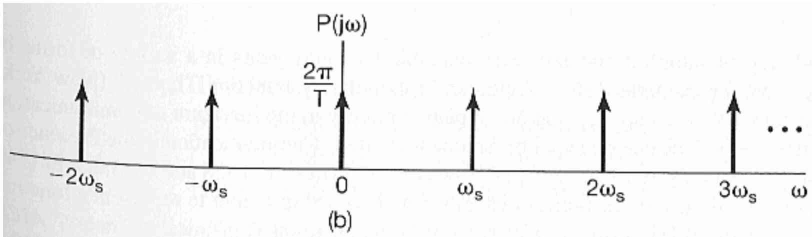
■ Impulse-Train Sampling:

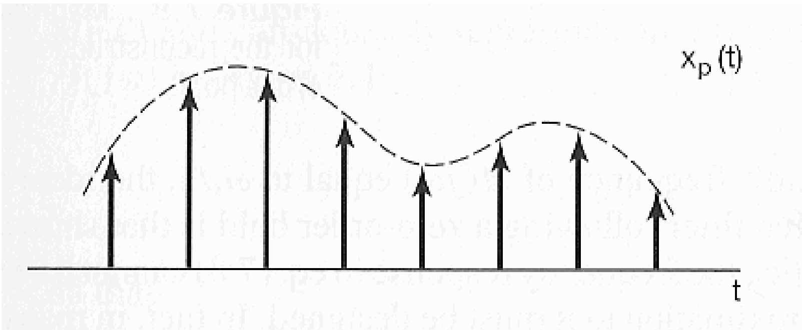




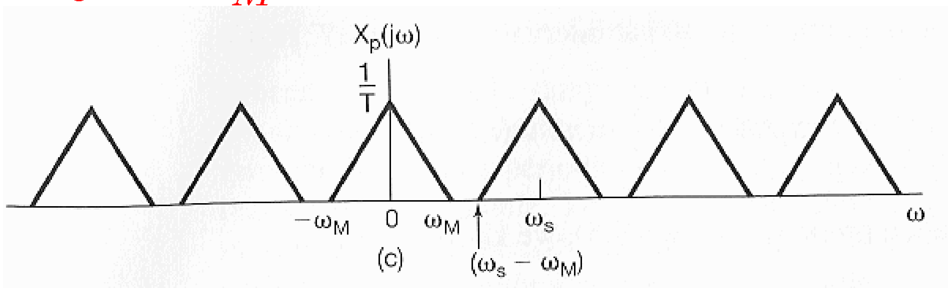




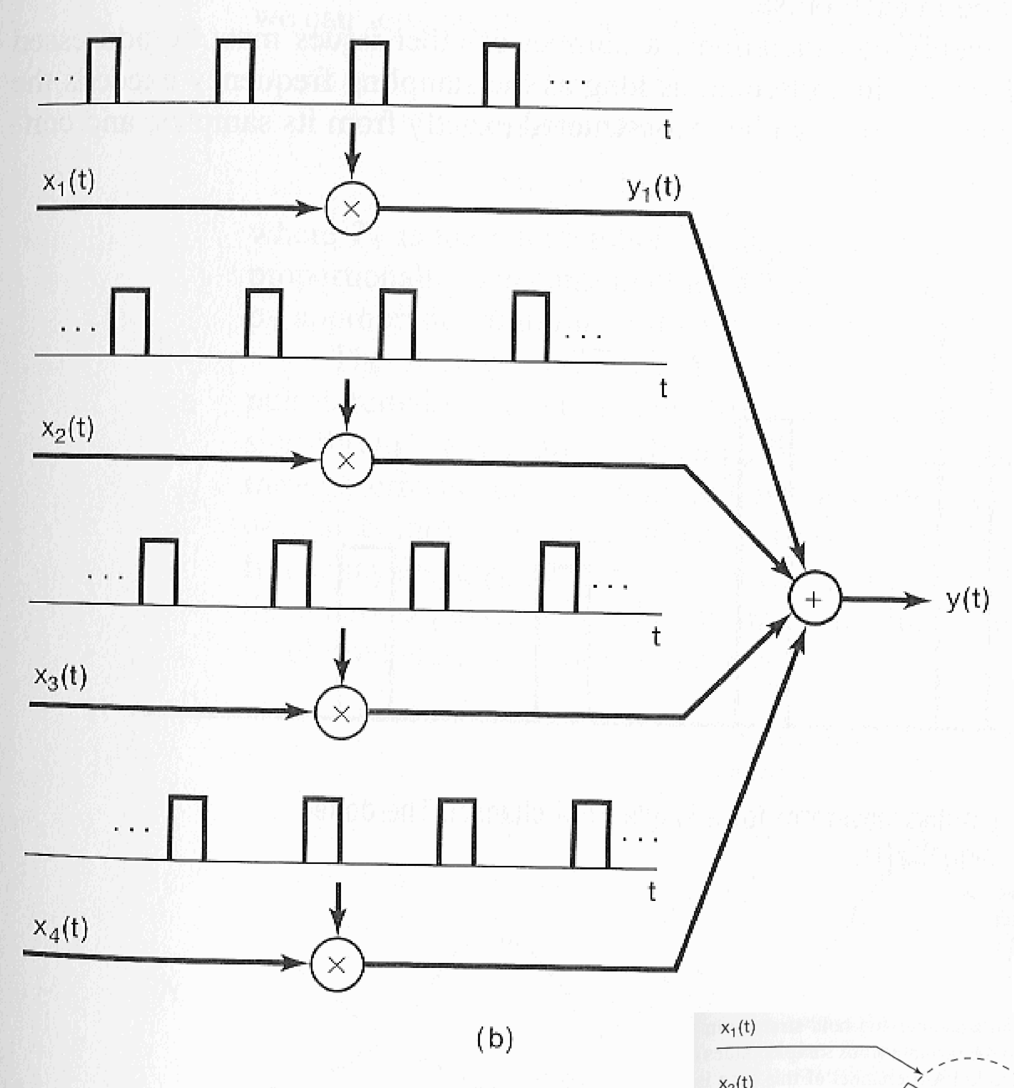


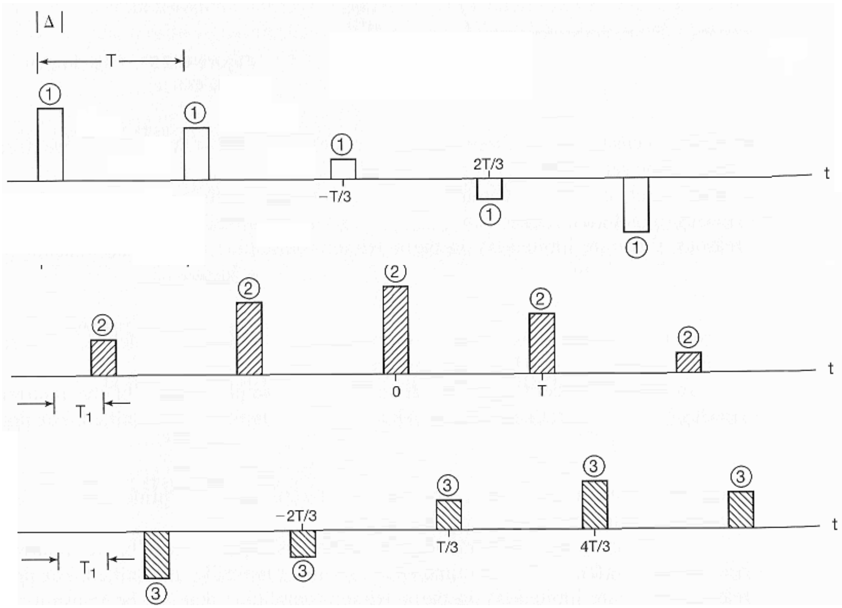


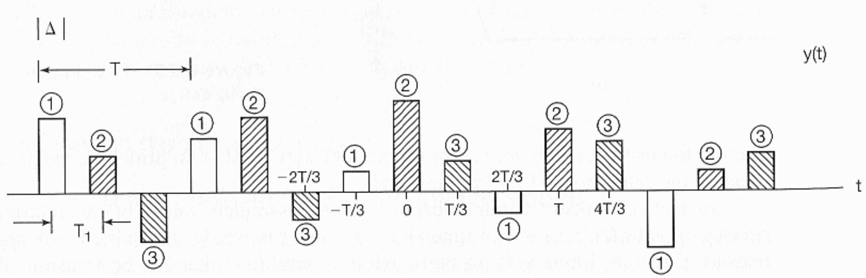
$w_s > 2w_M$

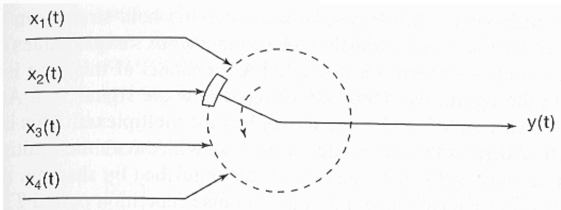


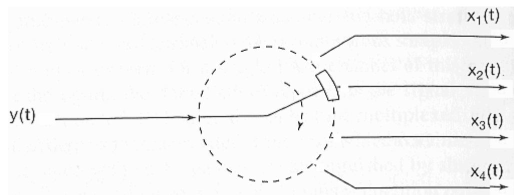
- Time-Division Multiplexing (TDM):











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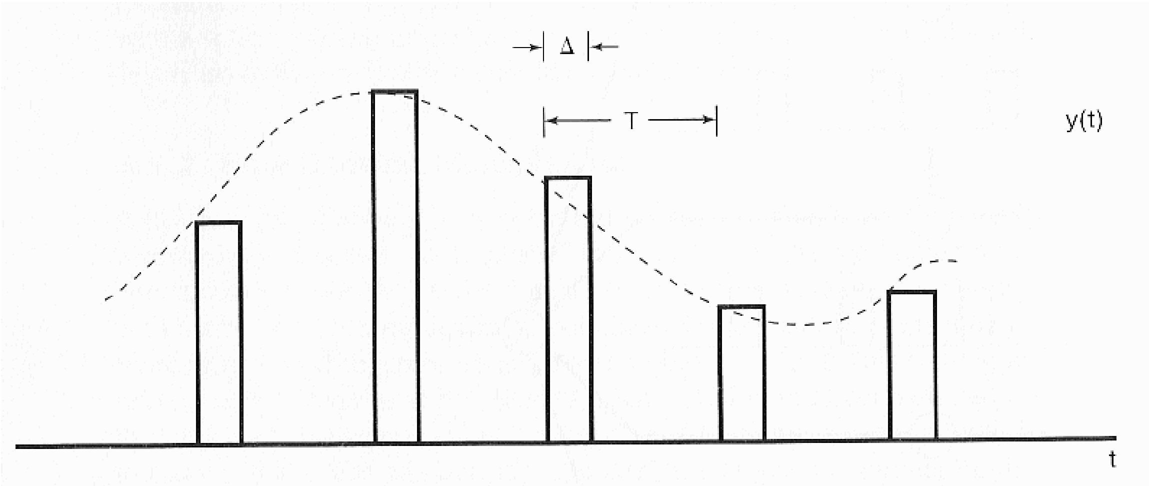
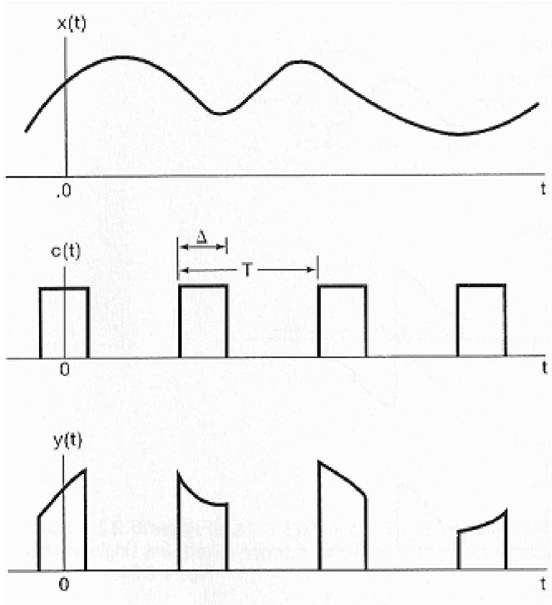
topics after line are not COVERED!

- Pulse-Amplitude Modulation
- Sinusoidal Frequency Modulation
- Discrete-Time Modulation

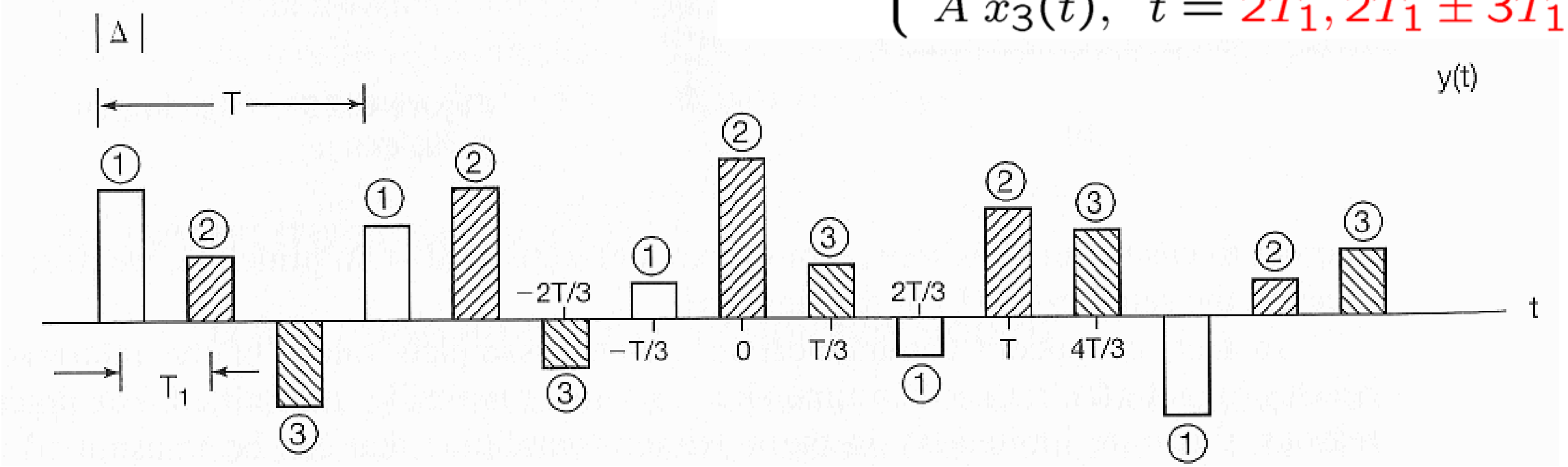
- Pulse-Amplitude Modulated Signals:



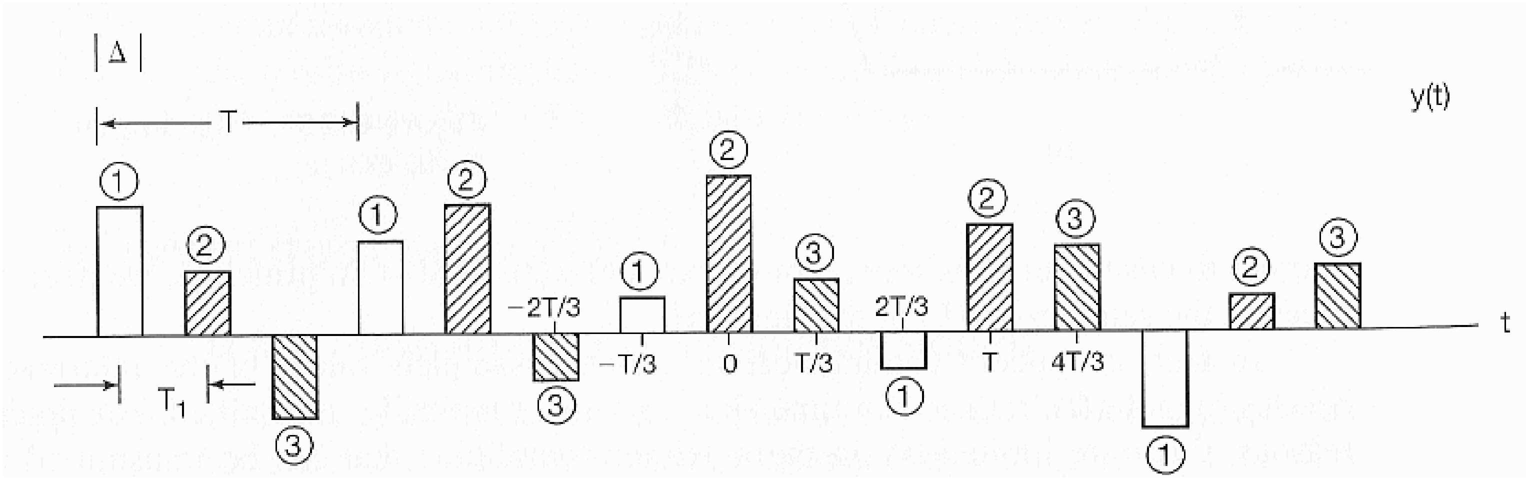


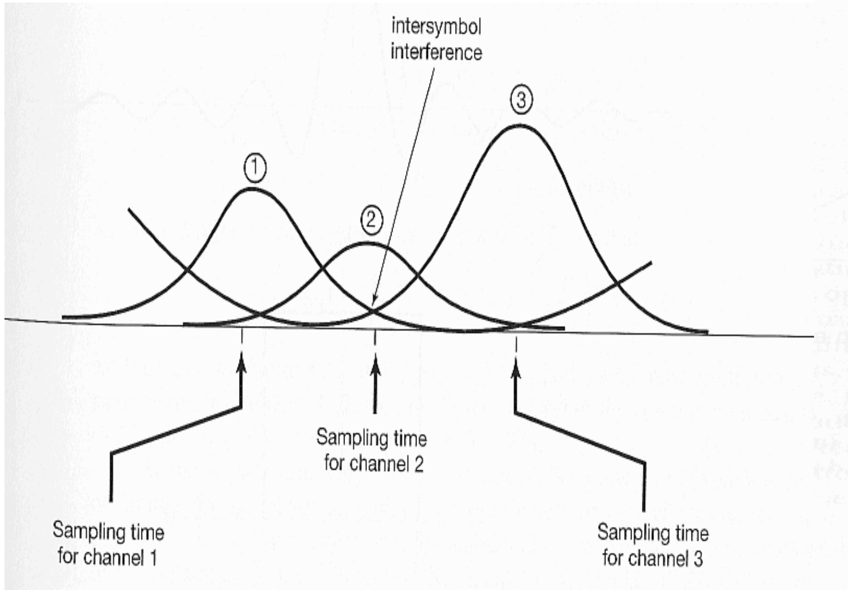
- TDM-PAM:

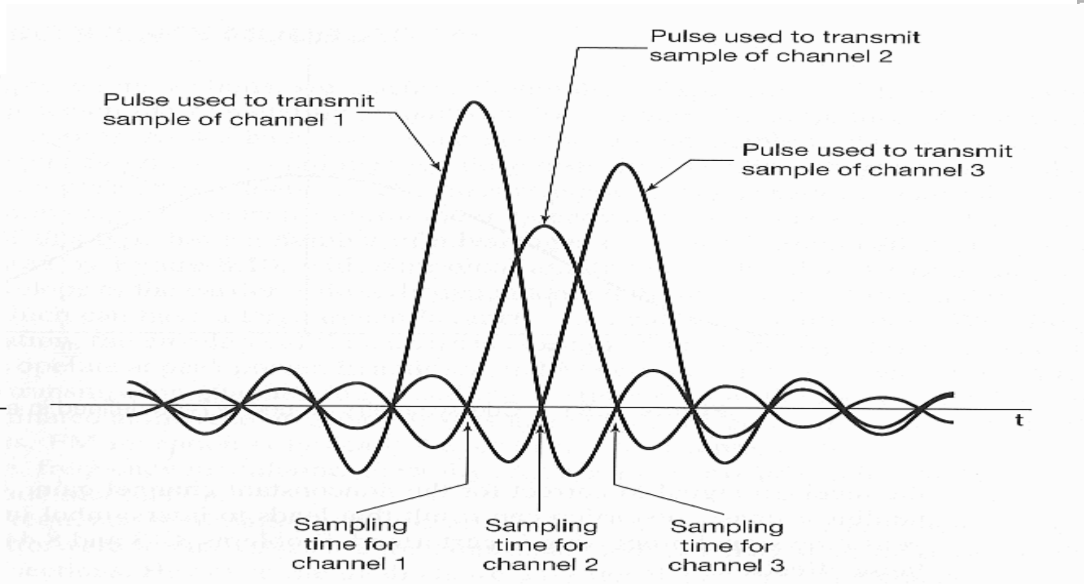
$$y(t) = \begin{cases} A\, x_1(t), & t = 0, \pm 3T_1, \cdots, \\ A\, x_2(t), & t = T_1, T_1 \pm 3T_1, \cdots, \\ A\, x_3(t), & t = 2T_1, 2T_1 \pm 3T_1, \cdots, \end{cases}$$



- Intersymbol Interference in PAM Systems:



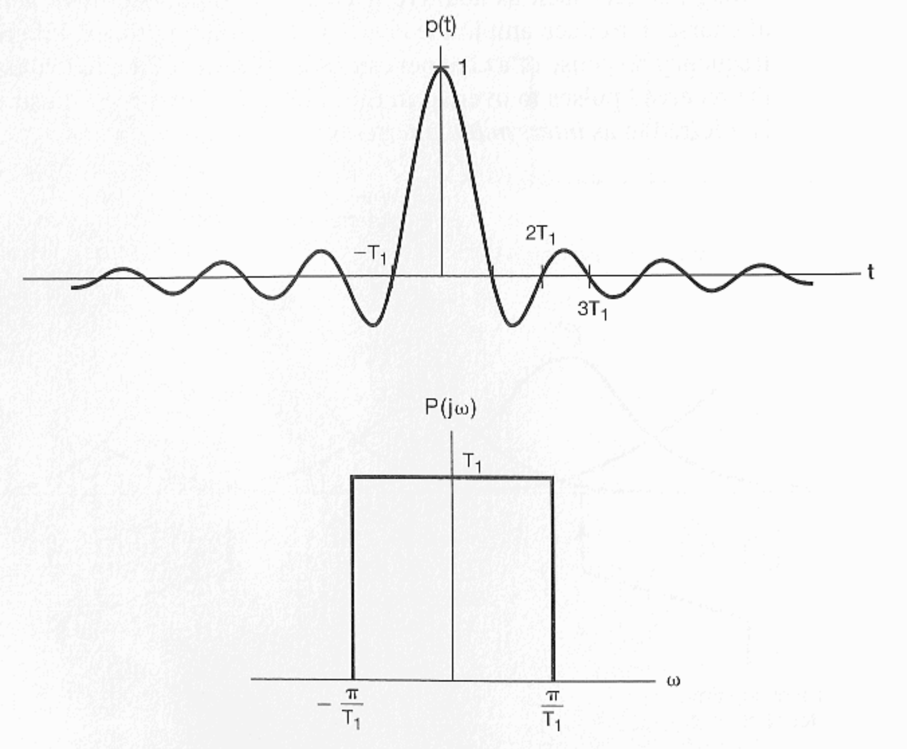


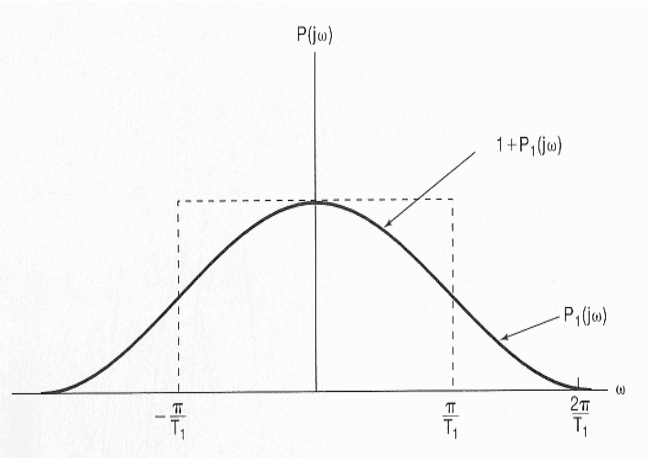


Avoiding Intersymbol Interference in PAM Systems:

$$p(t) = \frac{T_1 \ \sin(\pi \ t \ T_1)}{\pi t}$$

$$p(\pm T_1) = 0, \ p(\pm 2T_1) = 0, \ p(\pm 3T_1) = 0, \cdots$$

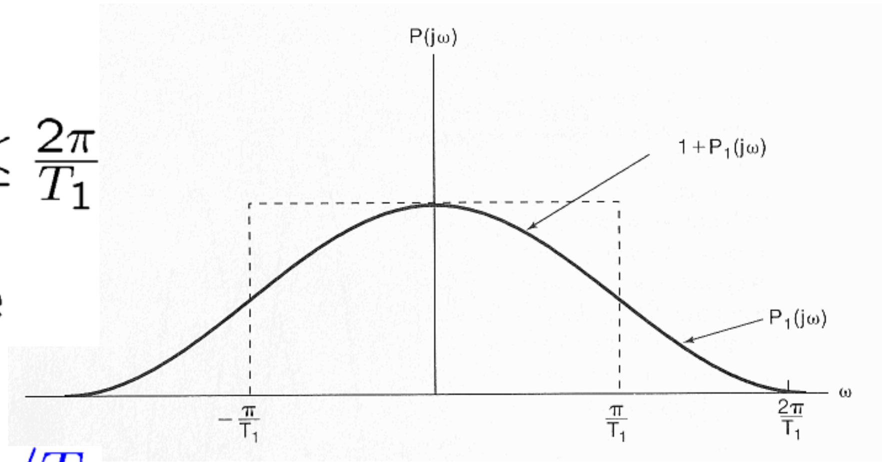




General Form of Band-Limited Pulses with Time-Domain Zero-Crossing at kT_1, $k \in Z$:

Problem 8.42

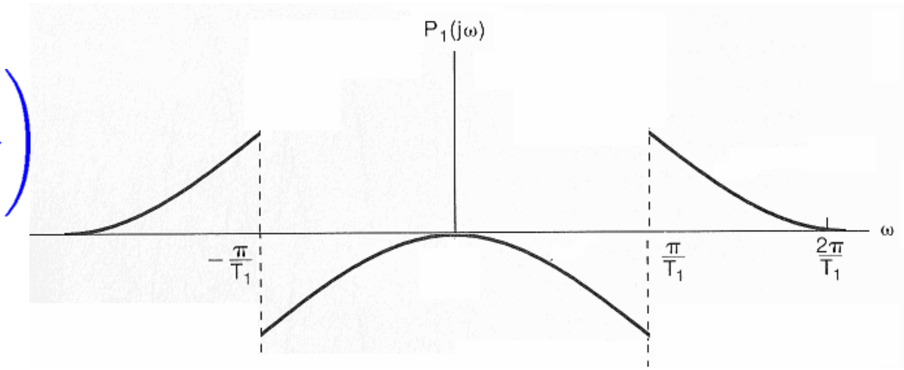
$$P(jw) = \begin{cases} 1 + P_1(jw) & |w| \le \frac{\pi}{T_1} \\ P_1(jw) & \frac{\pi}{T_1} < |w| \le \frac{2\pi}{T_1} \\ 0 & \text{otherwise} \end{cases}$$



$P_1(jw)$: odd symmetry around π/T_1

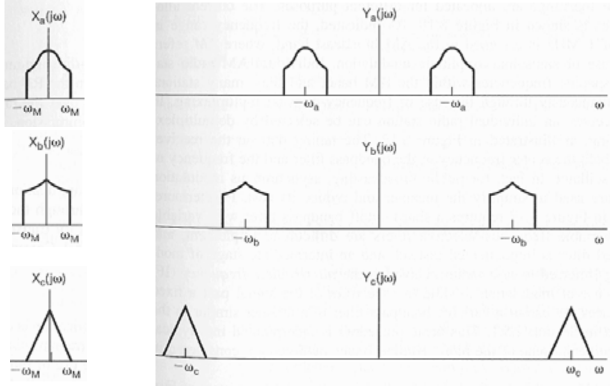
$$P_1\left(-jw + j\frac{\pi}{T_1}\right) = -P_1\left(jw + j\frac{\pi}{T_1}\right)$$

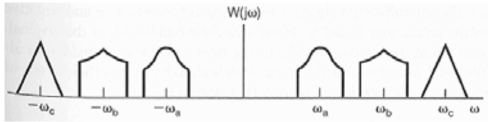
$$0 \le w \le \frac{\pi}{T_1}$$

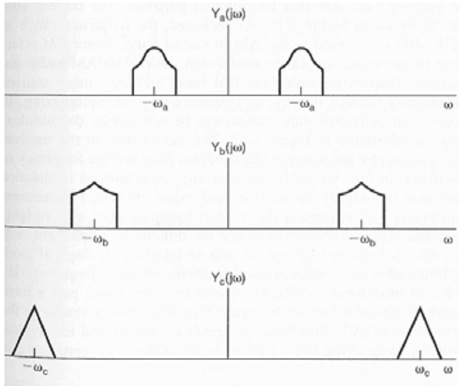


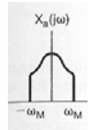
$\Rightarrow$ $p(t)$ has zero crossing at $\pm T_1, \pm 2T_1, \cdots$ i.e.,$p(\pm kT_1) = 0$

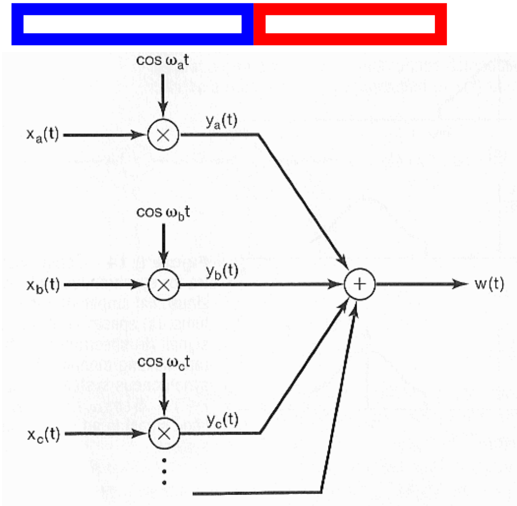
Modulation and Multiplexing

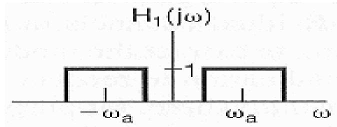


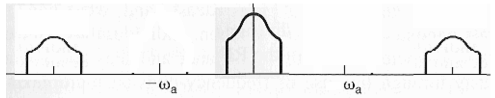


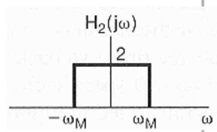


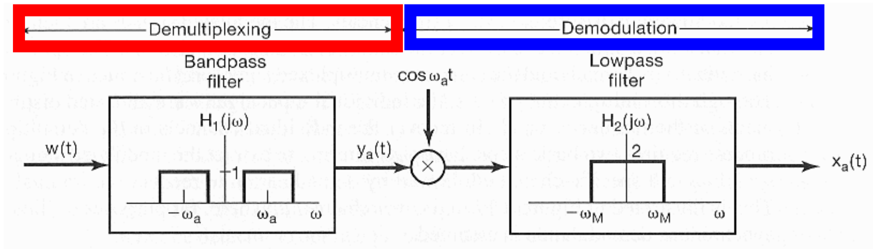


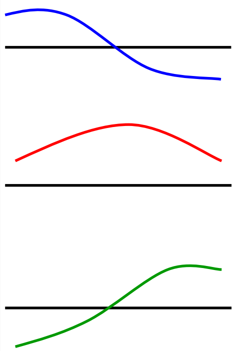


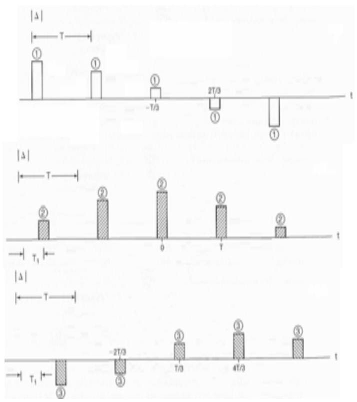


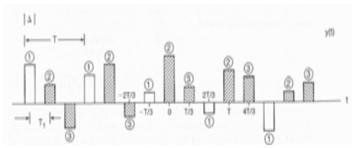


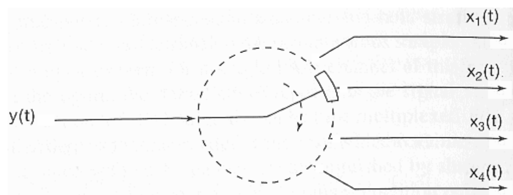












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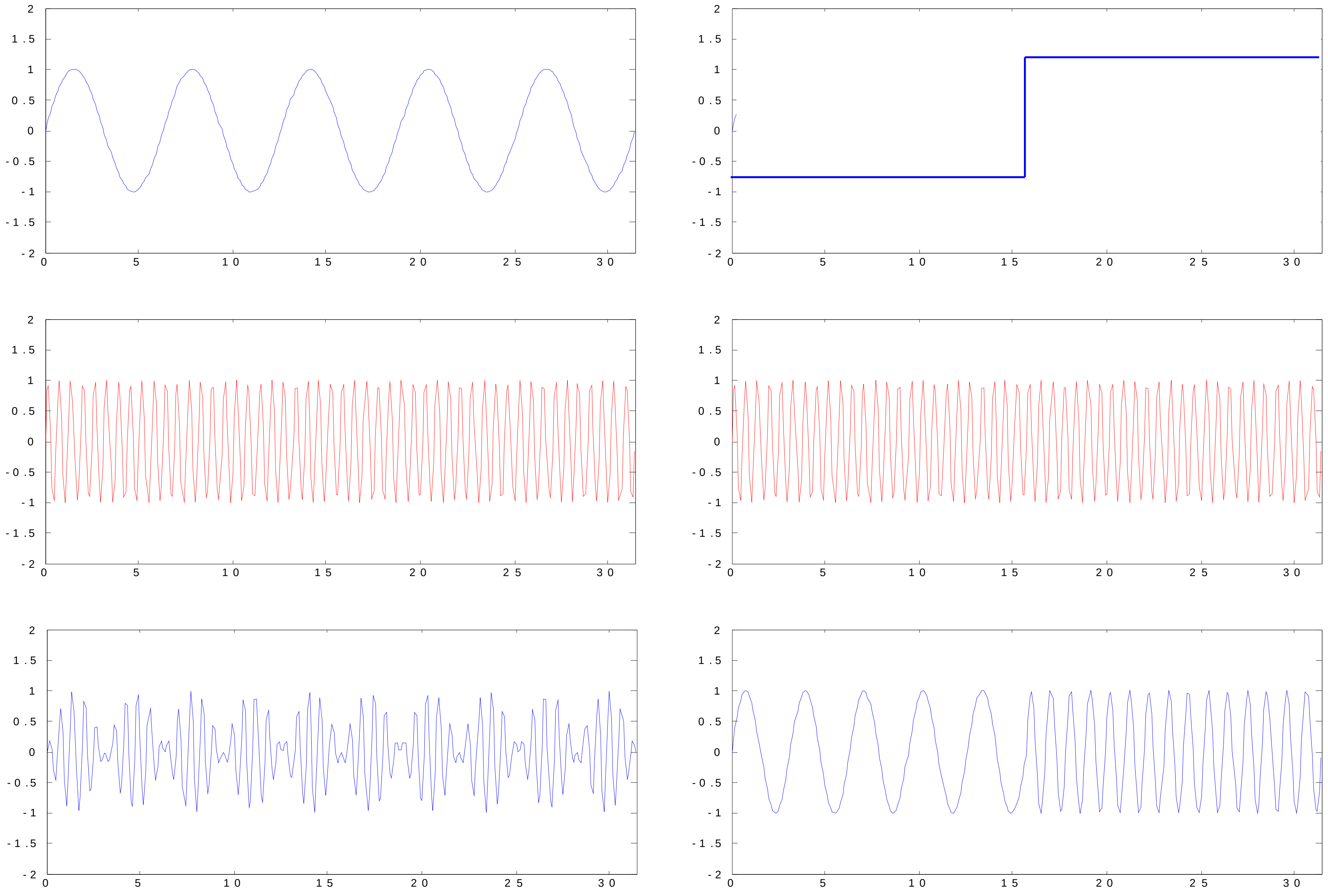
Frequency Modulation (FM):

- The modulating signals is used to control the frequency of a sinusoidal carrier
- With sinusoidal AM, the peak amplitude of the envelope of the carrier is directly dependent on the amplitude of the modulating signal $x(t)$, which can have a large dynamic range.
- With FM, the envelope of the carrier is constant
- An FM transmitter can always operate at peak power and amplitude variations introduced over a transmission channel due to additive disturbances or fading can be eliminated at the receiver
- FM generally requires greater bandwidth than does sinusoidal AM

Amplitude Modulation and Frequency Modulation



Angle Modulation:

$$c(t) = A \cos\left(w_c t + \theta_c\right) = A \cos\theta(t)$$

- Phase Modulation:
  - Use the modulating signal x(t) to vary the phase θ_c

$$y(t) = A \cos\left(\theta(t)\right) = A \cos\left(w_c t + \theta_c(t)\right)$$

$$\theta_c(t) = \theta_0 + k_p\, x(t)$$

$$\frac{d\theta(t)}{dt} = w_c + k_p \frac{dx(t)}{dt}$$

- Frequency Modulation:
  - Use the modulating signal x(t) to vary the derivative of the angle

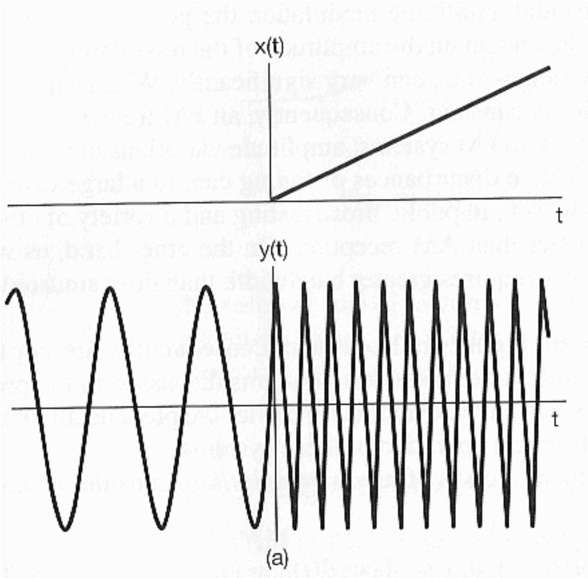
$$y(t) = A \cos\left(\theta(t)\right) \qquad \frac{d\theta(t)}{dt} = w_c + k_f\, x(t)$$

- Phase & Frequency Modulation:

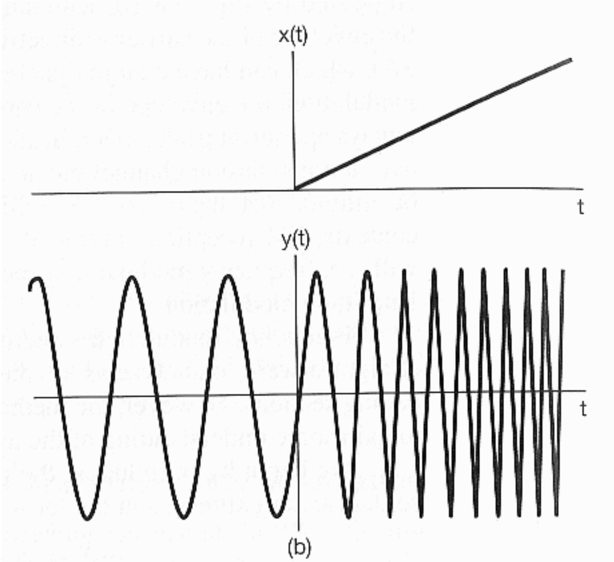
$$\frac{d\theta(t)}{dt} = w_c + k_p \frac{dx(t)}{dt}$$

$$\frac{d\theta(t)}{dt} = w_c + k_f \, x(t)$$

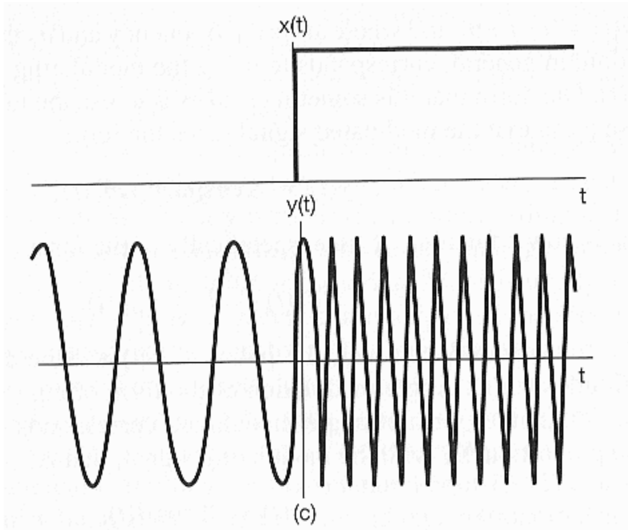
phase modulation | frequency modulation | frequency modulation



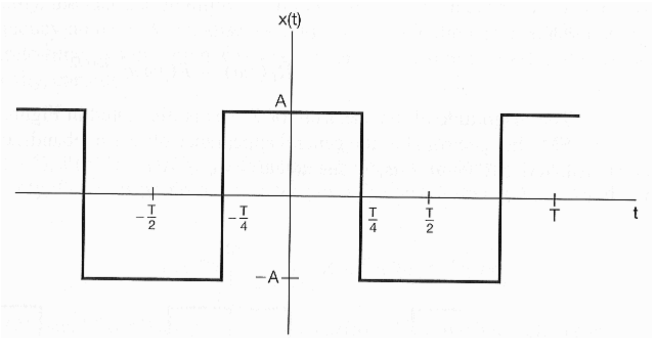
(a)

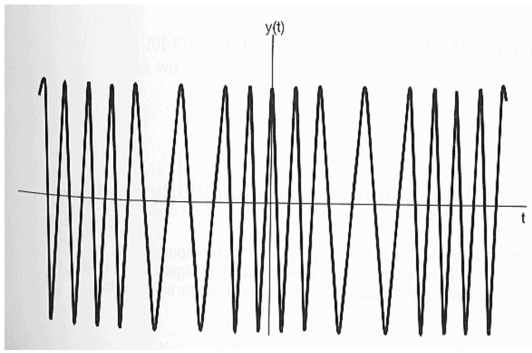


(b)



(c)





Instantaneous Frequency:

$$y(t) = A \cos\left(\theta(t)\right) \quad \Rightarrow w_i = \frac{d\theta(t)}{dt}$$

- If y(t) is truly sinusoidal:

$$\theta(t) = w_c t + \theta_0 \qquad w_i = w_c$$

- Phase Modulation:

$$w_i = \frac{d\theta(t)}{dt} = w_c + k_p \frac{dx(t)}{dt}$$

- Frequency Modulation:

$$w_i = \frac{d\theta(t)}{dt} = w_c + k_f\, x(t)$$

Narrowband FM:

- Frequency Modulation with $x(t) = A \cos\left(w_m t\right)$

- Instantaneous Frequency:

$$w_i(t) = \frac{d\theta(t)}{dt} = w_c + k_f A \cos\left(w_m t\right)$$

$$\Rightarrow w_c - k_f A \leq w_i(t) \leq w_c + k_f A$$

$$\Rightarrow \Delta w \triangleq k_f A$$

$$\Rightarrow w_i(t) = w_c + \Delta w \cos\left(w_m t\right)$$

- Narrowband FM:

$$x(t) = A\cos\left(w_m t\right)$$

$$y(t) = \cos\left(\theta(t)\right) = \cos\left(w_c t + \theta_c(t)\right)$$

$$\frac{d\theta(t)}{dt} = w_c + k_f\, x(t)$$

$$\Delta w \triangleq k_f\, A$$

$$\Rightarrow y(t) = \cos\left(w_c t + k_f \int x(t)dt\right)$$

$$= \cos\left(w_c t + \frac{\Delta w}{w_m}\sin(w_m t) + \theta_0\right)$$

let $\theta_0 = 0$

$$= \cos\left(w_c t + \frac{\Delta w}{w_m}\sin(w_m t)\right)$$

- Modulation Index for FM: $m \triangleq \dfrac{\Delta w}{w_m}$
- Which m is small $\Rightarrow$ narrowband FM

- Narrowband FM:

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

$$\Rightarrow \; y(t) = \cos\left(w_c t + m\sin(w_m t)\right)$$

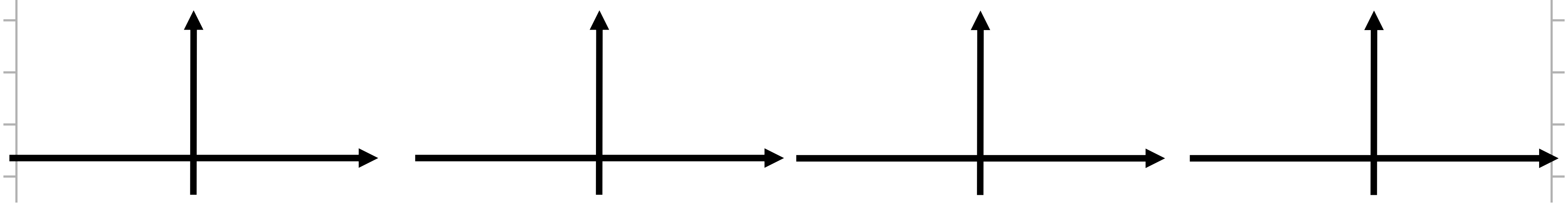
$$\text{or} \quad y(t) = \cos\left(w_c t\right)\cos\left(m\sin(w_m t)\right) - \sin\left(w_c t\right)\sin\left(m\sin(w_m t)\right)$$

- When m is sufficiently small (<< π/2)

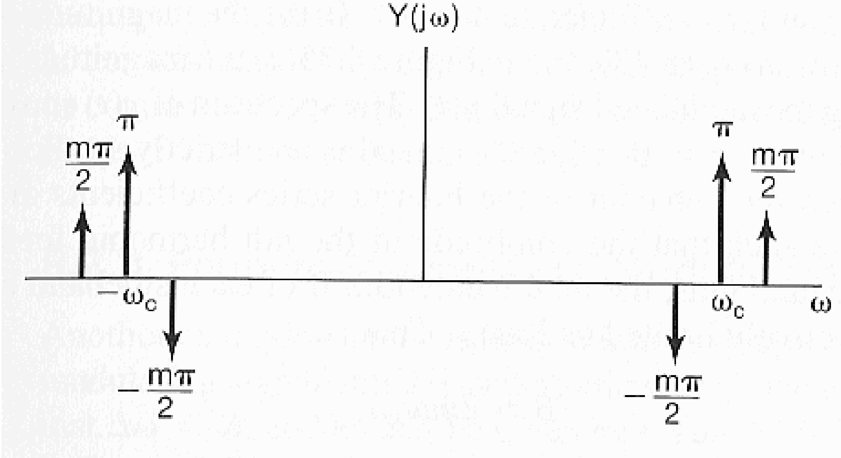
$$\text{if } 0 < \theta << 1 \quad \Rightarrow \quad \begin{aligned} \cos(\theta) &\approx 1 \\ \sin(\theta) &\approx \theta \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} \cos\left(m\sin(w_m t)\right) &\approx 1 \\ \sin\left(m\sin(w_m t)\right) &\approx m\sin(w_m t) \end{aligned}$$

$$\Rightarrow \; y(t) \approx \cos\left(w_c t\right) - m\sin(w_m t)\sin\left(w_c t\right)$$



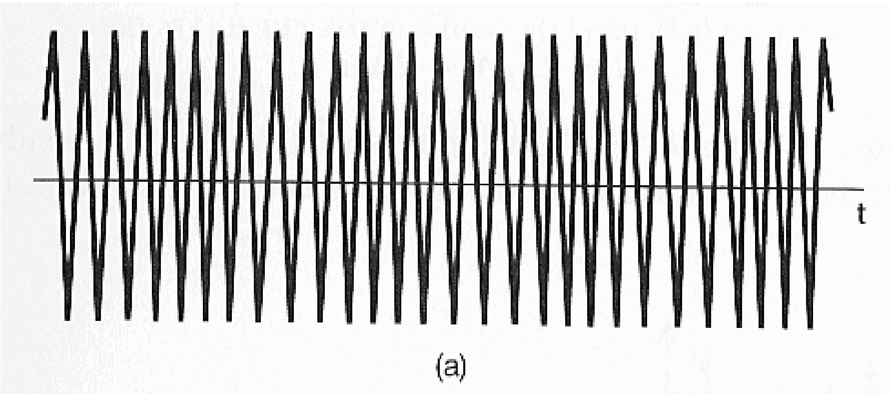
- Narrowband FM:



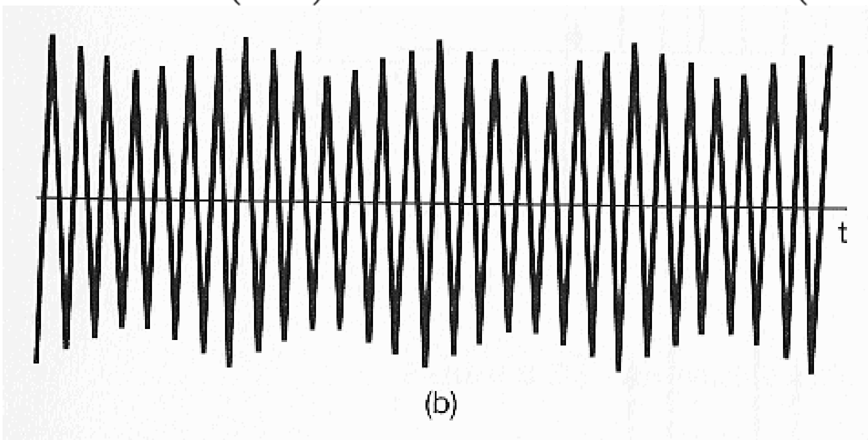
Approximate spectrum for narrowband FM

$$y(t) \approx \cos\left(w_c t\right) - m \sin(w_m t) \sin\left(w_c t\right)$$

$$y_2(t) = \cos\left(w_c t\right) + m \cos(w_m t) \cos\left(w_c t\right)$$



Narrowband FM



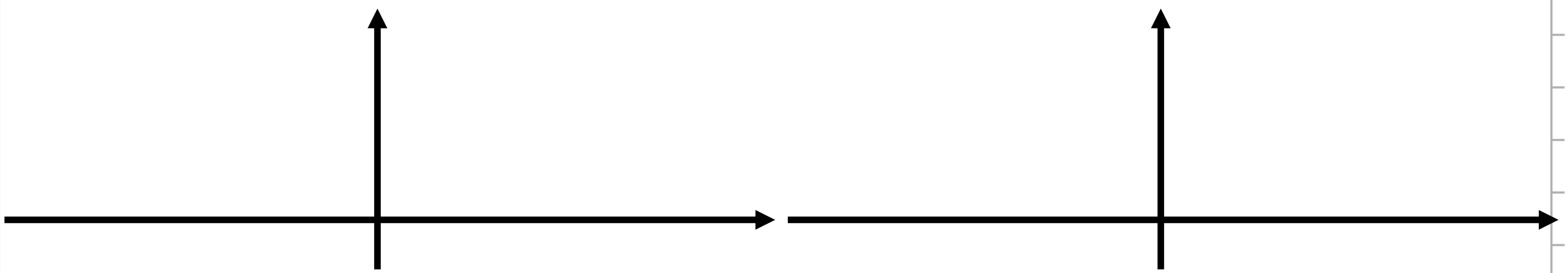
AM-Double Sideband/with carrier

Wideband FM:

- When m is large

$$y(t) = \cos\left(w_c t\right) \cos\left(m \sin(w_m t)\right) - \sin\left(w_c t\right) \sin\left(m \sin(w_m t)\right)$$

Periodic signals with fundamental frequency ω_m



- Magnitude of Spectrum of Wideband FM:

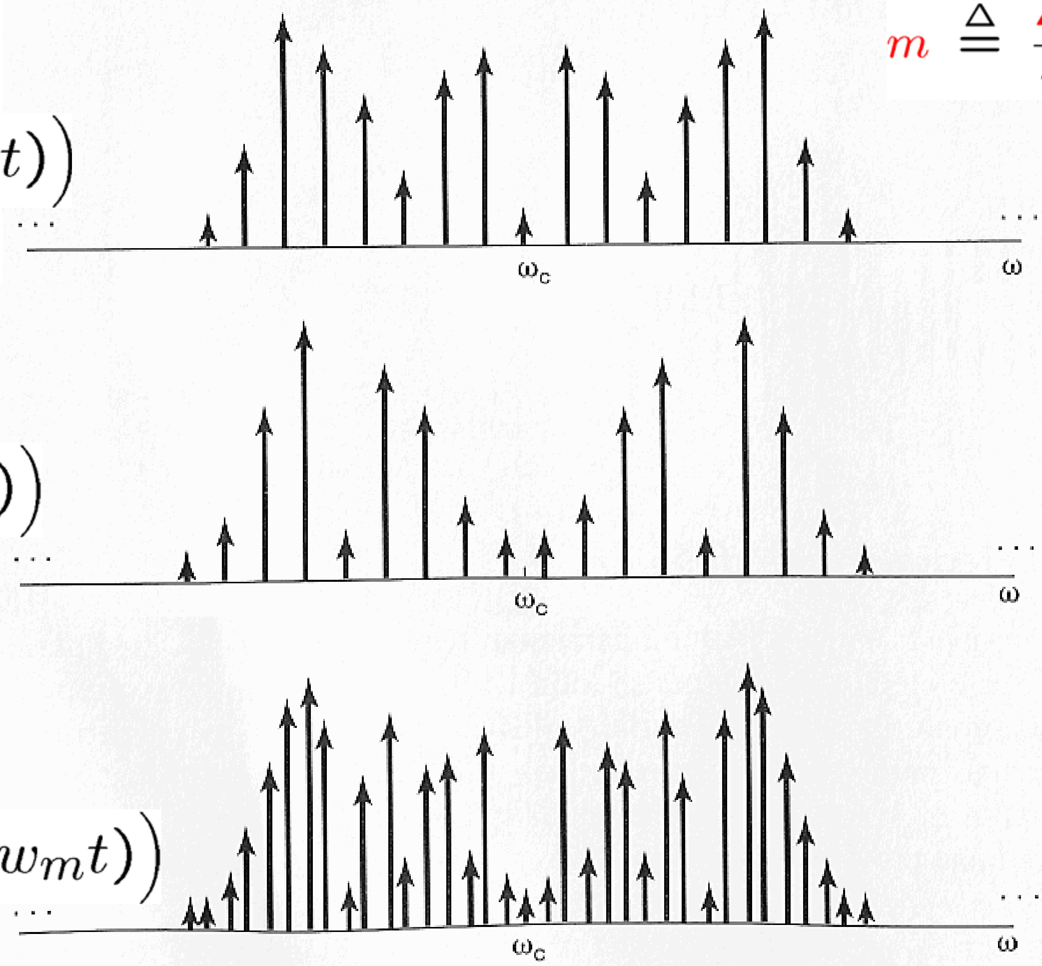
$$\Delta w \triangleq k_f \, A$$

$$m \triangleq \frac{\Delta w}{w_m}$$

$$\cos\left(w_c t\right)\ \cos\left(m \sin(w_m t)\right)$$

$$\sin\left(w_c t\right)\ \sin\left(m \sin(w_m t)\right)$$

$$y(t) = \cos\left(w_c t \ + \ m \sin(w_m t)\right)$$



$$\pm w_c \ + \ n \, w_m$$

$$\Rightarrow \ B \approx 2 \, m \, w_m \ = \ 2 \, k_f \, A \ = \ 2 \, \Delta w$$

Periodic Square-Wave Modulating Signal:

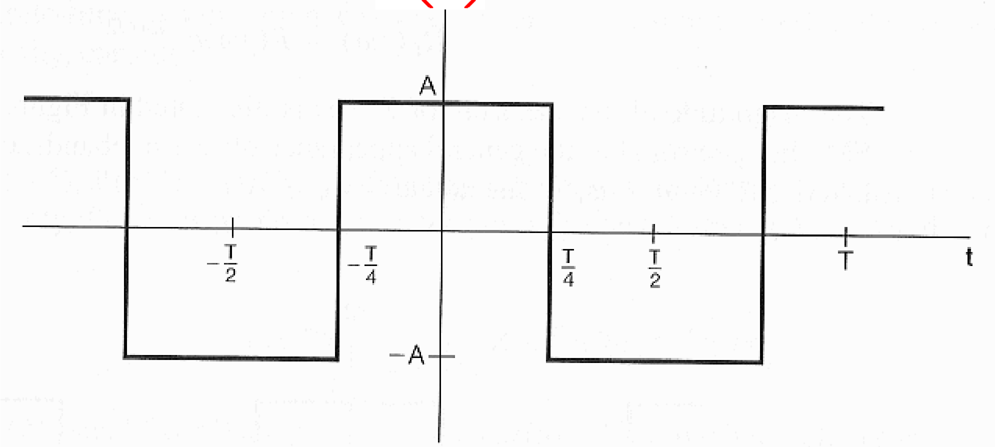
$$\Delta w \triangleq k_f A$$

$$m \triangleq \frac{\Delta w}{w_m}$$

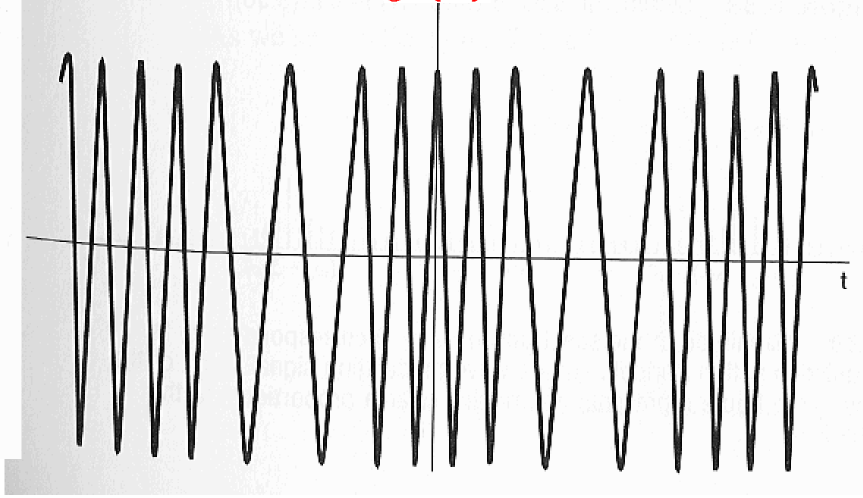
$$w_i(t) = w_c + k_f x(t) \qquad k_f = 1 \Rightarrow \Delta w = A$$

- When $x(t) > 0$, $w_i(t) = w_c + \Delta w$
- When $x(t) < 0$, $w_i(t) = w_c - \Delta w$

$x(t)$



$y(t)$

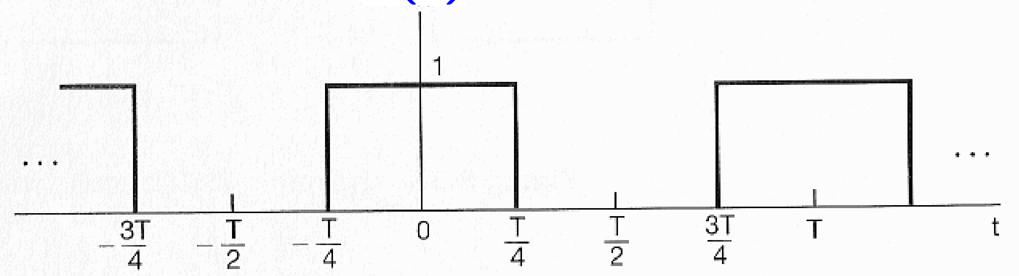


$$\Rightarrow \; y(t) = r(t)\cos\left((w_c + \Delta w)t\right) + r\left(t - \frac{T}{2}\right)\cos\left((w_c - \Delta w)t\right)$$

$$\Rightarrow \; Y(jw) = \frac{1}{2}\left[R\left(jw + jw_c + j\Delta w\right) + R\left(jw - jw_c - j\Delta w\right)\right]$$
$$+ \frac{1}{2}\left[R_T\left(jw + jw_c - j\Delta w\right) + R_T\left(jw - jw_c + j\Delta w\right)\right]$$

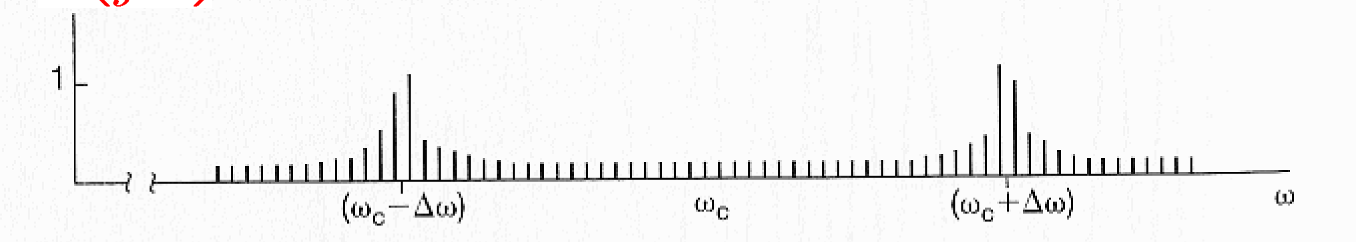
$$R\left(jw\right) = \sum_{k=-\infty}^{\infty} \frac{2}{2k+1}(-1)^k \delta\left(w - \frac{2\pi(2k+1)}{T}\right) + \pi\delta(w)$$

$r(t)$



$$R_T\left(jw\right) = R\left(jw\right)e^{-jwT/2}$$

$Y(jw)$

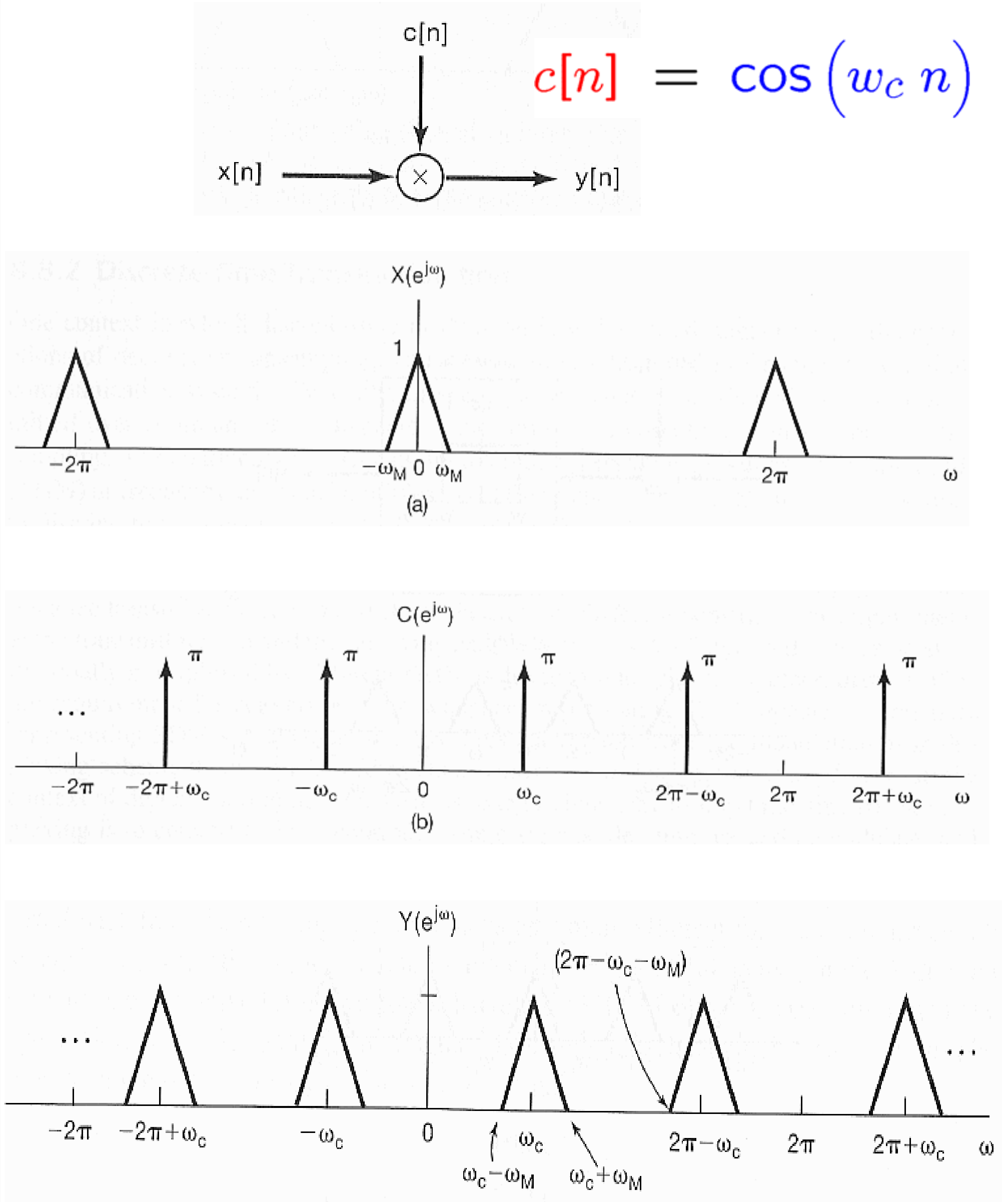


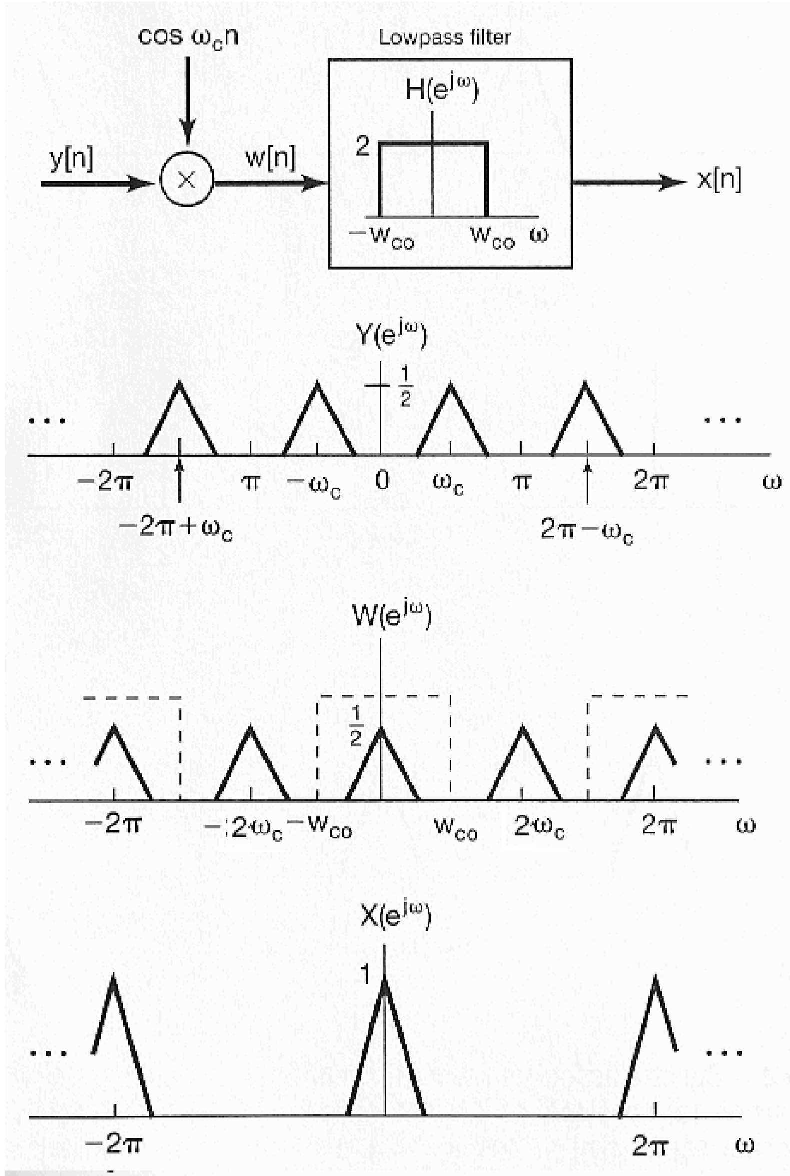
Outline

- Complex Exponential & Sinusoidal Amplitude Modulation & Demodulation
- Frequency-Division Multiplexing
- Single-Sideband Sinusoidal Amplitude Modulation
- Amplitude Modulation with a Pulse-Train Carrier
- Pulse-Amplitude Modulation
- Sinusoidal Frequency Modulation
- Discrete-Time Modulation

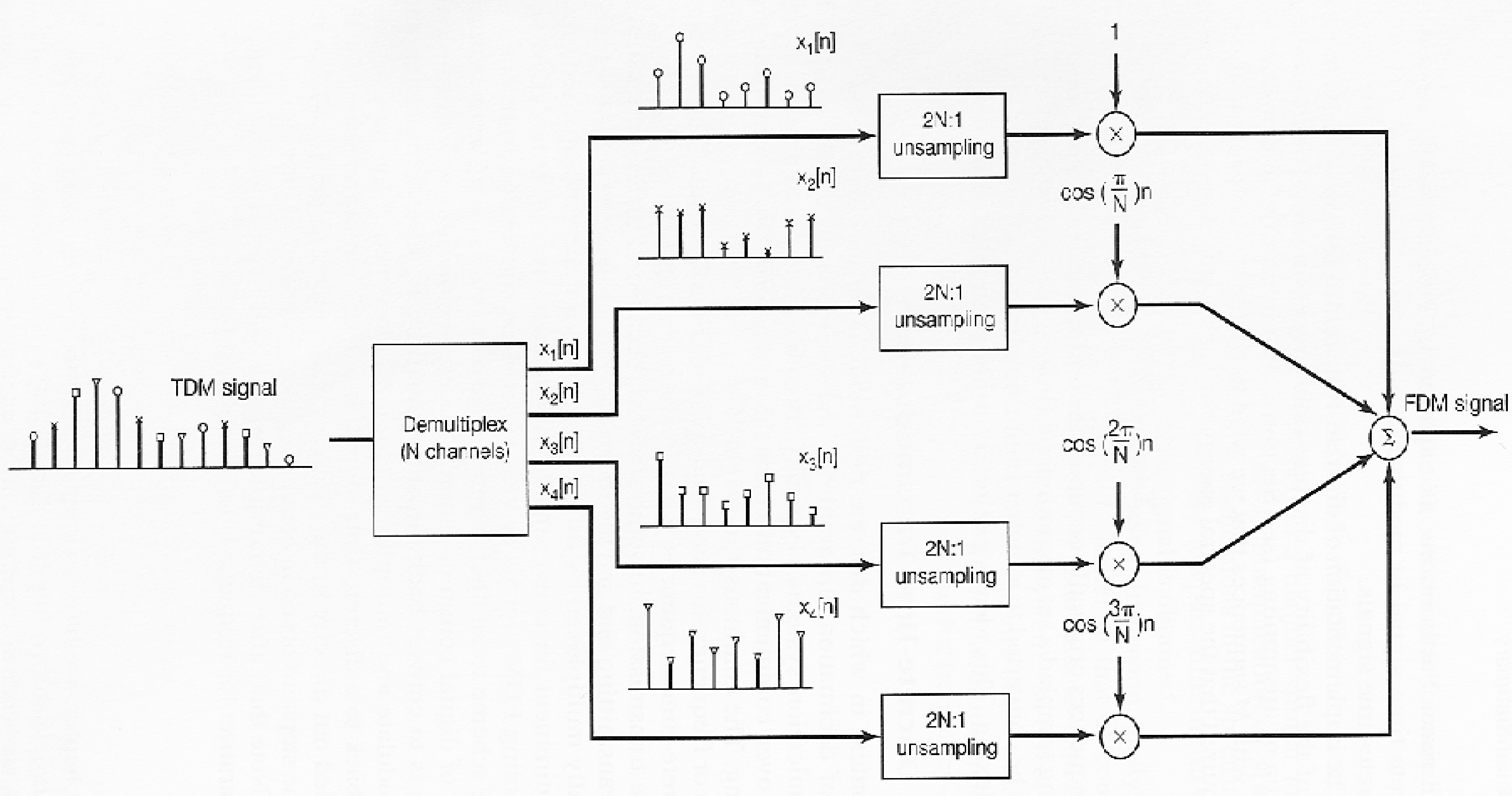
DT Sinusoidal AM:

$$c[n] = \cos(w_c\, n)$$

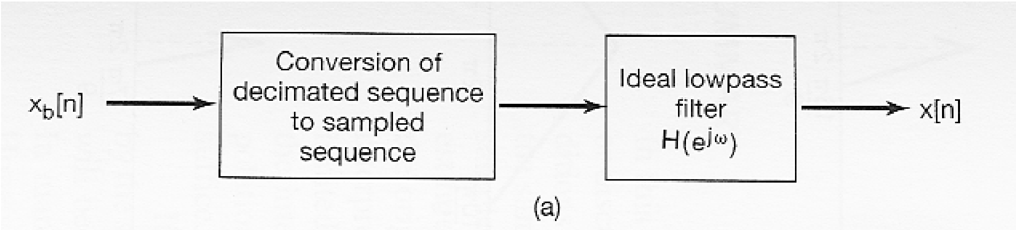


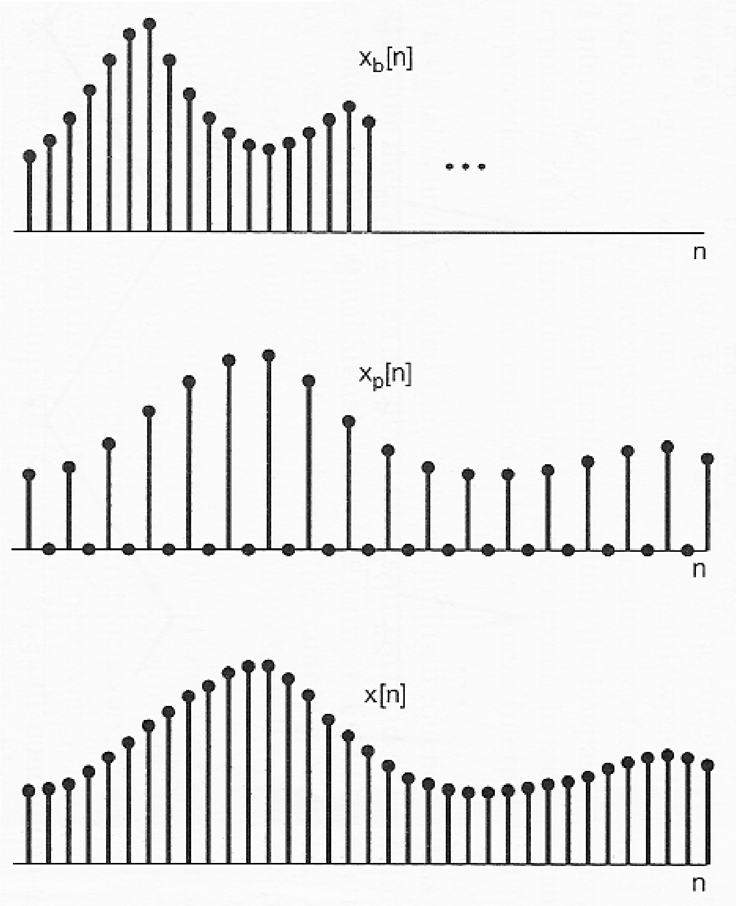


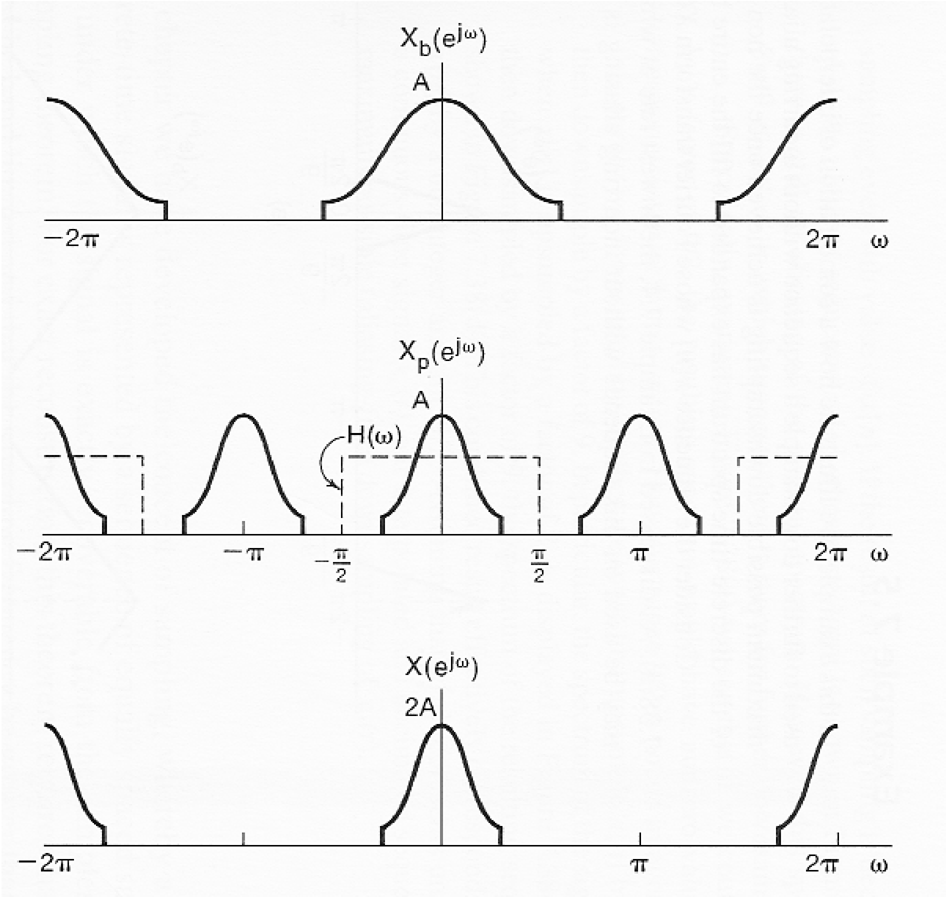
- Transmodulation or Transmultiplexing:
  - TDM to FDM



▪ Higher Equivalent Sampling Rate: Up-sampling







Spring 2011

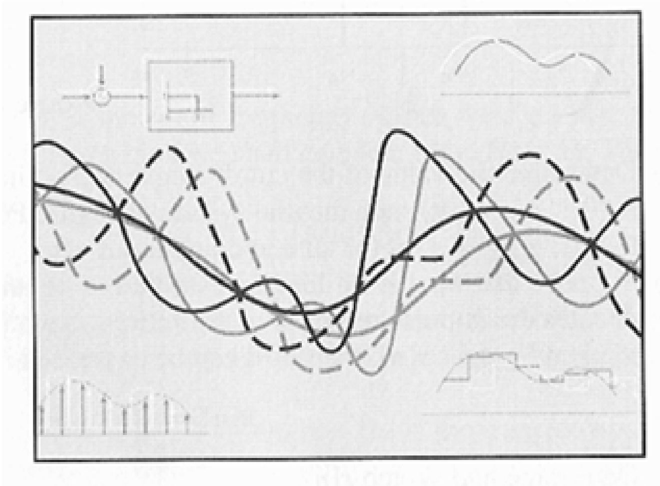
信號與系統
Signals and Systems

Chapter SS-7
Sampling

Feng-Li Lian

NTU-EE

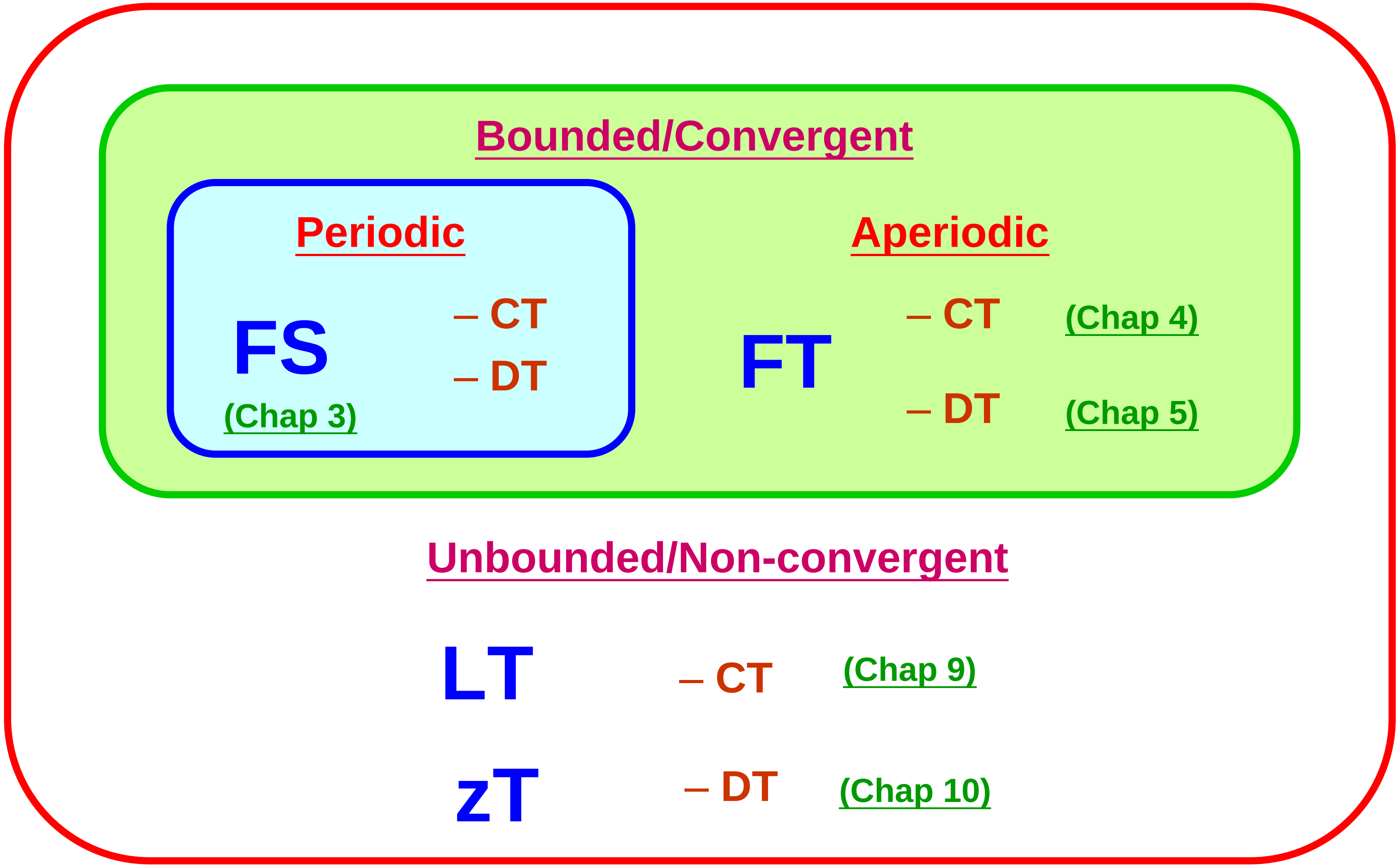
Feb11 – Jun11



Figures and images used in these lecture notes are adopted from
“Signals & Systems” by Alan V. Oppenheim and Alan S. Willsky, 1997

Flowchart

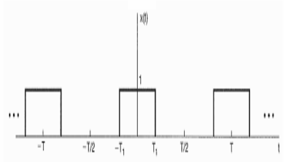
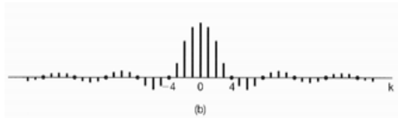
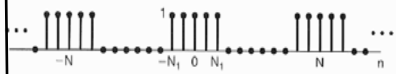
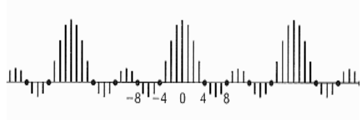
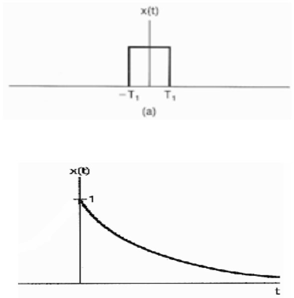
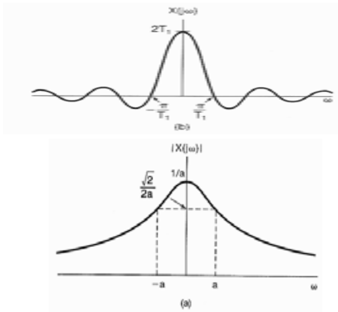
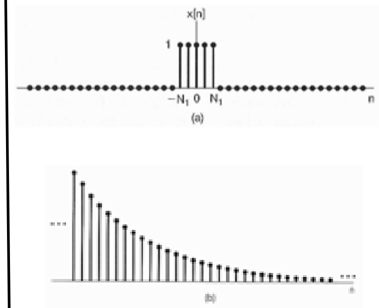
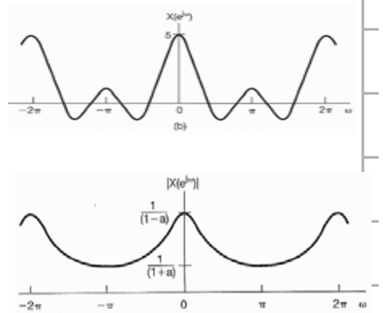
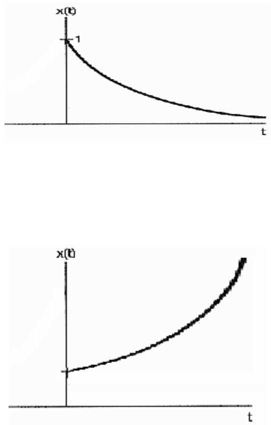
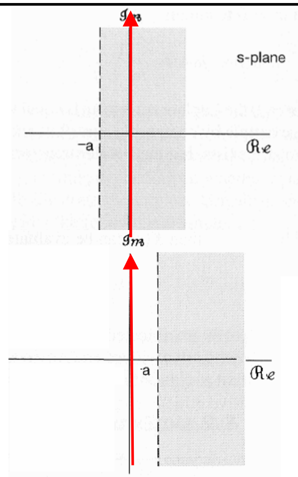
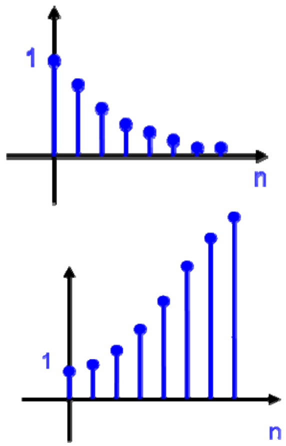
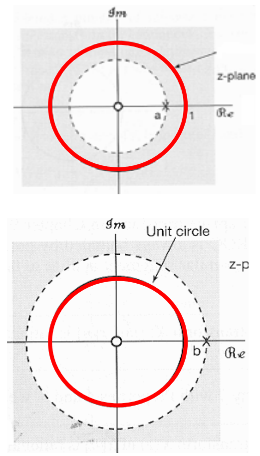
Introduction (Chap 1) LTI & Convolution (Chap 2)



Time-Frequency (Chap 6) Communication (Chap 8) Digital Signal Processing (dsp-8)

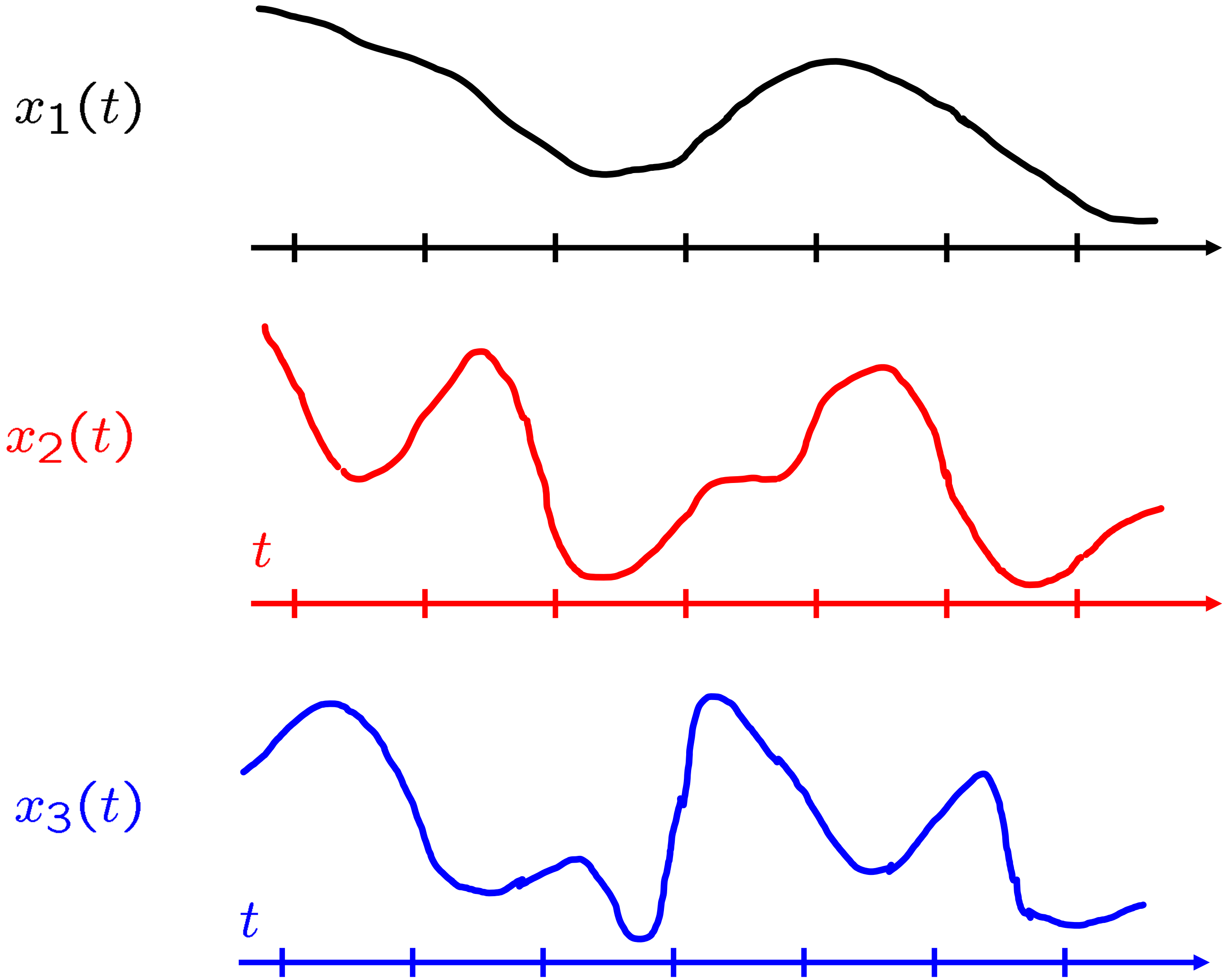
CT-DT (Chap 7) Control (Chap 11)

Fourier Series, Fourier Transform, Laplace Transform, z-Transform

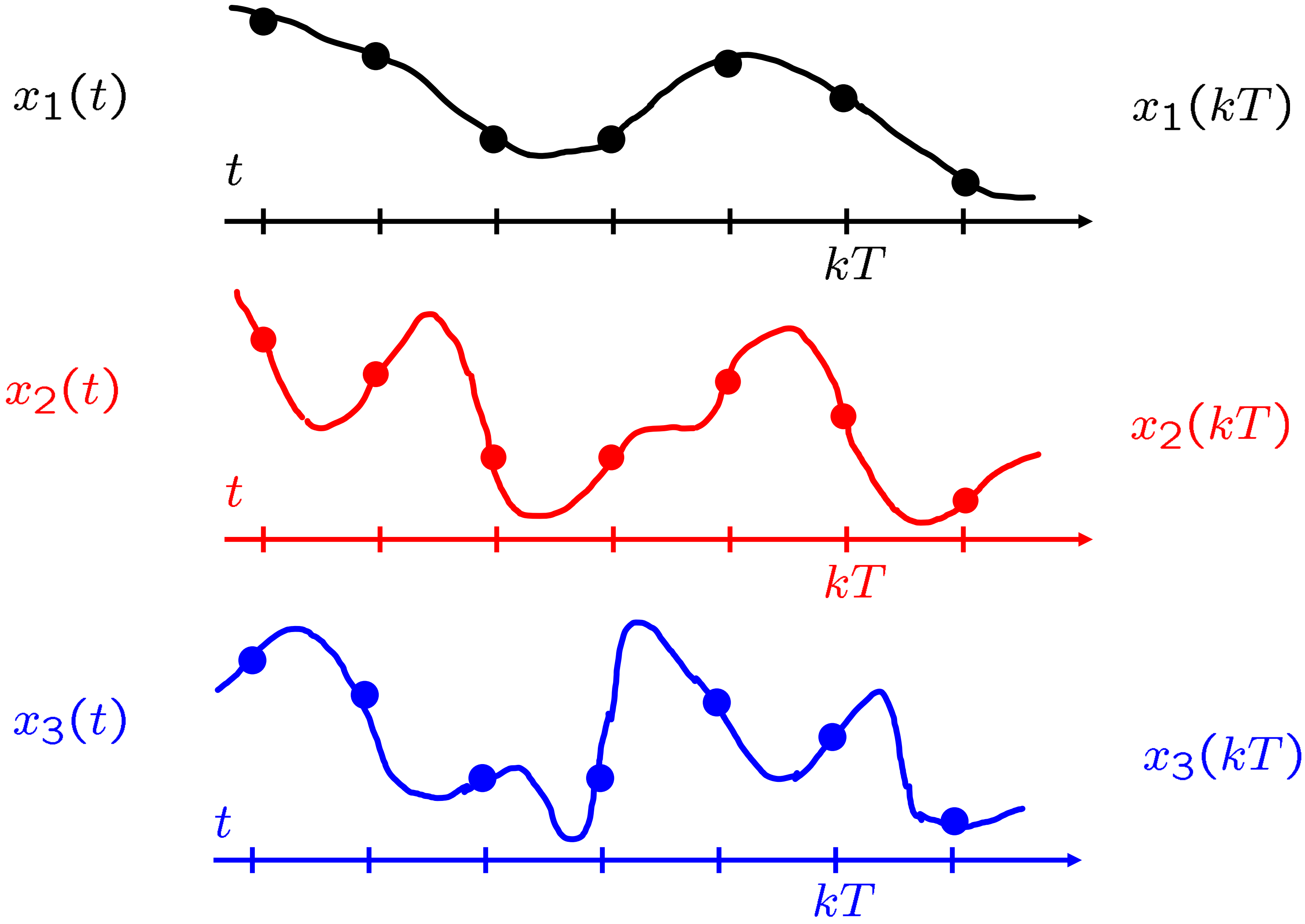
	CT		DT	
	time	frequency	time	frequency
FS				
FT				
LT/zT				

- Representation of of a Continuous-Time Signal by Its Samples: The Sampling Theorem
- Reconstruction of of a Signal from Its Samples Using Interpolation
- The Effect of Under-sampling: Aliasing
- Discrete-Time Processing of Continuous-Time Signals
- Sampling of Discrete-Time Signals

- Representation of CT Signals by its Samples

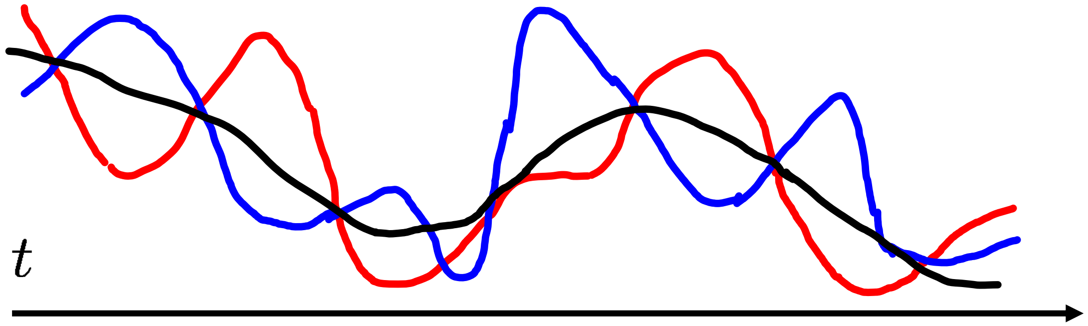


Representation of CT Signals by its Samples

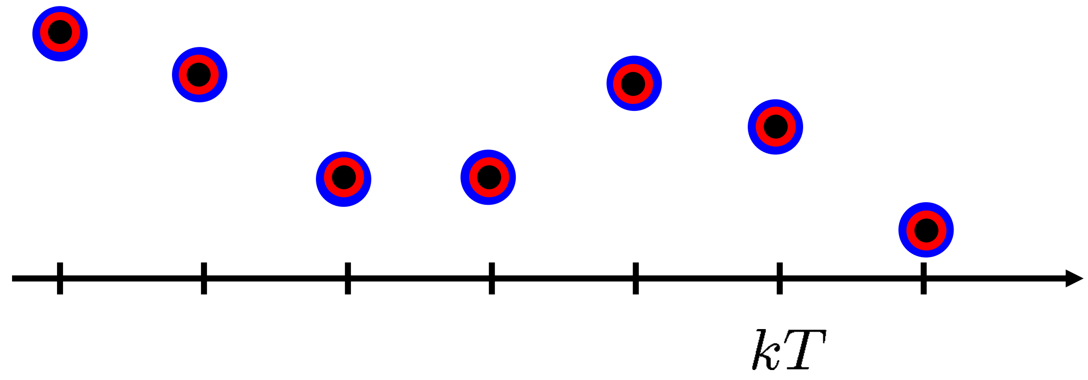


Representation of CT Signals by its Samples

$$x_1(t) \neq x_2(t) \neq x_3(t)$$



$$x_1(kT) = x_2(kT) = x_3(kT)$$

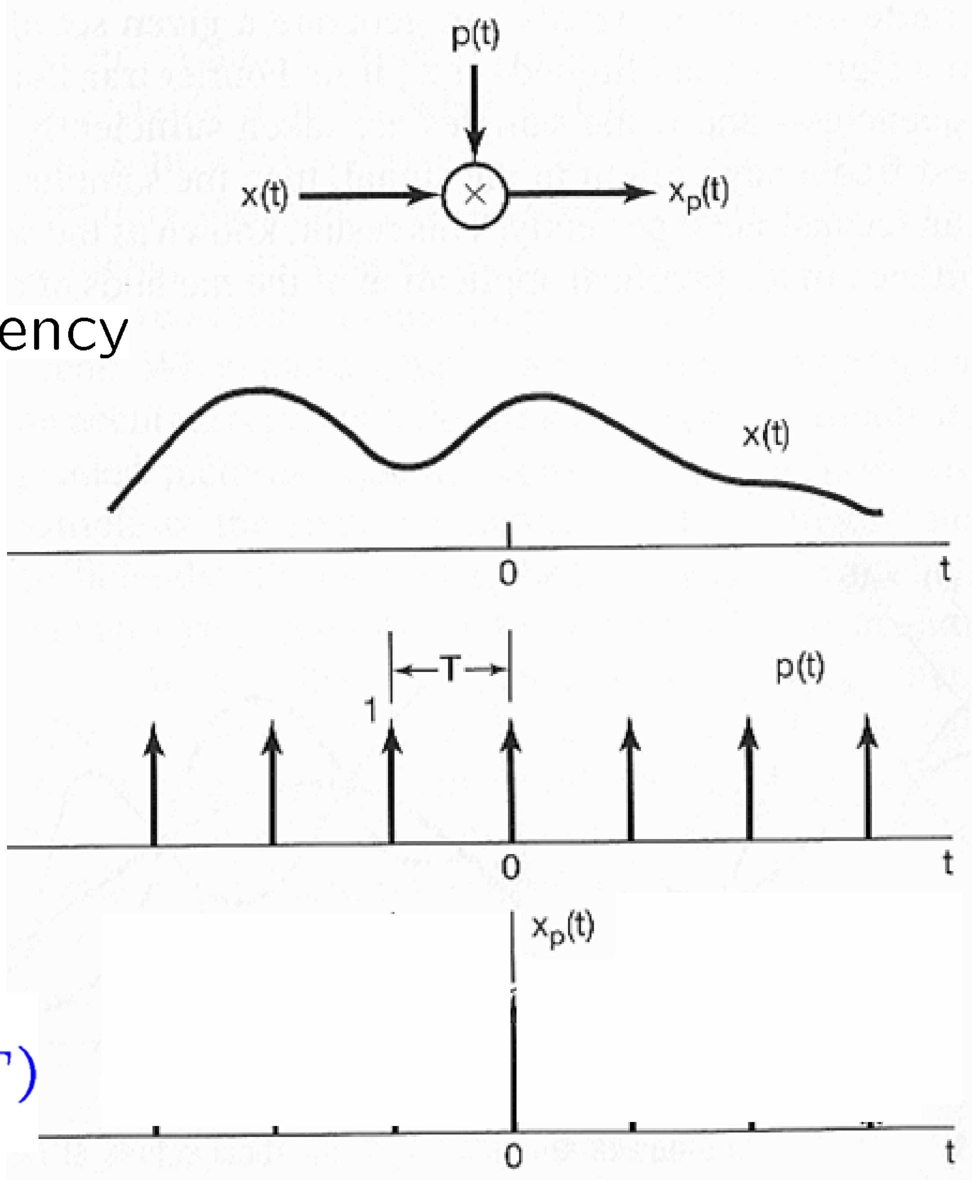


▪ Impulse-Train Sampling:

$p(t)$: sampling function

T : sampling period

$w_s = \dfrac{2\pi}{T}$: sampling frequency



$$\Rightarrow \; x_p(t) = x(t)\, p(t)$$

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT)\delta(t - nT)$$

Impulse-Train Sampling:

From multiplication property,

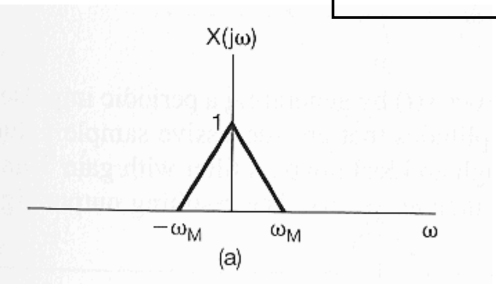
$$x_p(t) = x(t)\,p(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X_p(jw) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)P(j(w-\theta))d\theta$$

Eq 4.70, p. 322

$$= \frac{1}{T}\sum_{k=-\infty}^{+\infty} X(j(w-kw_s))$$

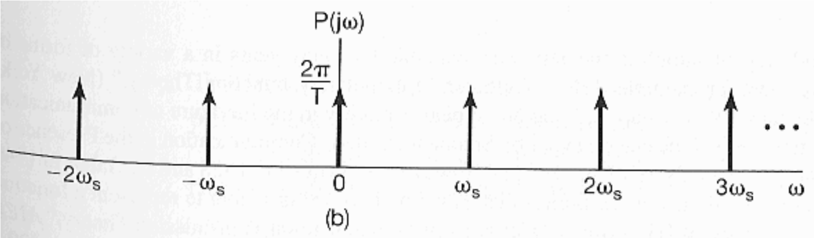
Ex 4.21, p. 323

$$x(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X(jw)$$



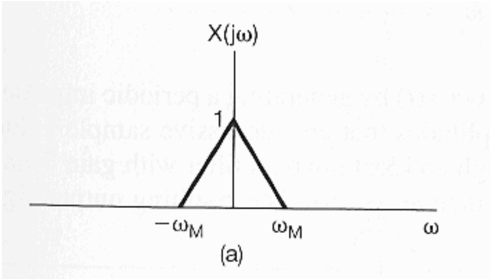
$$p(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad P(jw) = \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty} \delta(w-kw_s)$$

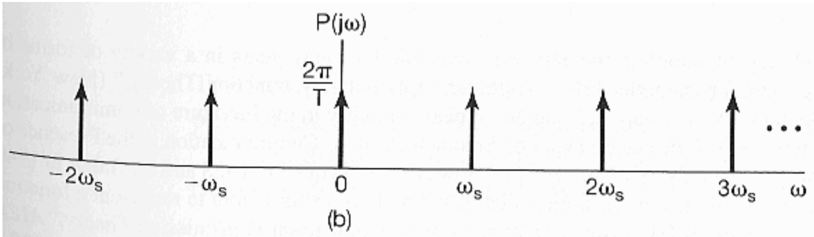
Ex 4.8, pp. 299-300



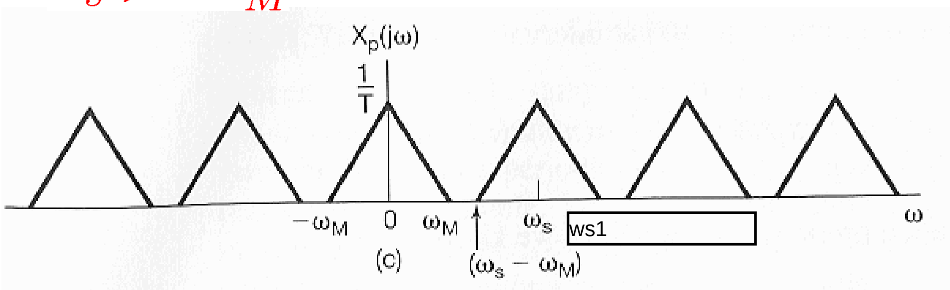
Impulse-Train Sampling:

Ex 4.21, 4.22, pp. 323-4

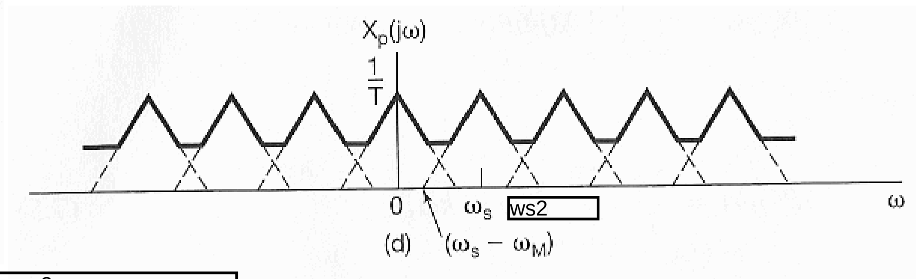




$w_s > 2w_M$



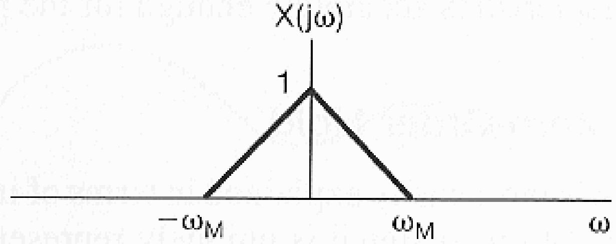
$w_s < 2w_M$



ws1 > ws2

aliasing

▪ The Sampling Theorem:



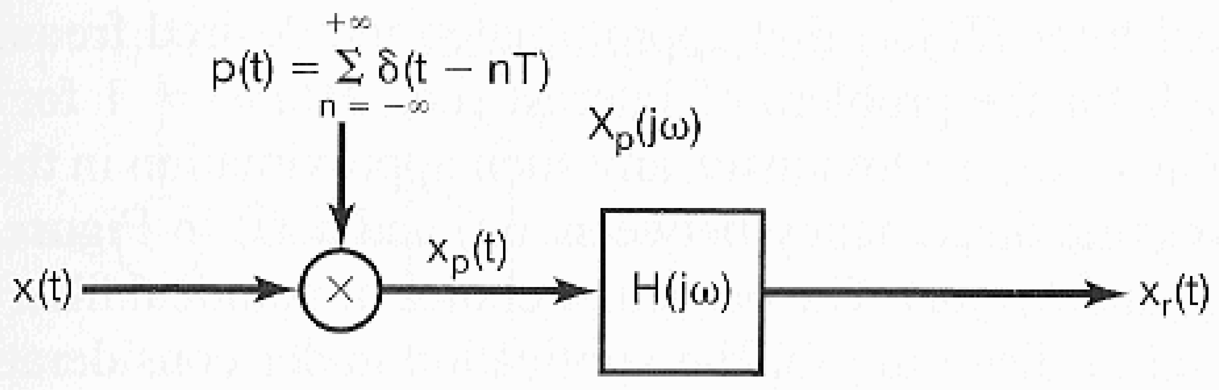
(b)

$x(t)$: a band-limited signal

with $X(jw) = 0$ for $|w| > w_M$

if $w_s > 2w_M$ where $w_s = \dfrac{2\pi}{T}$

$\Rightarrow$ $x(t)$ is uniquely determined by $x(nT), n = 0, \pm 1, \pm 2, \ldots,$

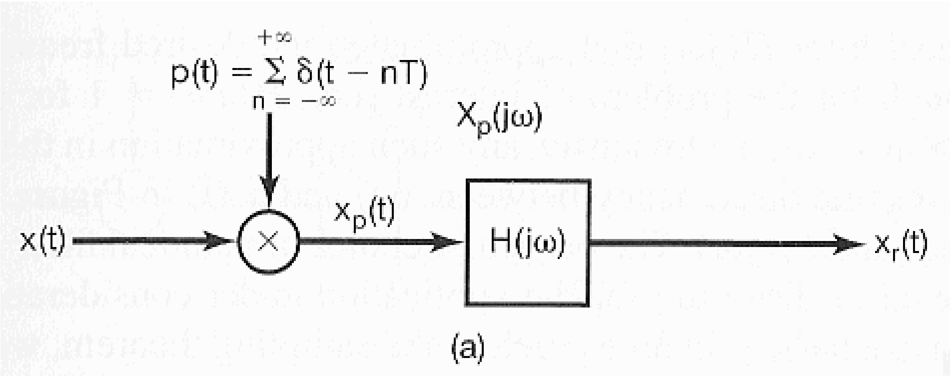


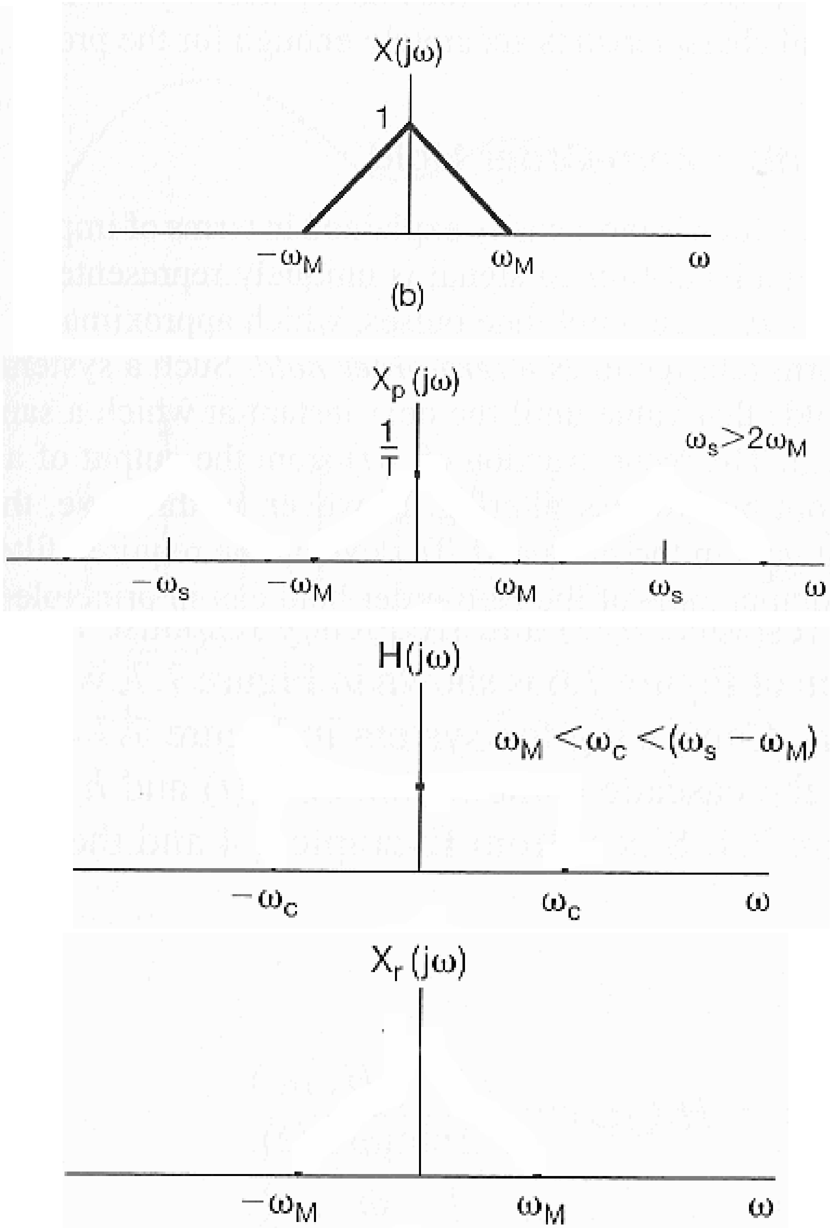
(a)

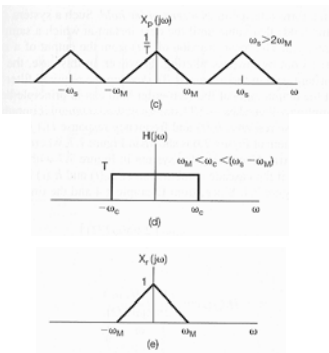
$\Rightarrow$ $2w_M$: Nyquist rate

w_M : Nyquist frequency

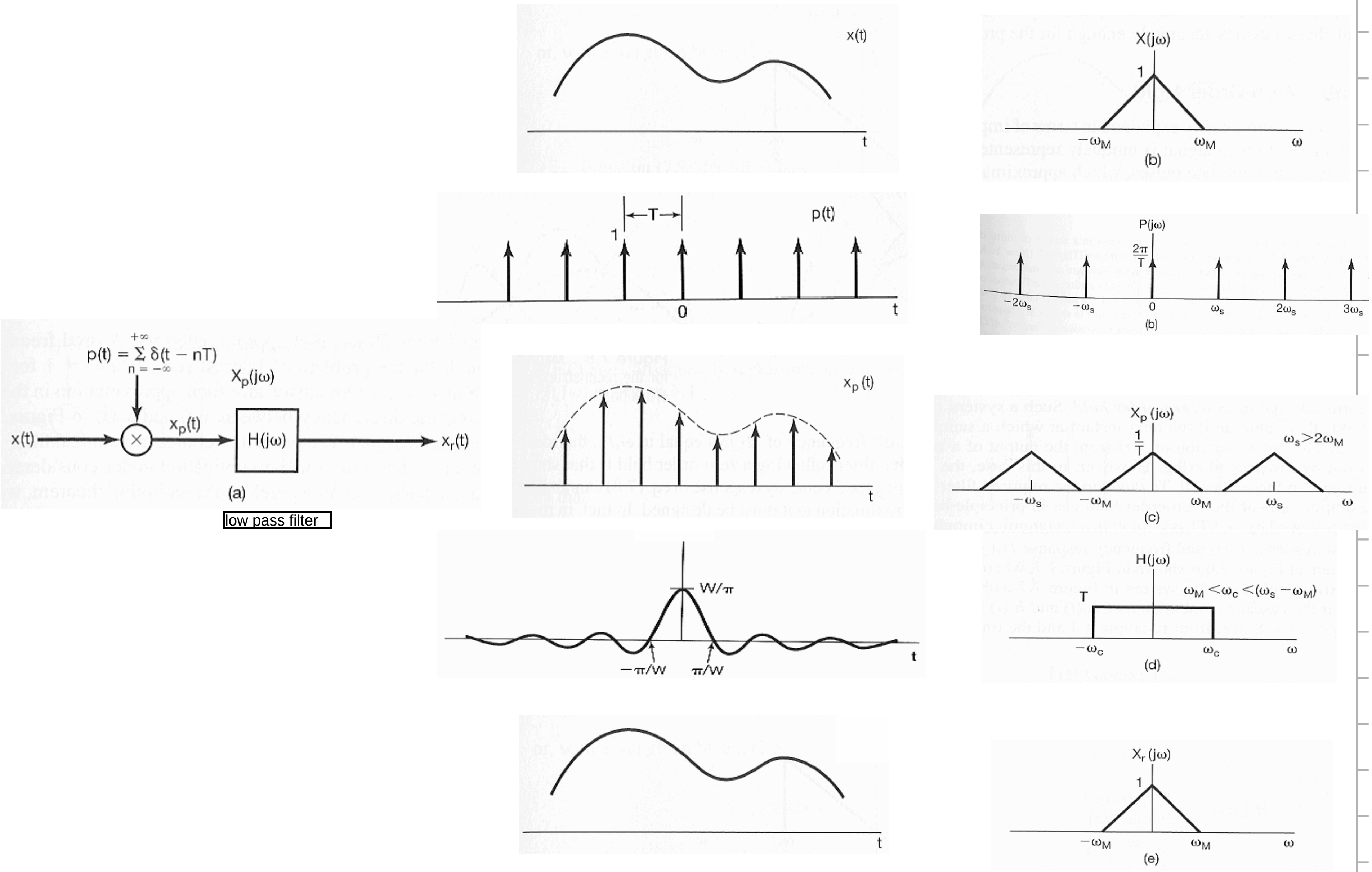
Exact Recovery by an Ideal Lowpass Filter:



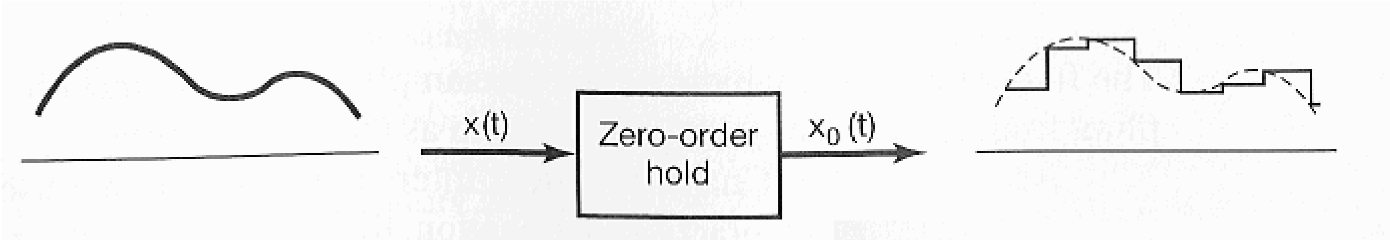


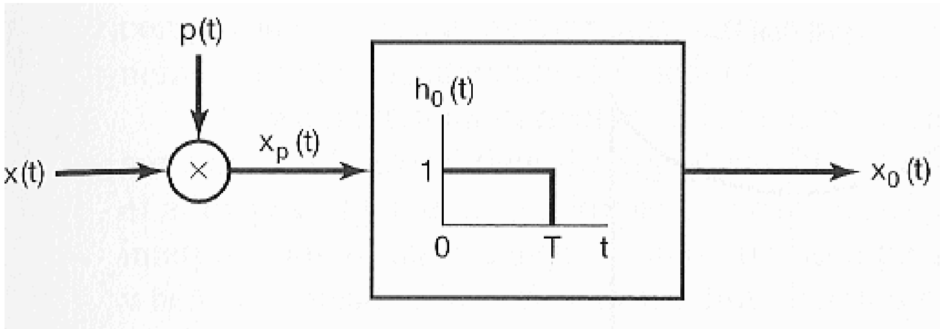


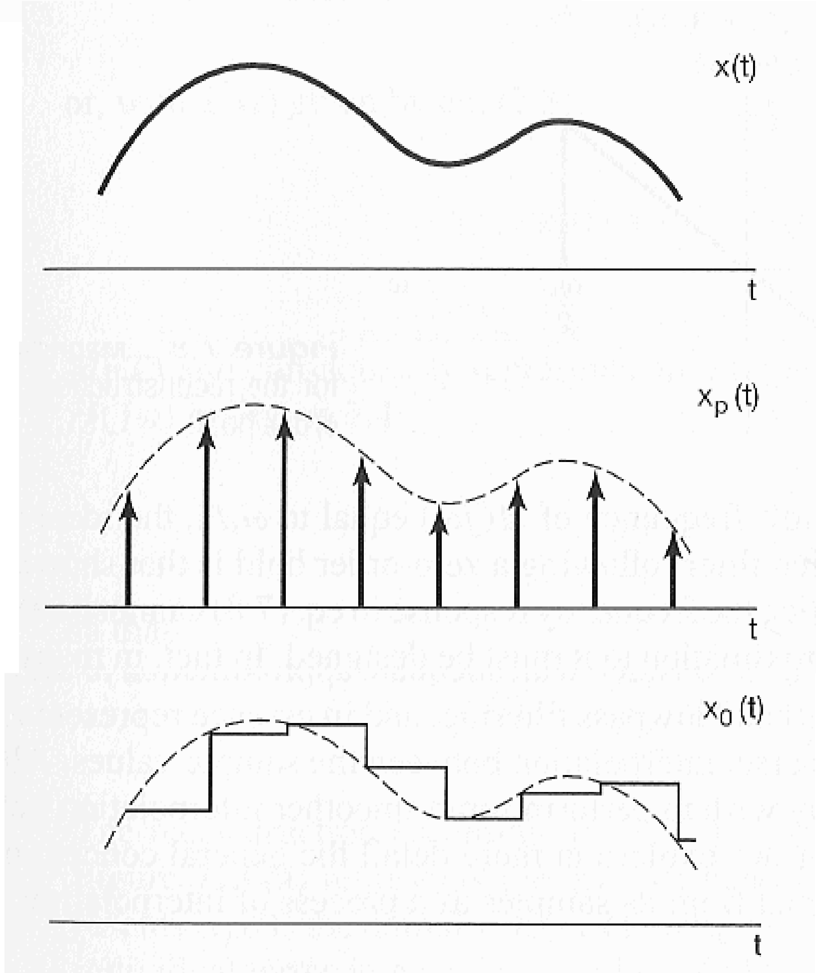
▪ Exact Recovery by an Ideal Lowpass Filter:



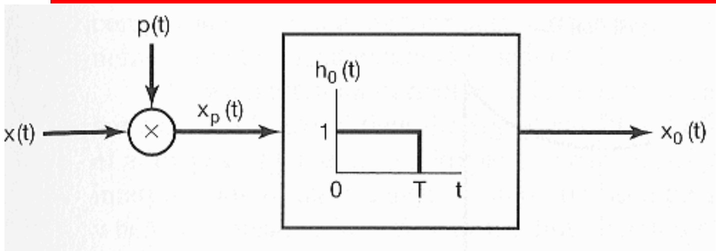
- Sampling with Zero-Order Hold:







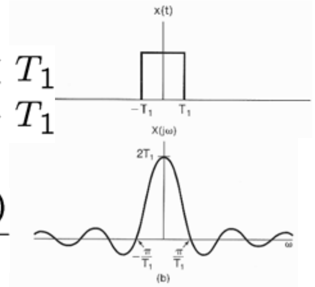
■ Sampling with Zero-Order Hold:



Ex 4.4, p. 293

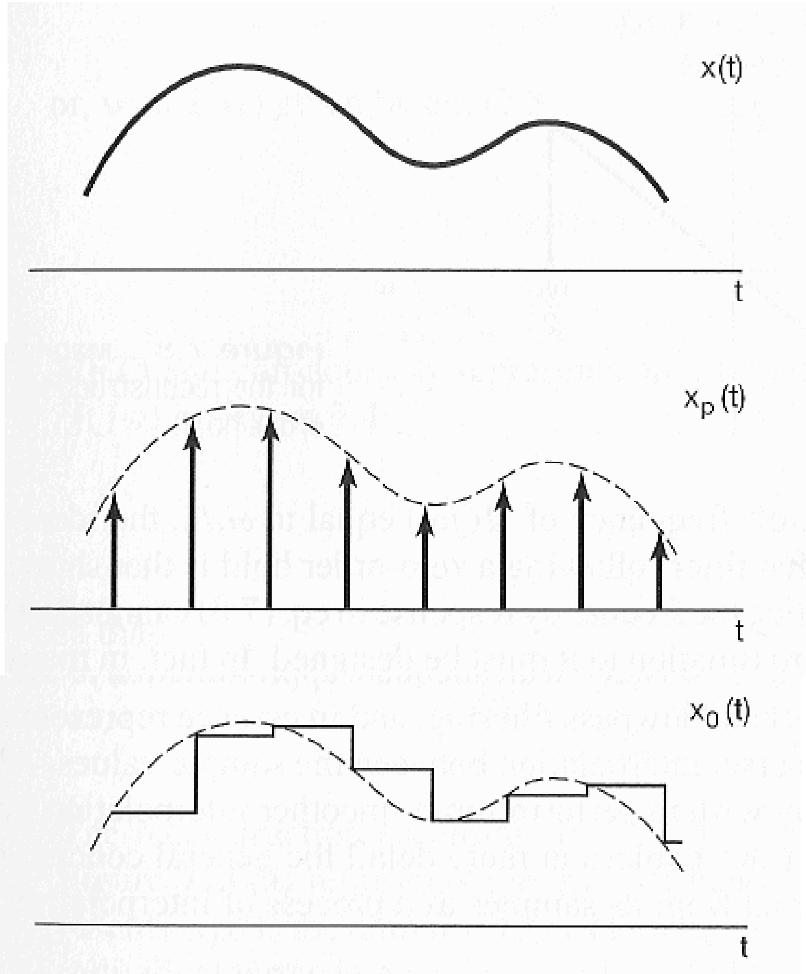
$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$

$$X(jw) = 2\frac{\sin(wT_1)}{w}$$

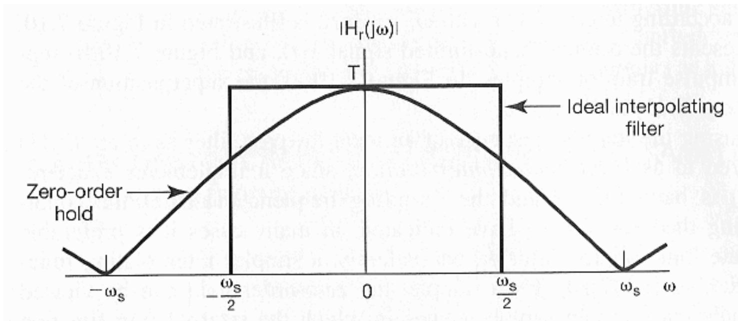


Eq 4.27, p. 301

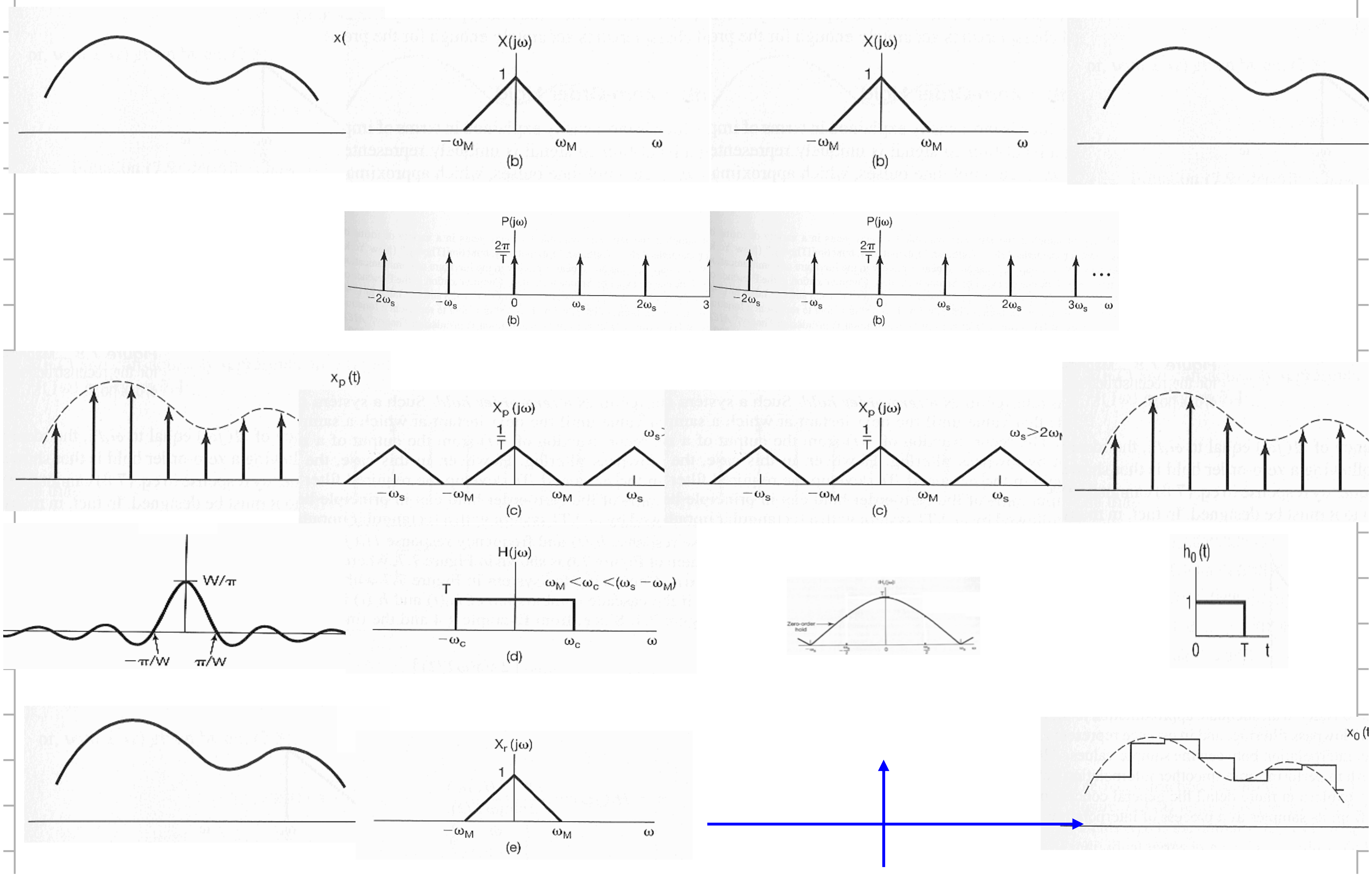
$$x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-jwt_0}X(jw)$$



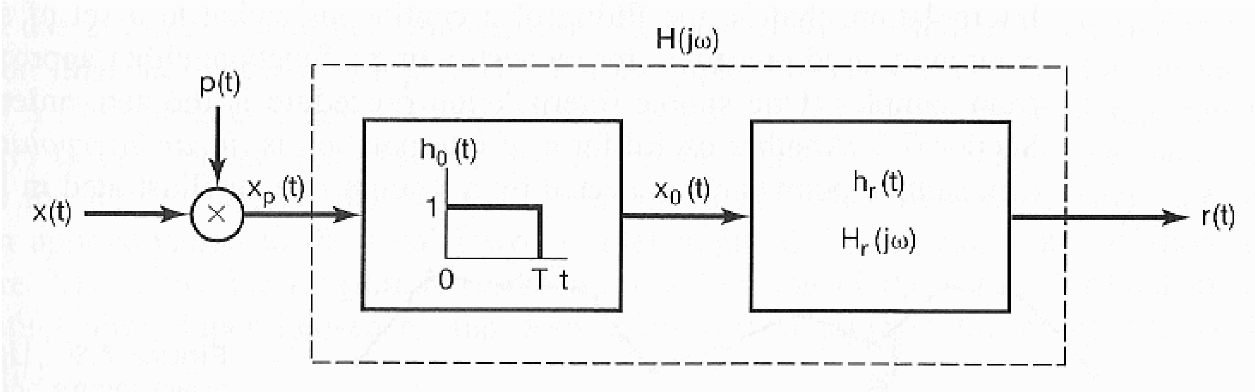
$$H_0(jw) = e^{-jwT/2}\left[\frac{2\sin(wT/2)}{w}\right]$$



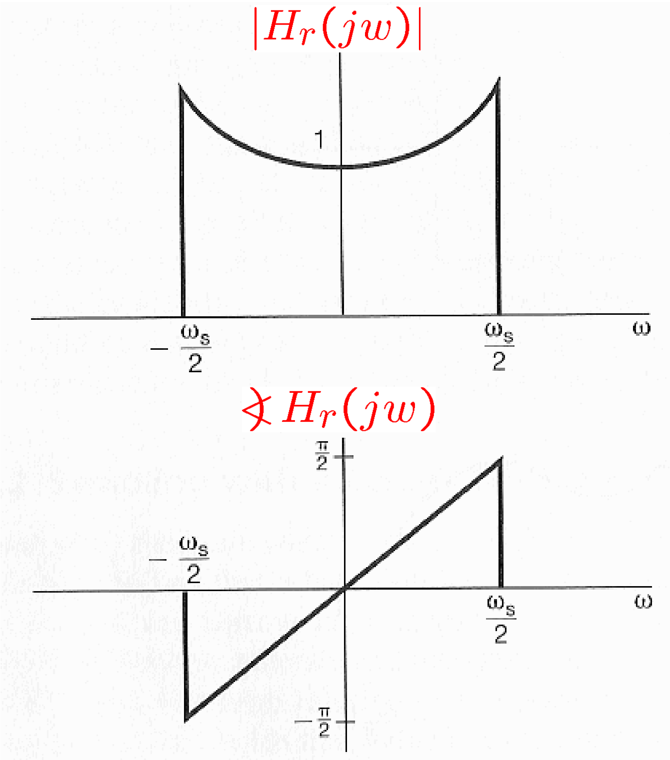
With Ideal Lowpass Filter & with Zero-Order Hold:



Sampling with Zero-Order Hold:



$$H_0(jw) = e^{-jwT/2}\left[\frac{2\sin(wT/2)}{w}\right]$$

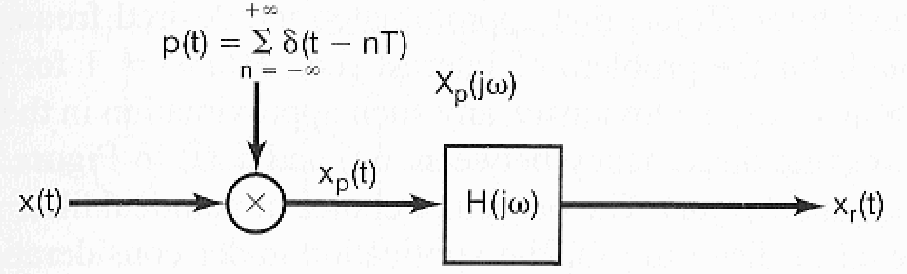


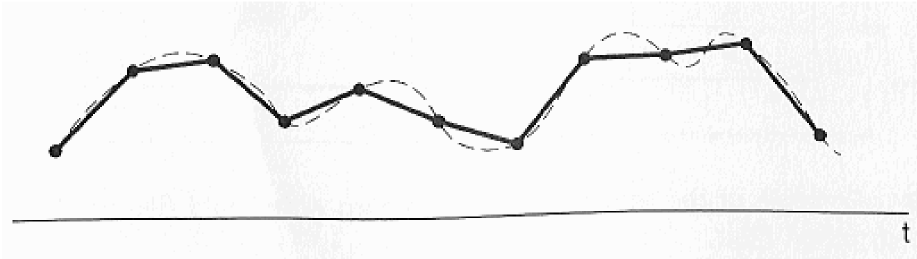
$$H(jw) = H_0(jw)H_r(jw)$$

$$\Rightarrow \quad H_r(jw) = \frac{e^{jwT/2}H(jw)}{\frac{2\sin(wT/2)}{w}}$$

- Representation of of a Continuous-Time Signal by Its Samples: The Sampling Theorem
- Reconstruction of of a Signal from Its Samples Using Interpolation
- The Effect of Under-sampling: Aliasing
- Discrete-Time Processing of Continuous-Time Signals
- Sampling of Discrete-Time Signals

- Exact Interpolation:





$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT)\delta(t - nT)$$

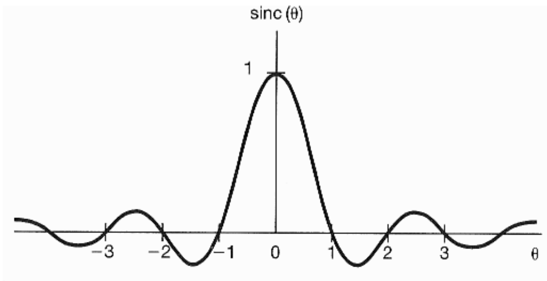
ideal lowpass filter

with a magnitude of T

$$x_r(t) = x_p(t) * h(t)$$

$$h(t) = T\,\frac{w_c}{\pi}\,\frac{\sin(w_c t)}{w_c t}$$

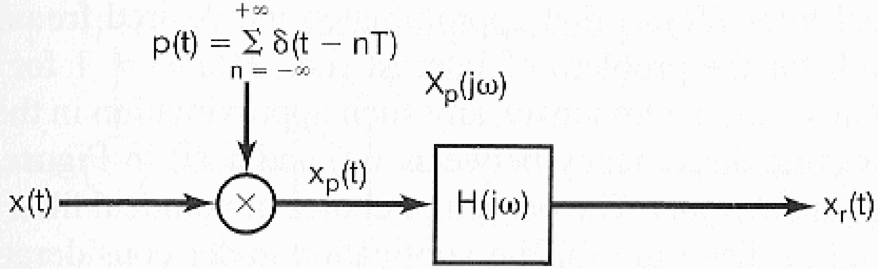
Ex 2.11, p. 110 $\quad x(t - t_0) = x(t) * \delta(t - t_0)$



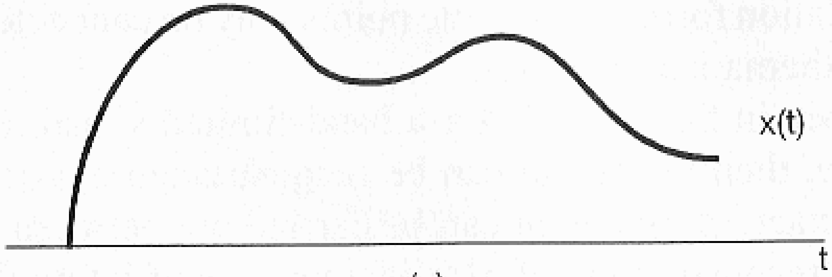
$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT)h(t - nT)$$

$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT)\,\frac{w_c T}{\pi}\,\frac{\sin(w_c(t - nT))}{w_c(t - nT)}$$

- Exact Interpolation:



(a)

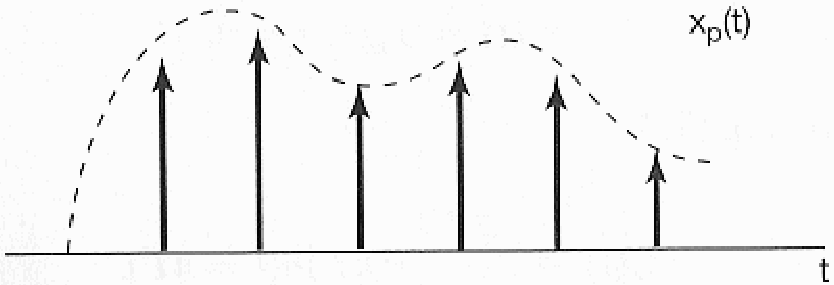


(a)

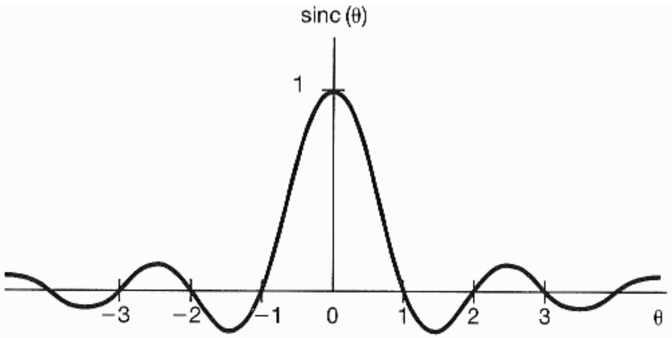
$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT)\,\frac{w_c T}{\pi}\,\frac{\sin(w_c(t-nT))}{w_c(t-nT)}$$

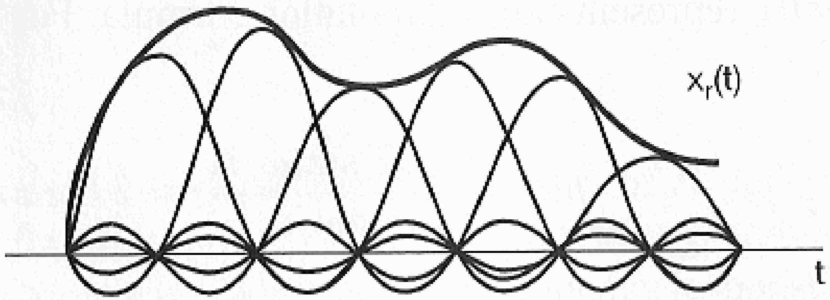
$$\frac{w_c}{\pi}\,\frac{2\pi}{w_s}\,\frac{\sin \pi(w_c(t-nT)/\pi)}{\pi w_c(t-nT)/\pi}$$

$$\frac{2w_c}{w_s}\,\mathrm{sinc}(w_c(t-nT)/\pi)$$



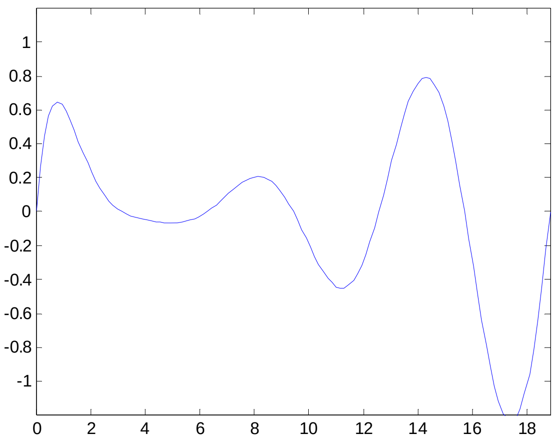
(b)

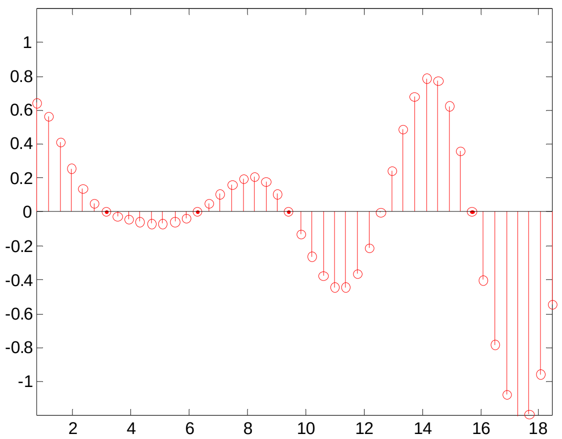


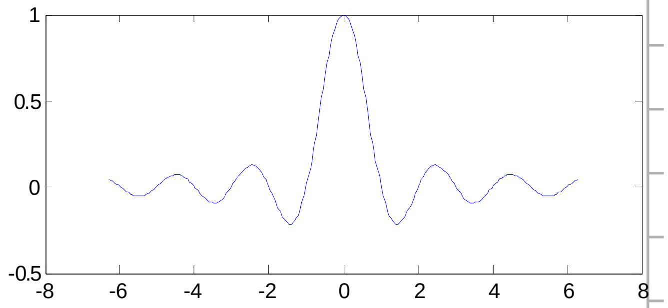


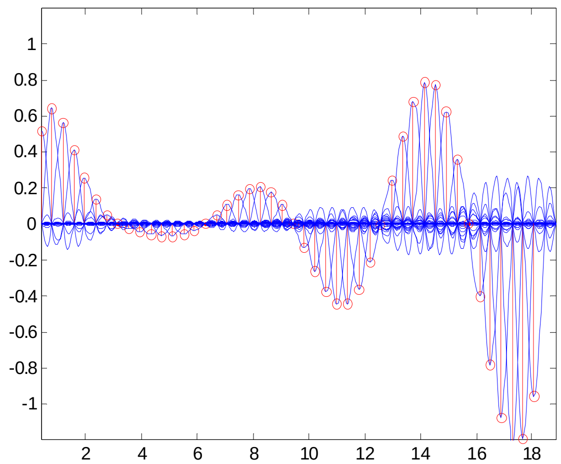
(c)

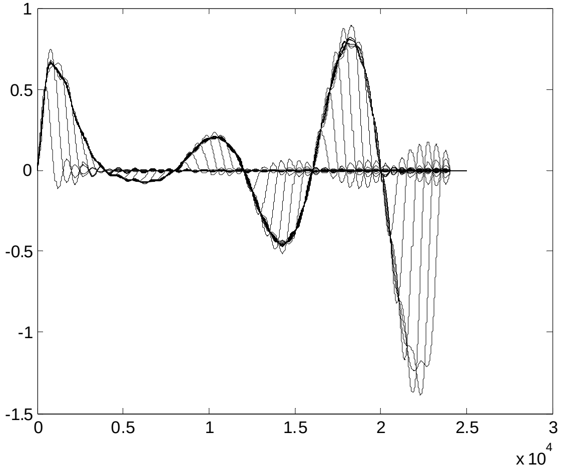
Reconstruction of a Signal from its Samples Using Interpolation

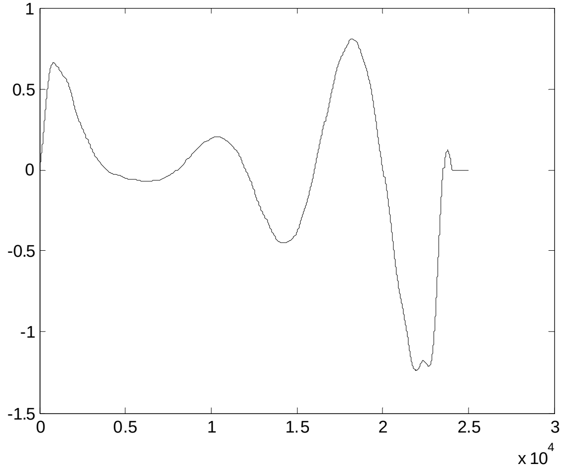




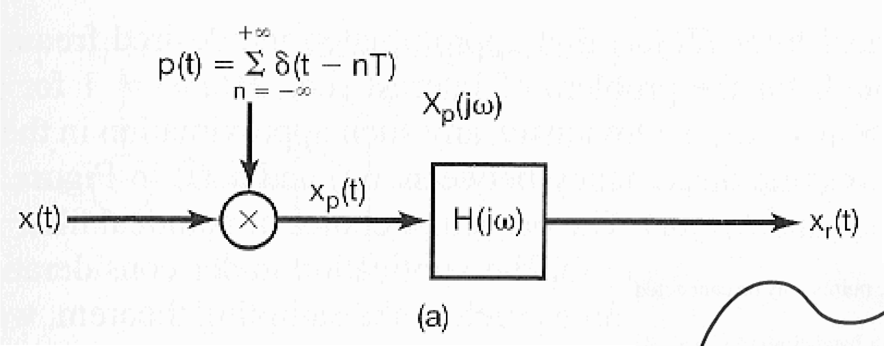


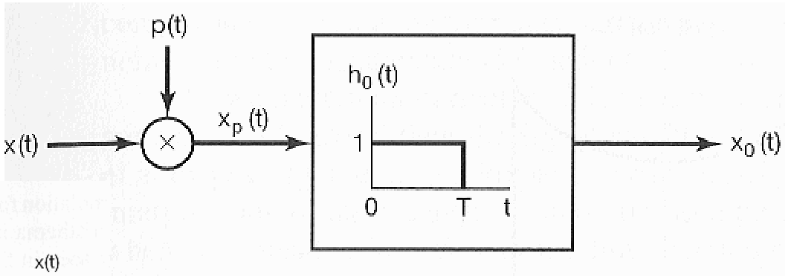


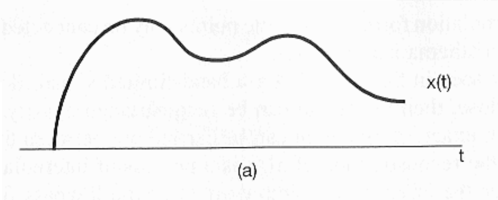


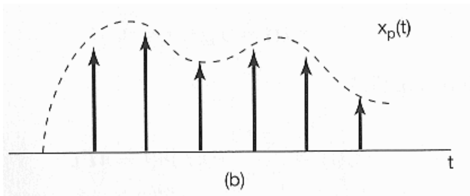


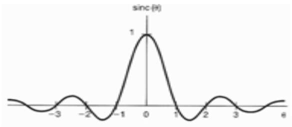
▪ Ideal Interpolating Filter & The Zero-Order Hold:

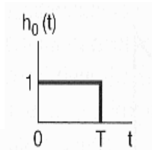


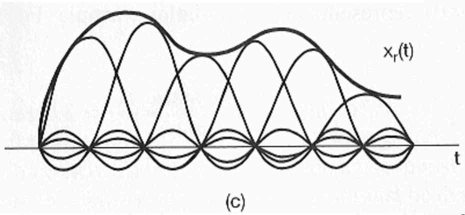


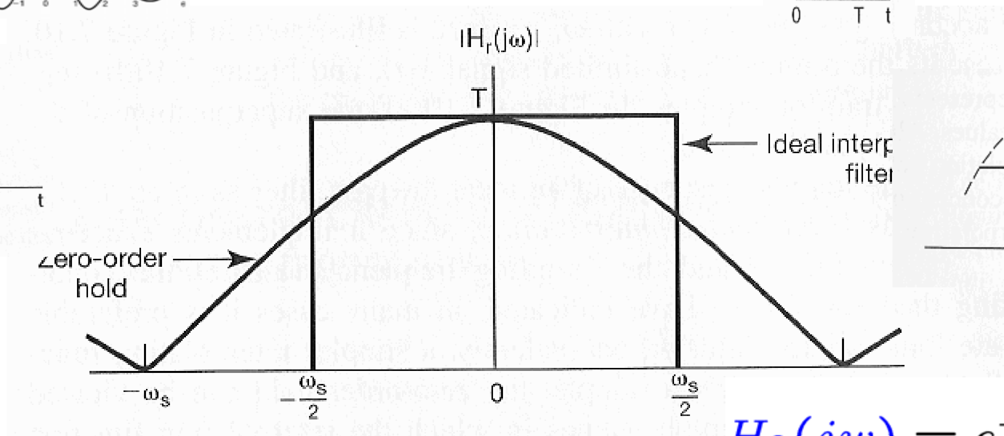


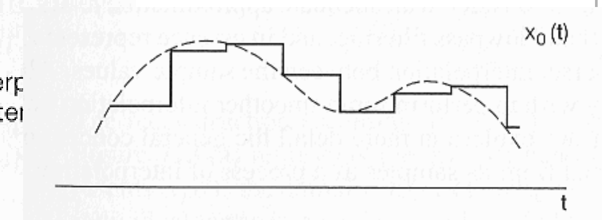










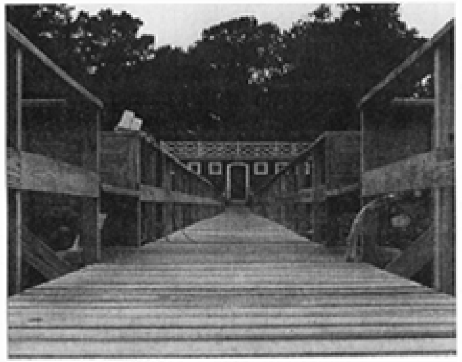


$$H_0(jw) = e^{-jwT/2}\left[\frac{2\sin(wT/2)}{w}\right]$$

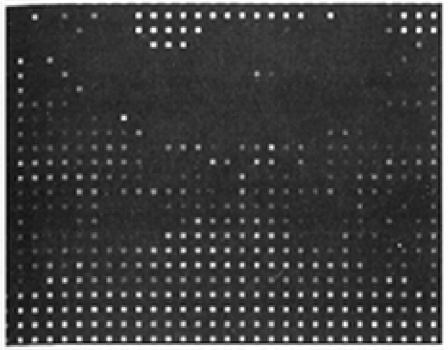
Sampling & Interpolation of Images:

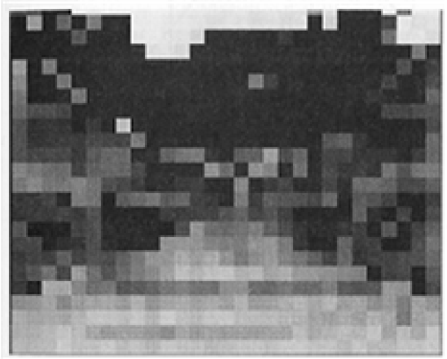
orignal image | impulse sampling | zero-order hold | zero-order hold

4 : 1

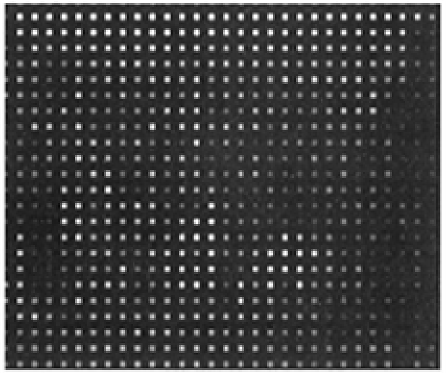


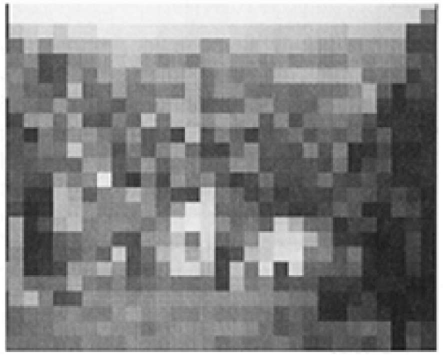
(a)



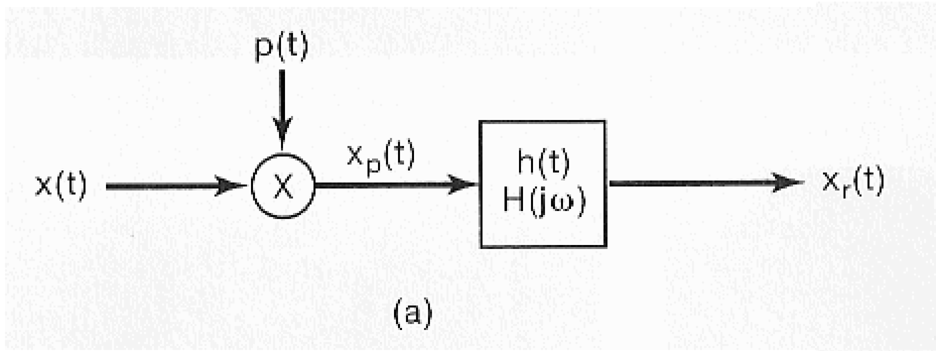


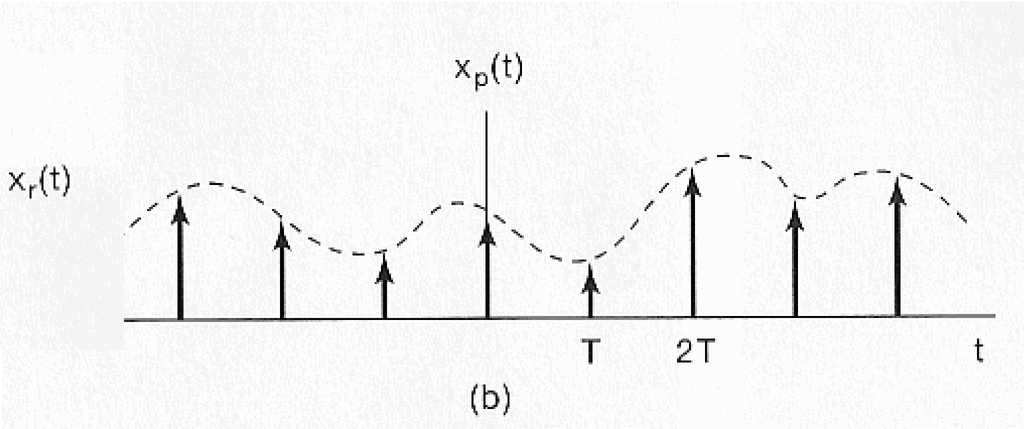
(g)

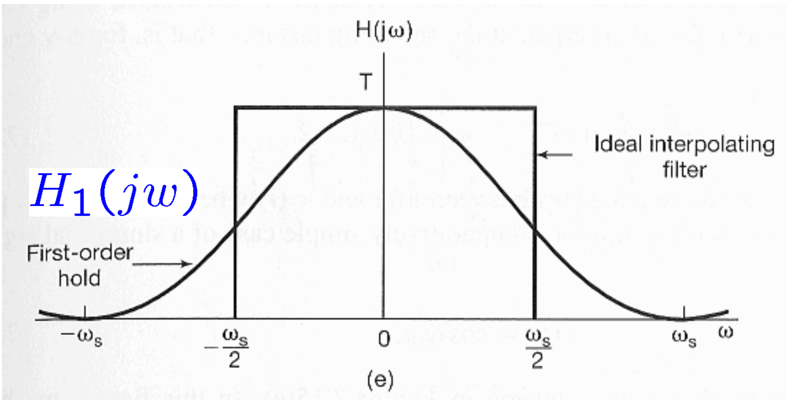


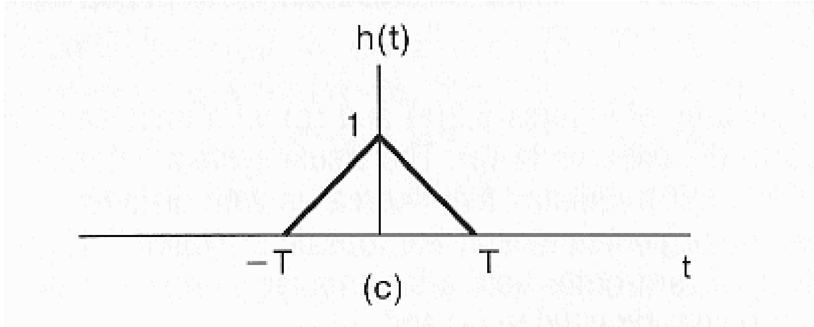


Higher-Order Holds:

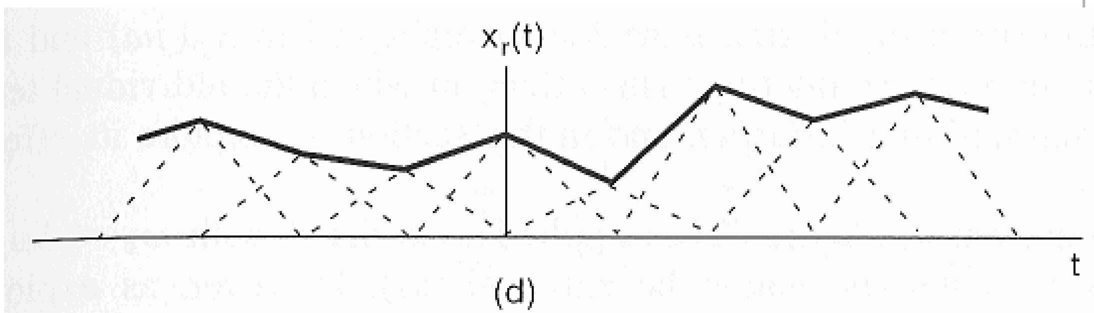








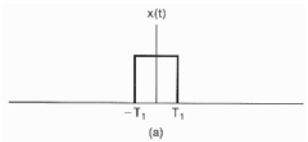
$$H_1(jw) = \frac{1}{T}\left[\frac{\sin(wT/2)}{w/2}\right]^2$$



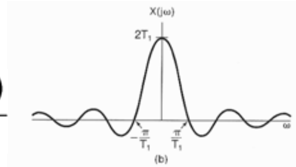
- Higher-Order Holds:

Ex 4.4, p. 293

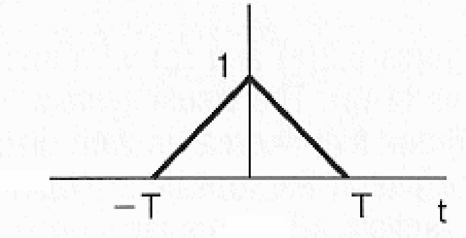
$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$



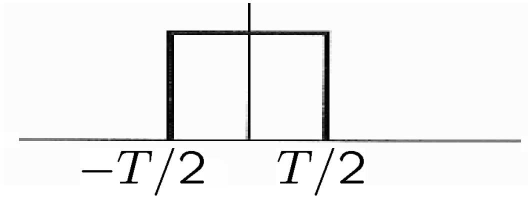
$$X(jw) = 2\frac{\sin(wT_1)}{w}$$

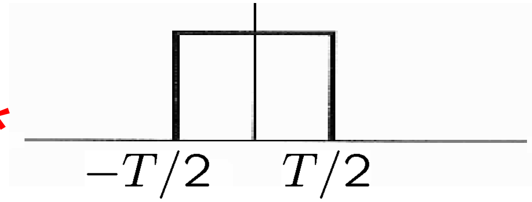


$h_1(t)$



$$= \frac{1}{T} \quad h_{sq}(t) \quad * \quad h_{sq}(t)$$

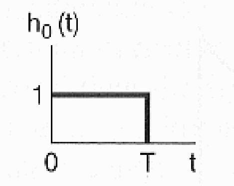




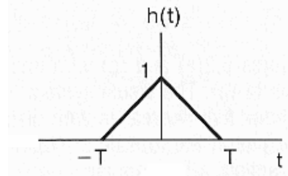
$$H_1(jw) = \frac{1}{T} \quad 2\frac{\sin(wT/2)}{w} \quad \text{X} \quad 2\frac{\sin(wT/2)}{w}$$

$$= \frac{1}{T}\left[\frac{\sin(wT/2)}{w/2}\right]^2$$

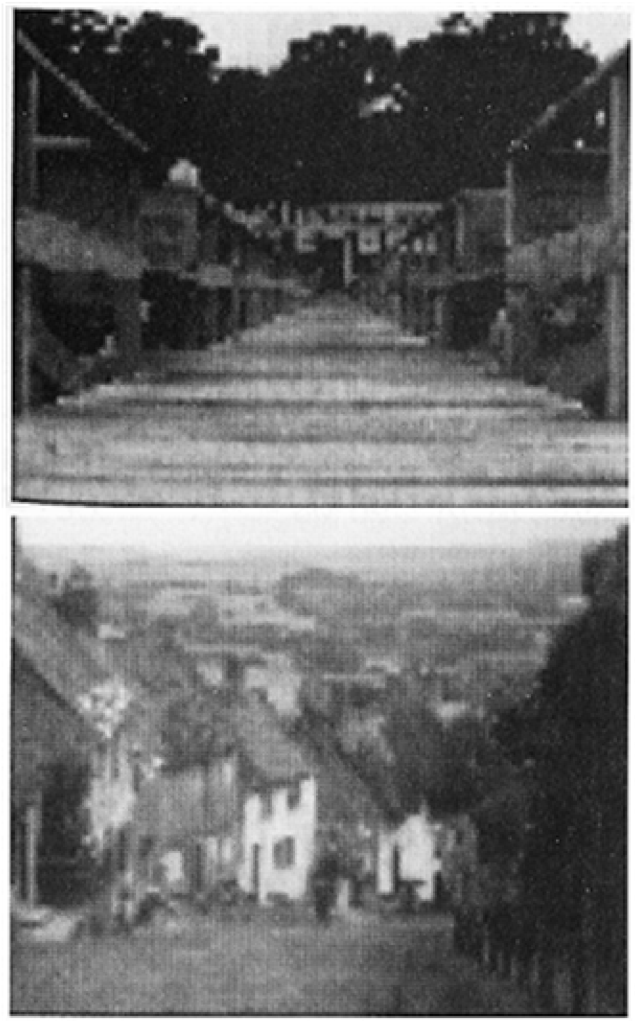
First-Order Hold on Image Processing:



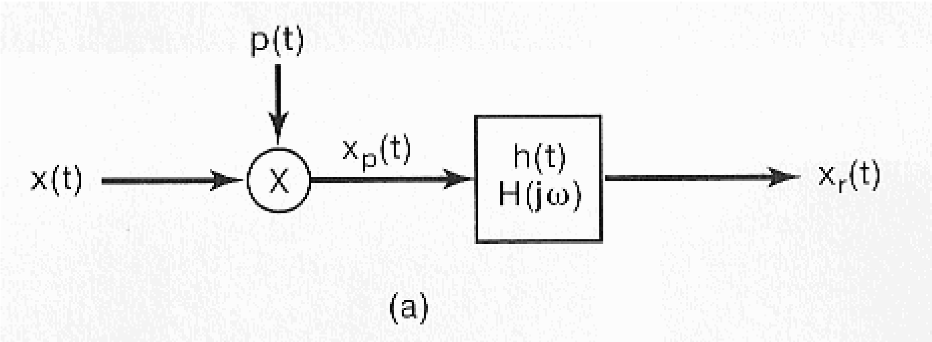
zero-order hold



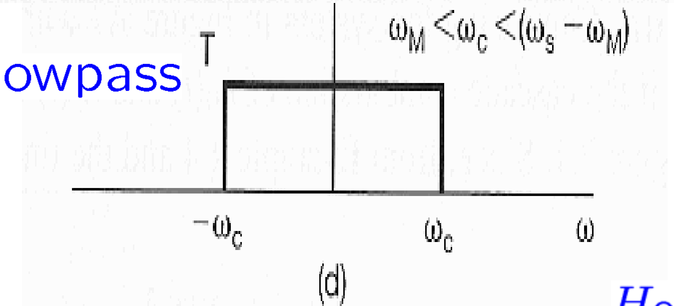
first-order hold

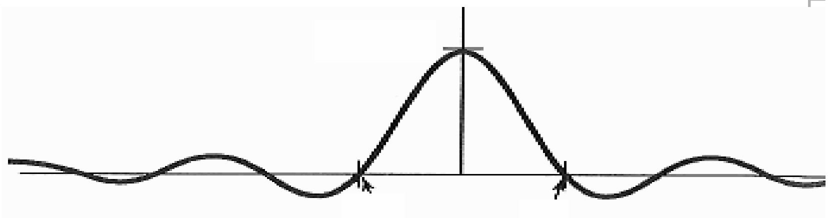


▪ Three Filters: Ideal Lowpass, Zero-Order, First-Order

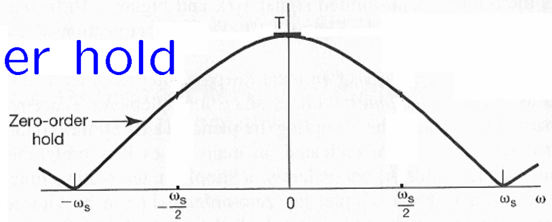


ideal lowpass

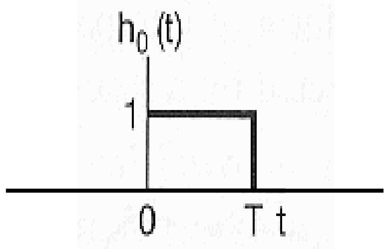




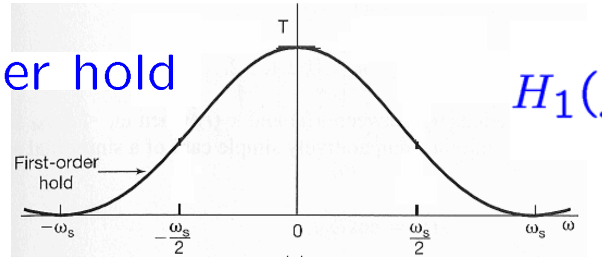
zero-order hold



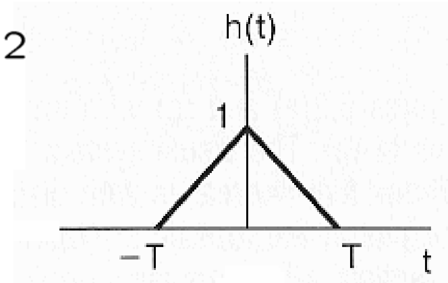
$$H_0(jw) = e^{-jwT/2}\left[\frac{2\sin(wT/2)}{w}\right]$$



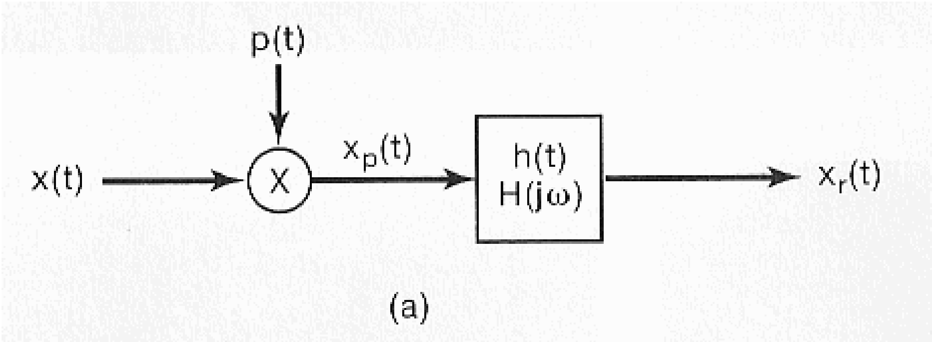
first-order hold

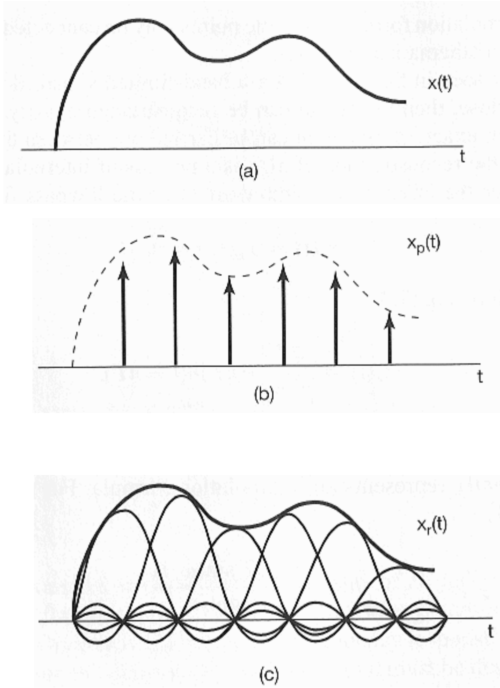


$$H_1(jw) = \frac{1}{T}\left[\frac{\sin(wT/2)}{w/2}\right]^2$$

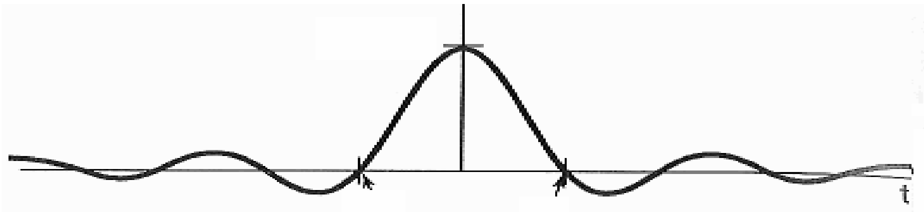


- Three Filters: Ideal Lowpass, Zero-Order, First-Order

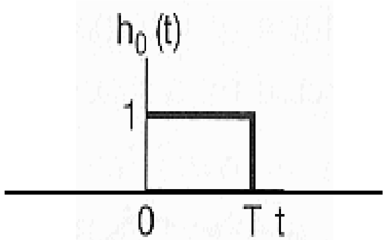


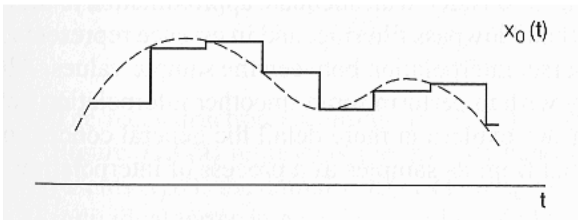


ideal lowpass

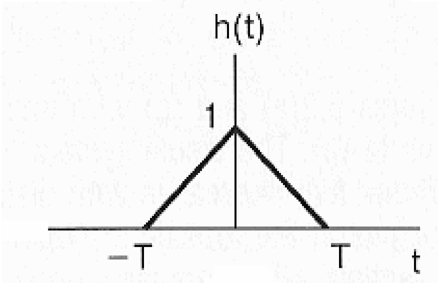


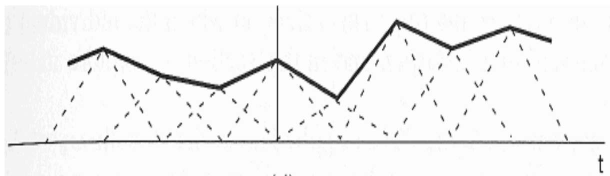
zero-order hold





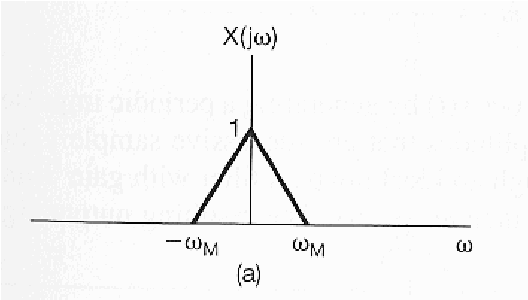
first-order hold



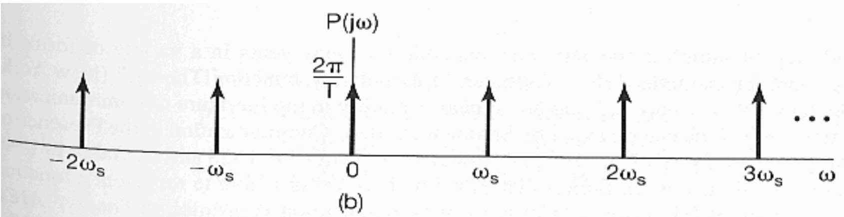


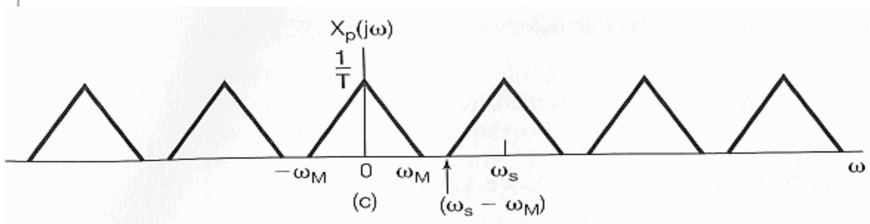
- Representation of of a Continuous-Time Signal by Its Samples: The Sampling Theorem
- Reconstruction of of a Signal from Its Samples Using Interpolation
- The Effect of Under-sampling: Aliasing
- Discrete-Time Processing of Continuous-Time Signals
- Sampling of Discrete-Time Signals

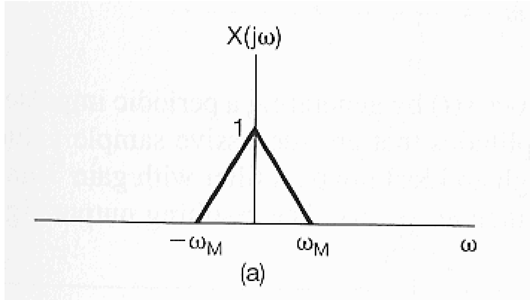
- <u>Overlapping in Frequency-Domain: Aliasing</u>



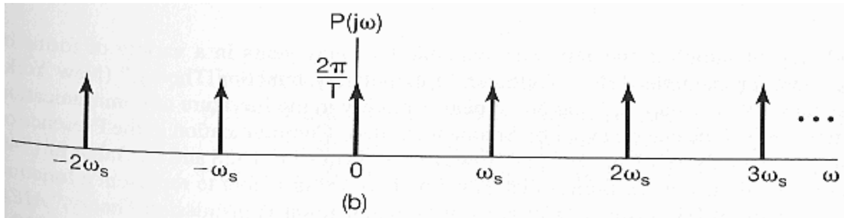
$w_s > 2w_M$

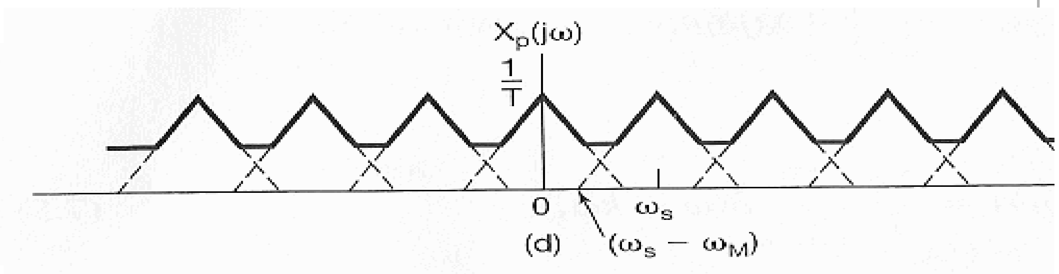




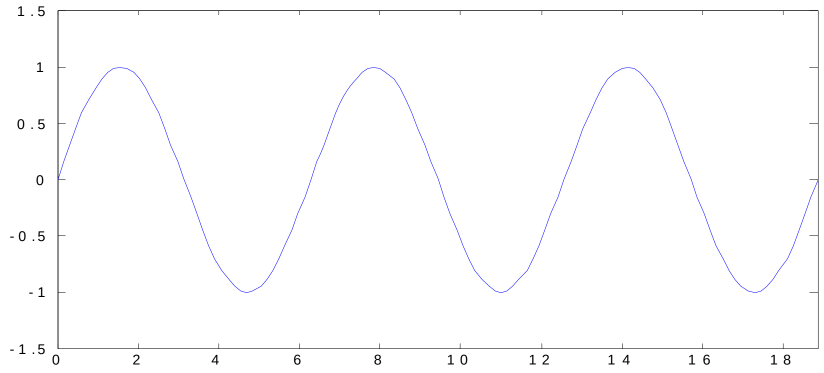


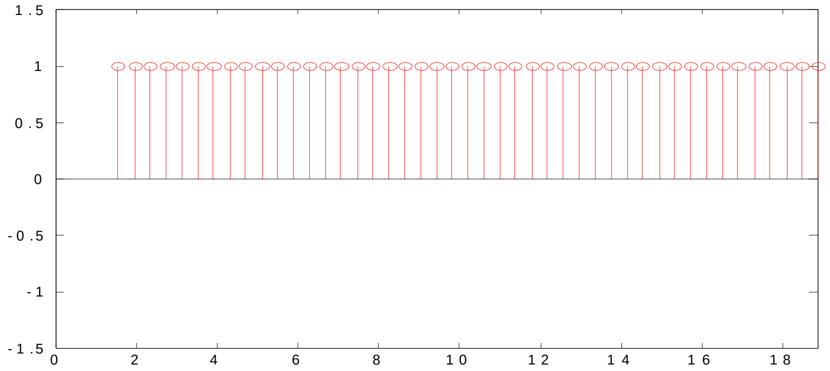
$w_s < 2w_M$

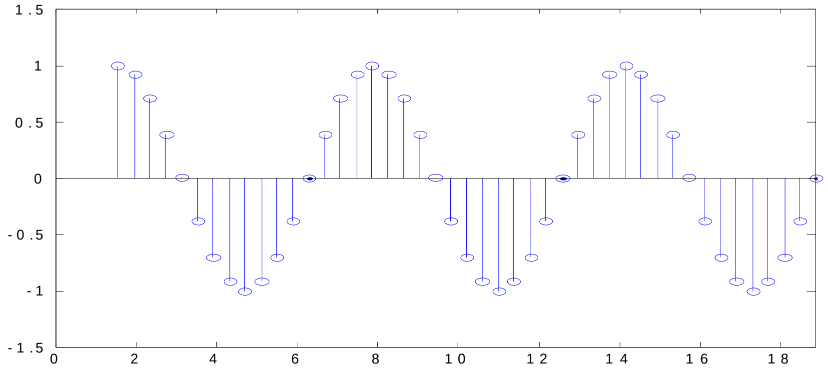


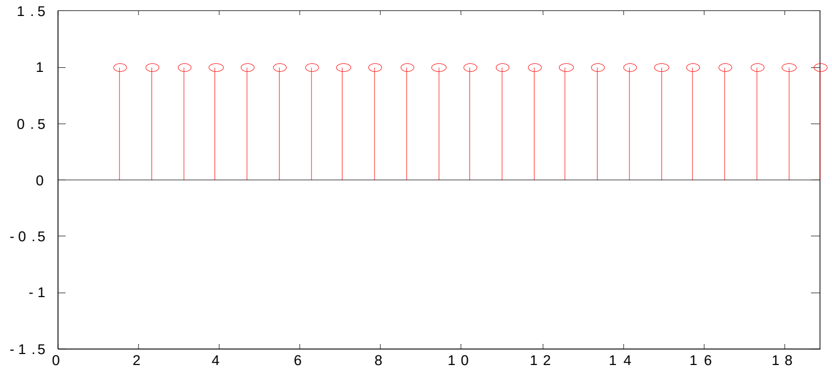


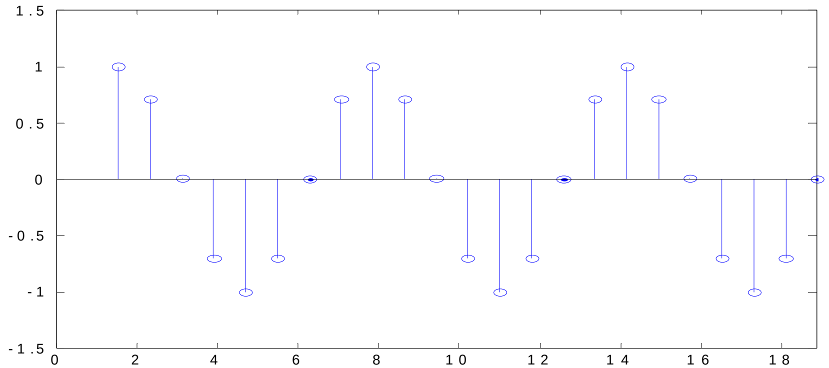
- Overlapping in Frequency-Domain: Aliasing



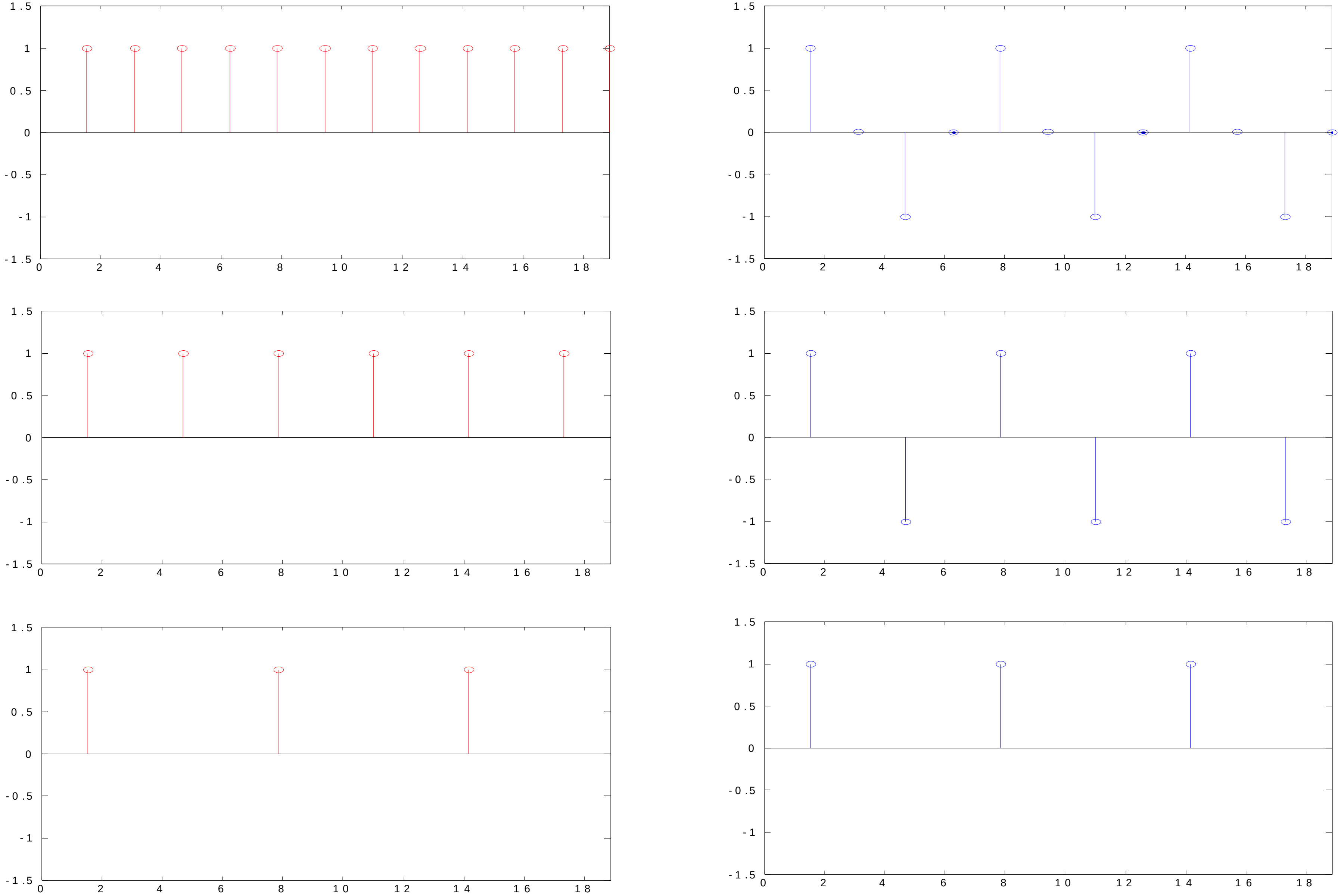




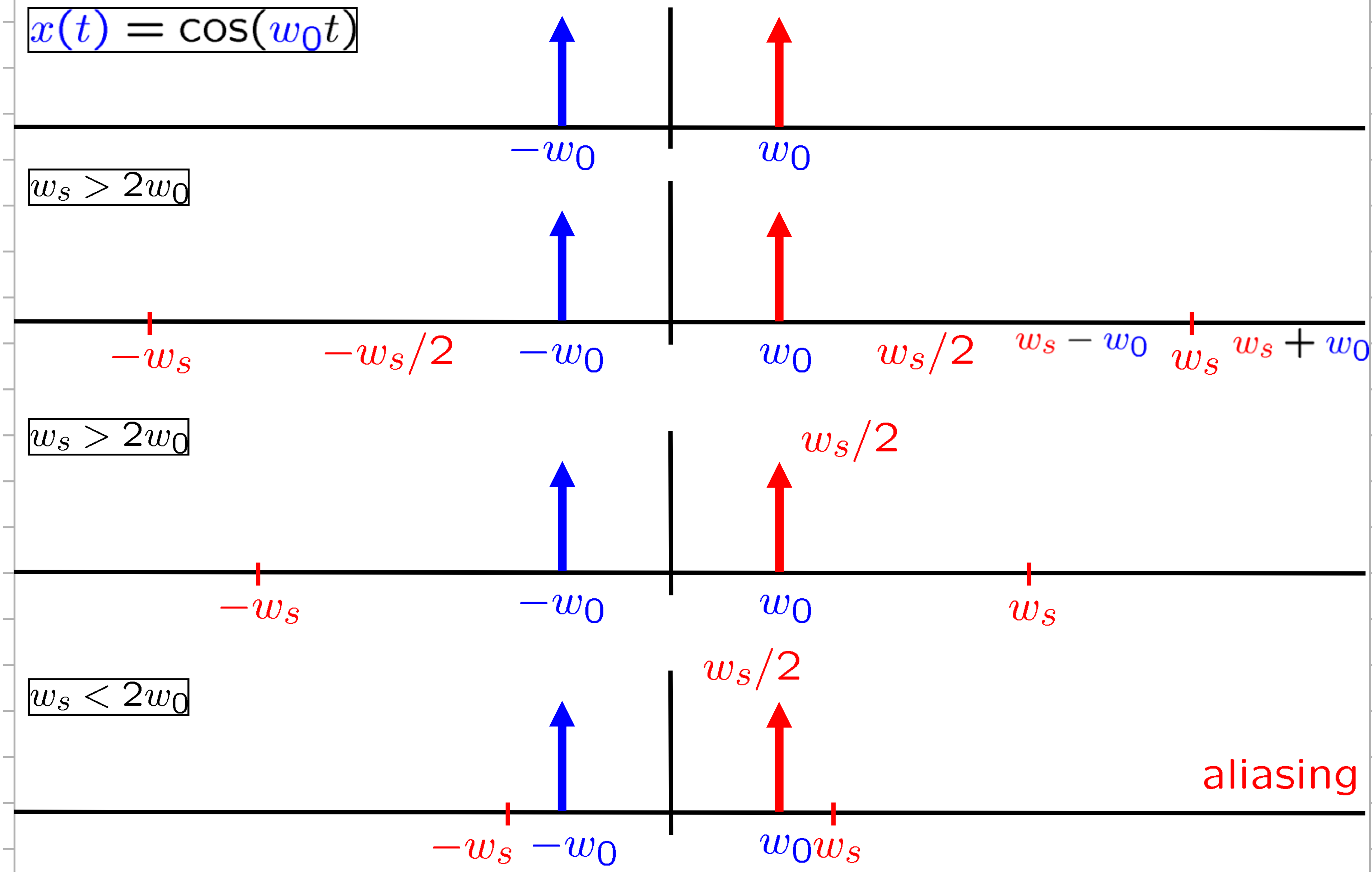




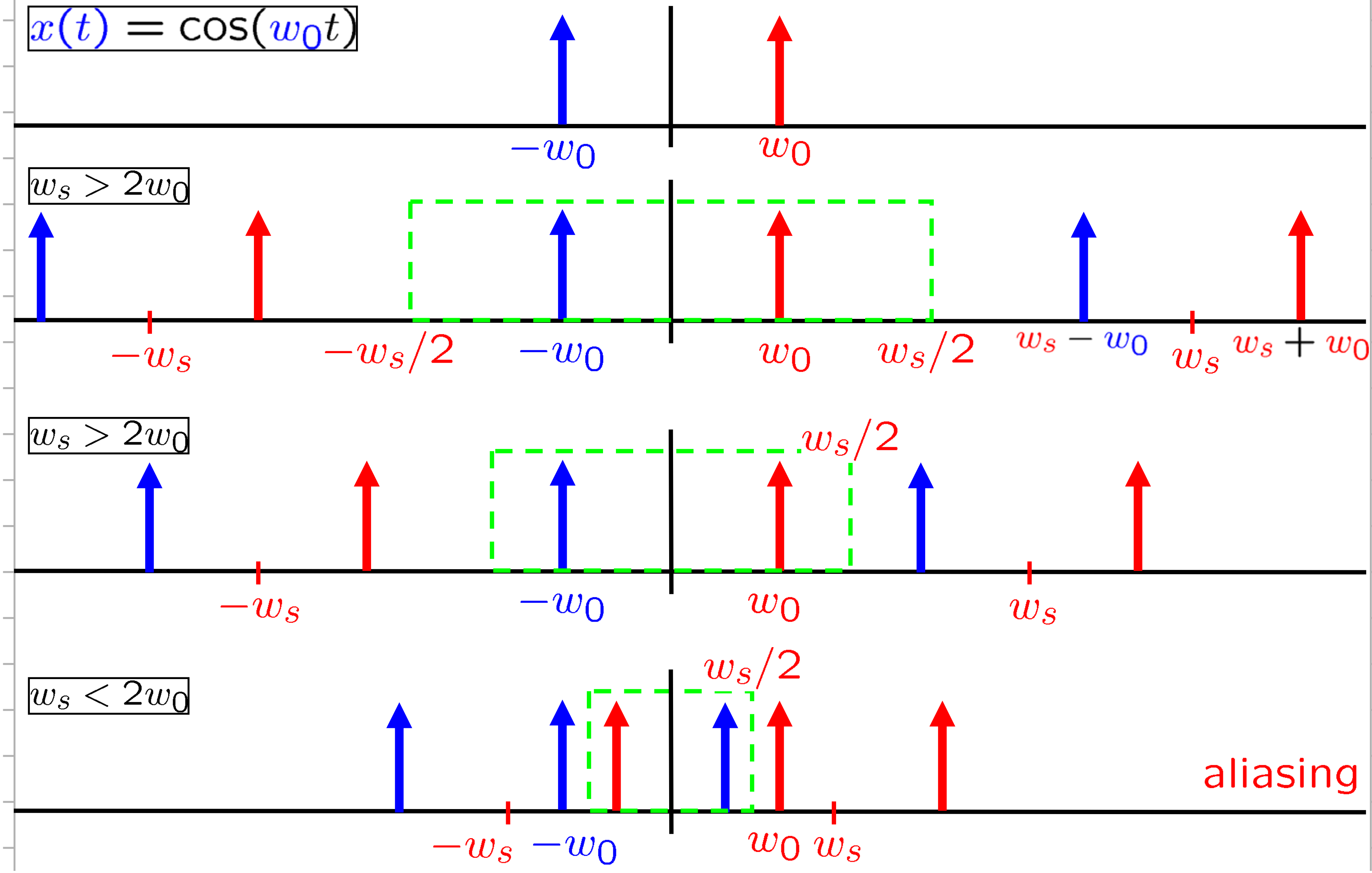
- Overlapping in Frequency-Domain: Aliasing



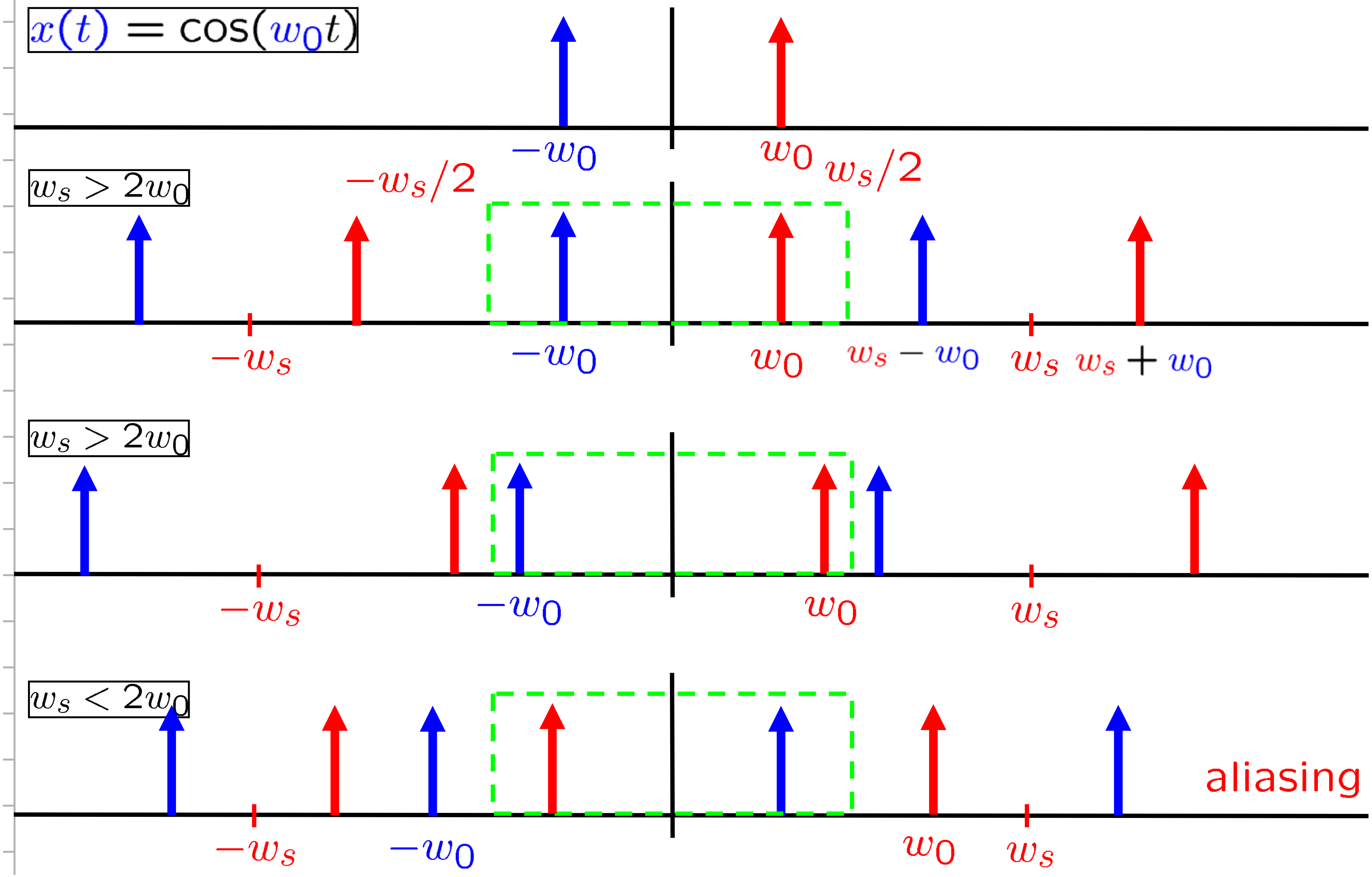
- Overlapping in Frequency-Domain: Aliasing



- Overlapping in Frequency-Domain: Aliasing

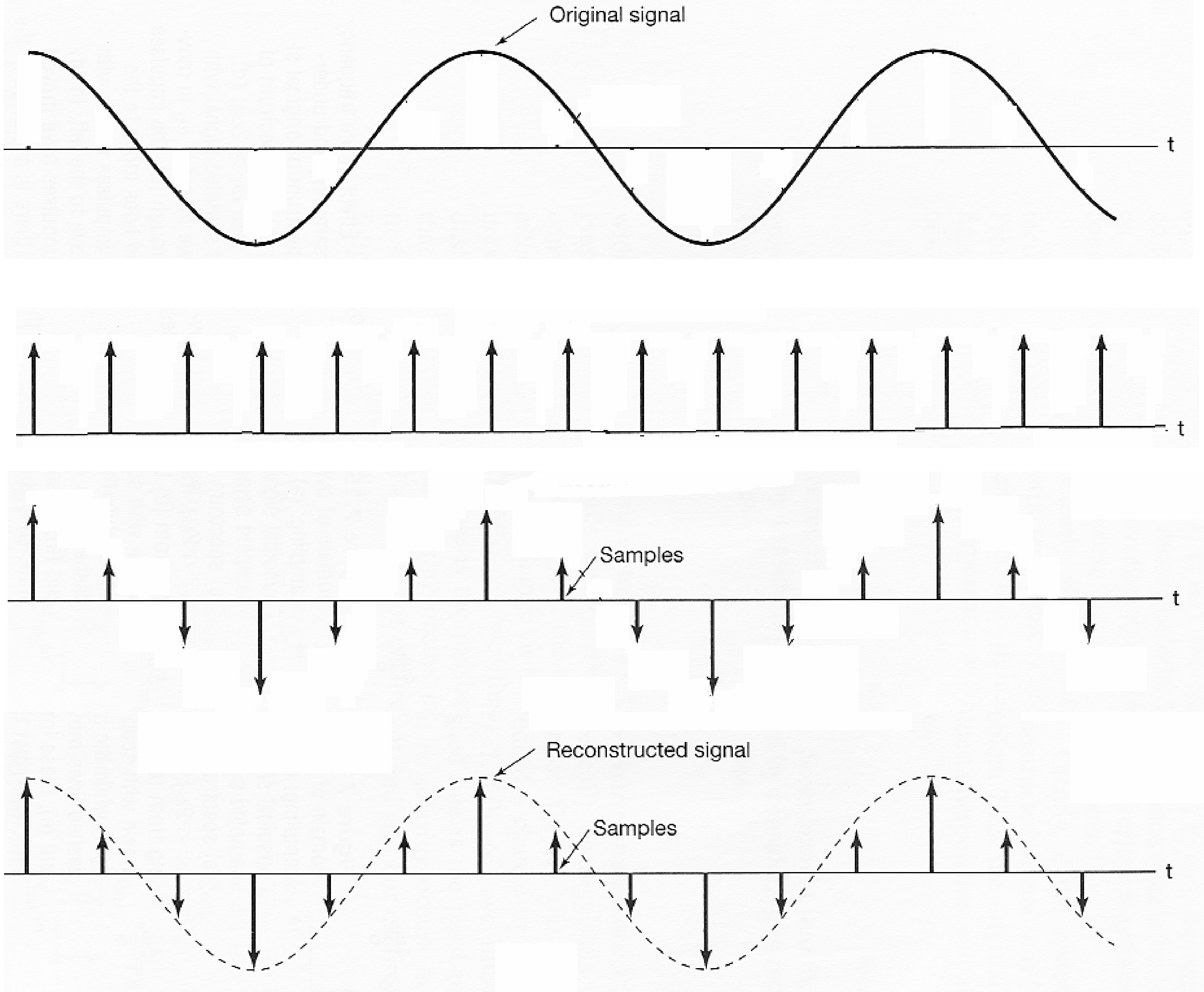


- Overlapping in Frequency-Domain: Aliasing

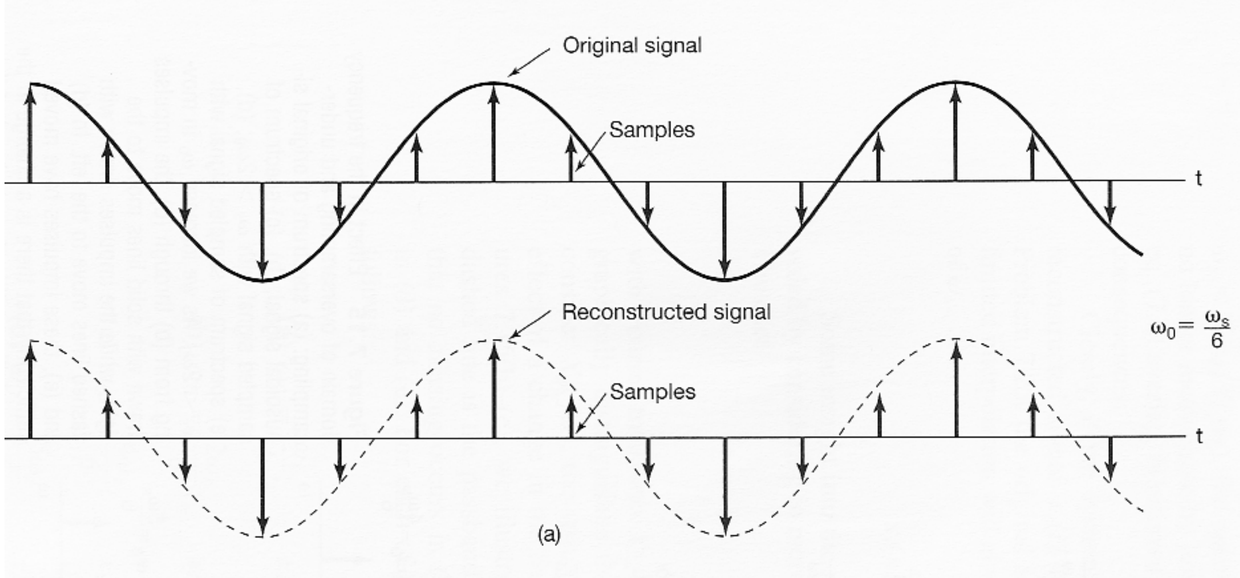


- Overlapping in Frequency-Domain: Aliasing

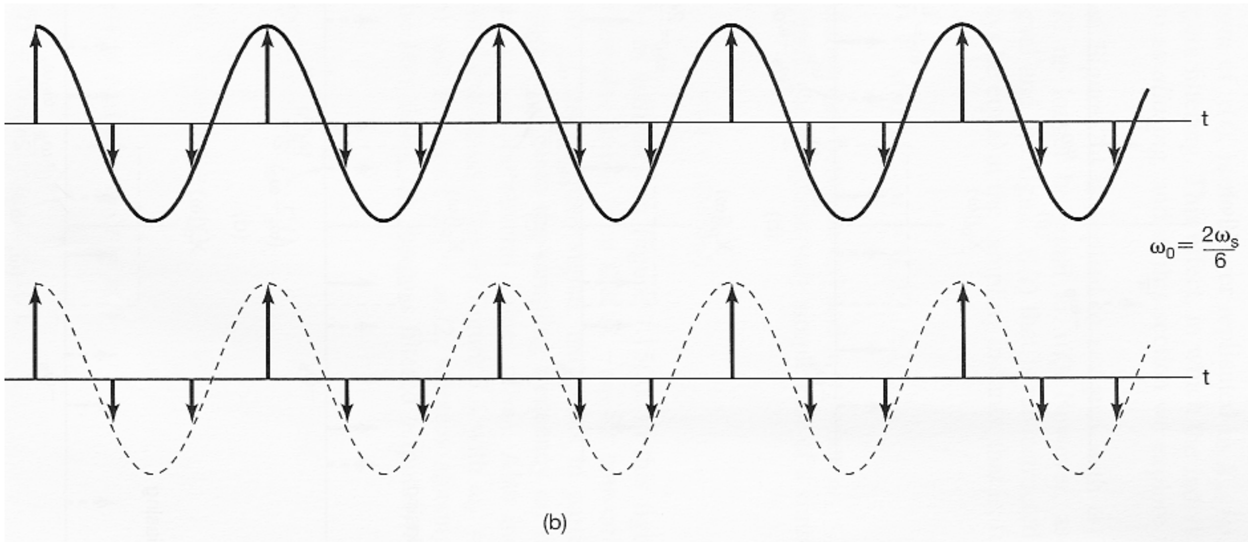
$$w_0 = \frac{w_s}{6}$$



Overlapping in Frequency-Domain: Aliasing

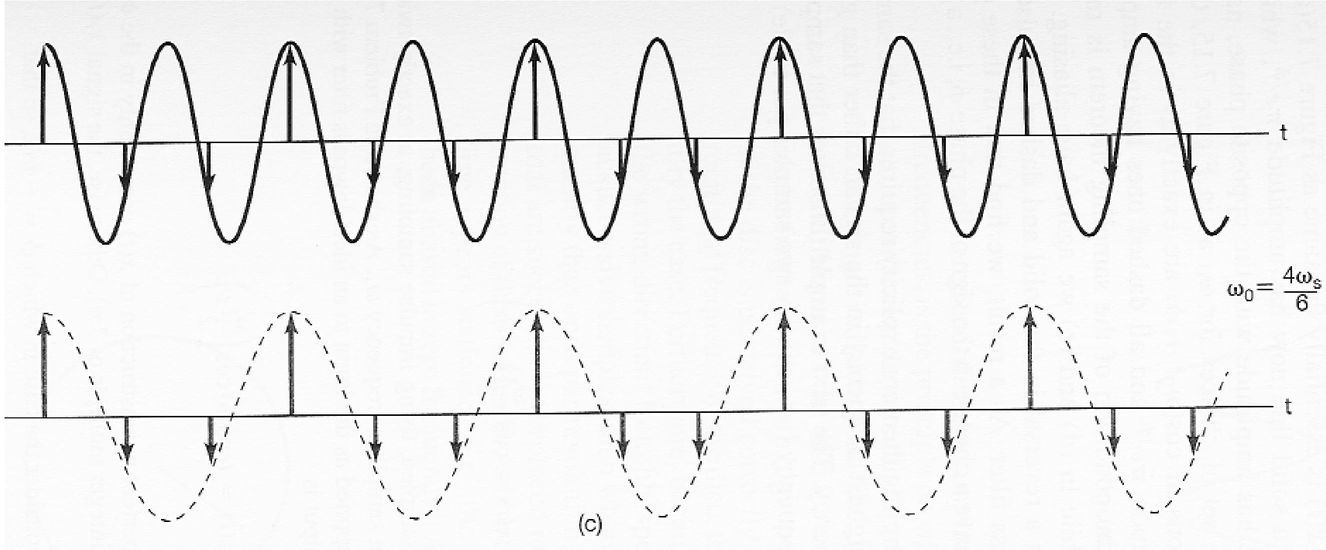


$$w_0 = \frac{w_s}{6}$$

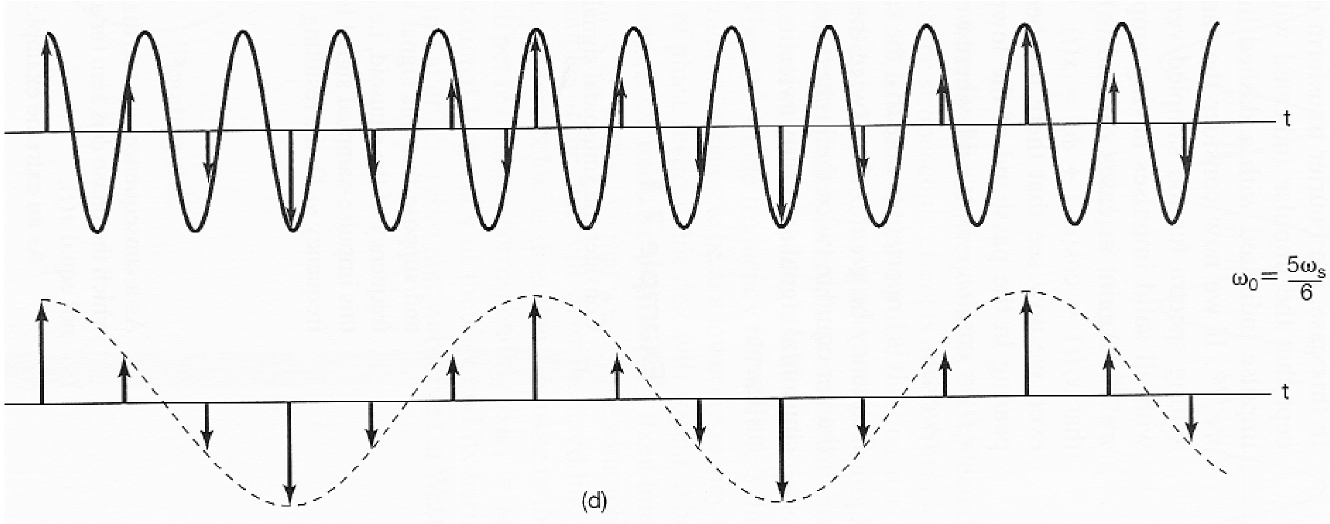


$$w_0 = \frac{2w_s}{6}$$

- Overlapping in Frequency-Domain: Aliasing

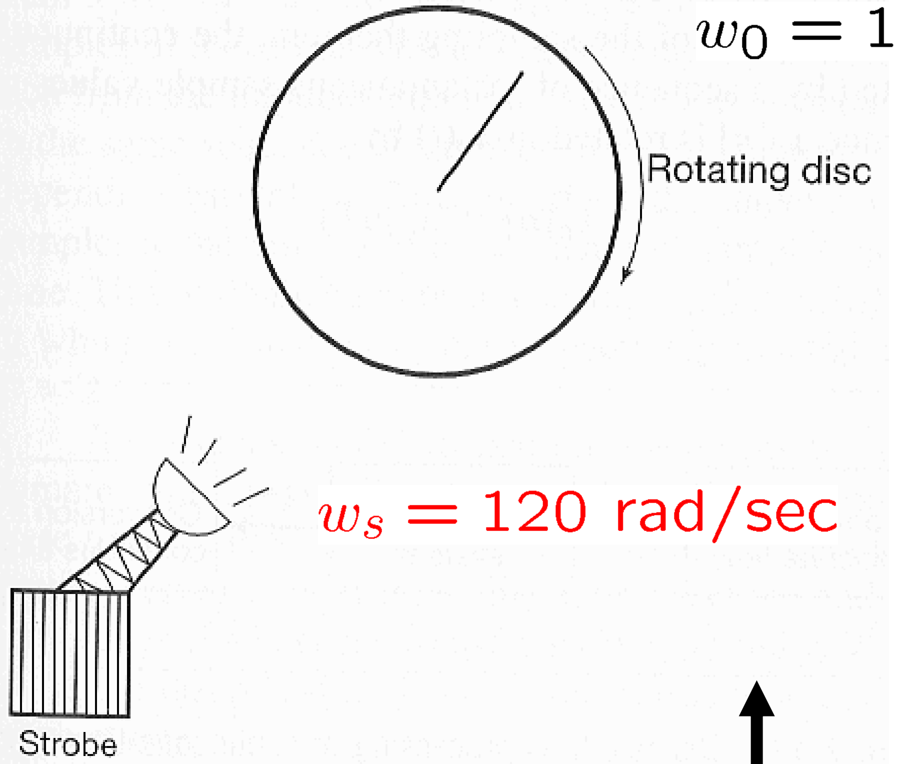


$$w_0 = \frac{4w_s}{6}$$



$$w_0 = \frac{5w_s}{6}$$

- Strobe Effect:

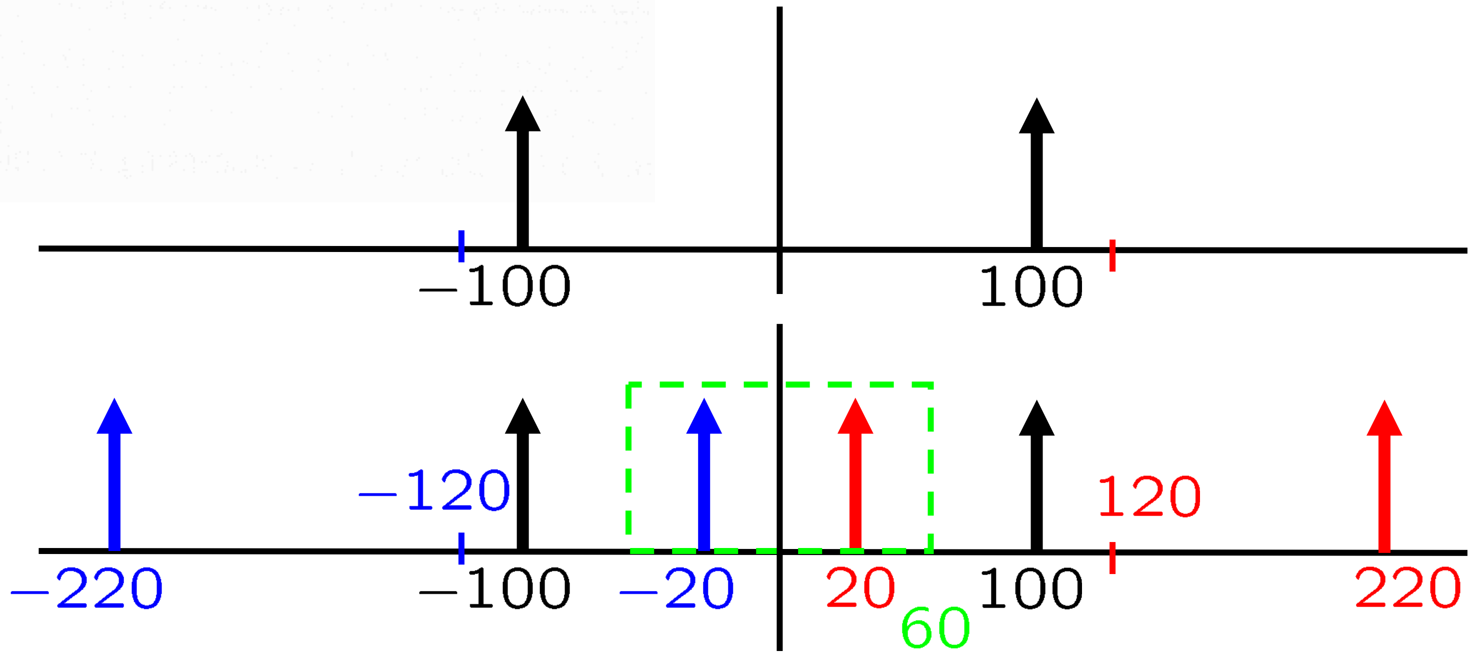


$w_0 = 100$ rad/sec

$w_s = 120$ rad/sec

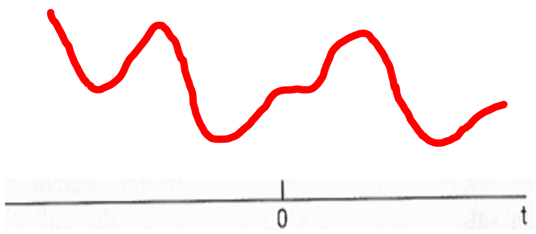
$\Rightarrow w = \pm w_s \pm w_0$

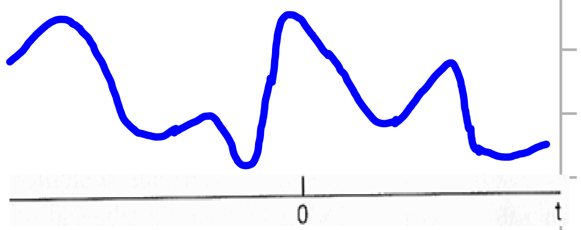
$= +20, -20$

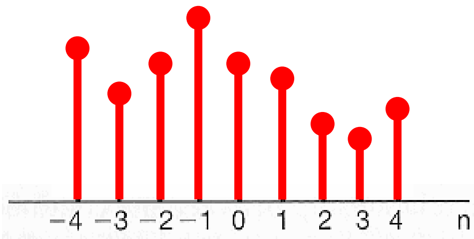


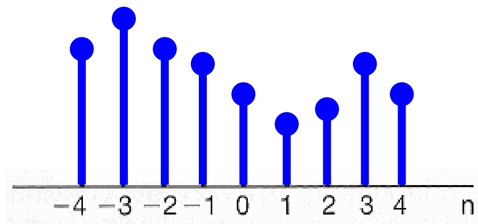
- Representation of of a Continuous-Time Signal by Its Samples: The Sampling Theorem
- Reconstruction of of a Signal from Its Samples Using Interpolation
- The Effect of Under-sampling: Aliasing
- Discrete-Time Processing of Continuous-Time Signals
- Sampling of Discrete-Time Signals

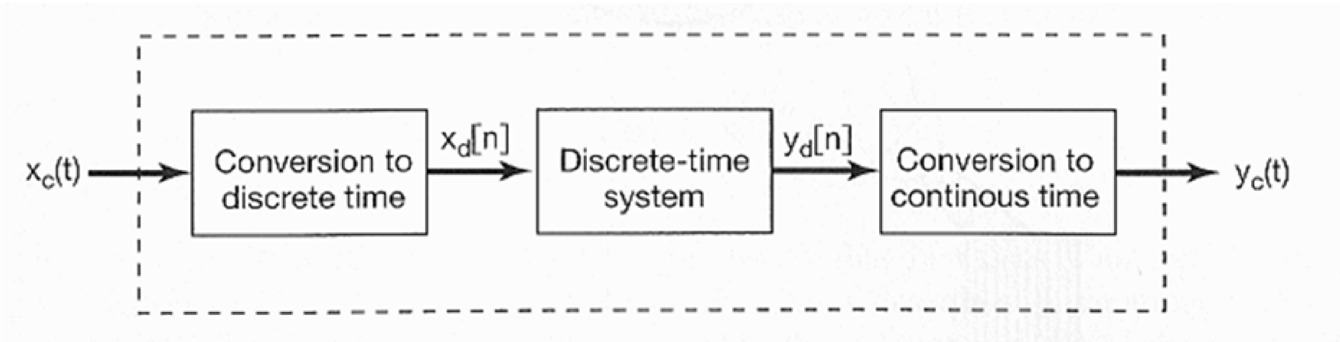
- Discrete-Time Processing of CT Signals:



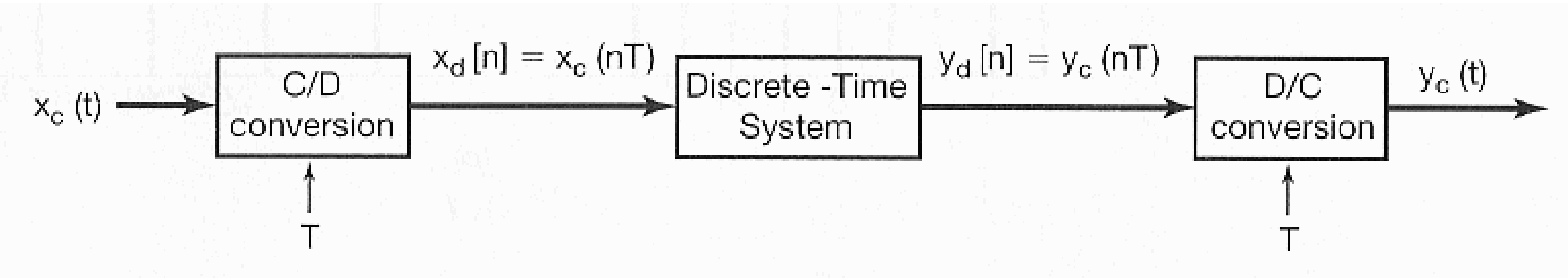


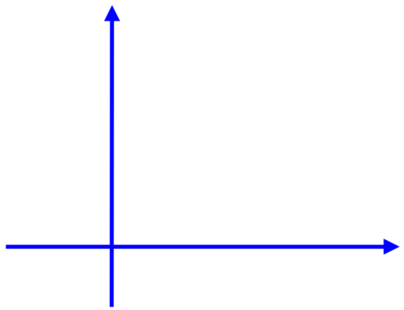






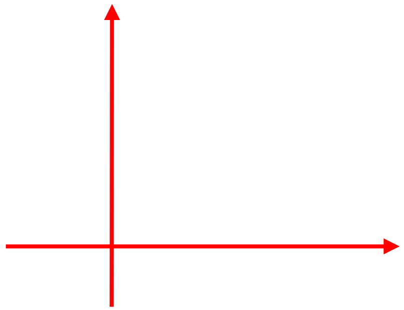
- C/D or A-to-D (ADC) and D/C or D-to-A (DAC):





C/D: continous-to-discrete-time conversion

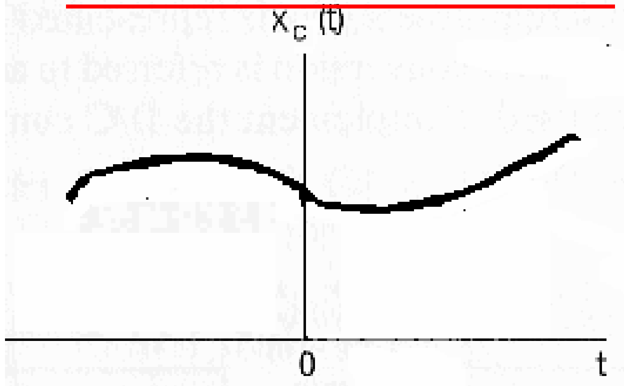
A-to-D: analog-to-digital converter

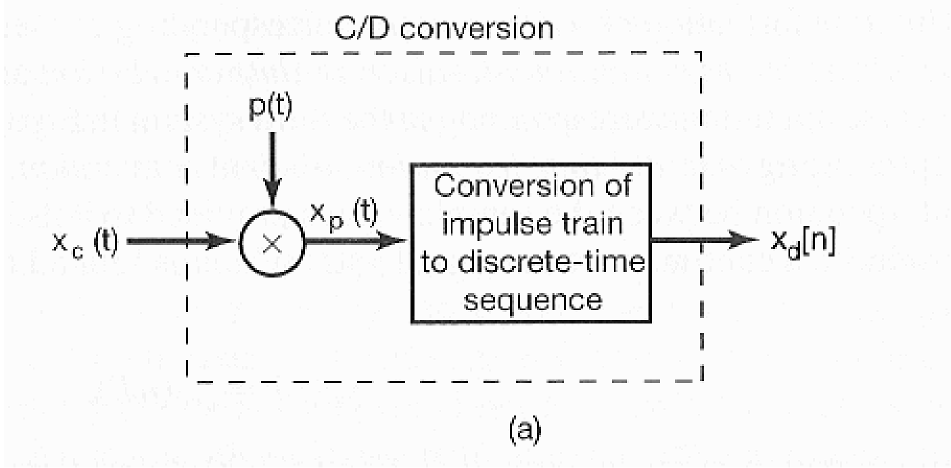


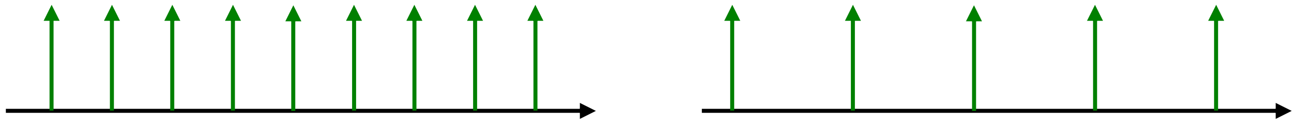
D/C: discrete-to-continous-time conversion

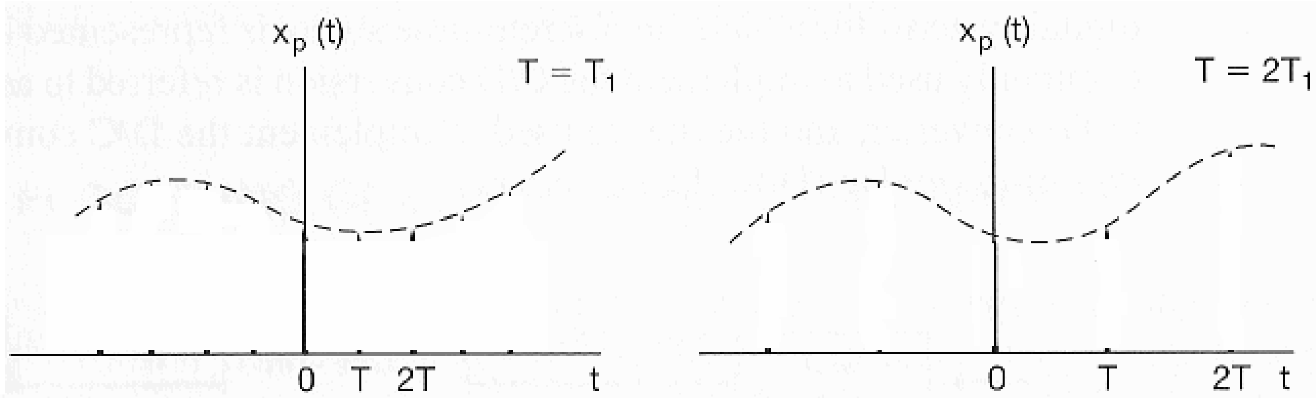
D-to-A: digital-to-analog converter

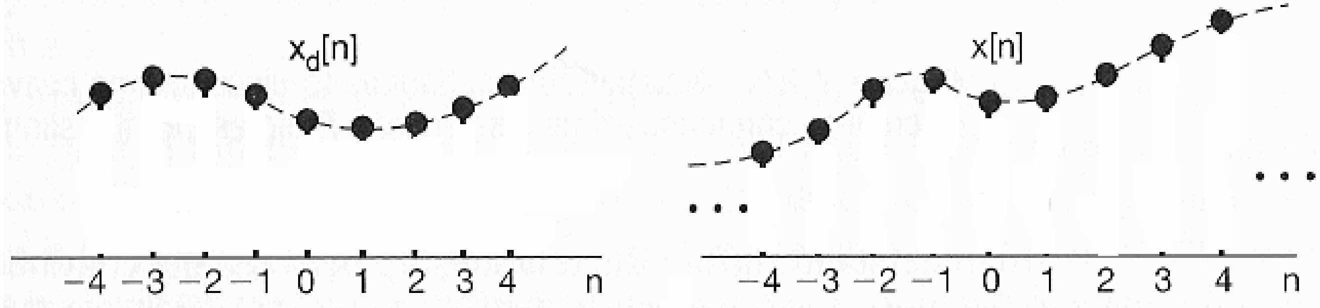
- C/D Conversion:

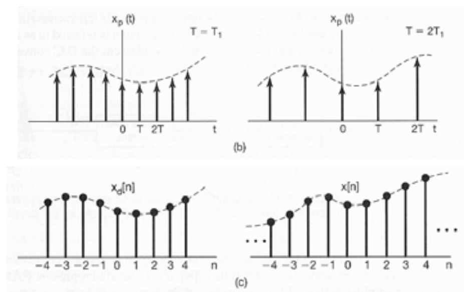




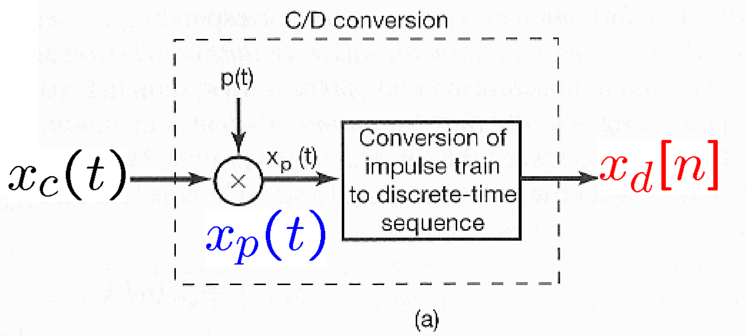


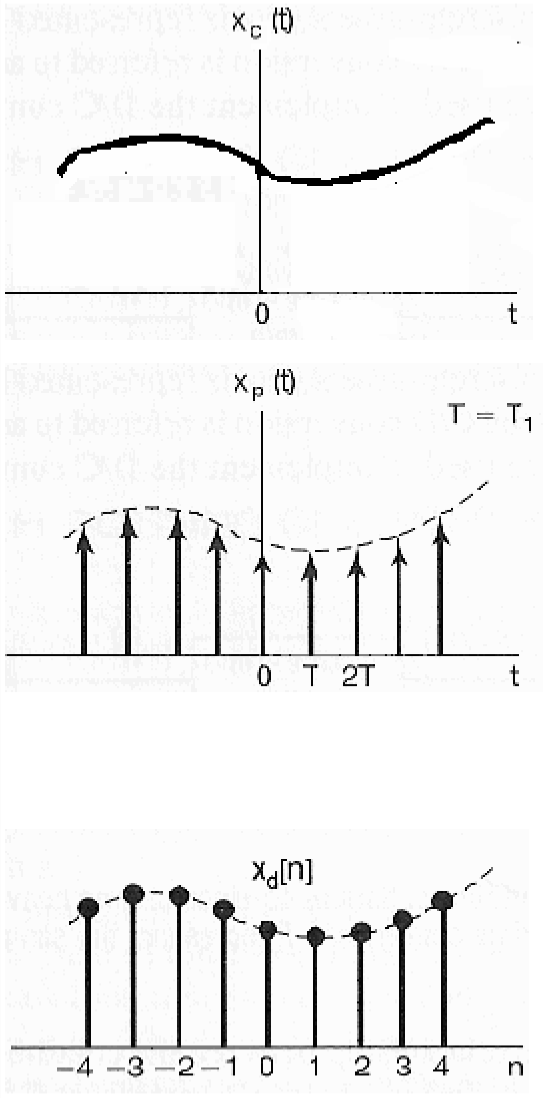


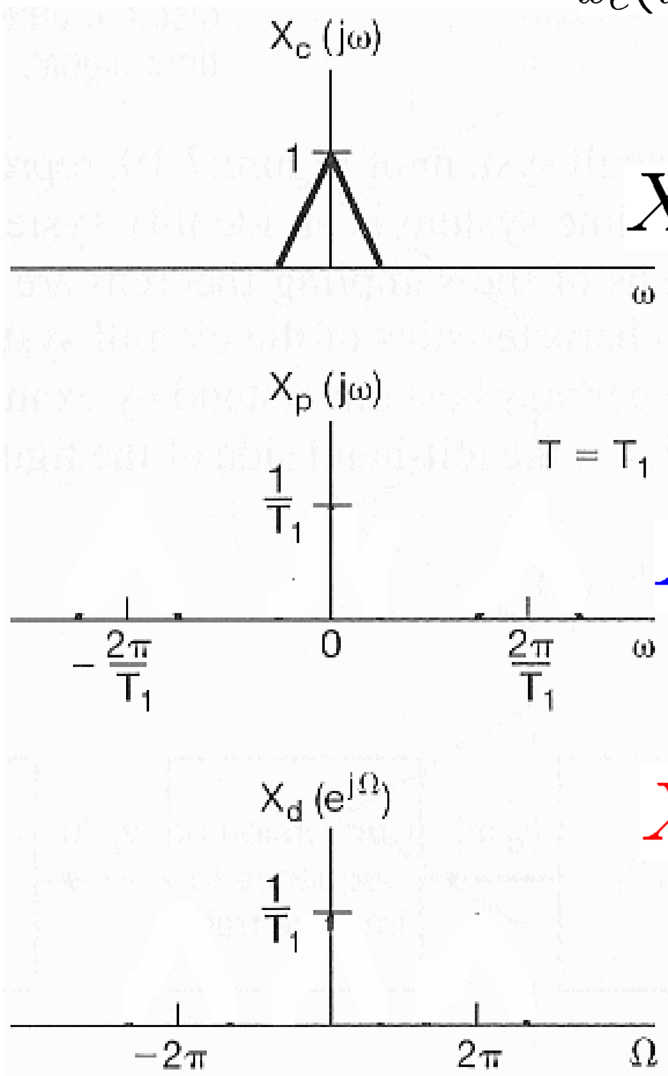




C/D Conversion:



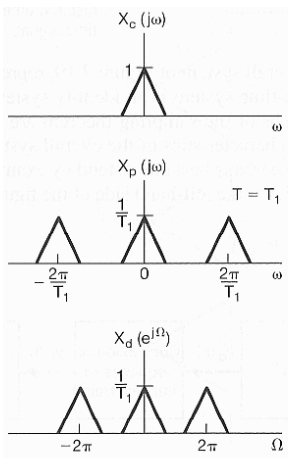




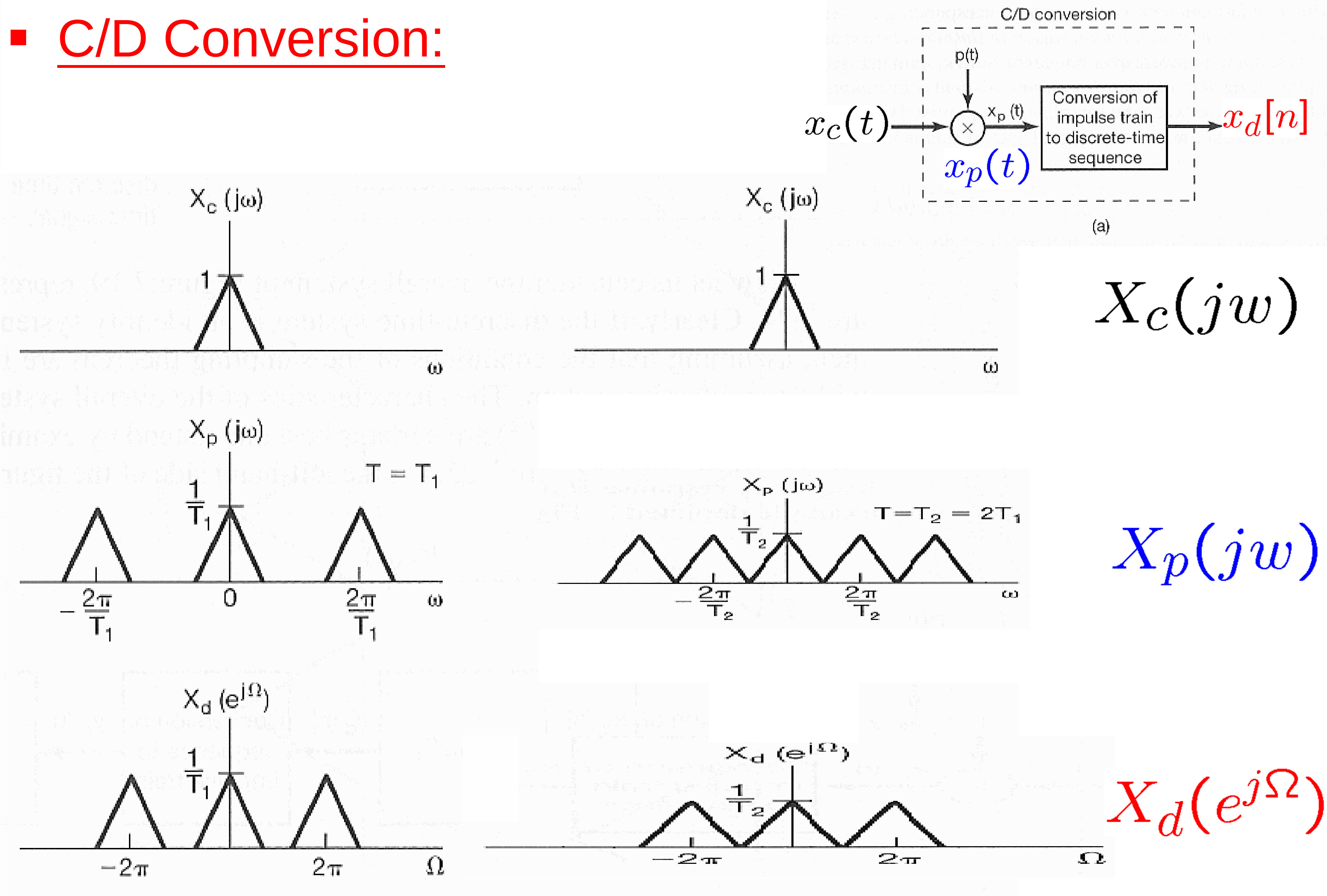
$X_c(jw)$

$X_p(jw)$

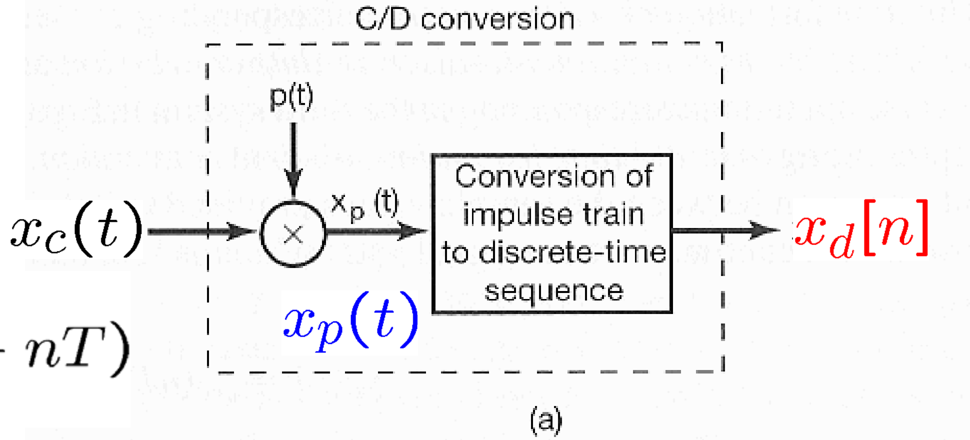
$X_d(e^{j\Omega})$



- ## C/D Conversion:



C/D Conversion:



(a)

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x_c(nT)\delta(t-nT)$$

Table 4.2, p. 329

$$\delta(t-t_0) \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-jwt_0}$$

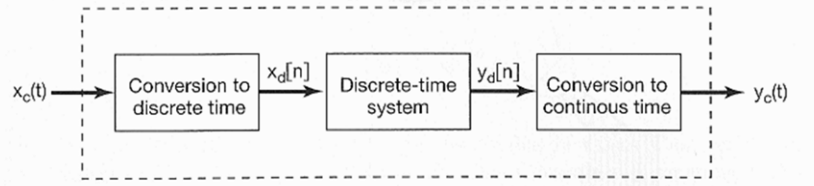
Eq 7.3, 7.6, p. 517

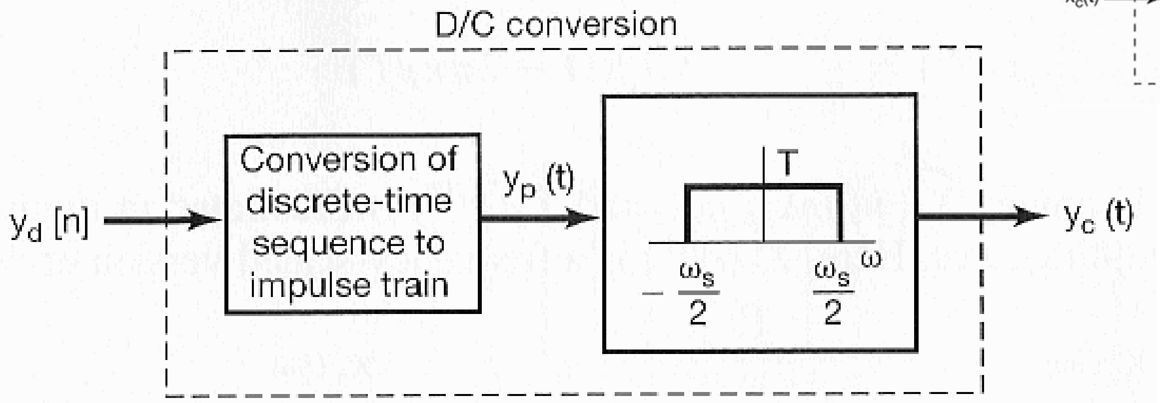
$$X_p(jw) = \sum_{n=-\infty}^{+\infty} x_c(nT)e^{-jwnT} = \frac{1}{T}\sum_{K=-\infty}^{+\infty} X_c(j(w-kw_s))$$

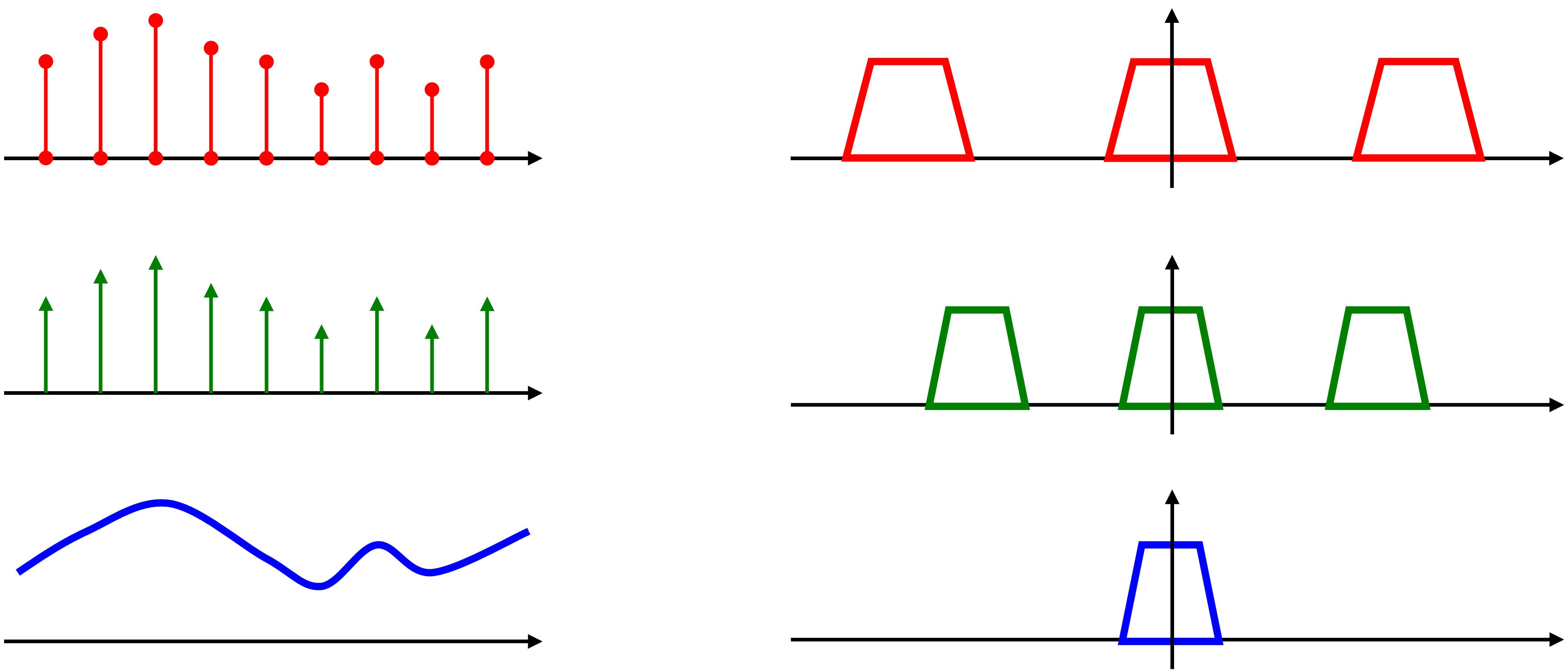
$$X_d(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x_d[n]e^{-j\Omega n} = \sum_{n=-\infty}^{+\infty} x_c(nT)e^{-j\Omega n}$$

$$\Rightarrow X_d(e^{j\Omega}) = X_p\left(j\frac{\Omega}{T}\right) = \frac{1}{T}\sum_{K=-\infty}^{+\infty} X_c\left(j\left(\frac{\Omega}{T}-k\frac{2\pi}{T}\right)\right)$$

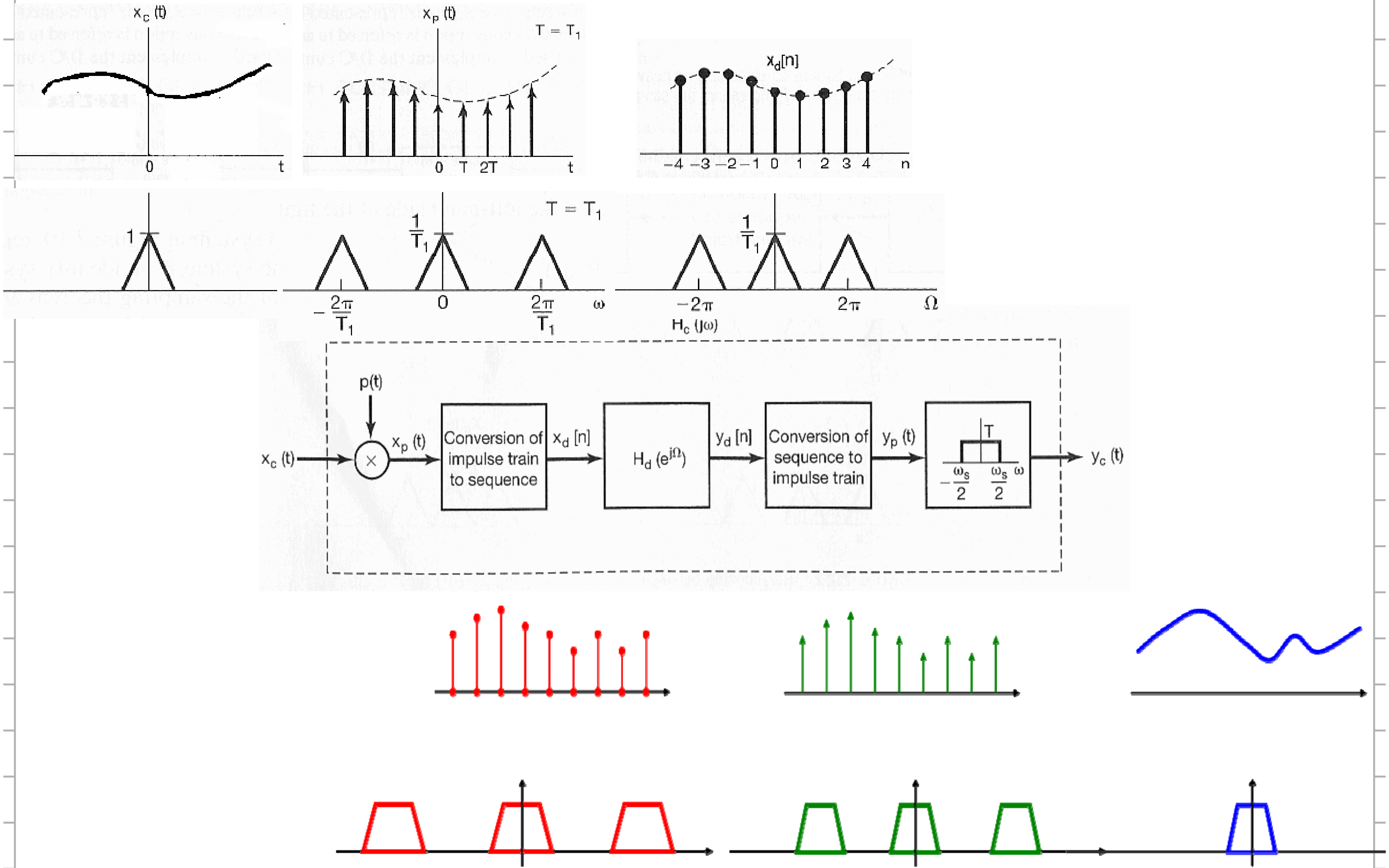
■ D/C Conversion:



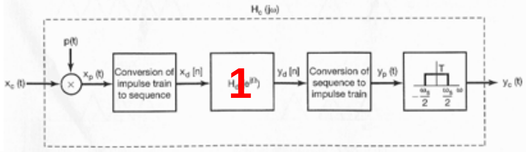




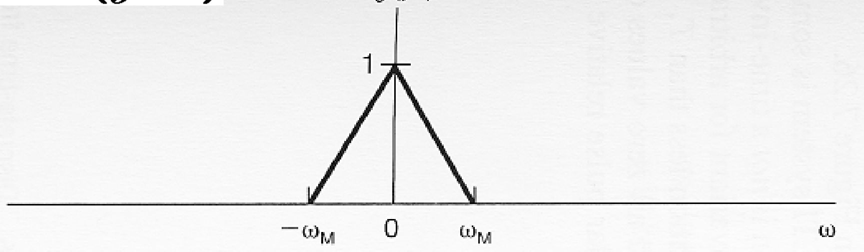
- Overall System:



- Frequency-Domain Illustration:

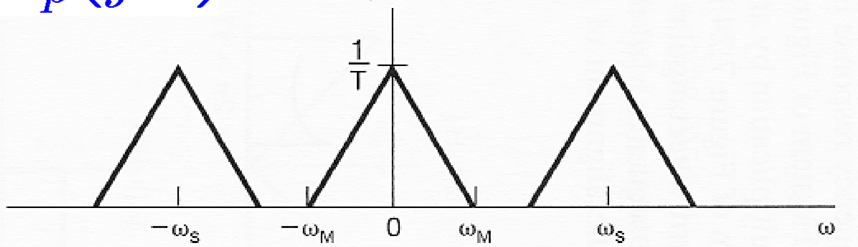


$X_c(jw)$



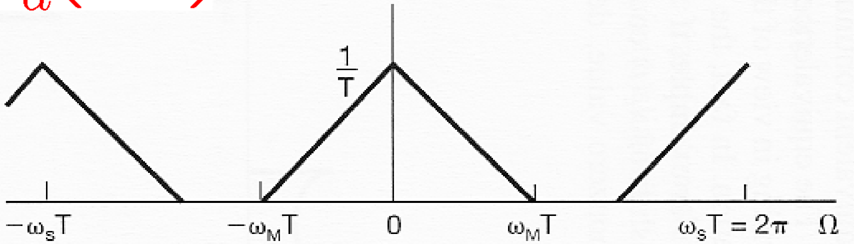
(a)

$X_p(jw)$



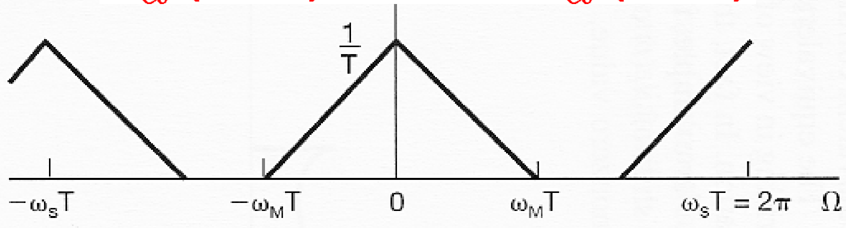
(b)

$X_d(e^{j\Omega})$



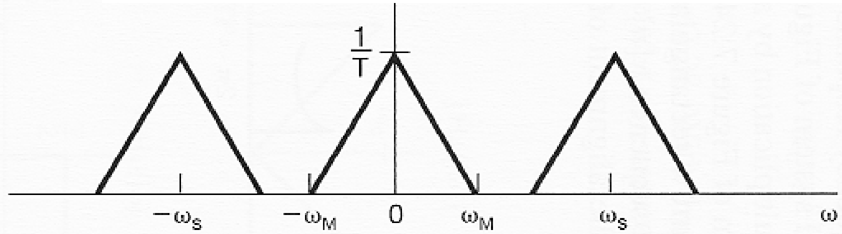
(c)

$Y_d(e^{j\Omega}) = 1\ X_d(e^{j\Omega})$



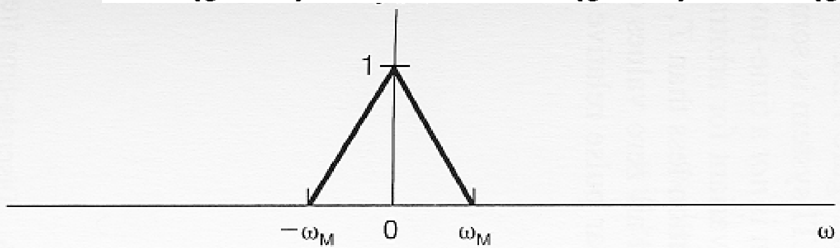
(c)

$Y_p(jw) \ = \ H_p(jw)X_p(jw)$



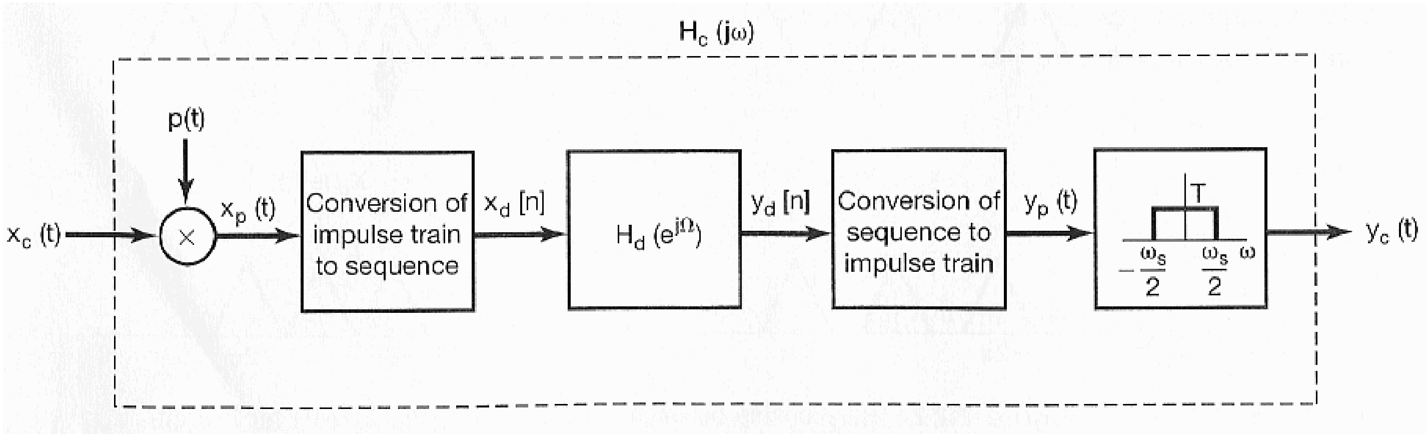
(b)

$Y_c(jw) \ = \ H_c(jw)X_c(jw)$

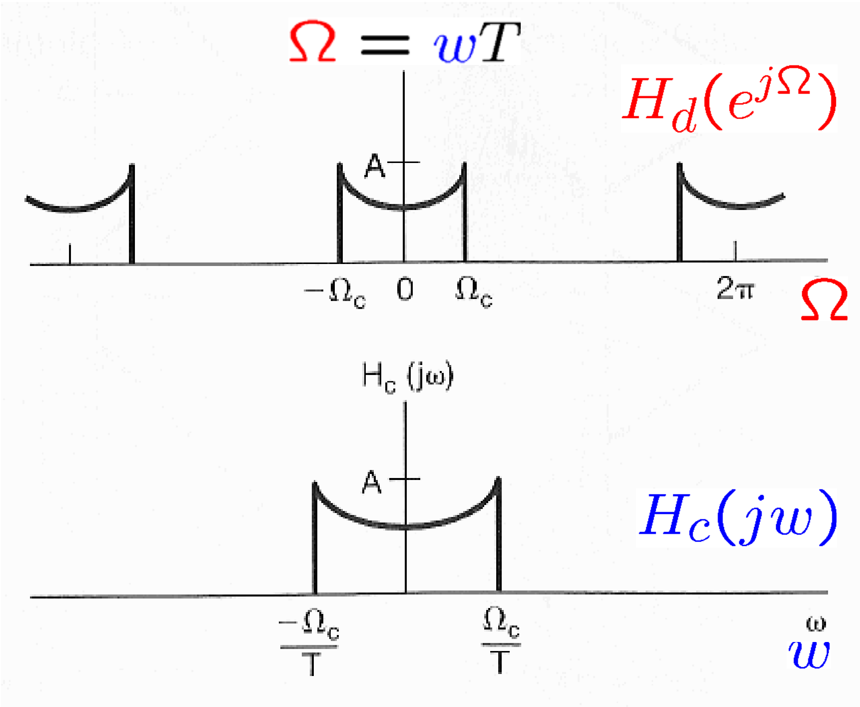


(a)

CT & DT Frequency Responses:

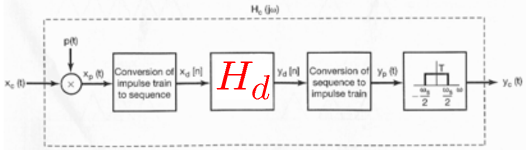


$$Y_c(jw) = X_c(jw)H_c(jw)$$

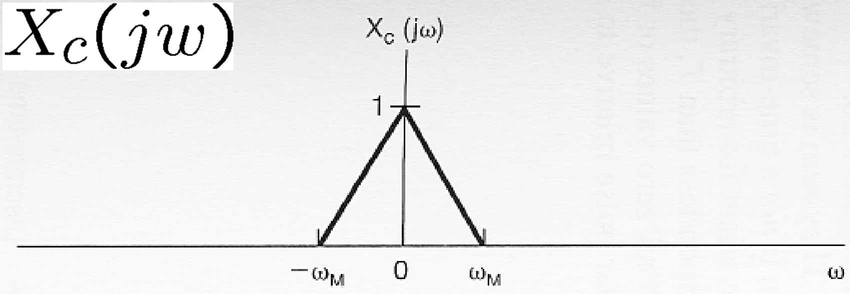


$$H_c(jw) = \begin{cases} H_d(e^{jwT}), & |w| < w_s/2 \\ 0, & |w| > w_s/2 \end{cases}$$

▪ Frequency-Domain Illustration:

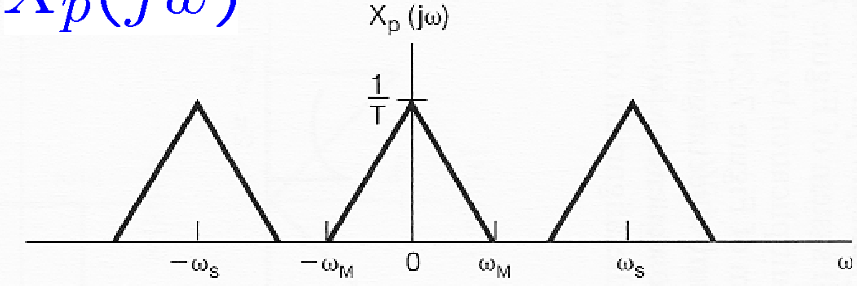


$X_c(jw)$



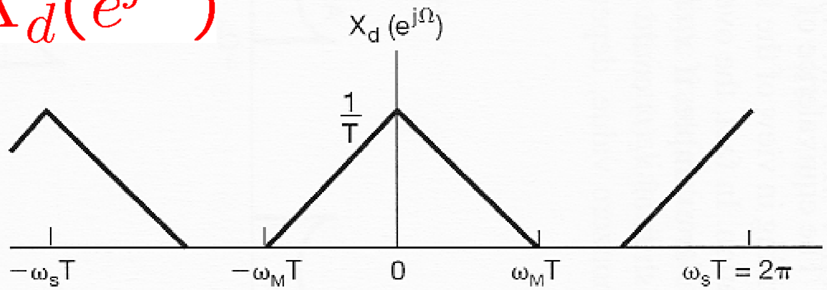
(a)

$X_p(jw)$



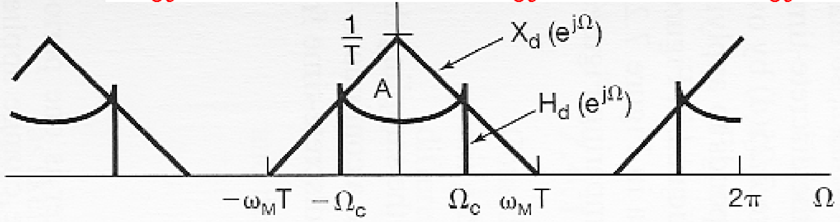
(b)

$X_d(e^{j\Omega})$



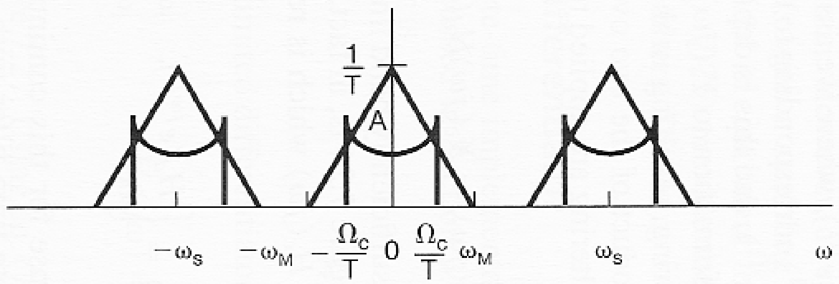
(c)

$Y_d(e^{j\Omega}) = H_d(e^{j\Omega})X_d(e^{j\Omega})$



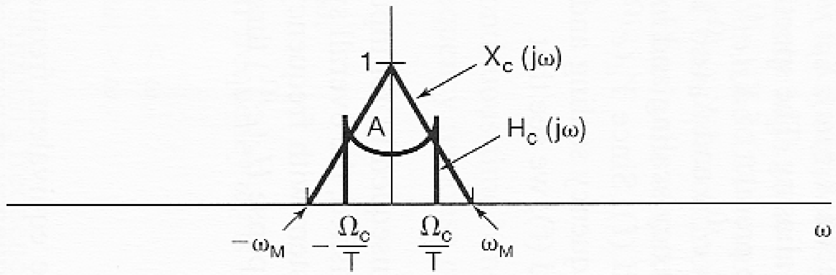
(d)

$Y_p(jw) = H_p(jw)X_p(jw)$



(e)

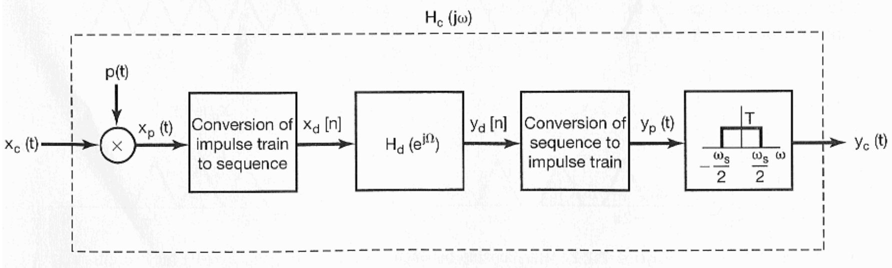
$Y_c(jw) = H_c(jw)X_c(jw)$



(f)

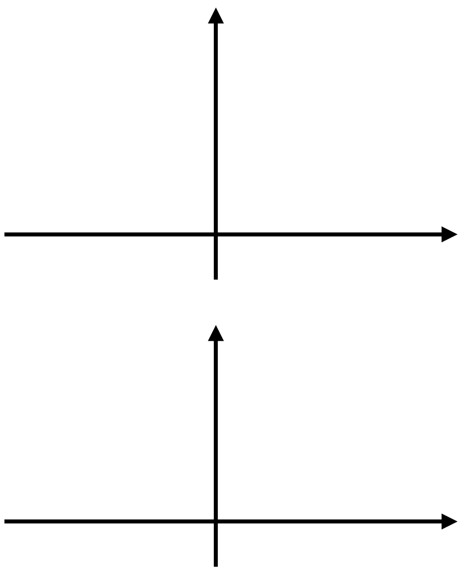
Digital Differentiator:

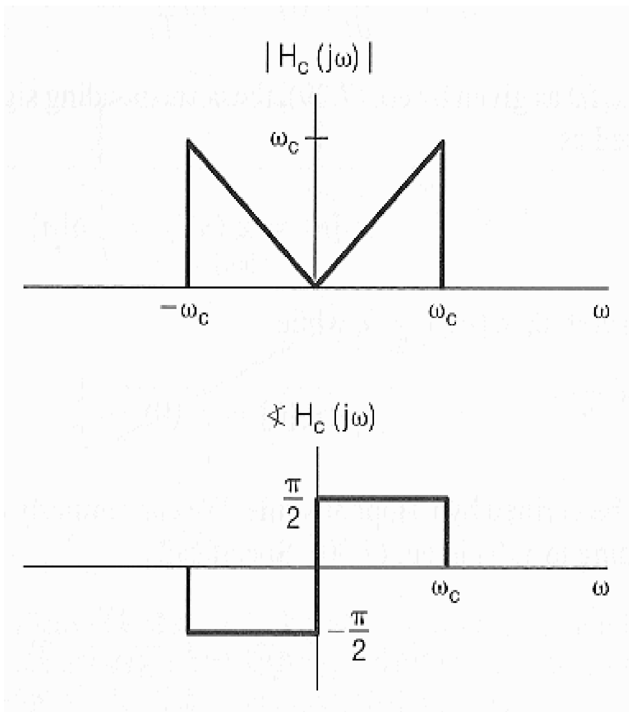
Ex 4.16, p. 317

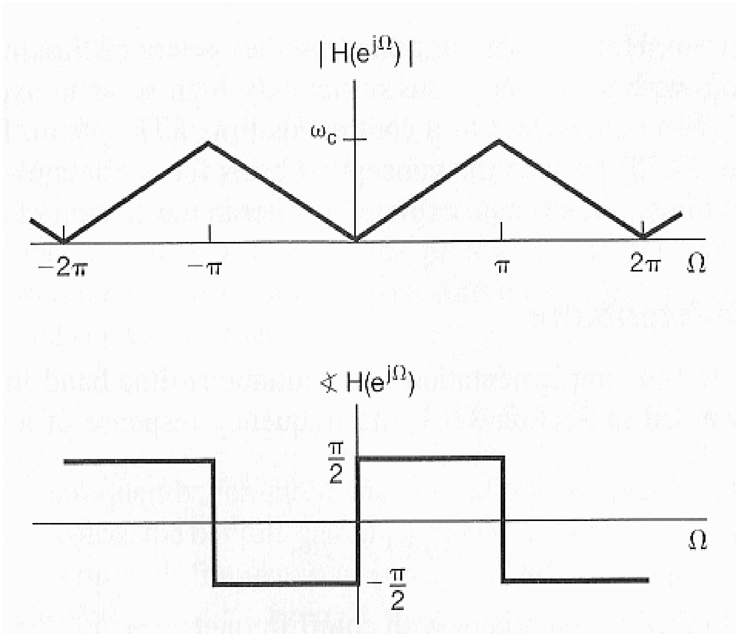


$$H_c(jw) = \begin{cases} jw, & |w| < w_c \\ 0, & |w| > w_c \end{cases}$$

$$H_d(e^{j\Omega}) = j\left(\frac{\Omega}{T}\right), \quad |\Omega| < \pi$$



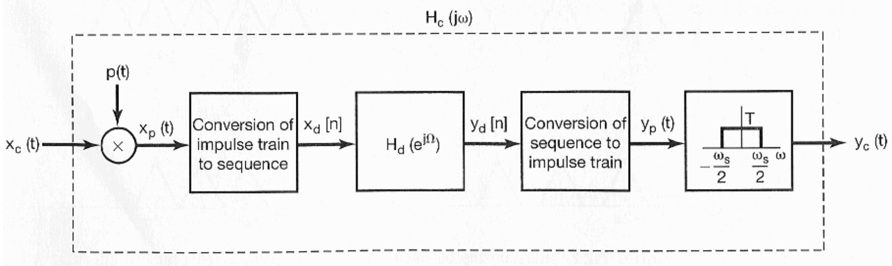




$$\Omega = wT, \; w_s = 2w_c$$

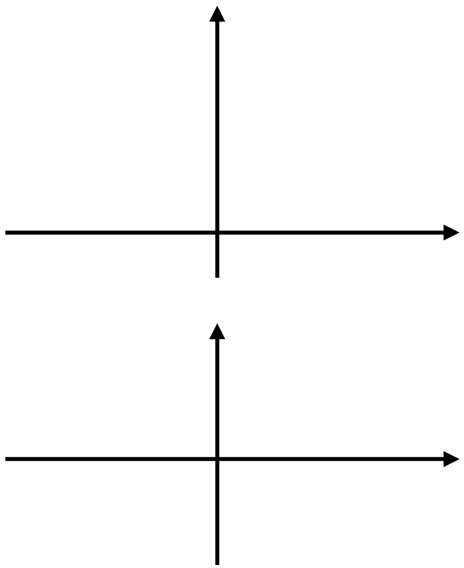
- ## Half-Sample Delay:

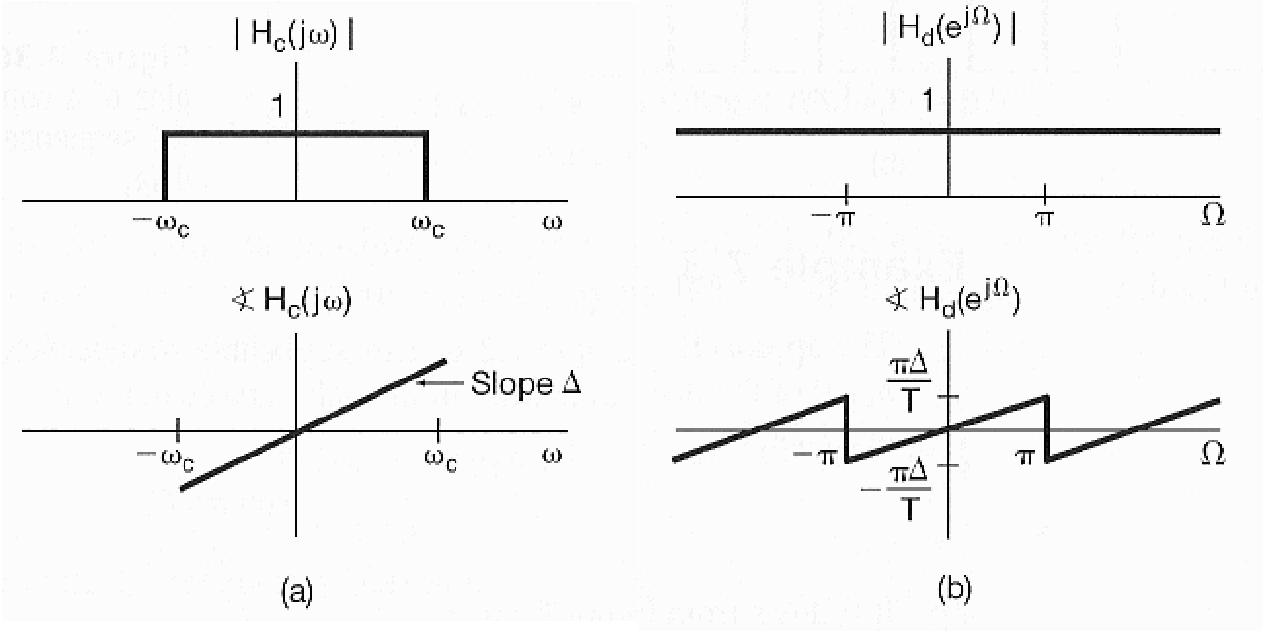
Ex 4.15, p. 317



$$H_c(jw) = \begin{cases} e^{-jw\Delta}, & |w| < w_c \\ 0, & |w| > w_c \end{cases}$$

$$H_d(e^{j\Omega}) = e^{-j\Omega\Delta/T}, \quad |\Omega| < \pi$$





$$\Omega = wT, \;\; w_s = 2w_c$$

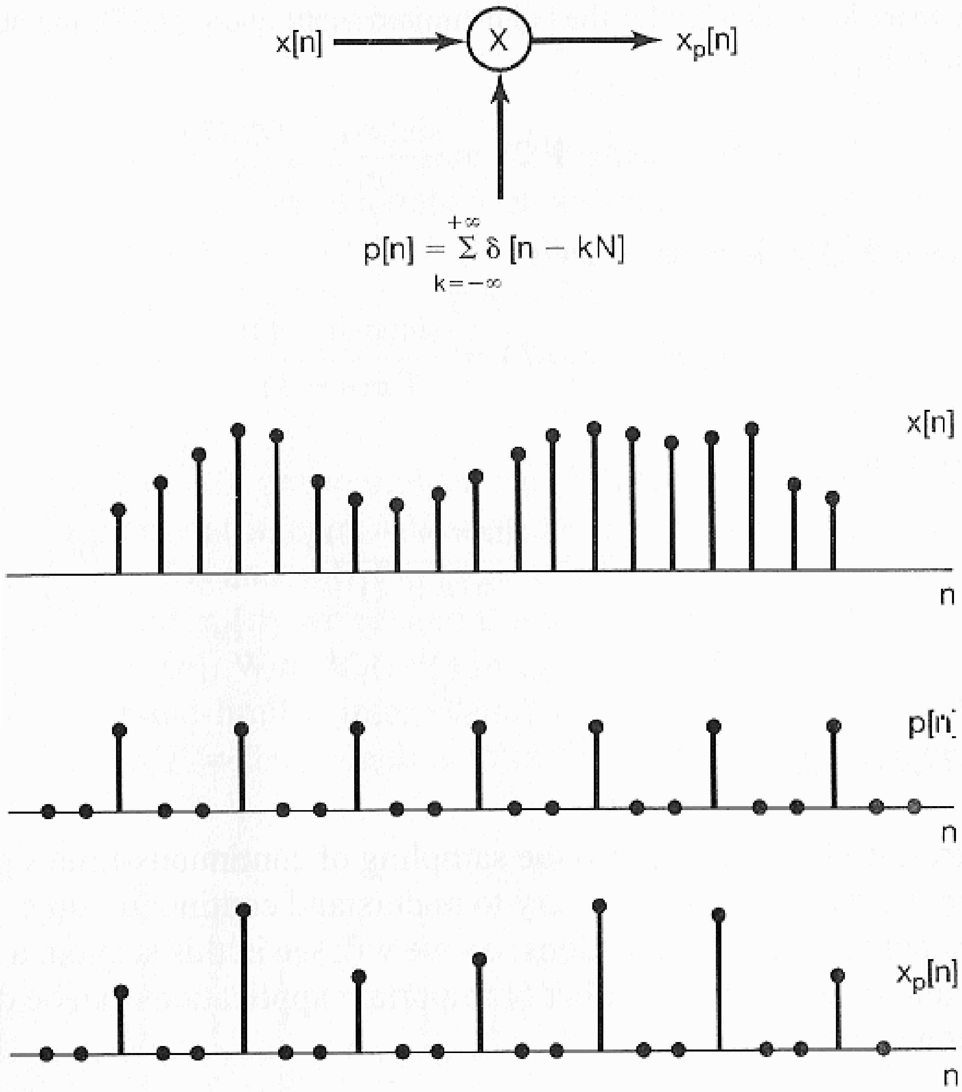
- Representation of of a Continuous-Time Signal by Its Samples: The Sampling Theorem
- Reconstruction of of a Signal from Its Samples Using Interpolation
- The Effect of Undersampling: Aliasing
- Discrete-Time Processing of Continuous-Time Signals
- Sampling of Discrete-Time Signals

- Impulse-Train Sampling:

$$x_p[n] = \begin{cases} x[n], & \text{if } n = kN \\ 0, & \text{otherwise} \end{cases}$$

$$x_p[n] = x[n]\, p[n]$$

$$= \sum_{k=-\infty}^{+\infty} x[kN]\, \delta[n-kN]$$

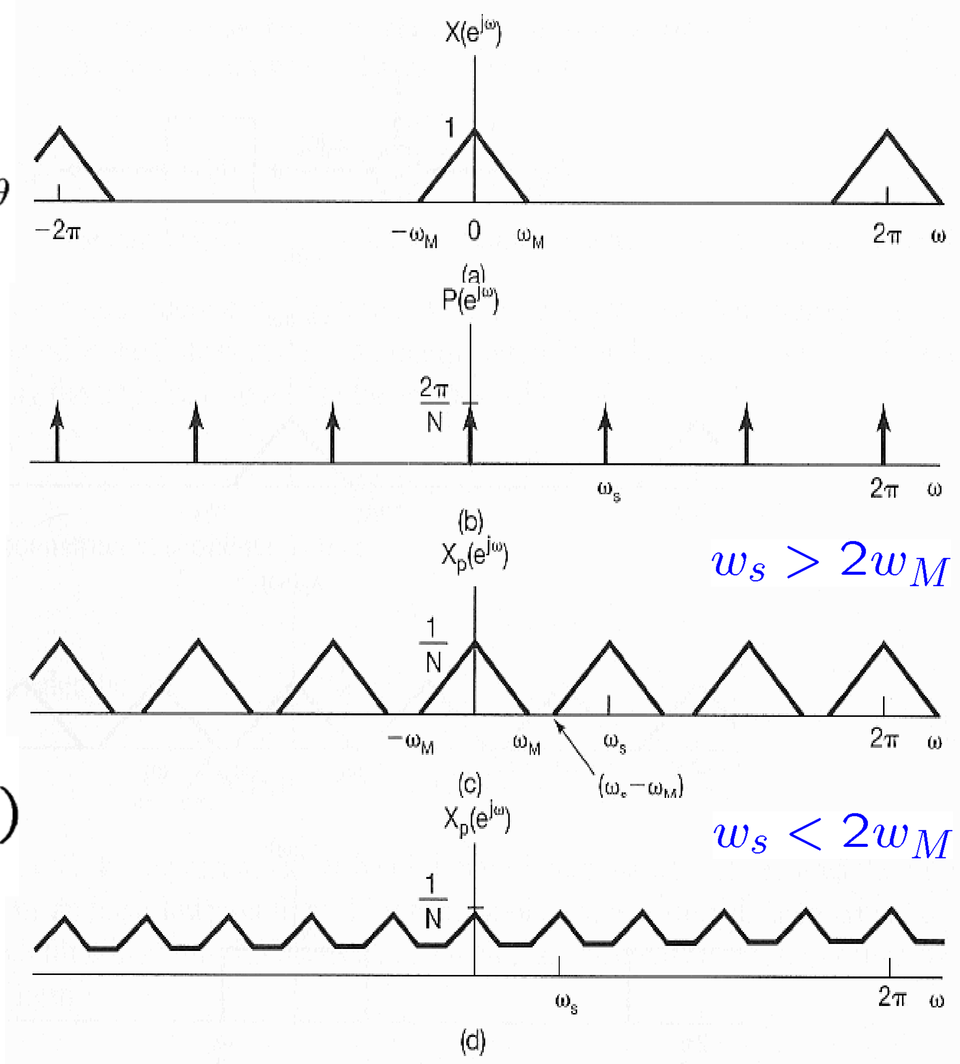


- Impulse-Train Sampling:

$$X_p(e^{jw}) = \frac{1}{2\pi}\int_{2\pi} P(e^{j\theta})X(e^{j(w-\theta)})d\theta$$

$$P(e^{jw}) = \frac{2\pi}{N}\sum_{k=-\infty}^{+\infty}\delta(w - kw_s)$$

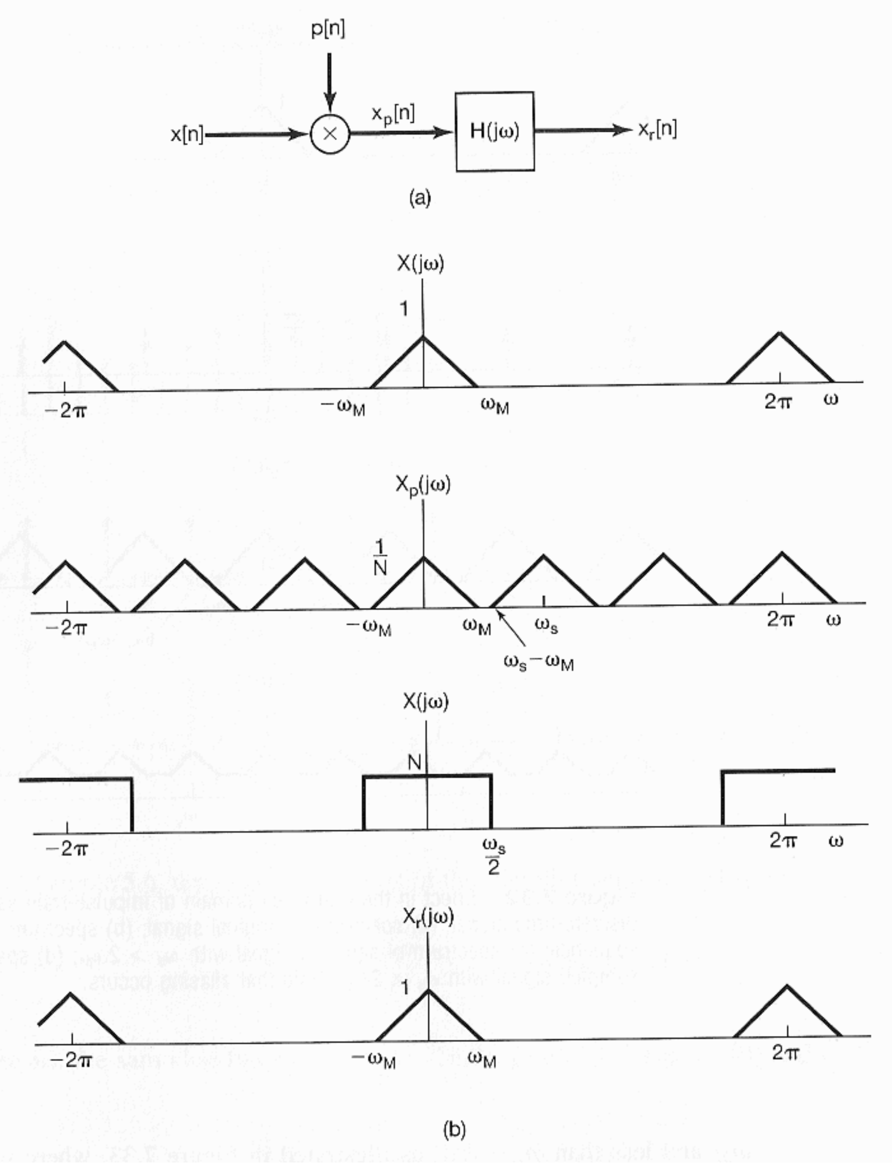
$$\Rightarrow \quad X_p(e^{jw}) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(w-kw_s)})$$



$w_s > 2w_M$

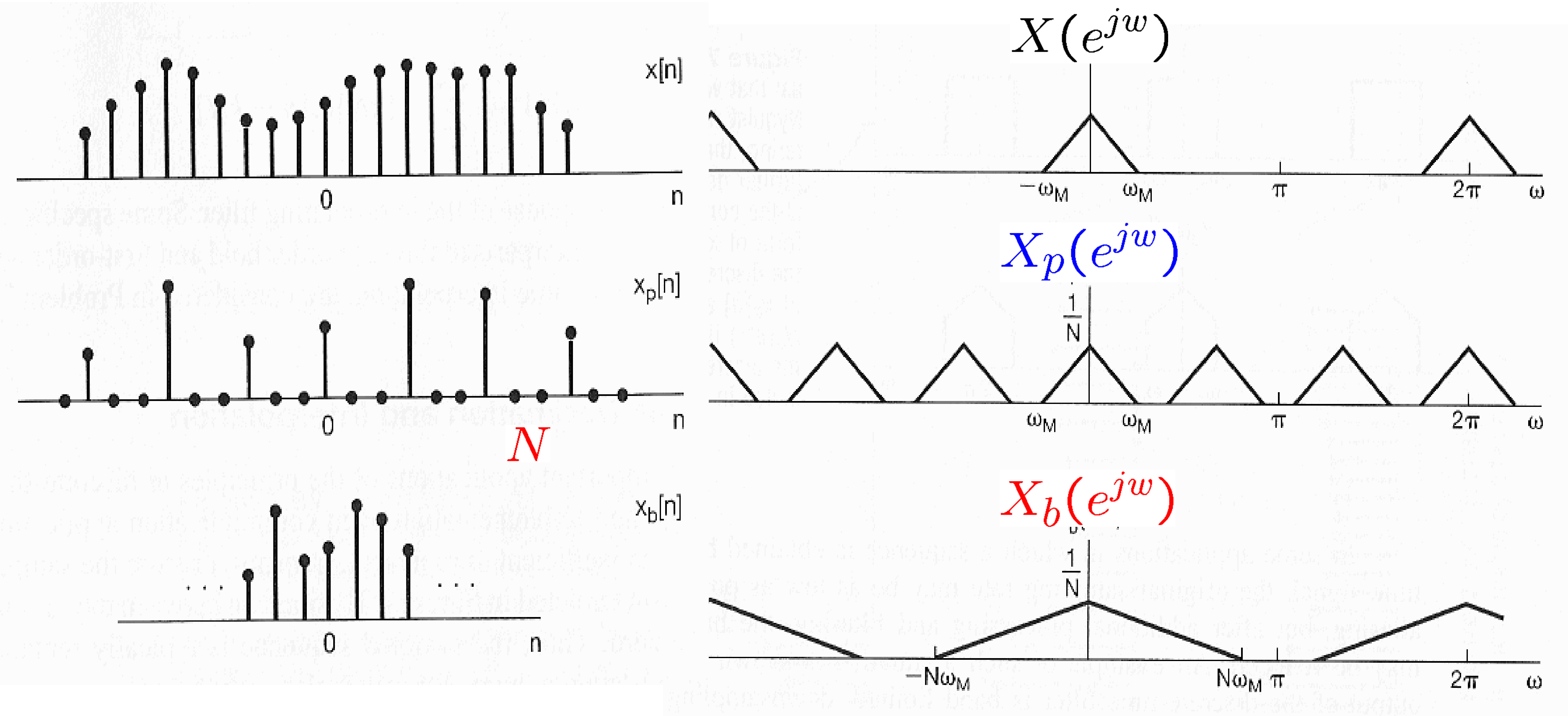
$w_s < 2w_M$

- Exact Recovery Using Ideal Lowpass Filter:



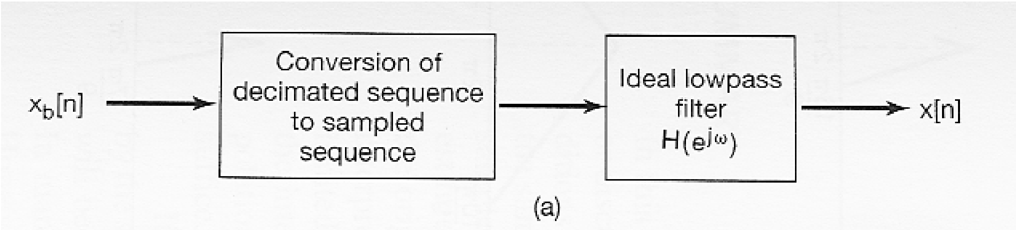
DT Decimation & Interpolation: Down-sampling

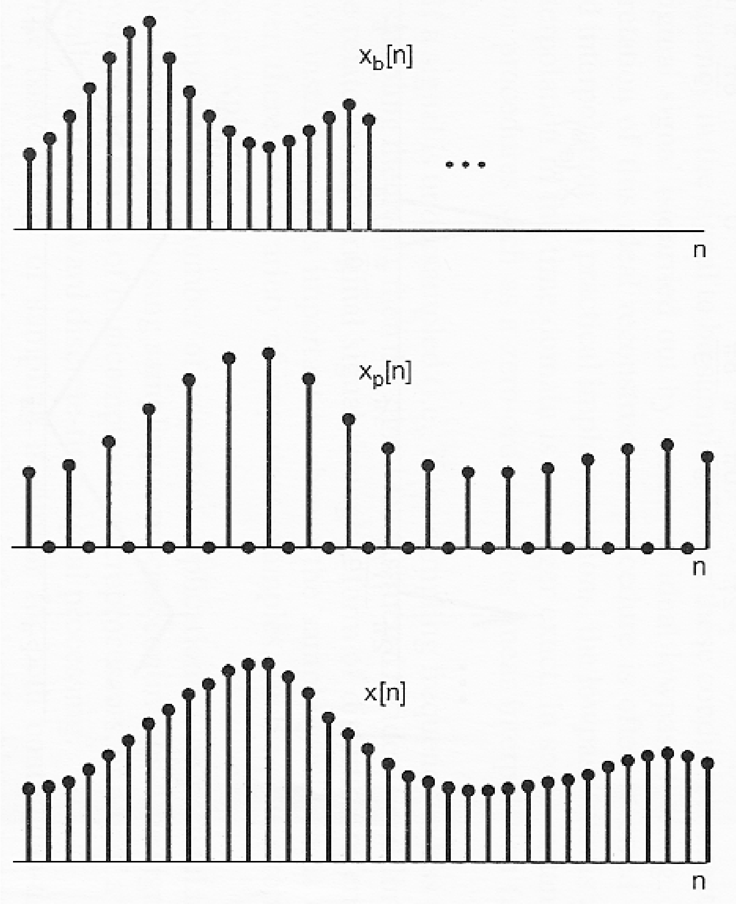
Eq 5.45, p. 378: Time expansion

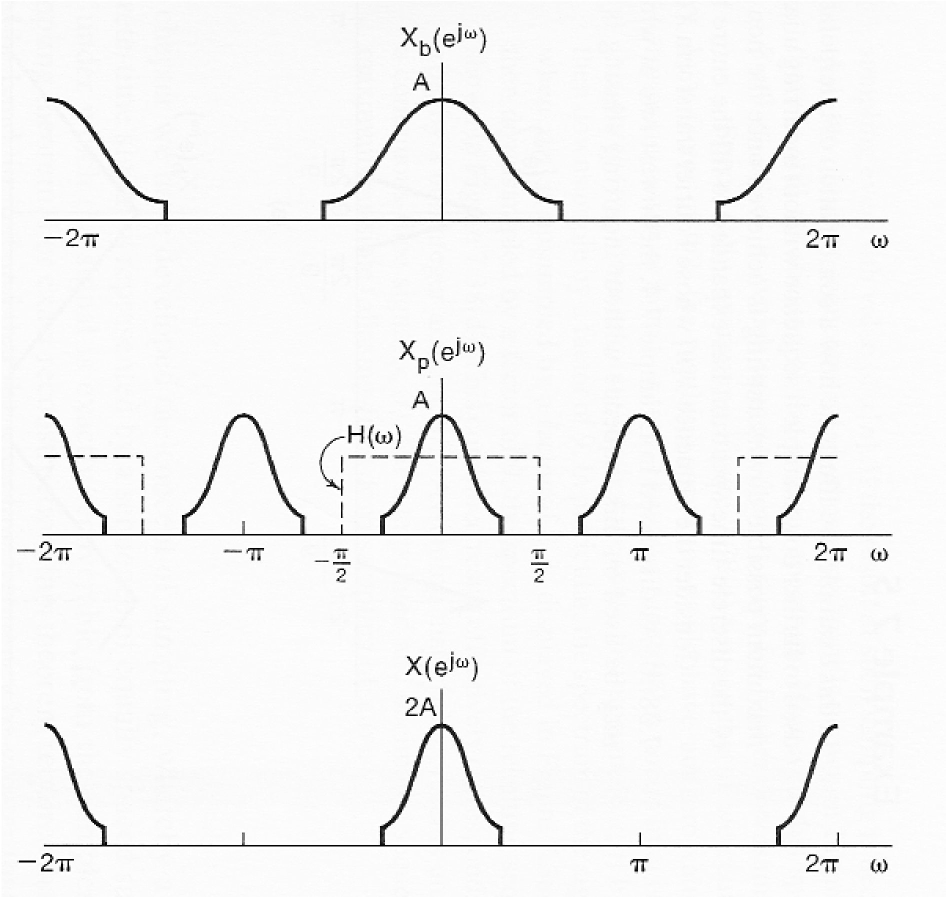


$$X_b(e^{jw}) = X_p(e^{jw/N})$$

- Higher Equivalent Sampling Rate: Up-sampling







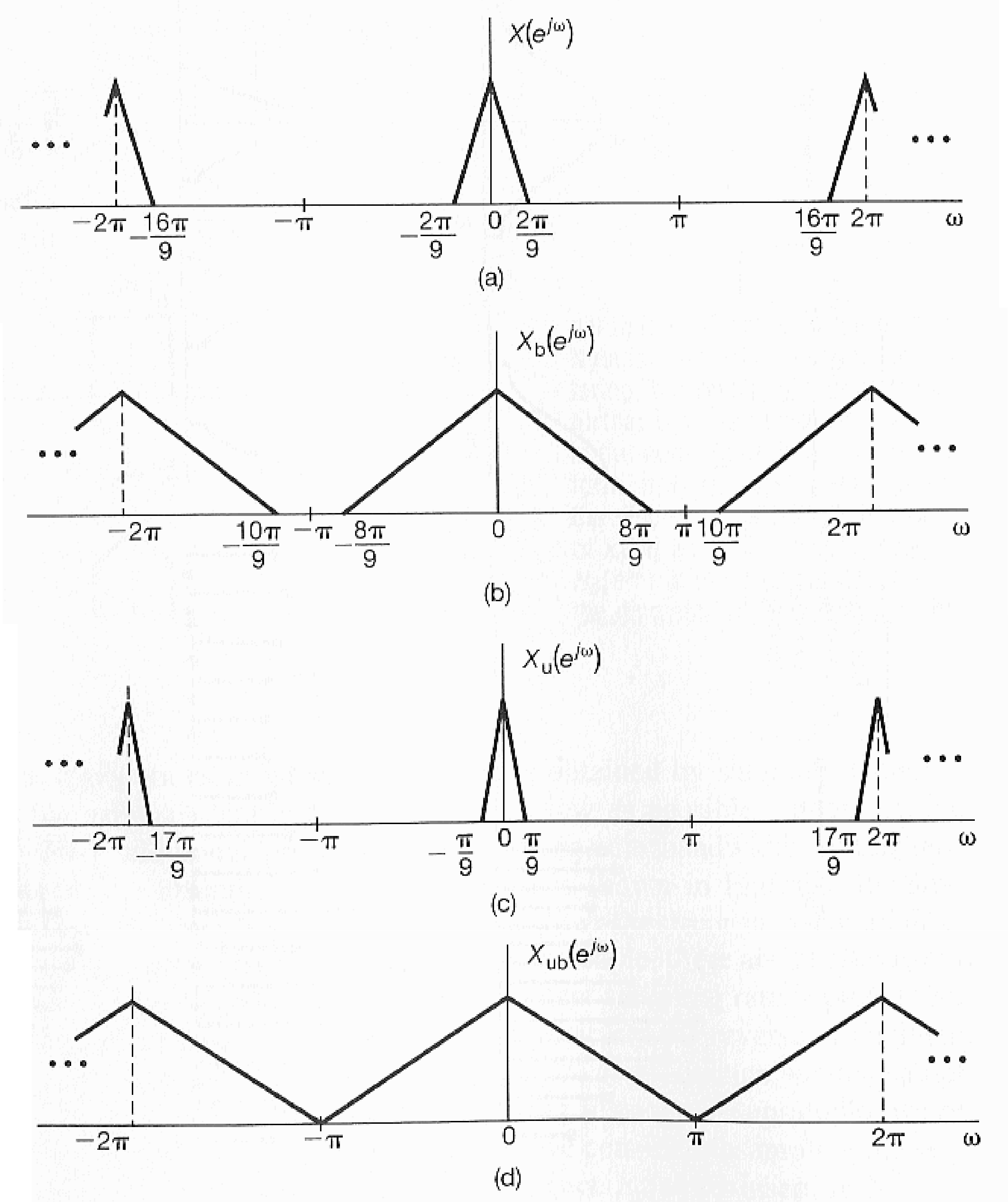
- Down-sampling
  + Up-sampling:

$$\frac{2\pi}{9} \times 4 < \pi$$

$$\frac{2\pi}{9} \times \frac{9}{2} = \pi$$

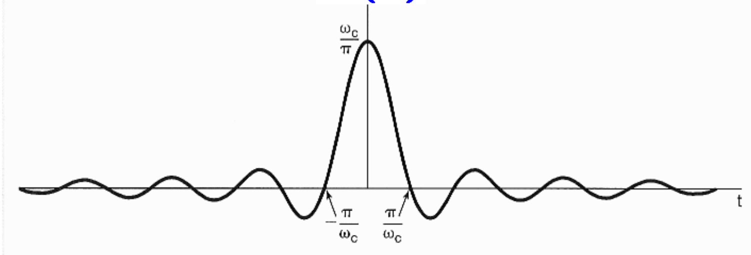
$$\frac{2\pi}{9} \times \frac{1}{2} = \frac{\pi}{9}$$

$$\frac{\pi}{9} \times 9 = \pi$$

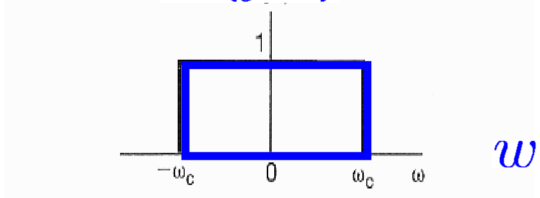


In Summary

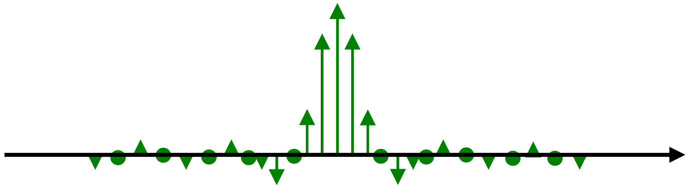
$h(t)$

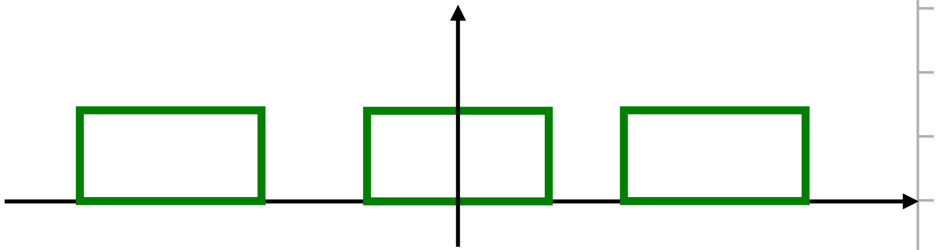


$H(jw)$



w



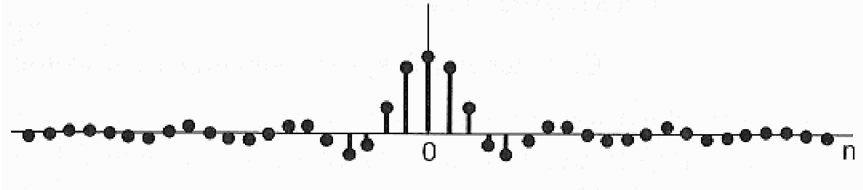


$$h(t) \xleftrightarrow{\mathcal{C.T.F.T.}} H(jw)$$

$$w_s = \frac{2\pi}{T} \qquad\qquad \Omega = wT$$

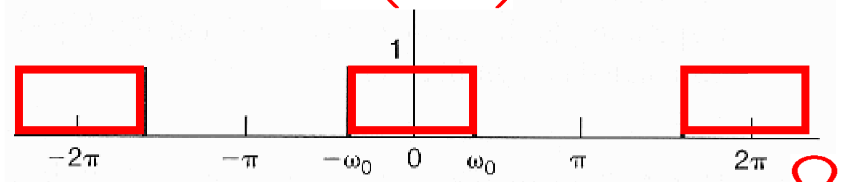
$$h[n] \xleftrightarrow{\mathcal{D.T.F.T.}} H(e^{j\Omega})$$

$h[n]$



(b)

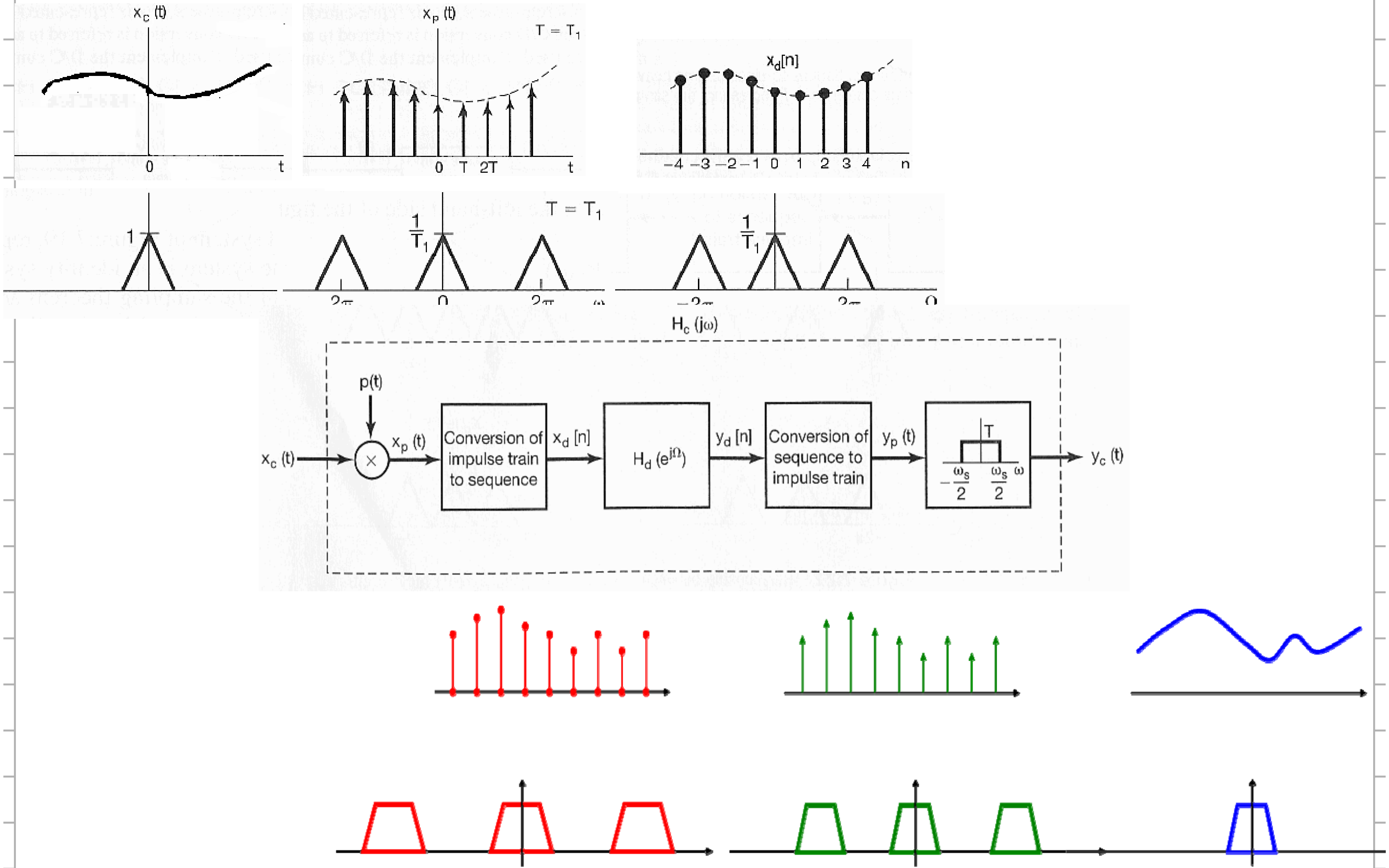
$H(e^{j\Omega})$



(a)

Ω

Discrete-Time Processing of CT Signals



Introduction (Chap 1)

LTI & Convolution (Chap 2)

Bounded/Convergent

Periodic

FS – CT – DT (Chap 3)

Aperiodic

FT
– CT (Chap 4)
– DT (Chap 5)

Unbounded/Non-convergent

LT – CT (Chap 9)

zT – DT (Chap 10)

Time-Frequency (Chap 6)

CT-DT (Chap 7)

Communication (Chap 8)

Control (Chap 11)

Digital Signal Processing (dsp-8)

Spring 2011

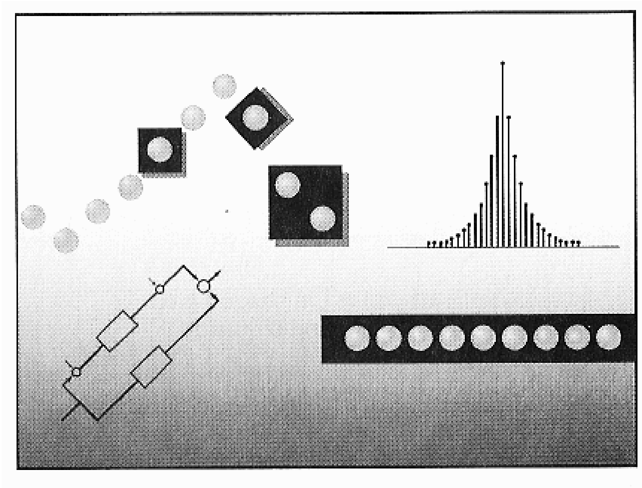
信號與系統
Signals and Systems

Chapter SS-5
The Discrete-Time Fourier Transform

Feng-Li Lian

NTU-EE

Feb11 – Jun11

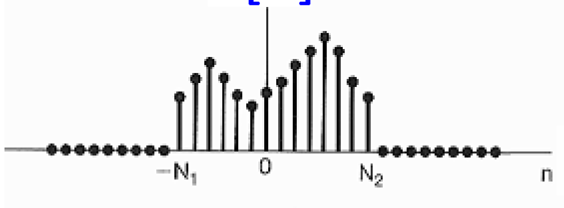


Figures and images used in these lecture notes are adopted from
"Signals & Systems" by Alan V. Oppenheim and Alan S. Willsky, 1997

- Representation of Aperiodic Signals: the Discrete-Time Fourier Transform
- The Fourier Transform for Periodic Signals
- Properties of Discrete-Time Fourier Transform
- The Convolution Property
- The Multiplication Property
- Duality
- Systems Characterized by Linear Constant-Coefficient Difference Equations

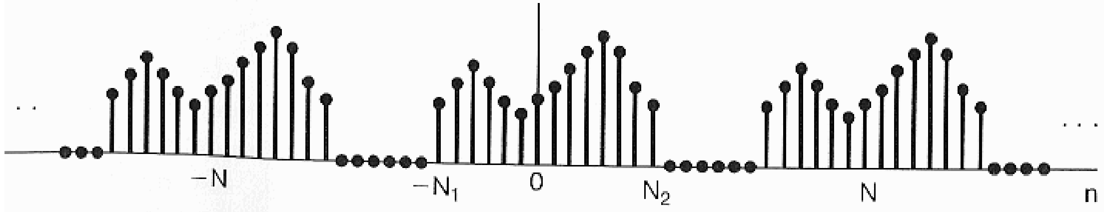
DT Fourier Transform of an Aperiodic Signal:

$x[n]$



an aperiodic signal

$\tilde{x}[n]$



a periodic signal

$$w_0 = \frac{2\pi}{N}$$

$$\tilde{x}[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n} = \sum_{k=<N>} a_k e^{jk(w_0)n}$$

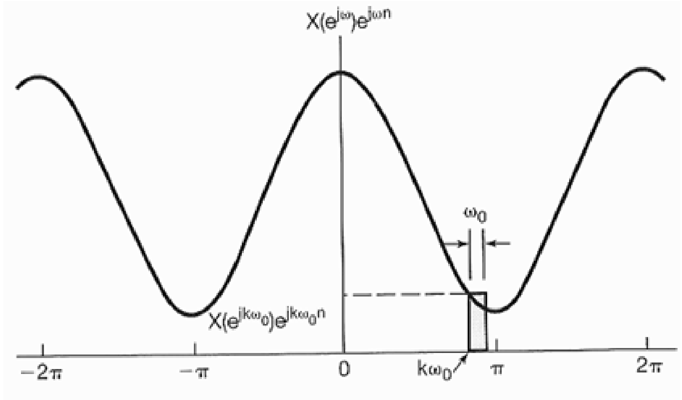
$$a_k = \frac{1}{N} \sum_{n=<N>} \tilde{x}[n] e^{-jk(2\pi/N)n} = \frac{1}{N} \sum_{n=<N>} \tilde{x}[n] e^{-jk(w_0)n}$$

$$\Rightarrow \ a_k = \frac{1}{N} \sum_{n=-N_1}^{N_2} x[n] e^{-jk(w_0)n} = \frac{1}{N} \sum_{n=-\infty}^{+\infty} x[n] e^{-jk(w_0)n}$$

DT Fourier Transform of an Aperiodic Signal:

- Define $X(e^{jw})$:

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn}$$



- Then,

$$a_k = \frac{1}{N} X(e^{jkw_0})$$

$$w = kw_0$$

- Hence,

$$\boxed{w_0 = \frac{2\pi}{N}}$$

$$\boxed{\frac{1}{N} = \frac{1}{2\pi} w_0}$$

$$\boxed{w_0 N = 2\pi}$$

$$\tilde{x}[n] = \sum_{k=<N>} \frac{1}{N} X(e^{jkw_0}) e^{jkw_0 n}$$

$$= \frac{1}{2\pi} \sum_{k=<N>} X(e^{jkw_0}) e^{jkw_0 n} w_0$$

▪ DT Fourier Transform of an Aperiodic Signal:

- As $N \to \infty$, $\tilde{x}[n] \to x[n]$

$$\boxed{w_0 N = 2\pi}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw$$

- inverse Fourier transform eqn
- synthesis eqn

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}$$

- $X(e^{jw})$: Fourier transform of $x[n]$
  spectrum
- analysis eqn

$$a_k = \frac{1}{N}X(e^{jw})\Big|_{w=kw_0}$$

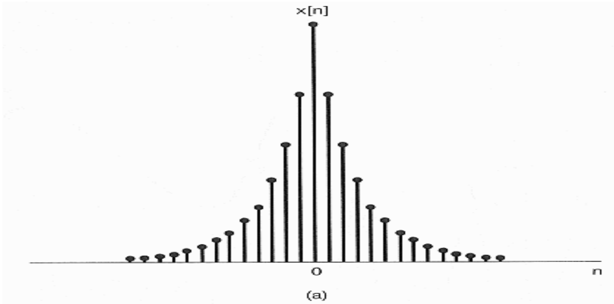
$$\boxed{w_0 = \frac{2\pi}{N}}$$

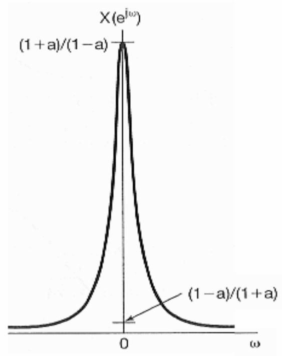
Periodicity of DT Fourier Transform:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw$$

is a periodic continues time function,w

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}$$





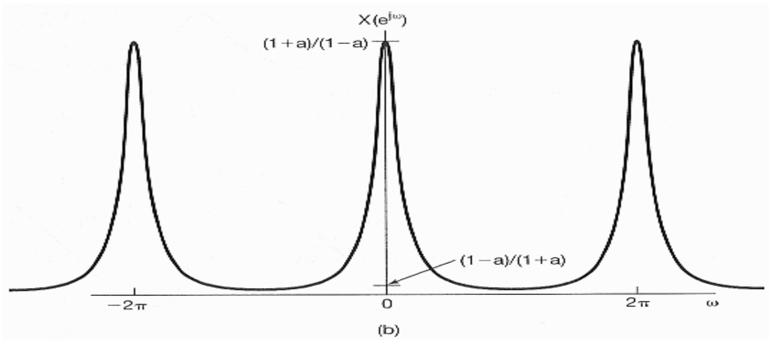
$$X(e^{j(w+2\pi)}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j(w+2\pi)n}$$

$$= \sum_{n=-\infty}^{+\infty} x[n]e^{-j(w)n}e^{-j(2\pi)n}$$

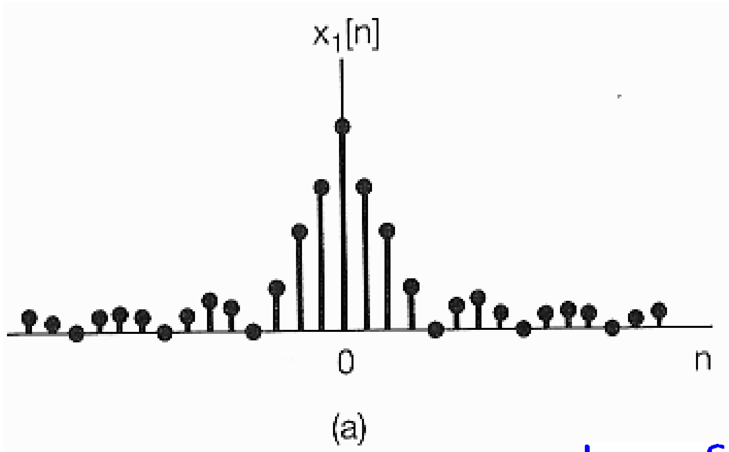
$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

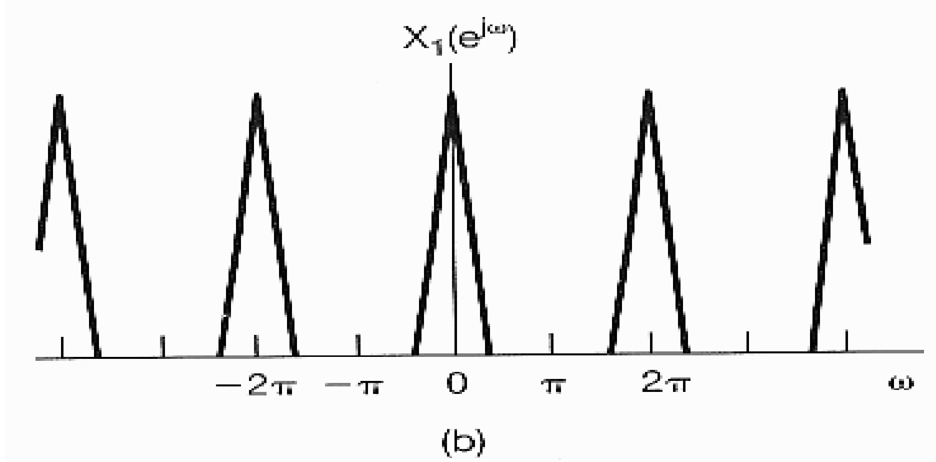
$$= \sum_{n=-\infty}^{+\infty} x[n]e^{-j(w)n}$$

$$= X(e^{jw})$$

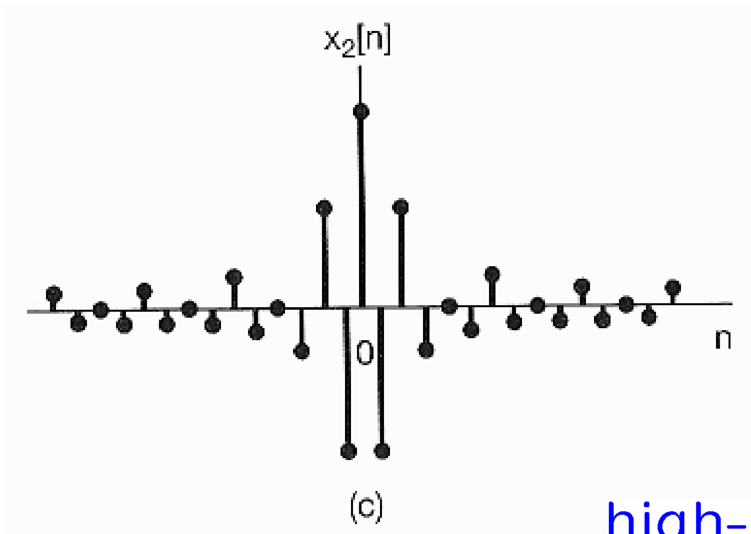


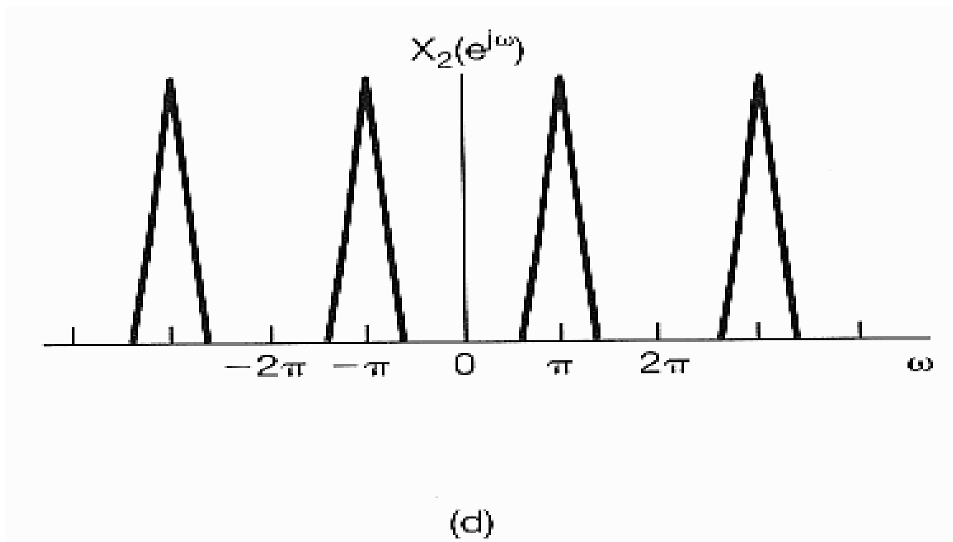
- High-Frequency & Low-Frequency Signals:





low-frequency signal



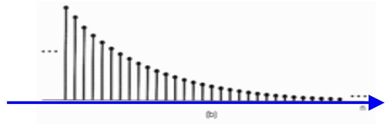


high-frequency signal

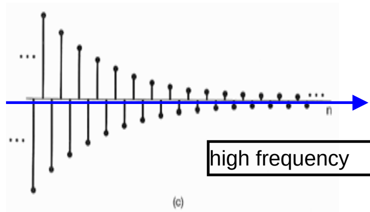
Feng-Li Lian © 2011

- **Example 5.1:**

$0 < a < 1$



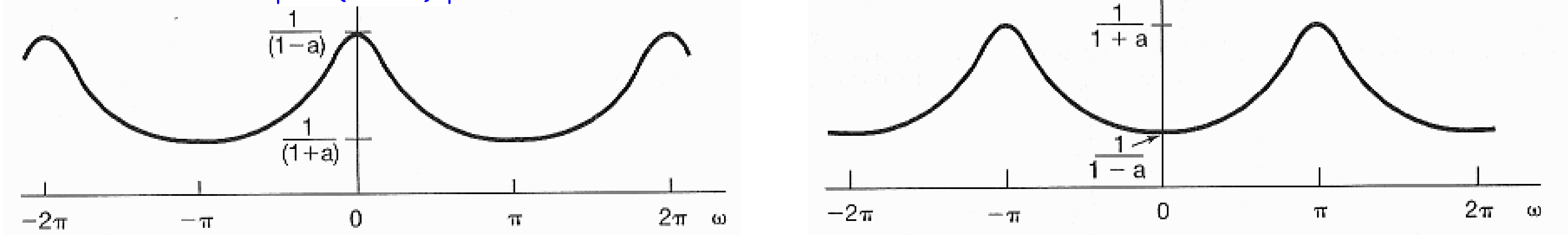
$-1 < a < 0$



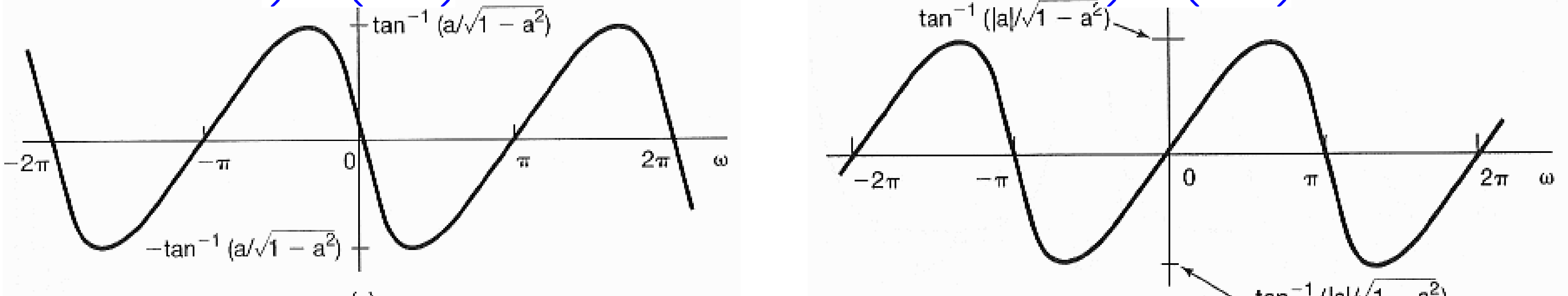
$$x[n] = a^n \, u[n], \quad |a| < 1$$

$$\Rightarrow \; X(e^{jw}) = \sum_{n=-\infty}^{+\infty} a^n \, u[n] \, e^{-jwn} = \sum_{n=0}^{\infty} (ae^{-jw})^n = \frac{1}{1 - ae^{-jw}}$$

$0 < a < 1$ $\quad |X(e^{jw})|$ $\qquad -1 < a < 0$ $\quad |X(e^{jw})|$



$\sphericalangle X(e^{jw})$ $\qquad \sphericalangle X(e^{jw})$



(a) $\qquad$ (b)

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn}$$

- **Example 5.2:** $x[n] = a^{|n|}, \quad 0 < a < 1$

  $-1 < a < 0$

$$\Rightarrow X(e^{jw}) = \sum_{n=-\infty}^{+\infty} a^{|n|} e^{-jwn}$$

$$= \sum_{n=0}^{+\infty} a^{n} e^{-jwn} + \sum_{n=-\infty}^{-1} a^{-n} e^{-jwn}$$

$$= \sum_{n=0}^{+\infty} (ae^{-jw})^{n} + \sum_{m=1}^{\infty} (ae^{jw})^{m}$$

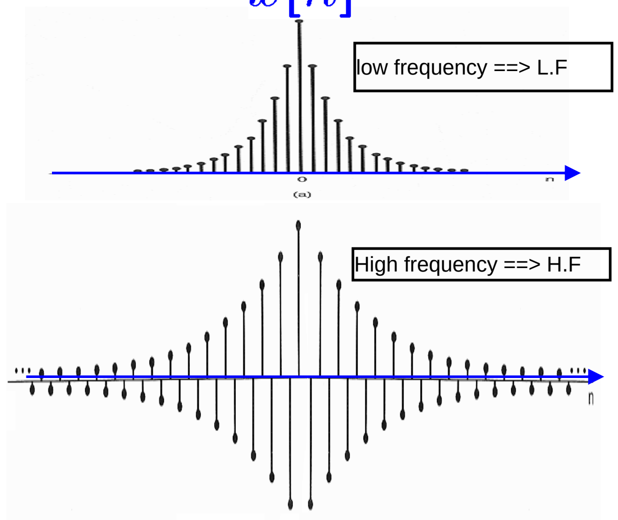
$$= \frac{1}{1 - ae^{-jw}} + \frac{ae^{jw}}{1 - ae^{jw}}$$

$$= \frac{1 - a^2}{1 - 2a\cos w + a^2}$$

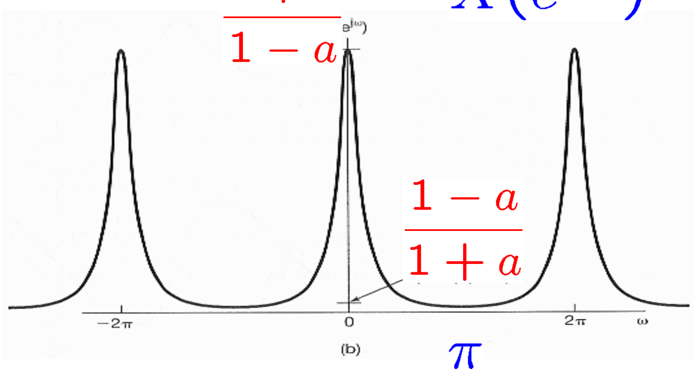
$$= \frac{1 - a^2}{(1 - a)^2}$$

$$= \frac{1 - a^2}{(1 + a)^2}$$

$x[n]$



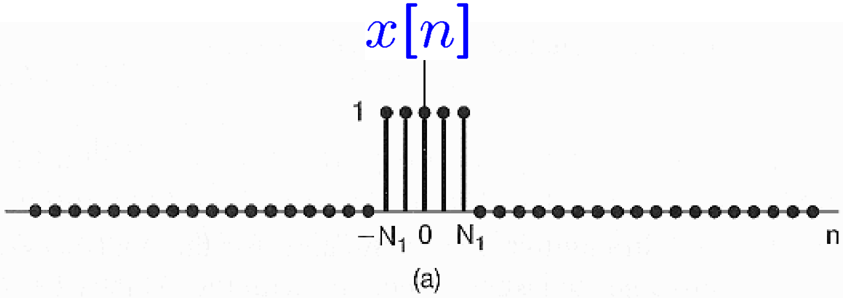
$X(e^{jw})$



- Example 5.3:

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}$$

$$x[n] = \begin{cases} 1, & |n| \leq N_1 \\ 0, & |n| > N_1 \end{cases}$$



(a)

$$\Rightarrow X(e^{jw}) = \sum_{n=-N_1}^{N_1} e^{-jwn}$$

$$= e^{-jw(-N_1)} + \cdots + e^{-jw(N_1)} \quad = e^{-jw(-N_1)} \left(\frac{1-(e^{-jw})^{2N_1+1}}{1-(e^{-jw})} \right)$$

$$= e^{jw(N_1)} \left(\frac{(e^{-jw})^{N_1+1/2}\left((e^{jw})^{N_1+1/2}-(e^{-jw})^{N_1+1/2}\right)}{(e^{-jw/2})\left((e^{jw/2})-(e^{-jw/2})\right)} \right)$$

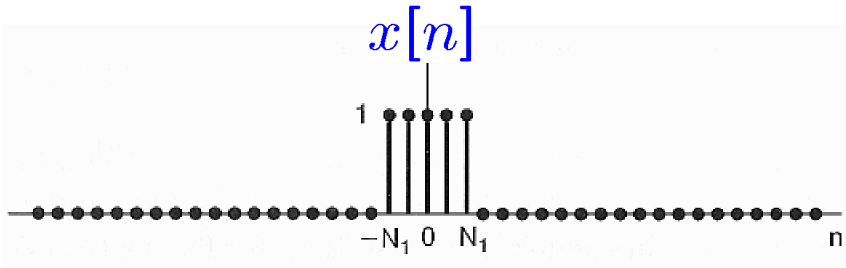
$$= \frac{\sin\left(w(N_1+\frac{1}{2})\right)}{\sin(w/2)}$$

$$\boxed{\begin{aligned} & 1 - e^{-j\theta} \\ &= e^{-j\theta/2}e^{j\theta/2} - e^{-j\theta/2}e^{-j\theta/2} \\ &= e^{-j\theta/2}\left(e^{j\theta/2} - e^{-j\theta/2}\right) \end{aligned}}$$

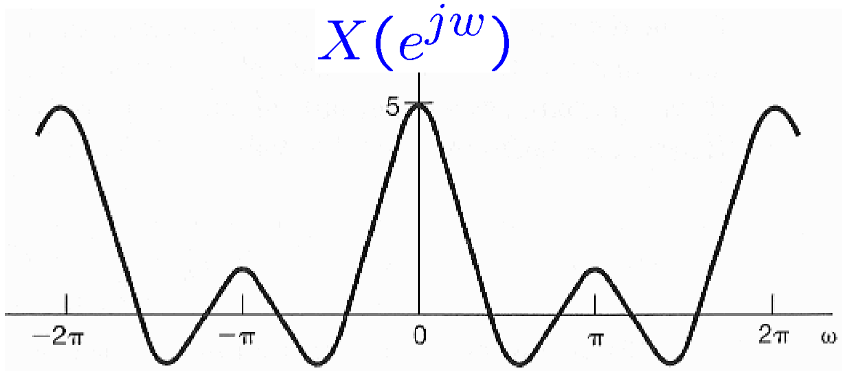
- Example 5.3:

$$\Rightarrow \; X(e^{jw}) = \sum_{n=-N_1}^{N_1} e^{-jwn}$$

$$= \frac{\sin\left(w(N_1+\frac{1}{2})\right)}{\sin(w/2)}$$



(a)



(b)

- Convergence of DT Fourier Transform:

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{jw}) e^{jwn} dw$$

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn}$$

- The analysis equation will converge:
  - Either if x[n] is absolutely summable, that is,

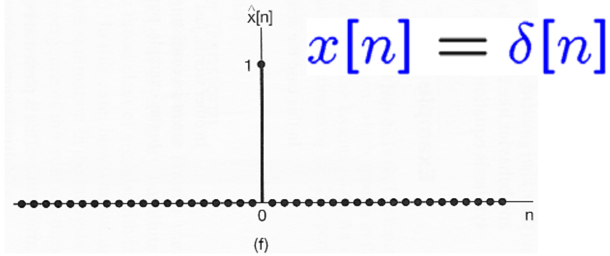
$$\sum_{n=-\infty}^{+\infty} |x[n]| < \infty$$

  - Or if x[n] has finite energy, that is,

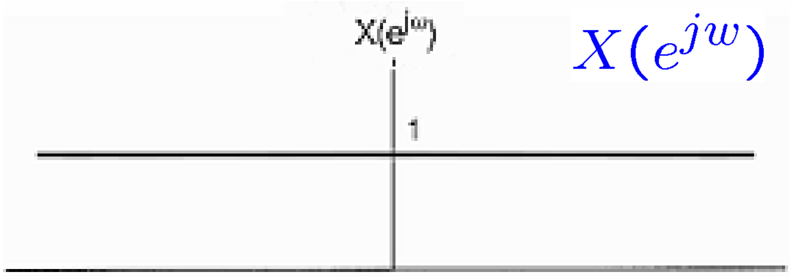
$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 < \infty$$

- **Example 5.4:**

$$x[n] = \delta[n], \quad \text{i.e., unit impulse}$$

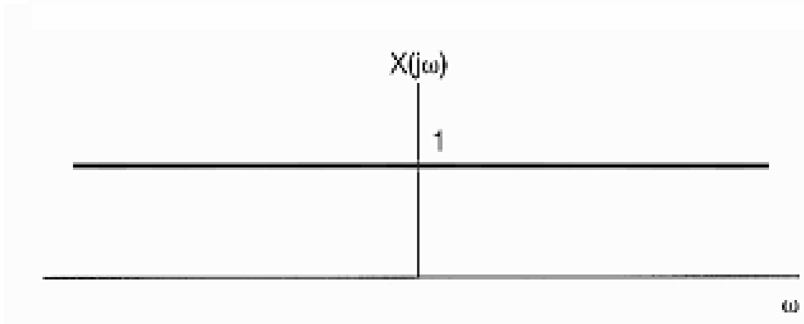


$$\Rightarrow \; X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn} = 1$$



$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{jw}) e^{jwn} dw$$

$$= \frac{1}{2\pi} \int_{2\pi} e^{jwn} dw$$

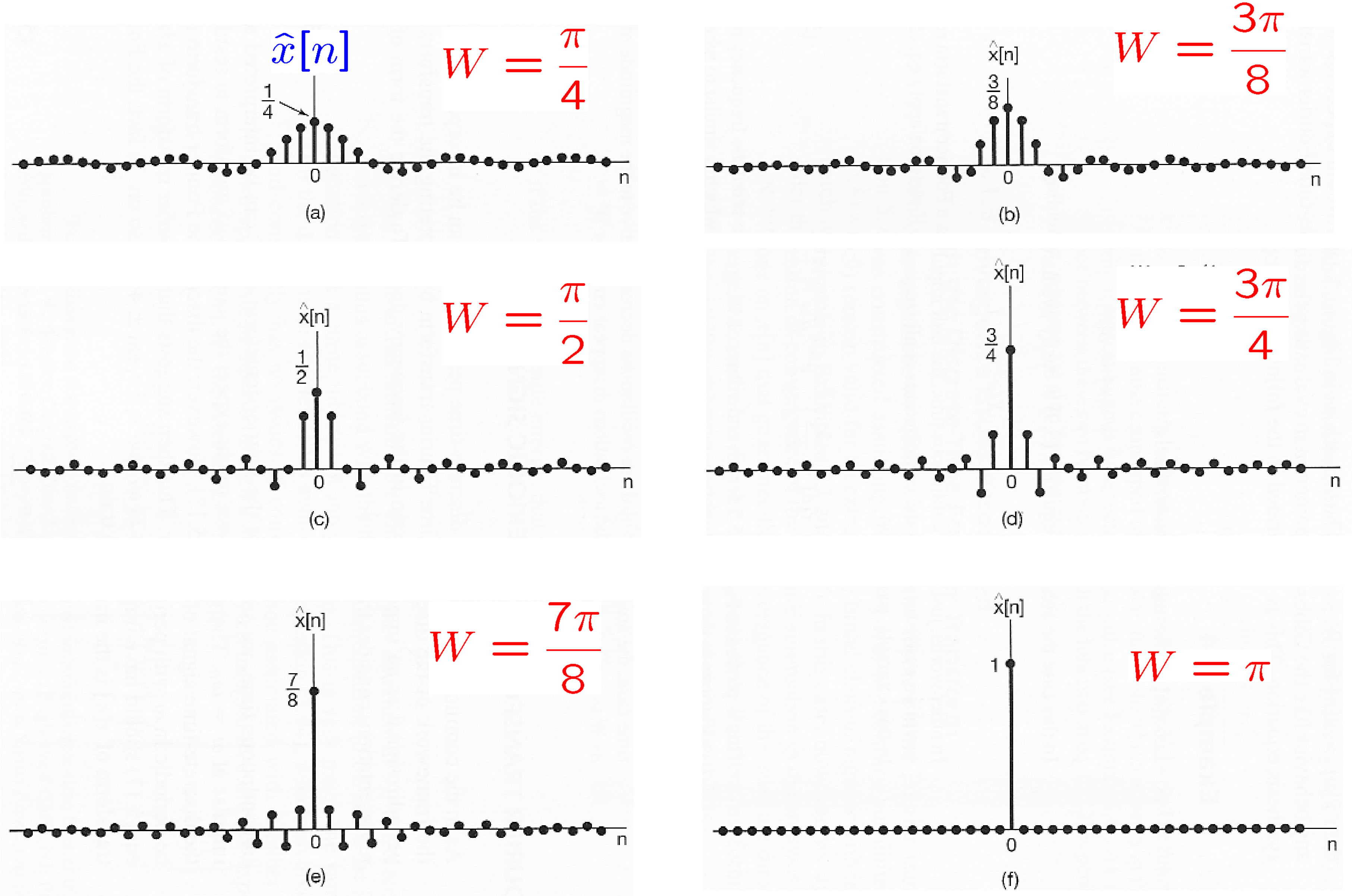


- **Approximation**

$$\hat{x}[n] = \frac{1}{2\pi} \int_{-W}^{+W} X(e^{jw}) e^{jwn} dw = \frac{\sin Wn}{\pi n}$$

Approximation of an Aperiodic Signal:

$$\boxed{\hat{x}[n] = \frac{\sin Wn}{\pi n}}$$



- Representation of Aperiodic Signals: the Discrete-Time Fourier Transform
- The Fourier Transform for Periodic Signals
- Properties of Discrete-Time Fourier Transform
- The Convolution Property
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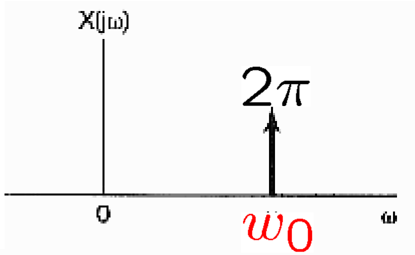
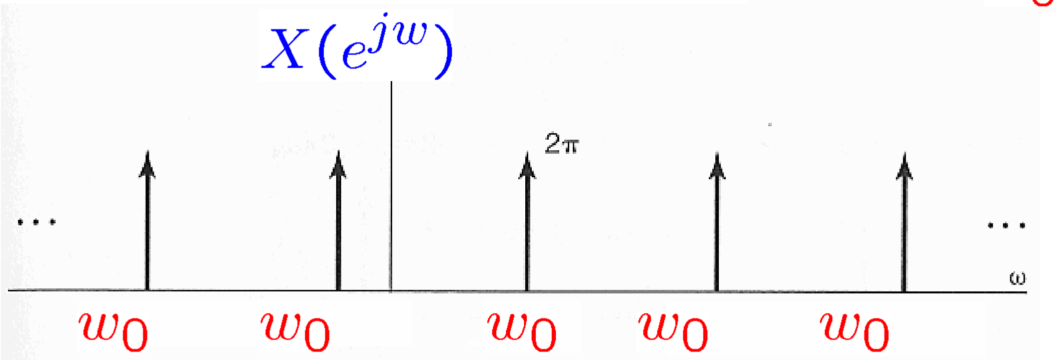
- Systems Characterized by Linear Constant-Coefficient Difference Equations

▪ Fourier Transform from Fourier Series:

$$x(t) = e^{jw_0 t} \xleftrightarrow{\mathcal{CTFT}} X(jw) = 2\pi\delta(w - w_0)$$



$$x[n] = e^{jw_0 n} \xleftrightarrow{\mathcal{DTFT}}$$



$$X(e^{jw}) = \cdots + 2\pi\delta(w - w_0 + 2\pi) + 2\pi\delta(w - w_0) + 2\pi\delta(w - w_0 - 2\pi) + \cdots$$

$$= \sum_{l=-\infty}^{+\infty} 2\pi\delta(w - w_0 - 2\pi l)$$

$$\boxed{w_0 = \frac{2\pi}{N}}$$

$$\frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw = \frac{1}{2\pi}\int_{2\pi}\sum_{l=-\infty}^{+\infty} 2\pi\delta(w - w_0 - 2\pi l)e^{jwn}\, dw$$

$$= e^{j(w_0 + 2\pi r)n} = e^{jw_0 n}$$

$$w_0 = \frac{2\pi}{N}$$

▪ Fourier Transform from Fourier Series:

- more generally,

$$x[n] = \sum_{k=<N>} a_k \, e^{jk\left(\frac{2\pi}{N}\right)n} = \sum_{k=<N>} a_k \, e^{jk(w_0)n}$$

$$X(e^{jw}) = \sum_{k=-\infty}^{+\infty} 2\pi \, a_k \, \delta\left(w - k\frac{2\pi}{N}\right) = \sum_{k=-\infty}^{+\infty} 2\pi \, a_k \, \delta\left(w - kw_0\right)$$

- If $k = 0, 1, ..., N-1$

$$x[n] = a_0 + a_1 \, e^{j\,1\,\left(\frac{2\pi}{N}\right)n} + a_2 \, e^{j\,2\,\left(\frac{2\pi}{N}\right)n} + \cdots + a_{N-1} \, e^{j\,(N-1)\,\left(\frac{2\pi}{N}\right)n}$$

$$= x_0 + x_1 + x_2 + \cdots + x_{N-1}$$

a linear combination of signals

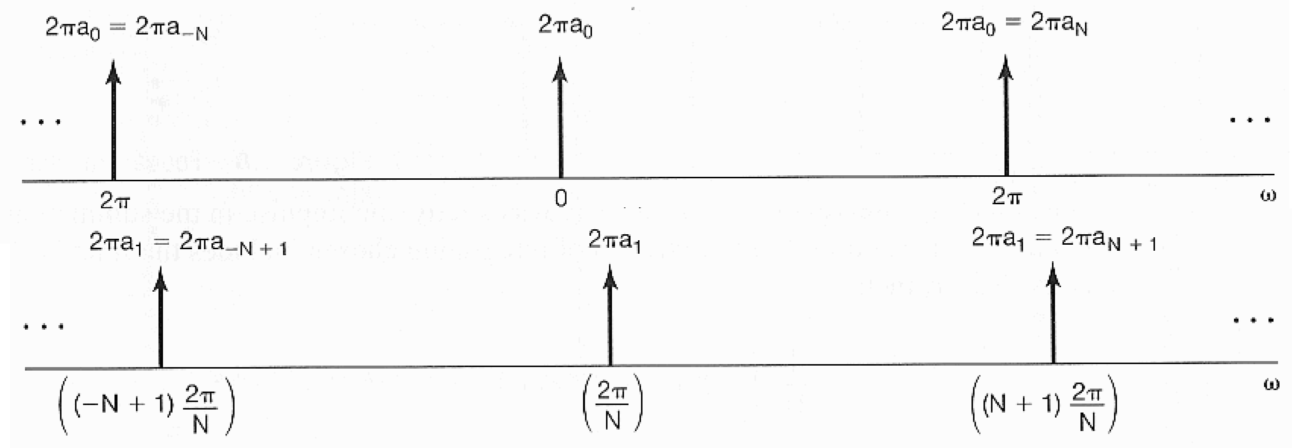
with $w_0 = 0, \frac{2\pi}{N}, \frac{2\cdot 2\pi}{N}, \cdots, \frac{(N-1)\cdot 2\pi}{N}$

▪ Fourier Transform from Fourier Series:

$$\boxed{w_0 = \frac{2\pi}{N}}$$

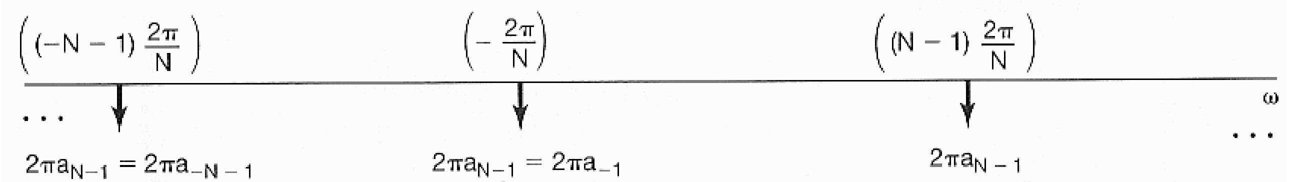
$$x_0 \overset{\mathcal{F}}{\longleftrightarrow}$$

$$x_1 \overset{\mathcal{F}}{\longleftrightarrow}$$

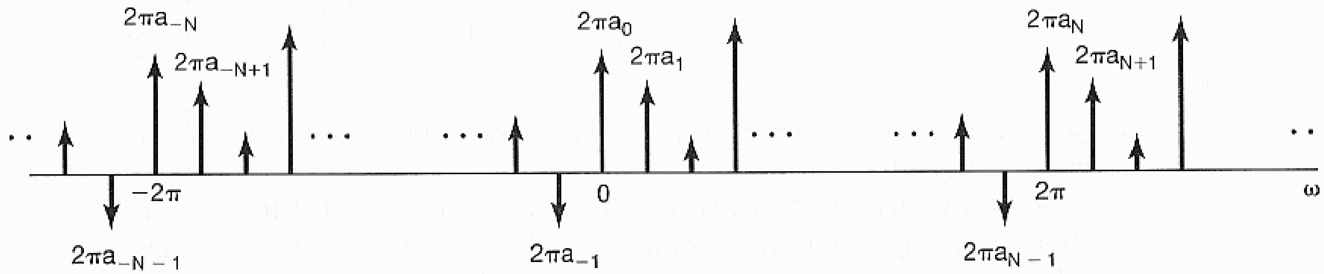


(b)

$$x_{N-1} \overset{\mathcal{F}}{\longleftrightarrow}$$



$$X(e^{jw}) \quad =$$



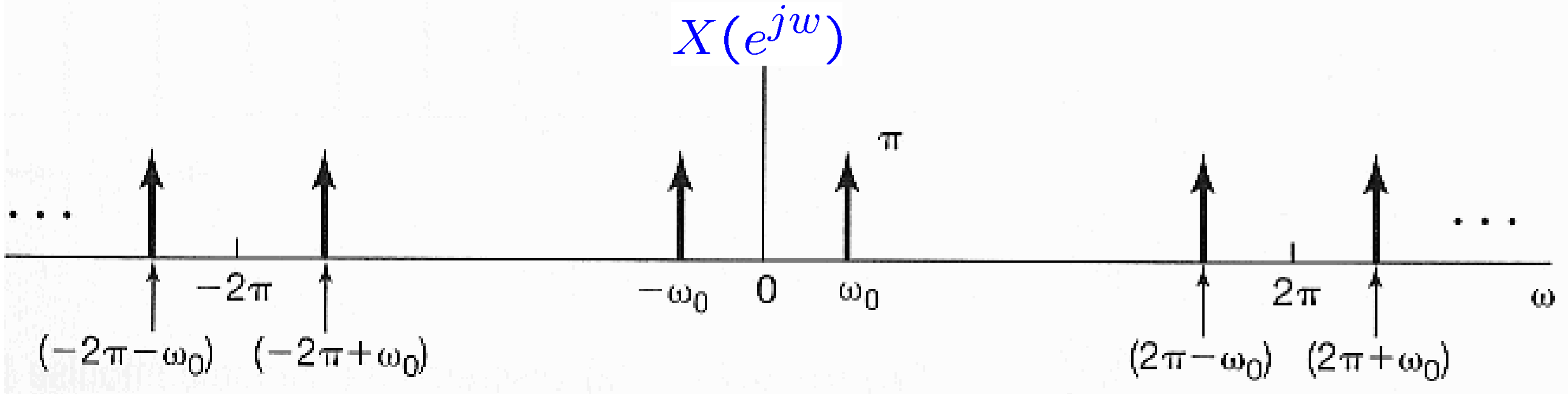
(d)

- Example 5.5:

$$x[n] = \cos(w_0 n) \quad = \frac{e^{jw_0 n} + e^{-jw_0 n}}{2} \qquad \text{with } w_0 = \frac{2\pi}{5}$$

$$X(e^{jw}) = \sum_{l=-\infty}^{+\infty} \pi\,\delta\left(w - \frac{2\pi}{5} - 2\pi l\right) + \sum_{l=-\infty}^{+\infty} \pi\,\delta\left(w + \frac{2\pi}{5} - 2\pi l\right)$$

$$= \pi\,\delta\left(w - \frac{2\pi}{5}\right) + \pi\,\delta\left(w + \frac{2\pi}{5}\right), \quad -\pi \le w < \pi$$



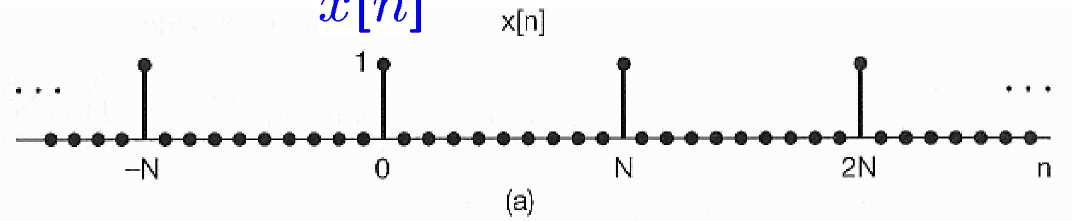
- **Example 5.6:**

$$x[n] = \sum_{k=<N>} a_k \, e^{jk(2\pi/N)n}$$

$$X(e^{jw}) = \sum_{k=-\infty}^{+\infty} 2\pi \, a_k \, \delta\left(w - \frac{2\pi k}{N}\right)$$

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk(2\pi/N)n}$$

$x[n]$



(a)

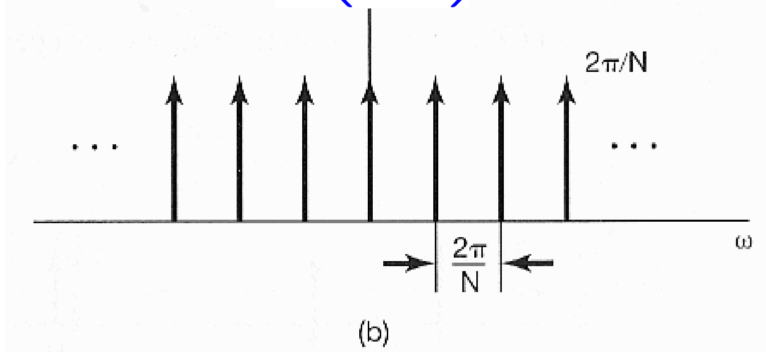
$$x[n] = \sum_{k=-\infty}^{+\infty} \delta[n - kN]$$

choose $0 \leq n \leq N-1$

$$\Rightarrow \; a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk(2\pi/N)n} = \frac{1}{N}$$

$$\Rightarrow \; X(e^{jw}) = \frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(w - k\frac{2\pi}{N}\right)$$

$X(e^{jw})$



(b)

- Representation of Aperiodic Signals: the Discrete-Time Fourier Transform
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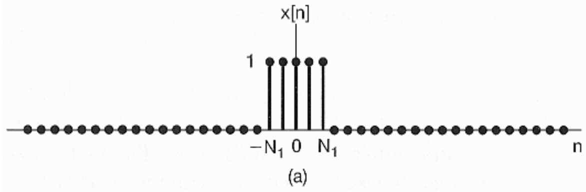
Outline

Section	**Property**
5.3.2	Linearity
5.3.3	Time Shifting
5.3.3	Frequency Shifting
5.3.4	Conjugation
5.3.6	Time Reversal
5.3.7	Time Expansion
5.4	Convolution
5.5	Multiplication
5.3.5	Differencing in Time
5.3.5	Accumulation
5.3.8	Differentiation in Frequency
5.3.4	Conjugate Symmetry for Real Signals
5.3.4	Symmetry for Real and Even Signals
5.3.4	Symmetry for Real and Odd Signals
5.3.4	Even-Odd Decomposition for Real Signals
5.3.9	Parseval's Relation for Aperiodic Signals

Outline

Property	CTFS	DTFS	CTFT	DTFT	LT	zT
Linearity	3.5.1		4.3.1	5.3.2	9.5.1	10.5.1
Time Shifting	3.5.2		4.3.2	5.3.3	9.5.2	10.5.2
Frequency Shifting (in s, z)			4.3.6	5.3.3	9.5.3	10.5.3
Conjugation	3.5.6		4.3.3	5.3.4	9.5.5	10.5.6
Time Reversal	3.5.3		4.3.5	5.3.6		10.5.4
Time & Frequency Scaling	3.5.4		4.3.5	5.3.7	9.5.4	10.5.5
(Periodic) Convolution			4.4	5.4	9.5.6	10.5.7
Multiplication	3.5.5	3.7.2	4.5	5.5		
Differentiation/First Difference		3.7.2	4.3.4, 4.3.6	5.3.5, 5.3.8	9.5.7, 9.5.8	10.5.7, 10.5.8
Integration/Running Sum (Accumulation)			4.3.4	5.3.5	9.5.9	10.5.7
Conjugate Symmetry for Real Signals	3.5.6		4.3.3	5.3.4		
Symmetry for Real and Even Signals	3.5.6		4.3.3	5.3.4		
Symmetry for Real and Odd Signals	3.5.6		4.3.3	5.3.4		
Even-Odd Decomposition for Real Signals			4.3.3	5.3.4		
Parseval's Relation for (A)Periodic Signals	3.5.7	3.7.3	4.3.7	5.3.9		
Initial- and Final-Value Theorems					9.5.10	10.5.9

- **Fourier Transform Pair:**

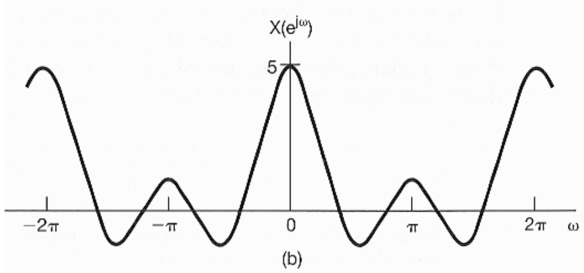


  - Synthesis equation:

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{jw}) e^{jwn} dw$$

  - Analysis equation:

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn}$$



  - Notations:

$$|a| < 1$$

$$X(e^{jw}) = \mathcal{F}\{x[n]\} \qquad \frac{1}{1 - ae^{jw}} = \mathcal{F}\{a^n u[n]\}$$

$$x[n] = \mathcal{F}^{-1}\{X(e^{jw})\} \qquad a^n u[n] = \mathcal{F}^{-1}\{\frac{1}{1 - ae^{jw}}\}$$

$$x[n] \xleftrightarrow{\mathcal{DTFT}} X(e^{jw}) \qquad a^n u[n] \xleftrightarrow{\mathcal{DTFT}} \frac{1}{1 - ae^{jw}}$$

Periodicity of DT Fourier Transform:

$$X(e^{j(w+2\pi)}) = X(e^{jw})$$

e^(j pi n) * x[n] --> X(e^(j(w-pi))

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw$$

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}$$

Linearity:

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{jw})$$

$$y[n] \overset{\mathcal{F}}{\longleftrightarrow} Y(e^{jw})$$

$$\Rightarrow a\, x[n] + b\, y[n] \overset{\mathcal{F}}{\longleftrightarrow} a\, X(e^{jw}) + b\, Y(e^{jw})$$

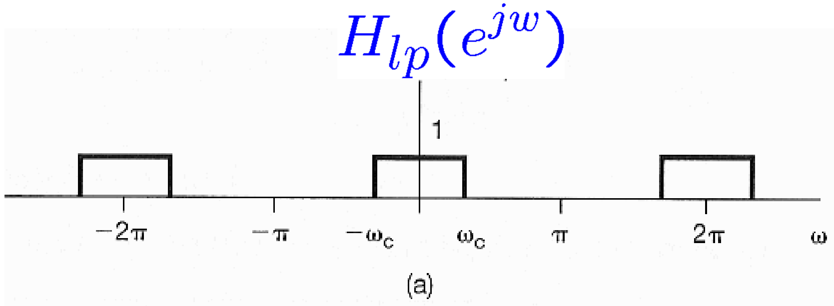
Time & Frequency Shifting:

sin(w0n) * x[n] ==> 1/2j (e^jw0n-e^-jw0n)cos(w0n) * x[n] ==> 1/2 (e^jw0n+e^-jw0n)we use time

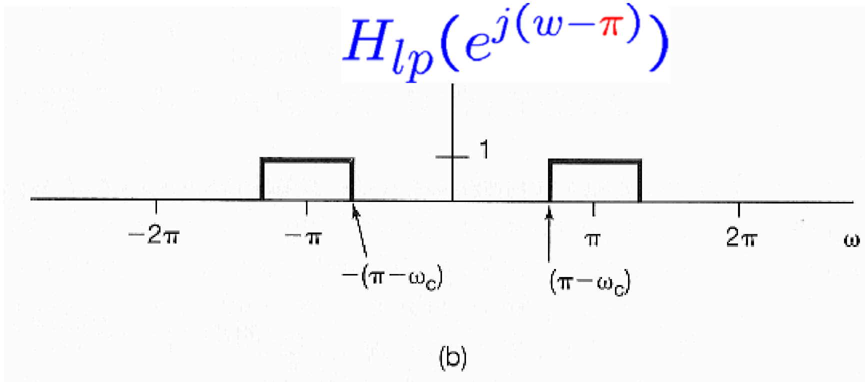
$$\Rightarrow x[n-n_0] \overset{\mathcal{F}}{\longleftrightarrow} e^{-jwn_0}X(e^{jw})$$

$$\Rightarrow e^{jw_0n}x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j(w-w_0)})$$

Example 5.7:



(a)



(b)

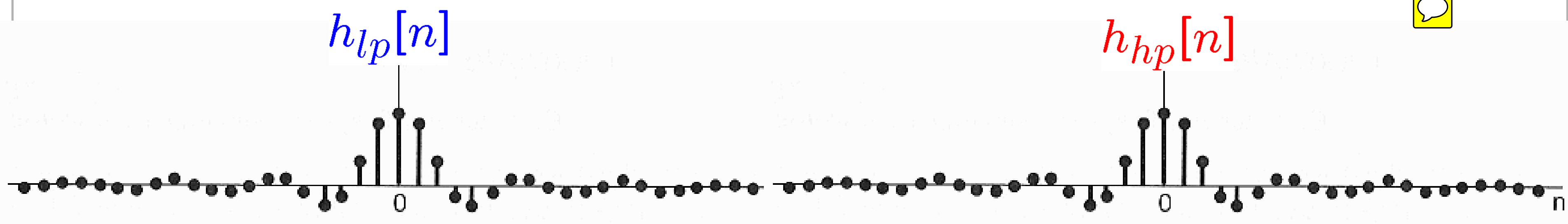
$$H_{hp}(e^{jw}) = H_{lp}(e^{j(w-\pi)})$$

$$\Rightarrow \; h_{hp}[n] = e^{j\pi n} \; h_{lp}[n]$$

$$e^{j\pi n} = \cos(\pi n) + j\sin(\pi n)$$

= -1^n

$$= (-1)^n h_{lp}[n]$$



Conjugation & Conjugate Symmetry:

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn}$$

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{jw}) \qquad x^*[n] \overset{\mathcal{F}}{\longleftrightarrow} X^*(e^{-jw})$$

- $x[n] = x^*[n] \Rightarrow X(e^{-jw}) = X^*(e^{jw})$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{jw}) e^{jwn} dw$$

$x[n]$ is real $\Rightarrow$ $X(e^{jw})$ is conjugate symmetric

- $x[n] = x^*[n]$ & $x[-n] = x[n]$

$$\Rightarrow X(e^{-jw}) = X^*(e^{jw}) \quad \& \quad X(e^{-jw}) = X(e^{jw})$$

$$\Rightarrow X(e^{jw}) = X^*(e^{jw})$$

$x[n]$ is real & even $\Rightarrow$ $X(e^{jw})$ are real & even

- $x[n]$ is real & odd $\Rightarrow$ $X(e^{jw})$ are purely imaginary & odd

- Conjugation & Conjugate Symmetry:

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{jw})$$

$$\mathcal{E}v\{x[n]\} \xleftrightarrow{\mathcal{F}} \mathcal{R}e\{X(e^{jw})\}$$

$$\mathcal{O}d\{x[n]\} \xleftrightarrow{\mathcal{F}} j\,\mathcal{I}m\{X(e^{jw})\}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw$$

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}$$

- Differencing & Accumulation:

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{jw})$$

$$x[n] - x[n-1] \stackrel{\mathcal{F}}{\longleftrightarrow} \left(1 - e^{-jw}\right)X(e^{jw})$$

$$X(e^{jw}) \qquad e^{-jw}X(e^{jw})$$

$$\sum_{m=-\infty}^{n} x[m] \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - e^{-jw}}X(e^{jw}) + \pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(w - 2\pi k)$$

dc or average value

$$y[n] = \sum_{m=-\infty}^{n} x[m] \qquad \Rightarrow \; y[n] - y[n-1] = x[n]$$

$$y[n-1] = \sum_{m=-\infty}^{n-1} x[m] \qquad \Rightarrow \; \left(1 - e^{-jw}\right)Y(e^{jw}) = X(e^{jw})$$

- Differentiation in Frequency:

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{jw})$$

$$\frac{1}{j} n x[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{d}{dw} X(e^{jw})$$

$$n x[n] \overset{\mathcal{F}}{\longleftrightarrow} j \frac{d}{dw} X(e^{jw})$$

$$\frac{d}{dw} X(e^{jw}) = \frac{d}{dw} \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn}$$

$$= \sum_{n=-\infty}^{+\infty} (-jn) x[n] e^{-jwn}$$

$$= (-j) \sum_{n=-\infty}^{+\infty} \Big[n x[n] \Big] e^{-jwn}$$

- Time Reversal:

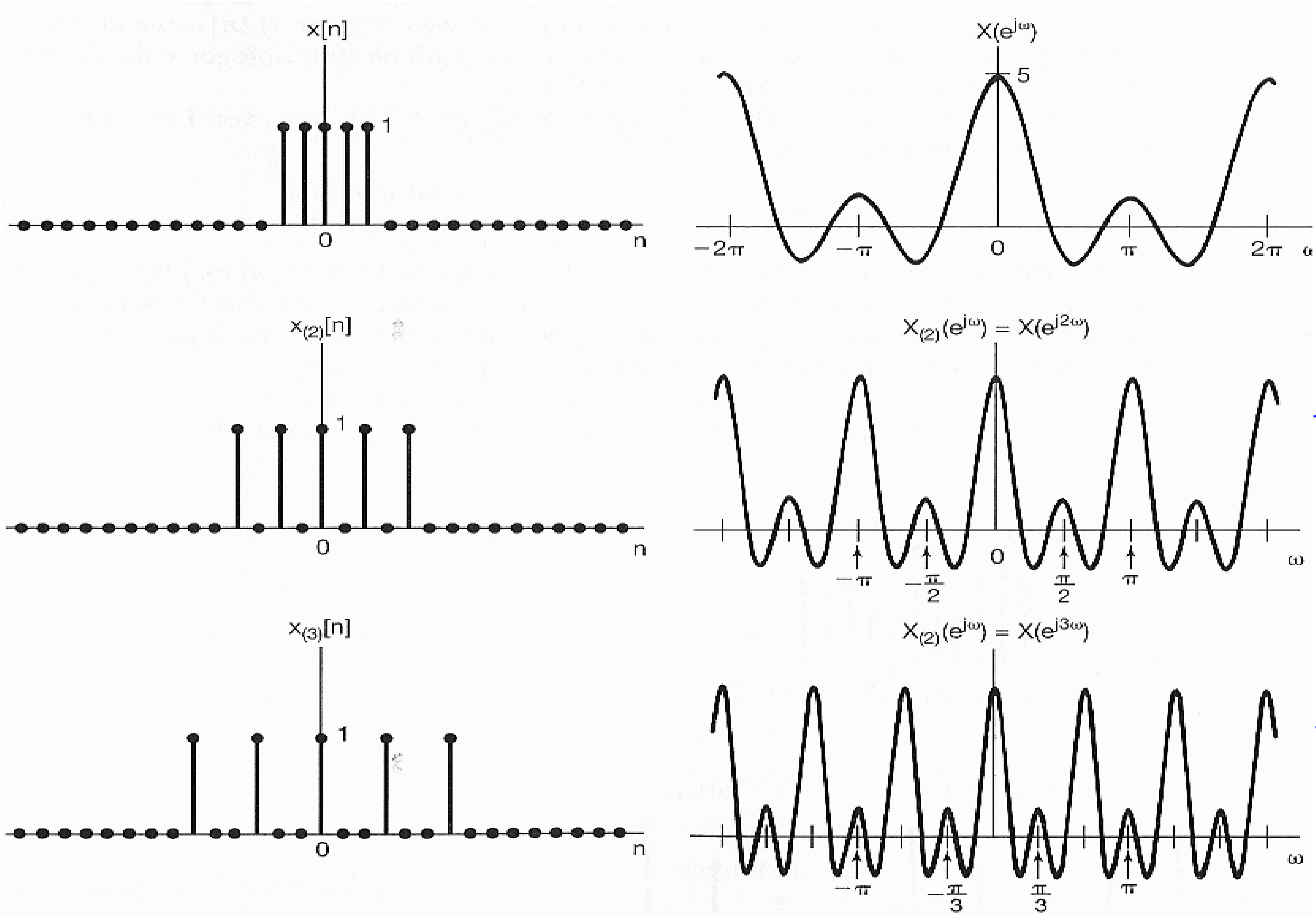
$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{jw})$$

$$x[-n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{-jw})$$

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn}$$

$$X(e^{j(-w)})) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j(-w)n}$$

- Time Expansion:

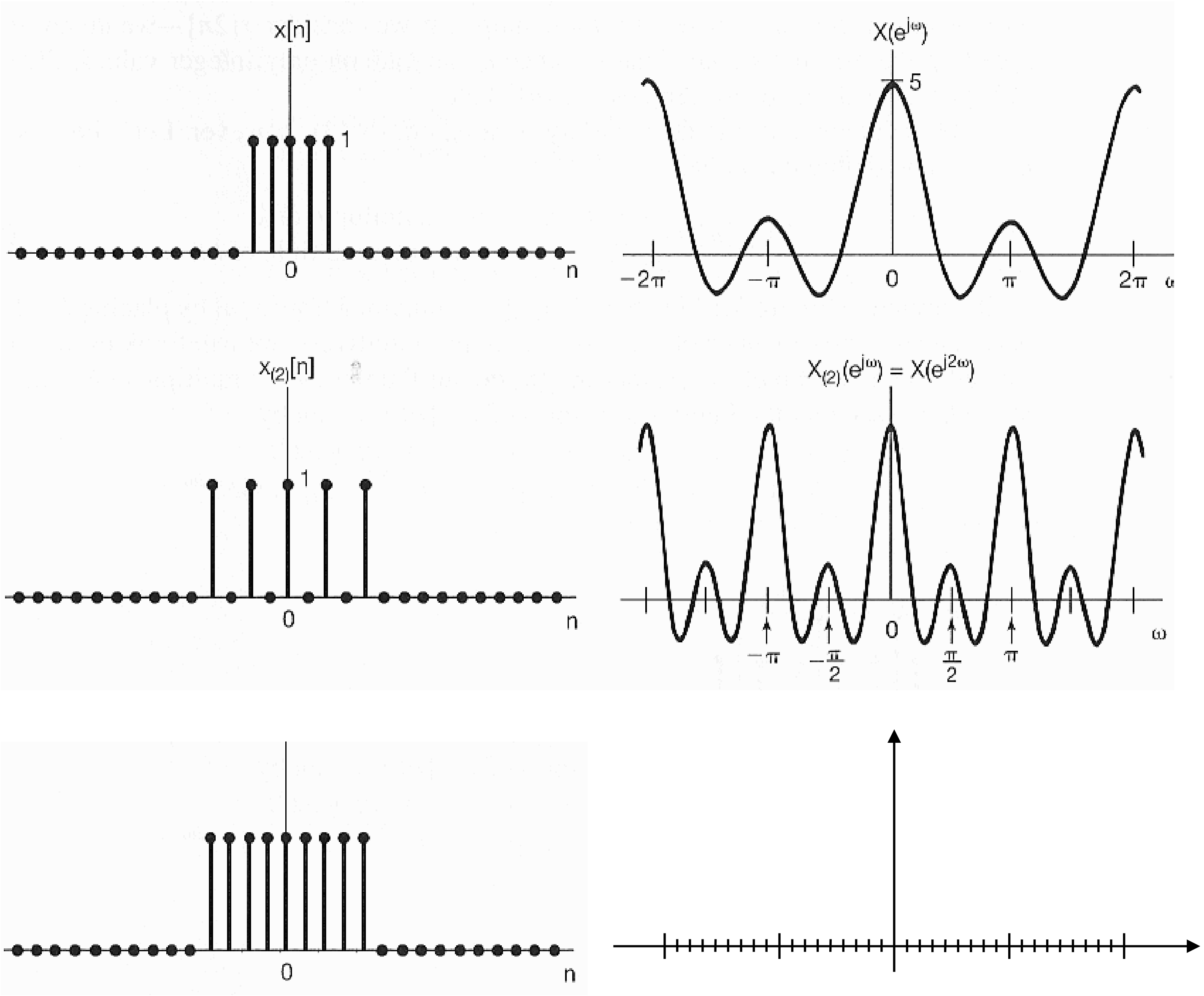


$X(e^{jw})$

$X(e^{j2w})$

$X(e^{j3w})$

- Time Expansion:

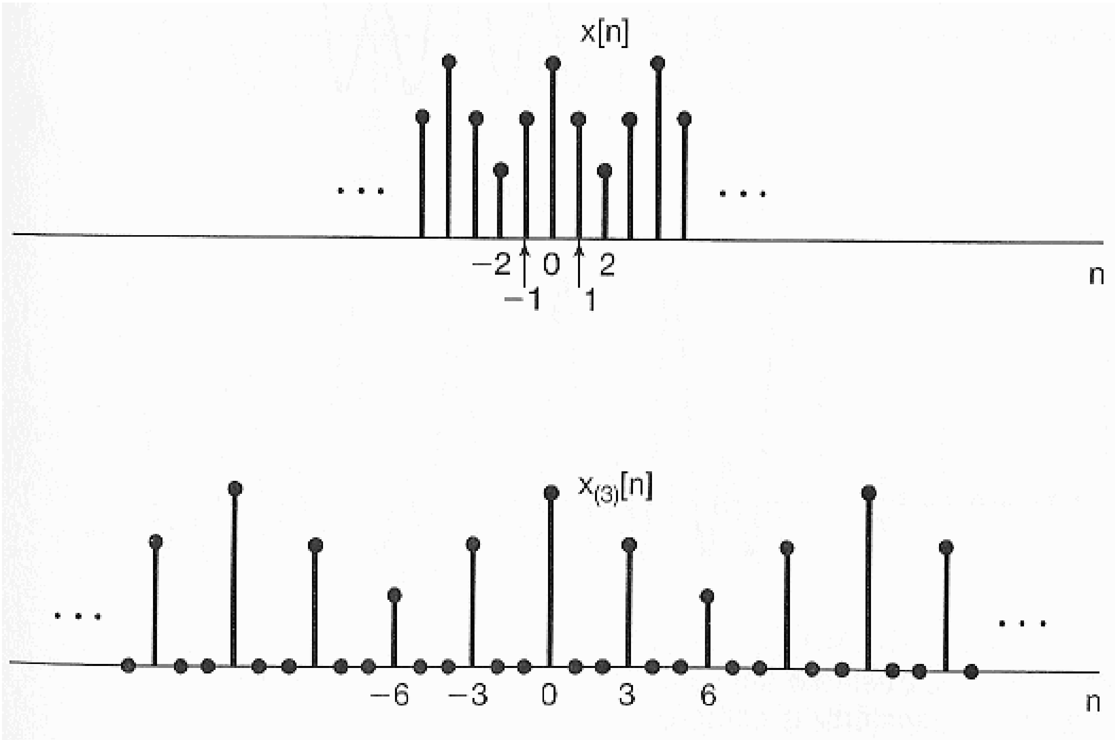


$X(e^{jw})$

$X(e^{j2w})$

Time Expansion:

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k \end{cases}$$



- Time Expansion:

$$\Rightarrow X_{(k)}(e^{jw}) = \sum_{n=-\infty}^{+\infty} x_{(k)}[n]e^{-jwn}$$

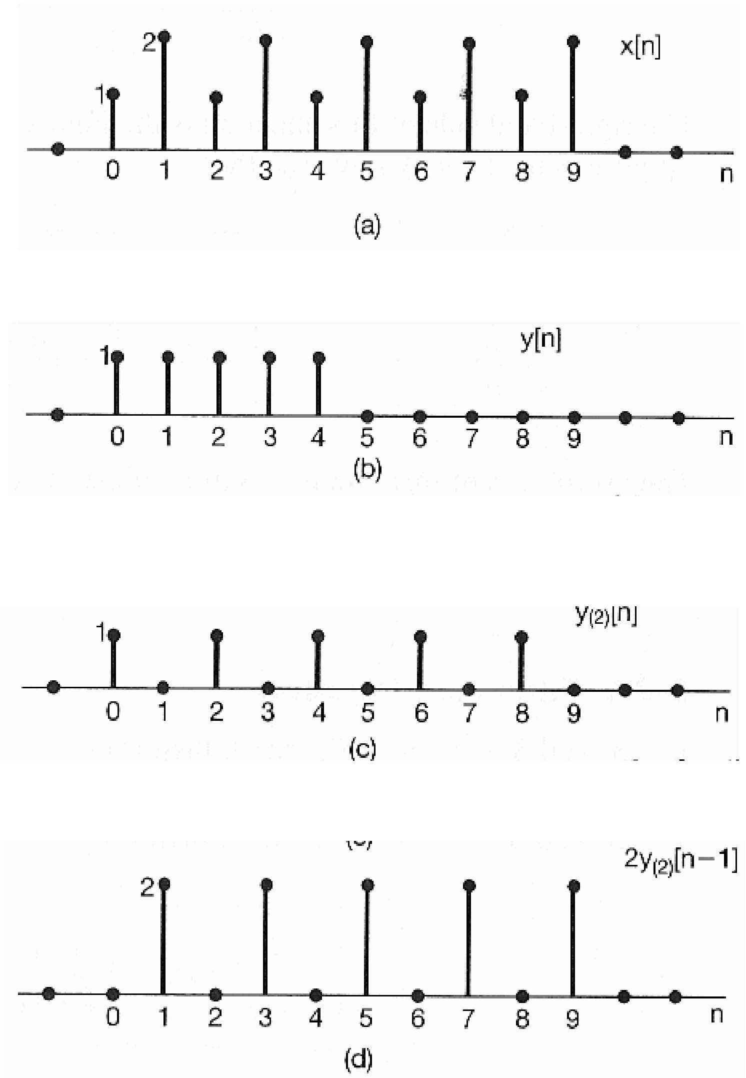
$$= \sum_{r=-\infty}^{+\infty} x_{(k)}[rk]e^{-jwrk}$$

$$= \sum_{r=-\infty}^{+\infty} x[r]e^{-j(kw)r} \qquad x_{(k)}[rk] = x[r]$$

$$= X(e^{jkw})$$

$$x_{(k)}[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{jkw})$$

- Example 5.9:



$$x[n] = y_{(2)}[n] + 2y_{(2)}[n-1]$$

$$Y(e^{jw}) = e^{-j2w}\frac{\sin(5w/2)}{\sin(w/2)}$$

$$y_{(2)}[n] = \begin{cases} y[n/2], & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

$$y_{(2)}[n] \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-j4w}\frac{\sin(5w)}{\sin(w)}$$

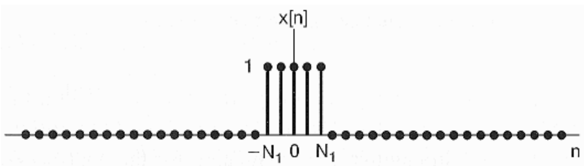
$$2y_{(2)}[n-1] \stackrel{\mathcal{F}}{\longleftrightarrow} 2e^{-jw}e^{-j4w}\frac{\sin(5w)}{\sin(w)}$$

$$X(e^{jw}) = (1 + 2e^{-jw}) \cdot e^{-j4w} \cdot \frac{\sin(5w)}{\sin(w)}$$

- Parseval's relation:

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{jw})$$

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{jw})|^2 dw$$



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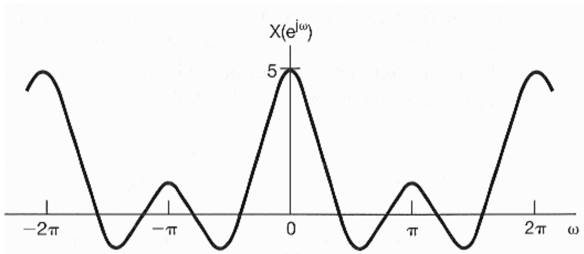
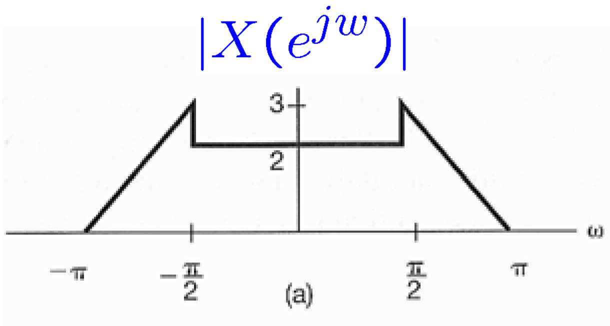
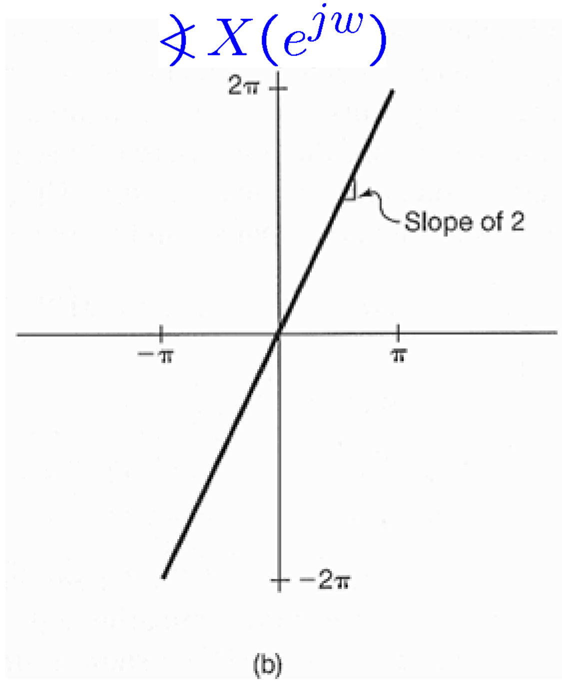


(b)

Example 5.10:





- $x[n]$ is periodic, real, even,

  and/or of finite energy?

$\rightarrow$ $X(e^{jw}) \neq 0$ $\Rightarrow$ $x[n]$ is NOT periodic

$\rightarrow$ even magnitude, odd phase $\Rightarrow$ $x[n]$ is real

$\rightarrow$ $X(e^{jw})$ is NOT real $\Rightarrow$ $x[n]$ is NOT even

$\rightarrow$ $X(e^{jw})$ is finite $\Rightarrow$ $x[n]$ is finite

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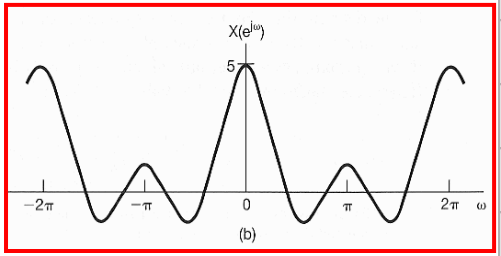
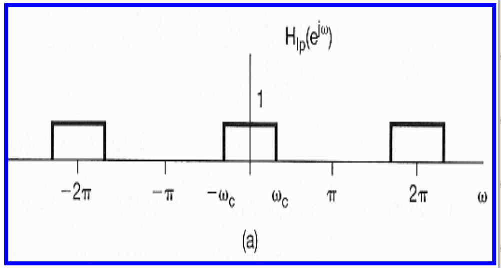
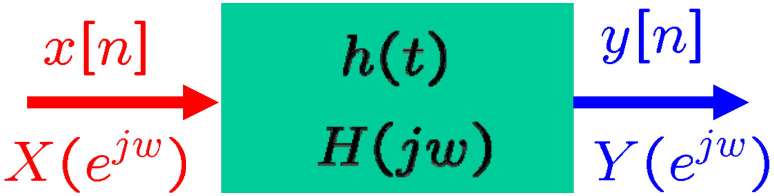
Convolution Property:

$$y[n] = x[n] * h[n] \xleftrightarrow{\mathcal{F}} Y(e^{jw}) = X(e^{jw})H(e^{jw})$$

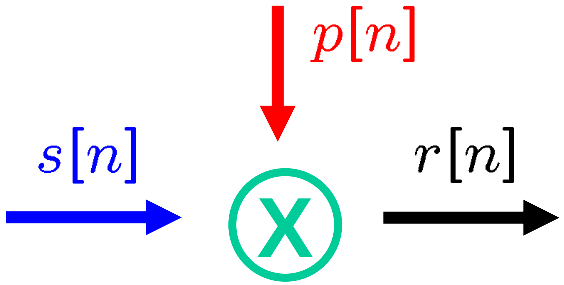


$$= \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$





Multiplication Property:

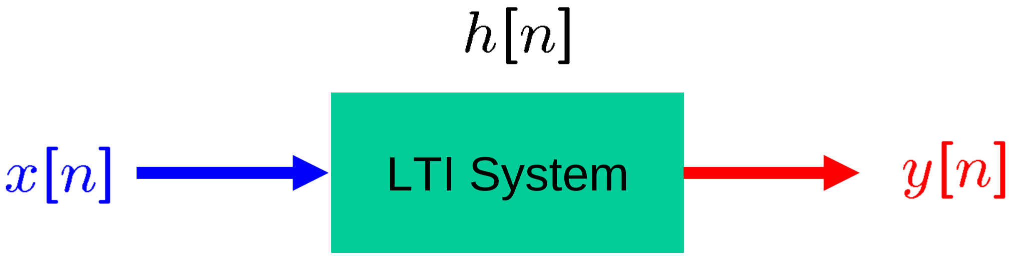


$$r[n] = s[n]p[n] \xleftrightarrow{\mathcal{F}} R(e^{jw}) = \frac{1}{2\pi}\int_{2\pi} S(e^{j\theta})P(e^{j(w-\theta)})d\theta$$

- Example 5.11:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw$$

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}$$



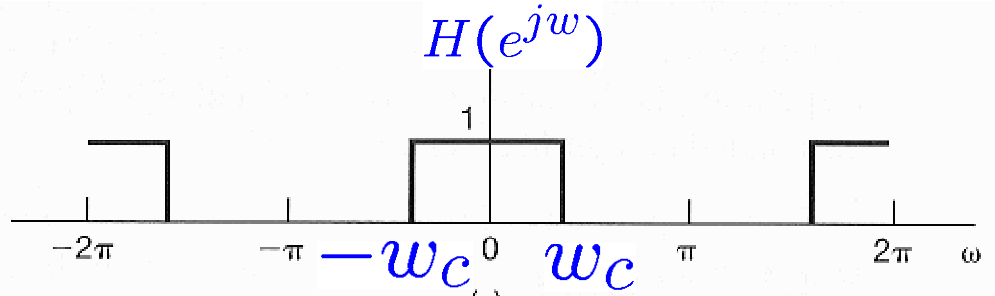
$$h[n] = \delta[n-n_0]$$

$$\Rightarrow H(e^{jw}) = \sum_{n=-\infty}^{+\infty} \delta[n-n_0]e^{-jwn} = e^{-jwn_0}$$

$$\Rightarrow Y(e^{jw}) = H(e^{jw})X(e^{jw})$$

$$= e^{-jwn_0}\, X(e^{jw}) \qquad \Rightarrow y[n] = x[n-n_0]$$

- Example 5.12:



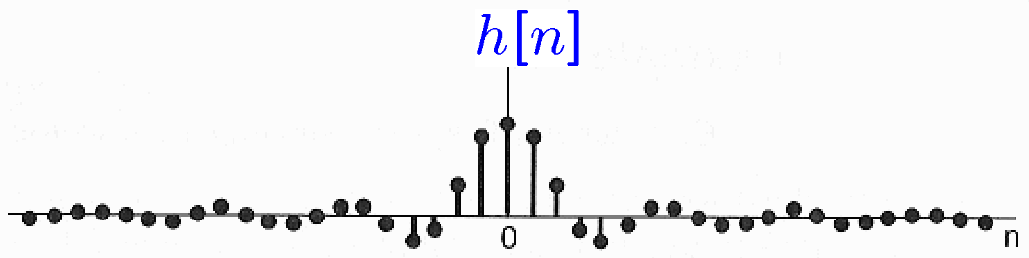
(a)

wc = cut off frequency

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{jw})e^{jwn}dw$$

$$= \frac{1}{2\pi}\int_{-w_c}^{w_c} e^{jwn}dw$$

$$= \frac{\sin w_c n}{\pi n}$$

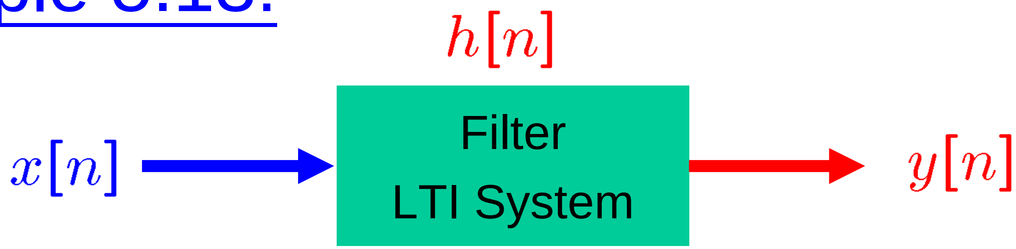


(b)

- not causal
- oscillatory

Convolution Property

- Example 5.13:



$$h[n] = a^n u[n], \quad |a| < 1 \qquad \Rightarrow \quad H(e^{jw}) = \frac{1}{1 - ae^{-jw}}$$

$$x[n] = b^n u[n], \quad |b| < 1 \qquad \Rightarrow \quad X(e^{jw}) = \frac{1}{1 - be^{-jw}}$$

$$\Rightarrow \; Y(e^{jw}) = H(e^{jw}) X(e^{jw})$$

$$= \frac{1}{1 - ae^{-jw}} \, \frac{1}{1 - be^{-jw}}$$

- Example 5.13:

if $a \neq b$ $$Y(e^{jw}) = \left[\left(\frac{a}{a-b}\right) \frac{1}{1-ae^{-jw}} + \left(\frac{-b}{a-b}\right) \frac{1}{1-be^{-jw}} \right]$$

$$\Rightarrow \ y[n] = \left(\frac{a}{a-b}\right) a^n u[n] - \left(\frac{b}{a-b}\right) b^n u[n]$$

if $a = b$ $$Y(jw) = \left(\frac{1}{1-ae^{-jw}}\right)^2 = \frac{j}{a} e^{jw} \frac{d}{dw} \left(\frac{1}{1-ae^{-jw}}\right)$$

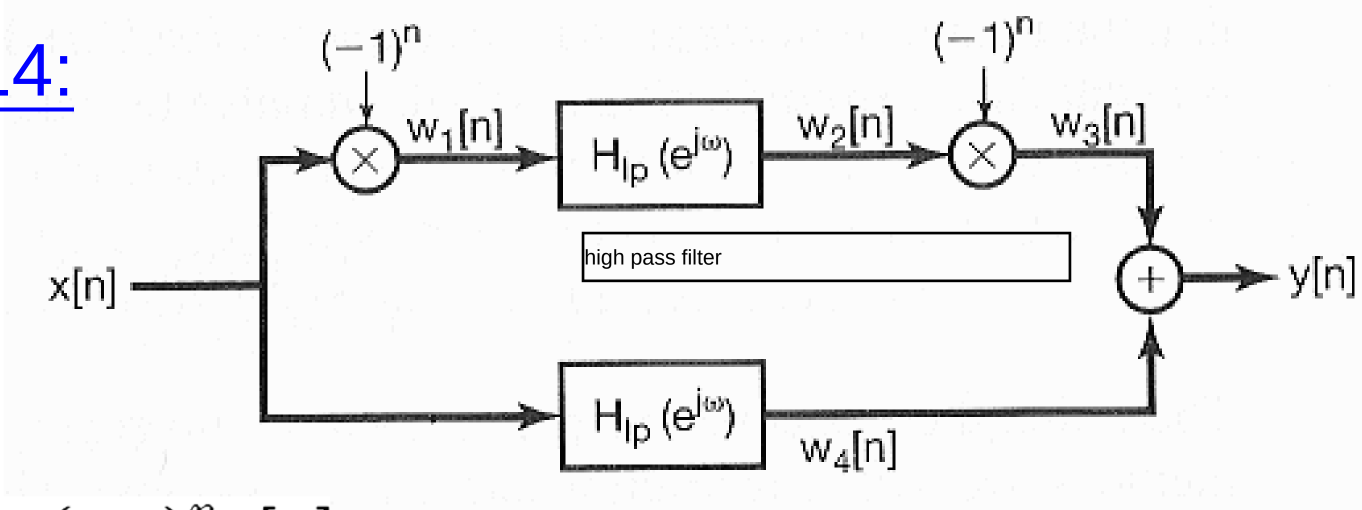
since $$a^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-ae^{-jw}}$$

and $$n\, a^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} j \frac{d}{dw} \left[\frac{1}{1-ae^{-jw}}\right]$$

and $$(n+1)\, a^{n+1} u[n+1] \overset{\mathcal{F}}{\longleftrightarrow} j\, e^{jw} \frac{d}{dw} \left[\frac{1}{1-ae^{-jw}}\right]$$

$$\Rightarrow \ y[n] = (n+1) a^n u[n+1]$$

- Example 5.14:



(a)

$$w_1[n] = e^{j\pi n} x[n] = (-1)^n x[n]$$

$$\Rightarrow \quad W_1(e^{jw}) = X(e^{j(w-\pi)})$$

$$W_4(e^{jw}) = H_{lp}(e^{jw})\ X(e^{jw})$$

$$W_2(e^{jw}) = H_{lp}(e^{jw})\ X(e^{j(w-\pi)})$$

$$w_3[n] = e^{j\pi n} w_2[n] = (-1)^n w_2[n]$$

$$\Rightarrow \quad W_3(e^{jw}) = W_2(e^{j(w-\pi)}) = H_{lp}(e^{j(w-\pi)})\ X(e^{j(w-2\pi)})$$

$$= H_{lp}(e^{j(w-\pi)})\ X(e^{jw})$$

- Example 5.14:

$$Y(e^{jw}) = W_3(e^{jw}) + W_4(e^{jw})$$

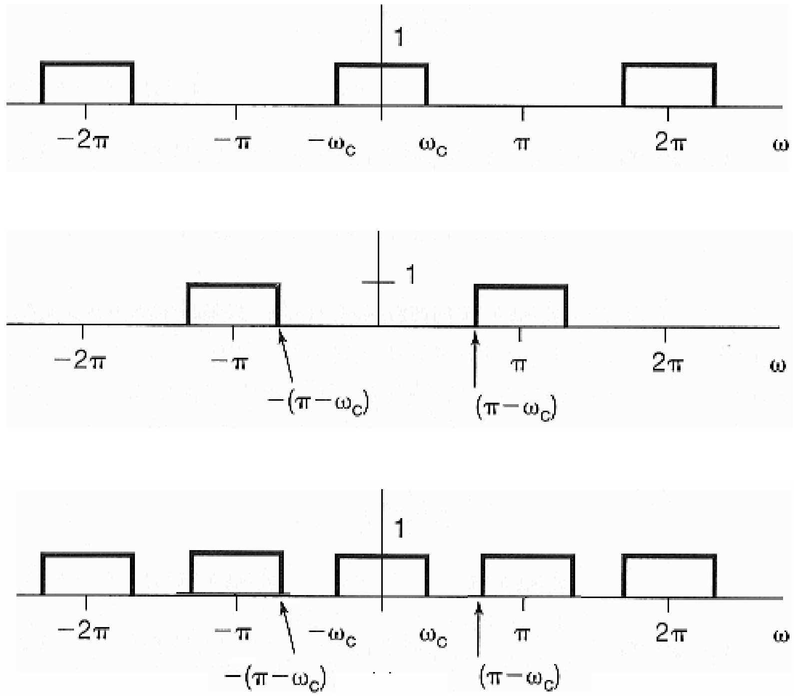
$$= H_{lp}(e^{j(w-\pi)})\; X(e^{jw}) + H_{lp}(e^{jw})\; X(e^{jw})$$

$$= \left[H_{lp}(e^{j(w-\pi)}) \; + H_{lp}(e^{jw})\right] \; X(e^{jw})$$

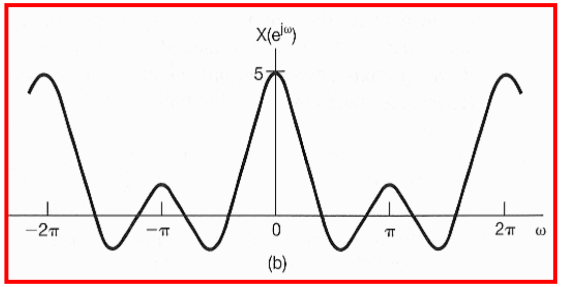
$$H(e^{jw}) = H_{lp}(e^{j(w-\pi)}) \; + H_{lp}(e^{jw})$$

highpass + lowpass

$\rightarrow$ bandstop

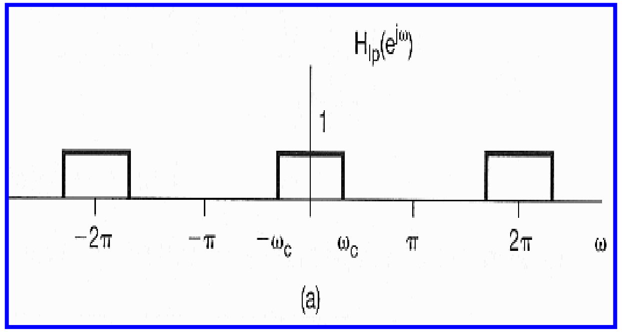


- Convolution Property:



$$y[n] = x[n] * h[n] \xleftrightarrow{\mathcal{F}} Y(e^{jw}) = X(e^{jw})H(e^{jw})$$

$$= \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$



- Multiplication Property:

$$r[n] = s[n]p[n] \xleftrightarrow{\mathcal{F}} R(e^{jw}) = \frac{1}{2\pi}\int_{2\pi} S(e^{j\theta})P(e^{j(w-\theta)})d\theta$$

Multiplication Property

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw$$

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}$$

$$r[n] = s[n]p[n]$$

$$\Rightarrow \quad R(e^{jw}) = \sum_{n=-\infty}^{+\infty} r[n]e^{-jwn}$$

$$= \sum_{n=-\infty}^{+\infty} s[n]p[n]e^{-jwn}$$

$$= \sum_{n=-\infty}^{+\infty} s[n]\left\{\frac{1}{2\pi}\int_{2\pi} P(e^{j\theta})e^{j\theta n}d\theta\right\}e^{-jwn}$$

$$= \frac{1}{2\pi}\int_{2\pi} P(e^{j\theta})\left[\sum_{n=-\infty}^{+\infty} s[n]e^{-j(w-\theta)n}\right]d\theta$$

$$= \frac{1}{2\pi}\int_{2\pi} P(e^{j\theta})S(e^{j(w-\theta)})d\theta \quad = \frac{1}{2\pi}\int_{2\pi} P(e^{j(w-\theta)})S(e^{j\theta})d\theta$$

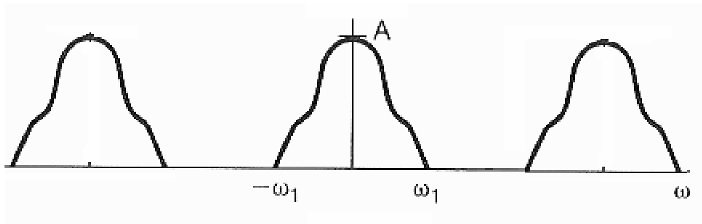
Multiplication Property

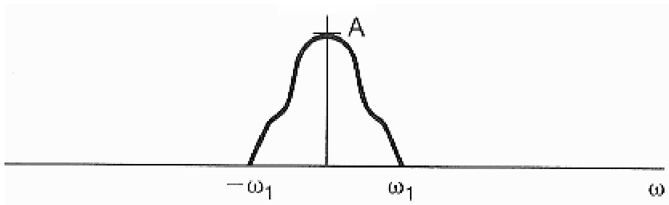
$$y(\theta) = \int_{\Theta} x(\theta)h(\theta-\tau)d\tau$$

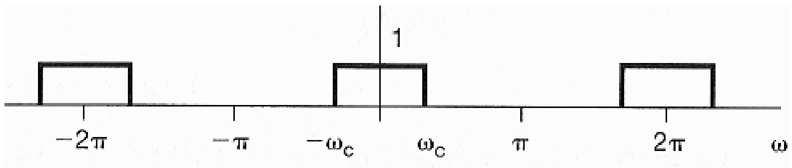
periodic convolution

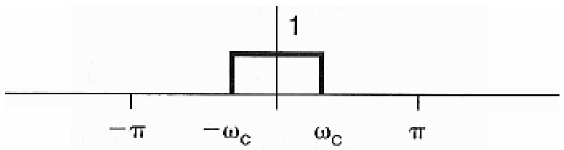
$$y(\theta) = \int_{-\infty}^{+\infty} x(\tau)h(\theta-\tau)d\tau$$

aperiodic convolution







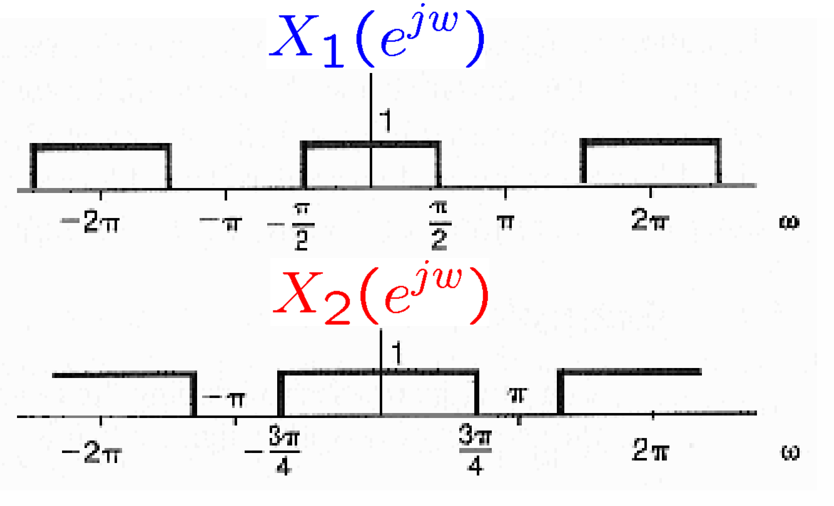


Example 5.15:

$$x[n] = x_1[n]x_2[n]$$

$$x_1[n] = \frac{\sin(\frac{\pi}{2}n)}{\pi n}$$

$$x_2[n] = \frac{\sin(\frac{3\pi}{4}n)}{\pi n}$$



$$X(e^{jw}) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X_1(e^{j\theta})X_2(e^{j(w-\theta)})d\theta$$

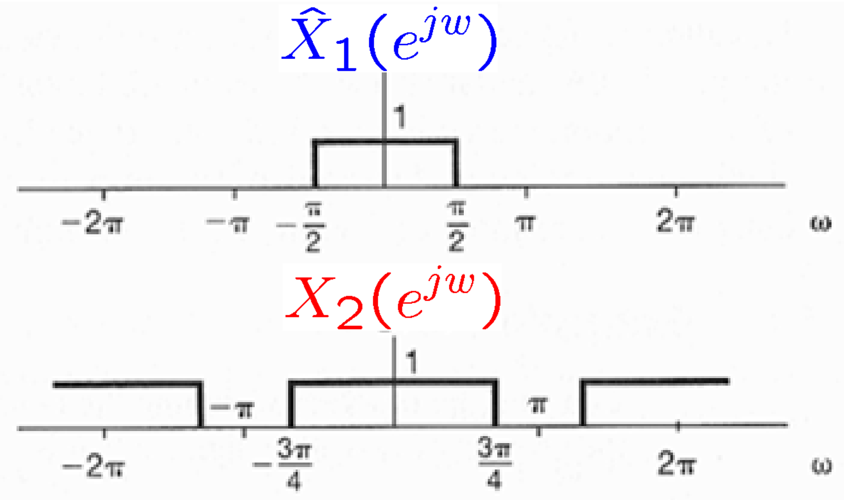
$$\hat{X}_1(e^{jw}) = \begin{cases} X_1(e^{jw}), & \text{for } -\pi < w \leq \pi \\ 0, & \text{otherwise} \end{cases}$$

$$X(e^{jw}) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \hat{X}_1(e^{j\theta})X_2(e^{j(w-\theta)})d\theta$$

- Example 5.15:

$$X(e^{jw}) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \hat{X}_1(e^{j\theta})X_2(e^{j(w-\theta)})d\theta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{X}_1(e^{j\theta})X_2(e^{j(w-\theta)})d\theta$$



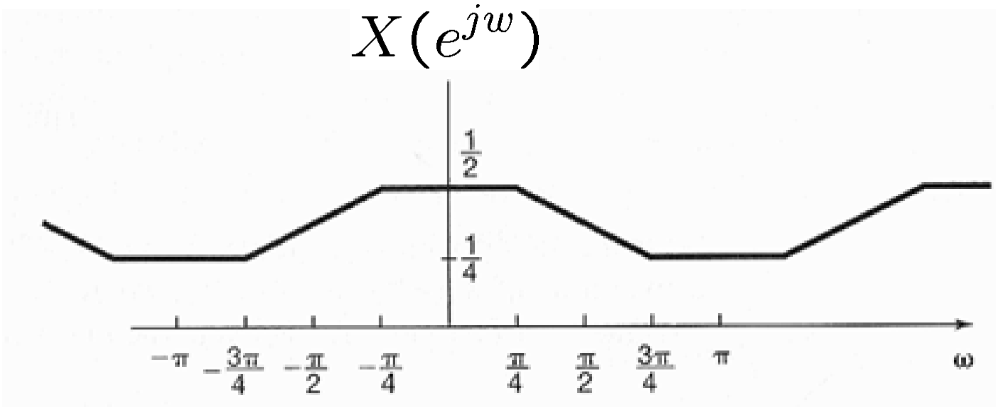


TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$ $y[n]$	$\left.\begin{matrix} X(e^{j\omega}) \\ Y(e^{j\omega}) \end{matrix}\right\}$ periodic with period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{R}e\{X(e^{j\omega})\} = \mathcal{R}e\{X(e^{-j\omega})\} \\ \mathcal{I}m\{X(e^{j\omega})\} = -\mathcal{I}m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ [$x[n]$ real] $x_o[n] = \mathcal{O}d\{x[n]\}$ [$x[n]$ real]	$\mathcal{R}e\{X(e^{j\omega})\}$ $j\mathcal{I}m\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals		

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

TABLE 5.2 BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \dots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & N_1 < \lvert n\rvert \le N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \; k \neq 0, \pm N, \pm 2N, \dots$ $a_k = \frac{2N_1 + 1}{N}, \; k = 0, \pm N, \pm 2N, \dots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k

$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \leq N_1 \\ 0, & N_1 < \lvert n\rvert \leq N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \ k \neq 0, \pm N, \pm 2N, \ldots$ $a_k = \frac{2N_1 + 1}{N}, \ k = 0, \pm N, \pm 2N, \ldots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n] \begin{cases} 1, & \lvert n\rvert \leq N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \leq \lvert\omega\rvert \leq W \\ 0, & W < \lvert\omega\rvert \leq \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n+1)a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
$\frac{(n+r-1)!}{n!(r-1)!} a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—

- Representation of Aperiodic Signals: the Discrete-Time Fourier Transform
- The Fourier Transform for Periodic Signals
- Properties of Discrete-Time Fourier Transform
- The Convolution Property
- The Multiplication Property
- Duality
- Systems Characterized by Linear Constant-Coefficient Difference Equations

DT Fourier Series Pair of Periodic Signals:

- $x[n] \xleftrightarrow{\mathcal{FS}} a_k$: DT Fouries series pair

$$x[n] = \sum_{k=<N>} a_k e^{jkw_0 n} = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jkw_0 n} = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk(2\pi/N)n}$$

IF $f[k] = \frac{1}{N} \sum_{n=<N>} g[n] e^{-jk(2\pi/N)n}$ $\quad g[n] \xleftrightarrow{\mathcal{FS}} f[k]$

$$f[n] = \sum_{k=<N>} \frac{1}{N} g[-k] e^{jk(2\pi/N)n} \qquad f[n] \xleftrightarrow{\mathcal{FS}} \frac{1}{N} g[-k]$$

LET $k = n, n = -k$ $\quad a_k :=: \frac{1}{N} x[-n]$

- Duality in DT Fourier Series:

$$x[n] \xleftrightarrow{\mathcal{FS}} a_k$$

$$x[n-n_0] \xleftrightarrow{\mathcal{FS}} a_k \, e^{-jk(2\pi/N)n_0}$$

$$e^{+jm(2\pi/N)n} \, x[n] \xleftrightarrow{\mathcal{FS}} a_{k-m}$$

$$\sum_{r=<N>} x[r] \, y[n-r] \xleftrightarrow{\mathcal{FS}} N a_k \, b_k$$

$$x[n] \, y[n] \xleftrightarrow{\mathcal{FS}} \sum_{l=<N>} a_l \, b_{k-l}$$

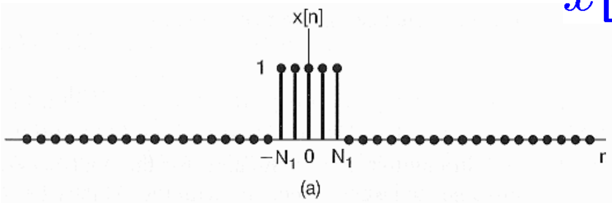
▪ Duality between DT-FT & CT-FS:

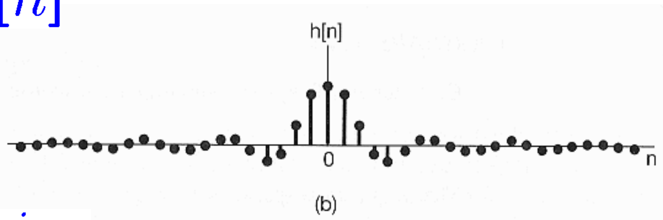
$$x[n] \xleftrightarrow{\mathcal{DTFT}} X(e^{jw}) \qquad\qquad x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

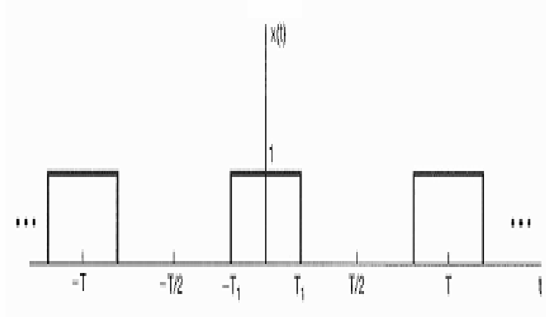
$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw \qquad\qquad a_k = \frac{1}{T}\int_{T} x(t)e^{-jkw_0 t}dt$$

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn} \qquad\qquad x(t) = \sum_{k=}^{+\infty} a_k e^{jkw_0 t}$$

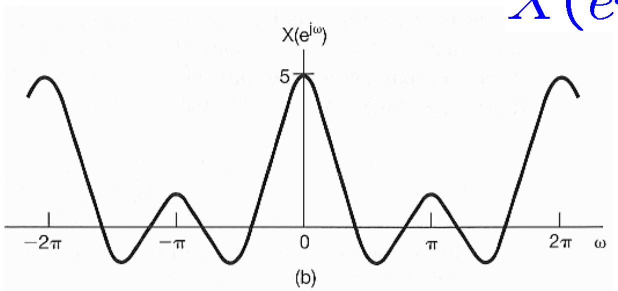
$x[n]$

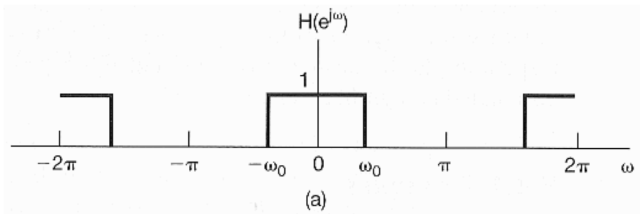


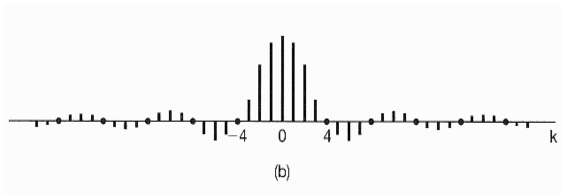




$X(e^{jw})$







Duality

TABLE 5.3 SUMMARY OF FOURIER SERIES AND TRANSFORM EXPRESSIONS

	Continuous time		Discrete time	
	Time domain	**Frequency domain**	**Time domain**	**Frequency domain**
Fourier Series	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ continuous time periodic in time	$a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t}$ discrete frequency aperiodic in frequency	$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$ discrete time periodic in time	$a_k = \frac{1}{N} \sum_{k=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}$ discrete frequency periodic in frequency
Fourier Transform	$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$ continuous time aperiodic in time	$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$ continuous frequency aperiodic in frequency	$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n}$ discrete time aperiodic in time	$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$ continuous frequency periodic in frequency

duality

duality

duality

- Representation of Aperiodic Signals: the Discrete-Time Fourier Transform
- The Fourier Transform for Periodic Signals
- Properties of Discrete-Time Fourier Transform
- The Convolution Property
- The Multiplication Property
- Duality
- Systems Characterized by Linear Constant-Coefficient Difference Equations

Systems Characterized by Linear Constant-Coefficient Difference Equations

- <u>A useful class of DT LTI systems:</u>

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N]$$

$$= b_0 x[n] + b_1 x[n-1] + \cdots + b_{M-1} x[n-M+1] + b_M x[n-M]$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$x[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow y[n]$$

$$Y(e^{jw}) = X(e^{jw}) H(e^{jw}) \qquad H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})}$$

$$\boxed{x[n-n_0] \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-jwn_0}X(e^{jw})}$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\sum_{k=0}^{N} a_k \quad y[n-k] = \sum_{k=0}^{M} b_k \quad x[n-k]$$

$$\sum_{k=0}^{N} a_k \; e^{-jkw} \; Y(e^{jw}) = \sum_{k=0}^{M} b_k \; e^{-jkw} \; X(e^{jw})$$

$$\Rightarrow \quad H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{\sum_{k=0}^{M} b_k e^{-jkw}}{\sum_{k=0}^{N} a_k e^{-jkw}}$$

$$= \frac{b_0 + b_1 e^{-jw} + \cdots + b_M e^{-jMw}}{a_0 + a_1 e^{-jw} + \cdots + a_N e^{-jNw}}$$

- Examples 5.18 & 5.19:

$x[n] \rightarrow$ LTI System $\rightarrow y[n]$

$|a| < 1$

$$y[n] - ay[n-1] = x[n] \quad \Rightarrow \quad H(e^{jw}) = \frac{1}{1 - ae^{-jw}}$$

$Y(\cdot) \qquad e^{-jw}Y(\cdot)$

$$\Rightarrow \quad h[n] = a^n u[n]$$

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$$

$$\Rightarrow \quad H(e^{jw}) = \frac{2}{1 - \frac{3}{4}e^{-jw} + \frac{1}{8}e^{-j2w}} = \frac{2}{(1 - \frac{1}{2}e^{-jw})(1 - \frac{1}{4}e^{-jw})}$$

$$= \frac{4}{(1 - \frac{1}{2}e^{-jw})} - \frac{2}{(1 - \frac{1}{4}e^{-jw})}$$

$$\Rightarrow \quad h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]$$

- Example 5.20:

$$h[n] \overset{\mathcal{F}}{\longleftrightarrow} H(e^{jw}) = \frac{2}{(1-\frac{1}{2}e^{-jw})(1-\frac{1}{4}e^{-jw})}$$

$$x[n] = \left(\frac{1}{4}\right)^n u[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow y[n] = ???$$

$$= x[n] * h[n]$$

$$\Rightarrow \quad Y(e^{jw}) = X(e^{jw}) H(e^{jw})$$

$$= \left[\frac{1}{1-\frac{1}{4}e^{-jw}}\right] \left[\frac{2}{(1-\frac{1}{2}e^{-jw})(1-\frac{1}{4}e^{-jw})}\right]$$

$$= \frac{2}{(1-\frac{1}{2}e^{-jw})(1-\frac{1}{4}e^{-jw})^2}$$

$$= \frac{8}{(1-\frac{1}{2}e^{-jw})} - \frac{4}{(1-\frac{1}{4}e^{-jw})} - \frac{2}{(1-\frac{1}{4}e^{-jw})^2}$$

$$\Rightarrow \quad y[n] = \left\{ 8\left(\frac{1}{2}\right)^n - 4\left(\frac{1}{4}\right)^n - 2(n+1)\left(\frac{1}{4}\right)^n \right\} u[n]$$

- Representation of Aperiodic Signals: the DT FT
- The FT for Periodic Signals
- Properties of the DT FT
  - Linearity
  - Time Shifting
  - Frequency Shifting
  - Conjugation
  - Time Reversal
  - Time Expansion
  - Convolution
  - Multiplication
  - Differencing in Time
  - Accumulation
  - Differentiation in Frequency
  - Conjugate Symmetry for Real Signals
  - Symmetry for Real and Even Signals & for Real and Odd Signals
  - Even-Odd Decomposition for Real Signals
  - Parseval's Relation for Aperiodic Signals
- The Convolution Property
- The Multiplication Property
- Duality
- Systems Characterized by Linear Constant-Coefficient Difference Equations

Signals & Systems (Chap 1) LTI & Convolution (Chap 2)

Bounded/Convergent

Periodic

FS (Chap 3)
– CT
– DT

Aperiodic

FT
– CT (Chap 4)
– DT (Chap 5)

Unbounded/Non-convergent

LT – CT (Chap 9)

zT – DT (Chap 10)

Time-Frequency (Chap 6) Communication (Chap 8)

CT-DT (Chap 7) Control (Chap 11)