دانشگاه صنعتی امیرکبیر
(پلی تکنیک تهران)

Signals and Systems

Assignment 1

Spring 2021

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Question 1 (4 pts)

Determine the values of P_∞ and E_∞ for each of the following signals.

(a) $x(t) = \frac{1}{2}^{-3|t|}$

(b) $x[n] = 3^n u[-n]$

(c) $x[n] = e^{\pi jn + \dfrac{\pi}{4}}$

(d) $x(t) = cos(\omega t)$ (what is the relationship between P_∞ and ω? why?)

(e) $x(t) = cos(2t) + jcos(t)$

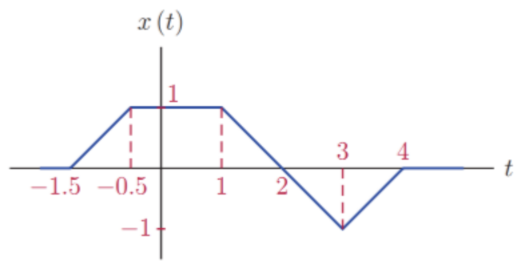
(f) $x[n] = e^{jn + n}$

Question 2 (1 pts)

For signal x(t) shown in the figure, plot the following (step-by-step):

(a) $x_1(t) = x(\frac{t+1}{5})$

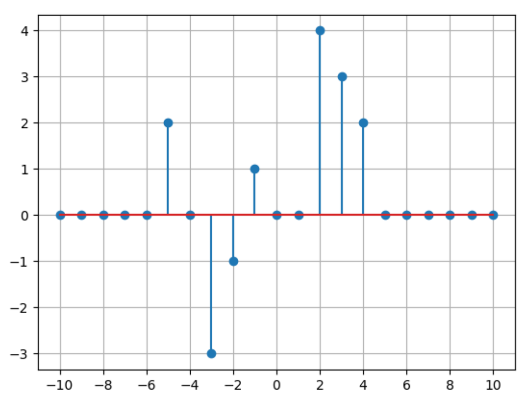
(b) $x_2(t) = x(-2t+1)$



For signal x[n] shown in the figure, plot the following (step-by-step):

(c) $x_3[n] = x[2n+1]$

(d) $x_4[n] = \begin{cases} x[\frac{n}{2}] & ,n \quad even \\ 0 & ,n \quad odd \end{cases}$



Question 3 (1.5 pts)

For each of the signals listed below, find the even and odd components $Ev\{x(t)\}$ and $Od\{x(t)\}$.

(a) $x(t) = cos(t)u(t)$

(b) $x(t) = e^{-2|t|}sin(t)$

(c) $x(t) = 2\Pi(t + 3.5)$ (solve by sketching)

(d) $x[n] = \delta[n] + u[n]$ (plot the results)

(e) $x[n] = (n + 1)^2$

note: $\Pi(t) = rect(t) = unitpulse = u(t + 0.5) - u(t - 0.5)$

Question 4 (8 pts)

Determine if each signal is periodic. If so, determine the fundamental period and the fundamental frequency.

(a) $x(t) = e^{j\pi t}$

(b) $x(t) = e^{(1+\pi j)t}$

(c) $x(t) = e^{(-2+3j)t} \times e^{(2-2j)t}$

(d) $x[n] = e^{j\frac{\pi}{3}n}$

(e) $x[n] = e^{j4n}$

(f) $x(t) = \int_{-\infty}^{t} \delta(\tau)d\tau + \displaystyle\int_{-\infty}^{-t} \delta(\tau)d\tau$

(g) $x(t) = cos(6\pi t) + sin(4\pi t)$

(h) $x[n] = cos(\frac{2\pi}{7}n) + e^{j\frac{5\pi}{3}n}$

(i) $x[n] = \displaystyle\sum_{k=-\infty}^{\infty} \delta[n-5k] + \delta[n-1-5k] + 2\delta[n+2-5k]$

(j) $x[n] = cos^2(\frac{2\pi}{9}n + \frac{\pi}{6}) + sin(\frac{5\pi}{3}) + 2sin(\frac{3\pi}{5}n)$

(k) $x(t) = Od\{cos(\pi t)u(t)\}$

(l) $x[n] = cos(\frac{\pi}{8}n^2)$

Question 5 (8 pts)

Determine whether or not the following systems are memoryless, causal, time-invariant, stable and linear:

(a) $y(t) = cos(t)$

(b) $y[n] = x[n-1]$

(c) $y[n] = \sum_{-K}^{+K} x[n-k]$

(d) $y(t) = cos(t)x(t)$

(e) $y(t) = e^{tx(t)}$

(f) $y(t) = \int_{-\infty}^{t+1} x(\tau)d\tau$

(g) $y[n] = x[3n+2]$

(h) $y[n] = sin(x[n])$

(i) $y(t) = x(cos(t) - 1)$

Question 6 (3 pts)

Determine if each of the given systems is invertible. If so, find the invert system.

(a) $y(t) = x(2t)$

(b) $y[n] = x^2[n]$

(c) $y[n] = x[n-1]x[n-3]$

(d) $y(t) = \begin{cases} x(t) & , t \geq 1 \\ x(t-2) & , t < 1 \end{cases}$

(e) $y[n] = \begin{cases} x[n] & , n \geq 0 \\ x[1] & , -2 \leq n < 0 \\ x[n+2] & , n < -2 \end{cases}$

(f) $y(t) = \dfrac{d^2x(t)}{dt^2}$

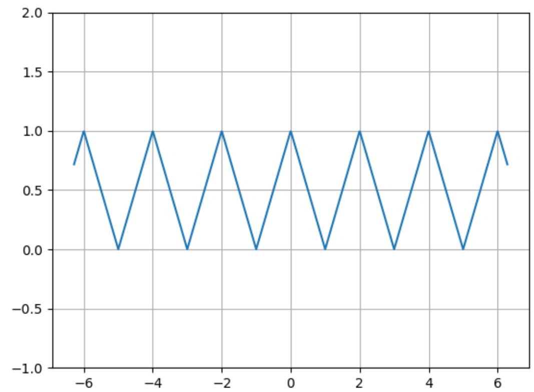
Programming Assignment (5 pts)

Plot the following signals within interval $-2\pi \leq t \leq 2\pi$.

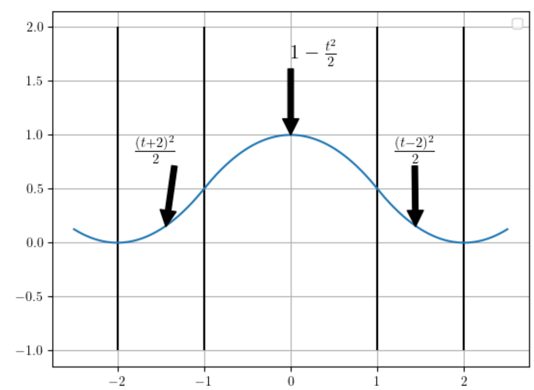
(a) $x(t) = sin(3t)$

(b) $x(t) = sin(\frac{\pi}{2}t)$

(c)



(d)



(e) $x[n] = \sum_{k=-\infty}^{\infty} \delta[n-5k] + \delta[n-1-5k] + 2\delta[n+2-5k]$ (use loops and conditionals)

Sabject: حل تمرین سری 1 سیگنال
Year: Month: Date:

1) (a) $x(t) = \frac{1}{2}^{-3|t|} \Rightarrow E_\infty = \int_{-\infty}^{\infty} \left(\left|\frac{1}{2}\right|^{-3|t|}\right)^2 dt = \int_{-\infty}^{\infty} \left(\frac{1}{2}\right)^{-6|t|} dt$

$= \int_{-\infty}^{0} \frac{1}{2}^{6t} dt + \int_{0}^{\infty} \frac{1}{2}^{-6t} dt = \infty$

$P_\infty = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left(\frac{1}{2}\right)^{-6|t|} dt = \infty$

1) (b) $x[n] = 3^n u[-n] \Rightarrow E_\infty = \sum_{-\infty}^{\infty} (3^n u[-n])^2$

$= \sum_{-\infty}^{0} 9^n = \frac{1}{1-\frac{1}{9}} = \frac{9}{8} \Rightarrow P_\infty = 0$

1) (c) $x[n] = e^{j\pi n + \frac{\pi}{4}}$

$\Rightarrow E_\infty = \sum_{-\infty}^{\infty} \left| e^{j\pi n} \times e^{\frac{\pi}{4}} \right|^2$

$= \sum_{-\infty}^{\infty} e^{\frac{\pi}{2}} = \infty$, $P_\infty = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{-N}^{N} e^{\frac{\pi}{2}} = e^{\frac{\pi}{2}}$

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1) (d) $x(t) = \cos(\omega t) \Rightarrow E_\infty = \int_{-\infty}^{\infty} \cos^2(\omega t)\, dt = \boxed{\infty}$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \frac{1+\cos(2\omega t)}{2}\, dt = \lim_{T\to\infty} \frac{1}{2T} \left[\frac{t}{2} \Big|_{-T}^{T} + \underbrace{\frac{1}{4\omega} \sin(2\omega t) \Big|_{-T}^{T}}_{\text{finite}} \right]$$

$$= \lim_{T\to\infty} \frac{1}{2T} \left[T + \frac{\sin(2\omega T)}{2\omega} \right] = \boxed{\frac{1}{2}}$$

P_∞ مستقل از ω است.

علت این است که $\cos(\omega t)$ یک سیگنال متناوب است. برای سیگنال
های متناوب P_∞ همان توان یک دوره تناوب است. یعنی انرژی سیگنال
تقسیم بر دوره تناوب. برای این سیگنال متناوب خاص، یعنی $\cos(\omega t)$ با تغییر
دوره تناوب، انرژی سیگنال (در یک دوره تناوب) به همان نسبت تغییر می‌کند و اثر تغییرات دوره
تناوب را خنثی می‌کند برای همین P_∞ نسبت به ω مستقل است. علت
تغییرات متناسب انرژی با دوره تناوب هم این است که طبق این فرمول:
$\cos^2(\omega t) = \frac{1+\cos(2\omega t)}{2}$ توان 2 $\cos(\omega t)$ را می‌توان به صورت ترکیبی خطی از
سیگنال‌ها با همان دوره تناوب باید نوشت (یعنی $\cos(2\omega t)$)

Sabject:
Year: Month: Date:

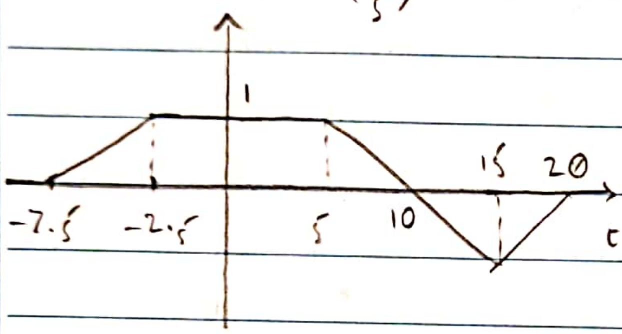
1) (e) $x(t) = \cos(2t) + j\cos(t) \Rightarrow E_\infty = \int_{-\infty}^{\infty} \cos^2(2t) + \cos^2(t)\, dt = \infty$

$\xrightarrow{\text{طبق تمرین حل}} P_\infty = \frac{1}{2} + \frac{1}{2} = 1$

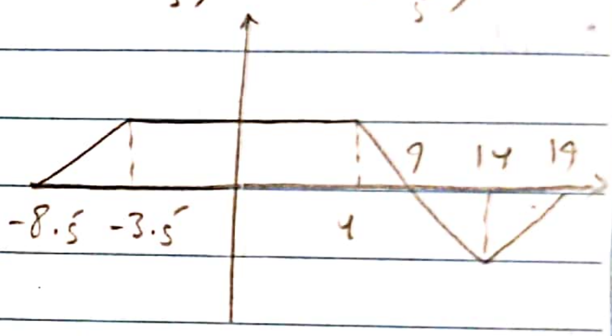
1) f) $x[n] = e^{jn+n} \Rightarrow E_\infty = \sum_{-\infty}^{\infty} |e^{jn+n}|^2 = \sum_{-\infty}^{\infty} e^{2n} = \infty$

$P_\infty = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{-N}^{N} e^{2n} = \infty$

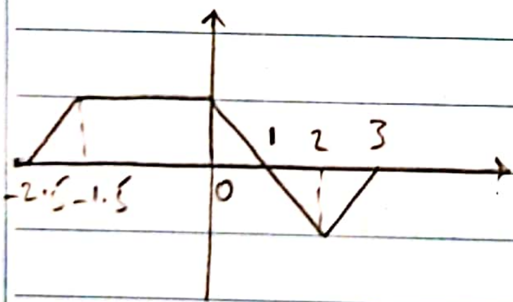
2) (a) $x(t) \longrightarrow x(\frac{t}{5})$



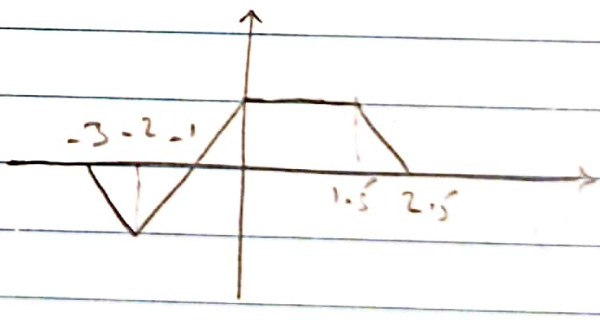
$x(\frac{t}{5}) \longrightarrow x(\frac{t+1}{5})$



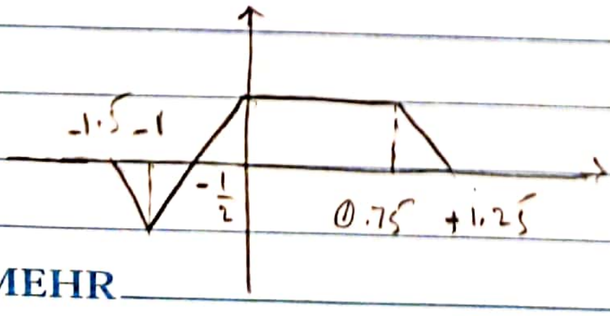
2) (b) $x(t) \longrightarrow x(t+1)$



$x(t+1) \longrightarrow x(-t+1)$



$x(-t+1) \longrightarrow x(-2t+1)$



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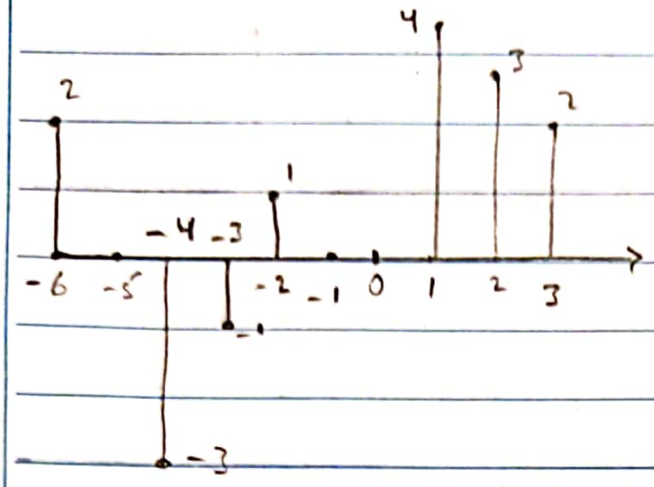
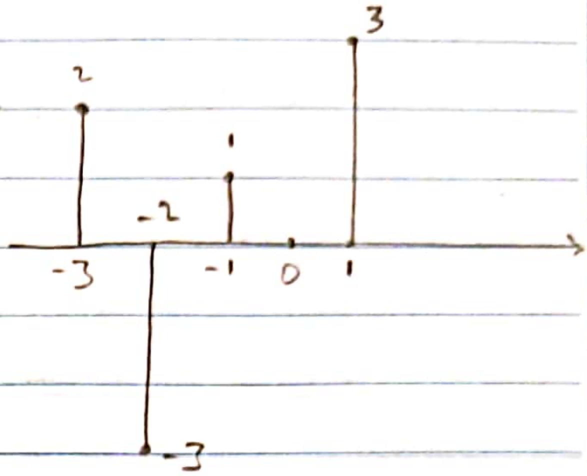
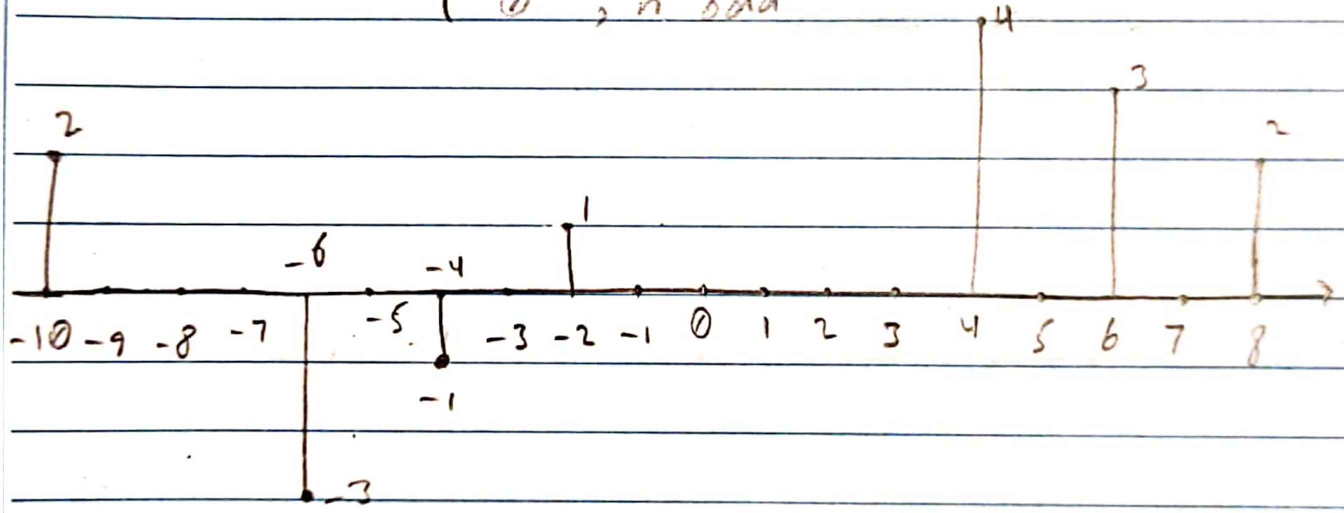
2) (c) $x[n] \longrightarrow x[n+1]$ $\qquad$ $x[n+1] \longrightarrow x[2n+1]$





2) (d) $x[n] \longrightarrow \begin{cases} x[\frac{n}{2}], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$



3) (a) $x_e(t) = \dfrac{\cos(t)u(t) + \cos(t)u(-t)}{2} = \cos(t)$

$x_o(t) = \dfrac{\cos(t)u(t) - \cos(-t)u(-t)}{2}$

3) (b) $x_e(t) = \dfrac{e^{-2|t|}\sin(t) + e^{-2|t|}\sin(-t)}{2} = 0$

$x_o(t) = x(t)$

Sabject:
Year: Month: Date:

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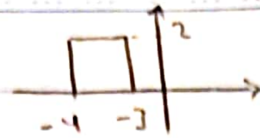
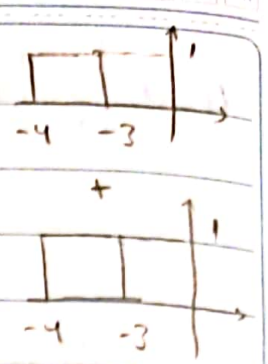
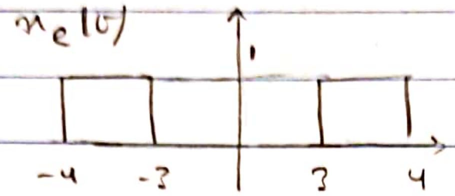
3) (c) $x(t) =$ 

اولین کاری که به ذهن
می‌رسد ساده است این
است که سیگنال را به
دو قسمت مساوی تقسیم کنیم.

$x(t) =$ 

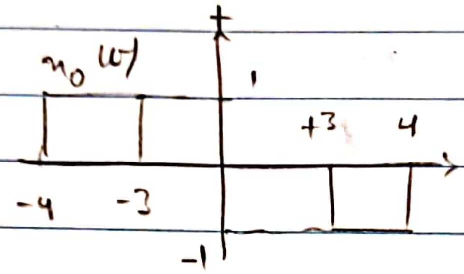
$x_e(t)$



اما هیچ یک از این دو قسمت نه
فرد هستند نه زوج، باید به نحوی
زوج و فرد بودنشان را تعیین کنیم
که سیگنال را عوض نکنند.

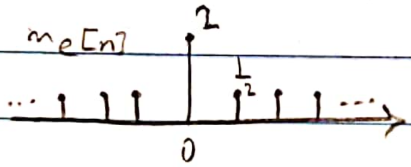
$x(t) =$

$x_o(t)$

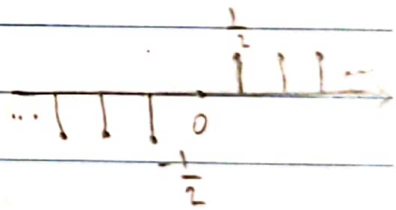


3) (d) $x[n] = \delta[n] + u[n] \Rightarrow x_e[n] = \frac{\delta[n] + u[n] + \delta[-n] + u[-n]}{2}$

$x_e[n]$



$= 2\delta[n] + \frac{1}{2}$

$x_o[n] = \frac{\delta[n] + u[n] - \delta[-n] - u[-n]}{2} =$ 

3) (e) $x_e[n] = \frac{(n+1)^2 + (n-1)^2}{2} = n^2 + 1$

$x_o[n] = \frac{(n+1)^2 - (n-1)^2}{2} = 2n$

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Year: Month: Date:

4) (a) $x(t) = e^{j\pi t} \Rightarrow \omega_0 = \pi \, , \, T_0 = \frac{2\pi}{\pi} = 2$

4) (b) $x(t) = e^{(1+\pi j)t} = \underbrace{e^{t}}_{\text{نامتناوب}} \times \underbrace{e^{\pi j t}}_{\text{متناوب}}$ → نامتناوب

4) (c) $x(t) = e^{jt} \Rightarrow \omega_0 = 1 \, , \, T_0 = 2\pi$

4) (d) $x[n] = e^{j\frac{\pi}{3}n} \Rightarrow \omega_0 = \frac{\pi}{3} \Rightarrow N_0 = \frac{2\pi}{\frac{\pi}{3}} = 6$

4) (e) $x[n] = e^{j4n} \Rightarrow \omega_0 = 4 \neq k\pi \Rightarrow$ نامتناوب

4) (f) $x(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau + \int_{-\infty}^{-t} \delta(\tau)\, d\tau = u(t) + u(-t) = 1$

متناوب با دوره تناوب پایه تعریف نشده

4) (g) $x(t) = \underbrace{\cos(6\pi t)}_{\omega_1} + \underbrace{\sin(4\pi t)}_{\omega_2}$ جمع دو متناوب همیشه متناوب است

$\left.\begin{array}{l} \omega_1 = 6\pi \Rightarrow T_1 = \frac{1}{3} \quad \times 3 = 1 \\ \omega_2 = 4\pi \Rightarrow T_2 = \frac{1}{2} \quad \times 2 = 1 \end{array}\right\} \Rightarrow T_0 = 1$

4) (h) $x[n] = \cos\left(\frac{2\pi}{7}n\right) + e^{j\frac{5\pi}{3}n}$

$\left.\begin{array}{l} \omega_1 = \frac{2\pi}{7} \Rightarrow N_1 = 7 \\ \omega_2 = \frac{5\pi}{3} \Rightarrow N_2 = \frac{6}{5} \rightarrow 6 \end{array}\right\} \Rightarrow N_0 = 6 \times 7 = 42$

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Sabject:
Year: Month: Date:

4) (i) $x[n+5] = \sum_{k=-\infty}^{\infty} \delta[n+5-5k] + \delta[n-1+5-5k] + 2\delta[n+2+5-5k]$

$= \sum_{k=-\infty}^{\infty} \delta[n-5(k-1)] + \delta[n-1-5(k-1)] + 2\delta[n+2-5(k-1)]$

$k-1=m \rightarrow x[n+5] = \sum_{m=-\infty}^{\infty} \delta[n-5m] + \delta[n-1-5m] + \delta[n+2-5m]$

$= x[n] \Rightarrow \boxed{N_0 = 5}$

4) (j) $x[n] = \cos^2\left(\frac{2\pi}{9}n + \frac{\pi}{6}\right) + \sin\left(\frac{5\pi}{3}\right) + 2\sin\left(\frac{3\pi}{5}n\right)$

$= \frac{1+\cos\left(\frac{4\pi}{9} + \frac{\pi}{3}\right)}{2} + \sin\left(\frac{5\pi}{3}\right) + 2\sin\left(\frac{3\pi}{5}n\right)$

$\left.\begin{array}{l} \omega_1 = \frac{4\pi}{9} \Rightarrow N_1 = \frac{9}{2} \rightarrow 9 \\ \omega_2 = \frac{3\pi}{5} \Rightarrow N_2 = \frac{10}{3} \rightarrow 10 \end{array}\right\} \Rightarrow \boxed{N_0 = 90}$

4) (k) $x(t) = Od\{\cos(\pi t)u(t)\} = \frac{\cos(\pi t)u(t) - \cos(-\pi t)u(-t)}{2}$

$= \frac{\cos(\pi t)u(t) - \cos(\pi t)u(-t)}{2} = \begin{cases} \frac{\cos(\pi t)}{2}, & t > 0 \\ -\frac{\cos(\pi t)}{2}, & t < 0 \\ 0, & t = 0 \end{cases}$

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Subject:
Year: Month: Date:

4) (l) $x[n] = \cos(\frac{\pi}{8} n^2)$

$\cos(\frac{\pi}{8}(n+N)^2) = \cos(\frac{\pi}{8} n^2) = \cos(\frac{\pi}{8} n^2 + \frac{\pi}{8} N^2 + \frac{\pi}{4} N n)$

$\Rightarrow \begin{cases} \frac{\pi}{8} N^2 = 2k\pi \Rightarrow N^2 = 16k \\ \frac{\pi}{4} N = 2k'\pi \Rightarrow N = 8k' \end{cases} \Rightarrow \boxed{N_0 = 8}$

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5) (a) $y(t) = \cos(x(t))$ memoryless ✓ Causal ✓

$y(t-t_0) = \cos(x(t-t_0)) \Rightarrow$ time-invariant ✓

$-1 \leq \cos(x(t)) \leq 1 \Rightarrow$ stable ✓

$\cos(x_1(t) + x_2(t)) \neq \cos(x_1(t)) + \cos(x_2(t)) \Rightarrow$ linear ✗

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5) (b) $y[n] = x[n-1]$ memoryless ✗ causal ✓

$y[n-n_0] = x[n-n_0-1] \Rightarrow$ time-invariant ✓

$\underbrace{|x[n]| < K}_{\text{bounded input}} \Rightarrow \underbrace{|x[n-1]| < k}_{\text{bounded output}} \Rightarrow$ stable ✓

$y_1 = x_1[n-1]$

$y_2 = x_2[n-1]$

5) (c) $y[n] = \sum_{k=-K}^{K} x[n-k]$, memoryless ✗ causal ✗

$y[n-n_0] = \sum_{-K}^{K} x[n-n_0-k]$ time-invariant ✓

$\underbrace{|x[n]| < N}_{\text{bounded input}} \Rightarrow \underbrace{\sum_{-k}^{K} |x[n-k]| \leq (2K+1)N}_{\text{bounded output}}$ stable ✓

$\left.\begin{array}{l} y_1[n] = \sum_{-k}^{K} x_1[n-k] \\ y_2[n] = \sum_{-k}^{K} x_2[n-k] \end{array}\right\} \Rightarrow y_1[n] + y_2[n] = \sum_{-K}^{K} x_1[n-k] + x_2[n-k]$

$\Rightarrow$ linear ✓

5) (d) $y(t) = \cos(t)\, x(t)$ memoryless ✓ causal ✓

$\cos(t)\, x(t-t_0) \neq y(t-t_0) \Rightarrow x(t-t_0) \not\to y(t-t_0) \Rightarrow$ TI ✗

$\underbrace{|x(t)| < k}_{\text{bounded input}} \Rightarrow \underbrace{|\cos(t)\, x(t)|}_{\text{bounded output}} < k \Rightarrow$ stable ✓

$\cos(t)[x_1(t) + x_2(t)] = \cos(t)\, x_1(t) + \cos(t)\, x_2(t) \Rightarrow$ linear ✓

Year: Month: Date:

5)(e) $y(t) = e^{t\,x(t)}$ memoryless ✓ causal ✓

$e^{t\,x(t-t_0)} \neq y(t-t_0) \Rightarrow x(t-t_0) \not\rightarrow y(t-t_0) \Rightarrow$ time-invariant X

$\underbrace{x(t)=1}_{\text{bounded input}} \longrightarrow \underbrace{e^{t}}_{\text{unbounded output}} \rightarrow$ stable X

$e^{t[x_1(t)+x_2(t)]} \neq e^{t\,x_1(t)} + e^{t\,x_2(t)} \rightarrow$ linear X

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5)(f)

$$y(t) = \int_{-\infty}^{t+1} x(\tau)\,d\tau$$ memoryless X $y(0) = \int_{-\infty}^{1} x(\tau)\,d\tau \Rightarrow$ causal X

$$\left.\begin{array}{l}\displaystyle\int_{-\infty}^{t+1} x(\tau-t_0)\,d\tau \\ u=\tau-t_0 \Rightarrow du=d\tau\end{array}\right\} = \int_{-\infty}^{t+1-t_0} x(u)\,du = y(t-t_0) \Rightarrow \text{time-invariant} ✓$$

stable X

$$\int_{-\infty}^{t+1} x_1(\tau)+x_2(\tau)\,d\tau = \int_{-\infty}^{t+1} x_1(\tau)\,d\tau + \int_{-\infty}^{t+1} x_2(\tau)\,d\tau$$

$\Rightarrow$ linear ✓

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5)(g) $y[n] = x[3n+2]$ memoryless X causal X

stable ✓ linear ✓

$x_1[n] \longrightarrow x_1[3n+2] = y_1[n]$

$x_2[n] = x_1[n-n_0]$ $x_2[n] \longrightarrow x_2[3n+2] = y_2[n] = x_1[3n+2-n_0]$

$\Rightarrow$ time-invariant X

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Subject:
Year: Month: Date:

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5) (h) $y[n] = \sin(x[n])$ memoryless ✓ causal ✓

time-invariant ✓ stable ✓ $\sin(x_1[n] + x_2[n]) \neq \sin(x_1[n]) + \sin(x_2[n])$

linear ✗

5) (i) $y(t) = x(\cos(t) - 1)$ $y(-2\pi) = x(0)$

⇒ memoryless ✗

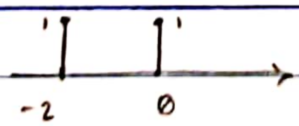
Causal ✗

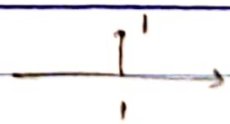
$y(t - t_0) = x(\cos(t - t_0) - 1) \neq x(\cos(t) - t_0)$ ⇒ time-invariant ✗

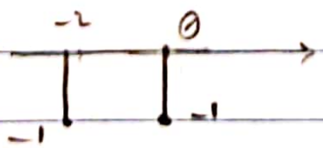
stable ✓ linear ✓

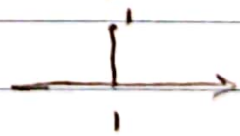
6) (a) $y^{-1}(t) = x(\frac{t}{2})$ invertible ✓

6) (b) $x[n] = 1 \Rightarrow y[n] = 1$, $x[n] = -1 \Rightarrow y[n] = 1$ ⇒ invertible ✗

6) (c) $x_1 =$ 

⇒ $y_1[n] =$ 

$x_2 =$ 

⇒ $y_2[n] =$ 

⇒ invertible ✗

MEHR

Subject:
Year: Month: Date:

Sa Su Mo Tu We Th

6) (d) چون بخشی از سیگنال ورودی بین (۱- تا ۱) از دست می‌رود، معکوس‌پذیر نیست

6) (e)
$$y[n] = \begin{cases} x[n] & , n \geq 0 \\ x[1] & , -2 \leq n < 0 \\ x[n+2] & , n < -2 \end{cases}$$

invertible ✓

$$y^{-1}[n] = \begin{cases} x[n] & , n \geq 0 \\ x[n-2] & , n < 0 \end{cases}$$

6) (f) invertible ✗

دانشگاه صنعتی امیرکبیر
(پلی تکنیک تهران)

Signals and Systems

Assignment 2

Spring 2021

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Question1 (5 pts)

Convolve the following pairs of signals.

(a) $\begin{cases} x_1(t) = e^t \\ x_2(t) = u(t) \end{cases}$

(b) $\begin{cases} x_1[n] = (\frac{1}{2})^{2n-1} \\ x_2[n] = u[-n-1] - u[-n-7] \end{cases}$

(c) $\begin{cases} x_1[n] = 2^n u[n] \\ x_2[n] = u[n+3] - u[n-5] \end{cases}$

(d) $\begin{cases} x_1[n] = 3^n (u[n] - u[n-6]) \\ x_2[n] = u[n+1] - u[n-10] \end{cases}$

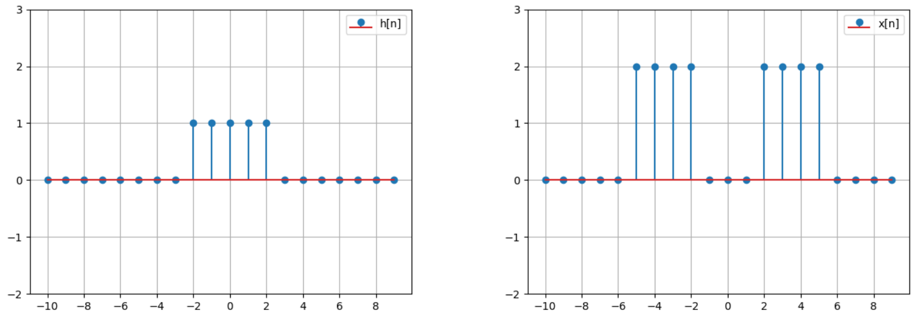
(e) $\begin{cases} x_1(t) = e^{-3t} u(t-1) \\ x_2(t) = u(t+1) - u(t-4) \end{cases}$

(f) $\begin{cases} x_1(t) = e^{2t} u(2-t) \\ x_2(t) = u(t) + u(t-2) \end{cases}$

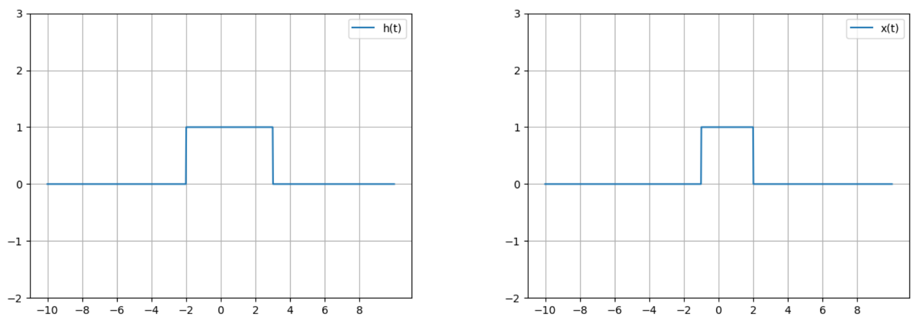
Question2 (5 pts)

For each pair of impulse response and input, determine LTI system's output. Sketch the results.

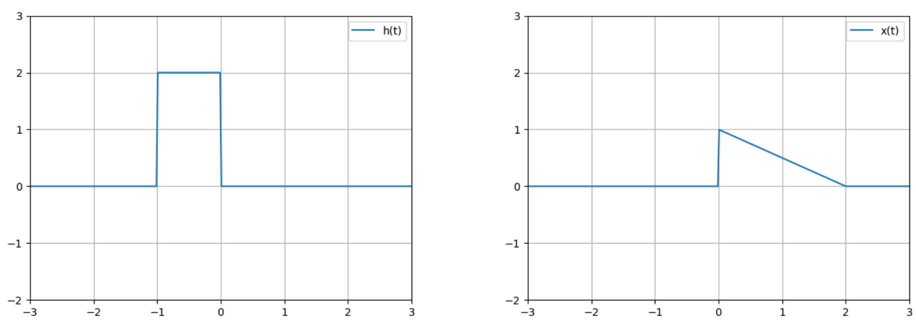
(a)



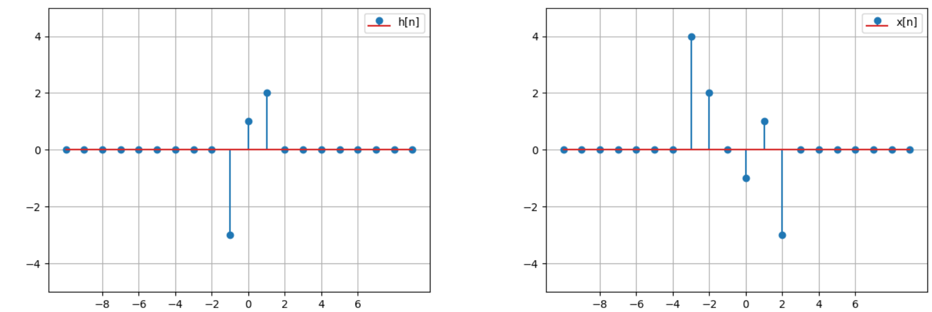
(b)



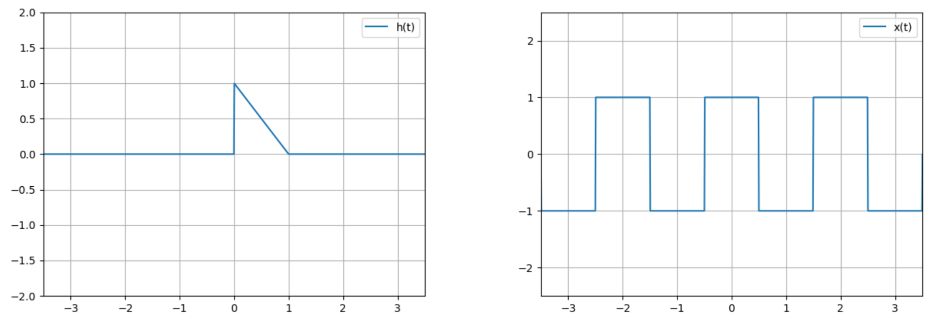
(c)



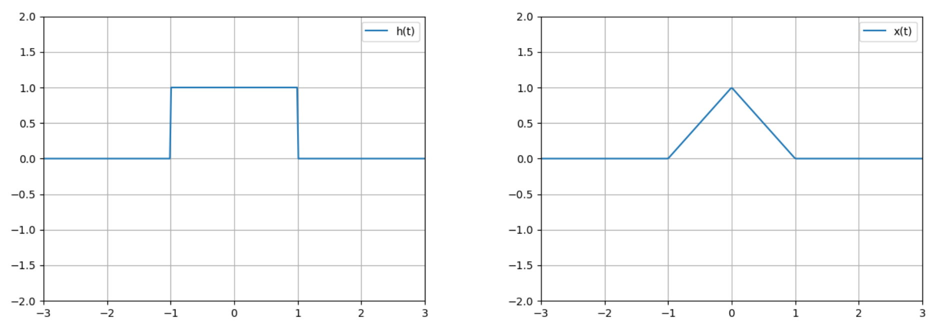
(d)



(e)



(f)



Question3 (3 pts)

For each of the following impulse responses, determine whether the corresponding LTI system is memoryless, casual and stable. Justify your answers.

(a) $h[n] = 2^{-3|n|}$

(b) $h(t) = te^{-t}u(t)$

(c) $h(t) = cos(t)u(t+1)$

(d) $h(t) = u(t+1) * \frac{d}{dt}u(t-2)$

(e) $h(t) = tan^{-1}(cos(t))$

(f) $h(t) = (t^2+1)\delta(t)$

(g) $h[n] = cos(\frac{\pi}{2}n)u[n+2] + \delta[n+2]$

Question4 (2 pts)

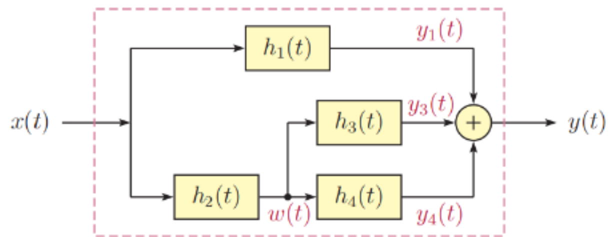
Consider the cascaded interconnection of LTI systems shown below where:

$$h_1(t) = sin(\frac{\pi}{3}t)$$

$$h_2(t) = e^{2t}u(1-t)$$

$$h_3(t) = u(t)$$

$$h_4(t) = \delta(t+1)$$



(a) Determine the impulse response h_{eq} of the equivalent system.

(b) Let the input signal be $x(t) = u(t+1) - u(t-1)$. Determine the signals $w(t)$, $y_1(t)$, $y_3(t)$ and $y_4(t)$.

Question5 (2 pts)

For each pair of impulse responses below, determine whether the corresponding systems are inverses of each other.

(a) $h_1[n] = (\frac{1}{5})^n u[n] \quad , \quad h_2[n] = \delta[n] - \frac{1}{5}\delta[n-1]$

(b) $h_1(t) = e^{-t}u(t) \quad , \quad h_2(t) = \delta(t) + \delta'(t)$

Question6 (3 pts)

Find the step response for systems with the following impulse responses.

(a) $h(t) = \delta(t+1) - \delta(t-1)$

(b) $h[n] = e^n(\frac{1}{3})^n u[n+2]$

(c) $h(t) = 2\delta^2(t)$

(d) $h(t) = \dfrac{1}{1+t^2}$

Note: *Step Response* of a system is the system's output given unit step as input.

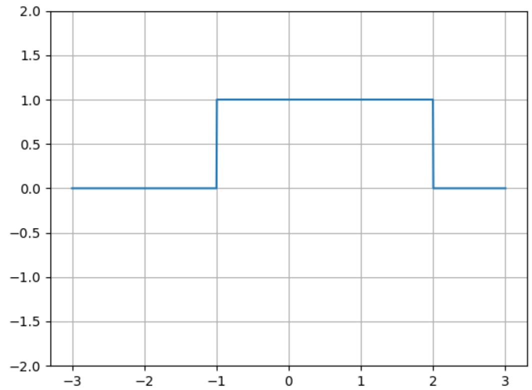
Question7 (2 pts)

(a) Consider an LTI system with input and output related through the equation

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau - 2) d\tau$$

What is the impulse response h(t) for this system?

(b) Determine the response of the system when the input x(t) is as shown in the figure:



Programming Assignment 1 (3 pts)

(a) Using either python or Matlab, convolve the following pair of signals and plot the result:

$$\begin{cases} x_1(t) = 2.5 \times sin(\frac{2t}{3}) \\ x_2(t) = 1.25 \times cos(\frac{2t}{3}) \end{cases} \qquad x_1 \text{ and } x_2 \text{ within interval } -10 \leq t \leq 10$$

(b) What is the output value for $t = 2.31$? You might have to multiply the output with dt.

(c) Compute the convolution integral within interval $-10 \leq \tau \leq 10$ for $t = 2.31$, i.e.,

$$\int_{-10}^{10} x_1(2.31 - \tau)x_2(\tau)d\tau$$

Is the answer the same with the previous part? If not, why?

(d) Convolve the following pairs of signals and plot the result:

$$\begin{cases} x_1[n] = (0.64)^n u[n] \\ x_2[n] = u[n] - u[n-4] \end{cases} \qquad x_1 \text{ and } x_2 \text{ within interval } -5 \leq n \leq 15$$

Note: You can set the size of the output of the convolution function by doing the following:

- If you are using *python*, use the *mode* argument of *convolve()*.
- If you are using *Matlab*, use the *shape* argument of *conv()*.

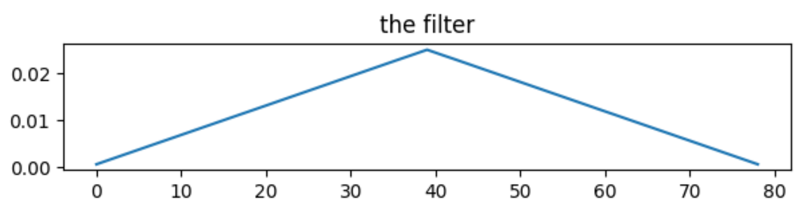
Programming Assignment 2 (8 bonus pts)

One of the applications of convolution in signal processing is applying filters. In this assignment we are going to apply two filters to an audio signal.

(a) First we are going to apply a triangular filter to our signal and see how it affects the sound.

(i) Open and read the data of *giggle_mono.wav* audio file provided with this assignment. You can use the *wave* module which is a standard python module. Use *open()* and *readframes()* functions in order to open and read the data of the audio file respectively. Here is the link to *wave*'s documentation.

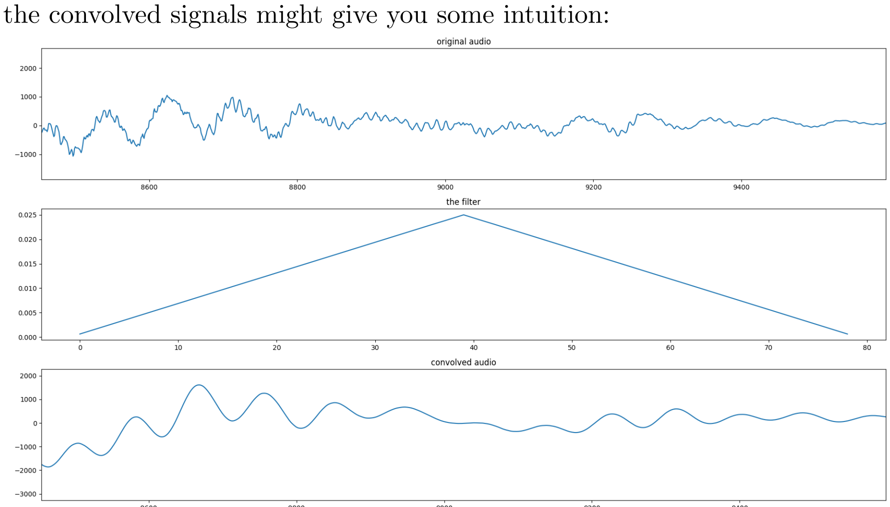
(ii) Create and plot a triangular signal similar to the one depicted below:



(iii) Convolve the audio signal with the filter. Note that you might have to scale the result so it has the same loudness as the original signal.

(iv) Using the *wave* module again, write the result of the convolution to a new audio file named *result1.wav*. Don't forget to set the parameters of the *Wave_Write* object using *setparams()* function before writing it into the new audio file.

(v) Using your knowledge about convolution and its mathematical definition, explain how convolving the triangular signal with the original audio signal has this *smoothing effect* on it. Plotting the original and the convolved signals might give you some intuition:



(b) Now we are going to add echo to the sound using attenuated version of the sound at regularly spaced intervals. Consequently, an often-used model for this phenomenon is an LTI system with an impulse response consisting of a train of impulses, i.e.,

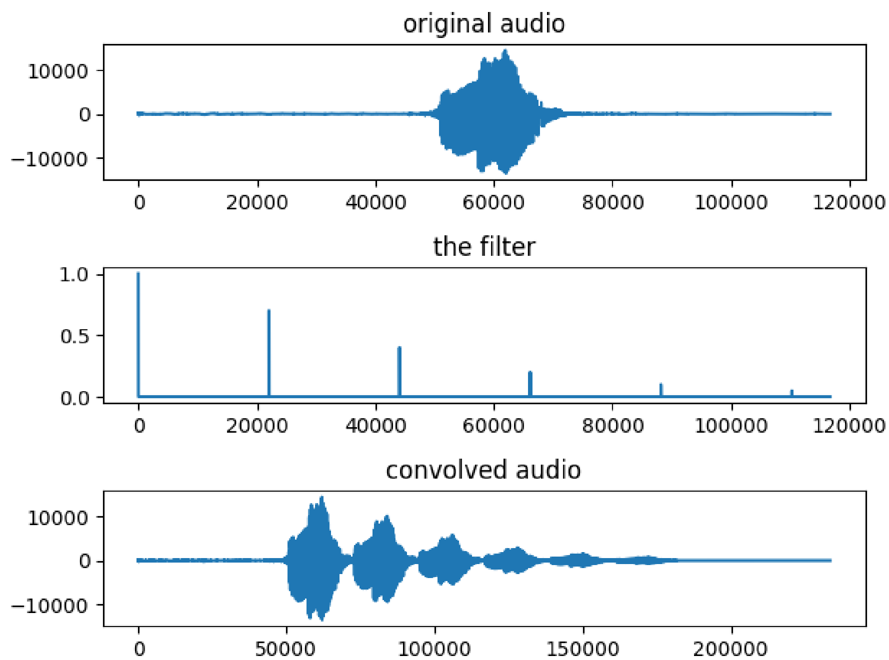
$$h(t) = \sum_{k=0}^{M} h_k \delta(t - kT)$$

Here M echoes occur T seconds apart, and h_k represents the gain factor on the kth echo resulting from an initial acoustic impulse.

(i) Open and read the data of *hello_mono.wav*.

(ii) Create an impulse train and plot the result. Note that *hello_mono.wav* has 44100 Hz sampling frequency. That means each second has 44100 audio data points (samples) in it.

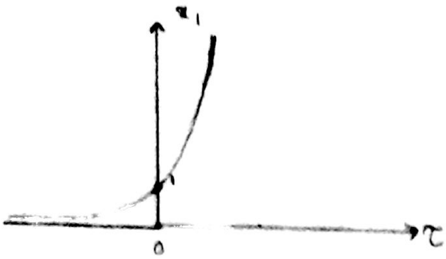
(iii) Convolve the audio signal with the filter and save the result in a new audio file named *result2.wav*. Here is an example with $M = 5$ and $T = 0.5s$:



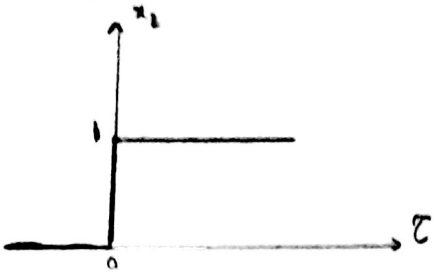
Question 1.

(a) $\begin{cases} x_1(t) = e^{t} \\ x_2(t) = u(t) \end{cases}$

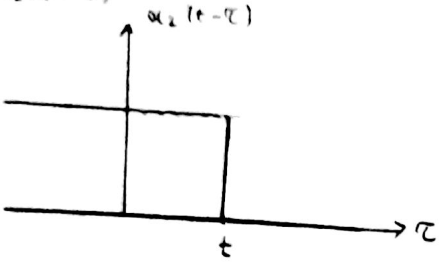
$x_1(\tau)$:



$x_2(\tau)$



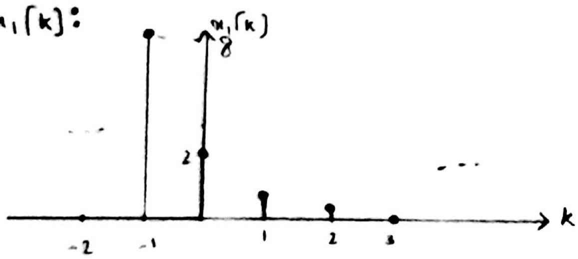
$x_2(t-\tau)$



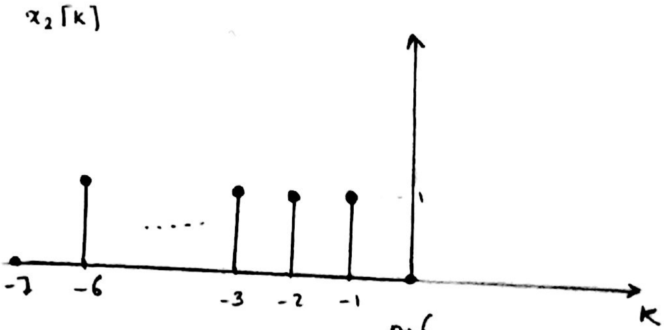
$$x_1 * x_2 = \int_{-\infty}^{+\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau = \int_{-\infty}^{t} e^{\tau} d\tau = e^{\tau}\Big/_{-\infty}^{t} = \boxed{e^{t}}$$

(b) $\begin{cases} x_1[n] = \left(\frac{1}{2}\right)^{2n-1} \\ x_2[n] = u[-n-1] - u[-n-7] \end{cases}$

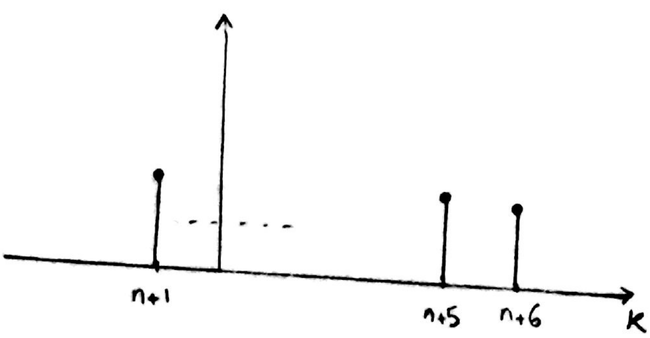
$x_1[k]$:



$x_2[k]$



$x_2[n-k]$:



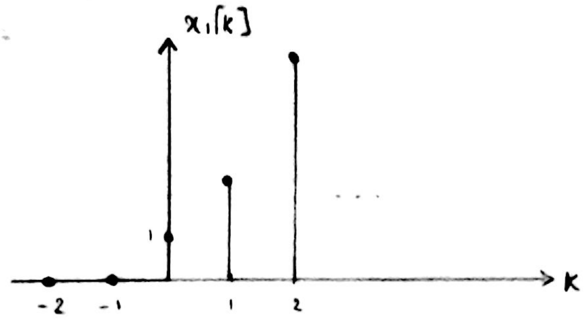
$$x_1 * x_2 = \sum_{k=-\infty}^{+\infty} x_1[k]\, x_2[n-k] = \sum_{k=n+1}^{n+6} \left(\frac{1}{2}\right)^{2k-1}$$

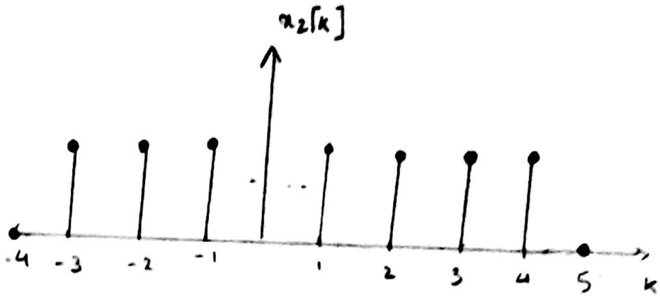
$$= 2 \sum_{k=n+1}^{n+6} \left(\frac{1}{4}\right)^{k} = 2 \times \frac{1-\left(\frac{1}{4}\right)^{n+6-(n+1)+1}}{1-\frac{1}{4}} \times \left(\frac{1}{4}\right)^{n+1}$$

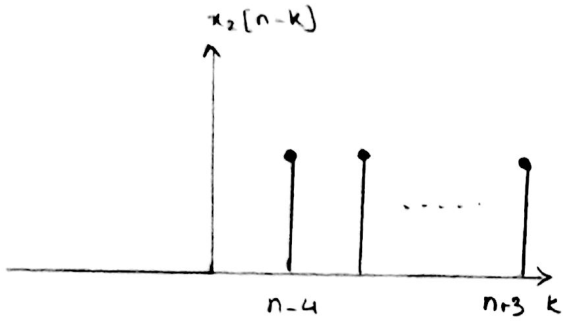
$$= 2 \times \frac{1-\left(\frac{1}{4}\right)^{6}}{\frac{3}{4}} \times \left(\frac{1}{4}\right)^{n+1}$$

$$= \frac{8}{3}\left(1-\left(\frac{1}{4}\right)^{6}\right) \times \left(\frac{1}{4}\right)^{n+1}$$

(C) $\begin{cases} x_1[n] = 2^n u[n] \\ x_2[n] = u[n+3] - u[n-5] \end{cases}$







$n < -3$:

$$x_1 * x_2 = 0$$

$-3 \leqslant n < 4$:

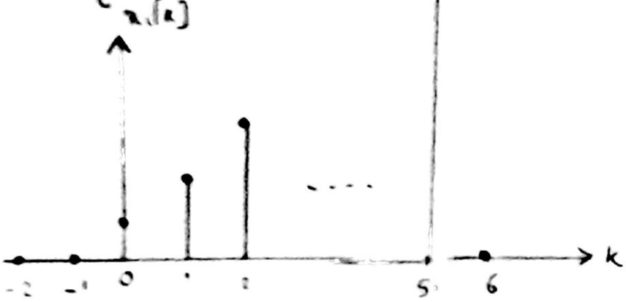
$$x_1 * x_2 = \sum_{k=0}^{n+3} 2^k \cancel{u[k]}^{\,1} = \sum_{k=0}^{n+3} 2^k \overset{\text{دنباله هندسی}}{=\!=} \frac{1-2^{n+3+1}}{1-2} = 2^{n+4} - 1$$

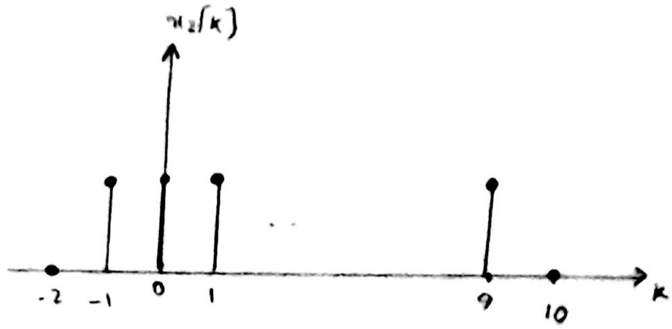
$4 \leqslant n$:

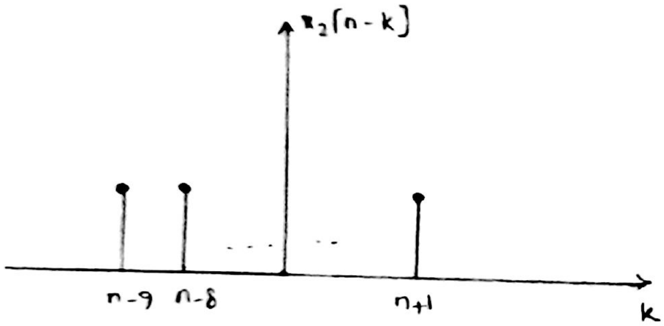
$$x_1 * x_2 = \sum_{k=n-4}^{n+3} 2^k \cancel{u[k]}^{\,1} = \sum_{k=n-4}^{n+3} 2^k \overset{\text{دنباله هندسی}}{=\!=} \frac{1-2^{(n+3)-(n-4)+1}}{1-2} \times 2^{n-4} = \frac{255}{16} \cdot 2^n$$

$\Longrightarrow$ جواب نهایی : $x_1[n] * x_2[n] = \begin{cases} 0 & n < -3 \\ 2^{n+4} - 1 & -3 \leqslant n < 4 \\ \frac{255}{16} \times 2^n & 4 \leqslant n \end{cases}$

(d) $\begin{cases} x_1[n] = 3^n (u[n] - u[n-6]) \\ x_2[n] = u[n+1] - u[n-10] \end{cases}$







$n < -1$:

$x_1 * x_2 = 0$

$-1 \leqslant n < 5$

$$x_1 * x_2 = \sum_{k=0}^{n+1} 3^k (u[k] - u[k-6]) = \sum_{k=0}^{n+1} 3^k \xlongequal{\text{دنباله هندسی}} \frac{1-3^{n+2}}{1-3} = \frac{3^{n+2}-1}{2}$$

$5 \leqslant n < 9$:

$$x_1 * x_2 = \sum_{k=0}^{5} 3^k \xlongequal{\text{دنباله هندسی}} \frac{1-3^6}{1-3} = \frac{3^6-1}{2} = 364$$

$9 \leqslant n < 15$:

$$x_1 * x_2 = \sum_{k=n-9}^{5} 3^k \xlongequal{\text{دنباله هندسی}} \frac{1-3^{5-(n-9)+1}}{1-3} \times 3^{n-9} = \frac{3^6 - 3^{n-9}}{2}$$

$n \geqslant 15$:

$x_1 * x_2 = 0$

$$\Longrightarrow \text{جواب نهایی}: \; x_1[n] * x_2[n] = \begin{cases} 0 & n < -1 \\ \frac{3^{n+2}-1}{2} & -1 \leqslant n < 5 \\ 364 & 5 \leqslant n < 9 \\ \frac{3^6 - 3^{n-9}}{2} & 9 \leqslant n < 15 \\ 0 & 15 \leqslant n \end{cases}$$

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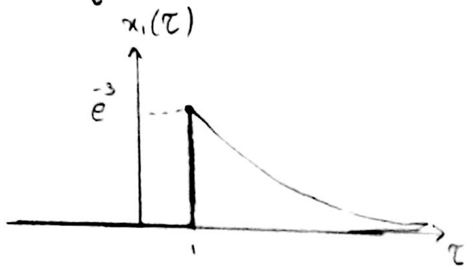
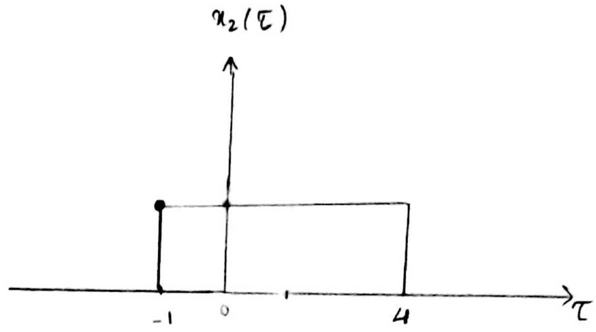
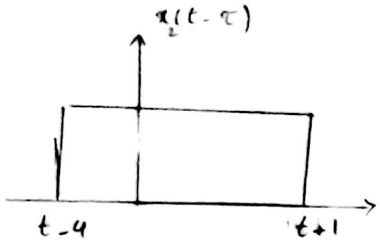
(ج) $\begin{cases} x_1(t) = e^{-3t}\, u(t-1) \\ x_2(t) = u(t+1) - u(t-4) \end{cases}$







$t < 0$:

$$x_1 * x_2 = 0$$

$0 \leq t < 5$

$$x_1 * x_2 = \int_1^{t+1} e^{-3\tau}\, u(\tau-1)\, d\tau = \int_1^{t+1} e^{-3\tau}\, d\tau = -\frac{1}{3} e^{-3\tau}\Big|_1^{t+1} = \frac{1}{3}\left(e^{-3} - e^{-3(t+1)}\right)$$

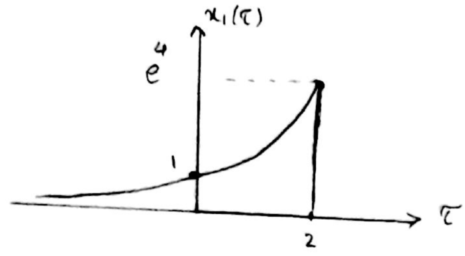
$5 \leq t$:

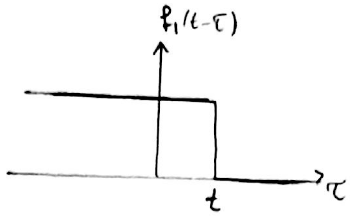
$$x_1 * x_2 = \int_{t-4}^{t+1} e^{-3\tau}\, u(\tau-1)\, d\tau = \int_{t-4}^{t+1} e^{-3\tau}\, d\tau = -\frac{1}{3} e^{-3\tau}\Big|_{t-4}^{t+1} = \frac{1}{3}\left(e^{12-3t} - e^{-3-3t}\right)$$

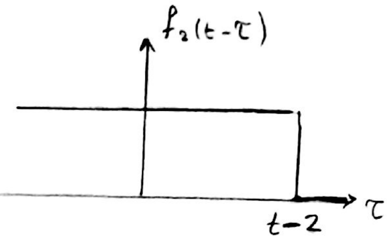
$\Longrightarrow$ جواب نهایی : $x_1(t) * x_2(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{3}\left(e^{-3} - e^{-3(t+1)}\right) & 0 \leq t < 5 \\ \frac{1}{3}\left(e^{12-3t} - e^{-3-3t}\right) & t \geq 5 \end{cases}$

(f) $\begin{cases} x_1(t) = e^{2t}\, u(2-t) \\ x_2(t) = u(t) + u(t-2) \end{cases}$

$$x_1 * x_2 = x_1 * (\underbrace{u(t)}_{f_1(t)} + \underbrace{u(t-2)}_{f_2(t)}) = x_1(t) * f_1(t) + x_1 * f_2(t)$$







$t < 2$:

$$x_1 * f_1 = \int_{-\infty}^{t} e^{2\tau} \overset{1}{\cancel{u(2-\tau)}}\, d\tau = \int_{-\infty}^{t} e^{2\tau} d\tau = \frac{e^{2\tau}}{2}\Big|_{-\infty}^{t} = \frac{e^{2t}}{2}$$

$t \geq 2$:

$$x_1 * f_1 = \int_{-\infty}^{2} e^{2\tau} \overset{1}{\cancel{u(2-\tau)}}\, d\tau = \int_{-\infty}^{2} e^{2\tau} d\tau = \frac{e^{2\tau}}{2}\Big|_{-\infty}^{2} = \frac{e^4}{2}$$

$$x_1(t) * f_1(t) = \begin{cases} \dfrac{e^{2t}}{2} & t < 2 \\ \dfrac{e^4}{2} & t \geq 2 \end{cases}$$

$t < 4$:

$$x_1 * f_2 = \int_{-\infty}^{t+2} e^{2\tau} d\tau = \frac{e^{2\tau}}{2}\Big|_{-\infty}^{t-2} = \frac{e^{2t-4}}{2}$$

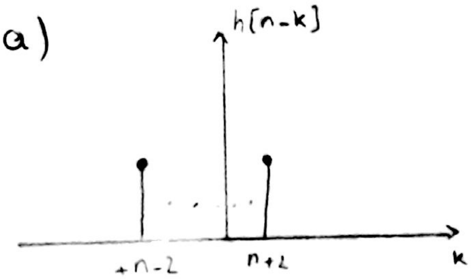
$t \geq 4$:

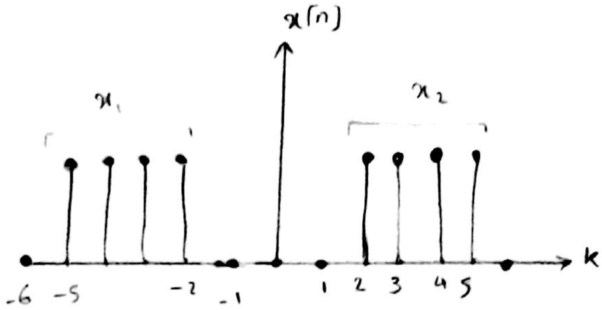
$$x_1 * f_2 = \int_{-\infty}^{2} e^{2\tau} d\tau = \frac{e^{2\tau}}{2}\Big|_{-\infty}^{2} = \frac{e^4}{2}$$

$$\Rightarrow x_1 * (f_1 + f_2) = x_1(t) * x_2(t) = \begin{cases} \dfrac{e^{2t} + e^{2t-4}}{2} & t < 2 \\ \dfrac{e^{2t-4} + e^4}{2} & 2 \leq t < 4 \\ e^4 & t \geq 4 \end{cases}$$

Question 2.

(a)





$$y[n] = x_1[n] * h[n] = (x_1[n] + x_2[n]) * h[n] = x_1 * h + x_2 * h$$

$n < -7$:

$$x_1 * h = 0$$

$-7 \leq n < -3$

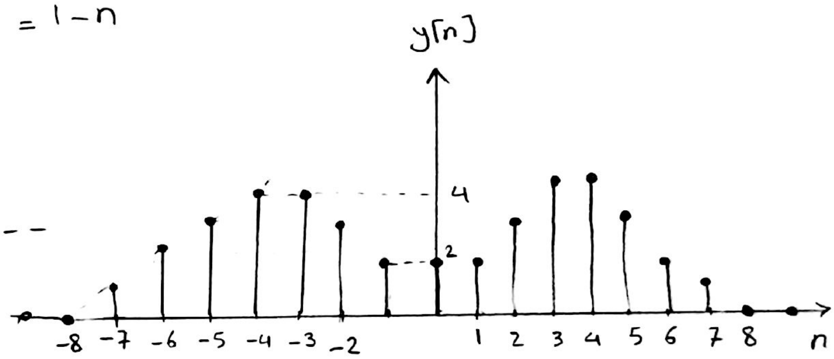
$$x_1 * h = \sum_{k=-5}^{n+2} 1 = n+2-(-5)+1 = n+8$$

$-3 \leq n < 1$

$$x_1 * h = \sum_{k=n-2}^{-2} 1 = -2-(n-2)+1 = 1-n$$

$n \geq 1$

$$x_1 * h = 0$$



$n < 0$:

$$x_2 * h = 0$$

$0 \leq n < 4$:

$$x_2 * h = \sum_{k=2}^{n+2} 1 = n+\cancel{2}-\cancel{2}+1 = n+1$$

$4 \leq n < 8$

$$x_2 * h = \sum_{k=n-2}^{5} 1 = 5-(n-2)+1 = 8-n$$

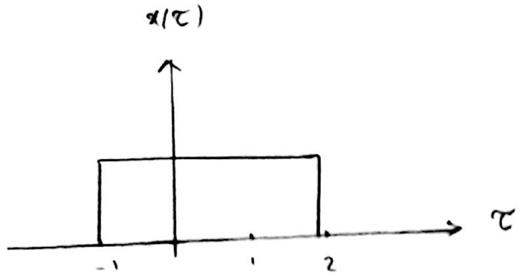
$n \geq 8$:

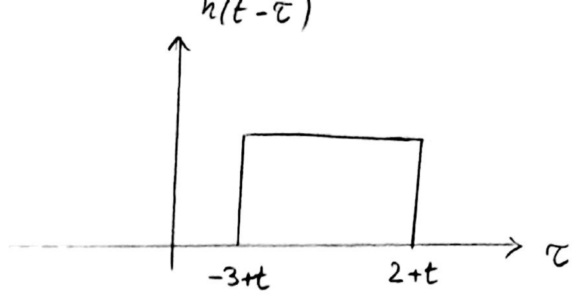
$$x_2 * h = 0$$

جواب نهایی:

$$x_1[n] * h[n] = \begin{cases} 0 & n < -7 \\ n+8 & -7 \leq n < -3 \\ 1-n & -3 \leq n < 0 \\ 2 & 0 \leq n < 1 \\ n+1 & 1 \leq n < 4 \\ 8-n & 4 \leq n < 8 \\ 0 & 8 \leq n \end{cases}$$

(b)





$y(t) = x(t) * h(t)$

$t < -3$:

$x * h = 0$

$-3 \leqslant t < 0$:

$x * h = \int_{-1}^{2+t} d\tau = 3 + t$

$0 \leqslant t < 2$

$x * h = \int_{-1}^{2} d\tau = 3$

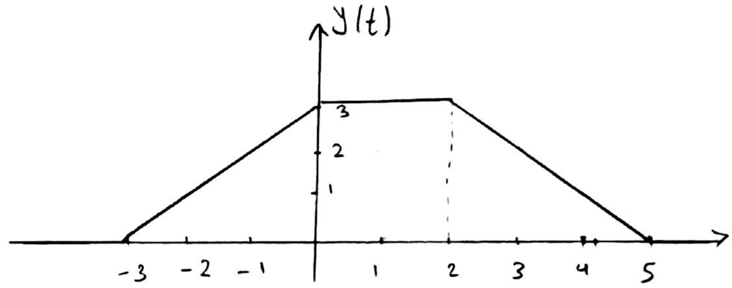
$2 \leqslant t < 5$

$x * h = \int_{t-3}^{2} d\tau = 5 - t$

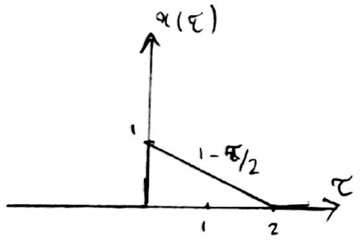
$t \geqslant 5$

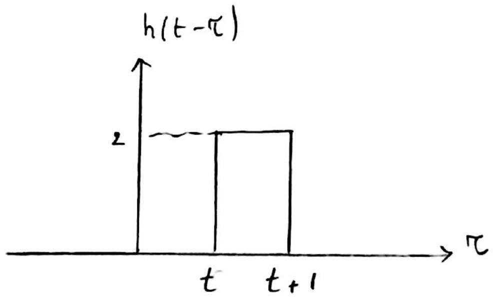
$x * h = 0$

$$y(t) = \begin{cases} 0 & t < -3 \\ 3+t & -3 \leqslant t < 0 \\ 3 & 0 \leqslant t < 2 \\ 5-t & 2 \leqslant t < 5 \\ 0 & t \geqslant 5 \end{cases}$$



(c)





$t < -1$:

$y(t) = x * h = 0$

$-1 \leqslant t < 0$

$x * h = 2\int_{0}^{t+1} (1 - \frac{\tau}{2}) d\tau = 2(\tau - \frac{\tau^2}{4}) \Big|_{0}^{t+1} = 2(t+1) - \frac{(t+1)^2}{2}$

$0 \leqslant t < 1$:

$x * h = 2\int_{t}^{t+1} (1 - \frac{\tau}{2}) d\tau = 2 - \frac{2t+1}{2}$

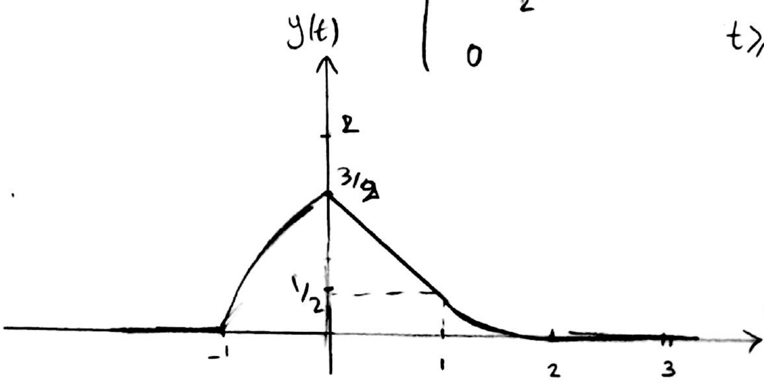
$1 \leqslant t < 2$:

$x * h = 2\int_{t}^{2} (1 - \frac{\tau}{2}) d\tau = \frac{(t-2)^2}{2}$

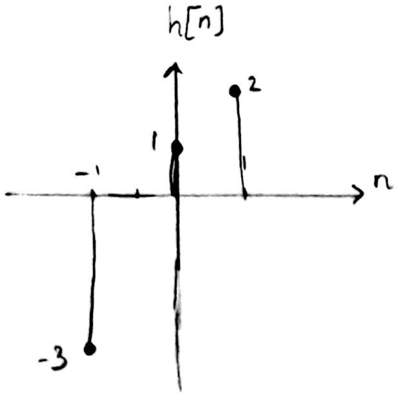
$t \geqslant 2$:

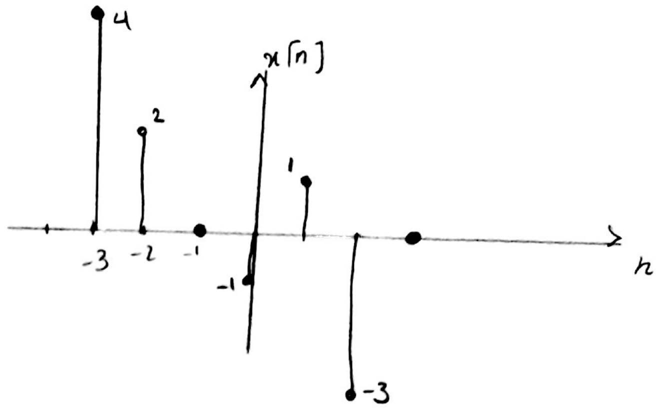
$x * h = 0$

$$y(t) = x * h = \begin{cases} 0 & t < -1 \\ \frac{(3-t)(t+1)}{2} & -1 \leqslant t < 0 \\ \frac{3-2t}{2} & 0 \leqslant t < 1 \\ \frac{(t-2)^2}{2} & 1 \leqslant t < 2 \\ 0 & t \geqslant 2 \end{cases}$$



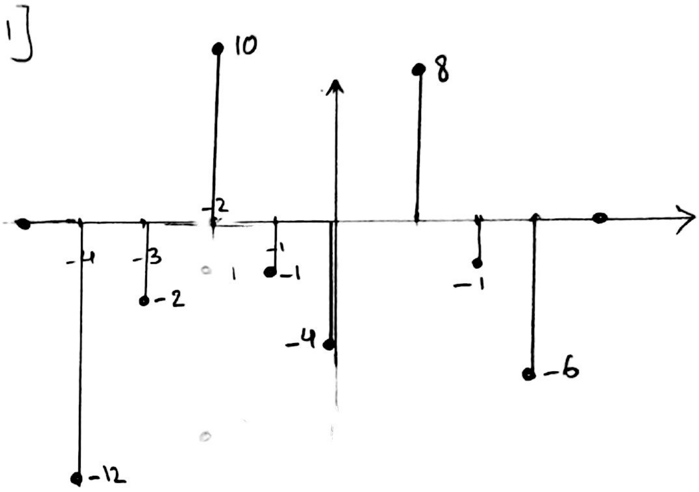
(d)





$$h[n] = \delta[n] + 2\delta[n-1] - 3\delta[n+1]$$

$$x[n] * h[n] = x[n] + 2x[n-1] - 3x[n+1]$$



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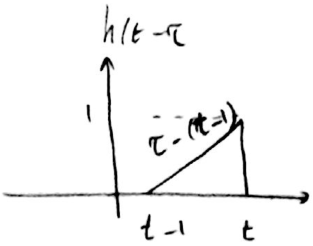
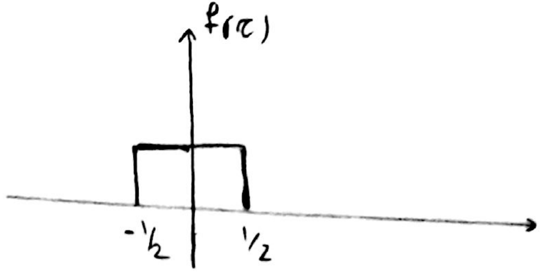
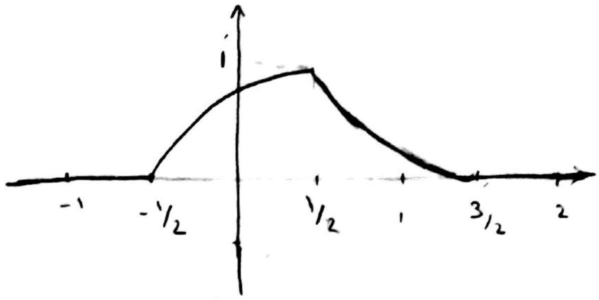
(c)





$w(t) = h(t) * f(t)$:



$t < -1/2$:

$$h(t) * f(t) = 0$$

$-1/2 < t < 1/2$:

$$h(t) * f(t) = \int_{-1/2}^{t} (\tau - (t+1))\, d\tau = \frac{\tau^2}{2} - \tau(t-1)\Big|_{-1/2}^{t} = \frac{t^2}{2} - \frac{1}{8} - (t+\tfrac{1}{2})(t-1)$$

$$= -\frac{1}{2}t^2 + \frac{1}{2}t + \frac{3}{8}$$

$\frac{1}{2} < t < 3/2$

$$h(t) * f(t) = \int_{t-1}^{1/2} (\tau - (t-1))\, d\tau = \frac{\tau^2}{2} - \tau(t-1)\Big|_{t-1}^{1/2} = \frac{1}{8} - \frac{(t-1)^2}{2} + (t-1)(t-3/2)$$

$$= \frac{t^2}{2} - \frac{3}{2}t + \frac{9}{8}$$

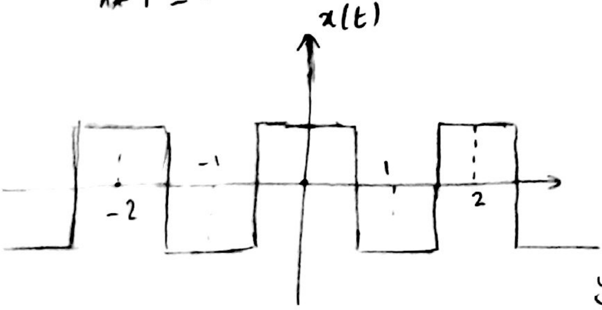
$t \geq 3/2$:

$$h * f = 0$$

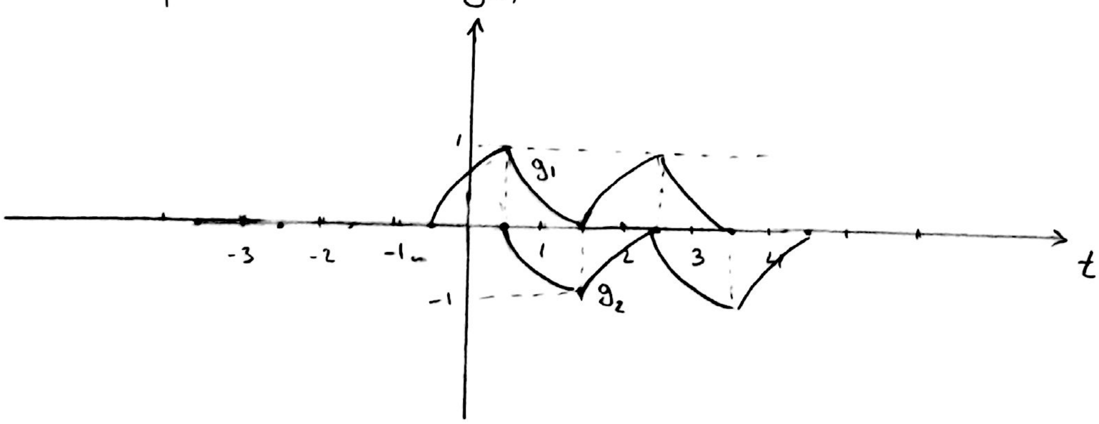
جواب نهائی: $w(t) = h * f = \left(-\frac{t^2}{2} + 1/2\, t + 3/8\right) u(t + 1/2)$

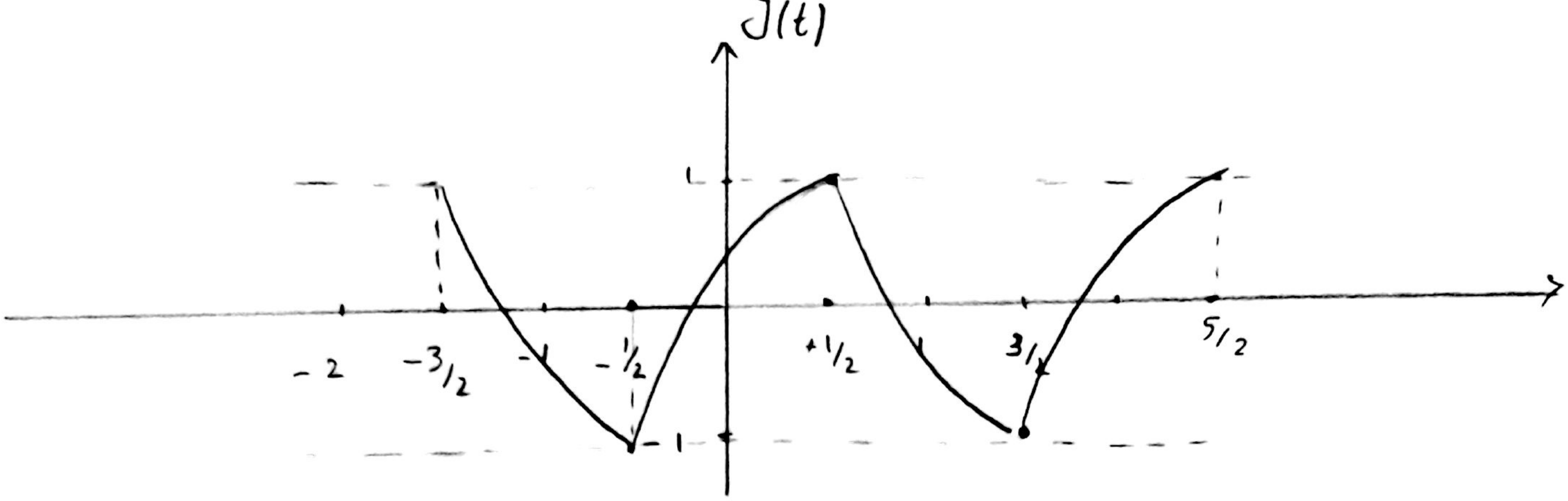
$$+ \left(t^2 - 2t + \frac{3}{4}\right) u(t - 1/2)$$

$$- \left(\frac{t^2}{2} - \frac{3}{2} + \frac{9}{8}\right) u(t - 3/2)$$

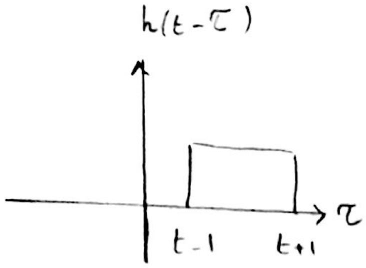


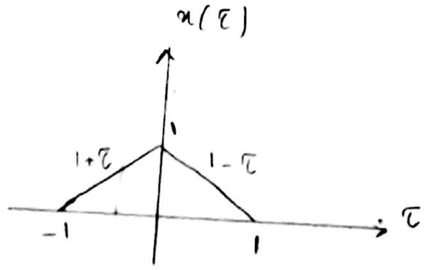
$$y(t) = \underbrace{\sum_{k=-\infty}^{+\infty} w(t-2k)}_{g_1(t)} - \underbrace{\sum_{k=-\infty}^{+\infty} w(t-(2k+1))}_{g_2(t)}$$




y(t)
1
-1
-2
-3/2
-1
-1/2
+1/2
3/2
5/2

(f)





$t<-2$:

$$x*h=0$$

$-2\leqslant t<-1$

$$x*h=\int_{-1}^{t-1}(1+\tau)\,d\tau=\tau+\frac{\tau^2}{2}\Big|_{-1}^{t+1}=2+t+\frac{(t+1)^2-1}{2}=2+t+\frac{(t+2)t}{2}=\frac{(t+2)^2}{2}$$

$-1\leqslant t<0$:

$$x*h=\int_{-1}^{0}(1+\tau)\,d\tau+\int_{0}^{t+1}(1-\tau)\,d\tau=\tau+\frac{\tau^2}{2}\Big|_{-1}^{0}+\tau-\frac{\tau^2}{2}\Big|_{0}^{t+1}=+\frac{1}{2}+t+1-\frac{(t+1)^2}{2}$$

$$=\frac{1}{2}+\frac{(t+1)(1-t)}{2}$$

$0\leqslant t<1$:

$$x*h=\int_{t-1}^{0}(1+\tau)\,d\tau+\int_{0}^{1}(1-\tau)\,d\tau=\tau+\frac{\tau^2}{2}\Big|_{t-1}^{0}+\frac{1}{2}=-t+1-\frac{(t-1)^2}{2}+\frac{1}{2}=\frac{(t-1)(-1-t)}{2}+\frac{1}{2}$$

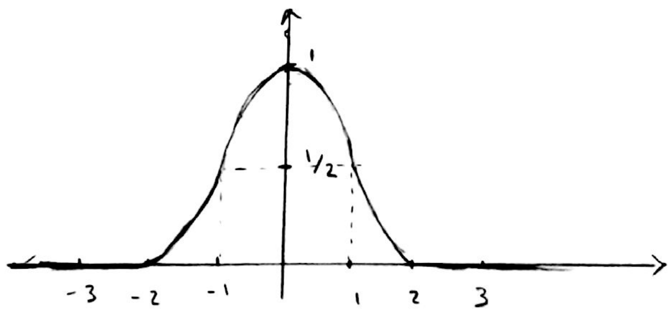
$1\leqslant t<2$:

$$x*h=\int_{t-1}^{1}(1-\tau)\,d\tau=\tau-\frac{\tau^2}{2}\Big|_{t-1}^{1}=2-t-\frac{1}{2}+\frac{(t-1)^2}{2}=\frac{(t-2)^2}{2}$$

$t\geqslant 2$:

$$x*h=0$$

$$y(t)=x*h=\begin{cases}0 & t<-2\\ \dfrac{(t+2)^2}{2} & -2\leqslant t<-1\\ \dfrac{1}{2}+\dfrac{(t+1)(1-t)}{2} & -1\leqslant t<0\\ \dfrac{1}{2}+\dfrac{(1-t)(t+1)}{2} & 0\leqslant t<1\\ \dfrac{(t-2)^2}{2} & 1\leqslant t<2\\ 0 & 2\leqslant t\end{cases}$$



Question 3.

(a) $h[n] = 2^{-3|n|}$

(1) System is with memory: $h[0] = 2^{-3\times 0} = 2^{0} = 1 \neq 0$

(2) System is not causal: $n=-1 \Rightarrow h[-1] = 2^{-3\times 1} = \frac{1}{8} \neq 0$

(3) System is stable:

$$\sum_{k=-\infty}^{+\infty} |h[k]| = \sum_{k=-\infty}^{+\infty} |2^{-3|k|}| = \sum_{k=0}^{+\infty} 2^{-3k} + \sum_{k=-\infty}^{-1} 2^{3k} = \left(1+\frac{1}{8}+\left(\frac{1}{8}\right)^2+\dots\right)+\left(\frac{1}{8}+\left(\frac{1}{8}\right)^2+\dots\right)$$

$$= \frac{1}{1-\frac{1}{8}} + \frac{\frac{1}{8}}{1-\frac{1}{8}} = \frac{8}{7}+\frac{1}{7} = \frac{9}{7} < \infty$$

(b) $h(t) = te^{-t}u(t)$

(1) System is memoryless: $h(0) = 0\times e^{0}\times 1 = 0$.

(2) System is causal: $\forall t<0.\ u(t)=0 \Rightarrow h(t)=0$

(3) System is stable:

$$\int_{-\infty}^{+\infty} |h(\tau)|\,d\tau = \int_{-\infty}^{+\infty} |\tau| e^{-\tau} u(\tau)d\tau = \int_{0}^{+\infty} \tau e^{-\tau} d\tau = -\tau e^{-\tau} - e^{-\tau}\Big|_{0}^{+\infty} = \lim_{\tau\to+\infty} \frac{\tau}{e^{\tau}} - 1$$

$$= -1 < \infty$$

(c) $h(t) = \cos(t)u(t+1)$

(1) System is with memory: $h(0) = \cos(0)u(1) = 1 \neq 0$

(2) System is not causal: $t=-\frac{1}{2} \Rightarrow h(-\frac{1}{2}) = \cos(-\frac{1}{2})u(\frac{1}{2}) = \cos(-\frac{1}{2}) \neq 0$

(3) System is not stable:

$$\int_{-\infty}^{+\infty} |h(\tau)|\,d\tau = \int_{-\infty}^{+\infty} |\cos(\tau)|\,u(\tau+1)d\tau = \int_{-1}^{+\infty} |\cos(\tau)|\,d\tau = +\infty$$

(d) $u(t+1) * \frac{d}{dt} u(t-2)$

$$\frac{d}{dt} u(t-2) = \delta(t-2) \Rightarrow h(t) = u(t+1) * \delta(t-2) = u(t-1)$$

(1) System is memoryless: $h(0) = u(-1) = 0$

(2) System is causal: $\forall t < 0 .\ u(t-1) = 0 \Rightarrow h(t) = 0$

(3) System is not stable: $\int_{-\infty}^{+\infty} |h(\tau)| d\tau = \int_{-\infty}^{+\infty} u(\tau-1) d\tau = \int_{1}^{+\infty} d\tau = \infty$

(e) $h(t) = \tan^{-1}(\cos(t))$

(1) System is with memory: $h(0) = \tan^{-1}(\cos(0)) = \tan^{-1}(1) = \frac{\pi}{4}$

(2) System is not causal: $t = -2\pi \Rightarrow h(-2\pi) = \tan^{-1}(\cos(-2\pi)) = \tan^{-1}(1) = \frac{\pi}{4}$

(3) System is not stable:

پاسخ ضربه $h(t)$ یک سیگنال متناوب است با $T = 2\pi$.

$$\int_0^T |h(\tau)| d\tau = \int_0^{2\pi} |\tan^{-1}(\cos t)| d\tau > 0 \Rightarrow \int_{-\infty}^{+\infty} |h(\tau)| d\tau = +\infty$$

(f) $h(t) = (t^2+1)\delta(t)$

(1) System is with memory: $h(0) = (0+1)\delta(0) = 1 \neq 0$

(2) System is causal: $\forall t < 0 .\ \delta(t) = 0 \Rightarrow h(t) = (t^2+1)\delta(t) = 0$

(3) System is stable:

$$\int_{-\infty}^{+\infty} |h(\tau)| d\tau = \int_{-\infty}^{+\infty} (\tau^2+1)\delta(\tau) d\tau = \tau^2+1 \Big|_{\tau=0} = 1 < \infty$$

(g) $h[n] = \cos(\frac{\pi}{2} n) u[n+2] + \delta[n+2]$

(1) System is with memory: $h[0] = \cos(0) u[2] + \delta[2] = 1 \neq 0$

(2) System is causal: if $n = -2 \Rightarrow h[-2] = \cos(\pi) u[0] + \delta[0] = -1 + 1 = 0$

else if $n = 1 \Rightarrow h[-1] = \cos(-\pi/2) u[1] + \delta[1] = 0$

else $(n < -2) \Rightarrow h[n] = \cos(\frac{\pi}{2} n) \overset{0}{u[n+2]} + \overset{0}{\delta[n+2]} = 0$

(3) System is not stable:

$$\sum_{k=-\infty}^{+\infty} |h[k]| = \sum_{k=-2}^{+\infty} |\cos(\pi/2\, k) u[k+2] + \delta[k+2]| \overset{h[-2]=0}{=} \sum_{k=-1}^{+\infty} |\cos(\frac{\pi}{2} k)| = +\infty$$

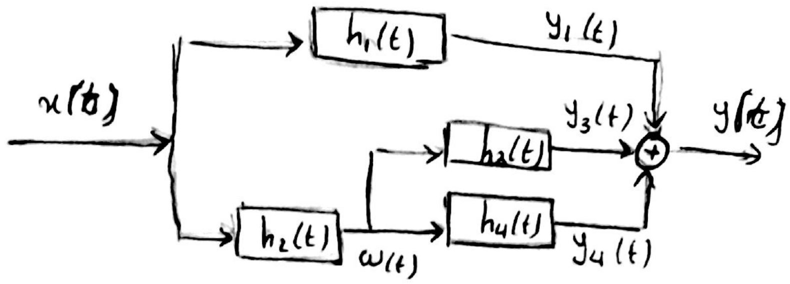
Question 4.

$h_1(t) = \sin\left(\frac{\pi}{3}t\right)$

$h_2(t) = e^{2t}u(1-t)$

$h_3(t) = u(t)$

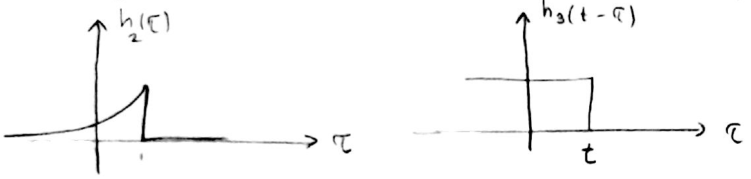
$h_4(t) = \delta(t+1)$



(a) $x(t) = \delta(t)$

$y_1(t) = \delta(t) * h_1(t) = h_1(t)$

$y_3(t) = \delta(t) * h_2(t) * h_3(t) = h_2(t) * h_3(t)$



$t < 1: \quad h_2 * h_3 = \int_{-\infty}^{t} e^{2\tau}\, d\tau = \frac{e^{2\tau}}{2}\Big|_{-\infty}^{t} = \frac{e^{2t}}{2}$

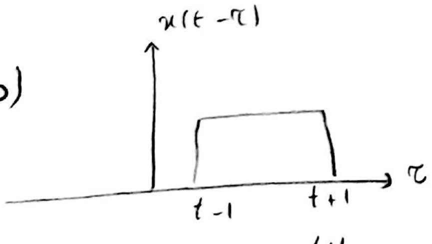
$t \geq 1: \quad h_2 * h_3 = \int_{-\infty}^{1} e^{2\tau}\, d\tau = \frac{e^{2\tau}}{2}\Big|_{-\infty}^{1} = \frac{e^{2}}{2}$

$\Rightarrow h_2 * h_3 = \frac{e^2}{2} + \left(\frac{e^{2t} - e^2}{2}\right)u(1-t)$

$y_4(t) = \delta(t) * h_2(t) * h_4(t) = h_2(t) * h_4(t) = h_2(t) * \delta(t+1) = h_2(t+1) = e^{2(t+1)}u(-t)$

$\Rightarrow h_{eq} = y_1 + y_3 + y_4 = \sin(\pi/3\, t) + \frac{e^2}{2} + \left(\frac{e^{2t} - e^2}{2}\right)u(1-t) + e^{2(t+1)}u(-t)$

(b)



$y_1 = h_1(t) * x_1(t) = \int_{t-1}^{t+1} \sin(\pi/3\, \tau)\, d\tau = -\frac{3}{\pi}\cos\left(\frac{\pi}{3}\tau\right)\Big|_{t-1}^{t+1} = -\frac{3}{\pi}\left[\cos\left(\pi/3(t+1)\right) - \cos\left(\pi/3(t-1)\right)\right]$

$y_2 = h_2 * x$

$t < 0:$

$h_2 * x = \int_{t-1}^{t+1} e^{2\tau}\, d\tau = \frac{e^{2\tau}}{2}\Big|_{t-1}^{t+1} = \frac{e^{2(t+1)} - e^{2(t-1)}}{2}$

$0 \leqslant t < 2:$

$$x * h_2 = \int_{t-1}^{1} e^{2\tau} d\tau = \frac{e^{2\tau}}{2}\Big|_{t-1}^{1} = \frac{e^2 - e^{t-1}}{2}$$

$2 \leqslant t:$

$x * h_2 = 0$

$$\omega_{(t)} = \left(\frac{e^{2(t+1)} - e^{-2(t-1)}}{2}\right)u(-t) + \left(\frac{e^2 - e^{t-1}}{2}\right)\left(u(t) - u(t-2)\right)$$

$$y_4(t) = \omega_{(t)} * h_4(t) = \omega_{(t)} * \delta(t+1) = \omega(t+1)$$

$$\Rightarrow y_4(t) = \left(\frac{e^{2(t+2)} - e^{-2t}}{2}\right)u(-t-1) + \frac{e^2 - e^t}{2}\left(u(t+1) - u(t-1)\right)$$

$y_3(t) = \omega(t) * u(t):$

$t < 0$

$$y_3(t) = \int_{-\infty}^{t} \frac{e^{2(\tau+1)} - e^{-2(\tau-1)}}{2} d\tau = \frac{e^2}{2}\int_{-\infty}^{t} e^{2\tau} - \frac{e^2}{2}\int_{-\infty}^{t} e^{-2\tau} d\tau$$

$$= \frac{e^2}{2} \times \frac{e^{2t}}{2} + \frac{e^2}{2}\left(\frac{e^{-2t}}{2}\right) = \frac{e^2}{4}\left(e^{2t} - e^{-2t}\right)$$

$0 \leqslant t < 2$

$$y_3(t) = \int_{-\infty}^{0} \frac{e^{2(\tau+1)} - e^{-2(\tau-1)}}{2} d\tau + \int_{0}^{t} \frac{e^2 - e^{\tau}}{2} d\tau$$

$$= 0 + \frac{e^2}{2}t - \frac{e^{\tau}}{2}\Big|_0^t = \frac{e^2}{2}t - \left(\frac{e^t}{2} - \frac{1}{2}\right) = \frac{e^2 t - e^t + 1}{2}$$

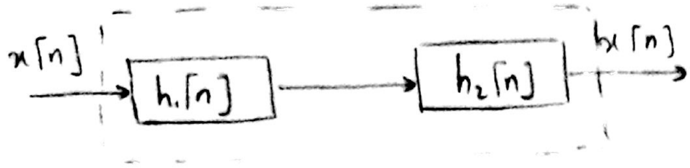
$t \geqslant 2:$

$$y_3(t) = \int_{-\infty}^{0} \frac{e^{2(\tau+1)} - e^{-2(\tau-1)}}{2} d\tau + \int_{0}^{2} \frac{e^2 - e^{\tau}}{2} d\tau = 0 + \frac{2e^2 - e^2 + 1}{2}$$

$$= \frac{e^2 + 1}{2}$$

Question 5.

(a) $h_1[n] = (\frac{1}{5})^n u[n]$, $h_2[n] = \delta[n] - \frac{1}{5}\delta[n-1]$



Inverse System: $h_1[n] * h_2[n] = \delta[n]$

$$h_1[n] * h_2[n] = (\tfrac{1}{5})^n u[n] * \left(\delta[n] - \tfrac{1}{5}\delta[n-1]\right) = (\tfrac{1}{5})^n u[n] * \delta[n] - (\tfrac{1}{5})^{n+1} u[n] * \delta[n-1]$$

$$= (\tfrac{1}{5})^n u[n] - (\tfrac{1}{5})^n u[n-1] = (\tfrac{1}{5})^n \left(u[n] - u[n-1]\right) = \delta[n]$$

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(b) $h_1(t) = e^{-t}u(t)$, $h_2(t) = \delta(t) + \delta'(t)$

$$h_1 * h_2 = e^{-t}u(t) * \delta(t) + e^{-t}u(t) * \delta'(t) = e^{-t}u(t) + \frac{d}{dt}\left(e^{-t}u(t)\right)$$

$$\frac{d}{dt}\left(e^{-t}u(t)\right) = e^{-t}\frac{d(u(t))}{dt} + u(t)\frac{d}{dt}\left(e^{-t}\right) = e^{-t}\delta(t) - e^{-t}u(t)$$

$$\Rightarrow h_1 * h_2 = \cancel{e^{-t}u(t)} + e^{-t}\delta(t) - \cancel{e^{-t}u(t)} = e^{-t}\delta(t) = \delta(t)$$

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$$e^{-t}\delta(t) = \begin{cases} e^{0}\delta(0) = \delta(0) & t = 0 \\ 0 & t \neq 0 \end{cases} = \delta(t)$$

Question 6.

(a) $h(t) = \delta(t+1) - \delta(t-1) \qquad x(t) = u(t)$

$y(t) = x * h = u(t) * (\delta(t+1) - \delta(t-1)) = u(t) * \delta(t+1) - u(t) * \delta(t-1)$

$= u(t+1) - u(t-1)$

(b) $h[n] = e^n \left(\frac{1}{3}\right)^n u[n+2]$

$n < -2$:

$y[n] = 0$

$-2 \leqslant n$:

$$y[n] = \sum_{k=-2}^{n} \left(\frac{e}{3}\right)^k = \left(\frac{e}{3}\right)^{-2} \times \frac{1 - \left(\frac{e}{3}\right)^{n+3}}{1 - \frac{e}{3}}$$

$$\Rightarrow y[n] = \left(\frac{e}{3}\right)^{-2} \times \frac{1 - \left(\frac{e}{3}\right)^{n+3}}{1 - e/3} u[n-2]$$

(c) $h(t) = 2\delta^2(t)$

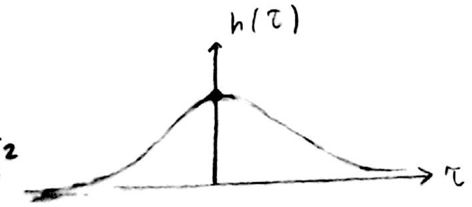
$t < 0$:

$y(t) = 0$

$t \geqslant 0$

$$y(t) = \int_{-\infty}^{t} 2\delta^2(\tau) u(t-\tau) d\tau = 2u(t)$$

(d) $h(t) = \frac{1}{1+t^2}$



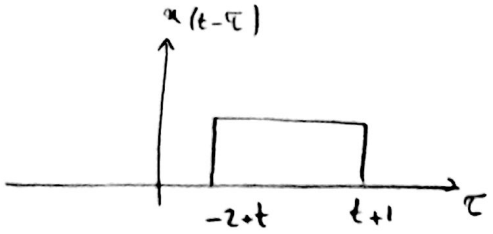
$$y(t) = x * h = \int_{-\infty}^{t} \frac{1}{1+\tau^2} d\tau = \tan^{-1}(\tau)\Big|_{-\infty}^{t} = \tan^{-1}(t) - \frac{\pi}{2}$$

Question 7.

(a) $$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)}\, x(\tau-2)\, d\tau \underset{\substack{\tau'=\tau-2 \\ d\tau'=d\tau}}{=} \int_{-\infty}^{t} e^{-(t-\tau'-2)}\, x(\tau')\, d\tau'$$

$$= \int_{-\infty}^{+\infty} e^{-(t-\tau'-2)}\, u(t-\tau')\, x(\tau')\, d\tau' \Rightarrow h(t) = e^{-(t-2)}\, u(t)$$

(b)



$t < -1$:

$$x * h = 0$$

$-1 \leq t < 2$

$$x * h = e^{2}\int_{0}^{t+1} e^{-\tau}\, d\tau = e^{2} \times \left. -e^{-\tau}\right|_{0}^{t+1} = e^{2}\left(1 - e^{-(t+1)}\right) = e^{2} - e^{1-t}$$

$t \geq 2$:

$$x * h = e^{2}\int_{t-2}^{t+1} e^{-\tau}\, d\tau = e^{2} \times \left. -e^{-\tau}\right|_{t-2}^{t+1} = e^{2} \times \left(e^{2-t} - e^{-1-t}\right)$$

$$y(t) = \begin{cases} 0 & t < -1 \\ e^{2} - e^{1-t} & 1 \leq t < 2 \\ e^{2}\left(e^{2-t} - e^{-1-t}\right) & t \geq 2 \end{cases}$$

Signals and Systems

Assignment 3

Spring 2021

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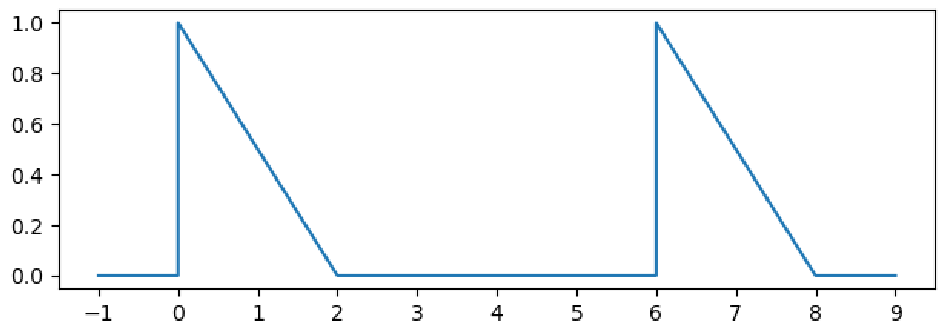
Question 1

Determine the Fourier Series coefficients a_k for the following periodic signals:

(a) $x(t) = 3cos(\frac{2\pi t}{3} + \frac{\pi}{3}) + 5sin(\frac{2\pi t}{18})$

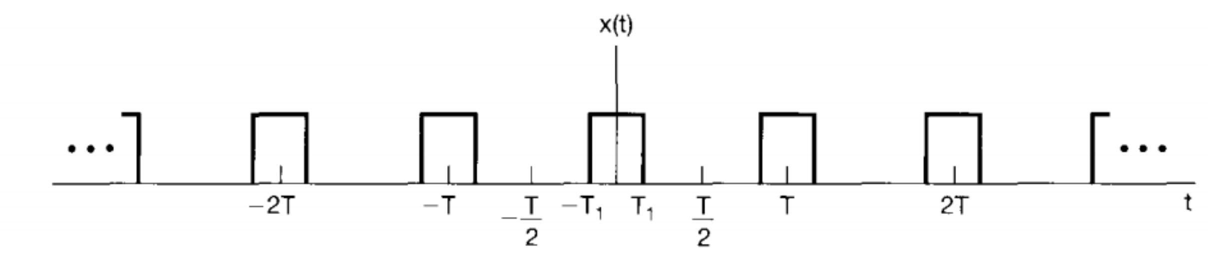
(b) $x(t) = 2sin(\frac{2\pi t}{3} + \frac{\pi}{6})$

(c) (Using the definition integral)

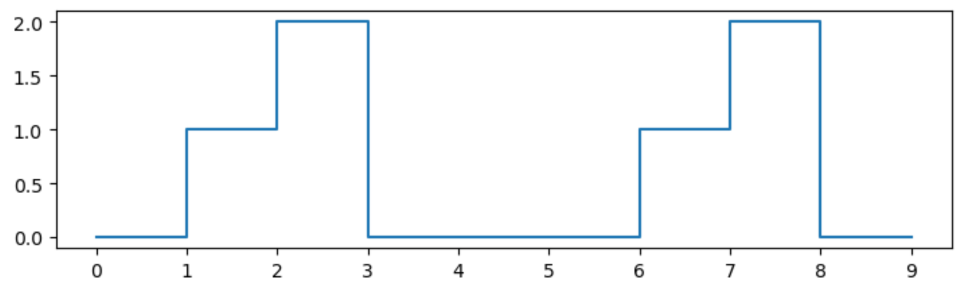


Question 2

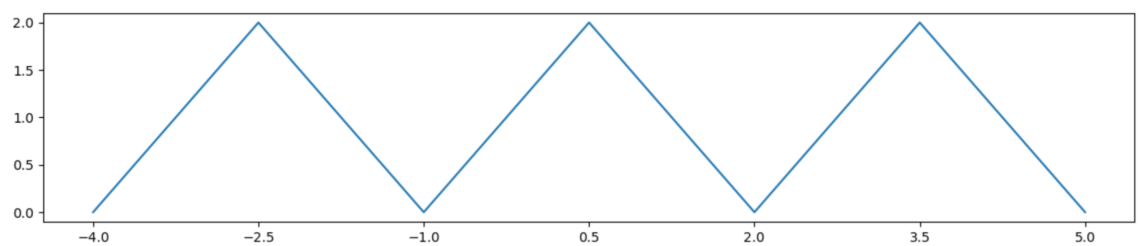
Determine the Fourier Series coefficients a_k for $x(t)$:



(a) .



(b) .



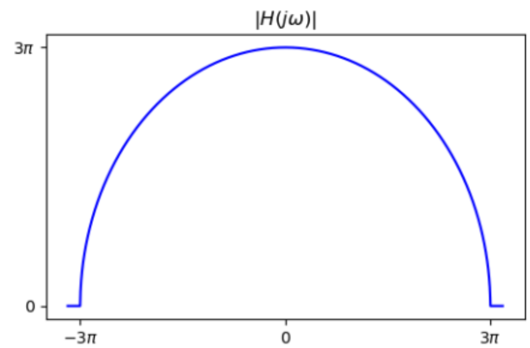
Question 3

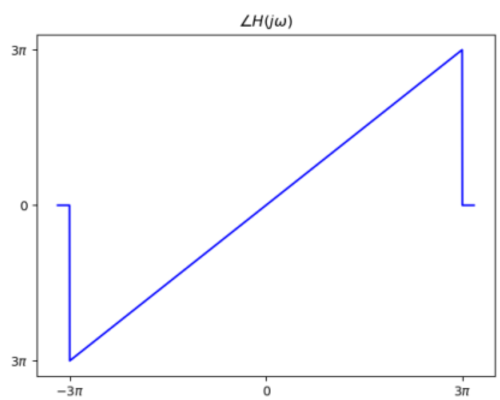
(Textbook Section 3.8 - Fourier Series and LTI Systems)

Consider a signal $x(t)$ with Fourier Series representation like this:

$$a_{-2} = a_2 = \frac{1}{16}$$
$$a_{-1} = a_1 = \frac{1}{8}$$
$$a_0 = 1$$

And otherwise $a_k = 0$. Keep in mind that $T = 4$. Consider an LTI System with frequency response $H(j\omega)$ as plotted below.





(a) Determine the output $y(t)$, and its Fourier Series coefficients b_k, if we apply $x(t)$ as input.

(b) Using Parseval's relation, determine the average power of $y(t)$.

Subject:
Date

Q1: a) $x(t) = 3\cos\left(\frac{2\pi t}{3} + \frac{\pi}{3}\right) + 5\sin\left(\frac{2\pi t}{18}\right)$

$\omega_1 = \frac{2\pi}{3} \rightarrow T_1 = 3 \qquad \omega_2 = \frac{2\pi}{18} \rightarrow T_2 = 18 \qquad T = \text{lcm}(3,18) = 18$

$\omega = \frac{2\pi}{18} = \frac{\pi}{9}$

$x(t) = \frac{3}{2}\left(e^{j\left(\frac{2\pi}{3}t + \frac{\pi}{3}\right)} + e^{-j\left(\frac{2\pi}{3}t + \frac{\pi}{3}\right)}\right) + \frac{5}{2j}\left(e^{j\frac{2\pi}{18}t} - e^{-j\frac{2\pi}{18}t}\right) = \sum a_k e^{jk\frac{\pi}{9}t}$

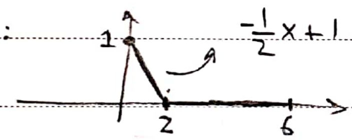
$a_6 = \frac{3}{2}e^{j\frac{\pi}{3}} \qquad a_{-6} = \frac{3}{2}e^{-j\frac{\pi}{3}} \qquad a_1 = \frac{5}{2j} \qquad a_{-1} = \frac{-5}{2j}$

b) $x(t) = 2\sin\left(\frac{2\pi}{3}t + \frac{\pi}{6}\right)$

$x(t) = \frac{2}{2j}\left(e^{j\left(\frac{2\pi}{3}t + \frac{\pi}{6}\right)} - e^{-j\left(\frac{2\pi}{3}t + \frac{\pi}{6}\right)}\right) = \sum_{-1}^{1} a_k e^{jk\frac{2\pi}{3}t}$

$a_1 = \frac{1}{j}e^{j\frac{\pi}{6}} \qquad a_{-1} = \frac{-1}{j}e^{-\frac{\pi}{6}}$

c) one period:



$T = 6 \Rightarrow \omega_0 = \frac{2\pi}{6} = \frac{\pi}{3}$

$a_0 = \frac{1}{6}\int_0^6 x(t)\,dt = \frac{1}{6}$

$a_k = \frac{1}{6}\int_0^6 x(t)\,e^{-jk\omega_0 t} = \frac{1}{6}\int_0^1 \left(\frac{-1}{2}t + 1\right)e^{-jk\frac{\pi}{3}t}\,dt$

$= \frac{-1}{12}\int_0^1 t\,e^{-jk\frac{\pi}{3}t}\,dt + \frac{1}{6}\int_0^1 e^{-jk\frac{\pi}{3}t}\,dt = \frac{-1}{12}I_1 + \frac{1}{6}I_2$

I_1: $\int_0^1 u\,dv = uv\Big|_0^1 - \int_0^1 v\,du$

$u = t \Rightarrow du = dt \qquad dv = e^{-jk\frac{\pi}{3}t}dt \Rightarrow v = \frac{-3}{jk\pi}e^{-jk\frac{\pi}{3}t}$

Subject:
Date

$$uv\Big|_0^1 = \frac{-3t}{jk\pi} e^{-jk\frac{\pi}{3}t}\Big|_0^1 = \frac{-3}{jk\pi} e^{-jk\frac{\pi}{3}}$$

$$\int_0^1 v\,du = \frac{-3}{jk\pi}\left(\frac{-3}{jk\pi} e^{-jk\frac{\pi}{3}t}\Big|_0^1\right) = \frac{-9}{k^2\pi^2}\left(e^{-jk\frac{\pi}{3}} - 1\right)$$

$$I_1 = \frac{-3}{jk\pi} e^{-jk\frac{\pi}{3}} + \frac{9}{k^2\pi^2}\left(e^{-jk\frac{\pi}{3}} - 1\right)$$

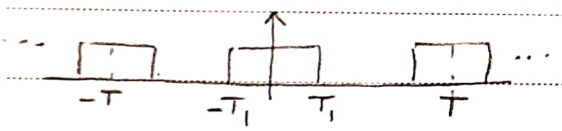
I_2: $$I_2 = \int_0^1 e^{-jk\frac{\pi}{3}t}\,dt = \frac{-3}{jk\pi}\left(e^{-jk\frac{\pi}{3}} - 1\right)$$

$$a_k = \frac{-1}{12} I_1 + \frac{1}{6} I_2 = \frac{1}{6}\left(\frac{-1}{2} I_1 + I_2\right) \qquad e^{-jk\frac{\pi}{3}} = m$$ تغییر متغیر برای سادگی نوشتار

$$a_k = \frac{1}{6}\left(\frac{-1}{2}\left(\frac{-3}{jk\pi} m + \frac{9}{k^2\pi^2}(m-1)\right) + \frac{-3}{jk\pi}(m-1)\right)$$

Q2:

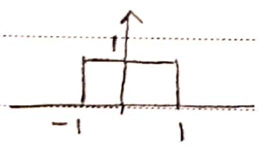
a) Oppenheim pages 193 – 194



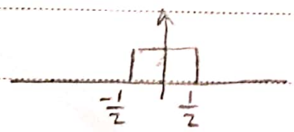
$$\xleftrightarrow{FS} \quad \frac{\sin(k\omega_0 T_1)}{k\pi} \qquad k \neq 0$$

$$\frac{2T_1}{T} \qquad k = 0$$

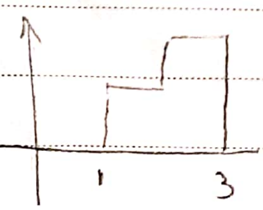
b)



$T_1 = 1$
$T = 5$ $\xleftrightarrow{FS}$ $a_k = \dfrac{\sin(k\frac{2\pi}{5})}{k\pi} \xrightarrow[(2)]{\text{shift right}} C_k = e^{-jk\frac{4\pi}{5}} a_k$

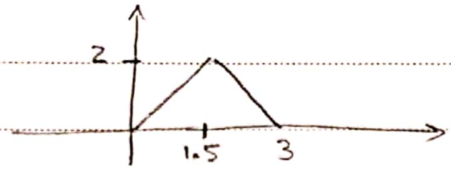
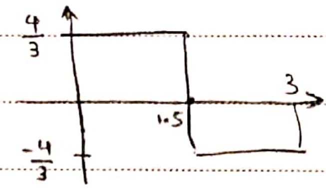


$T_1 = \frac{1}{2}$
$T = 5$ $\xleftrightarrow{FS}$ $b_k = \dfrac{\sin(k\frac{\pi}{5})}{k\pi} \xrightarrow[(2.5)]{\text{shift right}} d_k = e^{-jk\pi} b_k$



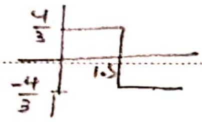
$$\longrightarrow E_k = C_k + d_k = e^{-jk\frac{4\pi}{5}} a_k + e^{-jk\frac{\pi}{5}} b_k$$

Subject:
Date

C.)  derivative $\longrightarrow$ 

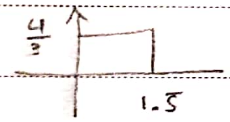
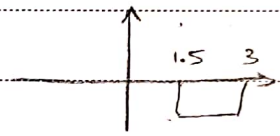
FS properties: $x(t) \longleftrightarrow a_k$

$$\int_{\infty}^{t} x(t)\,dt \longleftrightarrow \frac{1}{jk\omega_0} a_k \quad (\text{if } a_0 = 0)$$

in this case, $x(t) =$  $\xleftrightarrow{FS} a_k$ and we want to compute FS

coeffs for $\int_{-\infty}^{t} x(t)\,dt$. Since $a_0 = \frac{1}{3}(2-2) = 0$, it is possible to find a_k

and then compute $\frac{1}{jk\omega_0} a_k$ $\left(\omega_0 = \frac{2\pi}{3}\right)$

$x(t) =$  $+$ 

$T_1 = \frac{3}{4}$ $\qquad$ $T_1 = \frac{3}{4}$

$T = 3$ $\qquad$ $T = 3$

$$\Rightarrow a_k = \frac{4}{3} e^{-jk\frac{2\pi}{3}\left(\frac{3}{4}\right)} \frac{\sin\left(k\frac{2\pi}{3}\frac{3}{4}\right)}{k\pi} - \frac{4}{3} e^{-jk\frac{2\pi}{3}\left(\frac{9}{4}\right)} \frac{\sin\left(k\frac{2\pi}{3}\frac{3}{4}\right)}{k\pi}$$

$$ANS = \frac{1}{jk\frac{2\pi}{3}} a_k$$

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Q3: $T = 4 \Rightarrow \omega_0 = \frac{\pi}{2}$

$$x(t) = \sum_{k=-2}^{2} a_k e^{jk\frac{\pi}{2}t} = a_{-2} e^{j(-2)\frac{\pi}{2}t} + a_{-1} e^{-j(-1)\frac{\pi}{2}t} + a_1 e^{j\frac{\pi}{2}t} + a_2 e^{j2(\frac{\pi}{2})t}$$

a) $$y(t) = \sum_{k=-2}^{2} \underbrace{a_k H(jk\omega_0)}_{=b_k} e^{jk\omega_0 t}$$

$$\rightarrow b_{-2} = a_{-2} H(j(-2(\tfrac{\pi}{2}))) = a_{-2} \left| H(j(-\pi)) \right| e^{j\angle H(j(-2\pi))}$$

$b_{-1} = \dots$

$b_1 = \dots$

$b_2 = \dots$

b) Avg power of $y(t) = \sum_{k=-2}^{2} |b_k|^2$

دانشگاه صنعتی امیرکبیر
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Signals and Systems

Assignment 4

Spring 2021

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Question 1

Given $x(t)$ with Fourier Transform $X(j\omega)$, determine the Fourier Transform for the following signals in terms of $X(j\omega)$

(a) $x_1(t) = x(3+t) - x(-t+2)$

(b) $x_2(t) = x(2t+3)$

(c) $x_3(t) = \frac{d^2}{dt^2}x(t-2)$

(d) $x_4(t) = tx(t-1)$

Question 2

Determine the Fourier Transform for the following signals:

(a) $x(t) = 3 + sin(5\pi t + \frac{\pi}{6})$

(b) $x(t) = te^{-4t}cos(2t)u(t)$

(c) $x(t) = t\frac{sin(3t)}{\pi t}$

(d) $x(t) = \frac{4t}{(1+t^2)^2}$

(e) $x(t) = e^{-2|t|}cos(2t)$

Question 3

Determine the inverse Fourier Transform for the following signals:

(a) $X(j\omega) = 6\delta(\omega - 4)$

(b) $X(j\omega) = \frac{-j\omega+5}{-\omega^2+10j\omega+24}$

(c) $X(j\omega) = \pi e^{-2|\omega|}$

Question 4

The input and the output of a stable causal LTI system are related by the following differential equation:

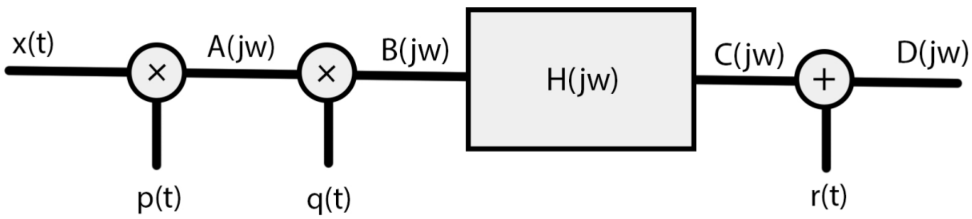
$$\frac{d^2}{dt^2}y(t) + 8\frac{d}{dt}y(t) + 15y(t) = \frac{d}{dt}x(t) + 4x(t)$$

(a) Find the impulse response of this system.

(b) Determine $y(t)$ if $x(t) = e^{-4t}u(t)$

Question 5

Consider the following system (Do NOT mistake the plus sign with multiplication!). Determine $A(j\omega), B(j\omega), C(j\omega), D(j\omega)$



$$x(t) = p(t) = \frac{sin(1.5\pi t)}{\pi t}$$

$$q(t) = cos(3\pi t)$$

$$H(j\omega) = 2\Big(u(\omega + 3\pi) - u(\omega - 3\pi)\Big)$$

$$r(t) = \frac{sin(\pi t)}{\pi t}$$

Question 6

Given the following frequency response for the LTI and stable system S, determine the differential equation that relates the input $x(t)$ and the output $y(t)$ of S.

$$H(j\omega) = \frac{j\omega + 10}{98 - \omega^2 + 21j\omega}$$

(1) (a) $x_1(t) = x(t+3) - x(-t+2)$

$\boxed{\text{Part 1}}$ $x(t) \longleftrightarrow X(j\omega)$

$\xrightarrow{t \leftarrow t+3} x(t+3) \leftrightarrow e^{j\omega 3} X(j\omega)$

$\boxed{\text{part 2}}$ $x(t) \longleftrightarrow X(j\omega)$

$\xrightarrow{t \leftarrow t+2} x(t+2) \longleftrightarrow e^{j\omega 2} X(j\omega)$

$\xrightarrow{t \leftarrow -t} x(-t+2) \leftrightarrow e^{-j\omega 2} X(-j\omega)$

$\Rightarrow X_1(j\omega) = e^{j\omega 3} X(j\omega) - e^{-j\omega 2} X(-j\omega)$

(b) $x_2(t) = x(2t+3)$

$x(t) \leftrightarrow X(j\omega)$

$\xrightarrow{t \leftarrow t+3} x(t+3) \leftrightarrow e^{j\omega 3} X(j\omega)$

$\xrightarrow{t \leftarrow 2t} x(2t+3) \longleftrightarrow \frac{1}{2} e^{j\omega \frac{3}{2}} X(j\frac{\omega}{2}) = X_2(j\omega)$

(c) $x_3(t) = \frac{d^2}{dt^2} x(t-2)$

$x(t) \leftrightarrow X(j\omega)$

$x(t-2) \leftrightarrow e^{-j\omega 2} X(j\omega)$

$\frac{d}{dt} x(t-2) \leftrightarrow (j\omega) e^{-j\omega 2} X(j\omega)$

$\frac{d^2}{dt^2} x(t-2) \leftrightarrow (j\omega)^2 e^{-j\omega 2} X(j\omega) = X_3(j\omega)$

(d) $x_4(t) = t x(t-1)$

$p(t) = x(t-1) \Rightarrow P(j\omega) = e^{-j\omega} X(j\omega)$

$t p(t) \leftrightarrow \frac{d}{d\omega} P(j\omega) = \frac{d}{d\omega}\left(e^{-j\omega} X(j\omega)\right) = X_4(j\omega)$

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(2) (a) $x(t) = 3 + \sin(5\pi t + \frac{\pi}{6})$

periodic $\rightarrow \omega_0 = 5\pi \Rightarrow T = \frac{2\pi}{5\pi} = 0.4$

$$x(t) = 3 + \frac{1}{2j}\left(e^{j(\omega_0 t + \frac{\pi}{6})} - e^{-j(\omega_0 t + \frac{\pi}{6})}\right)$$

$$\Rightarrow a_0 = 3,\ a_1 = \frac{1}{2j}e^{j\frac{\pi}{6}},\ a_{-1} = \frac{-1}{2j}e^{-j\frac{\pi}{6}}$$

$$x(t) = \sum a_k e^{jk\omega_0 t} \Rightarrow X(j\omega) = \sum 2\pi a_k \delta(\omega - k\omega_0)$$

$$= 6\pi\delta(\omega) + \frac{2\pi}{2j}e^{j\frac{\pi}{6}}\delta(\omega-\omega_0) - \frac{2\pi}{2j}e^{-j\frac{\pi}{6}}\delta(\omega+\omega_0)$$

(b) $x(t) = te^{-4t}\cos(2t)u(t)$

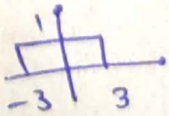
$$= \underbrace{(te^{-4t}u(t))}_{p(t)}\underbrace{(\cos(2t))}_{s(t)}$$

$$p(t) \leftrightarrow P(j\omega) = \frac{1}{(4+j\omega)^2}$$

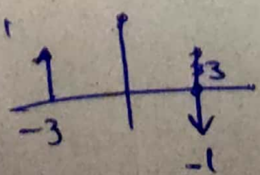
$$s(t) \leftrightarrow S(j\omega) = \pi\left(\delta(\omega-2) + \delta(\omega+2)\right)$$

$$x(t) = p(t)s(t) \Rightarrow X(j\omega) = \frac{1}{2\pi}\left(P(j\omega) * S(j\omega)\right)$$

$$= \frac{1}{2\pi}\pi\left(P(j(\omega-2)) + P(j(\omega+2))\right)$$

(c) $x(t) = t\frac{\sin(3t)}{\pi t}$ 　 می‌دانیم : $r(t) = \frac{\sin(3t)}{\pi t} \leftrightarrow$  $= R(j\omega)$

$$x(t) = tr(t) \Rightarrow X(j\omega) = j\frac{d}{d\omega}R(j\omega)$$

$= j \times$ 

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(d) $x(t) = \dfrac{4t}{(1+t^2)^2}$

Duality: $f(t) \leftrightarrow F(j\omega)$

$$F(t) \leftrightarrow 2\pi f(-\omega)$$

می‌دانیم : $e^{-|t|} \leftrightarrow \dfrac{2}{1+\omega^2}$

$\xrightarrow{\text{duality}}$ $\dfrac{2}{1+t^2} \leftrightarrow 2\pi e^{-|\omega|}$

$\xrightarrow{\text{مشتق}}$ $\dfrac{4t}{(1+t^2)^2} \leftrightarrow j\omega\, 2\pi e^{-|\omega|}$

(e) $x(t) = \underbrace{e^{-2|t|}}_{s(t)} \underbrace{\cos(2t)}_{p(t)}$

$$s(t) = e^{-2|t|} \leftrightarrow S(j\omega) = \frac{4}{4+\omega^2}$$

$$p(t) = \cos(2t) \leftrightarrow P(j\omega) = \pi\left(\delta(\omega-2) + \delta(\omega+2)\right)$$

ادامه حل مشابه قسمت b است

3 (a) $X(j\omega) = \delta(\omega-4) \times 6$

$$1 \leftrightarrow 2\pi\delta(\omega)$$

$$e^{j4t} \leftrightarrow 2\pi\delta(\omega+4)$$

$$\frac{1}{\pi}e^{j4t} \leftrightarrow 2\delta(\omega+4)$$

$$\frac{3}{\pi}e^{j4t} \leftrightarrow 6\delta(\omega+4)$$

(b) $X(j\omega) = \dfrac{-j\omega + 5}{-\omega^2 + 10j\omega + 24}$

$$= \frac{-j\omega + 5}{(j\omega)^2 + 10j\omega + 24} = \frac{-j\omega + 5}{(j\omega + 6)(j\omega + 4)} = \frac{A}{j\omega + 6} + \frac{B}{j\omega + 4}$$

با گرفتن مخرج مشترک و حل دو معادله، دو مجهول مقادیر B و A محاسبه می شوند:

$$\Rightarrow x(t) = Ae^{-6t}u(t) + Be^{-4t}u(t)$$

(c) $X(j\omega) = \pi e^{-2|\omega|}$

می دانیم: $e^{-a|t|} \longleftrightarrow \dfrac{2a}{a^2 + \omega^2}$

$$\xrightarrow{\text{duality}} \frac{2a}{a^2 + t^2} \longleftrightarrow 2\pi e^{-a|-\omega|} = 2\pi e^{-a|\omega|}$$

$$\xrightarrow{a=2} \frac{4}{4 + t^2} \longleftrightarrow 2\pi e^{-2|\omega|}$$

$$\longrightarrow \frac{2}{2 + t^2} \longleftrightarrow \pi e^{-2|\omega|}$$

4 (a) $y''(t) + 8y'(t) + 15y(t) = x'(t) + 4x(t)$

$$\xrightarrow{FT} (j\omega)^2 Y(j\omega) + 8j\omega Y(j\omega) + 15Y(j\omega) = j\omega X(j\omega) + 4X(j\omega)$$

$$\longrightarrow H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega + 4}{(j\omega)^2 + 8j\omega + 15} = \frac{j\omega + 4}{(j\omega + 5)(j\omega + 3)}$$

$$\Rightarrow H(j\omega) = \frac{A}{j\omega + 5} + \frac{B}{j\omega + 3} \Rightarrow \begin{cases} A + B = 1 \\ 3A + 5B = 4 \end{cases} \Rightarrow \begin{cases} A = 1/2 \\ B = 1/2 \end{cases}$$

$$\Rightarrow h(t) = \frac{1}{2}e^{-5t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

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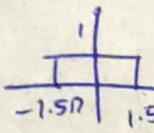
(b) $x(t) = e^{-4t} u(t)$

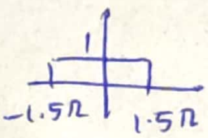
$\Rightarrow X(j\omega) = \dfrac{1}{4+j\omega}$

$Y(j\omega) = X(j\omega) H(j\omega) = \dfrac{1}{(j\omega+5)(j\omega+3)} = \dfrac{A}{j\omega+5} + \dfrac{B}{j\omega+3}$

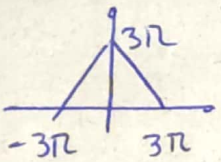
$\Rightarrow y(t) = Ae^{-5t} u(t) + Be^{-3t} u(t)$

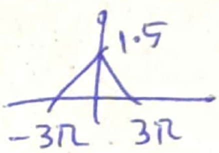
5 $a(t) = x(t)\,p(t) = \dfrac{\sin(1.5\pi t)}{\pi t} \cdot \dfrac{\sin(1.5\pi t)}{\pi t}$

$A(j\omega) = \dfrac{1}{2\pi}\Big($ 

$*$ 

$\Big)$

$= \dfrac{1}{2\pi}\Big($ 

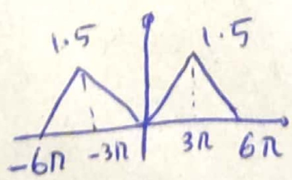
$\Big) =$ 

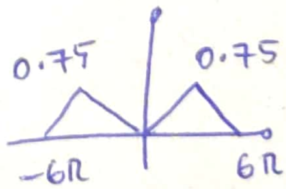
$= A(j\omega)$

$b(t) = a(t)\,q(t) \Rightarrow B(j\omega) = \dfrac{1}{2\pi}\left(A(j\omega) * Q(j\omega)\right)$

$= \dfrac{1}{2\pi}\pi\Big(A(j\omega) * \big(\delta(\omega-3\pi) + \delta(\omega+3\pi)\big)\Big)$

$= \dfrac{1}{2}\big(A(j(\omega-3\pi)) + A(j(\omega+3\pi))\big)$

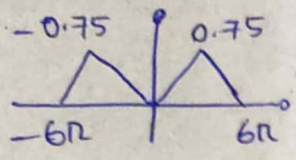
$= \dfrac{1}{2}\Big($ 

$\Big) =$ 

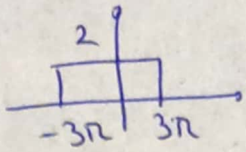
$= B(j\omega)$

$C(j\omega) = B(j\omega)\,H(j\omega)$

$=$



$\times$



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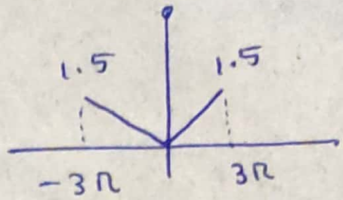
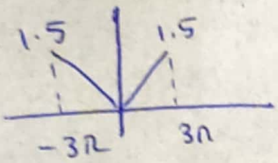


$D(j\omega) = C(j\omega) + R(j\omega)$

$=$



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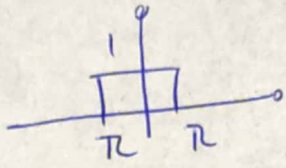

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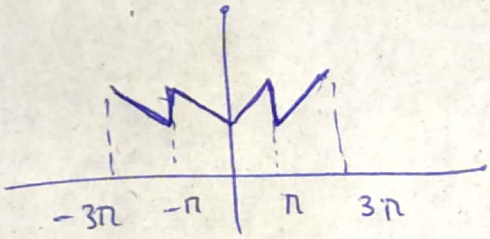

(6) $H(j\omega) = \dfrac{j\omega + 10}{98 - \omega^2 + 21j\omega} = \dfrac{j\omega + 10}{(j\omega)^2 + 21j\omega + 98} = \dfrac{Y(j\omega)}{X(j\omega)}$

$\Rightarrow (j\omega)^2 Y(j\omega) + 21j\omega\, Y(j\omega) + 98\, Y(j\omega) = j\omega\, X(j\omega) + 10\, X(j\omega)$

$\Rightarrow y''(t) + 21 y'(t) + 98 y(t) = x'(t) + 10 x(t)$

دانشگاه صنعتی امیرکبیر
(پلی تکنیک تهران)

Signals and Systems

Assignment 5

Spring 2021

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Question1

Let $x(t)$ be a signal with Nyquist rate ω_n. Determine the Nyquist rate for the following signals:

(a) $\int_{-\infty}^{t} x(t)dt$

(b) $x(-3t)$

(c) $x^3(t)$

(d) $x(t)sin(\frac{\pi}{2}t)$

Question2

Determine whether or not each of the following signals is band-limited and if it is, determine its Nyquist rate.

(a) $x(t) = \frac{1}{5}te^{-2t}u(t)$

(b) $x(t) = \dfrac{sin(\omega_1)}{\pi t}cos(\omega_2 t)$

(c) $x(t) = \begin{cases} 1, & |t| \leq T \\ 0, & |t| > T \end{cases}$

(d) $x(t) = cos^2(\frac{2\pi}{3}t) + cos(\pi t)sin(\frac{\pi}{4}t)$

(e) $x(t) = \dfrac{sin^2(\pi t)}{\pi t^2}$

Question3

Let $x[n]$ be a periodic signal with fundamental period N and Fourier series coefficients a_k. Determine the Fourier series coefficients for the following signals:

(a) $x[2-n]$

(b) $x^*[-n]$

(c) $x^2[n]$

(d) $\sum_{r=<N>} x[r]x[n+l-r]$

(e) $e^{-j\frac{8\pi}{N}n}x[n]$

(f) $x[n+1]-x[n]+x[n-2]$

Question4

Determine the Fourier series coefficients for the following signals:

(a) $sin\left(\frac{2\pi}{N}n\right) + cos\left(\frac{2\pi}{N}n + \frac{\pi}{4}\right)$

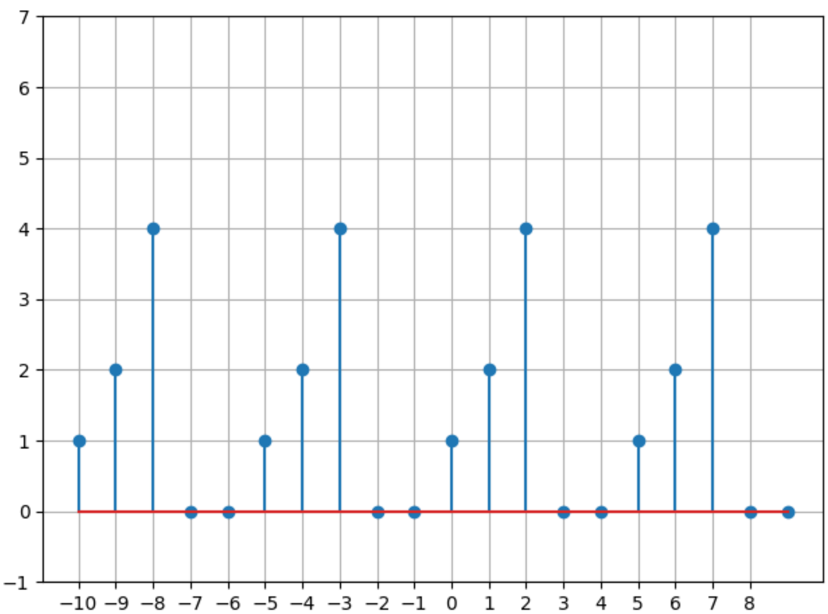
(b) $2 + 3cos\left(\frac{2\pi}{3}n\right) + sin\left(\frac{\pi}{3}n\right)$

(c) $(-1)^n + cos^2\left(\frac{\pi}{5}n + \frac{\pi}{4}\right)$

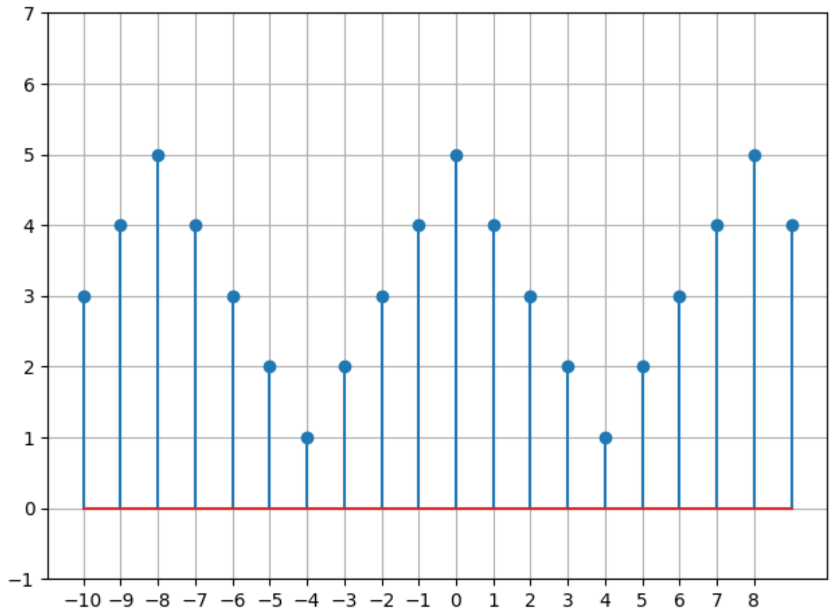
(d) $\sum_{k=-\infty}^{\infty} \delta[n-3k]$

(e) $\hat{x}[n] = \begin{cases} 1, & |n| \leq N_1 \\ 0, & N_1 < |n| \leq \frac{N}{2} \end{cases}$

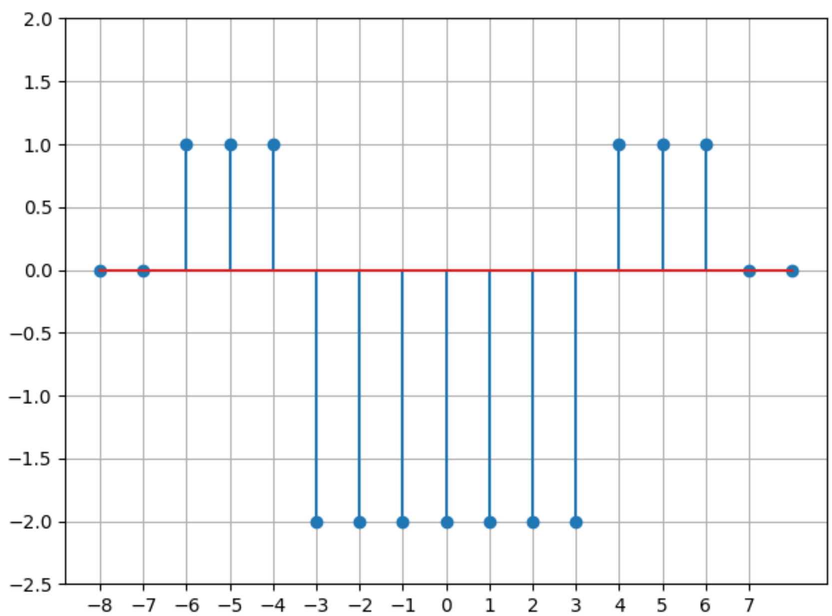
(f)



(g)



(h) Use the result from part e. (N = 10)



Question5

Let $x[n]$ be a real and odd periodic signal with period $N = 5$ and Fourier coefficients a_k. Given that

$$a_{11} = j, a_{13} = 3j, a_{17} = \frac{j}{2},$$

determine the values of a_0, a_{-1}, a_{-2}, and a_{-3}.

پاسخ تمرین سری پنجم

1) a) $\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{F.T.} \frac{1}{j\omega}X(j\omega)+\pi X(0)\,\delta(\omega) \Rightarrow \boxed{\text{nyquist rate} = \omega_n}$

1) b) $x(-3t) \xleftrightarrow{F.T.} \frac{1}{3}X\left(-\frac{j\omega}{3}\right) \Rightarrow \boxed{\text{nyquist rate} = 3\omega_n}$

1) c) $x^3(t) \xleftrightarrow{F.T.} \frac{1}{4\pi^2}X(j\omega)*X(j\omega)*X(j\omega) \Rightarrow \boxed{\text{nyquist rate} = 4\omega_n}$

چون هر کانولوشن، بازه‌ی فرکانسی دو برابر می‌شود.

1) d) $x(t)\sin\left(\frac{\pi}{2}t\right) \xleftrightarrow{F.T.} X(j\omega)*\frac{\pi}{j}\left[\delta\left(\omega-\frac{\pi}{2}\right)-\delta\left(\omega+\frac{\pi}{2}\right)\right] = X\left(j\left(\omega-\frac{\pi}{2}\right)\right)\times\frac{\pi}{j}-\frac{\pi}{j}X\left(j\left(\omega+\frac{\pi}{2}\right)\right)$

$\Rightarrow \text{nyquist rate} = 2\left(\frac{1}{2}\omega_n+\frac{\pi}{2}\right)=\omega_n+\pi$ |

چون قبلاً نرخ نایکوییست ω_n بوده پس حداکثر فرکانس حاضر در سیگنال $\frac{1}{2}\omega_n$ بوده

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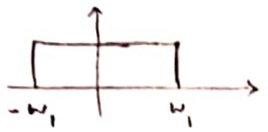
2) a) $\frac{1}{5}te^{-2t}u(t) \xleftrightarrow{F.T.} \frac{1}{5}\,\frac{1}{(2+j\omega)^2}$

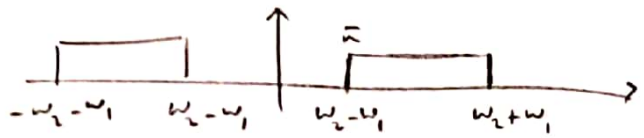
بازه‌ی فرکانسی نامحدود است

2) b) $\frac{\sin(\omega_1 t)}{\pi t} \xleftrightarrow{F.T.}$



$\Rightarrow \text{nyquist rate} = (\omega_1+\omega_2)\times 2$ |

$\frac{\sin(\omega_1 t)}{\pi t}\times\cos(\omega_2 t) \xleftrightarrow{F.T.}$



2) c) $x(t) \xleftrightarrow{F.T.} \frac{2\sin\omega T}{\omega}$

بازه‌ی فرکانسی نامحدود

2) d) $\text{nyquist rate} = \max\left\{\frac{4\pi}{3},\frac{5\pi}{4}\right\} = \frac{4\pi}{3}\times 2 = \frac{8\pi}{3}$ |

2) e) $\text{nyquist rate} = 2\times(\pi\times 2) = 4\pi$ |

---

3) a) $n \to n+2 : a_k e^{jk\frac{2\pi}{N}\times 2}$

$n \to -n : a_{-k} e^{-jk\frac{2\pi}{N}\times 2}$

3) b) a_k^*

3) c) $\sum_{\ell=\langle N\rangle} a_\ell a_{k-\ell}$

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3) d) $N a_k^2 e^{jk\frac{2\pi}{N} \times l}$

3) e) a_{k+4}

3) f) $\left(e^{jk\frac{2\pi}{N}} - 1 + e^{-jk\frac{4\pi}{N}}\right) a_k$

---

4) a) $\sin\left(\frac{2\pi}{N} n\right) + \cos\left(\frac{2\pi}{N}n + \frac{\pi}{4}\right) = \frac{1}{2j}\left(e^{j\frac{2\pi}{N}n} - e^{-j\frac{2\pi}{N}n}\right) + \frac{1}{2}\left(e^{j\left(\frac{2\pi}{N}n + \frac{\pi}{4}\right)} + e^{-j\left(\frac{2\pi}{N}n + \frac{\pi}{4}\right)}\right)$

$= \frac{1}{2j} e^{j\frac{2\pi}{N}n} - \frac{1}{2j} e^{-j\frac{2\pi}{N}n} + \frac{1}{2} e^{j\frac{\pi}{4}} e^{-j\frac{2\pi}{N}} + \frac{1}{2} e^{-j\frac{\pi}{4}} e^{-j\frac{2\pi}{N}}$

$= \underbrace{\left(\frac{1}{2j} + \frac{1}{2} e^{j\frac{\pi}{4}}\right)}_{a_1} e^{j\frac{2\pi}{N}n} + \underbrace{\left(-\frac{1}{2j} + \frac{1}{2} e^{-j\frac{\pi}{4}}\right)}_{a_{-1}} e^{-j\frac{2\pi}{N}n}$

---

4) b) $N = \frac{2\pi}{\frac{\pi}{3}} = 6 \Rightarrow 2 + 3\cos\left(\frac{2\pi}{3} n\right) + \sin\left(\frac{\pi}{3} n\right) = 2 + \frac{3}{2}\left(e^{2j\frac{2\pi}{6}n} + e^{-2j\frac{2\pi}{6}n}\right)$

$+ \frac{1}{2j}\left(e^{j\frac{2\pi}{6}n} - e^{-j\frac{2\pi}{6}n}\right) \Rightarrow a_0 = 2,\ a_2 = a_{-2} = \frac{3}{2},\ a_1 = a_{-1} = \frac{1}{2j}$

4) c) $N = 2 \times 5 = 10 \Rightarrow (-1)^n + \frac{1}{2} + \cos\left(\frac{2\pi}{5}n + \frac{\pi}{2}\right)/2 = e^{j\pi n} + \frac{1}{2} + \frac{1}{2}\left(e^{j\left(\frac{2\pi}{5}n + \frac{\pi}{2}\right)} + e^{-j\left(\frac{2\pi}{5}n + \frac{\pi}{2}\right)}\right)$

$= e^{5j\frac{2\pi}{10}n} + \frac{1}{2} + \frac{1}{2} e^{j\frac{\pi}{2}} e^{2j\frac{2\pi}{10}n} + \frac{1}{2} e^{-j\frac{\pi}{2}} e^{-2j\frac{2\pi}{10}n} \Rightarrow a_0 = \frac{1}{2},\ a_5 = 1,\ a_2 = \frac{e^{j\frac{\pi}{2}}}{2},\ a_{-2} = \frac{e^{-j\frac{\pi}{2}}}{2}$

4) d)

$$a_k = \begin{cases} \frac{\sin[(2k\pi/N)(N_1 + \frac{1}{2})]}{N \sin[2k\pi/2N]}, & k \neq 0, \pm N, \pm 2N, \dots \\ \frac{2N_1 + 1}{N}, & k = 0, \pm N, \pm 2N, \dots \end{cases}$$

4) f) $a_k = \frac{1}{N} \sum_{n = \langle N \rangle} x[n] e^{-jk\frac{2\pi}{N}n},\ N = 5 \Rightarrow a_k = \frac{1}{5} \sum_{n=0}^{4} x[n] e^{-jk\frac{2\pi}{5}n}$

$\Rightarrow a_0 = \frac{1}{5} \times 1 \times 1 + \frac{1}{5} \times 2 \times 1 + \frac{1}{5} \times 4 \times 1 + \frac{1}{5} \times 0 + \frac{1}{5} \times 0 = \frac{7}{5}$

$\Rightarrow a_1 = \frac{1}{5} \times 1 \times 1 + \frac{1}{5} \times 2 \times e^{-j\frac{2\pi}{5} \times 1} + \frac{1}{5} \times 4 \times e^{-j\frac{2\pi}{5} \times 2}$

$\Rightarrow a_2 = \frac{1}{5} + \frac{1}{5} \times 2 \times e^{-j \times 2 \times \frac{2\pi}{5}} + \frac{1}{5} \times 4 \times e^{-j \times 2 \times \frac{2\pi}{5} \times 2}$

$\Rightarrow a_3 = \frac{1}{5} + \frac{1}{5}\times 2\times e^{-j\times 3\times \frac{2\pi}{5}} + \frac{1}{5}\times 4\times e^{-j\times 3\times \frac{2\pi}{5}\times 2}$

$\Rightarrow a_4 = \frac{1}{5} + \frac{1}{5}\times 2\times e^{-j\times 4\times \frac{2\pi}{5}} + \frac{1}{5}\times 4\times e^{-j\times 4\times \frac{2\pi}{5}\times 2}$

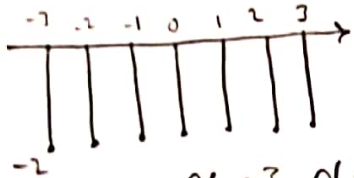
4)g) $N=8 \Rightarrow a_k = \frac{1}{8}\sum_{n=0}^{7} x[n]\, e^{-jk\frac{2\pi}{N}n}$

$\Rightarrow a_0 = \frac{1}{8}(5+4+3+2+1+2+3+4) = 3$

$a_1 = \frac{5}{8} + \frac{4}{8} e^{-j\times\frac{2\pi}{8}} + \frac{3}{8} e^{-j\frac{2\pi}{8}\times 2} + \frac{2}{8} e^{-j\times\frac{2\pi}{8}\times 3} + \frac{1}{8} e^{-j\frac{2\pi}{8}\times 4} + \frac{2}{8} e^{-j\frac{2\pi}{8}\times 5} + \frac{3}{8} e^{-j\frac{2\pi}{8}\times 6} + \frac{4}{8} e^{-j\frac{2\pi}{8}\times 7}$

$\vdots$

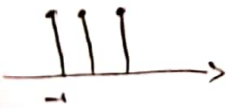
4)h)



$N_1 = 3,\ N = 10$

$\Rightarrow a_k = \begin{cases} \dfrac{-2\sin\left[\frac{7}{2}\times\frac{2k\pi}{10}\right]}{10\sin(2k\pi/20)}, & k \neq 10n \\ -\dfrac{14}{10}, & k = 10n \end{cases}$

$f[n]$



$N_1 = 1,\ N = 10$

$\Rightarrow b_k = \begin{cases} \sin\left[\frac{3}{2}\times\frac{2k\pi}{10}\right], & k \neq 10n \\ \dfrac{3}{10}, & k = 10n \end{cases}$

$c_k = b_k e^{jk\frac{2\pi}{10}\times 5} \Rightarrow \boxed{d_k = a_k + c_k}$

---

5) $x[n] \longleftrightarrow a_k$ $x[n]$ real and odd $\Rightarrow a_k$ purely imaginary and odd

$\Rightarrow a_0 = 0$

$a_{11} = a_{2\times5+1} = a_1 = -a_{-1} \Rightarrow a_{-1} = -j$

$a_{13} = a_{2\times5+3} = a_3 = -a_{-3} \Rightarrow a_{-3} = -3j$

$a_{17} = a_{3\times5+2} = a_2 = -a_{-2} \Rightarrow a_{-2} = -\frac{j}{2}$

دانشگاه صنعتی امیرکبیر
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Signals and Systems

Assignment 6

Spring 2021

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Question 1

Determine the Fourier Transform for the following signals:

(a) $x_1[n] = 4 + cos(\frac{\pi}{6}n + \frac{\pi}{8}) + 2sin(\frac{\pi}{4}n)$

(b) $x_2[n] = u[n-4] - u[n-15]$

(c) $x_3[n] = \frac{sin(\frac{\pi}{3}n)}{\pi n}$

(d) $x_4[n] = \left(\frac{1}{2}\right)^{|n|} u[-n-5]$

(e) $x_5[n] = 3^n sin(\frac{\pi}{8}n) u[-n]$

Question 2

Determine the Fourier Transform for the following signals in terms of $X(e^{j\omega})$:

(a) $x[3-n] + x[-3+n]$

(b) $x^*[-n]$ (the signal is real)

Question 3

Determine the inverse Fourier Transform for the following signals:

(a) $\hat{X}(e^{j\omega}) = \begin{cases} 1 & \frac{\pi}{3} < |\omega| < \frac{3\pi}{4} \\ 0 & otherwise \end{cases}$

(b) $X(e^{j\omega}) = 3 + 3e^{-j5\omega} + 5e^{j2021\omega}$

(c) $X(e^{j\omega}) = \frac{e^{-j\omega}+2}{e^{-j2\omega}+2e^{-j\omega}-8}$

Question 4

Consider a system consisting of the cascade of two LTI systems with frequency responses

$$H_1(e^{j\omega}) = \frac{3 - 2e^{-j5\omega}}{1 + e^{-j\omega}}$$

and

$$H_2(e^{j\omega}) = \frac{1}{1 - \frac{1}{8}e^{-j\omega} + \frac{1}{16}e^{-j3\omega}}$$

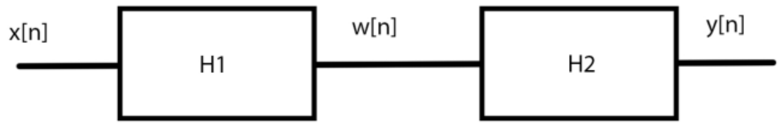
Find the difference equation describing the overall system.

Question 5

Determine the output of an LTI system with impulse response $h[n] = \frac{sin(\frac{\pi}{3}n)sin(\frac{\pi}{6}n)}{\pi^2 n^2}$ if the input is $x[n] = sin(\frac{\pi}{12}n) - 16sin(\frac{2\pi}{3}n)$

Question 6

Consider the following system:



$$w[n] = 2x[n]$$

$$h_2[n] = \frac{sin(\frac{\pi}{3}n)}{\pi n}$$

$$x[n] = cos(0.3\pi n) + sin(0.4\pi n) + 2$$

(a) Determine $W(e^{j\omega})$ in terms of $X(e^{j\omega})$

(b) Determine $H_1(e^{j\omega})$

(c) Determine $H_{eq}(e^{j\omega})$

(d) Determine $X(e^{j\omega})$

(e) Determine $y[n]$

Question 7

Consider a causal LTI system described by the difference equation

$$y[n] - \frac{1}{3}y[n-1] = 2x[n]$$

(a) Determine the frequency response $H(e^{j\omega})$ of this system.

(b) Determine the output if the input is $x[n] = \left(\frac{1}{3}\right)^n u[n]$

(c) Determine the output if the input is $x[n] = \left(\frac{-1}{3}\right)^n u[n]$

(d) Determine the output if the input has the following Fourier Transform

$$X(e^{j\omega}) = \frac{1 - \frac{1}{4}e^{-j\omega}}{1 + \frac{1}{2}e^{-j\omega}}$$

Question 1.

(a) $x_1[n] = 4 + \cos\left(\frac{\pi}{6}n + \frac{\pi}{8}\right) + 2\sin\left(\frac{\pi}{4}n\right)$

$\omega_0 = \frac{\pi}{12}$

$N = 24$

سری فوریه: $x[n] = 4 + \frac{1}{2}\left(e^{j(\pi/6 n + \frac{\pi}{8})} + e^{-j(\pi/6 n + \pi/8)}\right) + \frac{2}{2j}\left(e^{j\frac{\pi}{4}n} - e^{-j\pi/4 n}\right)$

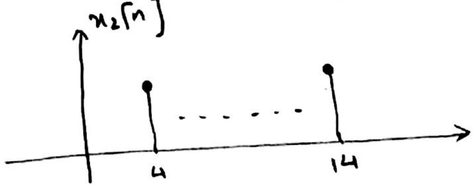
$$\Rightarrow \begin{cases} a_0 = 4 \\ a_2 = \frac{1}{2}e^{j\pi/8} \\ a_{-2} = \frac{1}{2}e^{-j\pi/8} \\ a_3 = \frac{1}{j} \\ a_{-3} = -\frac{1}{j} \end{cases}$$

چون x متناوب است، $X(e^{j\omega})$ متناوب و پیوسته است و دوره تناوب آن 2π است
$X(e^{j\omega})$ را در یک دوره تناوب با $\hat{X}(e^{j\omega})$ نمایش می‌دهیم

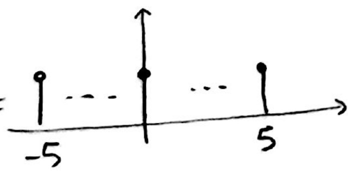
$$\hat{X}(e^{j\omega}) = 2\pi \sum_{\langle N \rangle} a_k \delta(\omega - k\omega_0) = 2\pi \sum_{k=-10}^{13} a_k \delta\left(\omega - \frac{\pi}{12}k\right)$$

$$= 8\pi\,\delta(\omega) + \pi e^{j\pi/8}\delta\left(\omega - \frac{\pi}{6}\right) + \pi e^{-j\pi/8}\delta(\omega + \pi/6) + \frac{2\pi}{j}\delta\left(\omega - \frac{\pi}{4}\right) - \frac{2\pi}{j}\delta\left(\omega + \frac{\pi}{4}\right)$$

(b) $x_2[n] = u[n-4] - u[n-15]$



$y[n] =$



$\longleftrightarrow \; Y(e^{j\omega}) = \dfrac{\sin\left(\frac{11}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)}$

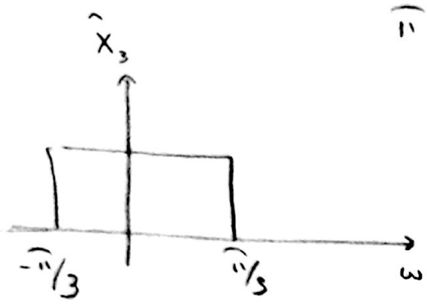
$x_2[n] = y[n-9] \longleftrightarrow X_2(e^{j\omega}) = e^{-9j\omega}\,Y(e^{j\omega})$

$$\Rightarrow X_2(e^{j\omega}) = e^{-9j\omega}\,\frac{\sin\left(\frac{11}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)}$$

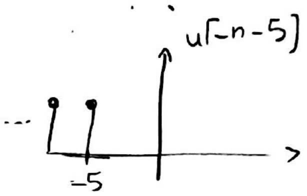
(c) $x_3[n] = \dfrac{\sin(\frac{\pi}{3}n)}{\pi n}$

$$x[n] = \frac{\sin(Wn)}{\pi n} \longleftrightarrow \hat{X}(e^{j\omega}) = \begin{cases} 1 & 0 \leqslant |\omega| \leqslant W \\ 0 & W \leqslant |\omega| \leqslant \pi \end{cases}$$

$$\Rightarrow x_3[n] = \frac{\sin(\pi/3\, n)}{\pi n} \longleftrightarrow \hat{X}_3(e^{j\omega}) = \begin{cases} 1 & 0 \leqslant |\omega| \leqslant \pi/3 \\ 0 & \pi/3 \leqslant |\omega| \leqslant \pi \end{cases}$$



(d) $x_4[n] = \left(\frac{1}{2}\right)^{|n|} u[-n-5]$



$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \left(\frac{1}{2}\right)^{|n|} u[-5-n]\, e^{-jn\omega}$$

$$= \sum_{n=-\infty}^{-5} \left(\frac{1}{2}\right)^{|n|} e^{-jn\omega}$$

$$= \sum_{n=-\infty}^{-5} \left(\frac{1}{2}\right)^{-n} e^{-j\omega n} = \sum_{n=-\infty}^{-5} \left(\frac{1}{2} e^{j\omega}\right)^{-n} = \sum_{n=5}^{+\infty} \left(\frac{1}{2} e^{j\omega}\right)^{n}$$

سری هندسی $\quad \left(\frac{1}{2} e^{j\omega}\right)^5 \dfrac{1}{1 - \frac{1}{2} e^{j\omega}} = \dfrac{e^{5j\omega}}{32} \times \dfrac{1}{1 - \frac{1}{2} e^{j\omega}}$

(e) $x_5[n] = 3^n \sin\left(\frac{\pi}{8}n\right) u[-n]$

$\alpha^n u[n] \quad |\alpha| < 1 \longleftrightarrow \frac{1}{1-\alpha e^{-j\omega}}$

$y[n] = x_5[-n] = 3^{-n} \sin\left(-\frac{\pi}{8}n\right) u[n] = \underbrace{-3^{-n} u[n]}_{s[n]} \underbrace{\sin\left(\pi/8 \, n\right)}_{r[n]}$

$r[n] = -3^{-n} u[n] \longleftrightarrow R(e^{j\omega}) = \frac{-1}{1-\frac{1}{3}e^{-j\omega}}$

$s[n] = \sin\left(\frac{\pi}{8}n\right) \longleftrightarrow \hat{S}(e^{j\omega}) = \frac{\pi}{j}\left(\delta\left(\omega-\frac{\pi}{8}\right) - \delta\left(\omega+\frac{\pi}{8}\right)\right)$

$y[n] = r[n]\, s[n] \longleftrightarrow Y(e^{j\omega}) = R(e^{j\omega}) * S(e^{j\omega}) \times \frac{1}{2\pi}$

$\Rightarrow Y(e^{j\omega}) = \frac{1}{2j}\left(R(e^{j(\omega-\pi/8)}) - R(e^{j(\omega+\pi/8)})\right)$

$X(e^{j\omega}) = Y(e^{-j\omega}) = \frac{1}{2j}\left(R(e^{j(-\omega-\pi/8)}) - R(e^{j(-\omega+\pi/8)})\right)$

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Question 2.

(a) $y[n] = x[3-n] + x[-3+n]$

$x[n] \longleftrightarrow X(e^{j\omega})$

$x[n+3] \longleftrightarrow e^{3j\omega} X(e^{j\omega})$

$x[-n+3] \longleftrightarrow e^{-3j\omega} X(e^{-j\omega})$

$x[n] \longleftrightarrow X(e^{j\omega})$

$x[n-3] \longleftrightarrow e^{-3j\omega} X(e^{j\omega})$

$Y(e^{j\omega}) = e^{-3j\omega} X(e^{j\omega}) + e^{3j\omega} X(e^{j\omega})$

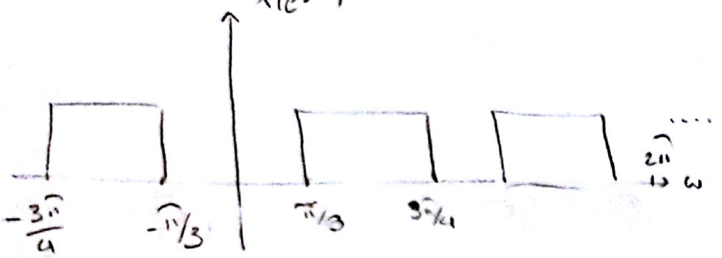
(b) $x^*[-n]$ (the signal is real)

$x^*[-n]$ real $\longrightarrow x^*[-n] = x[-n]$

$x[-n] \longleftrightarrow X(e^{-j\omega})$

Question 3.

(a) $\hat{X}(e^{j\omega}) = \begin{cases} 1 & \pi/3 < |\omega| < 3\pi/4 \\ 0 & \text{otherwise} \end{cases}$



$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{jn\omega}\,d\omega = \frac{1}{2\pi}\left[\int_{-3\pi/4}^{-\pi/3} e^{jn\omega}\,d\omega + \int_{+\pi/3}^{3\pi/4} e^{jn\omega}\,d\omega\right]$$

$$= \frac{1}{2\pi nj}\left(e^{jn\omega}\Big|_{-3\pi/4}^{-\pi/3} + e^{jn\omega}\Big|_{\pi/3}^{3\pi/4}\right) = \frac{1}{\pi n}\left(\frac{e^{-\frac{\pi}{3}n} - e^{\frac{\pi}{3}n}}{2j} + \frac{e^{\frac{3\pi}{4}n} - e^{-\frac{3\pi}{4}n}}{2j}\right)$$

$$= \frac{1}{\pi n}\left(\sin\left(\frac{3\pi}{4}n\right) - \sin\left(\frac{\pi}{3}n\right)\right)$$

(b) $X(e^{j\omega}) = 3 + 3e^{-j5\omega} + 5e^{j2021\omega}$

$\rightarrow x[n] = 3\delta[n] + 3\delta[n-5] + 5\delta[2021+n]$

(c) $X(e^{j\omega}) = \dfrac{2 + e^{-j\omega}}{e^{-2j\omega} + 2e^{j\omega} - 8}$

$$X(e^{j\omega}) = \frac{2 + e^{-j\omega}}{(e^{-j\omega}+4)(e^{-j\omega}-2)} = \frac{A}{e^{-j\omega}+4} + \frac{B}{e^{-j\omega}-2}$$

$\begin{cases} A + B = 1 \\ 4B - 2A = 2 \end{cases}$ $\quad B = \frac{2}{3} \quad A = \frac{1}{3}$ $\quad \Rightarrow X(e^{j\omega}) = \dfrac{1/3}{e^{-j\omega}+4} + \dfrac{\frac{2}{3}}{e^{-j\omega}-2}$

$$X(e^{j\omega}) = \frac{\frac{1}{12}}{1+\frac{e^{-j\omega}}{4}} - \frac{\frac{1}{3}}{1-\frac{e^{-j\omega}}{2}}$$

$$x_1[n] = \frac{1}{12}\left(-\frac{1}{4}\right)^n u[n] - \frac{1}{3}\left(\frac{1}{2}\right)^n u[n]$$

Question 4.

$$H_1(e^{j\omega}) = \frac{3-2e^{-j5\omega}}{1+e^{-j\omega}}, \quad H_2(e^{j\omega}) = \frac{1}{1-\frac{1}{8}e^{-j\omega}+\frac{1}{16}e^{-j3\omega}}$$

$$H(e^{j\omega}) = \frac{3-2e^{j\omega}}{(1+e^{-j\omega})(1-\frac{1}{8}e^{-j\omega}+\frac{1}{16}e^{-j\omega 3})}$$

$$= \frac{3-2e^{j\omega}}{1-\frac{1}{8}e^{-j\omega}+\frac{1}{16}e^{-3j\omega}+e^{-j\omega}-\frac{1}{8}e^{-2j\omega}+\frac{1}{16}e^{-4j\omega}}$$

$$= \frac{3-2e^{-j\omega}}{1+\frac{7}{8}e^{-j\omega}-\frac{1}{8}e^{-2j\omega}+\frac{1}{16}e^{-3j\omega}+\frac{1}{16}e^{-4j\omega}} = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

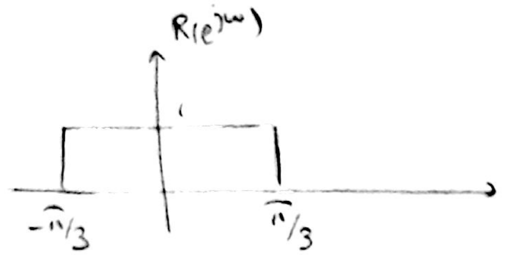
$$3X(e^{j\omega}) - 2e^{-j\omega}X(e^{j\omega}) = Y(e^{j\omega}) + \frac{7}{8}Y(e^{j\omega})e^{-j\omega} - \frac{1}{8}Y(e^{j\omega})e^{-2j\omega} + \frac{1}{16}e^{-3j\omega}Y(e^{j\omega}) + \frac{1}{16}e^{-4j\omega}Y(e^{j\omega})$$

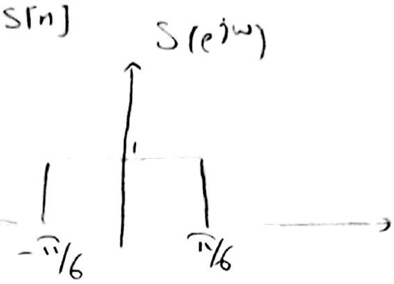
$$3x[n] - 2x[n-1] = y[n] + \frac{7}{8}y[n-1] - \frac{1}{8}y[n-2] + \frac{1}{16}y[n-3] + \frac{1}{16}y[n-4]$$

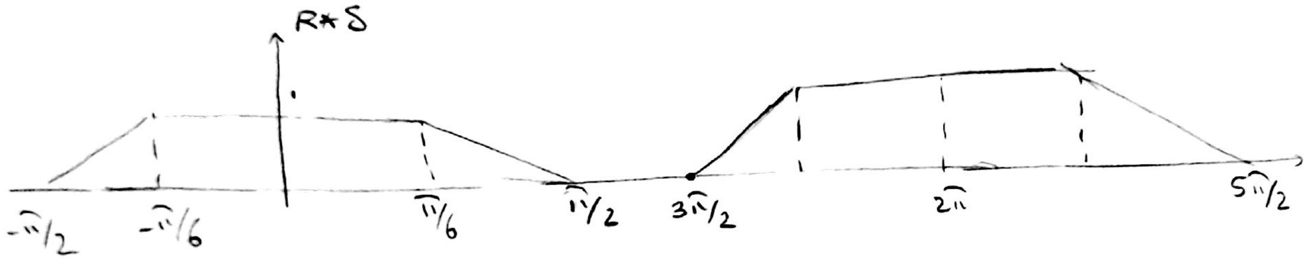
Question 5.

$$h[n] = \frac{\sin(\frac{\pi}{3}n)\sin(\frac{\pi}{6}n)}{\pi^2 n^2}, \quad x[n] = \sin(\frac{\pi}{12}n) - 16\sin(\frac{2\pi}{3}n)$$

$$h[n] = \underbrace{\frac{\sin(\frac{\pi}{3}n)}{\pi n}}_{r[n]} \times \underbrace{\frac{\sin(\frac{\pi}{6}n)}{\pi n}}_{s[n]} \qquad H(e^{j\omega}) = S(e^{j\omega}) * R(e^{j\omega})$$







$$x[n] = \sin(\frac{\pi}{12}n) - 16\sin(\frac{2\pi}{3}n) = \frac{1}{2j}\left(e^{j\frac{\pi}{12}n} - e^{-j\frac{\pi}{12}n} - 16e^{\frac{2\pi}{3}jn} + 16e^{-\frac{2\pi}{3}jn}\right)$$

$\omega_0 = \pi/12, \; N = 24$

$$\Rightarrow \begin{cases} a_1 = \frac{1}{2j} \\ a_{-1} = -\frac{1}{2j} \\ a_8 = -\frac{16}{2j} = -\frac{8}{j} \\ a_{-8} = \frac{+8}{j} \end{cases}$$

$$\hat{X}(e^{j\omega}) = 2\pi \sum_{\langle N \rangle} a_k \delta(\omega - k\omega_0) = 2\pi\left(\frac{1}{2j}\delta(\omega - \pi/12) - \frac{1}{2j}\delta(\omega + \pi/12) - \frac{8}{j}\delta(\omega - \frac{2\pi}{3}) + \frac{8}{j}\delta(\omega + \frac{2\pi}{3})\right)$$

$$\hat{Y}(e^{j\omega}) = \hat{H}(e^{j\omega}) \times \hat{X}(e^{j\omega}) = \frac{\pi}{j}\delta(\omega - \pi/12) - \frac{\pi}{j}\delta(\omega + \pi/12)$$

$$= \frac{2\pi}{2j}\left(\delta(\omega - \frac{\pi}{12}) - \delta(\omega + \pi/12)\right) \longleftrightarrow y[n] = \sin(\frac{\pi}{12})$$

Question 6.

$$w[n] = 2x[n],\ h_2[n] = \frac{\sin(\frac{\pi}{3}n)}{\pi n},\ x[n] = \cos(0.3\pi n) + \sin(0.4\pi n) + 2$$

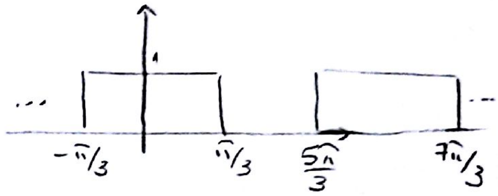
(a) $w[n] = 2x[n] \longleftrightarrow W(e^{j\omega}) = 2X(e^{j\omega})$

(b)

$$X(e^{j\omega}) \times H_1(e^{j\omega}) = W(e^{j\omega})$$

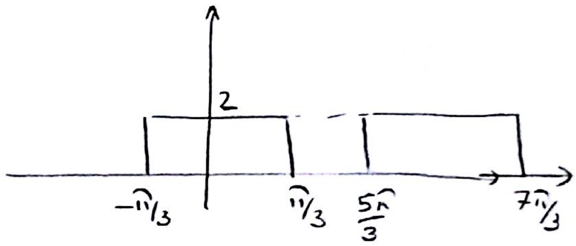
$$H_1(e^{j\omega}) = \frac{W(e^{j\omega})}{X(e^{j\omega})} = \frac{2X(e^{j\omega})}{X(e^{j\omega})} = 2$$

(c) $h_2[n] = \frac{\sin(\frac{\pi}{3}n)}{\pi n} \longleftrightarrow H(e^{j\omega}) =$



$$H(e^{j\omega}) = H_1(e^{j\omega}) \times H_2(e^{j\omega})$$

$$H_{eq}(e^{j\omega}) = \begin{cases} 2 & 0 \leq |\omega| \leq \pi/3 \\ 0 & \text{otherwise} \end{cases}$$



(d) $\hat{X}(e^{j\omega}) = \frac{\pi}{j}\left(\delta(\omega - \frac{2}{5}\pi) - \delta(\omega + \frac{2}{5}\pi)\right) + \pi\left(\delta(\omega - \frac{3\pi}{10}) + \delta(\omega + \frac{3\pi}{10})\right) + 2\delta(\omega)$

(e) $\hat{Y}(e^{j\omega}) = \hat{X}(e^{j\omega})\hat{H}(e^{j\omega}) = \pi\left(\delta(\omega - \frac{3\pi}{10}) + \delta(\omega + \frac{3\pi}{10})\right) + 2\delta(\omega)$

$$y[n] = \cos(0.3\pi) + 2$$

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Question 7.

$$y[n] - \frac{1}{3}y[n-1] = 2x[n]$$

(a) $Y(e^{j\omega}) - \frac{1}{3}e^{-j\omega}Y(e^{j\omega}) = 2X(e^{j\omega})$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{2}{1-\frac{1}{3}e^{-j\omega}}$$

(b) $x[n] = (\frac{1}{3})^n u[n] \longleftrightarrow X(e^{j\omega}) = \frac{1}{1-\frac{1}{3}e^{-j\omega}}$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = \frac{2}{1-\frac{1}{3}e^{-j\omega}} \times \frac{1}{1-\frac{1}{3}e^{-j\omega}} = \frac{2}{(1-\frac{1}{3}e^{-j\omega})^2}$$

$R(e^{j\omega}) = \frac{1}{1-\frac{1}{3}e^{-j\omega}}$

$\Rightarrow Y(e^{j\omega}) = -\frac{6}{j}\frac{d}{d\omega}R(e^{j\omega}) \rightarrow y[n] = -\frac{6}{j} \times \frac{n}{j} r[n]$

$\frac{dR(e^{j\omega})}{d\omega} = \frac{-j/3}{(1-1/3e^{-j\omega})^2}$

$\Rightarrow y[n] = 6n \times (\frac{1}{3})^n u[n]$

(c) $x[n] = (-1/3)^n u[n] \longleftrightarrow X(e^{j\omega}) = \frac{1}{1+\frac{1}{3}e^{-j\omega}}$

$$Y(e^{j\omega}) = \frac{1}{1+\frac{1}{3}e^{-j\omega}} \times \frac{2}{1-\frac{1}{3}e^{-j\omega}} = \frac{A}{1-\frac{1}{3}e^{-j\omega}} + \frac{B}{1+\frac{1}{3}e^{-j\omega}}$$

$\begin{cases} A+B=2 \\ A-B=0 \end{cases}$ $A=1$, $B=1$

$$Y(e^{j\omega}) = \frac{1}{1-\frac{1}{3}e^{-j\omega}} + \frac{1}{1+\frac{1}{3}e^{-j\omega}}$$

$$y[n] = ((\tfrac{1}{3})^n + (\tfrac{1}{3})^{-n})u[n]$$

(d) $Y(e^{j\omega}) = \frac{2}{1-\frac{1}{3}e^{-j\omega}} \times \frac{1-\frac{1}{4}e^{-j\omega}}{1+\frac{1}{2}e^{-j\omega}} = \frac{2}{1-\frac{1}{3}e^{-j\omega}} \times \left(-1/2 + \frac{3/2}{1+\frac{1}{2}e^{-j\omega}}\right)$

$$\Rightarrow Y(e^{j\omega}) = \frac{-1}{1-\frac{1}{3}e^{-j\omega}} + \frac{3}{1-\frac{1}{3}e^{-j\omega}} \times \frac{1}{1+\frac{1}{2}e^{-j\omega}}$$

$$Y(e^{j\omega}) = \frac{-1}{1-\frac{1}{3}e^{-j\omega}} + \frac{A}{1-\frac{1}{3}e^{-j\omega}} + \frac{B}{1+\frac{1}{2}e^{-j\omega}}$$

$A + B = 3$ $\qquad A = \frac{6}{5} \quad B = \frac{9}{5}$

$\frac{1}{2}A - \frac{1}{3}B = 0 \rightarrow 3A = 2B$

$$Y(e^{j\omega}) = \frac{-1}{1-\frac{1}{3}e^{-j\omega}} + \frac{\frac{6}{5}}{1-\frac{1}{3}e^{-j\omega}} + \frac{\frac{9}{5}}{1+\frac{1}{2}e^{-j\omega}} \Rightarrow$$

$$y[n] = -\left(\frac{1}{3}\right)^n u[n] + \frac{6}{5}\left(\frac{1}{3}\right)^n u[n] + \frac{9}{5}\left(-\frac{1}{2}\right)^n u[n]$$