MAT B41 — Techniques of the Calculus of Several Variables I

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This is a compilation of the notes from Professor Kathleen Smith's MAT B41 lectures. The page and section references in parentheses occurring after definitions, theorems, other facts, and section titles refer to the textbook *Multivariable Calculus, 9th ed., Stewart, Clegg & Watson.* Certain graphs/figures are from this textbook, while others have been made using the GeoGebra Calculator Suite or the PGFPLOTS LaTeX package. Each of the facts (definitions, theorems, etc.) are numbered for cross-referencing purposes.

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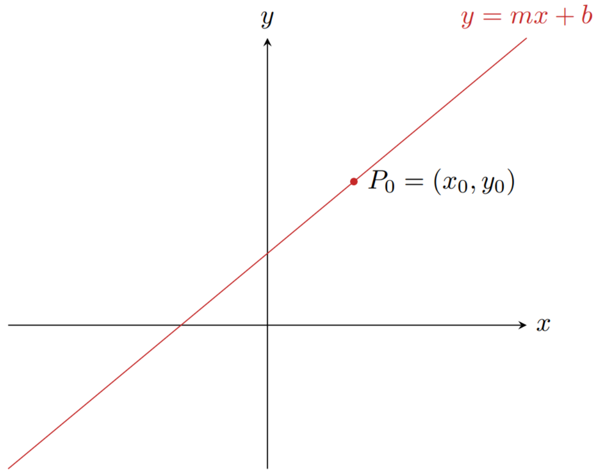
Chapter 1 Geometry of Euclidean Space

1.1 Equations of Lines & Planes and Parametric Equations (§§10.1, 12.5)



In $\mathbb{R}^2$, the equation of a line L is $y = mx + b$. Alternatively, in point-slope form, $y - y_0 = m(x - x_0)$ (given $P_0, P_1 \in L$ with $P_0 \neq P_1$ to compute m).

Definition 1.1 — Parametric Equations (p. 662)

Suppose that x and y are real-valued functions of t on an interval $I \subseteq \mathbb{R}$. That is, $x = f(t)$ and $y = g(t)$ with $t \in I$. These equations are called *parametric equations* with *parameter* t.

The set of points of x and y as t varies over I

$$\big\{(x, y) : x = f(t) \wedge y = g(t) \wedge t \in I\big\}$$

is the graph of the parametric equations of the *parametric curve*.

Example 1.2 *(Parametric Line)*: Consider $x = t - 1$ and $y = 2t + 4$, where $t \in \mathbb{R}$.

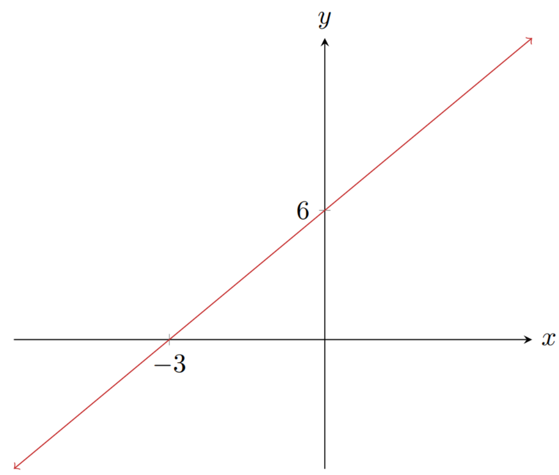
(a) Sketch the parametric curve.

(b) Write the parametric curve in the form $y = f(x)$.

(a) A table of values with some points as follows:

t	"$x(t)$"	"$y(t)$"
-1	-2	2
0	-1	4
1	0	6

Using these points, the curve is



(b) Since $x = t - 1 \iff t = x + 1$, we have

$$y = 2t + 4 = 2(x + 1) + 4 = 2x + 6$$

Therefore, $y = 2x + 6$.

Alternatively, we could proceed by noting that $y = 2t + 4 \iff t = \frac{1}{2}(y - 4)$, so

$$x + 1 = \frac{1}{2}(y - 4) \iff y = 2x + 6$$

Example 1.3: What curve/function in $\mathbb{R}^2$ is given by

$$x = \cos(t) \qquad y = \sin(t) \qquad t \in [0, 2\pi]$$

(in the form $y = f(x)$ or $f(x, y) = 0$)?

For all $t \in [0, 2\pi]$, $x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1$. Thus, the given curve is

$$f(x, y) = x^2 + y^2 - 1 = 0$$

1.2 More Lines in $\mathbb{R}^3$ (§12.5)

Definition 1.4 — Position Vector

A *position vector* represents a vector's components as a point with respect to the origin. In $\mathbb{R}^3$, a position vector $\mathbf{v}$ is denoted by $\mathbf{v} = \langle a, b, c\rangle$ for $a, b, c \in \mathbb{R}$.

Remark: The angled brackets for position vectors as in definition 1.4 are used to distinguish between ordered tuples representing points in space and vectors (e.g. $\langle a, b, c\rangle$ instead of (a, b, c) in $\mathbb{R}^3$).

Theorem 1.5 — Derivation of Line Equation in $\mathbb{R}^3$

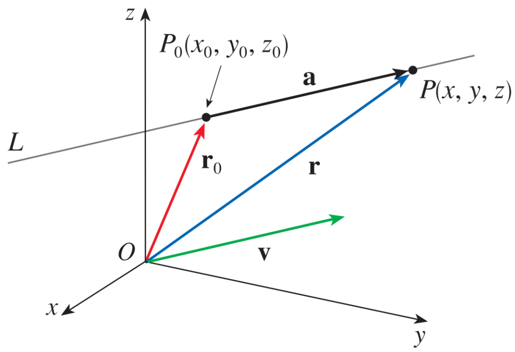
A line L in $\mathbb{R}^3$ may be determined by one of the following:

- Points $P_0, P \in L$ with $P_0 \neq P$
- A point $P_0 \in L$ and a *direction vector* — some $\mathbf{v} \in \mathbb{R}^3$ such that $\mathbf{v} \parallel L$

Given a point $P_0 = (x_0, y_0, z_0) \in L$ and a direction vector $\mathbf{v} = \langle a, b, c\rangle \in \mathbb{R}^3$ for L,

$$x = x_0 + ta \qquad y = y_0 + tb \qquad z = z_0 + tc \qquad t \in \mathbb{R}$$

are the parametric equations of L in $\mathbb{R}^3$. Here, $t \in \mathbb{R}$ satisfies $\overrightarrow{P_0P} = t\mathbf{v}$ for some arbitrary $P = (x, y, z) \in L$.



PROOF: We will derive the line L given a point and a direction vector. Let $P_0 = (x_0, y_0, z_0) \in L$, $\mathbf{v}$ be some direction vector for L, and $P = (x, y, z) \in L$ be an arbitrary point. Define position vectors $\mathbf{r}_0$ and $\mathbf{r}$ from the origin to P_0 and P (respectively), and let $\mathbf{a} = \overrightarrow{P_0P}$.

We have $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}$. Note that $\mathbf{a} \parallel \mathbf{v}$, so $\mathbf{a} = t\mathbf{v}$ for some $t \in \mathbb{R}$. Therefore, $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$. Now let $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$, $\mathbf{r} = \langle x, y, z \rangle$, and $\mathbf{v} = \langle a, b, c \rangle$. It follows that

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v} \iff \langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + \langle ta, tb, tc \rangle \iff \langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

Therefore, $x = x_0 + ta$, $y = y_0 + tb$, and $z = z_0 + tc$ for $t \in \mathbb{R}$ are the parametric equations of L in $\mathbb{R}^3$. ■

Definition 1.6 — Vector Equation of a Line (p. 865)

In the context of the derivation of the parametric equations of a line in $\mathbb{R}^3$ (the proof of theorem 1.5), the *vector equation of* L is

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

or equivalently,

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

Definition 1.7 — Symmetric Equations of a Line in $\mathbb{R}^3$

The *symmetric equations* of a line L in $\mathbb{R}^3$ is

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

where a, b, and c must be non-zero and $\langle x_0, y_0, z_0 \rangle$ and $\langle a, b, c \rangle$ are the same vectors as in the derivation of the parametric equations of L (the proof of theorem 1.5). If $a = 0$, $x = x_0$, and similarly $y = y_0$ and $z = z_0$ for the cases where $b = 0$ and $c = 0$ (respectively).

Remark: The symmetric equations in definition 1.7 follow from theorem 1.5, where $x = x_0 + ta \iff t = \dfrac{x - x_0}{a}$ for $a \neq 0$, and similarly $t = \dfrac{y - y_0}{b}$ for $b \neq 0$ and $t = \dfrac{z - z_0}{c}$ for $c \neq 0$.

1.3 Equations of Planes in $\mathbb{R}^3$

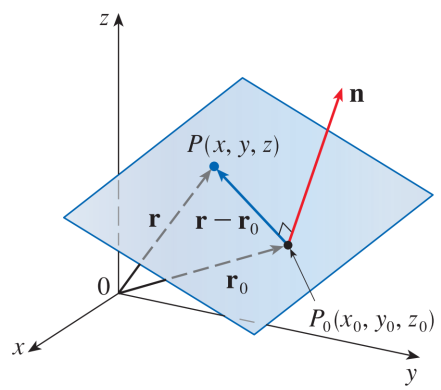
Theorem 1.8 — Derivation of Plane Equation in $\mathbb{R}^3$

The equation of a plane in $\mathbb{R}^3$ can be uniquely determined by one of the following:

- A point $P_0 = (x_0, y_0, z_0)$ in the plane and a *normal vector* to the plane
- A point $P_0 = (x_0, y_0, z_0)$ in the plane and a line L on the plane such that $P_0 \notin L$

Given a point $P_0 = (x_0, y_0, z_0)$ on the plane and a normal vector $\mathbf{n} = \langle a, b, c \rangle$ to the plane, the *scalar equation* of the plane is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$



PROOF: We will derive the plane's equation given a point and a normal vector. Let $P_0 = (x_0, y_0, z_0)$ be a point on the plane, $\mathbf{n}$ be a normal vector to the plane, and $P = (x, y, z)$ be an arbitrary point on the plane. Suppose that $\mathbf{a} = \overrightarrow{P_0P}$ and $\mathbf{r}_0$ and $\mathbf{r}$ are position vectors with respect to the points P_0 and P (respectively).

The set of all points P on the plane satisfy $\mathbf{n} \cdot \mathbf{a} = 0$ (as $\mathbf{n}$ is normal to the plane). Also, $\mathbf{a} = \mathbf{r} - \mathbf{r}_0$, so $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$. Now let $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$, $\mathbf{r} = \langle x, y, z \rangle$, and $\mathbf{n} = \langle a, b, c \rangle$. It follows that

$$\begin{aligned} \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 &\iff \langle a, b, c \rangle \cdot \big(\langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle\big) = 0 \\ &\iff \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0 \\ &\iff a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \end{aligned}$$

Thus, $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ is the plane's scalar equation. ■

Remark: Unlike in linear algebra, "normal" and "orthogonal" are synonymous in this course, both representing perpendicularity (but *not* unit length).

Definition 1.9 — Vector Equation of a Plane (p. 868)

In the context of the derivation of the scalar equation of a plane in $\mathbb{R}^3$ (the proof of theorem 1.8), the *vector equation of the plane* is

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

Corollary 1.10 — Plane Alternative Scalar Equation

In the setup of the derivation of a plane's scalar equation in $\mathbb{R}^3$ (the proof of theorem 1.8), an equivalent form of the scalar equation of the plane is

$$ax + by + cz + d = 0$$

for $\mathbf{n} = \langle a, b, c \rangle$ and $d = -(ax_0 + by_0 + cz_0)$.

Remark: The equivalent form of a plane's scalar equation in corollary 1.10 is a result of expanding the scalar equation (as in theorem 1.8's proof) and collecting like terms.

Example 1.11: Find the scalar equation of a plane containing the points $P = (1, 1, -2)$, $Q = (0, 2, 1)$, and $R = (-1, -1, 0)$.

We first find a normal vector $\mathbf{n}$ to the plane. Note that

$$\overrightarrow{QP} = \langle 1 - 0, 1 - 2, -2 - 1 \rangle = \langle 1, -1, -3 \rangle$$

$$\overrightarrow{QR} = \langle -1 - 0, -1 - 2, 0 - 1 \rangle = \langle -1, -3, -1 \rangle$$

Thus,

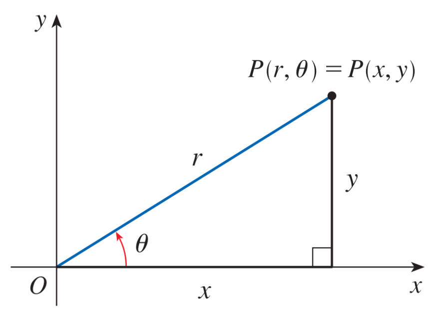
$$\mathbf{n} = \overrightarrow{QP} \times \overrightarrow{QR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -3 \\ -1 & -3 & -1 \end{vmatrix} = \mathbf{i}(1-9) - \mathbf{j}(-1-3) + \mathbf{k}(-3-1) = -8\mathbf{i} + 4\mathbf{j} - 4\mathbf{k} = \langle -8, 4, -4 \rangle$$

using cofactor expansion along the first row (note that the standard basis vectors **i**, **j**, and **k** are treated just as any other entry in the matrix). Therefore, taking $P_0 = (1, 1, -2)$ (any point on the plane works here), the scalar equation of the desired plane is

$$-8(x-1) + 4(y-1) - 4(z-2) = 0$$

by theorem 1.5.

1.4 Polar Coordinates (§10.3)



Definition 1.12 — Polar Coordinates (p. 686)

Each $(x, y) \in \mathbb{R}^2$ can be represented using *polar coordinates* (r, θ), where r is the *radial component* and θ is the *angular component*.

Let θ be the angle starting from the positive x-axis to the line segment between O and P. We have

$$x = r\cos\theta \qquad y = r\sin\theta \qquad \tan\theta = \frac{y}{x} \quad \text{provided } x \neq 0$$

where $r^2 = x^2 + y^2$.

Remark: If we restrict r and θ to $r > 0$ and $\theta \in [0, 2\pi)$, the polar representation is unique.

Example 1.13:

(a) Express $(1, -1)$ in polar coordinates such that $r > 0$ and $\theta \in [0, 2\pi)$.

(b) Convert $\left(2, \frac{3\pi}{2}\right)$ to rectangular coordinates.

(a) We have

$$r^2 = 1^2 + (-1)^2 = 2 \implies r = \sqrt{2}$$

as $r > 0$ and

$$\tan\theta = \frac{1}{-1} \implies \theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$$

as $\theta \in [0, 2\pi)$.

(b) We have

$$x = 2\cos\left(\frac{3\pi}{2}\right) = 2 \cdot 0 = 0$$

$$y = 2\sin\left(\frac{3\pi}{2}\right) = 2 \cdot (-1) = -2$$

Therefore, $(x, y) = (0, -2)$.

Chapter 2 Functions

2.1 Functions of Two Variables (§14.1)

Definition 2.1 — Two-Variable Function (p. 934)

A real *two-variable function* $z = f(x, y)$ is a rule that assigns to each $(x, y) \in D$ exactly one $z \in \mathbb{R}$. Here,

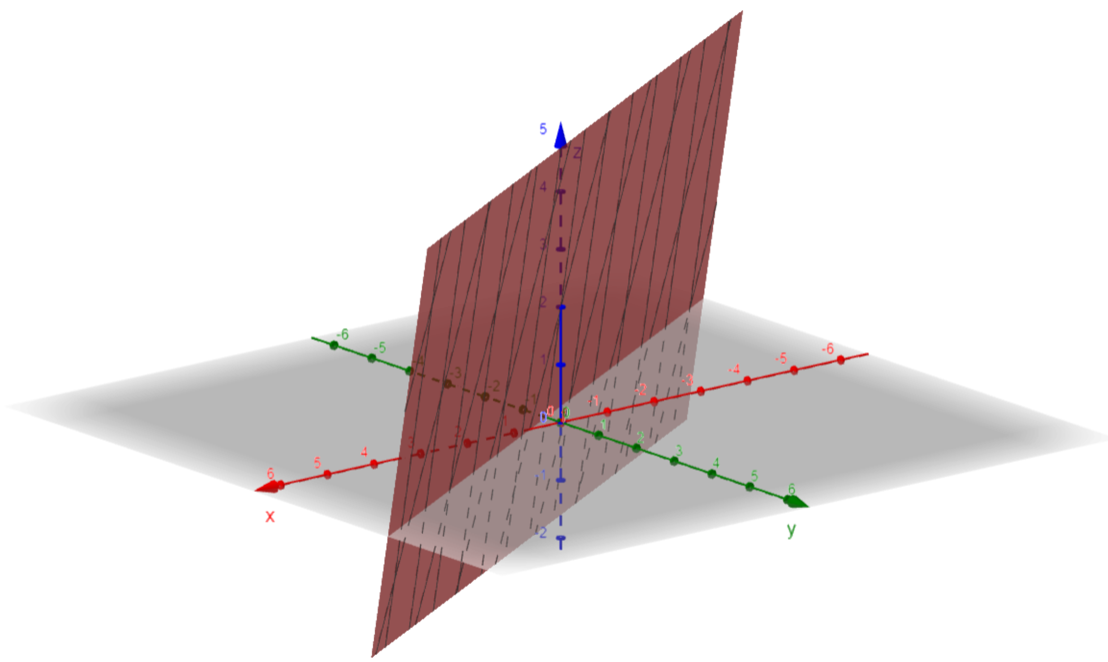
- $D = \{(x, y) \in \mathbb{R}^2 : z = f(x, y)\} = \operatorname{dom}(f)$ is the *domain* of f
- The set $\{z \in \mathbb{R} : z = f(x, y) \text{ for some } (x, y) \in D\}$ is the *range* of f

Example 2.2: Find the domain D and the range of the following functions and graph them:

(a) $f(x, y) = -3x + 5y + 2$

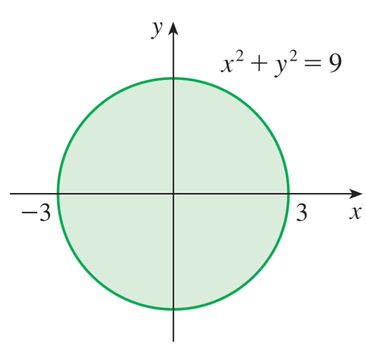
(b) $g(x, y) = \sqrt{9 - x^2 - y^2}$

(a) $\operatorname{dom}(f) = \{(x, y) \in \mathbb{R}^2 : f(x, y) = -3x + 5y + 2\} = \mathbb{R}^2$ as there are no x or y-values that make f undefined. For $k \in \mathbb{R}$, $-3x + 5y + 2 = k$ is the equation of a plane. Thus, $\operatorname{range}(f) = \{z \in \mathbb{R} : z = f(x, y)\} = \mathbb{R}^2$. The graph is as follows:



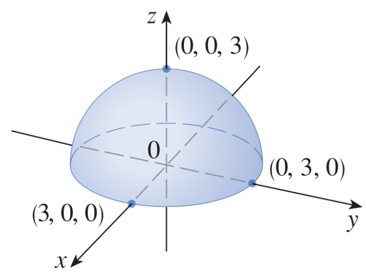
(b) We have $\operatorname{dom}(g) = \left\{(x, y) \in \mathbb{R}^2 : g(x, y) = \sqrt{9 - x^2 - y^2}\right\}$, where the function is defined if and only if

$$9 - x^2 - y^2 \geq 0 \iff 9 \geq x^2 + y^2$$



Thus, $\operatorname{dom}(g) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}$. Also, $\operatorname{range}(g) = \{z \in \mathbb{R} : z = f(x, y) \text{ for some } (x, y) \in \operatorname{dom}(f)\} =$

$[0,3]$ as $x^2+y^2 \leq 9 \implies 9-x^2-y^2 \geq 0 \implies \sqrt{9-x^2-y^2} \geq 0$ and $x^2+y^2 \geq 0 \implies 9-x^2-y^2 \leq 9 \implies \sqrt{9-x^2-y^2} \leq 3$. The graph is as follows:



Example 2.3: Consider $f(x,y) = \dfrac{(x^2+3y^2-9)(xy-1)}{x}$. For what $(x,y) \in \mathbb{R}^2$ is $f(x,y)$ zero, undefined, positive, and negative? Illustrate these points.

Since f is a quotient,

$$\operatorname{dom}(f) = \operatorname{dom}(x^2+3y^2-9) \cap \operatorname{dom}(xy-1) \cap \operatorname{dom}(x) \cap \left\{(x,y) \in \mathbb{R}^2 : x \neq 0\right\}$$

Note that x^2+3y^2-9, $xy-1$, and x are polynomials in two variables, so we get

$$\operatorname{dom}(f) = \mathbb{R}^2 \cap \mathbb{R}^2 \cap \mathbb{R}^2 \cap \left\{(x,y) \in \mathbb{R}^2 : x \neq 0\right\} = \left\{(x,y) \in \mathbb{R}^2 : x \neq 0\right\}$$

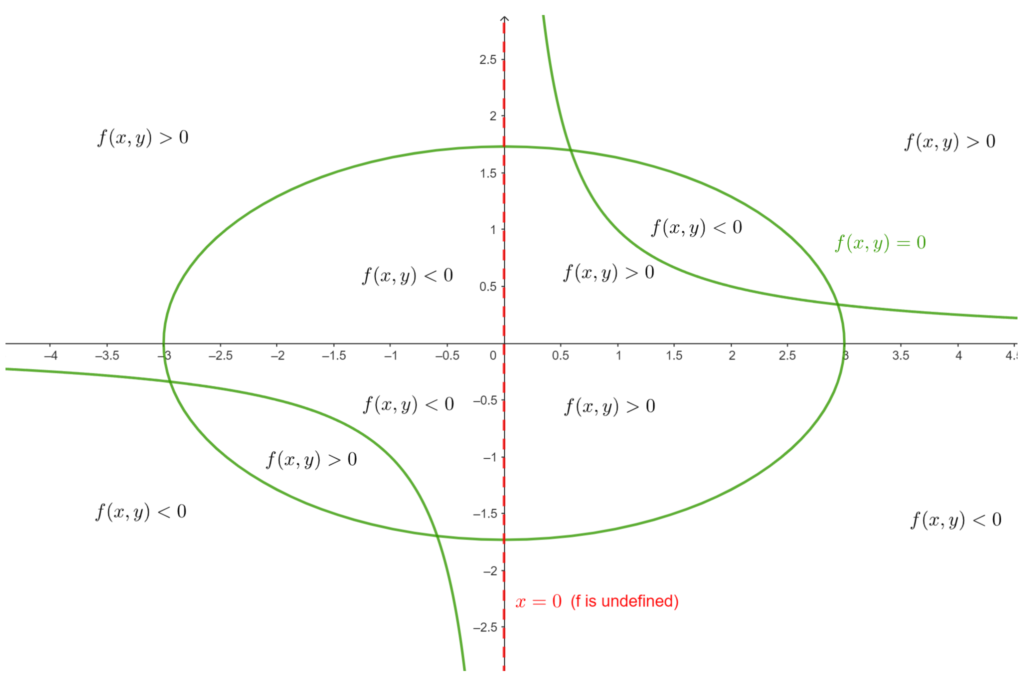
Thus, f is undefined for $\left\{(0,y) : y \in \mathbb{R}\right\}$ (i.e. the y-axis). Now observe that

$$f(x,y) = 0 \iff (x^2+3y^2-9)(xy-1) = 0 \iff x^2+3y^2-9 = 0 \vee xy-1 = 0$$

Here, $xy-1=0 \iff y = \dfrac{1}{x}$ and $x^2+3y^2-9=0 \iff \dfrac{x^2}{9}+\dfrac{y^2}{3} = 1 \iff \left(\dfrac{x}{3}\right)^2 + \left(\dfrac{y}{\sqrt{3}}\right)^2 = 1$. Thus,

$$f(x,y) = 0 \iff \left(\frac{x}{3}\right)^2 + \left(\frac{y}{\sqrt{3}}\right)^2 = 1 \vee y = \frac{1}{x}$$

We determine where f is positive and negative by substituting appropriate sample points.



Observe that

$$f(-4,2) = \frac{117}{4} > 0 \quad f(-1,1) = -10 < 0 \quad f\left(1, \frac{1}{2}\right) = \frac{13}{4} > 0 \quad f(2,1) = -1 < 0 \quad f(4,2) = \frac{91}{4} > 0$$

$$f(-4,-2) = -\frac{7}{4} < 0 \quad f(-2,-1) = 4 > 0 \quad f\left(-1, -\frac{1}{2}\right) = -\frac{19}{4} < 0 \quad f(1,-1) = 22 > 0 \quad f(4,-1) = -5 < 0$$

Remark: The intervals where a single-variable function is positive and negative can be determined by substituting sample points from each interval in the partition of $\mathbb{R}$ formed by the function's roots and undefined points. The same information can be obtained for functions in two variables using sample points from each region in the partition of $\mathbb{R}^2$ formed by the function's roots and undefined points.

2.2 Graphs of Two-Variable Functions

Definition 2.4 — Graph of a Two-Variable Function (p. 937)

If $z = f(x,y)$ has domain D, then the *graph* (or *surface*) of f is

$$\Big\{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D \wedge z = f(x,y)\Big\} \subseteq \mathbb{R}^3$$

Example 2.5: Sketch the graph in $\mathbb{R}^3$ given by:

(a) $z + x + y - 1 = 0$

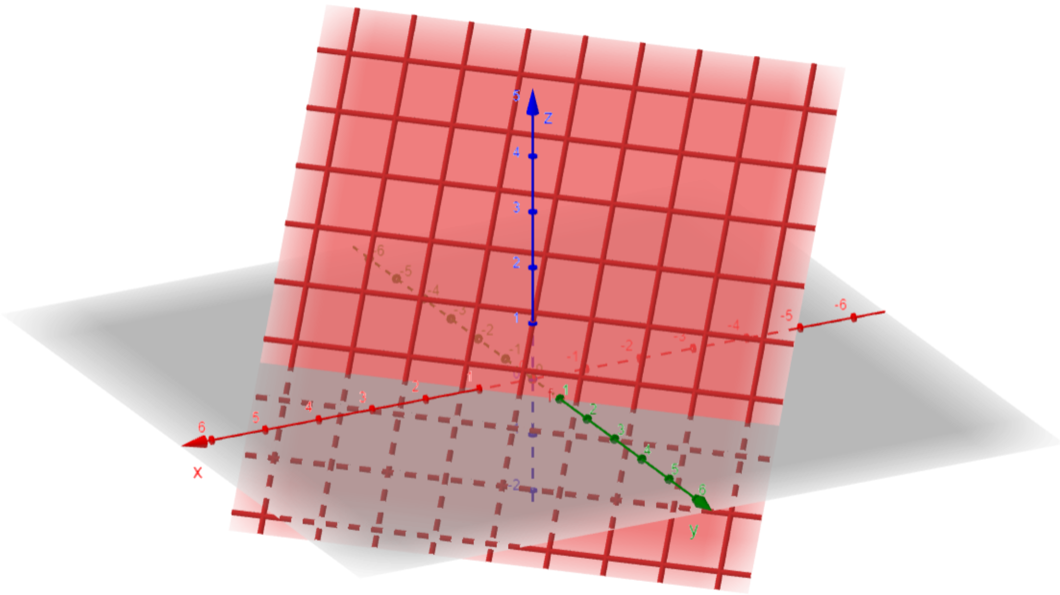
(b) $x^2 + y^2 - 4x + 2y + z^2 = 4$

(a) The given function is a plane. Note that $z + x + y - 1 = 0 \iff z = -x - y + 1$. Thus,

$$x = 0 \wedge y = 0 \implies z = 1 \implies P = (0,0,1) \in \operatorname{graph}(f)$$

$$x = 0 \wedge z = 0 \implies y = 1 \implies Q = (0,1,0) \in \operatorname{graph}(f)$$

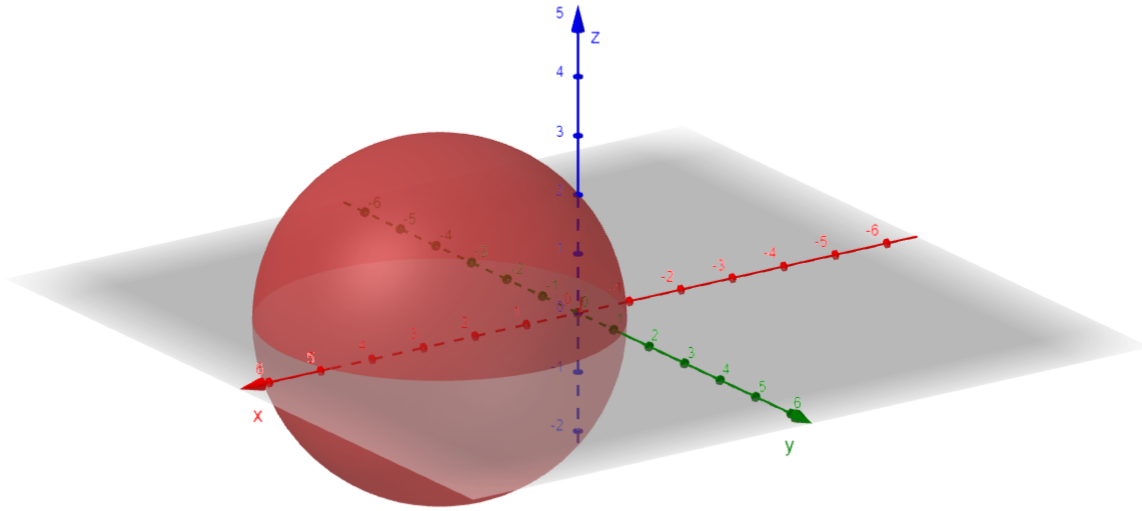
$$y = 0 \wedge z = 0 \implies x = 1 \implies R = (1,0,0) \in \operatorname{graph}(f)$$



(b) Observe that

$$x^2 + y^2 - 4x + 2y + z^2 = 4 \iff (x^2 - 4x + 4) + (y^2 + 2y + 1) + z^2 = 4 + 4 + 1 \iff (x-2)^2 + (y+1)^2 + z^2 = 9$$

which is the equation of a sphere with radius 3, centered at $(2, -1, 0)$. The graph is as follows:



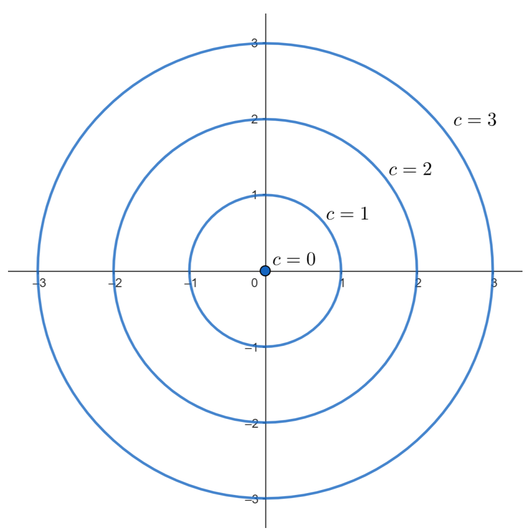
Definition 2.6 — Level Curve and Contour Map (p. 939)

Let $z = f(x, y)$ and $c \in \text{range}(f)$. The *level curve* (or *contour*) of f (for c) is the set of points $(x, y) \in \mathbb{R}^2$ that satisfy

$$f(x, y) = c$$

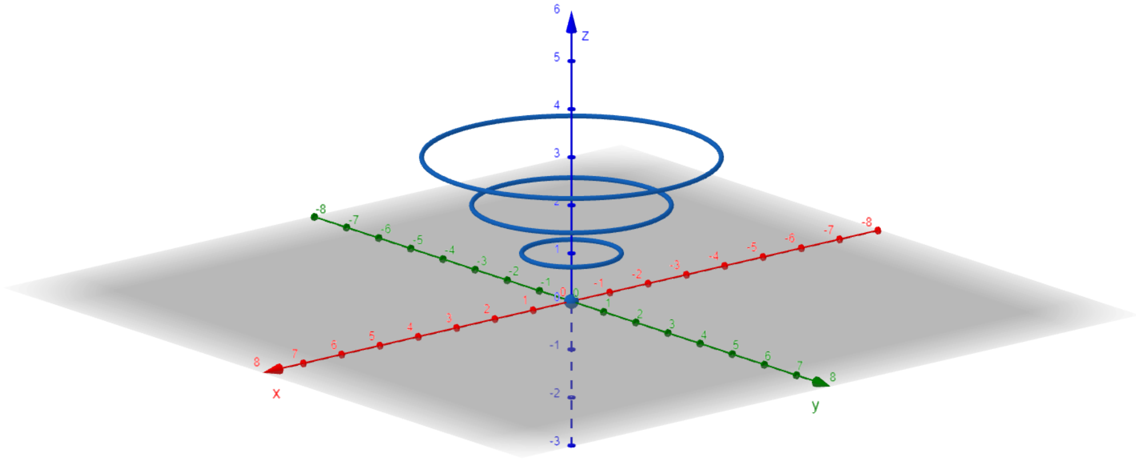
A collection of level curves is called a *contour diagram/map*.

Example 2.7: Consider the contour diagram

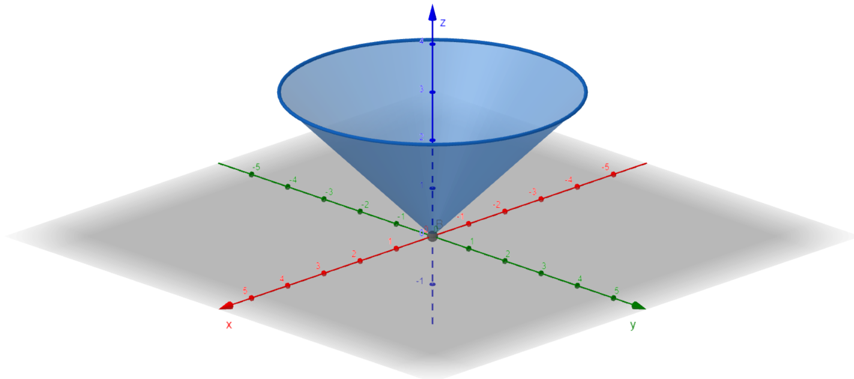


Provide a sketch of $z = f(x, y)$. Repeat this exercise for the same contour diagram, but with c values $c = 0$, $c = -1$, $c = -2$, and $c = -3$ in that order from the inner to outermost circle.

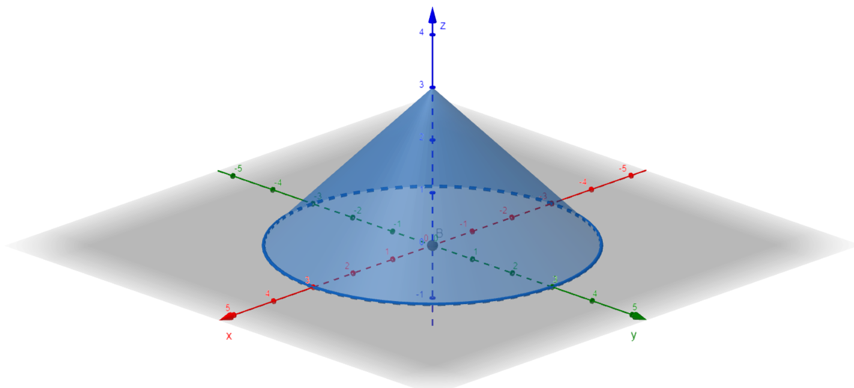
The given contours yield the graph



When the contours are interpolated, we obtain the following cone:



When the c values are reversed, we obtain the following cone:



Example 2.8: Draw the contour diagram of the graph for

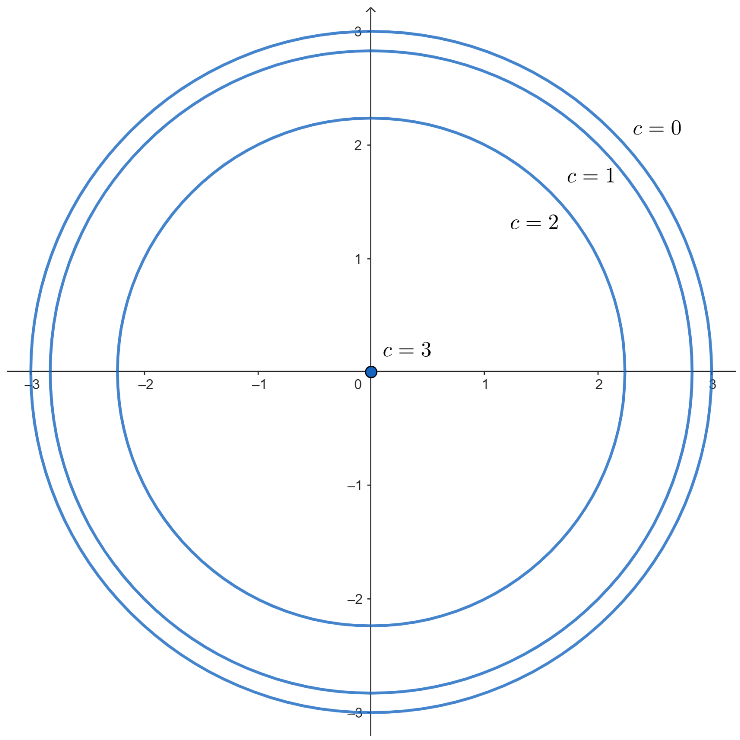
(a) $f(x,y) = \sqrt{9 - x^2 - y^2}$

(b) $z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$ for $a^2 \geq b^2 > 0$.

(a) From example 2.2, $\text{dom}(f) = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\right\}$, so $\text{range}(f) = [0,3]$ (as $0 \leq x^2 + y^2 \leq 9 \implies \sqrt{x^2+y^2} \leq 3$). For $c = 0$, we have

$$\sqrt{9 - x^2 - y^2} = 0 \iff 9 - x^2 - y^2 = 0 \iff x^2 + y^2 = 9$$

Similarly, for $c = 1$, $c = 2$, and $c = 3$, we have $f(x,y) = 1 \iff x^2 + y^2 = 8$, $f(x,y) = 2 \iff x^2 + y^2 = 5$, and $f(x,y) = 3 \iff x^2 + y^2 = 0$ (respectively). This gives us



(b) Let $c \in \mathbb{R}^{\geq 0}$. We have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = c \iff b^2 \cdot x^2 + a^2 \cdot y^2 = a^2 b^2 \cdot c \iff (bx)^2 + (ay)^2 = (ab)^2 c$$

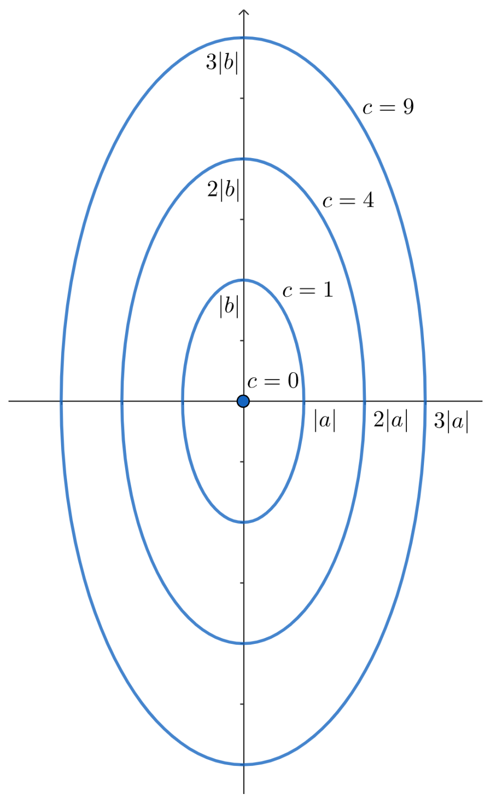
This is the equation of an ellipse centered at the origin. Setting $y = 0$ and solving for $|x|$ yields its radius along the x-axis, as follows:

$$(bx)^2 = (ab)^2 c - (ay)^2 = (ab)^2 c - (a \cdot 0)^2 = (ab)^2 c \implies x^2 = a^2 c \implies |x| = |a| \sqrt{c}$$

Similarly, setting $x = 0$ and solving for $|y|$ yields the ellipse's radius along the y-axis, as follows:

$$(ay)^2 = (ab)^2 c - (bx)^2 = (ab)^2 c - (b \cdot 0)^2 = (ab)^2 c \implies y^2 = b^2 c \implies |y| = |b| \sqrt{c}$$

Note that $c \geq 0$ in the previous calculations, so $\sqrt{c}$ is defined. For $c = 0$, $c = 1$, $c = 4$, and $c = 9$, we thus have ellipses centered at the origin with x and y-axis radii 0 and 0, $|a|$ and $|b|$, $2|a|$ and $2|b|$, and $3|a|$ and $3|b|$. This gives us the contour diagram



2.3 Multivariable Functions

Definition 2.9 — Multivariable Function (p. 945)

Let $n \in \mathbb{Z}^+$ and $D \subseteq \mathbb{R}^n$. A *function of n variables* is a rule which assigns to each point $(x-1, \ldots, x_n) \in D$ exactly one real number $z = f(x_1, \ldots, x_n)$.

1. $D = \Big\{(x_1, \ldots, x_n) \in \mathbb{R}^n : z = f(x_1, \ldots, x_n)\Big\}$ is the *domain* of f
2. $\Big\{z \in \mathbb{R} : z = f(x_1, \ldots, x_n) \text{ for some } (x_1, \ldots, x_n) \in D\Big\}$ is the *range* (or *codomain*) of f
3. The set of points $\Big\{\big(x_1, \ldots, x_n, f(x_1, \ldots, x_n)\big) \in \mathbb{R}^{n+1} : (x_1, \ldots, x_n) \in D\Big\}$ is the *graph* of f

Example 2.10: What the domain of $f(x, y, z) = \ln(z - y) + xy\sin(z)$?

Logarithms are defined on $\mathbb{R}^+$, so we need $z - y > 0 \iff z > y$. xy is defined for all $x, y \in \mathbb{R}$, while $\sin(z)$ is defined for $-1 \leq z \leq 1$. Therefore,

$$\operatorname{dom}(f) = \Big\{(x, y, z) \in \mathbb{R}^3 : y < z \land -1 \leq z \leq 1\Big\}$$

2.4 Limits (§14.2)

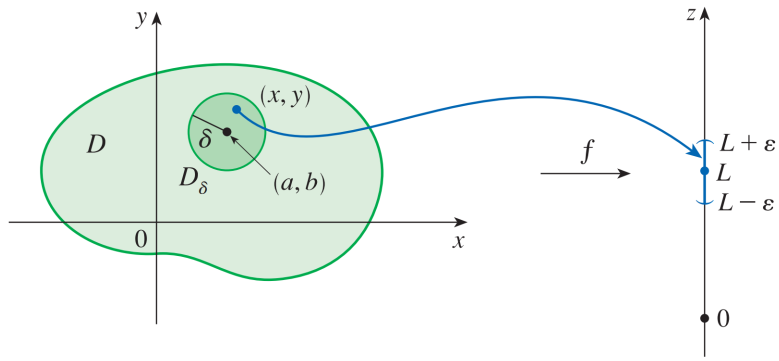
Let $a \in \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$. Recall that for single-variable functions, we say that the "limit of f as x approaches a", denoted by

$$\lim_{x \to a} f(x) = L$$

if and only if for all $\epsilon > 0$, there exists some $\delta > 0$ such that

$$0 < |x - a| < \delta \implies \big|f(x) - L\big| < \epsilon$$

where $x \in \operatorname{dom}(f)$ and $L \in \mathbb{R}$.



We now establish an analogous concept for multivariable functions.

Definition 2.11 — Two-Variable Function Limit (p. 952)

Let $(a, b) \in \mathbb{R}^2$ and $z = f(x, y)$. We say that the "limit of f as (x, y) approaches (a, b)", denoted by

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if and only if for all $\epsilon > 0$, there exists some $\delta > 0$ such that

$$0 < \big\|(x, y) - (a, b)\big\| < \delta \implies \big|f(x, y) - L\big| < \epsilon$$

where $x \in D = \text{dom}(f)$ and $L \in \mathbb{R}$.

Remark: In definition 2.11, the *Euclidean norm* is used, which is given by

$$\big\|(x, y) - (a, b)\big\| = \sqrt{(x - a)^2 + (y - b)^2}$$

Example 2.12: Prove that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2 + y^2}$ does not exist.

Let $f(x, y)$ be the given function and $D = \text{dom}(f)$. We want to choose a curve/path $C_1 \subseteq D$ for which $\displaystyle\lim_{\substack{(x,y)\to(a,b)\\(x,y)\in C_1}} f(x, y) = C_1$ and another path $C_2 \subseteq D$ for which $\displaystyle\lim_{\substack{(x,y)\to(a,b)\\(x,y)\in C_2}} f(x, y) = L_2$ such that $L_1 \neq L_2$.

Along the x-axis, $y = 0$, so we get

$$\lim_{(x,0)\to(0,0)} \frac{x^2}{x^2 + 0^2} = \lim_{x\to 0} \frac{x^2}{x^2} = \lim_{x\to 0} 1 = 1$$

for $x \neq 0$. Along the y-axis, $x = 0$, which gives us

$$\lim_{(0,y)\to(0,0)} \frac{0^2}{0^2 + y^2} = \lim_{y\to 0} 0 = 0$$

Since $0 \neq 1$, the limit depends on the path. Therefore, the limit does not exist.

Example 2.13: Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$ does not exist.

Along the x-axis, $y = 0$, so we get

$$\lim_{(x,0)\to(0,0)} \frac{x^2 \cdot 0}{x^4 + 0^2} = \lim_{x\to 0} 0 = 0$$

Along the curve $y = x^2$, we get

$$\lim_{(x,x^2)\to(0,0)} \frac{x^2 \cdot x^2}{x^4 + (x^2)^2} = \lim_{x\to 0} \frac{x^4}{2x^4} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

for $x \neq 0$. Since $0 \neq \frac{1}{2}$, the limit does not exist.

Example 2.14: Compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{-(x^2+y^2)}-1}{x^2+y^2}$, if it exists.

Let $x = r\cos\theta$ and $y = r\sin\theta$ for $r \in \mathbb{R}^{\geq 0}$ and $\theta \in [0, 2\pi)$. As $(x,y) \to (0,0)$,

$$r^2 = x^2 + y^2 \to 0 \iff r = \pm\sqrt{x^2+y^2} \to 0$$

Since $r \geq 0$, $r \to 0^+$. Thus, by L'Hopital's Rule (for $\frac{0}{0}$ indeterminate forms), the limit becomes

$$\lim_{r\to 0^+} \frac{e^{-r^2}-1}{r^2} = \lim_{r\to 0^+} \frac{e^{-r^2}\cdot(-2r)}{2r} = \lim_{r\to 0^+} -e^{-r^2} = -e^0 = -1$$

Example 2.15 *(One-Variable ϵ-δ)*: Let $a, b, c \in \mathbb{R}$. Prove, using ϵ-δ, that $\lim_{x\to c} f(x)$ exists for $f(x) = ax + b$.

We want to show that for some $L \in \mathbb{R}$, for all $\epsilon > 0$, there exists some $\delta > 0$ such that $0 < |x - c| < \delta \implies \big|f(x) - L\big| < \epsilon$. Let $L = ac + b \in \mathbb{R}$ and $\epsilon > 0$. Choose $\delta = \dfrac{\epsilon}{|a|+1} > 0$, and suppose that $0 < |x - c| < \delta$. We want to show that $\big|f(x) - (ac + b)\big| < \epsilon$. Now

$$\begin{aligned}
\big|f(x) - (ac+b)\big| &= \big|(ax+b) - (ac+b)\big| \\
&= \big|a(x-c)\big| \\
&= |a||x-c| && \text{(by the absolute value property } |AB| = |A||B|\text{)} \\
&< |a|\,\delta \\
&= |a| \cdot \frac{\epsilon}{|a|+1} \\
&< \frac{|a|+1}{|a|+1}\cdot\epsilon && \text{(by increasing the numerator)} \\
&= \epsilon
\end{aligned}$$

as desired.

Remark: The choice of $\delta = \dfrac{\epsilon}{|a|+1}$ in example 2.15 instead of $\delta = \dfrac{\epsilon}{|a|}$ was made to handle the case where $|A| = 0$, while still allowing us to use the same reasoning as with $\dfrac{\epsilon}{|a|}$ with the inequalities.

Example 2.16 *(Two-Variable ϵ-δ)*: Let $(a, b) \in \mathbb{R}^2$ and $A, B \in \mathbb{R}$. Prove, by definition (i.e. using ϵ-δ), that $\displaystyle\lim_{(x,y)\to(a,b)} f(x, y) = Aa + Bb$ for $f(x, y) = Ax + By$.

We want to show that for all $\epsilon > 0$, there exists some $\delta > 0$ such that $0 < \big\|(x, y) - (a, b)\big\| < \delta \implies \big|f(x, y) - (Aa + Bb)\big| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = \dfrac{\epsilon}{|A|+|B|+1} > 0$, and suppose that $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$. We have

$$\begin{aligned}
&\big|f(x,y) - (Aa+Bb)\big| \\
&= \big|(Ax+By) - (Aa+Bb)\big| \\
&= \big|A(x-a) + B(y-b)\big| \\
&\leq \big|A(x-a)\big| + \big|B(y-b)\big| && \text{(by the triangle inequality)} \\
&= |A||x-a| + |B||y-b| && \text{(by the absolute value multiplicative property)} \\
&= |A|\sqrt{(x-a)^2} + |B|\sqrt{(y-b)^2} && \text{(since } |\cdot| = \sqrt{\cdot^2}\text{)} \\
&\leq |A|\sqrt{(x-a)^2+(y-b)^2} + |B|\sqrt{(y-b)^2+(x-a)^2} && \text{(as the square root function is increasing and } \cdot^2 \geq 0\text{)} \\
&= \big(|A|+|B|\big)\sqrt{(x-a)^2+(y-b)^2}
\end{aligned}$$

$$
\begin{aligned}
&< \big(|A| + |B|\big)\delta \\
&= \big(|A| + |B|\big) \cdot \frac{\epsilon}{|A| + |B| + 1} \\
&< \frac{|A| + |B| + 1}{|A| + |B| + 1} \cdot \epsilon \\
&= \epsilon
\end{aligned}
$$

as desired.

Example 2.17: Prove by definition that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+y^2+1} = 0$.

Let $\epsilon > 0$ be arbitrary. Choose $\delta = \dfrac{\epsilon}{2} > 0$, and suppose that $0 < \sqrt{x^2+y^2} < \delta$. We want to show that $\left|\dfrac{x+y}{x^2+y^2+1} - 0\right| < \epsilon$. Now

$$
\begin{aligned}
\left|\frac{x+y}{x^2+y^2+1} - 0\right| &= \left|\frac{x+y}{x^2+y^2+1}\right| \\
&= \frac{|x+y|}{|x^2+y^2+1|} && \text{(by the absolute value property } \left|\tfrac{A}{B}\right| = \tfrac{|A|}{|B|}\text{)} \\
&= \frac{|x+y|}{x^2+y^2+1} && \text{(since } x^2+y^2+1 > 0 \text{ for all } (x,y) \in \mathbb{R}^2\text{)} \\
&\le |x+y| && \text{(as } x^2+y^2+1 \ge 1 \text{ for all } (x,y) \in \mathbb{R}^2\text{)} \\
&\le |x| + |y| && \text{(by the triangle inequality)} \\
&= \sqrt{x^2} + \sqrt{y^2} && \text{(since } |\cdot| = \sqrt{\cdot^2}\text{)} \\
&\le \sqrt{x^2+y^2} + \sqrt{y^2+x^2} && \text{(as } \cdot^2 \ge 0 \text{ and the square root function is increasing)} \\
&= 2\sqrt{x^2+y^2} \\
&< 2\delta \\
&= 2 \cdot \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}
$$

as desired.

Definition 2.18 — n-Variable Function Limit (p. 959)

Let $n \in \mathbb{Z}^+$, $\mathbf{a} \in \mathbb{R}^n$, and $f(\mathbf{x})$ be an n-variable function with domain $D \subseteq \mathbb{R}^n$. We say that

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$$

if for all $\epsilon > 0$, there exists some $\delta > 0$ such that

$$0 < \|\mathbf{x} - \mathbf{a}\| < \delta \implies \big|f(x) - L\big| < \epsilon$$

where $\mathbf{x} \in D$ and $L \in \mathbb{R}$.

Theorem 2.19 — Limit Laws (p. 955)

Let $(a,b) \in \mathbb{R}^2$ and $f(x,y)$ and $g(x,y)$ be defined for all $(x,y) \neq (a,b)$ in a neighbourhood (a disk) around (a,b).

If $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = L_1 \in \mathbb{R}$ and $\displaystyle\lim_{(x,y)\to(a,b)} g(x,y) = L_2 \in \mathbb{R}$ exist then

1. $\displaystyle\lim_{(x,y)\to(a,b)} \big(f(x,y) \pm g(x,y)\big) = L_1 \pm L_2$
2. $\displaystyle\lim_{(x,y)\to(a,b)} cf(x,y) = cL_1$ for all $c \in \mathbb{R}$
3. $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y)g(x,y) = L_1 L_2$
4. $\displaystyle\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L_1}{L_2}$ (provided that $L_2 \neq 0$)

Proof: Let $c \in \mathbb{R}$. Suppose that $f \to L_1$ and $g \to L_2$ as $(x,y) \to (a,b)$. To prove (1) and (2), we will show that $\displaystyle\lim_{(x,y)\to(a,b)} \big(f(x,y) + cg(x,y)\big) = L_1 + cL_2$ using the ϵ-δ definition. We want to show that for all $\epsilon > 0$, there exists some $\delta > 0$ such that

$$0 < \big\|(x,y) - (a,b)\big\| < \delta \implies \Big|\big(f(x,y) + cg(x,y)\big) - (L_1 + cL_2)\Big| < \epsilon$$

Let $\epsilon > 0$ be arbitrary. Since $f \to L_1$ as $(x,y) \to (a,b)$, there exists some $\delta_1 > 0$ such that

$$0 < \big\|(x,y) - (a,b)\big\| < \delta_1 \implies \big|f(x,y) - L\big| < \frac{\epsilon}{2}$$

Similarly, since $g \to L_2$ as $(x,y) \to (a,b)$, there exists some $\delta_2 > 0$ such that

$$0 < \big\|(x,y) - (a,b)\big\| < \delta_2 \implies \big|g(x,y) - L\big| < \frac{\epsilon}{2\big(|c| + 1\big)}$$

Choose $\delta = \min(\delta_1, \delta_2) > 0$, and suppose that $0 < \big\|(x,y) - (a,b)\big\| < \delta$. We want to show that $\big|f(x,y) + cg(x,y) - (L_1 + cL_2)\big| < \epsilon$. Now

$$\begin{aligned}
\big|f(x,y) + cg(x,y) - (L_1 + cL_2)\big| &= \Big|\big(f(x,y) - L_1\big) + c\big(g(x,y) - L_2\big)\Big| \\
&\leq \big|f(x,y) - L_1\big| + \Big|c\big(g(x,y) - L_2\big)\Big| && \text{(by the triangle inequality)} \\
&= \big|f(x,y) - L_1\big| + |c|\big|g(x,y) - L_2\big| && \text{(by an absolute value property)} \\
&< \frac{\epsilon}{2} + |c| \cdot \frac{\epsilon}{|c| + 1} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} && \left(\text{as } \tfrac{|c|}{|c|+1} < 1\right) \\
&= \epsilon
\end{aligned}$$

■

Example 2.20: Compute $\displaystyle\lim_{(x,y)\to(2,-1)} \frac{2x + 3y}{4x - 3y}$, if it exists.

Using limit laws, we obtain

$$\frac{\lim_{(x,y)\to(2,-1)}(2x + 3y)}{\lim_{(x,y)\to(2,-1)}(4x - 3y)} = \frac{2\lim_{(x,y)\to(2,-1)} x + 3\lim_{(x,y)\to(2,-1)} y}{4\lim_{(x,y)\to(2,-1)} x - 3\lim_{(x,y)\to(2,-1)} y} = \frac{1}{11}$$

Definition 2.21 — Two-Variable Rational Function

A *rational function of two variables* is a function of the form

$$\frac{p(x,y)}{q(x,y)}$$

where $p(x,y)$ and $q(x,y)$ are two-variable polynomials with $q(x,y) \neq 0$.

2.5 Continuity (§14.2)

Definition 2.22 — Two-Variable Function Continuity (p. 957)

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a function with domain $D \subseteq \mathbb{R}^2$ and $(a,b) \in \mathbb{R}^2$. f is *continuous at* (a,b) if and only if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

That is, for all $\epsilon > 0$, there exists some $\delta > 0$ such that

$$\left\|(x,y)-(a,b)\right\| < \delta \implies \left|f(x,y)-f(a,b)\right| < \epsilon$$

f is *continuous on* D if and only if f is continuous at (x,y) for all $(x,y) \in D$.

Example 2.23: Prove that $f(x,y) = \begin{cases} \dfrac{2x+3y}{4x-3y} & \text{if } (x,y) \neq (2,-1) \\ 0 & \text{if } (x,y) = (2,-1) \end{cases}$ is discontinuous at $(2,-1)$.

From example 2.20, we have

$$\lim_{(x,y)\to(2,-1)} f(x,y) = \lim_{(x,y)\to(2,-1)} \frac{2x+3y}{4x-3y} = \frac{1}{11}$$

so the limit exists. By the definition of f, $f(2,-1) = 0$, so f is defined at $(2,-1)$. Since $\frac{1}{11} \neq 0$ (i.e. $\lim\limits_{(x,y)\to(2,-1)} f(x,y) \neq f(2,-1)$), f is discontinuous at $(2,-1)$.

Theorem 2.24 — Continuity Properties (p. 957)

Let $f, g\colon \mathbb{R}^2 \to \mathbb{R}$ and $(a,b) \in \operatorname{dom}(f) \cap \operatorname{dom}(g)$. If f and g are continuous at (a,b), then each of the following functions are continuous at (a,b):

1. $f \pm g$
2. cf for all $c \in \mathbb{R}$
3. fg
4. $\dfrac{f}{g}$ (provided that $g(a,b) \neq 0$)

Theorem 2.25 — Continuity Composition (p. 958)

Let g be a two-variable function with domain $D \subseteq \mathbb{R}^2$ and range $R \subseteq \mathbb{R}$, and f be a a single-variable function. Suppose that $(a,b) \in D$ and $z = g(a,b)$. If

- g is continuous at (a,b) and
- f is continuous at $z \in \operatorname{dom}(f)$

then the two-variable composition $f \circ g$ is continuous at (a,b).

Example 2.26: Determine the set of points at which the function $H(x,y) = \dfrac{xy}{1+e^{x-y}}$ is continuous. Justify your answer.

H is a quotient.

- The numerator xy is a polynomial and thus continuous on its domain $\mathbb{R}^2$
- The denominator $1 + e^{x-y}$ is the sum of
  - 1, which is a polynomial and thus continuous on its domain $\mathbb{R}^2$
  - e^{x-y}, a composition of the polynomial $x - y$, which is continuous on its domain $\mathbb{R}^2$, and the single-variable function e^t that is continuous on its domain $\mathbb{R}$. Thus, e^{x-y} is continuous on $\mathbb{R}^2$

- $1 + e^{x-y} \neq 0$ for all $(x, y) \in \mathbb{R}^2$ since $e^t > 0$ for all $t \in \mathbb{R}$

Therefore, H is continuous on the "common points of continuity", namely

$$\mathbb{R}^2 \cap \mathbb{R}^2 - \left\{(x, y) \in \mathbb{R}^2 : 1 + e^{x-y} = 0\right\} = \mathbb{R}^2 \cap \mathbb{R}^2 - \emptyset = \mathbb{R}^2$$

Remark: In example 2.26, the fact that polynomials of two variables are continuous on their domains ($\mathbb{R}^2$) was proven in Q7 (c) of extra exercises 3.

Example 2.27 *(§14.2 Q43)*: Determine the set of points for which $f(x, y) = \dfrac{1 + x^2 + y^2}{1 - x^2 - y^2}$ is discontinuous. Justify your answer.

f is a quotient whose numerator and denominator are both polynomials and thus continuous on $\mathbb{R}^2$. Therefore, f is only discontinuous where its denominator is zero; that is,

$$1 - x^2 - y^2 = 0 \iff x^2 + y^2 = 1$$

so f is discontinuous on $\left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\right\}$ (all points on the unit circle).

Example 2.28: Find $\lim\limits_{(x,y)\to(-2,2)} e^{-xy} \cos(x + y)$, if it exists.

Using limit laws, we get

$$\left(\lim_{(x,y)\to(-2,2)} e^{-xy}\right)\left(\lim_{(x,y)\to(-2,2)} \cos(x + y)\right)$$

e^{-xy} is a composition of the polynomial $-xy$ (which is continuous on its domain $\mathbb{R}^2$) and the single-variable exponential e^t (which is continuous on its domain $\mathbb{R}$). Thus, e^{-xy} is continuous on $\mathbb{R}^2$. In particular, it is continuous at $(-2, 2) \in \mathbb{R}^2$.

$\cos(x + y)$ is a composition of the polynomial $x + y$ (which is continuous on its domain $\mathbb{R}^2$) and the single-variable function $\cos(t)$ (which is continuous on its domain $\mathbb{R}$). Therefore, $\cos(x + y)$ is continuous on $\mathbb{R}^2$ and thus continuous at $(-2, 2) \in \mathbb{R}^2$.

By continuity at $(-2, 2)$, the limit becomes

$$e^{-(-2)\cdot 2} \cdot \cos(-2 + 2) = e^4 \cdot \cos(0) = e^4$$

Chapter 3 Differentiation

3.1 Partial Derivatives (§14.3)

Recall that for a single-variable function $f\colon \mathbb{R} \to \mathbb{R}$, the derivative is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We now establish a similar concept for multivariable functions.

Definition 3.1 — Two-Variable Partial Derivatives

Let $z = f(x, y)$. The *partial derivative of f with respect to x*, denoted by

$$f_x \text{ or } \frac{\partial f}{\partial x} \text{ or } \frac{\partial z}{\partial x} \text{ or } \mathrm{D}_1 f$$

is

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

Similarly, the *partial derivative of f with respect to y*, denoted by

$$f_y \text{ or } \frac{\partial f}{\partial y} \text{ or } \frac{\partial z}{\partial y} \text{ or } \mathrm{D}_2 f$$

is

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

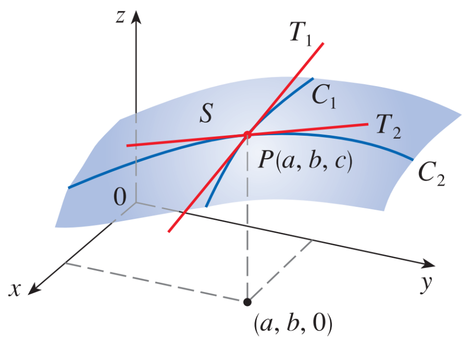
Remark: In the context of definition 3.1, suppose that $c = f(a, b)$ (for some $(a, b) \in \mathrm{dom}(f)$),

$$C_1 = \Big\{ \big(x, b, f(x, b)\big) : (x, b) \in \mathrm{dom}(f) \Big\}$$

and

$$C_2 = \Big\{ \big(a, y, f(a, y)\big) : (a, y) \in \mathrm{dom}(f) \Big\}$$

Let T_1 and T_2 be the tangents to C_1 and C_2 (respectively) at (a, b). The slopes of T_1 and T_2 are $f_x(a, b)$ and $f_y(a, b)$ (respectively), depicted in the following graph:



Example 3.2: Compute $f_x(0, 1)$ and $f_y(0, 1)$ by definition for $f(x, y) = x^2 - 3xy$.

By definition, $f_x(x, y) = \lim\limits_{h \to 0} \dfrac{f(x+h, y) - f(x, y)}{h}$, so

$$\begin{aligned} f_x(0, 1) &= \lim_{h \to 0} \frac{f(0+h, 1) - f(0, 1)}{h} \\ &= \lim_{h \to 0} \frac{(h^2 - 3h \cdot 1) - (0^2 - 3 \cdot 0 \cdot 1)}{h} && \text{(by the definition of } f\text{)} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{h^2 - 3h}{h} \\
&= \lim_{h\to 0} \frac{h(h-3)}{h} \\
&= \lim_{h\to 0} (h-3) && \text{(as } h \neq 0\text{)} \\
&= -3
\end{aligned}
$$

Also, $f_y(x, y) = \lim_{h\to 0} \dfrac{f(x, y+h) - f(x, y)}{h}$, so

$$
\begin{aligned}
f_y(0,1) &= \lim_{h\to 0} \frac{f(0, 1+h) - f(0,1)}{h} \\
&= \lim_{h\to 0} \frac{\big(0^2 - 3\cdot 0\cdot(1+h)\big) - 3\cdot 0\cdot(1+h)}{h} \\
&= \lim_{h\to 0} \frac{0}{1+h} \\
&= \lim_{h\to 0} 0 \\
&= 0
\end{aligned}
$$

Example 3.3: Let $f(x, y) = xy^3 - x^2\sqrt{y}$ for $y \neq 0$. Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Treating y as a constant, we get

$$f_x(x, y) = y^3 - 2x\sqrt{y}$$

Similarly, treating x as a constant yields

$$f_y(x, y) = x \cdot 3y^2 - x^2 \cdot \frac{1}{2} y^{-1/2} = 3xy^2 - \frac{x^2}{2\sqrt{y}}$$

Example 3.4: Let $\dfrac{x^2}{2} + \dfrac{y^2}{4} + \dfrac{z^2}{3} = 1$. Compute f_x.

Implicitly differentiating with respect to x while treating y as a constant, we get

$$
\begin{aligned}
\frac{2x}{2} + 0 + \frac{2z}{3} \cdot \frac{\partial z}{\partial x} = 0 \quad &\Longleftrightarrow \quad \frac{2}{3} z f_x = -x \\
&\Longleftrightarrow \quad f_x = -\frac{3x}{2z}
\end{aligned}
$$

Definition 3.5 — n-Variable Partial Derivatives (p. 966)

Let $z = f(x_1, \ldots, x_n)$, where $f\colon \mathbb{R}^n \to \mathbb{R}$ is a function. The *partial derivative of f with respect to x_i* is

$$\frac{\partial z}{\partial x_i} = \lim_{h\to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_n)}{h}$$

This is denoted as

$$\frac{\partial z}{\partial x_i} \text{ or } \frac{\partial f}{\partial x_i} \text{ or } \mathrm{D}_i f \text{ or } f_{x_i}$$

Definition 3.6 — Higher Order Partial Derivatives

Let $z = f(x, y)$. The second order partial derivatives of f are

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$
$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right)$$
$$(f_x)_y = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$
$$(f_y)_x = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

Example 3.7: Let $z = f(x, y) = xe^{-3y} + \sin(2x - 5y)$. Compute all second order partial derivatives of f.

Treating y as a constant, we get

$$f_x = e^{-3y} + \cos(2x - 5y) \cdot 2$$

Thus,

$$f_{xx} = \frac{\partial}{\partial x}\Big(e^{-3y} + 2\cos(2x - 5y)\Big) = 0 - 2\sin(2x - 5y) \cdot 2 = -4\sin(2x - 5y)$$

Now treating x as a constant, we get

$$f_{xy} = \frac{\partial}{\partial y}\Big(e^{-3y} + 2\cos(2x - 5y)\Big) = -3 \cdot e^{-3y} - 2\sin(2x - 5y) \cdot (-5) = -3e^{-3y} + 10\sin(2x - 5y)$$

Treating x as a constant yields

$$f_y = -3 \cdot xe^{-3y} + \cos(2x - 5y) \cdot (-5) = -3xe^{-3y} - 5\cos(2x - 5y)$$

Thus,

$$f_{yy} = \frac{\partial}{\partial y}\Big(-3xe^{-3y} - 5\cos(2x - 5y)\Big) = -3 \cdot (-3xe^{-3y}) + 5\sin(2x - 5y) \cdot (-5) = 9e^{-3y} - 25\sin(2x - 5y)$$

Treating y as a constant now yields

$$f_{yx} = \frac{\partial}{\partial x}\Big(-3xe^{-3y} - 5\cos(2x - 5y)\Big) = -3e^{-3y} + 5\sin(2x - 5y) \cdot 2 = -3e^{-3y} + 10\sin(2x - 5y)$$

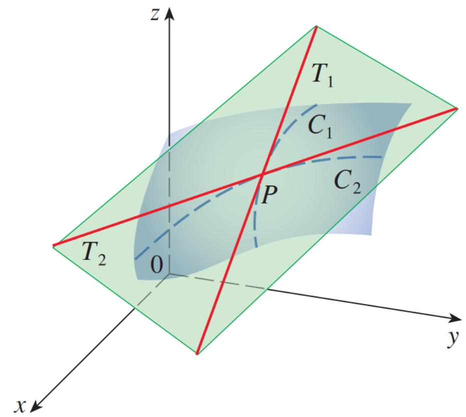
Theorem 3.8 — Clairaut's Theorem

Let $z = f(x, y)$ and $(a, b) \in \operatorname{dom}(f)$. Suppose there exists a disk $D \subseteq \operatorname{dom}(f)$ such that $(a, b) \in D$. If f_{xy} and f_{yx} are continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Remark: The proof of theorem 3.8 can be found in Appendix F of the textbook.

3.2 Tangent Planes (§14.4)



Let (x_0, y_0, z_0) be a point on a surface $z = f(x, y)$, $\mathbf{u} = \langle 0, 1, f_y(x_0, y_0) \rangle$, and $\mathbf{v} = \langle 1, 0, f_x(x_0, y_0) \rangle$. In the above graph, T_1 and T_2 are the tangent lines to f at $P = (x_0, y_0, z_0)$, with position vectors $\mathbf{u}$ and $\mathbf{v}$ (respectively). Also, $C_1 = \left\{ \big(x, y_0, f(x, y_0)\big) : (x, y_0) \in \operatorname{dom}(f) \right\}$ and $C_2 = \left\{ \big(x_0, y, f(x_0, y)\big) : (x_0, y) \in \operatorname{dom}(f) \right\}$.

Note that $\mathbf{u}$ and $\mathbf{v}$ span a plane. It can be shown that $\mathbf{n} = \mathbf{u} \times \mathbf{v} = \langle f_x(x_0, y_0), f_y(x_0, y_0), -1 \rangle$.

Definition 3.9 — Tangent Plane Equation (p. 975)

Let $z = f(x, y)$ and $(x_0, y_0) \in \operatorname{dom}(f)$. Suppose that f_x and f_y are continuous near (x_0, y_0). The *tangent plane to f at (x_0, y_0, z_0)* (where $z_0 = f(x_0, y_0)$) has the equation

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

Remark: Definition 3.9 uses the equation of a plane in $\mathbb{R}^3$, as in theorem 1.8.

Example 3.10: Find the tangent plane to $f(x, y) = x^3 y^{-2}$ at $(1, 1, 1)$.

We have $f_x = 3y^{-2}x^2$ and $f_y = -2x^3y^{-3}$. These are both rational functions, continuous where $y \neq 0$. In particular, they are continuous at $(1, 1)$. We have $f_x(1, 1) = 3$ and $f_y(1, 1) = -2$, so the plane's equation (by definition 3.9) is

$$3(x - 1) - 2(y - 1) - (z - 1) = 0 \iff 3x - 2y - z = 0$$

Example 3.11: Find the point where the tangent plane to $f(x, y) = e^{x-y}$ at $P_0 = (1, 1, 1)$ intersects the z-axis.

We have $f_x = e^{x-y} \cdot 1 = e^{x-y}$ and $f_y = e^{x-y} \cdot (-1) = -e^{x-y}$. These are compositions of the polynomial $x - y$, which is continuous on its domain $\mathbb{R}^2$, and a single-variable exponential $g(t) = \pm e^t$, which is continuous on its domain $\mathbb{R}$. Thus, the compositions are continuous on $\mathbb{R}^2$. In particular, f_x and f_y are continuous at $(1, 1)$. Therefore, the tangent plane at P_0 is

$$e^{1-1}(x - 1) - e^{1-1}(y - 1) - (z - 1) = 0 \iff (x - 1) - (y - 1) - (z - 1) = 0 \iff x - y - z + 1 = 0$$

The tangent plane intersects the z-axis where $x = 0$ and $y = 0$, which yields

$$0 - 0 - z + 1 = 0 \iff z = 1$$

Therefore, the tangent plane intersects the z-axis at $(0, 0, 1)$.

Definition 3.12 — Differentiability

Let $z = f(x, y)$ and $(a, b) \in \mathbb{R}^2$. f is *differentiable at* (a, b) if and only if each of the following conditions hold:

- f_x and f_y both exist at (a, b)
- $\displaystyle\lim_{(x,y)\to(a,b)} \frac{f(x,y) - \big(f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)\big)}{\big\|(x,y)-(a,b)\big\|} = 0$

Remark: In definition 3.12, the term subtracted from $f(x, y)$ in the limit's numerator is the expression

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(y - b)$$

obtained by isolating z in the equation for the tangent plane at (a, b) (as in definition 3.9).

Example 3.13: Let $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$. Is f differentiable at $(0, 0)$?

Consider $f_x(0, 0)$ and $f_y(0, 0)$. By definition,

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{1}{h} \cdot \left(\frac{2h \cdot 0}{h^2 + 0^2} - 0\right) = \lim_{h\to 0} \frac{1}{h} \cdot 0 = \lim_{h\to 0} 0 = 0$$

Also,

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{1}{h} \cdot \left(\frac{2 \cdot 0 \cdot h}{0^2 + h^2} - 0\right) = \lim_{h\to 0} \frac{1}{h} \cdot 0 = \lim_{h\to 0} 0 = 0$$

Now

$$\lim_{(x,y)\to(0,0)} \frac{\dfrac{2xy}{x^2+y^2} - \big(0 + 0\cdot(x-0) + 0\cdot(y-0)\big)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{\dfrac{2xy}{x^2+y^2}}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{2xy}{(x^2+y^2)^{3/2}}$$

Along the path $y = x$ for $x > 0$, the limit becomes

$$\lim_{(x,y)\to(0,0)} \frac{2x^2}{(2x^2)^{3/2}} = \lim_{x\to 0^+} \frac{2x^2}{2^{3/2}x^3} = \lim_{x\to 0^+} \frac{1}{\sqrt{2}x} = \infty$$

so the limit does not exist. By definition, f is not differentiable at $(0, 0)$.

Theorem 3.14 — Major Differentiability Theorems (p. 977)

Let $z = f(x, y)$.

1. If f is differentiable at $(a, b) \in \text{dom}(f)$, then f is continuous at (a, b).
2. If f_x and f_y both exist near (a, b) and f_x and f_y are continuous at (a, b), then f is differentiable at (a, b).

Example 3.15: Show that $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ as in example 3.13 is not differentiable at $(0, 0)$ using (1) in theorem 3.14.

Consider $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y)$. Along $y = x$, we have

$$\lim_{(x,x)\to(0,0)} \frac{2x \cdot x}{x^2 + x^2} = \lim_{x\to 0} \frac{2x^2}{2x^2} = \lim_{x\to 0} 1 = 1$$

Along $y = 0$, we get

$$\lim_{(x,0)\to(0,0)} \frac{2x \cdot 0}{x^2 + 0^2} = \lim_{x\to 0} 0 = 0$$

Since $0 \neq 1$, the limit does not exist, so f is not continuous at $(0,0)$. By the contrapositive of (1) in theorem 3.14, f is not differentiable at $(0,0)$.

Example 3.16: Show that the converses of both parts in theorem 3.14 are false using counterexamples.

Consider $f(x,y) = (x+y)^{1/3}$. This is a composition of the polynomial $x+y$, which is continuous on its domain $\mathbb{R}^2$, and the single-variable cube root function, which is continuous on its domain $\mathbb{R}$. Thus, f is continuous on $\mathbb{R}^2$, including $(0,0)$. Observe that

$$\begin{aligned} f_x(0,0) &= \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} && \text{(by definition)} \\ &= \lim_{h\to 0} \frac{h^{1/3} - 0^{1/3}}{h} \\ &= \lim_{h\to 0} \frac{1}{h^{2/3}} \end{aligned}$$

which does not exist. Thus, $f_x(0,0)$ does not exist, so f is not differentiable at $(0,0)$. Since f is continuous but not differentiable at $(0,0)$, $f(x,y) = (x+y)^{1/3}$ is a counterexample for the converse of (1) in theorem 3.14.

A counterexample for the converse of (2) in theorem 3.14 is

$$f(x,y) = \begin{cases} \left(x^2+y^2\right)\sin\left(\dfrac{1}{\sqrt{x^2+y^2}}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

It can be shown with some lengthy limit computations that this function is differentiable on $\mathbb{R}^2$, though its partial derivatives do not exist at $(0,0)$.

Example 3.17: Let $f(x,y) = x^3 y^{-2}$. Where is f differentiable? Justify your answer.

Treating y as a constant, we have $f_x = 3x^2y^{-2} = \dfrac{3x^2}{y^2}$. Similarly, $f_y = x^2 \cdot (-2y^{-3}) = -\dfrac{2x^3}{y^3}$ by treating x as a constant. f_x and f_y are rational functions and thus continuous on their domains. Now

$$\operatorname{dom}(f_x) = \mathbb{R}^2 - \big\{(x,0) : x \in \mathbb{R}\big\}$$

$$\operatorname{dom}(f_y) = \mathbb{R}^2 - \big\{(x,0) : x \in \mathbb{R}\big\}$$

Since f_x and f_y are rational functions, they are continuous on their domains. Therefore, f is differentiable on $\mathbb{R}^2 - \big\{(x,0) : x \in \mathbb{R}\big\}$.

3.3 Vector-Valued Functions

Definition 3.18 — Vector-Valued Function

Let $m, n \in \mathbb{Z}^+$. A *vector-valued function* $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is an n-variable function whose range is a set of vectors in $\mathbb{R}^m$. That is, if $\underline{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, then

$$f(\underline{x}) = \big\langle f_1(\underline{x}), f_2(\underline{x}), \ldots, f_m(\underline{x}) \big\rangle$$

for some functions $f_1, f_2, \ldots, f_m$.

Example 3.19: One vector-valued function $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ is

$$f(x,y) = \left\langle xy, \frac{x}{y}, x+y \right\rangle$$

Definition 3.20 — Vector-Valued Derivative

Let $m, n \in \mathbb{Z}^+$ and $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a function so that $f(\underline{x}) = \langle f_1(\underline{x}), \ldots, f_m(\underline{x})\rangle$. The *derivative of f at $\underline{a} \in \mathbb{R}^n$* is the $m \times n$ matrix of partial derivatives

$$\mathrm{D}f(\underline{x}) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \dfrac{\partial f_m}{\partial x_2} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix}_{m\times n}$$

Example 3.21: Find $\mathrm{D}f(\underline{x})$ where $\underline{x} = (1,1)$ and $f\colon \mathbb{R}^2 \to \mathbb{R}$ is given by $f(x,y) = x^3y^{-2}$.

From example 3.10, $f_x(1,1) = 3$ and $f_y(1,1) = -2$. Thus,

$$\mathrm{D}f(1,1) = \begin{bmatrix} \left.\dfrac{\partial f}{\partial x}\right|_{(1,1)} & \left.\dfrac{\partial f}{\partial y}\right|_{(1,1)} \end{bmatrix}_{1\times 2} = \begin{bmatrix} 3 & -2 \end{bmatrix}$$

Example 3.22: Let $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ be given by $f(x,y) = \left\langle xy, \dfrac{x}{y}, x+y \right\rangle$. Find $\mathrm{D}f(2,1)$.

We have $f_1(x,y) = xy$, $f_2 = \dfrac{x}{y}$, and $f_3(x,y) = x+y$, where each $f_i\colon \mathbb{R}^2 \to \mathbb{R}$. Thus, for $\underline{x} = (x,y)$,

$$\mathrm{D}f(x,y) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x}(\underline{x}) & \dfrac{\partial f_1}{\partial y}(\underline{x}) \\ \dfrac{\partial f_2}{\partial x}(\underline{x}) & \dfrac{\partial f_2}{\partial y}(\underline{x}) \\ \dfrac{\partial f_3}{\partial x}(\underline{x}) & \dfrac{\partial f_3}{\partial y}(\underline{x}) \end{bmatrix}_{3\times 2} = \begin{bmatrix} y & x \\ \dfrac{1}{y} & -\dfrac{x}{y^2} \\ 1 & 1 \end{bmatrix} \implies \mathrm{D}f(2,1) = \begin{bmatrix} 1 & 2 \\ 1 & -2 \\ 1 & 1 \end{bmatrix}$$

Theorem 3.23 — Differentiation Rules

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ and $g\colon \mathbb{R}^2 \to \mathbb{R}$ be functions and $\underline{x} = (x,y) \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$. If f and g are both differentiable at $\underline{x}$, then each of the following hold:

1. *Sum Rule:* $f+g$ is differentiable at $\underline{x}$ with

$$\mathrm{D}\big(f(\underline{x}) + g(\underline{x})\big) = \mathrm{D}f(\underline{x}) + \mathrm{D}g(\underline{x})$$

2. *Constant Multiple Rule:* For all $c \in \mathbb{R}$, cf is differentiable at $\underline{x}$ with

$$\mathrm{D}\big(cf(\underline{x})\big) = c\mathrm{D}f(\underline{x})$$

3. *Product Rule:* fg is differentiable at $\underline{x}$ with

$$\mathrm{D}\big(f(\underline{x})g(\underline{x})\big) = \mathrm{D}\big(f(\underline{x})\big)g(\underline{x}) + f(\underline{x})\mathrm{D}\big(g(\underline{x})\big)$$

4. *Quotient Rule:* $\dfrac{f}{g}$ is differentiable at $\underline{x}$, provided that $g(\underline{x}) \neq 0$, with

$$\mathrm{D}\left(\frac{f(\underline{x})}{g(\underline{x})}\right) = \frac{\mathrm{D}\big(f(\underline{x})\big)g(\underline{x}) - f(\underline{x})\mathrm{D}\big(g(\underline{x})\big)}{\big(g(\underline{x})\big)^2}$$

Example 3.24: Let $h(x, y, z) = zxe^{xy}$ (so that $h\colon \mathbb{R}^3 \to \mathbb{R}$). Find $\mathrm{D}h(x, y, z) = \mathrm{D}h(\underline{x})$

(a) By the definition of the derivative.

(b) Using the product rule.

(a) By definition,

$$\mathrm{D}h(\underline{x}) = \begin{bmatrix} h_x(\underline{x}) & h_y(\underline{x}) & h_z(\underline{x}) \end{bmatrix}_{1\times 3} = \begin{bmatrix} ze^{xy} + zxye^{xy} & zx^2e^{xy} & xe^{xy} \end{bmatrix}$$

(b) We have $h(\underline{x}) = g(\underline{x})f(\underline{x})$, where $g(\underline{x}) = zx$ ($g\colon \mathbb{R}^3 \to \mathbb{R}$) and $f(\underline{x}) = e^{xy}$ ($f\colon \mathbb{R}^3 \to \mathbb{R}$). Thus,

$$\begin{aligned} \mathrm{D}h(\underline{x}) &= \mathrm{D}\big(g(\underline{x})f(\underline{x})\big) \\ &= \mathrm{D}\big(g(\underline{x})\big)f(\underline{x}) + g(\underline{x})\mathrm{D}\big(f(\underline{x})\big) && \text{(by the product rule)} \\ &= \begin{bmatrix} z & 0 & x \end{bmatrix}_{1\times 3} e^{xy} + zx \begin{bmatrix} ye^{xy} & xe^{xy} & 0 \end{bmatrix}_{1\times 3} \\ &= \begin{bmatrix} ze^{xy} + zxye^{xy} & zx^2e^{xy} & xe^{xy} \end{bmatrix} \end{aligned}$$

3.4 The Chain Rule (§14.5)

Theorem 3.25 — Chain Rule: Case I (p. 985)

Let $z = f(x, y) = f\big(x(t), y(t)\big)$, where $x\colon \mathbb{R} \to \mathbb{R}$ and $y\colon \mathbb{R} \to \mathbb{R}$. If

- f is differentiable at (x, y) and
- $x(t)$ and $y(t)$ are differentiable at t

then z is differentiable and

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y} \cdot \frac{\mathrm{d}y}{\mathrm{d}t}$$

Example 3.26: Find $\frac{\mathrm{d}w}{\mathrm{d}t}$ for $w = \ln(x^2 + y^2 + z^2)$, where $x = \sin(t)$, $y = \cos(t)$, and $z = \tan(t)$,

(a) Using the multivariable chain rule (theorem 3.25).

(b) Using the single-variable chain rule (writing w as a function of t only).

(a) We have

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial w}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial w}{\partial y} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial w}{\partial z} \cdot \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{2x}{x^2 + y^2 + z^2} \cdot \cos(t) + \frac{2y}{x^2 + y^2 + z^2} \cdot \big(-\sin(t)\big) + \frac{2z}{x^2 + y^2 + z^2} \cdot \sec^2(t)$$

Note that

$$x^2 + y^2 + z^2 = \sin^2(t) + \cos^2(t) + \tan^2(t) = 1 + \tan^2(t) = \sec^2(t)$$

so the original expression becomes

$$\frac{2\sin(t)\cos(t)}{\sec^2(t)} - \frac{2\cos(t)\sin(t)}{\sec^2(t)} + \frac{2\tan(t)\sec^2(t)}{\sec^2(t)} = 2\tan(t)$$

(b) As shown earlier, $x^2 + y^2 + z^2 = \sec^2(t)$. Thus, $w = \ln\big(\sec^2(t)\big)$, so

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{1}{\sec^2(t)} \cdot 2\sec(t) \cdot \sec(t)\tan(t) = 2\tan(t)$$

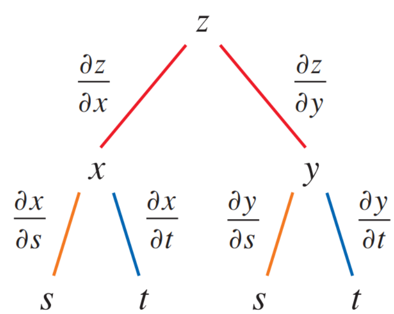
Theorem 3.27 — Chain Rule: Case II (p. 987)

Let $z = f(x,y) = f\big(x(s,t), y(s,t)\big)$. If

- f is differentiable at (x,y) and
- x and y are differentiable at (s,t)

then

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$



Example 3.28: Let $z = \tan^{-1}(xy)$, $x = st$, and $y = se^t$. Compute $\left.\frac{\partial z}{\partial s}\right|_{s=0}$.

We have

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} = \frac{1}{1+(xy)^2}\cdot y\cdot t + \frac{1}{1+(xy)^2}\cdot x\cdot e^t$$

At $s=0$, $x = 0\cdot t = 0$ and $y = 0\cdot e^t = 0$. Therefore,

$$\left.\frac{\partial z}{\partial s}\right|_{s=0} = \frac{1}{1+0^2}\cdot 0\cdot t + \frac{1}{1+0^2}\cdot 0\cdot e^t = 0$$

Theorem 3.29 — Chain Rule: General Case

Let $m,n,p \in \mathbb{Z}^+$, $g\colon \mathbb{R}^n \to \mathbb{R}^m$, and $U = \operatorname{dom}(g) \subseteq \mathbb{R}^n$. Also, let $f\colon \mathbb{R}^m \to \mathbb{R}^p$, where $g(U) \subseteq \operatorname{dom}(f)$. Suppose that $\underline{x} \in U$. If

- g is differentiable at $\underline{x}$ and
- f is differentiable at $g(\underline{x})$

then $f\circ g$ is differentiable at $\underline{x}$ with

$$\mathrm{D}(f\circ g)(\underline{x}) = \mathrm{D}f\big(g(\underline{x})\big)\mathrm{D}g(\underline{x})$$

Remark: In theorem 3.29, $\mathrm{D}f\big(g(\underline{x})\big)$ is a $p\times m$ matrix and $\mathrm{D}g(\underline{x})$ is an $m\times n$ matrix. The multiplication performed is matrix multiplication.

Example 3.30: Let $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $g(x,y) = \big\langle x+y^2, y+x^2\big\rangle$ and $f(u,v) = \langle e^u, uv\rangle$.

(a) Compute $\mathrm{D}(f\circ g)$ at $\underline{x} = (1,2)$ using the chain rule (general case).

(b) Compute $\mathrm{D}(f\circ g)$ at $(1,2)$ directly (writing $f\big(g(x,y)\big)$ as a vector-valued function of x and y only)

(a) We have $g(\underline{x}) = g(1,2) = \langle 1+2^2, 2+1^2 \rangle = \langle 5,3 \rangle$ and

$$\mathrm{D}g(x,y) = \begin{bmatrix} 1 & 2y \\ 2x & 1 \end{bmatrix} \implies \mathrm{D}g(1,2) = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}$$

Also,

$$\mathrm{D}f(u,v) = \begin{bmatrix} e^u & 0 \\ v & u \end{bmatrix} \implies \mathrm{D}f\big(g(1,2)\big) = \mathrm{D}f(5,3) = \begin{bmatrix} e^5 & 0 \\ 3 & 5 \end{bmatrix}$$

Thus, by the chain rule,

$$\mathrm{D}(f \circ g)(1,2) = \begin{bmatrix} e^5 & 0 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} e^5 & 4e^5 \\ 13 & 17 \end{bmatrix}$$

(b) We have $f\big(g(x,y)\big) = f(x+y^2, y+x^2) = \left\langle e^{x+y^2}, (x+y^2)\cdot(y+x^2) \right\rangle = \left\langle e^{x+y^2}, x^3+y^3+x^2y^2+xy \right\rangle$. Letting $f_1(x,y) = e^{x+y^2}$, $f_2(x,y) = x^3+y^3+x^2y^2+xy$, and $\underline{x} = (x,y)$, we get

$$\mathrm{D}f\big(g(\underline{x})\big) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x}(\underline{x}) & \dfrac{\partial f_1}{\partial y}(\underline{x}) \\ \dfrac{\partial f_2}{\partial x}(\underline{x}) & \dfrac{\partial f_2}{\partial y}(\underline{x}) \end{bmatrix}_{2\times 2} = \begin{bmatrix} e^{x+y^2} & e^{x+y^2}\cdot 2y \\ 3x^2+2xy^2+y & 3y^2+2x^2y+x \end{bmatrix}$$

Thus,

$$\mathrm{D}f\big(g(1,2)\big) = \begin{bmatrix} e^{1+2^2} & e^{1+2^2}\cdot 2\cdot 2 \\ 3\cdot 1^2+2\cdot 1\cdot 2^2+2 & 3\cdot 2^2+2\cdot 1^2\cdot 2+1 \end{bmatrix} = \begin{bmatrix} e^5 & 4e^5 \\ 13 & 17 \end{bmatrix}$$

3.5 Gradients and Directional Derivatives (§14.6)

Definition 3.31 — Gradient

Let $n \in \mathbb{Z}^+$, $f\colon \mathbb{R}^n \to \mathbb{R}$, and $\underline{x} = (x_1, \dots, x_n) \in \mathrm{dom}(f)$. The *gradient* of f, denoted $\mathrm{grad}(f)$ or ∇f, is a vector-valued function $\nabla f\colon \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\nabla f = \left\langle \frac{\partial f}{\partial x_1}(\underline{x}), \frac{\partial f}{\partial x_2}(\underline{x}), \dots, \frac{\partial f}{\partial x_n}(\underline{x}) \right\rangle$$

Example 3.32: Let $f(x,y,z) = \sqrt{x^2+y^2+z^2}$. Find ∇f.

By the gradient's definition,

$$\begin{aligned} \nabla f(x,y,z) &= \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle \\ &= \left\langle \frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2x, \frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2y, \frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2z \right\rangle \\ &= \left\langle \frac{x}{sqrtx^2+y^2+z^2}, \frac{y}{sqrtx^2+y^2+z^2}, \frac{z}{sqrtx^2+y^2+z^2} \right\rangle \\ &= \frac{\mathbf{r}}{r} \end{aligned}$$

where $\mathbf{r} = \langle x,y,z \rangle$ and $r = \sqrt{x^2+y^2+z^2}$. Note that in this example, ∇f is the unit vector in the direction of (x,y,z).

Definition 3.33 — Directional Derivative (p. 995)

Let $z = f(x, y)$, $(x, y) \in \text{dom}(f)$, and $\mathbf{u} = \langle a, b\rangle \in \mathbb{R}^2$. The *directional derivative* of f at (x, y) in the direction of $\mathbf{u}$ is

$$\mathrm{D}_{\mathbf{u}} f(x, y) = \lim_{h\to 0} \frac{f(x + ha, y + hb) - f(x, y)}{h}$$

if it exists.

Remark: Recall that for $f\colon \mathbb{R}^2 \to \mathbb{R}$ with $(x_0, y_0) \in \text{dom}(f)$, we have

$$f_x(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + h, y) - f(x_0, y_0)}{h}$$

which is simply the rate of change of f with respect to x *in the direction of* $\mathbf{i}$.

In general,

$$\mathrm{D}_{\mathbf{i}} f = f_x \qquad\qquad \mathrm{D}_{\mathbf{j}} = f_y$$

Example 3.34: Let $f(x, y) = x + y$ and $\mathbf{u} = \langle 1, 1\rangle$. Find $\mathrm{D}_{\mathbf{u}} f(x, y)$.

By definition,

$$\mathrm{D}_{\mathbf{u}} = \lim_{h\to 0} \frac{f(x + h\cdot 1, y + h\cdot 1) - f(x, y)}{h} = \lim_{h\to 0} \frac{(x + h) + (y + h) - (x + y)}{h} = \lim_{h\to 0} \frac{2h}{h} = \lim_{h\to 0} 2 = 2$$

Theorem 3.35 — Directional Derivatives as Dot Products (p. 996)

Let $z = f(x, y)$ and $(x, y) \in \text{dom}(f)$. If

- f is differentiable at (x, y) and
- $\mathbf{u} = \langle a, b\rangle$ is any unit vector in $\mathbb{R}^2$

then the directional derivative of f at (x, y) in the direction of $\mathbf{u}$ exists and

$$\mathrm{D}_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b$$

Remark: In theorem 3.35, notice that

$$f_x(x, y)a + f_y(x, y)b = \big\langle f_x(x, y), f_y(x, y)\big\rangle \cdot \langle a, b\rangle = \nabla f \cdot \mathbf{u}$$

Example 3.36: Compute the directional derivative of $f(x, y, z) = xyz^3$ at the point $p = (2, 1, -3)$ in the direction of $\mathbf{u} = \langle 2, -2, 1\rangle$.

We have

$$\nabla f(x, y, z) = \big\langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\big\rangle = \left\langle yz^3, xz^3, 3xyz^2\right\rangle$$

Note that $\|\mathbf{u}\| = \sqrt{2^2 + (-2)^2 + (-1)^2} = \sqrt{9} = 3$, so the unit vector in the direction of $\mathbf{u}$ is $\mathbf{v} = \frac{1}{3}\langle 2, -2, 1\rangle = \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right\rangle$. Thus,

$$\mathrm{D}_{\mathbf{v}} f(x, y, z) = \left\langle yz^3, xz^3, 3xyz^2\right\rangle \cdot \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right\rangle = \frac{2yz^3}{3} - \frac{2xz^3}{3} + xyz^2$$

so we have

$$\mathrm{D}_{\mathbf{v}} f(2, 1, -3) = \frac{2\cdot 1\cdot (-3)^3}{3} - \frac{2\cdot 2\cdot (-3)^3}{3} + 2\cdot 2\cdot (-3)^2 = -18 - (-36) + 18 = 36$$

3.6 Extrema of Two-Variable Functions (§14.7)

Definition 3.37 — Maximum and Minimum (p. 1008)

Let $z = f(x, y)$ and $\underline{x}_0 = (x, y) \in \operatorname{dom}(f)$.

f has a *local maximum* at $\underline{x}_0$ if there exists a neighbourhood V of $\underline{x}_0$ (i.e. an open disk centered at $\underline{x}_0$) such that

$$f(\underline{x}_0) \geq f(\underline{x})$$

for all $\underline{x} \in V$.

f has an *absolute maximum* at $\underline{x}_0$ if

$$f(\underline{x}_0) \geq f(\underline{x})$$

for all $\underline{x} \in \operatorname{dom}(f)$.

f has a *local minimum* at $\underline{x}_0$ if there exists a neighbourhood V of $\underline{x}_0$ such that

$$f(\underline{x}_0) \leq f(\underline{x})$$

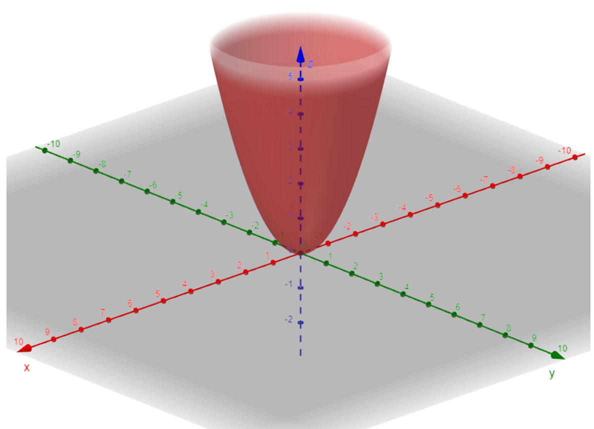
for all $\underline{x} \in V$.

f has an *absolute minimum* at $\underline{x}_0$ if

$$f(\underline{x}_0) \leq f(\underline{x})$$

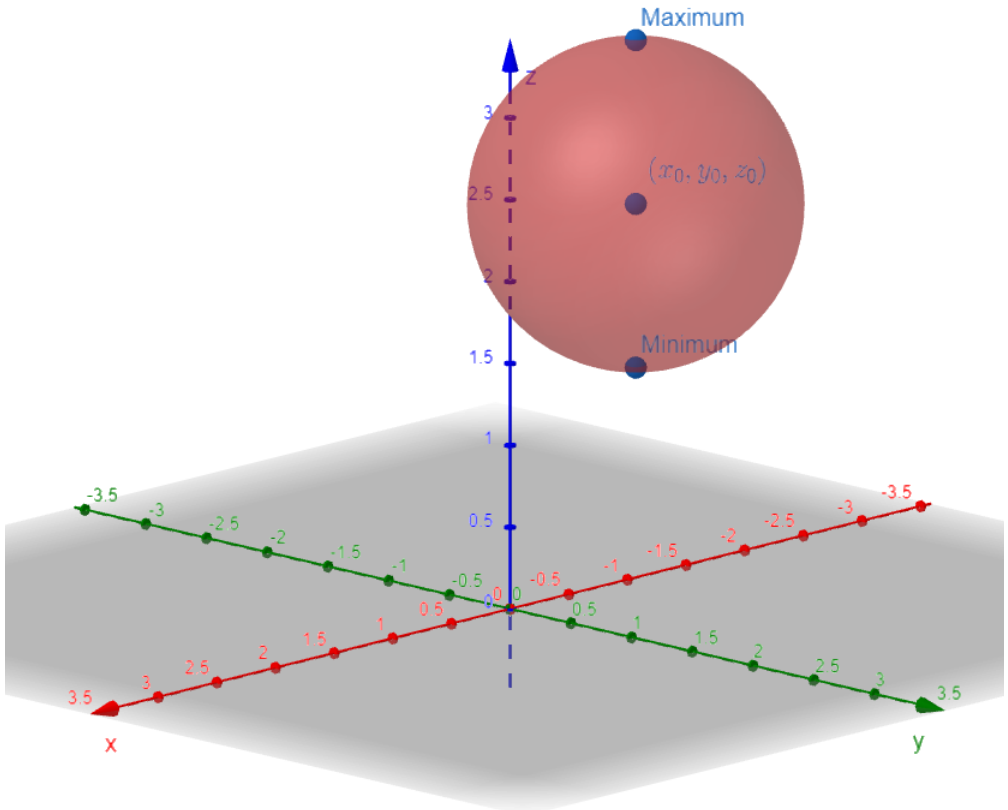
for all $\underline{x} \in \operatorname{dom}(f)$.

Example 3.38: Let $z = x^2 + y^2$.



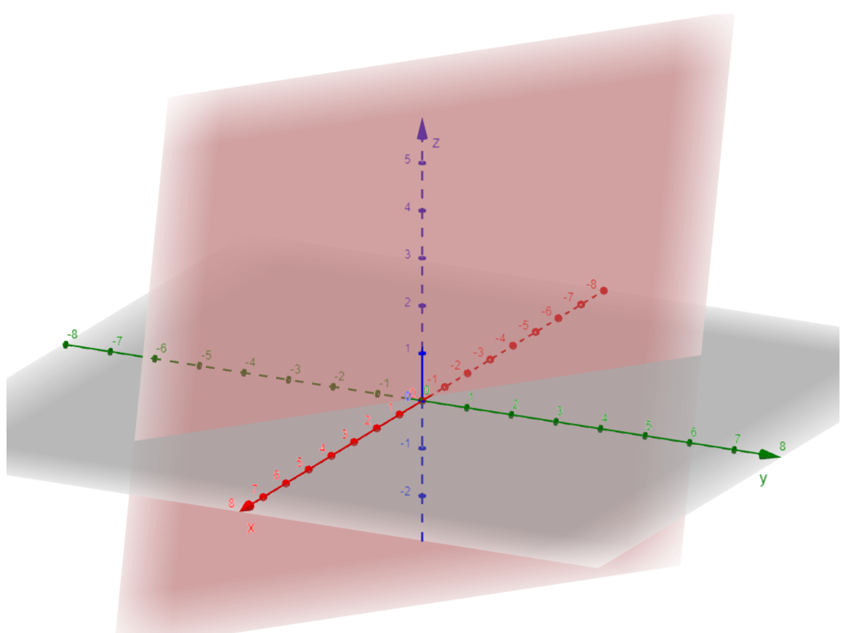
There is a local minimum at $(0, 0, 0)$. This is also the global minimum. The function has no local or absolute maxima.

Example 3.39: Consider $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$ for $r \in \mathbb{R}^+$ (and $(x_0, y_0, z_0) \in \mathbb{R}^3$).



There is a local maximum at $(x_0, y_0, z_0 + r)$, which is also the absolute maximum. Also, there is a local minimum at $(x_0, y_0, z_0 - r)$, which is also the absolute minimum.

Example 3.40: Let $z = 1 + 3x + 4y$.



This function has no extrema.

Theorem 3.41 — Fermat's Theorem for Two-Variable Functions (p. 1009)

Let $z = f(x, y)$, $\underline{x}_0 = (a, b) \in \text{dom}(f)$, and $\underline{x} = (x, y)$. If

- f has an extremum at $\underline{x}_0$ and
- $f_x(\underline{x}_0)$ and $f_y(\underline{x}_0)$ both exist

then $f_x(\underline{x}_0) = f_y(\underline{x}_0) = 0$.

Proof: Suppose that

1. f has a local minimum or maximum at $\underline{x}_0 = (a, b)$ and
2. $f_x(a, b)$ and $f_y(a, b)$ exist

Let $g(x) = f(x, b)$. By (1), g has a local minimum/maximum at a. Without loss of generality, suppose that g has a local maximum at a. For all $x > a$ "near" a (in an interval),

$$\begin{aligned} g(a) \geq g(x) &\iff 0 \geq g(x) - g(a) \\ &\iff 0 \geq \frac{g(x) - g(a)}{x - a} && \text{(as } x - a > 0) \\ &\implies 0 \geq \lim_{x \to a^+} \frac{g(x) - g(a)}{x - a} \\ &\qquad = \lim_{h \to 0^+} \frac{g(a + h) - g(a)}{h} && \text{(for } h = x - a) \\ &\qquad = \lim_{h \to 0^+} \frac{f(a + h, b) - f(a, b)}{h} && \text{(by the definition of } g) \\ &\qquad = f_x(a, b) && \text{from the right} \end{aligned}$$

By (2), this value exists, so $0 \geq f_x(a, b)$ as $h \to 0^+$. By similar reasoning, for $x < a$ "near" a, we have $0 \leq f_x(a, b)$ as $h \to 0^-$. Since $f_x(a, b)$ exists by (2), $0 \leq f_x(a, b) \leq 0$. Therefore, $f_x(a, b) = 0$. Using analogous arguments (with $g(y) = f(a, y)$), $f_y(a, b) = 0$. ■

Definition 3.42 — Critical Point (pp. 1009, 1014)

Let $z = f(x, y)$ and $\underline{x}_0 \in \text{dom}(f)$. The point $\underline{x}_0$ is said to be a *critical point* if $f_x(\underline{x}_0) = f_y(\underline{x}_0) = 0$, or at least one of these partial derivatives does not exist.

Example 3.43: Find all critical points of $f(x, y) = 3x - x^3 - 3xy^2$.

Since f is a polynomial, we have $\text{dom}(f) = \mathbb{R}^2$. Now $f_x(x, y) = 3 - 3x^2 - 3y^2$ and $f_y(x, y) = -6xy$. Thus,

$$f_y(x, y) = 0 \iff -6xy = 0 \iff x = 0 \vee y = 0$$

If $x = 0$,

$$f_x(x, y) = 0 \iff 3 - 3x^2 - 3y^2 = 0 \implies 3 - 3 \cdot 0^2 - 3y^2 = 0 \iff y^2 = 1 \iff y = \pm 1$$

so $(0, 1)$ and $(0, -1)$ are critical points. Similarly, if $y = 0$,

$$f_x(x, y) = 0 \iff 3 - 3x^2 - 3y^2 = 0 \implies 3 - 3x^2 - 3 \cdot 0^2 = 0 \iff x^2 = 1 \iff x = \pm 1$$

so $(1, 0)$ and $(-1, 0)$ are also critical points. Note that f_x and f_y are polynomials, so there are no critical points where f_x or f_y does not exist.

Definition 3.44 — Saddle Point (p. 1010)

Suppose that $z = f(x, y)$ and $\underline{x}_0 \in \text{dom}(f)$. If $\underline{x}_0$ is a critical point of f but f does *not* have an extremum at $\underline{x}_0$, then f is said to be a *saddle point.*

Example 3.45 *(The Pringle Chip Saddle Point™)*: The center of a pringle chip (before you eat it) is a saddle point.

Theorem 3.46 — Second Derivatives Test (p. 1010)

Let $z = f(x, y)$ and $\underline{x}_0 = (x, y) \in \text{dom}(f)$. Suppose that $\underline{x}_0$ is a critical point of f and all second-order partial derivatives of f are continuous in a disk centered at $\underline{x}$. Let

$$\Delta = \Delta(\underline{x}) = f_{xx}(\underline{x})f_{yy}(\underline{x}) - \big(f_{xy}(\underline{x})\big)^2$$

(a) If $\Delta(\underline{x}_0) > 0$ and $f_{xx}(\underline{x}_0) > 0$, then f has a local minimum at $\underline{x}_0$

(b) If $\Delta(\underline{x}_0) > 0$ and $f_{xx}(\underline{x}_0) < 0$, then f has a local maximum at $\underline{x}_0$

(c) If $\Delta(\underline{x}_0) < 0$, then f has a saddle point at $\underline{x}$

Remark: In theorem 3.46, if $\Delta(\underline{x}_0) = 0$, then the test is inconclusive.

Example 3.47: Classify all critical points of $f(x, y) = 3x - x^3 - 3xy^2$ as in example 3.43 (i.e. find all local extrema and saddle points of f).

From example 3.43, f has the critical points $(\pm 1, 0)$ and $(0, \pm 1)$, $f_x(x, y) = 3 - 3x^2 - 3y^2$, and $f_y(x, y) = -6xy$. Thus, $f_{xx}(x, y) = -6x$, $f_{xy}(x, y) = -6y$, $f_{yy}(x, y) = -6x$, and $f_{yx}(x, y) = -6y$. These are all polynomials and thus continuous on $\mathbb{R}^2$. We now have

Critical point	Δ at the critical point	f_{xx} at the critical point	Conclusion
$(0, 1)$	-36		Saddle point
$(1, 0)$	36	-6	Local maximum
$(0, -1)$	-36		Saddle point
$(-1, 0)$	36	6	Local minimum

Example 3.48: Consider $f(x, y) = 6xy^2 - 2x^3 - 3y^4$. Note that f has critical points at $(0, 0)$, $(1, 1)$, and $(1, -1)$. Also, the second derivatives test is inconclusive at $(0, 0)$, while the other points are local maxima. Determine if $(0, 0)$ is an extremum, and classify it if so.

Along $y = 0$, $f(x, y) = f(x, 0) = -2x^3$, which is a decreasing function of x. Thus, $(0, 0)$ cannot be a local minimum or maximum of f, so $(0, 0)$ is a saddle point.

Remark: The function in example 3.48 is known as a *monkey saddle*.

Definition 3.49 — Boundary Point (p. 1014)

Suppose that $A \subseteq \mathbb{R}^2$. $(x, y) \in \mathbb{R}^2$ is called a *boundary point* of A if every disk centered at (x, y) contains points in A and every disk centered at (x, y) contains points not in A.

The *set of boundary points* of A is denoted by ∂A.

Definition 3.50 — Closed Set (p. 1014)

$A \subseteq \mathbb{R}^2$ is said to be *closed* if it contains all of its boundary points.

Remark: Open sets do *not* refer to sets that are not closed; their definition differs in topology.

Definition 3.51 — Bounded Set (p. 1014)

Suppose that $A \subseteq \mathbb{R}^2$. A is *bounded* if there exists a disk $U \subseteq \mathbb{R}^2$ such that $A \subseteq U$.

Example 3.52: Identify whether each of the following sets are closed and/or bounded.

(a) $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$

(b) $A = [0, 1] \times [0, 1] = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \wedge 0 \leq y \leq 1\}$

(c) $A = \{(x, y) \in \mathbb{R}^2 : y \leq x\}$

(a) A is not closed, but it is bounded.

(b) A is both closed and bounded.

(c) A is closed but not bounded.

Theorem 3.53 — Extreme Value Theorem (p. 1014)

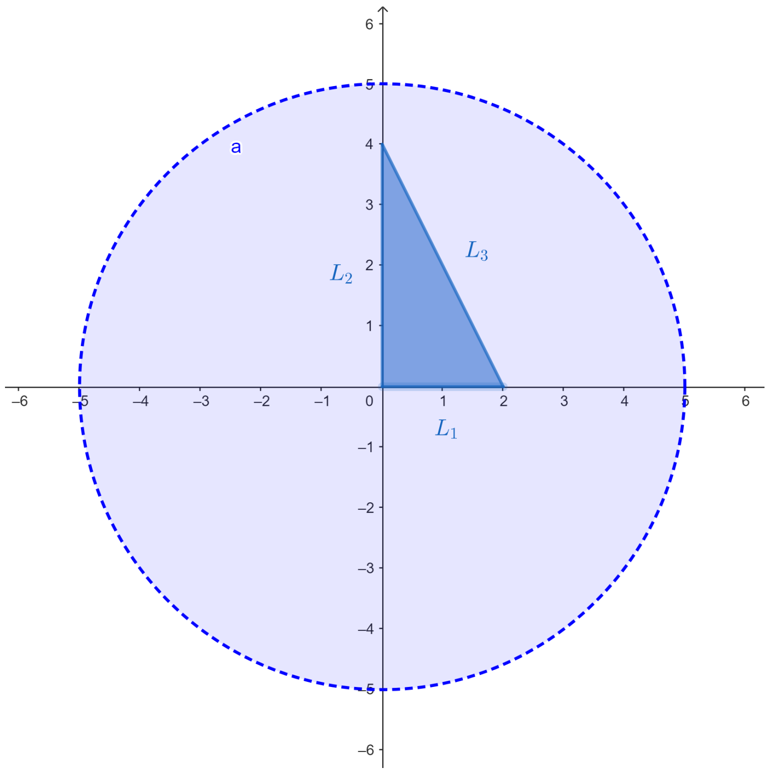
Let $A \subseteq \mathbb{R}^2$. If

- f is continuous on A and
- A is closed and bounded

then f attains both absolute minimum and maximum values in A.

Example 3.54: Find the absolute minimum and maximum values of $f(x, y) = xy - x - y + 3$ on the triangle A in the xy-plane with vertices $(0, 0)$, $(2, 0)$, and $(0, 4)$, if they exist.

f is a two-variable polynomial and thus continuous on its domain $\mathbb{R}^2$. Since $A \subseteq \mathbb{R}^2$, f is continuous on A.



A is bounded as $A \subseteq \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 5^2\}$. Also, $\partial A = L_1 \cup L_2 \cup L_3$. Observe that $\partial A \subseteq A$. By the extreme value theorem, f has absolute maximum and minimum values on A.

Note that $\operatorname{dom}(f) = \mathbb{R}^2$, $f_x(x, y) = y - 1$, and $f_y(x, y) = x - 1$. Thus,

$$f_x(x, y) = f_y(x, y) = 0 \iff (x, y) = (1, 1) \in U$$

We now find the extrema on ∂U.

On L_1, $y = 0$ for $0 \leq x \leq 2$. Thus,

$$f(x, 0) = -x + 3$$

which is decreasing. Therefore, f attains its extrema at the endpoints $(0,0),(2,0) \in \partial U$. On L_2, $x = 0$ for $0 \le y \le 4$. We get

$$f(0,y) = -y + 3$$

which is decreasing, so f attains its extrema at the endpoints $(0,0),(0,4) \in \partial U$. On L_3, $y = -2x+4$ for $0 \le x \le 2$. This yields

$$f(x,-2x+4) = -2x^2 + 5x + 3 = -2\left(x^2 - 2\cdot\frac{5}{4}x + \left(\frac{5}{4}\right)^2\right) + 3 + 2\left(\frac{5}{4}\right)^2 = -2\left(x - \frac{5}{4}\right)^2 + \frac{49}{8}$$

which is maximized at $x = \frac{5}{4}$ and minimized at one of the endpoints. Note that $x = 0 \implies f(0,-2\cdot 0+4) = 3$ and $x = 2 \implies f(2,-2\cdot 2+4) = 5$, so the minimum is at $x = 2$. Now $x = \frac{5}{4} \implies y = -2x+4 = \frac{3}{2}$ and $x = 2 \implies y = -2x+4 = 0$, so the maximum is $\left(\frac{5}{4},\frac{3}{2}\right) \in \partial U$ and the minimum is $(2,0) \in \partial U$.

In summary, the values of f at the points found in the previous steps are

$$f(1,1) = 2 \qquad f(0,0) = 3 \qquad f(2,0) = 1 \qquad f(0,4) = -1 \qquad f\left(\frac{5}{4},\frac{3}{2}\right) \approx 2.125$$

so the absolute maximum is 3 and the absolute minimum is -1.

Theorem 3.55 — Extended Closed Interval Method

The extended closed interval method for finding the absolute minimum and maximum values of a continuous function f on some closed and bounded region U is as follows:

1. Locate all critical points of f in U
2. Compute the value of f at all the critical points
3. Find the minimum and maximum values of f on ∂U
4. The largest of the values computed in steps (2) and (3) is the absolute maximum; similarly, the smallest of these values is the absolute minimum

3.7 Lagrange Multipliers (§14.8)

Theorem 3.56 — Lagrange Multiplier Method (p. 1021)

The minimum and maximum values of $f(x,y)$ subject to the constraint $g(x,y) = k$ (for some $k \in \mathbb{R}$) can be found using the following steps, assuming that $\nabla g \ne \mathbf{0}$:

1. Solve the following system for $x, y, \lambda \in \mathbb{R}$:

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = k \end{cases}$$

2. Compute f at all these points (x,y); the largest computed value is the absolute maximum, and the smallest is the absolute minimum

Remark: In theorem 3.56, λ is known as the *Lagrange multiplier*.

Example 3.57: Find the local extrema of $f(x,y) = 4xy$ subject to the constraint $x^2 = 1 - y^2$.

We have $\text{dom}(f) = \mathbb{R}^2$ as f is a polynomial. Note that $x^2 = 1 - y^2 \iff x^2 + y^2 = 1$, so we let $g(x,y) = x^2 + y^2$. Now

$$\nabla f(x,y) = \big\langle f_x(x,y), f_y(x,y)\big\rangle = \langle 4y, 4x\rangle$$

Also,

$$\nabla g(x,y) = \big\langle g_x(x,y), g_y(x,y)\big\rangle = \langle 2x, 2y\rangle$$

Thus, we get the following system, which we must solve for $x, y, \lambda \in \mathbb{R}$:

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 1 \end{cases} \implies \begin{cases} 4y = \lambda \cdot 2x \\ 4x = \lambda \cdot 2y \\ x^2 + y^2 = 1 \end{cases}$$

If $x \neq 0$, we get $\lambda = \dfrac{2y}{x}$. Similarly, $y \neq 0$, we get $\lambda = \dfrac{2x}{y}$. We obtain

$$\lambda = \lambda \iff \frac{2y}{x} = \frac{2x}{y} \iff 2y^2 = 2x^2 \iff y = \pm x$$

Now

$$x^2 + y^2 = 1 \implies x^2 + x^2 = 1 \implies 2x^2 = 1 \implies x = \pm \frac{1}{\sqrt{2}}$$

If $x = \frac{1}{\sqrt{2}}$, $x^2 + y^2 = 1 \implies y = \pm\sqrt{1 - x^2} = \pm\sqrt{1 - \frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$, which yields the points $\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$ and $\left(\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}\right)$.

If $x = -\frac{1}{\sqrt{2}}$, $y = \pm\sqrt{1 - x^2} = \pm\sqrt{1 - \frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$, which yields $\left(-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$ and $\left(-\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}\right)$.

Now if $x = 0$, $y = 0$ (as $y = \lambda \cdot 2x = \lambda \cdot 0 = 0$). Note that $x^2 + y^2 = 0^2 + 0^2 \neq 1$, so we discard the point $(0,0)$ (as it violates the constraint given by g).

We now get

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 2 \qquad \text{maximum value}$$
$$f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -2 \qquad \text{minimum value}$$
$$f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -2 \qquad \text{minimum value}$$
$$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = 2 \qquad \text{maximum value}$$

Example 3.58: A rectangular box without a top is to be constructed from 588m^2 of material. Find the maximum volume of such a box.

Let x, y, and z be the width, length, and height of the box (respectively). We want to find the maximum of $V(x,y,z) = xyz$ subject to the constraint $g(x,y,z) = xy + 2xz + 2yz = 588$ with $x \geq 0$, $y \geq 0$, and $z \geq 0$.

$$\nabla V(x,y,z) = \big\langle V_x(x,y,z), V_y(x,y,z), V_z(x,y,z) \big\rangle = \langle yz, xz, xy \rangle$$

and

$$\nabla g(x,y,z) = \big\langle g_x(x,y,z), g_y(x,y,z), g_z(x,y,z) \big\rangle = \langle y + 2z, x + 2z, 2x + 2y \rangle$$

We now solve the following system for $x, y, \lambda \in \mathbb{R}$:

$$\begin{cases} \nabla V(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) = 588 \end{cases} \implies \begin{cases} yz = \lambda \cdot (y + 2z) \\ xz = \lambda \cdot (x + 2z) \\ xy = \lambda \cdot (2x + 2y) \\ xy + 2xz + 2yz = 588 \end{cases} \implies \begin{cases} xyz = \lambda \cdot x(y + 2z) \\ xyz = \lambda \cdot y(x + 2z) \\ xyz = \lambda \cdot z(2x + 2y) \\ xy + 2xz + 2yz = 588 \end{cases}$$

Here,

$$\begin{cases} xyz = \lambda \cdot x(y+2z) \\ xyz = \lambda \cdot y(x+2z) \\ xyz = \lambda \cdot z(2x+2y) \end{cases} \implies \lambda(xy+2xz) = \lambda(xy+2yz) = \lambda(2xz+2yz) \implies xy+2xz = xy+2yz = 2xz+2yz$$

noting that $\lambda \neq 0$ (as $\lambda = 0 \implies yz = xz = xy = 0 \implies g(x,y,z) = xy+2xz+2yz = 0 \neq 588$). Now

$$xy+2xz = xy+2yz \implies xz = yz \implies x = y$$

using the fact that $z \neq 0$ (since $z = 0 \implies V(x,y,z) = xyz = 0$, though a larger volume can be achieved). Similarly,

$$xy+2yz = 2xz+2yz \implies xy = 2xz \implies y = 2z$$

as $x \neq 0$ (by similar reasoning as for why $z \neq 0$). Thus, the given constraint yields

$$xy+2xz+2yz = 588 \implies 2z\cdot 2z + 2\cdot 2z\cdot z + 2\cdot 2z\cdot z = 588 \implies 12z^2 = 588 \implies z^2 = 49 \implies z = 7$$

since $z \geq 0$. We obtain $x = y = 2z = 14$, so $(14,14,7)$ is an extremum of f. This gives us $V(x,y,z) = xyz = 1372$. Note that at least one other point resulting in a smaller volume; one example is $\left(\sqrt{\frac{588}{5}}, \sqrt{\frac{588}{5}}, \sqrt{\frac{588}{5}}\right)$, which satisfies the constraint

$$g\left(\sqrt{\frac{588}{5}}, \sqrt{\frac{588}{5}}, \sqrt{\frac{588}{5}}\right) = \sqrt{\frac{588}{5}}\cdot\sqrt{\frac{588}{5}} + 2\cdot\sqrt{\frac{588}{5}}\cdot\sqrt{\frac{588}{5}} + 2\cdot\sqrt{\frac{588}{5}}\cdot\sqrt{\frac{588}{5}} = 5\cdot\frac{588}{5} = 588$$

For this point, $V\left(\sqrt{\frac{588}{5}}, \sqrt{\frac{588}{5}}, \sqrt{\frac{588}{5}}\right) \approx 1275.3 < 1372$.

Therefore, the extremum found must be the maximum, so the maximum volume is 1372m^2.

Example 3.59: Find the minimum and maximum values of $f(x,y) = x^2+y^2-x-y+1$ on $\{(x,y)\in\mathbb{R}^2 : x^2+y^2 \leq 1\}$ using the Lagrange multiplier method.

Let $g(x,y) = x^2+y^2$. We first find the extrema of f subject to the constraint $g(x,y) = 1$. Now $f_x(x,y) = 2x-1$, $f_y(x,y) = 2y-1$, $g_x(x,y) = 2x$, and $g_y(x,y) = 2y$. We thus obtain the system

$$\begin{cases} \nabla f(x,y) = \lambda\nabla g(x,y) \\ g(x,y) = 1 \end{cases} \implies \begin{cases} 2x-1 = \lambda\cdot 2x \\ 2y-1 = \lambda\cdot 2y \\ x^2+y^2 = 1 \end{cases} \implies \begin{cases} 2x(1-\lambda) = 1 \\ 2y(1-\lambda) = 1 \\ x^2+y^2 = 1 \end{cases}$$

Since $2x(1-\lambda) = 1 \neq 0$, $1-\lambda \neq 0$. Thus,

$$2x(1-\lambda) = 1 = 2y(1-\lambda) \implies x = y$$

This yields

$$x^2+y^2 = 1 \implies x^2+x^2 = 1 \implies x^2 = \frac{1}{2} \implies x = \pm\frac{1}{\sqrt{2}}$$

so we have the points $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. Now

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 2-\sqrt{2} \qquad f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 2 \qquad f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = 2 \qquad f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = 2+\sqrt{2}$$

Therefore, the maximum and minimum values of f subject to $g(x,y) = 1$ are $2+\sqrt{2}$ and $2-\sqrt{2}$ (respectively).

Example 3.60: Find the extreme values of $f(x,y,z)=z$ subject to the constraints $x^2+y^2=z^2$ and $x+y+z=24$.
Let $g(x,y,z)=x^2+y^2-z^2$ and $h(x,y,z)=x+y+z$. We now solve the following system for $x,y,z,\lambda,\mu\in\mathbb{R}$:

$$\begin{cases}\nabla f(x,y,z)=\lambda\nabla g(x,y,z)+\mu\nabla h(x,y,z)\\ g(x,y,z)=0\\ h(x,y,z)=24\end{cases}\implies\begin{cases}\nabla f(x,y,z)=\lambda\nabla g(x,y,z)+\mu\nabla h(x,y,z)\\ g(x,y,z)=0\\ h(x,y,z)=24\end{cases}$$

$$\implies\begin{cases}0=\lambda\cdot 2x+\mu\cdot 1\\ 0=\lambda\cdot 2y+\mu\cdot 1\\ 1=\lambda\cdot(-2z)+\mu\cdot 1\\ x^2+y^2-z^2=0\\ x+y+z=24\end{cases}$$

$$\implies\begin{cases}\mu=-2\lambda x\\ \mu=-2\lambda y\\ \mu=2\lambda z+1\\ x^2+y^2-z^2=0\\ x+y+z=24\end{cases}$$

Thus, $-2\lambda x=-2\lambda y\implies\lambda(x-y)=0$, so $\lambda=0$ or $x=y$. If $\lambda=0$, $\mu=-2\lambda x=-2\cdot 0\cdot x=0$, which contradicts the fact that $\mu=2\lambda z+1=2\cdot 0\cdot z+1=1$. Thus, $x=y$, so

$$x+y+z=24\implies x+x+z=24\implies z=24-2x$$

Therefore,

$$x^2+y^2=z^2\implies x^2+x^2=(24-2x)^2\implies 2x^2=576-96x+4x^2\implies x^2-48x+288=0\implies x=24\pm 12\sqrt{2}$$

If $x=24+12\sqrt{2}$, $y=x=24+12\sqrt{2}$ and $z=24-2x=24-2(24+12\sqrt{2})=-24-24\sqrt{2}$. If $x=24-12\sqrt{2}$, $y=x=24-12\sqrt{2}$ and $z=24-2x=24-2(24-12\sqrt{2})=-24+24\sqrt{2}$. We thus have the points $\left(24+12\sqrt{2},24+12\sqrt{2},-24-24\sqrt{2}\right)$ and $\left(24-12\sqrt{2},24-12\sqrt{2},-24+24\sqrt{2}\right)$, for which

$$f\left(24+12\sqrt{2},24+12\sqrt{2},-24-24\sqrt{2}\right)=-24-24\sqrt{2}\qquad f\left(24-12\sqrt{2},24-12\sqrt{2},-24+24\sqrt{2}\right)=-24+24\sqrt{2}$$

Therefore, the maximum value is $24\sqrt{2}-24$ and the minimum value is $-24\sqrt{2}-24$.

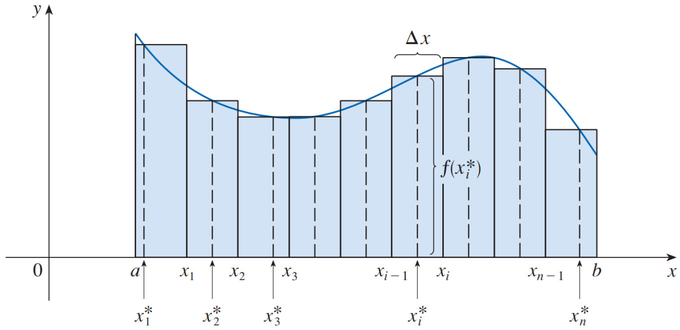
Chapter 4 Multiple Integrals

4.1 Double Integrals Over Rectangular Regions (§15.1)

Recall that for a single-variable function f with $[a,b] \subseteq \operatorname{dom}(f)$ for some $a, b \in \mathbb{R}$, the definite integral of f from a to b is defined using Riemann sums as

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

where each $x_i^* \in [x_{i-1}, x_i]$ and $\Delta x = \frac{b-a}{n}$ for $n \in \mathbb{Z}^+$. A typical choice for x_i^* would be $x_i^* = a + i\Delta x$ for $0 \le i \le n$.



We now generalize this notion to multivariable functions.

Definition 4.1 — Double Integral Over a Rectangle (p. 1040)

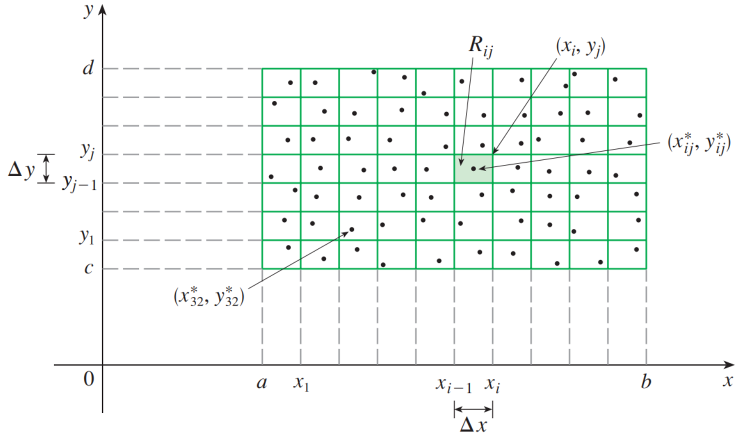
Let $z = f(x,y)$, $a,b,c,d \in \mathbb{R}$, $R = \left\{(x,y) \in \mathbb{R}^2 : x \in [a,b] \wedge y \in [c,d]\right\} \subseteq \operatorname{dom}(f)$, and $m, n \in \mathbb{Z}^+$. Define $\Delta x = \dfrac{b-a}{n}$, $\Delta y = \dfrac{d-c}{m}$, $x_i = a + i\Delta x$ for $0 \le i \le n$, and $y_j = c + j\Delta y$ for $0 \le j \le m$. The *double integral of f over the rectangle R* is

$$\iint\limits_R f(x,y)\,\mathrm{d}A = \lim_{m,n\to\infty} \sum_{j=1}^{m}\sum_{i=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta x \Delta y$$

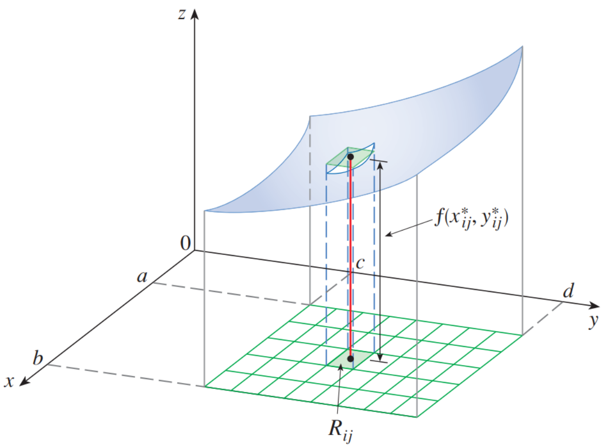
if this limit exists, where $\Delta A = \Delta x \Delta y$ and each $\left(x_{ij}^*, y_{ij}^*\right) \in [x_{i-1}, x_i] \times \left[y_{j-1}, y_j\right]$.

f is said to be *integrable on R* if this limit exists.

Remark: In definition 4.1, each x_i and y_j divides R into subrectangles R_{ij} as follows:



Each $\left(x_{ij}^*, y_{ij}^*\right)$ serves as a sample point in its subrectangle, depicted below:



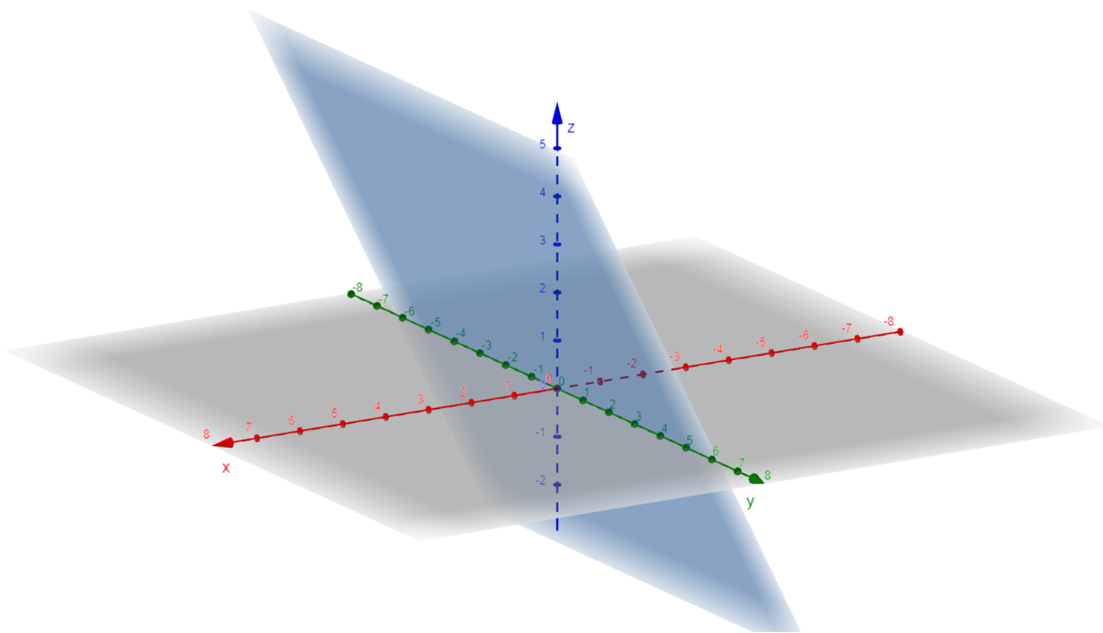
Geometrically, if $z = f(x, y) \geq 0$, then

$$\iint_R f(x, y)\,\mathrm{d}A = V$$

where V is the volume of the solid under $z = f(x, y)$ on R.

Example 4.2: Evaluate $\displaystyle\iint_R 2x\,\mathrm{d}A$ for $R = [0, 2] \times [0, 4]$, if it exists.

Note that $z = f(x, y) = 2x \geq 0$ on R as $x \in [0, 2]$. The graph of f is as follows:



Thus, we get the volume of a triangular prism

$$V = \frac{1}{2}bhl = \frac{1}{2} \cdot 2 \cdot (2 \cdot 2) \cdot 4 = 16$$

Theorem 4.3 — Continuity and Integrability

If f is continuous on a region R, then f is integrable on R.

If f is bounded on R and the set of points where f is discontinuous lie in a union of a finite number of continuous curves, then f is integrable on R.

Theorem 4.4 — Properties of Double Integrals Over Rectangles

Let $a, b, c, d \in \mathbb{R}$, $m, n \in \mathbb{Z}^+$, and $R = [a, b] \times [c, d]$. If $f(x, y)$ and $g(x, y)$ are integrable over R, then

(a) $f + g$ is integrable on R with

$$\iint\limits_R \big(f(x,y) + g(x,y)\big)\,\mathrm{d}A = \iint\limits_R f(x,y)\,\mathrm{d}A + \iint\limits_R g(x,y)\,\mathrm{d}A$$

(b) For all $c \in \mathbb{R}$, cf is integrable on R with

$$\iint\limits_R cf(x,y)\,\mathrm{d}A = c\iint\limits_R f(x,y)\,\mathrm{d}A$$

(c) If $f(x, y) \le g(x, y)$ for all $(x, y) \in R$, then

$$\iint\limits_R f(x,y)\,\mathrm{d}A \le \iint\limits_R g(x,y)\,\mathrm{d}A$$

This also holds for strict inequalities.

Proof: We will prove the summation property. Suppose that f and g are integrable on R. Let $\{x_i\}_{i=0}^n$ be a Riemann partition of $[a, b]$, and $\{y_j\}_{j=0}^m$ be a Riemann partition of $[c, d]$. By integrability,

$$\iint\limits_R f(x,y)\,\mathrm{d}A = \lim_{n,m\to\infty}\sum_{i=1}^n\sum_{j=1}^m f(x_{ij}^*, y_{ij}^*)\Delta x\Delta y$$

and

$$\iint\limits_R g(x,y)\,\mathrm{d}A = \lim_{n,m\to\infty}\sum_{i=1}^n\sum_{j=1}^m g(x_{ij}^*, y_{ij}^*)\Delta x\Delta y$$

where each $\left(x_{ij}^*, y_{ij}^*\right) \in [x_{i-1}, x_i] \times \left[y_{j-1}, y_j\right]$. Note that $\operatorname{dom}(f + g) = \operatorname{dom}(f) \cap \operatorname{dom}(g)$, $R \subseteq \operatorname{dom}(f)$, and $R \subseteq \operatorname{dom}(g)$ (by integrability). Thus, $R \subseteq \operatorname{dom}(f + g)$. Now

$$\begin{aligned}
\iint\limits_R \big(f(x,y) + g(x,y)\big)\,\mathrm{d}A &= \lim_{n,m\to\infty}\sum_{i=1}^n\sum_{j=1}^m\Big(f(x_{ij}^*, y_{ij}^*) + g(x_{ij}^*, y_{ij}^*)\Big)\Delta x\Delta y && \text{(by the Riemann double integral's definition)}\\
&= \lim_{n,m\to\infty}\sum_{i=1}^n\left(\sum_{j=1}^m f(x_{ij}^*, y_{ij}^*)\Delta x\Delta y + \sum_{j=1}^m g(x_{ij}^*, y_{ij}^*)\Delta x\Delta y\right) && \text{(using } \Sigma \text{ properties)}\\
&= \lim_{n,m\to\infty}\left(\sum_{i=1}^n\sum_{j=1}^m f(x_{ij}^*, y_{ij}^*)\Delta x\Delta y + \sum_{i=1}^n\sum_{j=1}^m g(x_{ij}^*, y_{ij}^*)\Delta x\Delta y\right) && \text{(using } \Sigma \text{ properties again)}\\
&= \lim_{n,m\to\infty}\sum_{i=1}^n\sum_{j=1}^m f(x_{ij}^*, y_{ij}^*)\Delta x\Delta y + \lim_{n,m\to\infty}\sum_{i=1}^n\sum_{j=1}^m g(x_{ij}^*, y_{ij}^*)\Delta x\Delta y && \text{(using limit laws)}\\
&= \iint\limits_R f(x,y)\,\mathrm{d}A + \iint\limits_R g(x,y)\,\mathrm{d}A \in \mathbb{R} && \text{(by the integrability of } f \text{ and } g\text{)}
\end{aligned}$$

Therefore, $f + g$ is integrable on R with $\displaystyle\iint\limits_R \big(f(x,y) + g(x,y)\big)\,\mathrm{d}A = \iint\limits_R f(x,y)\,\mathrm{d}A + \iint\limits_R g(x,y)\,\mathrm{d}A$. ■

Remark: In this course, defining a Riemann partition $\{x_i\}_{i=0}^n$ of an interval $[a, b]$ is equivalent to defining both $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$ for $0 \le i \le n$.

Definition 4.5 — Iterated Integral (p. 1043)

Suppose that $a, b, c, d \in \mathbb{R}$. The expressions

$$\int_a^b \left(\int_c^d f(x,y)\,\mathrm{d}y \right) \mathrm{d}x = \int_a^b \int_c^d f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

and

$$\int_c^d \left(\int_a^b f(x,y)\,\mathrm{d}x \right) \mathrm{d}y = \int_c^d \int_a^b f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

are called *iterated integrals*.

Example 4.6:

(a) Evaluate $\int_{-1}^{0} \int_{0}^{2} (4x^3 + 6xy^2)\,\mathrm{d}y\,\mathrm{d}x$.

(b) Evaluate $\int_{0}^{2} \int_{-1}^{0} (4x^3 + 6xy^2)\,\mathrm{d}x\,\mathrm{d}y$.

(a) We have

$$\begin{aligned}
\int_{-1}^{0} \int_{0}^{2} \left(4x^3 + 6xy^2\right) \mathrm{d}y\,\mathrm{d}x &= \int_{-1}^{0} \left(4x^3 y + 2xy^3\right)\Big|_{y=0}^{y=2} \mathrm{d}x \\
&= \int_{-1}^{0} \left(8x^3 + 2x \cdot 2^3 - (0+0)\right) \mathrm{d}x \\
&= \int_{-1}^{0} \left(8x^3 + 16x\right) \mathrm{d}x \\
&= \left(2x^4 + 8x^2\right)\Big|_{x=-1}^{x=0} \\
&= (0+0) - (2 \cdot 1 + 8 \cdot 1) \\
&= -10
\end{aligned}$$

(b) Now

$$\begin{aligned}
\int_{0}^{2} \int_{-1}^{0} \left(4x^3 + 6xy^2\right) \mathrm{d}x\,\mathrm{d}y &= \int_{0}^{2} \left(x^4 + 3x^2 y^2\right)\Big|_{x=-1}^{x=0} \mathrm{d}y \\
&= \int_{0}^{2} \left((0+0) - (1 + 3y^2)\right) \mathrm{d}y \\
&= \int_{0}^{2} \left(-3y^2 - 1\right) \mathrm{d}y \\
&= \left(-y^3 - y\right)\Big|_{y=0}^{y=2} \mathrm{d}y \\
&= (-8 - 2) - 0 \\
&= -10
\end{aligned}$$

Theorem 4.7 — Fubini's Theorem (p. 1044)

Let $a, b, c, d \in \mathbb{R}$ and $R = [a, b] \times [c, d]$. If $f(x, y)$ is continuous on R, then

$$\iint_R f(x,y)\,\mathrm{d}A = \int_a^b \int_c^d f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_c^d \int_a^b f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

Example 4.8: Evaluate $\iint_R 2x\,\mathrm{d}A$, where $R = [0,2] \times [0,2]$ (as in example 4.2), if it exists.

Note that $f(x, y) = 2x$ is a two-variable polynomial is thus continuous on its domain $\mathbb{R}^2$. Also, $R \subseteq \mathbb{R}^2$, so f is continuous on R. Now

$$\begin{aligned}\iint_R 2x\,\mathrm{d}A &= \int_0^4 \int_0^2 2x\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^4 x^2 \Big|_{x=0}^{x=2} \mathrm{d}y \\ &= \int_0^4 (4-0)\,\mathrm{d}y \\ &= \int_0^4 4\,\mathrm{d}y \\ &= 4y\Big|_{y=0}^{y=4} \\ &= 4\cdot 4 - 4\cdot 0 \\ &= 16\end{aligned}$$

4.2 Double Integrals Over General Regions (§15.2)

Definition 4.9 — Type I and II Regions

Let $a, b \in \mathbb{R}$ and $f_1\colon [a, b] \to \mathbb{R}$ and $f_2\colon [a, b] \to \mathbb{R}$ be continuous functions such that $f_1(x) \leq f_2(x)$ for all $x \in [a, b]$. A *type I* region is of the form

$$D = \left\{(x,y) \in \mathbb{R}^2 : a \leq x \leq b \wedge f_1(x) \leq y \leq f_2(x)\right\}$$

Let $c, d \in \mathbb{R}$ and $g_1\colon [c, d] \to \mathbb{R}$ and $g_2\colon [c, d] \to \mathbb{R}$ be continuous functions such that $g_1(y) \leq g_2(y)$ for all $y \in [c, d]$. A *type II* region is of the form

$$D = \left\{(x,y) \in \mathbb{R}^2 : c \leq y \leq d \wedge g_1(y) \leq x \leq g_2(y)\right\}$$

Remark: In the context of definition 4.9, assuming that f is continuous on D, by Fubini's theorem,

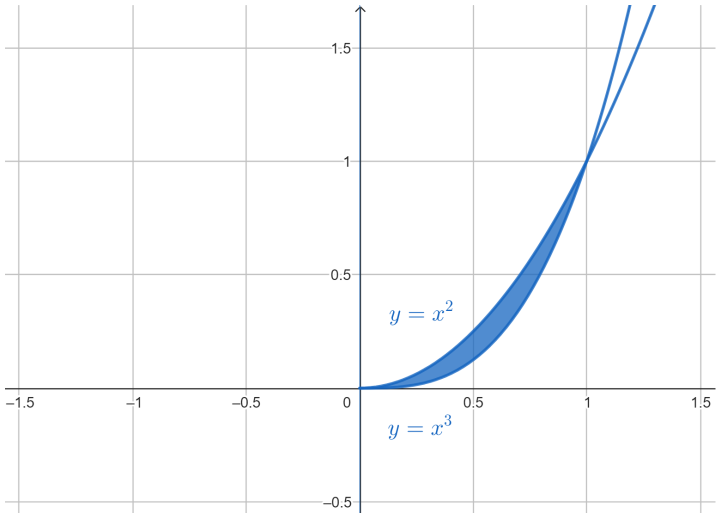
$$\iint_D f(x,y)\,\mathrm{d}A = \begin{cases} \displaystyle\int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x & \text{if } D \text{ is type I} \\ \displaystyle\int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y & \text{if } D \text{ is type II}\end{cases}$$

Example 4.10: Evaluate $\iint_D xy^2\,\mathrm{d}A$, where $D = \left\{(x,y) \in \mathbb{R}^2 : x \geq 0 \wedge y \geq 0 \wedge x^3 \leq y \leq x^2\right\}$. Do this

(a) With D as a type I region

(b) With D as a type II region

(a) Graphically, we have the following region:



Now

$$x^3 = x^2 \iff x^3 - x^2 = 0 \iff x^2(x-1) = 0$$

so the curves intersect when $x = 0$ or $x = 1$. This yields

$$\begin{aligned}
\iint\limits_D xy^2 \,\mathrm{d}A &= \int_0^1 \int_{x^3}^{x^2} xy^2 \,\mathrm{d}y \,\mathrm{d}x \\
&= \int_0^1 \frac{xy^3}{3}\bigg|_{y=x^3}^{y=x^2} \mathrm{d}x \\
&= \int_0^1 \frac{x}{3}\Big((x^2)^3 - (x^3)^3\Big) \,\mathrm{d}x \\
&= \frac{1}{3}\int_0^1 (x^7 - x^{10}) \,\mathrm{d}x \\
&= \frac{1}{3}\left(\frac{x^8}{8} - \frac{x^11}{11}\right)\bigg|_0^1 \\
&= \frac{1}{3}\left(\frac{1}{8} - \frac{1}{11} - (0-0)\right) \\
&= \frac{1}{88}
\end{aligned}$$

(b) As shown earlier, the curves $y = x^3$ and $y = x^2$ intersect where $x = 0$ or $x = 1$. Now $x = 0 \implies y = 0^3 = 0^2 = 0$ and $x = 1 \implies y = 1^3 = 1^2 = 1$. Furthermore,

$$y = x^2 \iff x = \pm\sqrt{y} \iff x = \sqrt{y}$$

as $x \geq 0$ and

$$y = x^3 \iff x = y^{1/3}$$

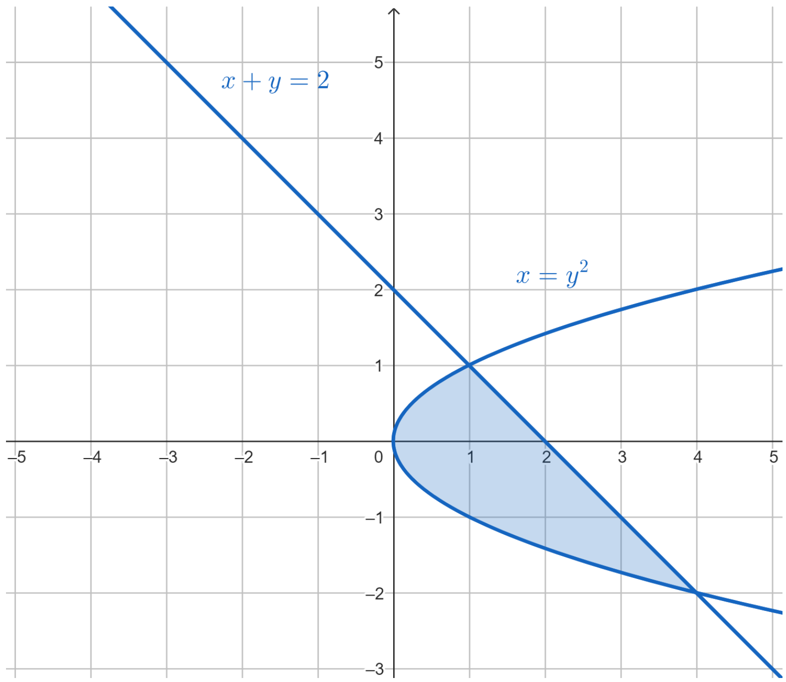
so we obtain

$$\iint\limits_D xy^2 \,\mathrm{d}A = \int_0^1 \int_{y^{1/2}}^{y^{1/3}} xy^2 \,\mathrm{d}x \,\mathrm{d}y$$

$$= \int_0^1 \frac{x^2y^2}{2}\Bigg|_{x=y^{1/2}}^{x=y^{1/3}} \mathrm{d}y$$
$$= \int_0^1 \frac{y^2}{2}\left((y^{1/3})^2 - (y^{1/2})^2\right) \mathrm{d}y$$
$$= \frac{1}{2}\int_0^1 \mathrm{d}y$$

Example 4.11: Consider $I = \iint\limits_D \left(6x+2y^2\right) \mathrm{d}A$, where D is a region bounded by $x = y^2$ and $x + y = 2$. Is it easier to evaluate I with D as a type I region or a type II region? What is I if D is regarded as a type I region?

The region D is as follows:



Note that if we were to take y as a function of x, the upper bound would be a piecewise function. This issue does not arise when taking x to be a function of y; in this case, we have $y^2 \le x \le 2-y$ and $y \in [c,d]$ for some $c, d \in \mathbb{R}$. Thus, regarding D as a type II region is easier for evaluating the integral.

Now
$$y^2 = 2-y \iff y^2+y-2=0 \iff (y+2)(y-1)=0$$
so the curves intersect where $y=-2$ or $y=1$. This yields
$$I = \int_{-2}^{1}\int_{y^2}^{2-y}\left(6x+2y^2\right)\mathrm{d}x\,\mathrm{d}y$$

If D is instead regarded as a type I region, note that the curves intersect at $y=1$ and $y=-2$. Now $y=1 \implies x = 1^2 = 1$, $y=-2 \implies x = (-2)^2 = 4$, and $x = y^2 \iff y = \pm\sqrt{x}$. Thus, we have $-\sqrt{x} \le x \le h(x)$ for

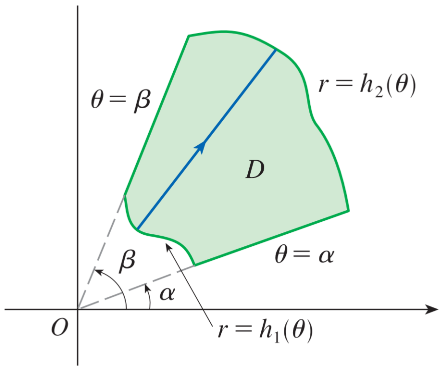
$$h(x) = \begin{cases} \sqrt{x} & \text{if } x \le 1 \\ -x+2 & \text{if } x > 1 \end{cases}$$

Therefore,

$$I = \int_0^4 \int_{-x^{1/2}}^{h(x)} \left(6x+2y^2\right)\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^1 \int_{-x^{1/2}}^{h(x)} \left(6x + 2y^2\right) \mathrm{d}y\, \mathrm{d}x + \int_1^4 \int_{-x^{1/2}}^{h(x)} \left(6x + 2y^2\right) \mathrm{d}y\, \mathrm{d}x$$
$$= \int_0^1 \int_{-x^{1/2}}^{x^{1/2}} \left(6x + 2y^2\right) \mathrm{d}y\, \mathrm{d}x + \int_1^4 \int_{-x^{1/2}}^{-x+2} \left(6x + 2y^2\right) \mathrm{d}y\, \mathrm{d}x$$

4.3 Double Integrals in Polar Coordinates (§15.3)



Theorem 4.12 — Double Integration in Polar Coordinates (p. 1065)

Let $\alpha, \beta \in \mathbb{R}$. If f is continuous on the *polar region*

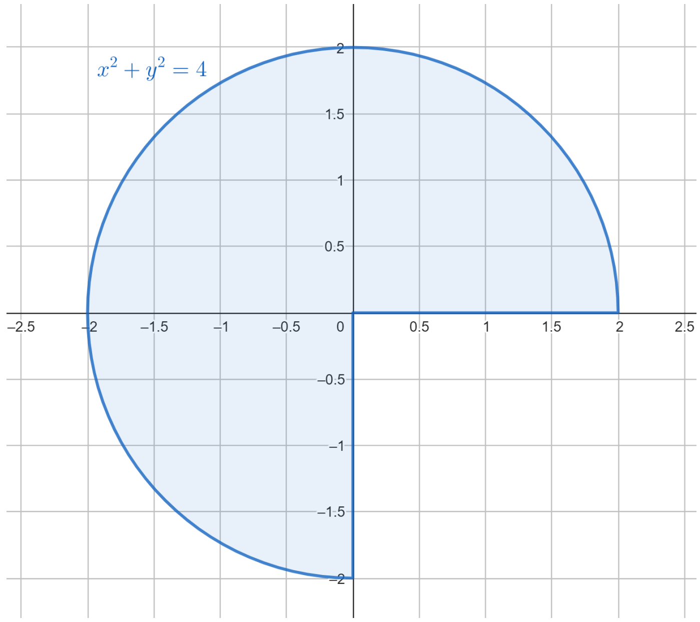
$$R = \left\{(r, \theta) : \alpha \le \theta \le \beta \wedge h_1(\theta) \le r \le h_2(\theta)\right\}$$

for some functions h_1 and h_2, then

$$\iint\limits_D f(x, y)\, \mathrm{d}A = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r, \theta) r\, \mathrm{d}r\, \mathrm{d}\theta$$

where $D \subseteq \mathbb{R}^2$ is the region in rectangular coordinates corresponding to R.

Example 4.13: Consider the following region D:



If f is continuous on D,

(a) Which is easier to evaluate $I = \iint\limits_D f(x,y)\,\mathrm{d}A$: D as a type I/type II or a polar region? Why?

(b) What is this integral?

(a) Regarding D as a polar region is easier as D is not a type I or type II region (the lower integration bounds would be piecewise functions which are discontinuous at certain points). Furthermore, each $(x,y) \in D$ can be written in the form (r,θ) for some $\theta \in [\alpha,\beta]$ and $r \in \big[h_1(\theta), h_2(\theta)\big]$ (for some functions h_1 and h_2).

(b) In polar coordinates, D becomes

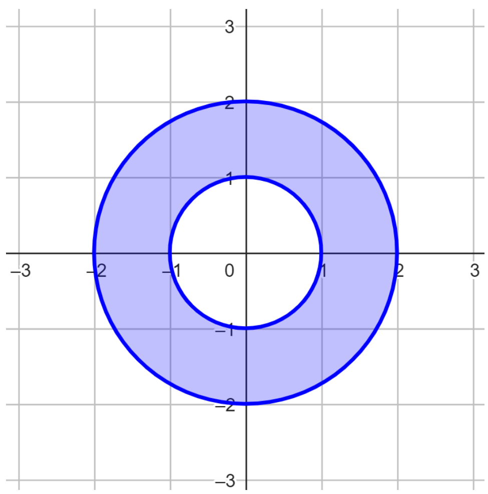
$$\left\{(r,\theta) : \theta \in \left[0, \frac{3\pi}{2}\right] \wedge r \in [0,2]\right\}$$

Therefore, the integral is

$$I = \int_0^{3\pi/2} \int_0^2 f(r,\theta) r\,\mathrm{d}r\,\mathrm{d}\theta$$

Example 4.14: Evaluate $I = \iint\limits_D \sin\big(x^2+y^2\big)\,\mathrm{d}A$, where $D = \big\{(x,y) \in \mathbb{R}^2 : 1 \le x^2+y^2 \le 4\big\}$, using polar coordinates.

We have the following region:



Here, $\theta \in [0,2\pi]$ and $r \in [1,2]$. Since $r^2 = x^2+y^2$, we have

$$I = \int_0^{2\pi} \int_1^2 \sin\left(r^2\right) r\,\mathrm{d}r\,\mathrm{d}\theta$$

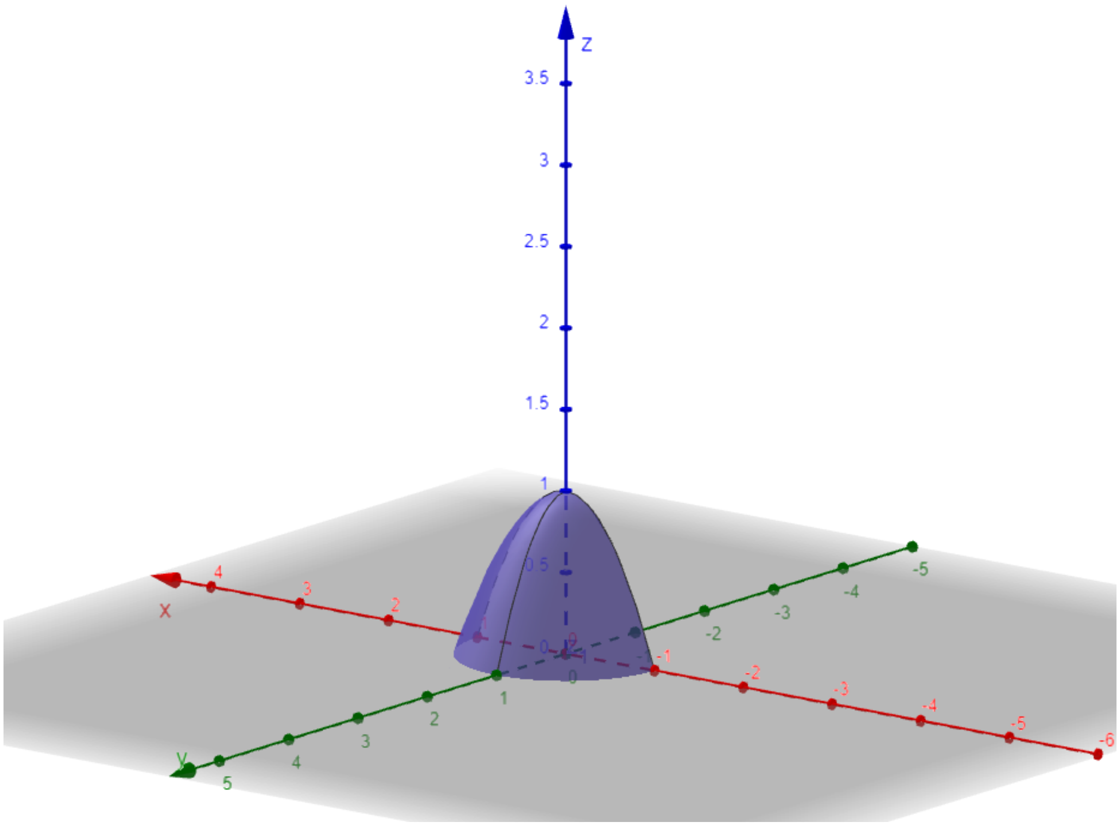
Let $u = r^2$ so that $\mathrm{d}u = 2r\,\mathrm{d}r \implies \frac{\mathrm{d}u}{2} = r\,\mathrm{d}r$. Now $r=1 \implies u = 1^2 = 1$ and $r = 2 \implies u = 2^2 = 4$, so the integral becomes

$$\begin{aligned}
\int_0^{2\pi} \int_1^4 \sin(u) \frac{\mathrm{d}u}{2}\,\mathrm{d}\theta &= \frac{1}{2} \int_0^{2\pi} \int_1^4 \sin(u)\,\mathrm{d}u\,\mathrm{d}\theta \\
&= \frac{1}{2} \int_0^{2\pi} -\cos(u)\Big|_{u=1}^{u=4} \mathrm{d}\theta \\
&= -\frac{1}{2} \int_0^{2\pi} \big(\cos(4) - \cos(1)\big)\,\mathrm{d}\theta \\
&= -\frac{\cos(4)-\cos(1)}{2}\,\theta\Big|_0^{2\pi} \\
&= -\frac{\cos(4)-\cos(1)}{2}(2\pi - 0) \\
&= -\big(\cos(4)-\cos(1)\big)\pi
\end{aligned}$$

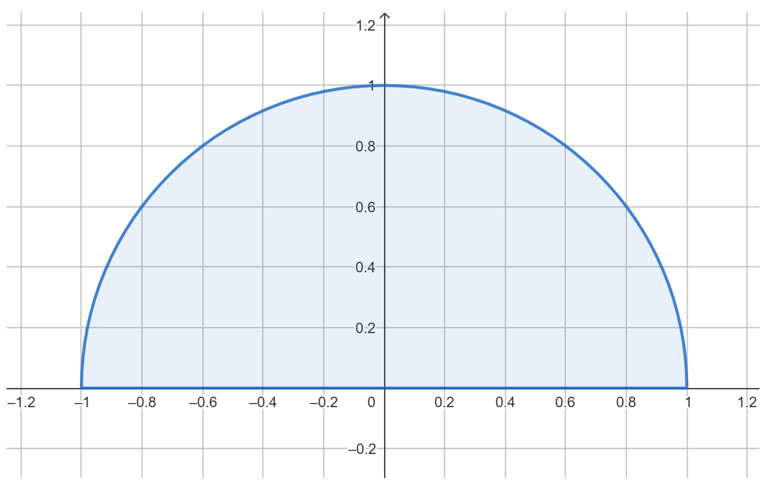
Remark: In this course, showing that an integrand is continuous is not required when integrating.

Example 4.15: Write down the volume integral in polar coordinates of the solid bounded by $z = 1 - \left(x^2 + y^2\right)$, $z = 0$, and $y \geq 0$.

We have the following volume:



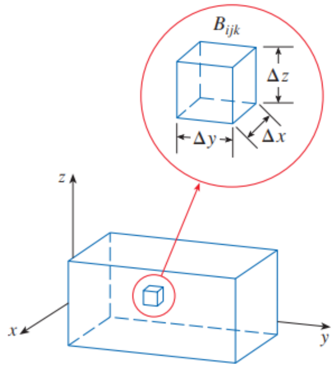
The region in the xy-plane is as follows:



Here, $\theta \in [0, \pi]$ and $r \in [0, 1]$. Since $r^2 = x^2 + y^2$, the volume is

$$V = \int_0^\pi \int_0^1 \left(1 - r^2\right) r \, \mathrm{d}r \, \mathrm{d}\theta$$

4.4 The Triple Integral (§15.6)



Definition 4.16 — Triple Integral (p. 1083)

Let $w = f(x, y, z)$, $B \subseteq \mathrm{dom}(f)$ be the box

$$B = \Big\{(x, y, z) \in \mathbb{R}^3 : x \in [a, b] \wedge y \in [c, d] \wedge z \in [e, f]\Big\} = [a, b] \times [c, d] \times [e, f]$$

where $a, b, c, d, e, f \in \mathbb{R}$. Let $\{x_i\}_{i=0}^{n}$, $\{y_j\}_{j=0}^{m}$, and $\{z_k\}_{k=0}^{l}$ be Riemann partitions of $[a, b]$, $[c, d]$, and $[e, f]$ (respectively). The *triple integral of f over B* is

$$\iiint\limits_B f(x, y, z)\,\mathrm{d}V = \lim_{n,m,l\to\infty} \sum_{k=1}^{l}\sum_{j=1}^{m}\sum_{i=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta x\Delta y\Delta z$$

if this limit exists, where $\Delta V = \Delta x\Delta y\Delta z$ and each $\left(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}\right) \in [x_{i-1}, x_i] \times \left[y_{j-1}, y_j\right] \times [z_{k-1}, z_k]$.

Remark: Similar to Riemann partitions for double integrals, defining a Riemann partition $\{x_i\}_{i=0}^{n}$ of an interval $[a, b]$ is equivalent to defining both $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$ for $0 \le i \le n$ in this course.

A few interpretations of the triple integrals are as follows. If $f(x, y, z) = 1$, $\iiint\limits_B 1\,\mathrm{d}V$ is the *volume* of B. Also, if $f(x, y, z)$ is the density of a solid at the point (x, y, z), then $\iiint\limits_B f(x, y, z)\,\mathrm{d}V$ is the *mass* of the solid.

Theorem 4.17 — Fubini's Theorem for Triple Integrals

Let $w = f(x, y, z)$ and $B = [a, b] \times [c, d] \times [e, f] \subseteq \mathrm{dom}(f)$ for some $a, b, c, d, e, f \in \mathbb{R}$. If f is continuous on B, then

$$\begin{aligned}
\iiint\limits_B f(x,y,z)\,\mathrm{d}V &= \int_a^b \int_c^d \int_e^f f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_a^b \int_e^f \int_c^d f(x,y,z)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x \\
&= \int_c^d \int_a^b \int_e^f f(x,y,z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_c^d \int_e^f \int_a^b f(x,y,z)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y \\
&= \int_e^f \int_a^b \int_c^d f(x,y,z)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z \\
&= \int_e^f \int_c^d \int_a^b f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z
\end{aligned}$$

Example 4.18: Let $f(x, y, z) = xy + yz$ and $B = [-1, 1] \times [2, 3] \times [0, 1]$. Find $I = \iiint\limits_B f(x,y,z)\,\mathrm{d}V$.

f is a three-variable polynomial and thus continuous on $\mathbb{R}^3$. In particular, f is continuous on $B \subset \mathbb{R}^3$. Now

$$\begin{aligned}
I &= \int_{-1}^1 \int_2^3 \int_0^1 (xy + yz)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{-1}^1 \int_2^3 \left(xyz + \frac{yz^2}{2} \right)\Bigg|_{z=0}^{z=1} \mathrm{d}y\,\mathrm{d}x \\
&= \int_{-1}^1 \int_2^3 \left(xy + \frac{y}{2} - 0 \right) \mathrm{d}y\,\mathrm{d}x \\
&= \int_{-1}^1 \left(\frac{xy^2}{2} + \frac{y^2}{4} \right)\Bigg|_{y=2}^{y=3} \mathrm{d}x \\
&= \int_{-1}^1 \left(\left(\frac{9x}{2} + \frac{9}{4} \right) - (2x + 1) \right) \mathrm{d}x \\
&= \int_{-1}^1 \left(\frac{5x}{2} + \frac{5}{4} \right) \mathrm{d}x \\
&= \left(\frac{5x^2}{4} + \frac{5x}{4} \right)\Bigg|_{-1}^{1} \\
&= \left(\frac{5}{4} + \frac{5}{4} \right) - \left(\frac{5}{4} - \frac{5}{4} \right) \\
&= \frac{5}{2}
\end{aligned}$$

Definition 4.19 — Triple Integral Elementary Regions (pp. 1084–1086)

Let $a, b, c, d, e, f \in \mathbb{R}$, g_1, g_2, h_1, and h_2 be continuous single-variable functions, and u_1 and u_2 be continuous two-variable functions.

A solid region is of *type 1* (also referred to as a *z-simple* region) if each of its points satisfies one of the following restrictions on x and y:

$$a \leq x \leq b \wedge g_1(x) \leq y \leq g_2(x)$$
$$c \leq y \leq d \wedge h_1(y) \leq x \leq h_2(y)$$

and

$$u_1(x, y) \leq z \leq u_2(x, y)$$

A solid region is of *type 2* (also referred to as an *x-simple* region) if each of its points satisfies one of the following restrictions on y and z:

$$c \leq y \leq d \wedge g_1(y) \leq z \leq g_2(y)$$
$$e \leq z \leq f \wedge h_1(z) \leq y \leq h_2(z)$$

and

$$u_1(y, z) \leq x \leq u_2(y, z)$$

A solid region is of *type 3* (also referred to as a *y-simple* region) if each of its points satisfies one of the following restrictions on x and z:

$$a \leq x \leq b \wedge g_1(x) \leq z \leq g_2(x)$$
$$e \leq z \leq f \wedge h_1(z) \leq x \leq h_2(z)$$

and

$$u_1(x, z) \leq y \leq u_2(x, z)$$

Example 4.20: Evaluate $\displaystyle\iiint_E z \, \mathrm{d}V$, where E is the solid in the first octant bounded by the surface $z = 12xy$ and the planes $y = x$ and $x = 1$.

In the first octant, $x \geq 0$, $y \geq 0$, and $z \geq 0$. We thus have

$$E = \left\{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq 1 \wedge 0 \leq y \leq x \wedge 0 \leq z \leq 12xy\right\}$$

This is z-simple. Therefore,

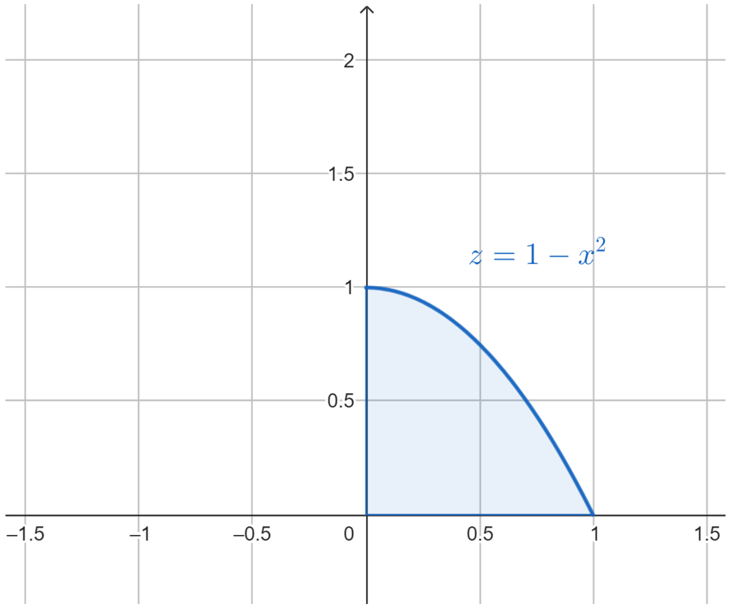
$$\begin{aligned}
\iiint_E z \, \mathrm{d}V &= \int_0^1 \int_0^x \int_0^{12xy} z \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_0^1 \int_0^x \left.\frac{z^2}{2}\right|_{z=0}^{z=12xy} \mathrm{d}y \, \mathrm{d}x \\
&= \int_0^1 \int_0^x \frac{1}{2}\Big((12xy)^2 - 0^2\Big) \, \mathrm{d}y \, \mathrm{d}x \\
&= \frac{12^2}{2} \int_0^1 \int_0^x x^2 y^2 \, \mathrm{d}y \, \mathrm{d}x \\
&= 72 \int_0^1 \left.\frac{x^2 y^3}{3}\right|_{y=0}^{y=x} \mathrm{d}x \\
&= 24 \int_0^1 x^2 \Big(x^3 - 0\Big) \, \mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= 24 \int_0^1 x^5 \, \mathrm{d}x \\
&= 24 \cdot \frac{x^6}{6} \bigg|_0^1 \\
&= 24\left(1^6 - 0^6\right) \\
&= 24
\end{aligned}
$$

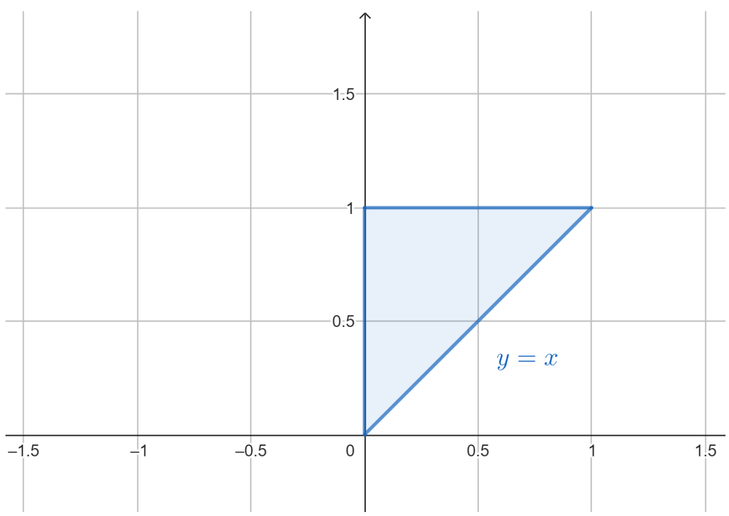
4.5 Changing the Order of Integration for Triple Integrals

Example 4.21: Suppose that $f(x, y, z)$ is continuous on $\mathbb{R}^3$. Express $I = \int_0^1 \int_0^y \int_0^{1-x^2} f(x, y, z) \, \mathrm{d}z \, \mathrm{d}x \, \mathrm{d}y$ as an iterated integral in the order $\mathrm{d}y \, \mathrm{d}x \, \mathrm{d}z$.

We have $0 \leq y \leq 1$, $0 \leq x \leq y$, and $0 \leq z \leq 1 - x^2$. Since we must integrate with respect to y first, we seek a y-simple region. Now $0 \leq x \leq y$ and $0 \leq y \leq 1$, so $0 \leq x \leq 1$. This gives us the following projection onto the xz-plane:



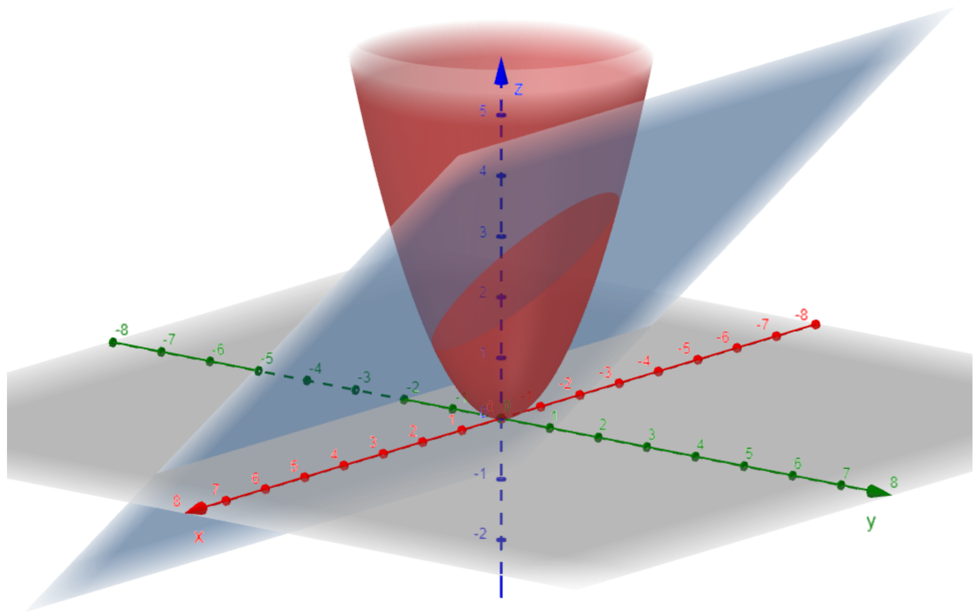
It follows that $0 \leq z \leq 1$. For $x \geq 0$, $z = 1 - x^2 \iff x^2 = 1 - z \iff x = \pm\sqrt{1 - z} \iff x = \sqrt{1 - z}$. Now projecting onto the xy-plane, we get



Thus, $x \le y \le 1$. Using the established bounds,

$$I = \int_0^1 \int_0^{\sqrt{1-z}} \int_x^1 f(x,y,z)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z$$

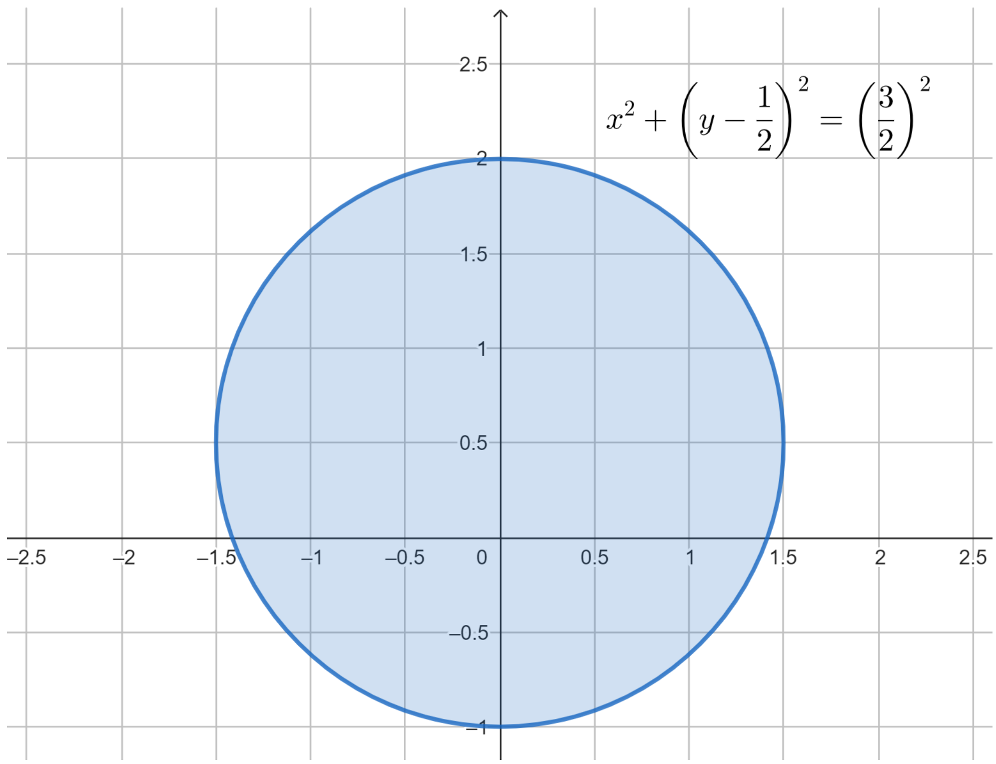
Example 4.22: Find the volume of the solid bounded by $z = x^2 + y^2$ and $z = y + 2$.



Since z is bounded by functions of x and y, it appears that a z-simple solid region should be found. We have

$$x^2 + y^2 = y + 2 \iff x^2 + y^2 - y = 2 \iff x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{3}{2}\right)^2$$

Projecting onto the xy-plane, we thus have



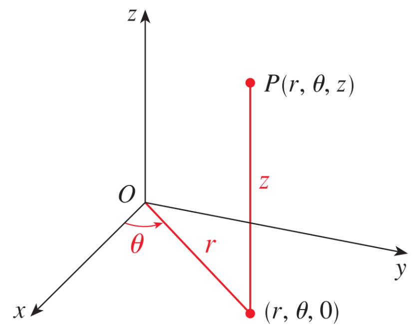
Therefore, we have the region

$$E = \begin{cases} x^2 + y^2 \le z \le y + 2 \\ |x| < \sqrt{\left(\frac{3}{2}\right)^2 - \left(y - \frac{1}{2}\right)^2} \\ -1 \le y \le 2 \end{cases}$$

This region is complicated for integration, so we instead seek an x-simple region.

Exercise.

4.6 Triple Integrals in Cylindrical Coordinates (§15.7)



Definition 4.23 — Cylindrical Coordinates (p. 1096)

Each $P = (x, y, z) \in \mathbb{R}^3$ can be expressed in *cylindrical coordinates* (r, θ, z), where

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

Example 4.24: Convert the cylindrical coordinates $\left(2, \dfrac{2\pi}{3}, 1\right)$ to rectangular coordinates.

We have $r = 2$, $\theta = \frac{2\pi}{3}$, and $z = 1$. Thus, $x = r\cos\theta = 2\cos\left(\frac{2\pi}{3}\right) = 2 \cdot \left(-\frac{1}{2}\right) = -1$ and $y = r\sin\theta = 2\sin\left(\frac{2\pi}{3}\right) = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$, so the corresponding rectangular coordinates are $\left(-1, \sqrt{3}, 1\right)$.

Theorem 4.25 — Triple Integration in Cylindrical Coordinates (p. 1097)

If $f(x, y, z)$ is continuous on a cylindrical region

$$E_C = \left\{(r, \theta, z) : \alpha \le \theta \le \beta \wedge h_1(\theta) \le r \le h_2(\theta) \wedge u_1(r\cos\theta, r\sin\theta) \le z \le u_2(r\cos\theta, r\sin\theta)\right\}$$

for some $\alpha, \beta \in \mathbb{R}$ and functions $h_1(\theta)$, $h_2(\theta)$, $u_1(x, y)$ and $u_2(x, y)$, then

$$\iiint_E f(x, y, z)\,\mathrm{d}V = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$

where $E \subseteq \mathbb{R}^3$ is the solid region in rectangular coordinates corresponding to E_C.

Example 4.26: Use cylindrical coordinates to rewrite $I = \displaystyle\iiint_E (x + y + z)\,\mathrm{d}V$, where E is the solid in the first octant that lies under $z = 4 - \left(x^2 + y^2\right)$.

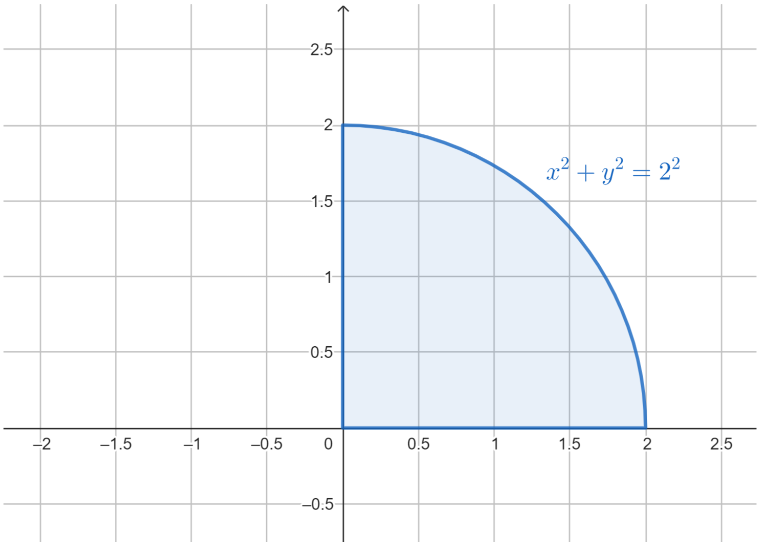
In the solid, we have

$$0 \le z \le 4 - \left(x^2 + y^2\right) \iff 0 \le z \le 4 - r^2$$

Now

$$0 = 4 - r^2 \iff r^2 = 4 \iff r = 2$$

assuming that $r \geq 0$. Thus, projecting onto the xy-plane, we get



Note that in the first octant, $0 \leq \theta \leq \dfrac{\pi}{2}$ as $x \geq 0$ and $y \geq 0$. Therefore, we have the solid region

$$E = \begin{cases} 0 \leq z \leq 4 - r^2 \\ 0 \leq r \leq 2 \\ 0 \leq \theta \leq \dfrac{\pi}{2} \end{cases}$$

Finally, this yields

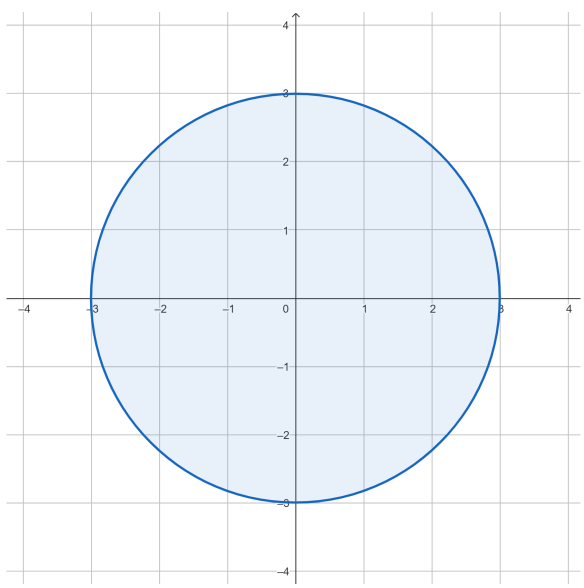
$$I = \int_0^{\pi/2} \int_0^2 \int_0^{4-r^2} (r\cos\theta + r\sin\theta + z) r \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta$$

Example 4.27: "How" to evaluate $I = \displaystyle\iiint_E x^2 \, \mathrm{d}V$, where E is the solid that lies within $x^2 + y^2 = 9$, above $z = 0$, and below $z = 25x^2 + 25y^2$?

We have

$$0 \leq z \leq \sqrt{25x^2 + 25y^2} \iff 0 \leq z \leq 5r$$

assuming that $r \geq 0$. Projecting onto the xy-plane, we have



Thus, $0 \le r \le 3$ and $0 \le \theta \le 2\pi$. Therefore, we have the region

$$E_C = \begin{cases} 0 \le z \le 5r \\ 0 \le r \le 3 \\ 0 \le \theta \le 2\pi \end{cases}$$

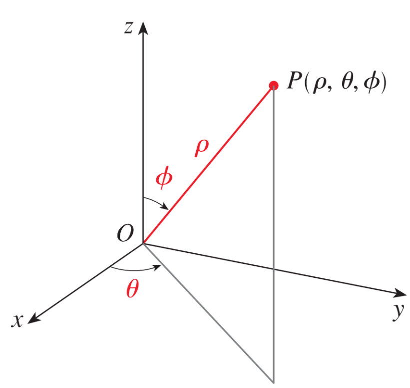
so the integral can be evaluated as

$$I = \int_0^{2\pi} \int_0^3 \int_0^{5r} r^2 \cos^2(\theta) r \,\mathrm{d}z \,\mathrm{d}r \,\mathrm{d}\theta$$

Evaluating the integral, we obtain

$$\begin{aligned}
I &= \int_0^{2\pi} \int_0^3 \int_0^{5r} r^3 \cos^2(\theta) \,\mathrm{d}z \,\mathrm{d}r \,\mathrm{d}\theta \\
&= \int_0^{2\pi} \int_0^3 r^3 \cos^2(\theta) z \Big|_{z=0}^{z=5r} \mathrm{d}r \,\mathrm{d}\theta \\
&= \int_0^{2\pi} \int_0^3 \left(r^3 \cos^2(\theta) \cdot 5r - 0\right) \mathrm{d}r \,\mathrm{d}\theta \\
&= \int_0^{2\pi} \int_0^3 5r^4 \cos^2(\theta) \,\mathrm{d}r \,\mathrm{d}\theta \\
&= \int_0^{2\pi} r^5 \cos^2(\theta) \Big|_{r=0}^{r=3} \mathrm{d}\theta \\
&= \int_0^{2\pi} \left(3^5 \cdot \cos^2(\theta) - 0\right) \mathrm{d}\theta \\
&= \int_0^{2\pi} 243 \cos^2(\theta) \,\mathrm{d}\theta \\
&= \int_0^{2\pi} 243 \cdot \frac{\cos(2\theta) + 1}{2} \,\mathrm{d}\theta && \left(\text{as } \cos(2\theta) = 2\cos^2\theta - 1 \implies \cos^2\theta = \tfrac{\cos(2\theta)+1}{2}\right) \\
&= \frac{243}{2} \int_0^{2\pi} \left(\cos(2\theta) + 1\right) \mathrm{d}\theta \\
&= \frac{243}{2} \left(\frac{\sin(2\theta)}{2} + \theta\right) \Big|_0^{2\pi} \mathrm{d}\theta \\
&= \frac{243}{2} \left((0 + 2\pi) - (0 + 0)\right) \\
&= 243\pi
\end{aligned}$$

4.7 Triple Integrals in Spherical Coordinates (§15.8)



Definition 4.28 — Spherical Coordinates p. 1102

Each $P = (x, y, z) \in \mathbb{R}^3$ can be expressed in *spherical coordinates* (ρ, θ, ϕ), where

$$x = \rho \sin(\phi)\cos(\theta) \qquad y = \rho\sin(\phi)\sin(\theta) \qquad z = \rho\cos\phi$$

where $\rho \geq 0$ and $0 \leq \phi \leq \pi$. Note that $\rho^2 = x^2 + y^2 + z^2$.

Example 4.29: Write the rectangular coordinates $(1, 1, 0)$ in spherical coordinates.

We have

$$\rho^2 = x^2 + y^2 + z^2 = 1^2 + 1^2 + 0^2 = 2 \implies \rho = \sqrt{2}$$

Also,

$$\tan(\theta) = \frac{y}{x} = \frac{1}{1} \implies \theta = \frac{\pi}{4}$$

Finally,

$$\cos\phi = \frac{z}{\rho} = \frac{0}{\sqrt{2}} = 0 \implies \phi = \frac{\pi}{2}$$

Therefore, the spherical coordinates are $\left(\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{2}\right)$.

Theorem 4.30 — Triple Integration in Spherical Coordinates

Let $a, b, \alpha, \beta, c, d \in \mathbb{R}$. If f is continuous on a *spherical wedge*

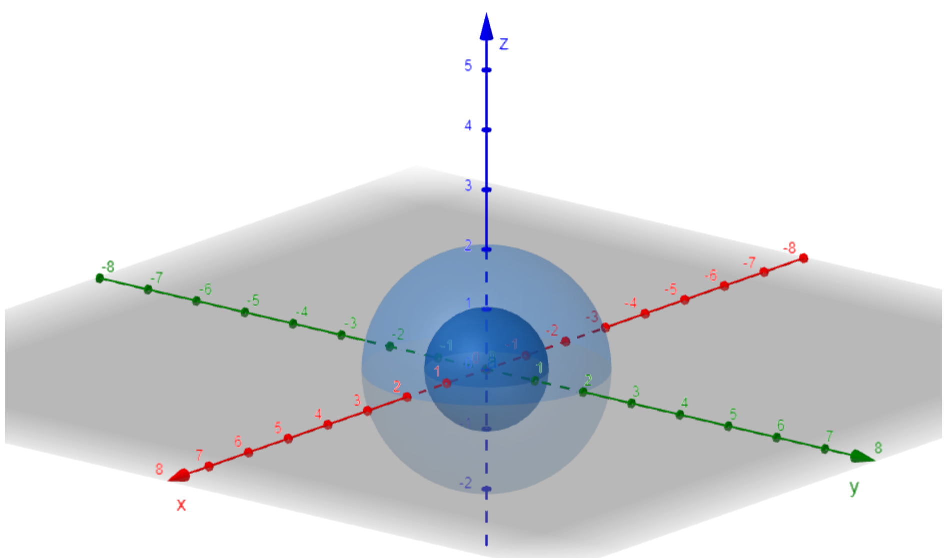
$$E = \left\{(\rho, \theta, \phi) : a \leq \rho \leq b \wedge \alpha \leq \theta \leq \beta \wedge c \leq \phi \leq d\right\}$$

then

$$\iiint\limits_E f(x, y, z)\,\mathrm{d}V = \int_c^d \int_\alpha^\beta \int_a^b f\big(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos\phi\big)\rho^2\sin(\phi)\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi$$

Example 4.31: Evaluate $\iiint\limits_E \left(x^2 + y^2\right)\mathrm{d}V$, where E is the region between $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 1$.

We have the following solid region:



From the diagram, we have spherical wedge

$$E = \left\{(\rho, \theta, \phi) : 1 \leq \rho \leq 2 \wedge 0 \leq \theta \leq 2\pi \wedge 0 \leq \phi \leq \pi\right\}$$

Thus, the integral becomes

$$\begin{aligned}
\int_0^{\pi}\int_0^{2\pi}\int_1^2 \left(\rho^2\sin^2(\phi)\right)\rho^2\sin(\phi)\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi &= \int_0^{\pi}\int_0^{2\pi}\int_1^2 \rho^4\sin^3(\phi)\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \int_0^{\pi}\int_0^{2\pi} \left.\frac{\rho^5\sin^3(\phi)}{5}\right|_{\rho=1}^{\rho=2}\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \int_0^{\pi}\int_0^{2\pi}\left(\frac{2^5\cdot\sin^3(\phi)}{5}-\frac{\sin^3(\phi)}{5}\right)\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \int_0^{\pi}\int_0^{2\pi}\frac{31\sin^3(\phi)}{5}\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \int_0^{\pi}\left.\frac{31\sin^3(\phi)}{5}\cdot\theta\right|_{\theta=0}^{\theta=2\pi}\mathrm{d}\phi \\
&= \int_0^{\pi}\left(\frac{31\sin^3(\phi)}{5}\cdot 2\pi - 0\right)\mathrm{d}\phi \\
&= \int_0^{\pi}\frac{62\pi\sin^3(\phi)}{5}\,\mathrm{d}\phi \\
&= \frac{62\pi}{5}\int_0^{\pi}\sin^2(\phi)\cdot\sin(\phi)\,\mathrm{d}\phi \\
&= \frac{62\pi}{5}\int_0^{\pi}\left(1-\cos^2(\phi)\right)\cdot\sin(\phi)\,\mathrm{d}\phi && \text{(by the Pythagorean identity)}
\end{aligned}$$

Now let $u=\cos(\phi)$. We have $\mathrm{d}u=-\sin(\phi)$, $\phi=0 \implies u=1$, and $\phi=\pi \implies u=-1$, so the integral becomes

$$\begin{aligned}
\frac{62\pi}{5}\int_1^{-1}\left(1-u^2\right)(-\,\mathrm{d}u) &= \frac{62\pi}{5}\int_1^{-1}\left(u^2-1\right)\mathrm{d}u \\
&= \frac{62\pi}{5}\left.\left(\frac{u^3}{3}-u\right)\right|_1^{-1} \\
&= \frac{62\pi}{5}\left(\left(-\frac{1}{3}-(-1)\right)-\left(\frac{1}{3}-1\right)\right) \\
&= \frac{62\pi}{5}\cdot\frac{4}{3} \\
&= \frac{248\pi}{15}
\end{aligned}$$

Theorem 4.32 — Independent Triple Integrals

Let $B=[a,b]\times[c,d]\times[e,f]$ for some $a,b,c,d,e,f\in\mathbb{R}$. If f, g, and h are continuous on $[a,b]$, $[c,d]$, and $[e,f]$ (respectively), then

$$\iiint_B f(x)g(y)h(z)\,\mathrm{d}V = \left(\int_a^b f(x)\,\mathrm{d}x\right)\left(\int_c^d g(x)\,\mathrm{d}y\right)\left(\int_e^f h(z)\,\mathrm{d}z\right)$$

This holds for *all permutations* of $\int_a^b f(x)\,\mathrm{d}x$, $\int_c^d g(x)\,\mathrm{d}y$, and $\int_e^f h(z)\,\mathrm{d}z$.

PROOF: We have

$$\iiint_B f(x)g(y)h(z)\,\mathrm{d}V = \int_a^b\int_c^d\int_e^f f(x)g(y)h(z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \qquad \text{(by theorem 4.17)}$$

$$= \int_a^b f(x) \left(\int_c^d g(y) \left(\int_e^f h(z) \, \mathrm{d}z \right) \mathrm{d}y \right) \mathrm{d}x$$

(as $f(x)$ is constant with respect to y and z, and $g(y)$ is constant with respect to z)

$$= \left(\int_a^b f(x) \, \mathrm{d}x \right) \left(\int_c^d g(x) \, \mathrm{d}y \right) \left(\int_e^f h(z) \, \mathrm{d}z \right) \qquad \blacksquare$$

Example 4.33: Consider $\iiint\limits_E (x^2 + y^2) \, \mathrm{d}V$, where E is the region between $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 1$ (this is the same example as example 4.31).

It was established that the region of integration is

$$E = \left\{ (\rho, \theta, \phi) : 1 \leq \rho \leq 2 \wedge 0 \leq \theta \leq 2\pi \wedge 0 \leq \phi \leq \pi \right\}$$

It is easier to integrate over this region using theorem 4.32, which yields

$$\begin{aligned} \int_0^\pi \int_0^{2\pi} \int_1^2 \left(\rho^2 \sin^2(\phi) \right) \rho^2 \sin(\phi) \, \mathrm{d}\rho \, \mathrm{d}\theta \, \mathrm{d}\phi &= \int_0^\pi \int_0^{2\pi} \int_1^2 \rho^4 \sin^3(\phi) \, \mathrm{d}\rho \, \mathrm{d}\theta \, \mathrm{d}\phi \\ &= \left(\int_0^\pi \sin^3(\phi) \, \mathrm{d}\phi \right) \left(\int_0^{2\pi} 1 \, \mathrm{d}\theta \right) \left(\int_1^2 \rho^4 \, \mathrm{d}\rho \right) \end{aligned}$$

Let $u = \cos\phi$ so that $\mathrm{d}u = -\sin(\phi) \, \mathrm{d}\phi$, $\phi = 0 \implies u = 1$, and $\phi = \pi \implies u = -1$. This yields

$$\begin{aligned} \left(\int_1^{-1} \left(1 - u^2 \right) (-\mathrm{d}u) \right) \left(\theta \Big|_0^{2\pi} \right) \left(\frac{\rho^5}{5} \Big|_1^2 \right) &= \left(\int_1^{-1} \left(u^2 - 1 \right) \mathrm{d}u \right) \left(\theta \Big|_0^{2\pi} \right) \left(\frac{\rho^5}{5} \Big|_1^2 \right) \\ &= \left(\frac{u^3}{3} - u \right) \Bigg|_1^{-1} \cdot 2\pi \cdot \left(\frac{2^5}{5} - \frac{1}{5} \right) \\ &= \left(\left(-\frac{1}{3} - (-1) \right) - \left(\frac{1}{3} - 1 \right) \right) \cdot 2\pi \cdot \frac{31}{5} \\ &= \frac{4}{3} \cdot \frac{62\pi}{5} \\ &= \frac{248\pi}{15} \end{aligned}$$

4.8 Change of Variables (§15.9)

Recall that for single integrals, a substitution would be of the form

$$\int_a^b f\big(g(x)\big) g'(x) \, \mathrm{d}x = \int_{g(a)}^{g(b)} f(u) \, \mathrm{d}u$$

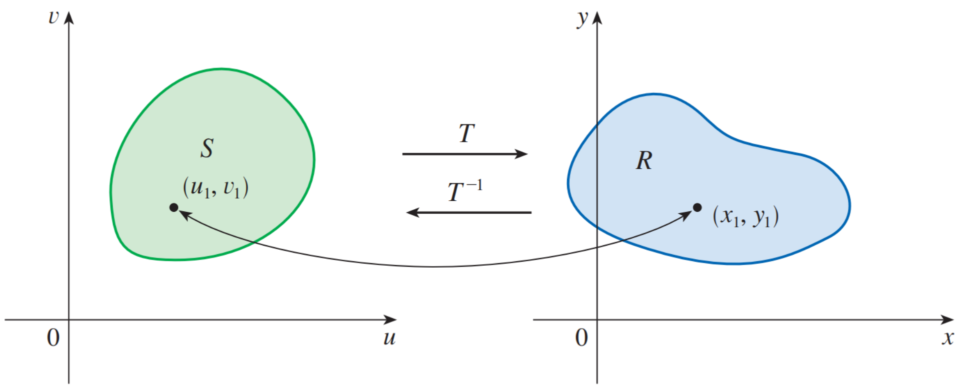
letting $u = g(x)$ so that $\mathrm{d}u = g'(x) \, \mathrm{d}x$. We now generalize this notion to multiple integrals.

Definition 4.34 — C^1 Transformation

Suppose that $T \colon S \to \mathbb{R}^2$ for some $S \subseteq \mathbb{R}^2$ is the transformation given by

$$T(u, v) = (x, y)$$

where $x = g(u, v)$ and $y = h(u, v)$ for some functions g and h. The map T is called a C^1 *transformation* or said to be *class* C^1 if the first-order partial derivatives of g and h are continuous on S.



Definition 4.35 — Image of a Transformation

The *image* of $S \subseteq \mathbb{R}^2$ under a transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is

$$T(S) = \{T(u, v) : (u, v) \in S\}$$

Definition 4.36 — Jacobian of a Transformation (p. 1111)

Given a transformation $T(u, v) = (x, y)$, where $x = g(u, v)$ and $y = h(u, v)$, the *Jacobian* of T is denoted

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

Example 4.37: Let $T = \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$. Compute $\dfrac{\partial(x,y)}{\partial(r,\theta)}$.

By definition,

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = \cos(\theta)\cdot r\cos\theta - (-r\sin\theta)\cdot\sin\theta = r\cos^2\theta + r\sin^2\theta = r\left(\cos^2\theta + \sin^2\theta\right) = r$$

Theorem 4.38 — Change of Variables (p. 1112)

Let $S, R \subseteq \mathbb{R}^2$ be type I or II regions and $T\colon S \to R$ be class C_1 and one-to-one, where $R = T(S)$. If f is continuous on R and S, then

$$\iint_R f(x,y)\,\mathrm{d}A = \iint_S f\big(x(u,v), y(u,v)\big)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\mathrm{d}u\,\mathrm{d}v$$

Remark: The vertical bars enclosing $\frac{\partial(x,y)}{\partial(u,v)}$ in theorem 4.38 denote the absolute value, whereas the vertical bars in definition 4.36 denote the matrix determinant.

Theorem 4.38 also generalizes to triple integrals.

Example 4.39: Find the area of the region $R \subseteq \mathbb{R}^2$ bounded by $xy = 1$, $xy = 3$, $xy^2 = 1$, $xy^2 = 2$, $x > 0$, and $y > 0$.

Let $u = xy$ and $v = xy^2$. We have the transformation T whose domain is $S = \big\{(u,v) : 1 \leq u \leq 3 \wedge 1 \leq v \leq 2\big\}$. Now

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

Note that

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ y^2 & 2xy \end{vmatrix} = 2xy^2 - xy^2 = xy^2$$

Since $\det\big(A^{-1}\big) = \det(A)$ for all linear transformations,

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{xy^2} = \frac{1}{v}$$

Therefore, using theorem 4.32, the desired area is

$$\int_1^2 \int_1^3 \frac{1}{v}\,\mathrm{d}u\,\mathrm{d}v = \left(\int_1^2 \frac{1}{v}\right)\left(\int_1^3 1\,\mathrm{d}u\right) = \left(\ln(v)\Big|_1^2\right)\left(u\Big|_1^3\right) = \big(\ln(2) - \ln(1)\big)(3-1) = 2\ln(2)$$