MAT B43 — Introduction to Analysis

Fall 2023

This is a compilation of the notes from the MAT B43 lectures. Each of the facts (definitions, theorems, axioms, etc.) are numbered for cross-referencing purposes.

Contents

Useful Facts

Chapter 1 Number Systems

1.1 Basic Set Theory

Definition 1.1 — Set

A *set* is a collection of objects, called *elements*.

Definition 1.2 — Relation

A *relation* of pairs of elements in a set S is a set of ordered pairs $(a, b) \in S \times S$.

Example 1.3: Consider $S = \{1, 2, 3\}$ and $\mathcal{R} = \big\{(1, 2), (2, 3), (3, 3)\big\}$.

Note that 1 is not related to 1, 2 is not related to 2, 3 is not related to 2, and 1 is not related to 3.

Example 1.4: Consider $S = \mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathcal{R} = \big\{(a, b) \in \mathbb{N} \times \mathbb{N} : |a - b| \text{ is a multiple of } 2\big\}$.

Note that $\mathcal{R} = \big\{(a, b) \in \mathbb{N} \times \mathbb{N} : a \text{ and } b \text{ have the same parity}\big\}$.

Definition 1.5 — Equivalence Relation

A relation $\mathcal{R}$ over a set S is an *equivalence relation* if the following conditions hold:

1. For all $x \in S$, $(x, x) \in \mathcal{R}$ (*reflexivity*)
2. If $(x, y) \in \mathcal{R}$, then $(y, x) \in \mathcal{R}$ (*symmetry/commutativity*)
3. If $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$, then $(x, z) \in \mathcal{R}$ (*transitivity*)

The symbol $\sim$ is used to denote equivalence between elements in a set S. For any $a, b \in S$, $a \sim b$ if and only if $(a, b) \in \mathcal{R}$.

Remark: Note that the relation in example 1.3 is not an equivalence relation, while the one in example 1.4 is an equivalence relation.

Definition 1.6 — Equivalence Class

For a set S, $C \subseteq S$ is said to be an *equivalence class* if C contains only

1. Equivalent elements, and
2. All such equivalent elements in S

More formally, given the equivalence relation $\sim$, the equivalence class of $c \in S$ is the set

$$[c] = \{x \in S : x \sim c\}$$

Proposition 1.7 — Properties of Equivalence Classes

For any set S, the following properties hold:

1. Any two distinct equivalence classes of S are disjoint
2. The union of all equivalence classes of S is S itself

Example 1.8: Consider $S = \big\{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1 \land 0 \le y \le 1\big\}$ and $A, B \in S$ with $A = (x, y)$ and $B = (x', y')$. Define the equivalence relation $\sim$ as $A \sim B$ if and only if $x = x'$.

Observe that

1. $A \sim A$ (as $x = x$)
2. $A \sim B \implies B \sim A$ (as $x = x' \implies x' = x$)

3. $A \sim B \land B \sim C \implies A \sim C$ (as for $C = (x'', y'') \in S$, $x = x' \land x' = x'' \implies x = x''$)

so S is an equivalence relation. The equivalence classes are the vertical line segments of the form $\{(x,y) : 0 \leq y \leq 1\}$ for some fixed $x \in [0,1]$.

Definition 1.9 — Quotient Set

For any set S, the *quotient set* is the set of all equivalence classes of S. We denote the quotient set of S by

$$S/\sim$$

where $\sim$ is the equivalence relation used to define the equivalence classes.

Example 1.10: Consider $S = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \land 0 \leq y \leq 1\}$ and $A, B \in S$ with $A = (x,y)$ and $B = (x', y')$. Define the equivalence relation $\sim$ as $A \sim B$ if and only if $x = x'$. (This is the same setup as in example 1.8.)

The quotient set of S is

$$Q = \left\{ l : l(x) = \bigcup_{0 \leq y \leq 1} (x,y) \right\}$$

which is indeed the union of the vertical lines l with their specified x-values.

1.2 Constructing the Natural Numbers

Axiom 1.11 — Peano Axioms

The principal axioms of the natural numbers (positive integers) are as follows:

1. $1 \in \mathbb{N}$
2. For all $n \in \mathbb{N}$, there exists a successor $\hat{n}$ of n which satisfies
   (a) For all $n \in \mathbb{N}$, $\hat{n} \neq 1$
   (b) $\hat{m} = \hat{n} \implies m = n$
3. If some property $Q(n)$ (for $n \in \mathbb{N}$) satisfies the following conditions:
   (a) $Q(1)$ is true
   (b) $Q(n) \implies Q(\hat{n})$

   then $Q(n)$ holds for all $n \in \mathbb{N}$.

Example 1.12: Let $Q(n)$ be the predicate $1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$.

$Q(1)$ holds as $1 = \frac{1 \cdot 2}{2}$. Now if $Q(n)$ holds for some $n \in \mathbb{N}$,

$$1 + 2 + \ldots + n + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)\left((n+1)+1\right)}{2}$$

so $Q(n+1)$ also holds. Therefore, $Q(n)$ holds for all $n \in \mathbb{N}$.

1.3 Constructing the Integers

Definition 1.13 — The Set of Integers

The set of *integers* $\mathbb{Z}$ is defined as

$$\mathbb{Z} = (\mathbb{N} \times \mathbb{N})/\sim$$

where $\sim$ is the equivalence relation given by $(a, b) \sim (n, m)$ if and only if $a+m = b+n$. Each equivalence class $[(a, b)]$ is denoted by $b - a$.

Remark: Observe that definition 1.13 only uses addition in $\mathbb{N}$, and the notation for the equivalence classes additionally uses the definition of subtraction in $\mathbb{Z}$ (though it may be defined using only subtraction in $\mathbb{N}$, as demonstrated in section A1.2 of the textbook).

Example 1.14: Consider the set $\mathbb{N} \times \mathbb{N}$ and relation $\sim$ given by $(a, b) \sim (n, m)$ if and only if $a + m = b + n$, as in definition 1.13. Prove that $\sim$ is an equivalence relation.

Proof: Since $a + b = b + a$, $(a, b) \sim (a, b)$ and thus $\sim$ is reflexive.

If $(a, b) \sim (n, m)$, $a + m = b + n$ and thus $n + b = m + a$, so $(n, m) \sim (a, b)$. Therefore, $\sim$ is symmetric.

Now if $(a, b) \sim (n, m)$ and $(n, m) \sim (c, d)$, we have $a + m = b + n$ and $n + d = m + c$. This yields $n = m + c - d$ and thus $a + m = b + n \implies a + m = b + (m + c - d) \implies a + d = b + c$. Thus, $(a, b) \sim (c, d)$, so $\sim$ is transitive.

Therefore, $\sim$ is an equivalence relation. ∎

Example 1.15: Consider the set $\mathbb{N}\times\mathbb{N}$ and the equivalence relation $\sim$ given by $(a, b) \sim (n, m)$ if and only if $a+m = b+n$. (This is the same setup as in definition 1.13.)

Note that $[(1, 2)] = [(2, 3)] = [(3, 4)] = \big\{(n, n + 1) : n \in \mathbb{N}\big\}$, which is denoted by 1, and $[(2, 1)] = [(3, 2)] = [(4, 3)] = \big\{(n, n - 1) : n \in \mathbb{N}\big\}$, which is denoted by -1.

Definition 1.16 — Addition and Multiplication on the Integers

Let $A, B \in \mathbb{Z}$ with $A = [(a, b)]$ and $B = [(n, m)]$ (expressed in $\mathbb{Z} = (\mathbb{N} \times \mathbb{N})/\sim$ as in definition 1.13).

Addition on the integers is defined as the operation $+ \colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by

$$A + B = [(a + n, b + m)]$$

Multiplication on the integers is defined as the operation $\cdot \colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by

$$A \cdot B = [(am + bn, bm + an)]$$

Remark: Note that the operations in definition 1.16 are defined using addition and multiplication in $\mathbb{N}$ only.

Definition 1.17 — The Set of Rational Numbers

The set of *rational* numbers $\mathbb{Q}$ is defined as

$$\mathbb{Q} = \big(\mathbb{Z} \times (\mathbb{Z} - \{0\})\big)/\sim$$

where $\sim$ is the equivalence relation given by $(a, b) \sim (n, m)$ if and only if $am = bn$. Each equivalence class $[(a, b)]$ is denoted by $\dfrac{a}{b}$.

Definition 1.18 — Addition and Multiplication on the Rationals

Let $A, B \in \mathbb{Q}$ with $A = [(a, b)]$ and $B = [(c, d)]$ (expressed in $\big(\mathbb{Z} \times (\mathbb{Z} - \{0\})\big)/\!\sim$ from definition 1.17).

Addition on the rationals is defined as the operation $+\colon \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ given by

$$A + B = [(ad + cb, bd)]$$

Multiplication on the rationals is defined as the operation $\cdot\colon \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ given by

$$A \cdot B = [(ac, bd)]$$

1.4 The Real Numbers

Many numbers are not rational, including $x = \sqrt{2} \notin \mathbb{Q}$. We consider the set $S = \big\{x \in \mathbb{Q} : x^2 < 2\big\} \subseteq \mathbb{Q}$. $\sqrt{2} \equiv S$ in the sense that it is the least upper bound of S. This is the idea used to construct $\mathbb{R}$ in this section.

Definition 1.19 — Order

The number a is said to be greater than b, denoted by $a > b$, if $a - b > 0$.

Remark: The $>$ in $a - b > 0$ refers to the comparison of Dedekind cuts (as in definition 1.32) representing the corresponding real numbers in the inequality.

Definition 1.20 — Upper Bound and Lower Bound

Let A be an ordered set and $S \subseteq A$ be non-empty.

$a \in A$ is an *upper bound* for S if for all $x \in S$, $x \leq a$.

$b \in A$ is a *lower bound* for S if for all $x \in S$, $x \geq b$.

Definition 1.21 — Least Upper Bound

Let A be an ordered set and $S \subseteq A$ be a set with an upper bound. $M \in A$ is said to be the *least upper bound* (or *supremum*) of S, denoted $M = \sup(S)$, if the following conditions are satisfied:

1. M is an upper bound for S
2. If $a \in A$ is an upper bound for S, then $M \leq a$

Proposition 1.22 — Least Upper Bound Uniqueness

If a set has a least upper bound, it is unique.

Proof: Let S be a set with least upper bounds M_1 and M_2. For all upper bounds l for S, $M_1 \leq l$ and $M_2 \leq l$ (by definition). Therefore, $M_1 \leq M_2$ and $M_2 \leq M_1$, so $M_1 = M_2$. ■

Definition 1.23 — Greatest Lower Bound

Let A be an ordered set and $S \subseteq A$ be a set with a lower bound. $m \in A$ is said to be the *greatest lower bound* (or *infimum*) for S, denoted $m = \inf(S)$, if the following conditions are satisfied:

1. m is a lower bound for S
2. If $b \in A$ is a lower bound for S, then $m \geq b$

Proposition 1.24 — Greatest Lower Bound Uniqueness

If a set has a greatest lower bound, it is unique.

Theorem 1.25 — Existence of $\mathbb{R}$

There exists a set $\mathbb{R}$ that satisfies each of the following conditions:

1. $\mathbb{R}$ is an ordered field with operations $+$ and $\times$
2. $\mathbb{R}$ contains $\mathbb{Q}$
3. Any bounded above subset $S \subseteq \mathbb{R}$ has a supremum in $\mathbb{R}$

$\mathbb{R}$ is the set of *real numbers.*

Example 1.26: Consider $S = \left\{x \in \mathbb{Q} : x^2 < 2\right\} \subseteq \mathbb{R}$.

S has a supremum in $\mathbb{R}$, namely $\sup(S) = \sqrt{2}$. Note that $\sup(S) \notin S$ and $\sup(S) \notin \mathbb{Q}$.

Example 1.27 *(Irrationality of $\sqrt{2}$)*: Show that for $S = \left\{x \in \mathbb{Q} : x^2 < 2\right\} \subseteq \mathbb{Q}$, $\sup(S) \notin \mathbb{Q}$.

Proof: By the set construction in the proof to proposition 1.28, $(\sup(S))^2 = 2$. Suppose, for the sake of contradiction, that $M = \frac{a}{b}$ with $a, b \in \mathbb{N}$ such that $\gcd(a, b) = 1$. It follows that

$$M^2 = 2 = \frac{a^2}{b^2} \implies a^2 = 2b^2 \implies 2 \mid a^2 \implies 2 \mid a$$

Now let $a = 2a_1$ for some $a_1 \in \mathbb{N}$. We obtain

$$(2a_1)^2 = 2b^2 \implies 4a_1^2 = 2b^2 \implies b^2 = 2a_1^2 \implies 2 \mid b^2 \implies 2 \mid b$$

We have shown that $2 \mid a$ and $2 \mid b$, which contradicts the fact that $\gcd(a, b) = 1$. Therefore, by contradiction, $M \notin \mathbb{Q}$. ■

Proposition 1.28 — Existence of the Square Root in $\mathbb{R}$

For all $x \in \mathbb{R}$ with $x > 0$, there exists some $y \in \mathbb{R}$ with $y > 0$ such that

$$y^2 = x$$

That is, y is the *square root* of x, denoted $y = \sqrt{x}$.

Proof: For any $x \in \mathbb{R}$ with $x > 0$, consider the set

$$S_x = \left\{w \in \mathbb{R} : w > 0 \wedge w^2 < x\right\}$$

Note that S_x is bounded above by x if $x > 1$ and 1 if $x \leq 1$.

If $x > 1$, $1^2 = 1 < x \implies 1 \in S_x$.

If $x = 1$, $\left(\frac{1}{2}\right)^2 = \frac{1}{4} < x \implies \frac{1}{2} \in S_x$.

If $x < 1$, $x^2 < x \implies x \in S_x$ (as $0 < x < 1 \implies x \cdot x < 1 \cdot x \implies x^2 < x$ using a multiplicative property of ordered fields).

In any case, S_x is non-empty. Now observe that if $w \in S_x$ and $z \in \mathbb{R}$ satisfies $0 < z < w$, then

$$z < w \implies z^2 < w^2 \implies z^2 < x \implies z \in S_x$$

so for all $z \in \mathbb{R}$ with $0 < z < w$, $z \in S_x$. We will now prove that $y = \sup(S_x)$ satisfies $y^2 = x$.

Assume, for the sake of contradiction, that $y^2 < x$. We will show that there exists some $\epsilon \in \mathbb{R}^+$ such that $(y + \epsilon)^2 < x$. Note that for $\epsilon < 1$, $\epsilon^2 < \epsilon$, so

$$(y + \epsilon)^2 = y^2 + 2y\epsilon + \epsilon^2 < y^2 + 2y\epsilon + \epsilon$$

Now

$$y^2 + 2y\epsilon + \epsilon < x \iff \epsilon(2y + 1) < x - y^2 \iff \epsilon < \frac{x - y^2}{2y + 1}$$

so for $\epsilon = \min\left(1, \frac{1}{2} \cdot \frac{x - y^2}{2y + 1}\right)$, $(y + \epsilon)^2 < x$. We thus have $y + \epsilon \in S_x$ and $y + \epsilon > y$, which contradicts the fact that $y = \sup(S_x)$. Therefore, $y^2 \geq x$ by contradiction.

Now suppose that $y^2 > x$, for the sake of contradiction. We will show that for some $\epsilon \in \mathbb{R}^+$, $(y-\epsilon)^2 \geq x$. Observe that

$$(y-\epsilon)^2 = y^2 - 2y\epsilon + \epsilon^2 > y^2 - 2y\epsilon - \epsilon = y^2 - \epsilon(2y+1)$$

Now

$$y^2 - \epsilon(2y+1) \geq x \iff y^2 - x \geq \epsilon(2y+1) \iff \epsilon \leq \frac{y^2 - x}{2y+1}$$

Choose $\epsilon = \dfrac{y^2-x}{2y+1}$. We have shown that $\epsilon \leq \frac{y^2-x}{2y+1} \implies (y-\epsilon)^2 \geq x$, so $y - \epsilon < y$ is an upper bound for S_x. This contradicts the fact that $y = \sup(S_x)$, so we cannot have $y^2 > x$.

Since $y^2 < x$ and $y^2 > x$ do not hold, $y^2 = \sup(S_x)$. ■

Remark: Proposition 1.28 does *not* hold in $\mathbb{Q}$; consider the counterexample $x = 2 \in \mathbb{Q}$, where $y^2 \neq 2$ for all $y \in \mathbb{Q}$.

Theorem 1.29 — Archimedean Property for $\mathbb{R}^+$

For all $a, b \in \mathbb{R}$ with $0 < a < b$, there exists some $n \in \mathbb{N}$ such that

$$na > b$$

Proof: Assume, for the sake of contradiction, that $na \leq b$ for all $n \in \mathbb{N}$. Consider the set

$$M_a = \{na : n \in \mathbb{N}\}$$

$M_a \neq \emptyset$ as $a \in M_a$, and M_a is bounded above by b. Thus, we let $y = \sup(M_a)$. There exists some $k \in \mathbb{N}$ such that

$$y - a < ka \leq y$$

(since y is the supremum and $ka \in M_a$), so $y < ka + a = (k+1)a$ and $(k+1)a \in M_a$. This contradicts the fact that y is an upper bound for M_a. By contradiction, there exists some $n \in \mathbb{N}$ such that $na > b$. ■

Theorem 1.30 — Density of $\mathbb{Q}$ in $\mathbb{R}$

For all $x, y \in \mathbb{R}$ with $x < y$, there exists some $q \in \mathbb{Q}$ such that $x < q < y$.

Proof: For simplicity, assume that $x, y \in \mathbb{R}^+$ (the cases where x or y is negative follow similarly). Let $q = \frac{n}{m}$ for some $n, m \in \mathbb{N}$. It follows that

$$x < q < y \iff x < \frac{n}{m} < y \iff mx < n < my$$

Now let $\epsilon = y - x$. By the Archimedean property, $m\epsilon > 10$ for some m, so

$$m\epsilon = m(y-x) = my - mx > 10 \implies my > mx + 10$$

There exists some n such that $mx < n < mx + 10$, for which we have

$$mx < n < mx + 10 < my \implies mx < n < my$$

Therefore, for the chosen m and n, $x < q = \frac{n}{m} < y$. ■

Definition 1.31 — Dedekind Cut

A *Dedekind cut* is a set $\mathcal{C} \subseteq \mathbb{Q}$ that satisfies each of the following conditions:

1. $\mathcal{C} \neq \emptyset$
2. $\mathcal{C}$ is bounded above
3. If $q \in \mathcal{C}$, then for all $q' < q$, $q' \in \mathcal{C}$
4. If $q \in \mathcal{C}$, there exists some $q^+ \in \mathcal{C}$ with $q^+ > q$

Remark: The idea of a cut (as in definition 1.31) is to construct an "interval" of rational numbers (only) before defining the real numbers (and intervals). That is, cuts construct

$$(-\infty, a) \cap \mathbb{Q}$$

for $a \in \mathbb{R}$ without relying on $\mathbb{R}$.

Definition 1.32 — $\mathbb{R}$ as a Set of Cuts

The field $\mathbb{R}$ is defined by

$$\mathbb{R} = \{\mathcal{C} : \mathcal{C} \text{ is a cut}\}$$

Let $\mathcal{C}_1, \mathcal{C}_2 \in \mathbb{R}$. The field has addition defined as

$$\mathcal{C}_1 + \mathcal{C}_2 = \{x + y : x \in \mathcal{C}_1 \wedge y \in \mathcal{C}_2\}$$

ordering defined as

$$\mathcal{C}_1 \leq \mathcal{C}_2 \iff \mathcal{C}_1 \subseteq \mathcal{C}_2$$

the additive identity defined as the set of all negative rationals $\hat{0} = \mathbb{Q}^-$, and multiplication defined as

1. $\mathcal{C}_1 \cdot \mathcal{C}_2 = \{q \in \mathbb{Q} : q < xy \text{ for some } x \in \mathcal{C}_1 \text{ and } y \in \mathcal{C}_2 \text{ with } x > 0 \wedge y > 0\}$ if $\mathcal{C}_1 > \hat{0}$ and $\mathcal{C}_2 > \hat{0}$
2. $-\big(\mathcal{C}_1 \cdot (-\mathcal{C}_2)\big)$ if $\mathcal{C}_1 > \hat{0}$ and $\mathcal{C}_2 < \hat{0}$
3. $-\big((-\mathcal{C}_1) \cdot \mathcal{C}_2\big)$ if $\mathcal{C}_1 < \hat{0}$ and $\mathcal{C}_2 > \hat{0}$
4. $(-\mathcal{C}_1) \cdot (-\mathcal{C}_2)$ if $\mathcal{C}_1 < \hat{0}$ and $\mathcal{C}_2 < \hat{0}$
5. $\hat{0}$ if $\mathcal{C}_1 = \hat{0}$ or $\mathcal{C}_2 = \hat{0}$

Remark: The set of cuts $\mathbb{R}$ (as in definition 1.32) is the set of real numbers, whose existence is asserted by theorem 1.25.

Theorem 1.33 — Least Upper Bound Property

Let $\mathcal{A}$ be a cut. For any bounded above collection of cuts $S = \{\mathcal{C} : \mathcal{C} \subset \mathcal{A}\}$ satisfies

$$\sup(S) = \bigcup_{\mathcal{C} \in S} \mathcal{C}$$

That is, the supremum exists.

Remark: Using theorem 1.33, each $a \in \mathbb{R}$ can now be uniquely identified as the supremum of a cut $\mathcal{C}$.

Chapter 2 Sequences

2.1 Cardinality

Definition 2.1 — Injective, Surjective, and Bijective

Let A and B be sets and $f\colon A \to B$ ($f(A) \subseteq B$).

We say that f is *one-to-one* (or *injective*) if $f(x) = f(y) \implies x = y$.

We say that f is *onto* (or *surjective*) if $f(A) = B$.

f is *bijective* if it is one-to-one and onto.

Example 2.2: There is no bijection from $\{1,2\}$ to $\{1,2,3\}$. Similarly, there is no bijection from $\{1,2\}$ to $\{3\}$.

Definition 2.3 — Cardinality Equality

Let A and B be sets. We say that A and B have the *same cardinality* if there exists a bijection $f\colon A \to B$. The cardinality of A is denoted by $\operatorname{card}(A)$ or $|A|$.

Example 2.4: Do $\mathbb{Z}$ and $\mathbb{N}$ have the same cardinality?

Consider $f\colon \mathbb{N} \to \mathbb{Z}$ given by

$$f(n) = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even} \\ -\dfrac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

f is a bijection that maps the even natural numbers to the positive integers and the odd natural numbers to the non-positive integers. Therefore, $|\mathbb{N}| = |\mathbb{Z}|$.

2.2 Sequences

Definition 2.5 — Sequence

Let A be a set. A *sequence* is a function $f\colon \mathbb{N} \to A$. We denote the outputs of sequences by

$$\{f(1), f(2), f(3), \ldots, f(n), \ldots\} \equiv \{f_1, f_2, f_3, \ldots, f_n, \ldots\}$$

A *sequence of real numbers* is a collection

$$(a_1, a_2, a_3, \ldots, a_n, \ldots)$$

where each $a_i \in \mathbb{R}$. We will use (a_n) to denote a sequence of real numbers.

Example 2.6: Some sequences are $\{1, 2, 3, \ldots\}$ and $\{2, 1, 3, 4, 5, \ldots\}$.

Definition 2.7 — Bounded Sequence

A sequence (a_n) is *bounded* if there exists some $M \in \mathbb{R}^+$ such that for all $n \in \mathbb{N}$,

$$|a_n| < M$$

The sequence is *bounded below* if there exists some $r \in \mathbb{R}$ such that for all $n \in \mathbb{N}$,

$$a_n > r$$

Similarly, the sequence is *bounded above* if there exists some $s \in \mathbb{R}$ such that for all $n \in \mathbb{N}$,

$$a_n < s$$

Definition 2.8 — Convergence

A sequence (a_n) *converges* to $\alpha \in \mathbb{R}$, denoted by

$$\lim_{n\to\infty} a_n = \alpha \text{ or } a_n \to \alpha$$

if for all $\epsilon > 0$, there exists some $N > 0$ such that

$$n > N \implies |a_n - \alpha| < \epsilon$$

Proposition 2.9 — Properties of Convergent Sequences

The following properties hold for any convergent sequence:

1. The limit of the sequence is unique
2. The sequence is bounded

PROOF: Let (a_n) be a convergent sequence. For (1), assume that the sequence has limits $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 \neq \alpha_2$ (for the sake of contradiction). Without loss of generality, suppose that $\alpha_2 > \alpha_1$. By definition, for all $\epsilon_1 > 0$, there exists some $N_1 > 0$ such that $n > N_1 \implies |a_n - \alpha_1| < \epsilon_1$. Similarly, for all $\epsilon_2 > 0$, there exists some $N_2 > 0$ such that $n > N_2 \implies |a_n - \alpha_2| < \epsilon_2$. Take $\epsilon = \dfrac{\alpha_2 - \alpha_1}{2}$ and $N = \max(N_1, N_2)$. We have

$$\alpha_1 + \epsilon < \alpha_2 - \epsilon$$

so for $n > N$ and $\epsilon_1 = \epsilon_2 = \epsilon$, both $|a_n - \alpha_1| < \epsilon$ and $|a_n - \alpha_2| < \epsilon$ cannot be satisfied. This is because $|a_n - \alpha_1| < \epsilon \implies a_n - \alpha_1 < \epsilon \implies a_n < \alpha_1 + \epsilon$, while $|a_n - \alpha_2| < \epsilon \implies a_n - \alpha_2 > -\epsilon \implies a_n > \alpha_2 - \epsilon$. By contradiction, the limit must be unique.

To prove (2), suppose that $(a_n) \to \alpha \in \mathbb{R}$. Let $\epsilon = 1$ with $|a_n - \alpha| < \epsilon$ when $n > N \in \mathbb{N}$. If $1 \le n \le N$,

$$\min\{a_1, \dots, a_N\} \le a_n \le \max\{a_1, \dots, a_N\}$$

For $n > N$, since $|a_n - \alpha| < \epsilon = 1$, we have

$$a_n - \alpha < 1 \implies a_n < \alpha + 1$$

and

$$a_n - \alpha > -1 \implies a_n > \alpha - 1$$

so $\alpha - 1 < a_n < \alpha + 1$. In either case, (a_n) is bounded both above and below, so (a_n) is bounded. ■

Example 2.10: Observe that $a_n = (-1)^n$ is bounded, but has no limit. This shows that the converse of (2) in proposition 2.9 does not hold in general.

Example 2.11: Consider the sequence given by $a_n = \dfrac{1}{n}(-1)^n$. We will show that $\frac{1}{n}(-1)^n \to 0$.

Let $\epsilon > 0$. We want to find some $N > 0$ such that for all $n > N$,

$$|a_n - 0| < \epsilon \iff \left|\frac{1}{n}(-1)^n\right| < \epsilon \iff \frac{1}{n} < \epsilon \iff n > \frac{1}{\epsilon}$$

Now choose $N = \dfrac{1}{\epsilon}$. As shown above, this choice guarantees that $n > N \implies |a_n - 0| < \epsilon$.

Example 2.12: Consider the sequence given by $a_n = \dfrac{1}{n^2}$. We will show that $a_n \to 0$.

Let $\epsilon > 0$. We will find some $N > 0$ such that for all $n > N$,

$$|a_n - 0| < \epsilon \iff \left|\frac{1}{n^2} - 0\right| < \epsilon \iff \frac{1}{n^2} < \epsilon \iff n^2 > \frac{1}{\epsilon} \iff n > \frac{1}{\sqrt{\epsilon}}$$

Choose $N = \dfrac{1}{\sqrt{\epsilon}}$. As shown above, this choice of N guarantees that $n > N \implies |a_n - 0| < \epsilon$.

Example 2.13 *($\mathbb{N}$ Is Not Bounded Above)*: Consider the sequence given by $a_n = n$. Assume that $a_n \to L \in \mathbb{R}$. For all $\epsilon > 0$, there exists some $N > 0$ such that $n > N \implies |a_n - L| < \epsilon$. Take $\epsilon = 1$ and N_1 be its corresponding value of N. For all $n > N_1$, we have

$$|a_n - L| < \epsilon \iff |n - L| < 1 \iff -1 < n - L < 1 \iff L - 1 < n < L + 1$$

so for all $n > N_1$, $n < L + 1$, which contradicts the fact that $a_n \to L$. By contradiction, a_n does not converge to any $L \in \mathbb{R}$.

Proposition 2.14 — Limit Properties

Let (a_n) and (b_n) be convergent sequences that converge to $a \in \mathbb{R}$ and $b \in \mathbb{R}$ (respectively). The following properties hold:

1. $(a_n + b_n) \to a + b$
2. For all $c \in \mathbb{R}$, $c \cdot a_n \to c \cdot a$
3. $a_n \cdot b_n \to a \cdot b$

Proof: We will prove properties (1) and (3).

By definition, for all $\epsilon > 0$, there exists some $N_1 > 0$ such that $n > N_1 \implies |a_n - a| < \epsilon$. Similarly, for all $\epsilon > 0$, there exists some $N_2 > 0$ such that $n > N_2 \implies |b_n - b| < \epsilon$.

To prove (1), let $\epsilon > 0$. For all $n > \max(N_1, N_2)$, we have

$$\big|(a_n + b_n) - (a + b)\big| = \big|(a_n - a) + (b_n) - b\big| \leq |a_n - a| + |b_n - b| < \epsilon + \epsilon = 2\epsilon$$

using the triangle inequality. Thus, if $N > 0$ corresponds to $\frac{\epsilon}{2}$, we have

$$\big|(a_n + b_n) - (a + b)\big| < 2 \cdot \frac{\epsilon}{2} = \epsilon$$

To prove (3), note that

$$\begin{aligned}
|a_n \cdot b_n - a \cdot b| &= |a_n b_n - a_n b + a_n b - ab| \\
&= \big|a_n(b_n - b) + b(a_n - a)\big| \\
&\leq \big|a_n(b_n - b)\big| + \big|b(a_n - a)\big| && \text{(by the triangle inequality)} \\
&= |a_n||b_n - b| + |b||a_n - a| \\
&< |a_n|\,\epsilon + |b|\,\epsilon \\
&= \epsilon\big(|a_n| + |b|\big) \\
&< \epsilon\big(M_1 + |b|\big) && \text{(for some } M_1 \in \mathbb{R}\text{, by the boundedness of } (a_n))
\end{aligned}$$

Take $N > 0$ to be a value corresponding to $\dfrac{\epsilon}{M_1 + |b|}$. We thus have

$$n > N \implies |a_n \cdot b_n - a \cdot b| < \frac{\epsilon}{\big(M_1 + |b|\big)} \cdot \big(M_1 + |b|\big) = \epsilon$$

■

Definition 2.15 — Cauchy Sequence

A sequence (a_n) is *Cauchy* if for all $\epsilon > 0$, there exists some $N > 0$ such that

$$n > N \wedge m > N \implies |a_n - a_m| < \epsilon$$

Remark: Cauchy sequences are used to prove convergence without finding limits.

Proposition 2.16 — Boundedness of Cauchy Sequences

Every Cauchy sequence is bounded.

PROOF: Let $\epsilon = 1$. For some $N > 0$, we have $|a_n - a_m| < 1$ for all $n, m > N$ by definition. Fix $m = m_0 > N$, and suppose that $n \geq N$. We have

$$|a_n - a_{m_0}| < 1$$

so a_n is bounded whenever $n \geq N$. If $1 \leq n < N$, we let $t = \min\left\{a_1, \ldots, a_{\lfloor N \rfloor}\right\}$ and $T = \max\left\{a_1, \ldots, a_{\lfloor N \rfloor}\right\}$ so that

$$t \leq a_n \leq T$$

In either case, $\min(t, a_{m_0} - 1)$ and $\max(T, a_{m_0} + 1)$ are lower and upper bounds for (a_n) (respectively). Therefore, (a_n) is bounded. ■

> **Example 2.17**: Consider the sequence given by $a_n = \dfrac{1}{n}$. We will show that (a_n) is Cauchy.
>
> Let $\epsilon > 0$. Note that
>
> $$\begin{aligned} |a_n - a_m| &= \left|\frac{1}{n} - \frac{1}{m}\right| \\ &\leq \left|\frac{1}{n}\right| + \left|-\frac{1}{m}\right| && \text{(by the triangle inequality)} \\ &= \frac{1}{n} + \frac{1}{m} \end{aligned}$$
>
> Choose $N = \dfrac{2}{\epsilon}$. If $n > N$ and $m > N$, then
>
> $$|a_n - a_m| \leq \frac{1}{n} + \frac{1}{m} < \frac{1}{2/\epsilon} + \frac{1}{2/\epsilon} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Theorem 2.18 — Convergence of Cauchy Sequences

A sequence is Cauchy if and only if it is convergent.

PROOF: Let (a_n) be a sequence.

($\Longrightarrow$) Suppose that $a_n \to L \in \mathbb{R}$. We have

$$\begin{aligned} |a_n - a_m| &= \big|(a_n - L) - (a_m - L)\big| \\ &\leq |a_n - L| + \big|-(a_m - L)\big| && \text{(by the triangle inequality)} \\ &= |a_n - L| + |a_m - L| \end{aligned}$$

There exists some $N > 0$ such that if $n > N$ and $m > N$, then $|a_n - L| < \frac{\epsilon}{2}$ and $|a_m - L| < \frac{\epsilon}{2}$, so

$$|a_n - L| + |a_m - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus, if a sequence converges, then it is Cauchy.

($\Longleftarrow$) Now suppose that (a_n) is Cauchy. We will show that (a_n) converges to the supremum of its "eventual" lower bounds. More formally, consider the set

$$S = \{x \in \mathbb{R} : n \geq k \implies a_i > x \text{ for some } k \in \mathbb{N}\}$$

Intuitively, k is the "eventual" index after which (a_n) is bounded below by x. Since Cauchy sequences are bounded, (a_n) has a lower bound, so $S \neq \emptyset$. Furthermore, (a_n) is bounded above as it is Cauchy. We will now show that $\lim\limits_{n\to\infty} a_n = \sup(S)$.

Suppose that for some $\epsilon > 0$, $|a_n - a_m| < \epsilon$ whenever $n > N$ and $m > N$ (for some $N > 0$). We fix m and let n vary. Note that

$$|a_n - a_m| < \epsilon \iff a_m - \epsilon < a_n < a_m + \epsilon$$

Thus, $a_m - \epsilon \in S$ (as it is a lower bound for the sequence when $n > N$), which yields $a_m - \epsilon \leq \sup(S)$. Also, $a_m + \epsilon \notin S$ (as it is not a lower bound for the sequence when $n > N$), so $\sup(S) \leq a_m + \epsilon$. It follows that

$$a_m - \epsilon \leq \sup(S) \leq a_m + \epsilon \implies \big|a_m - \sup(S)\big| \leq \epsilon$$

For any $\epsilon' > 0$, let $\epsilon = \dfrac{\epsilon'}{2}$. We have shown that for all $\epsilon' > 0$,

$$m > N \implies \left|a_m - \sup(S)\right| \le \epsilon = \frac{\epsilon'}{2} < \epsilon'$$

Therefore, $\lim\limits_{m\to\infty} a_m = \sup(S)$. ■

Remark: The idea of "eventual" lower bounds used in the proof for theorem 2.18 alludes to the idea of a lim inf, as covered later in definition 2.25.

Lemma 2.19 — Cauchy by Exponential Bound

Let (a_n) be a sequence. If $|a_{n+1} - a_n| \le \dfrac{1}{2^n}$ for all $n \in \mathbb{N}$, then (a_n) is Cauchy.

Proof: Fix $\epsilon > 0$, and let $n > m > N$ (where $N > 0$). We have

$$\begin{aligned}
|a_n - a_m| &= \left|(a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \ldots + (a_{m+1} - a_m)\right| \\
&\le |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \ldots + |a_{m+1} - a_m| && \text{(by the triangle inequality)} \\
&\le \frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \ldots + \frac{1}{2^m} \\
&= \frac{1}{2^m}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1-m}}\right) \\
&= \frac{1}{2^m} \cdot \frac{1 - (1/2)^{(n-1-m)+1}}{1 - (1/2)} && \text{(by the finite geometric series sum formula)} \\
&= \frac{1}{2^m} \cdot 2 \cdot \left(1 - \left(\frac{1}{2}\right)^{n-m}\right) \\
&= \frac{1}{2^{m-1}} - \frac{1}{2^{m-1}} \cdot \frac{1}{2^{n-m}} \\
&= \frac{1}{2^{m-1}} - \frac{1}{2^{n-1}} \\
&\le \frac{c}{2^{N-1}}
\end{aligned}$$

The last inequality holds for some $c \in \mathbb{R}^+$ as $n > m > N \implies \frac{1}{2^{n-1}} < \frac{1}{2^{m-1}} < \frac{1}{2^{N-1}}$. Thus, for $N = \log_2\left(\dfrac{2c}{\epsilon}\right)$, we have

$$|a_n - a_m| \le \frac{c}{2^{N-1}} = \frac{c}{2^{\log_2(2c/\epsilon)-1}} = \frac{2c}{2c/\epsilon} = \epsilon$$

■

Definition 2.20 — Subesequence

Let (a_n) be a sequence. (b_k) is a *subsequence* of (a_n) if for all $k \in \mathbb{N}$,

$$b_k = a_{i_k}$$

where each $i_k \in \mathbb{N}$ and $i_1 < i_2 < i_3 < \ldots$.

Example 2.21: Consider $(a_n) = (-1)^n$. We have $(a_n) = \{-1, 1, -1, 1, -1, 1, \ldots\}$, so for $i_k = 2k$,

$$a_{i_k} = a_{2k} = (-1)^{2k} = 1$$

Thus, $a_{i_k} \to 1$. Similarly, for $j_k = 2k+1$,

$$a_{j_k} = a_{2k+1} = (-1)^{2k+1} = -1$$

so $a_{j_k} \to -1$.

Theorem 2.22 — Bolzano–Weierstrass

Any bounded sequence has a *convergent* subsequence.

PROOF: Suppose that the sequence (a_n) is bounded with $|a_n| \leq M \in \mathbb{R}^{\geq 0}$ for all $n \in \mathbb{N}$. There are infinitely many terms of the sequence in the interval $[-M, M]$, so there must be infinitely many terms in $[-M, 0]$ or $[0, M]$. Without loss of generality, suppose that $[0, M]$ has infinitely many terms, and choose a_{i_1} to be one such term. Again, there must be infinitely many terms in $\left[0, \frac{M}{2}\right]$ or $\left[\frac{M}{2}, M\right]$. Without loss of generality, suppose that $\left[0, \frac{M}{2}\right]$ has infinitely many terms, and take a_{i_2} to be one such term with $i_2 > i_1$. Continuing this process, we obtain the subsequence $\left(a_{i_k}\right)$ with

$$\left|a_{i_{k+1}} - a_{i_k}\right| \leq \frac{M}{2^{k-1}}$$

for all $k \in \mathbb{N}$, as $a_{i_k}, a_{i_{k+1}} \in \left[0, \frac{M}{2^{k-1}}\right]$ (without loss of generality). By lemma 2.19, $\left(a_{i_k}\right)$ is Cauchy and thus converges. ■

Proposition 2.23 — Bounded Monotone Convergence

If a sequence $A = (a_n)$ is increasing and bounded above, then $a_n \to \sup(A)$.

Similarly, if the sequence is decreasing and bounded below, then $a_n \to \inf(A)$.

2.3 Limit Superior and Limit Inferior

Definition 2.24 — Supremum and Infimum of Unbounded Sets

Let $A \subseteq \mathbb{R}$. If A is not bounded above, we define $\sup(A) = \infty$. Similarly, we define $\inf(A) = -\infty$ if A is not bounded below.

Definition 2.25 — Limit Superior and Limit Inferior

Let (a_n) be a sequence and $A_k = \{a_k, a_{k+1}, a_{k+2}, \ldots\}$ for each $k \in \mathbb{N}$. The *limit superior* of (a_n) is defined as

$$\limsup a_n = \lim_{k\to\infty} \sup(A_k)$$

Similarly, the *limit inferior* of (a_n) is defined as

$$\liminf a_n = \lim_{k\to\infty} \inf(A_k)$$

Remark: In definition 2.25, the values of $\sup(A_k)$ form a decreasing sequence as the least upper bound of fewer terms in (a_n) is computed as $k \to \infty$. Similarly, the values of $\inf(A_k)$ form an increasing sequence as the greatest lower bound of fewer terms is computed as $k \to \infty$.

By proposition 2.23, $\limsup a_n$ and $\liminf a_n$ exist (they could be finite or infinite).

Example 2.26: Compute the $\limsup$ and $\liminf$ of the sequence given by $a_n = n \cdot \sin\left(n \cdot \frac{\pi}{2}\right)$.

By the periodicity of sin (as evaluated at integer multiples of $\frac{\pi}{2}$), $a_n = n \cdot \sin\left(n \cdot \frac{\pi}{2}\right)$ alternates between 0, n, 0, and $-n$. Thus, $\limsup a_n = \infty$ and $\liminf a_n = -\infty$.

Example 2.27: Show that if $\liminf a_n = \limsup a_n \in \mathbb{R}$ for some sequence (a_n), then (a_n) converges.

Let $A_k = \{a_k, a_{k+1}, a_{k+2}, \ldots\}$, where $k \in \mathbb{N}$. For all $a_n \in A_k$ (i.e. $n \geq k$), we have $\inf(A_k) \leq a_n \leq \sup(A_k)$. As $k \to \infty$, $n \to \infty$ for $n \geq k$. Thus,

$$\begin{aligned}
\lim_{k\to\infty} \inf(A_k) \leq \lim_{k\to\infty} a_n \leq \lim_{k\to\infty} \sup(A_k) \implies & \lim_{k\to\infty} \inf(A_k) \leq \lim_{n\to\infty} a_n \leq \lim_{k\to\infty} \sup(A_k) \\
\implies & \liminf a_n \leq \lim_{n\to\infty} a_n \leq \limsup a_n \\
\implies & \liminf a_n \leq \lim_{n\to\infty} a_n \leq \liminf a_n \qquad (\text{as } \liminf a_n = \limsup a_n)
\end{aligned}$$

$$\implies \qquad \lim_{n\to\infty} a_n = \liminf a_n$$

Since $\liminf a_n \in \mathbb{R}$, (a_n) converges.

Proposition 2.28 — Subsequence $\limsup$ and $\liminf$ Inequality

For any subsequence (a_{n_k}) of a sequence (a_n), we have

$$\liminf a_n \leq \liminf a_{n_k} \leq \limsup a_{n_k} \leq \limsup a_n$$

Furthermore, there exists a subsequence (b_{n_k}) of (a_n) such that

$$\lim_{k\to\infty} b_{n_k} = \limsup a_n$$

and another subsequence (c_{n_k}) of (a_n) such that

$$\lim_{k\to\infty} c_{n_k} = \liminf a_n$$

Example 2.29: Show that $a_n = n^{1/n} \to 1$ as $n \to \infty$.

Let $x_n = a_n - 1$ for all $n \in \mathbb{N}$. We have

$$\begin{aligned}
x_n = n^{1/n} - 1 \iff n^{1/n} &= x_n + 1 \\
\implies \quad n &= (x_n + 1)^n \\
&= 1 + nx_n + \frac{n(n-1)}{2}x_n^2 + \ldots + x_n^n && \text{(by the binomial theorem)} \\
&\geq \frac{n(n-1)}{2}x_n^2 \\
\implies \quad x_n^2 &\leq n \cdot \frac{2}{n(n-1)} && \text{(assuming that } n \neq 1\text{)} \\
&= \frac{2}{n-1} \\
\implies \quad x_n &\leq \sqrt{\frac{2}{n-1}} \to 0 && \text{as } n \to \infty
\end{aligned}$$

Since $x_n = a_n - 1 \to 0$, $a_n = n^{1/n} \to 1$ as $n \to \infty$.

Chapter 3 Series of Numbers

3.1 Convergence of Series

Definition 3.1 — Series

Let (a_n) be a sequence. We define the *series* of (a_n) to be the *infinite sum*

$$S = a_1 + a_2 + a_3 + \dots$$

Formally, the *partial sums* of (a_n) are given by

$$S_n = \sum_{i=1}^{n} a_i$$

where $n \in \mathbb{N}$. We have

$$S = \lim_{n\to\infty} S_n$$

Example 3.2: Consider the sequence given by $a_n = 1$ for all $n \in \mathbb{N}$. We have $S = a_1+a_2+a_3+\dots = 1+1+1+\dots = \infty$.

Definition 3.3 — Series Convergence and Divergence

Let (a_n) be a sequence and $S = a_1 + a_2 + a_3 + \dots$ be its series. The series *converges* if S is finite and *diverges* if S is infinite (or does not exist).

Example 3.4: Consider the sequence given by $a_n = \dfrac{1}{2^n}$. We have

$$\begin{aligned}
S = a_1 + a_2 + \dots + a_n + \dots &= \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n} + \dots \\
&= \lim_{n\to\infty}\left(\frac{1}{2} + \dots + \frac{1}{2^n}\right) \\
&= \frac{1}{2}\lim_{n\to\infty}\left(1 + \dots + \frac{1}{2^{n-1}}\right) \\
&= \frac{1}{2}\cdot\frac{1-(1/2)^n}{1-1/2} \\
&= \frac{1}{2}\cdot\frac{1-0}{1/2} && \left(\text{as } \tfrac{1}{2^n} \to 0 \text{ as } n\to\infty\right) \\
&= 1
\end{aligned}$$

Example 3.5: Consider the sequence given by $a_n = (-1)^n$. We may obtain

$$\sum_{i=1}^{\infty} a_i = -1+1-1+1-1+1+\dots = (-1+1)+(-1+1)+(-1+1)+\dots = 0+0+0+\dots = 0$$

or

$$\sum_{i=1}^{\infty} a_i = -1+1-1+1-1+\dots = -1+(1-1)+(1-1)\dots = -1+0+0+\dots = -1$$

We have $0 \neq -1$, so the series does not converge.

Remark: Example 3.5 incorrectly utilizes associativity with addition, even though the property may not hold for certain infinite sums. This is more of an intuitive reasoning why the series does not converge, rather than a rigorous justification.

Example 3.6: Consider the sequence given by $a_n = (-1)^n$, as in example 3.5. We have $S_1 = -1$, $S_2 = -1+1 = 0$,

$S_3 = -1 + 1 - 1 = -1$, and $S_4 = -1 + 1 - 1 + 1 = 0$. In general,

$$S_n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Thus, the series does not converge.

Proposition 3.7 — Vanishing Condition

Let (a_n) be a sequence. If the series of (a_n) converges, then $a_n \to 0$ as $n \to \infty$.

PROOF: Since the series of (a_n) converges, the sequence of its partial sums (S_n) converges, so (S_n) is a Cauchy sequence. Thus, for all $\epsilon > 0$, there exists some $N > 0$ such that for all $m > N$ and $n > N$, $|S_m - S_n| < \epsilon$. Taking $m = n + 1 > N$, we obtain

$$|S_{n+1} - S_n| < \epsilon \iff |a_{n+1}| < \epsilon \iff |a_{n+1} - 0| < \epsilon$$

Thus, for all $n > N + 1$, $|a_n - 0| < \epsilon$, so $\lim_{n\to\infty} a_n = 0$. ■

Remark: The converse of proposition 3.7 does not hold in general.

Theorem 3.8 — Cauchy Criterion for Series

Let (a_n) be a sequence and (S_n) be its partial sums. The series of (a_n) converges if and only if for all $\epsilon > 0$, there exists some $N > 0$ such that for all $m > N$ and $n > N$,

$$|S_m - S_n| < \epsilon$$

Proposition 3.9 — Non-Negative Sequence Bounded Series

Suppose that (a_n) is a sequence with $a_n \geq 0$ for all $n \in \mathbb{N}$. Let (S_n) be its partial sums. The series of (a_n) converges if and only if (S_n) is bounded.

PROOF: For all $n \in \mathbb{N}$,

$$S_{n+1} = a_1 + \ldots + a_n + a_{n+1} = S_n + a_{n+1} \geq S_n$$

as $a_{n+1} \geq 0$. Thus, (S_n) is increasing, so it converges if and only if it is bounded by proposition 2.23. ■

3.2 Series Convergence Tests

Theorem 3.10 — Cauchy Condensation Test

Let (a_n) be a sequence with $a_1 \geq a_2 \geq a_3 \geq \ldots \geq 0$. The series of (a_n) converges if and only if the series

$$\sum_{k=1}^{\infty} 2^k \cdot a_{2^k}$$

converges.

PROOF: Let $S = \sum_{n=1}^{\infty} a_n$ and $S' = \sum_{n=0}^{\infty} 2^n \cdot a_{2^n}$. We have

$$\begin{aligned} S &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + \ldots + a_{15} + a_{16} + \ldots \\ &\leq a_1 + a_2 + a_2 + a_4 + a_4 + a_4 + a_4 + a_8 + a_8 + \ldots + a_8 + a_{16} + \ldots = S' \end{aligned}$$

since $a_3 \leq a_2$, $a_5 \leq a_4$, $a_6 \leq a_4$, and so on. Also,

$$\begin{aligned} S &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + \ldots + a_{15} + a_{16} + \ldots \\ &\geq a_2 + a_4 + a_4 + a_8 + a_8 + a_8 + a_8 + a_{16} + \ldots + a_{16} + a_{32} + \ldots \\ &= \frac{1}{2}\left(2a_2 + 2^2 a_{2^2} + 2^3 a_{2^3} + \ldots\right) \end{aligned}$$

$$= \frac{1}{2}(S' - a_1)$$

The inequality holds since $a_1 \geq a_2$, $a_2 \geq a_4$, $a_3 \geq a_4$, $a_5 \geq 8$, and so on. We have shown that

$$S' \geq S \geq \frac{1}{2}(S' - a_1)$$

so S is bounded if and only if S' is bounded. Furthermore, S and S' are monotonic, so they converge if and only if they are bounded (by proposition 2.23). Thus, S converges if and only if S' converges. ∎

Example 3.11 *(Harmonic Series Divergence)*: The harmonic series $S = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$ diverges.

Proof: Let $a_n = \frac{1}{n}$. We have $a_n > 0$ for all $n \in \mathbb{N}$ and

$$\sum_{k=1}^{\infty} 2^k \cdot a_{2^k} = 1 + 2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} + 8 \cdot \frac{1}{8} + \ldots = 1 + 1 + 1 + 1 + \ldots = \infty$$

Thus, the series diverges (by theorem 3.10). ∎

Proposition 3.12 — Comparison Test

Let (a_n) be a sequence and $\sum_{k=1}^{\infty} b_k$ be a convergent series. If $|a_n| \leq b_n$ for all $n \in \mathbb{N}$, then the series of (a_n) converges.

Proof: Since the series of (b_n) is convergent, it is Cauchy. Thus, for all $\epsilon > 0$, there exists some $N > 0$ such that if $m, n \in \mathbb{N}$ satisfy $n \geq m > N$, then $\left|\sum_{k=m}^{n} b_k\right| < \epsilon$. Note that $b_n \geq |a_n|$ for all $n \in \mathbb{N}$, so each $b_n \geq 0$ and thus $\left|\sum_{k=m}^{n} b_k\right| = \sum_{k=m}^{n} b_k$. We now obtain

$$\sum_{k=m}^{n} a_k \leq \sum_{k=m}^{n} |a_k| \leq \sum_{k=m}^{n} b_k = \left|\sum_{k=m}^{n} b_k\right| < \epsilon$$

Thus, the series of (a_n) is Cauchy, so it converges. ∎

Remark: In proposition 3.12, the fact that $|a_n| \leq b_n$ for all $n \in \mathbb{N}$ ensures that $b_n \geq 0$ for all $n \in \mathbb{N}$.

Theorem 3.13 — Ratio Test

Let (a_n) be a sequence. If

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$$

then the series of (a_n) converges.

Proof: Let $\beta \in \mathbb{R}$ with $\beta < 1$. Suppose that for all $\epsilon > 0$ such that $\gamma = \beta + \epsilon < 1$, there exists some $N \in \mathbb{N}$ such that

$$n > N \implies \left|\frac{a_{n+1}}{a_n}\right| \leq \gamma$$

We have $|a_{n+1}| \leq \gamma \cdot |a_n|$, so

$$\begin{aligned}
|a_{N+1}| &\leq \gamma |a_N| \\
|a_{N+2}| &\leq \gamma |a_{N+1}| \leq \gamma \cdot \gamma |a_N| = \gamma^2 |a_N| \\
&\vdots \\
|a_{N+k}| &\leq \gamma^k |A_N|
\end{aligned}$$

for all $k \in \mathbb{N}$. Thus, by the triangle inequality,

$$|a_N + a_{N+1} + \ldots + a_{N+k}| \leq |a_N| + |a_{N+1}| + \ldots + |a_{N+k}| \leq |a_N| \cdot (1 + \gamma + \ldots + \gamma^k)$$

The partial sums $1 + \gamma + \ldots + \gamma^k$ form a geometric series. Since $\gamma < 1$, this series converges. Therefore, (a_n) converges by proposition 3.12. ■

Remark: Both theorem 3.16 and theorem 3.13 do not require computations involving partial sums. Note that the lim sups must be strictly less than 1 in both tests for the conclusions to follow (equality cannot hold).

Example 3.14: Consider the sequence given by $a_n = \dfrac{1}{n!}$. For all $n \in \mathbb{N}$, $a_n \geq 0$, so

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{a_{n+1}}{a_n} = \frac{1/(n+1)!}{1/n!} = \frac{1}{n+1}$$

Thus, $\limsup\limits_{n\to\infty} a_n = 0 < 1$, so the series converges by theorem 3.13.

Example 3.15: Consider the sequence given by $a_n = \dfrac{2^n}{n!}$. For all $n \in \mathbb{N}$, $a_n \geq 0$, so

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}/(n+1)!}{2^n/n!} = \frac{n! \cdot 2^n \cdot 2}{n! \cdot (n+1) \cdot 2^n} = \frac{2}{n+1} \to 0$$

as $n \to \infty$. Thus, $\limsup\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$, so the series converges (by theorem 3.13).

Theorem 3.16 — Root Test

Suppose that (a_n) is a sequence. If

$$\limsup_{n\to\infty} |a_n|^{1/n} < 1$$

then the series of (a_n) converges.