MAT B43 — Introduction to Analysis

COMPILED BY ALI RAJAN

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This is a compilation of the notes from Professor Stefanos Aretakis' MAT B43 lectures. Each of the facts (definitions, theorems, axioms, etc.) are numbered for cross-referencing purposes.

Contents

Fact List

Chapter 1 Number Systems

1.1 Basic Set Theory

Definition 1.1 — Set

A *set* is a collection of objects, called *elements*.

Definition 1.2 — Relation

A *relation* of pairs of elements in a set S is a set of ordered pairs $(a, b) \in S \times S$.

Example 1.3: Consider $S = \{1, 2, 3\}$ and $\mathcal{R} = \big\{(1, 2), (2, 3), (3, 3)\big\}$.

Note that 1 is not related to 1, 2 is not related to 2, 3 is not related to 2, and 1 is not related to 3.

Example 1.4: Consider $S = \mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathcal{R} = \big\{(a, b) \in \mathbb{N} \times \mathbb{N} : |a - b| \text{ is a multiple of } 2\big\}$.

Note that $\mathcal{R} = \big\{(a, b) \in \mathbb{N} \times \mathbb{N} : a \text{ and } b \text{ have the same parity}\big\}$.

Definition 1.5 — Equivalence Relation

A relation $\mathcal{R}$ over a set S is an *equivalence relation* if the following conditions hold:

1. For all $x \in S$, $(x, x) \in \mathcal{R}$ (*reflexivity*)
2. If $(x, y) \in \mathcal{R}$, then $(y, x) \in \mathcal{R}$ (*symmetry/commutativity*)
3. If $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$, then $(x, z) \in \mathcal{R}$ (*transitivity*)

The symbol $\sim$ is used to denote equivalence between elements in a set S. For any $a, b \in S$, $a \sim b$ if and only if $(a, b) \in \mathcal{R}$.

Remark: Note that the relation in example 1.3 is not an equivalence relation, while the one in example 1.4 is an equivalence relation.

Definition 1.6 — Equivalence Class

For a set S, $C \subseteq S$ is said to be an *equivalence class* if C contains only

1. Equivalent elements, and
2. All such equivalent elements in S

More formally, given the equivalence relation $\sim$, the equivalence class of $c \in S$ is the set

$$[c] = \{x \in S : x \sim c\}$$

Proposition 1.7 — Properties of Equivalence Classes

For any set S, the following properties hold:

1. Any two distinct equivalence classes of S are disjoint
2. The union of all equivalence classes of S is S itself

Example 1.8: Consider $S = \big\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \land 0 \leq y \leq 1\big\}$ and $A, B \in S$ with $A = (x, y)$ and $B = (x', y')$. Define the equivalence relation $\sim$ as $A \sim B$ if and only if $x = x'$.

Observe that

1. $A \sim A$ (as $x = x$)
2. $A \sim B \implies B \sim A$ (as $x = x' \implies x' = x$)

3. $A \sim B \wedge B \sim C \implies A \sim C$ (as for $C = (x'', y'') \in S$, $x = x' \wedge x' = x'' \implies x = x''$)

so S is an equivalence relation. The equivalence classes are the vertical line segments of the form $\{(x, y) : 0 \leq y \leq 1\}$ for some fixed $x \in [0, 1]$.

Definition 1.9 — Quotient Set

For any set S, the *quotient set* is the set of all equivalence classes of S. We denote the quotient set of S by

$$S/\sim$$

where $\sim$ is the equivalence relation used to define the equivalence classes.

Example 1.10: Consider $S = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \wedge 0 \leq y \leq 1\}$ and $A, B \in S$ with $A = (x, y)$ and $B = (x', y')$. Define the equivalence relation $\sim$ as $A \sim B$ if and only if $x = x'$. (This is the same setup as in example 1.8.)

The quotient set of S is

$$Q = \left\{ l : l(x) = \bigcup_{0 \leq y \leq 1} (x, y) \right\}$$

which is indeed the union of the vertical lines l with their specified x-values.

1.2 Constructing the Natural Numbers

Axiom 1.11 — Peano Axioms

The principal axioms of the natural numbers (positive integers) are as follows:

1. $1 \in \mathbb{N}$
2. For all $n \in \mathbb{N}$, there exists a successor $\hat{n}$ of n which satisfies
   (a) For all $n \in \mathbb{N}$, $\hat{n} \neq 1$
   (b) $\hat{m} = \hat{n} \implies m = n$
3. If some property $Q(n)$ (for $n \in \mathbb{N}$) satisfies the following conditions:
   (a) $Q(1)$ is true
   (b) $Q(n) \implies Q(\hat{n})$

   then $Q(n)$ holds for all $n \in \mathbb{N}$.

Example 1.12: Let $Q(n)$ be the predicate $1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$.

$Q(1)$ holds as $1 = \frac{1 \cdot 2}{2}$. Now if $Q(n)$ holds for some $n \in \mathbb{N}$,

$$1 + 2 + \ldots + n + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)\left((n+1)+1\right)}{2}$$

so $Q(n+1)$ also holds. Therefore, $Q(n)$ holds for all $n \in \mathbb{N}$.

1.3 Constructing the Integers

Definition 1.13 — The Set of Integers

The set of *integers* $\mathbb{Z}$ is defined as

$$\mathbb{Z} = (\mathbb{N} \times \mathbb{N})/\!\sim$$

where $\sim$ is the equivalence relation given by $(a, b) \sim (n, m)$ if and only if $a+m = b+n$. Each equivalence class $[(a, b)]$ is denoted by $b - a$.

Remark: Observe that definition 1.13 only uses addition in $\mathbb{N}$, and the notation for the equivalence classes additionally uses the definition of subtraction in $\mathbb{Z}$ (though it may be defined using only subtraction in $\mathbb{N}$, as demonstrated in section A1.2 of the textbook).

Example 1.14: Consider the set $\mathbb{N} \times \mathbb{N}$ and relation $\sim$ given by $(a, b) \sim (n, m)$ if and only if $a + m = b + n$, as in definition 1.13. Prove that $\sim$ is an equivalence relation.

Proof: Since $a + b = b + a$, $(a, b) \sim (a, b)$ and thus $\sim$ is reflexive.

If $(a, b) \sim (n, m)$, $a + m = b + n$ and thus $n + b = m + a$, so $(n, m) \sim (a, b)$. Therefore, $\sim$ is symmetric.

Now if $(a, b) \sim (n, m)$ and $(n, m) \sim (c, d)$, we have $a + m = b + n$ and $n + d = m + c$. This yields $n = m + c - d$ and thus $a + m = b + n \implies a + m = b + (m + c - d) \implies a + d = b + c$. Thus, $(a, b) \sim (c, d)$, so $\sim$ is transitive.

Therefore, $\sim$ is an equivalence relation. ■

Example 1.15: Consider the set $\mathbb{N}\times\mathbb{N}$ and the equivalence relation $\sim$ given by $(a, b) \sim (n, m)$ if and only if $a+m = b+n$. (This is the same setup as in definition 1.13.)

Note that $[(1, 2)] = [(2, 3)] = [(3, 4)] = \big\{(n, n + 1) : n \in \mathbb{N}\big\}$, which is denoted by 1, and $[(2, 1)] = [(3, 2)] = [(4, 3)] = \big\{(n, n - 1) : n \in \mathbb{N}\big\}$, which is denoted by -1.

Definition 1.16 — Addition and Multiplication on the Integers

Let $A, B \in \mathbb{Z}$ with $A = [(a, b)]$ and $B = [(n, m)]$ (expressed in $\mathbb{Z} = (\mathbb{N} \times \mathbb{N})/\!\sim$ as in definition 1.13).

Addition on the integers is defined as the operation $+\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by

$$A + B = [(a + n, b + m)]$$

Multiplication on the integers is defined as the operation $\cdot\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by

$$A \cdot B = [(am + bn, bm + an)]$$

Remark: Note that the operations in definition 1.16 are defined using addition and multiplication in $\mathbb{N}$ only.

Definition 1.17 — The Set of Rational Numbers

The set of *rational* numbers $\mathbb{Q}$ is defined as

$$\mathbb{Q} = \big(\mathbb{Z} \times (\mathbb{Z} - \{0\})\big)/\!\sim$$

where $\sim$ is the equivalence relation given by $(a, b) \sim (n, m)$ if and only if $am = bn$. Each equivalence class $[(a, b)]$ is denoted by $\dfrac{a}{b}$.

Definition 1.18 — Addition and Multiplication on the Rationals

Let $A, B \in \mathbb{Q}$ with $A = [(a, b)]$ and $B = [(c, d)]$ (expressed in $\big(\mathbb{Z} \times (\mathbb{Z} - \{0\})\big)/\sim$ from definition 1.17).

Addition on the rationals is defined as the operation $+ \colon \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ given by

$$A + B = [(ad + cb, bd)]$$

Multiplication on the rationals is defined as the operation $\cdot \colon \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ given by

$$A \cdot B = [(ac, bd)]$$

1.4 The Real Numbers

Many numbers are not rational, including $x = \sqrt{2} \notin \mathbb{Q}$. We consider the set $S = \left\{x \in \mathbb{Q} : x^2 < 2\right\} \subseteq \mathbb{Q}$. $\sqrt{2} \equiv S$ in the sense that it is the least upper bound of S. This is the idea used to construct $\mathbb{R}$ in this section.

Definition 1.19 — Order

The number a is said to be greater than b, denoted by $a > b$, if $a - b > 0$.

Remark: The $>$ in $a - b > 0$ refers to the comparison of Dedekind cuts (as in definition 1.32) representing the corresponding real numbers in the inequality.

Definition 1.20 — Upper Bound and Lower Bound

Let A be an ordered set and $S \subseteq A$ be non-empty.

$a \in A$ is an *upper bound* for S if for all $x \in S$, $x \leq a$.

$b \in A$ is a *lower bound* for S if for all $x \in S$, $x \geq b$.

Definition 1.21 — Least Upper Bound

Let A be an ordered set and $S \subseteq A$ be a set with an upper bound. $M \in A$ is said to be the *least upper bound* (or *supremum*) of S, denoted $M = \sup(S)$, if the following conditions are satisfied:

1. M is an upper bound for S
2. If $a \in A$ is an upper bound for S, then $M \leq a$

Proposition 1.22 — Least Upper Bound Uniqueness

If a set has a least upper bound, it is unique.

Proof: Let S be a set with least upper bounds M_1 and M_2. For all upper bounds l for S, $M_1 \leq l$ and $M_2 \leq l$ (by definition). Therefore, $M_1 \leq M_2$ and $M_2 \leq M_1$, so $M_1 = M_2$. ■

Definition 1.23 — Greatest Lower Bound

Let A be an ordered set and $S \subseteq A$ be a set with a lower bound. $m \in A$ is said to be the *greatest lower bound* (or *infimum*) for S, denoted $m = \inf(S)$, if the following conditions are satisfied:

1. m is a lower bound for S
2. If $b \in A$ is a lower bound for S, then $m \geq b$

Proposition 1.24 — Greatest Lower Bound Uniqueness

If a set has a greatest lower bound, it is unique.

Theorem 1.25 — Existence of $\mathbb{R}$

There exists a set $\mathbb{R}$ that satisfies each of the following conditions:

1. $\mathbb{R}$ is an ordered field with operations $+$ and $\times$
2. $\mathbb{R}$ contains $\mathbb{Q}$
3. Any bounded above subset $S \subseteq \mathbb{R}$ has a supremum in $\mathbb{R}$

$\mathbb{R}$ is the set of *real numbers*.

Example 1.26: Consider $S = \left\{x \in \mathbb{Q} : x^2 < 2\right\} \subseteq \mathbb{R}$.

S has a supremum in $\mathbb{R}$, namely $\sup(S) = \sqrt{2}$. Note that $\sup(S) \notin S$ and $\sup(S) \notin \mathbb{Q}$.

Example 1.27 *(Irrationality of $\sqrt{2}$)*: Show that for $S = \left\{x \in \mathbb{Q} : x^2 < 2\right\} \subseteq \mathbb{Q}$, $\sup(S) \notin \mathbb{Q}$.

Proof: By the set construction in the proof to proposition 1.28, $(\sup(S))^2 = 2$. Suppose, for the sake of contradiction, that $M = \frac{a}{b}$ with $a, b \in \mathbb{N}$ such that $\gcd(a, b) = 1$. It follows that

$$M^2 = 2 = \frac{a^2}{b^2} \implies a^2 = 2b^2 \implies 2 \mid a^2 \implies 2 \mid a$$

Now let $a = 2a_1$ for some $a_1 \in \mathbb{N}$. We obtain

$$(2a_1)^2 = 2b^2 \implies 4a_1^2 = 2b^2 \implies b^2 = 2a_1^2 \implies 2 \mid b^2 \implies 2 \mid b$$

We have shown that $2 \mid a$ and $2 \mid b$, which contradicts the fact that $\gcd(a, b) = 1$. Therefore, by contradiction, $M \notin \mathbb{Q}$. ■

Proposition 1.28 — Existence of the Square Root in $\mathbb{R}$

For all $x \in \mathbb{R}$ with $x > 0$, there exists some $y \in \mathbb{R}$ with $y > 0$ such that

$$y^2 = x$$

That is, y is the *square root* of x, denoted $y = \sqrt{x}$.

Proof: For any $x \in \mathbb{R}$ with $x > 0$, consider the set

$$S_x = \left\{w \in \mathbb{R} : w > 0 \wedge w^2 < x\right\}$$

Note that S_x is bounded above by x if $x > 1$ and 1 if $x \leq 1$.

If $x > 1$, $1^2 = 1 < x \implies 1 \in S_x$.

If $x = 1$, $\left(\frac{1}{2}\right)^2 = \frac{1}{4} < x \implies \frac{1}{2} \in S_x$.

If $x < 1$, $x^2 < x \implies x \in S_x$ (as $0 < x < 1 \implies x \cdot x < 1 \cdot x \implies x^2 < x$ using a multiplicative property of ordered fields).

In any case, S_x is non-empty. Now observe that if $w \in S_x$ and $z \in \mathbb{R}$ satisfies $0 < z < w$, then

$$z < w \implies z^2 < w^2 \implies z^2 < x \implies z \in S_x$$

so for all $z \in \mathbb{R}$ with $0 < z < w$, $z \in S_x$. We will now prove that $y = \sup(S_x)$ satisfies $y^2 = x$.

Assume, for the sake of contradiction, that $y^2 < x$. We will show that there exists some $\epsilon \in \mathbb{R}^+$ such that $(y + \epsilon)^2 < x$. Note that for $\epsilon < 1$, $\epsilon^2 < \epsilon$, so

$$(y + \epsilon)^2 = y^2 + 2y\epsilon + \epsilon^2 < y^2 + 2y\epsilon + \epsilon$$

Now

$$y^2 + 2y\epsilon + \epsilon < x \iff \epsilon(2y + 1) < x - y^2 \iff \epsilon < \frac{x - y^2}{2y + 1}$$

so for $\epsilon = \min\left(1, \frac{1}{2} \cdot \frac{x - y^2}{2y + 1}\right)$, $(y + \epsilon)^2 < x$. We thus have $y + \epsilon \in S_x$ and $y + \epsilon > y$, which contradicts the fact that $y = \sup(S_x)$. Therefore, $y^2 \geq x$ by contradiction.

Now suppose that $y^2 > x$, for the sake of contradiction. We will show that for some $\epsilon \in \mathbb{R}^+$, $(y-\epsilon)^2 \geq x$. Observe that

$$(y-\epsilon)^2 = y^2 - 2y\epsilon + \epsilon^2 > y^2 - 2y\epsilon - \epsilon = y^2 - \epsilon(2y+1)$$

Now

$$y^2 - \epsilon(2y+1) \geq x \iff y^2 - x \geq \epsilon(2y+1) \iff \epsilon \leq \frac{y^2-x}{2y+1}$$

Choose $\epsilon = \dfrac{y^2-x}{2y+1}$. We have shown that $\epsilon \leq \frac{y^2-x}{2y+1} \implies (y-\epsilon)^2 \geq x$, so $y-\epsilon < y$ is an upper bound for S_x. This contradicts the fact that $y = \sup(S_x)$, so we cannot have $y^2 > x$.

Since $y^2 < x$ and $y^2 > x$ do not hold, $y^2 = \sup(S_x)$. ■

Remark: Proposition 1.28 does *not* hold in $\mathbb{Q}$; consider the counterexample $x = 2 \in \mathbb{Q}$, where $y^2 \neq 2$ for all $y \in \mathbb{Q}$.

Theorem 1.29 — Archimedean Property for $\mathbb{R}^+$

For all $a, b \in \mathbb{R}$ with $0 < a < b$, there exists some $n \in \mathbb{N}$ such that

$$na > b$$

Proof: Assume, for the sake of contradiction, that $na \leq b$ for all $n \in \mathbb{N}$. Consider the set

$$M_a = \{na : n \in \mathbb{N}\}$$

$M_a \neq \emptyset$ as $a \in M_a$, and M_a is bounded above by b. Thus, we let $y = \sup(M_a)$. There exists some $k \in \mathbb{N}$ such that

$$y - a < ka \leq y$$

(since y is the supremum and $ka \in M_a$), so $y < ka + a = (k+1)a$ and $(k+1)a \in M_a$. This contradicts the fact that y is an upper bound for M_a. By contradiction, there exists some $n \in \mathbb{N}$ such that $na > b$. ■

Theorem 1.30 — Density of $\mathbb{Q}$ in $\mathbb{R}$

For all $x, y \in \mathbb{R}$ with $x < y$, there exists some $q \in \mathbb{Q}$ such that $x < q < y$.

Proof: For simplicity, assume that $x, y \in \mathbb{R}^+$ (the cases where x or y is negative follow similarly). Let $q = \frac{n}{m}$ for some $n, m \in \mathbb{N}$. It follows that

$$x < q < y \iff x < \frac{n}{m} < y \iff mx < n < my$$

Now let $\epsilon = y - x$. By the Archimedean property, $m\epsilon > 10$ for some m, so

$$m\epsilon = m(y-x) = my - mx > 10 \implies my > mx + 10$$

There exists some n such that $mx < n < mx + 10$, for which we have

$$mx < n < mx + 10 < my \implies mx < n < my$$

Therefore, for the chosen m and n, $x < q = \frac{n}{m} < y$. ■

Definition 1.31 — Dedekind Cut

A *Dedekind cut* is a set $\mathcal{C} \subseteq \mathbb{Q}$ that satisfies each of the following conditions:

1. $\mathcal{C} \neq \emptyset$
2. $\mathcal{C}$ is bounded above
3. If $q \in \mathcal{C}$, then for all $q' < q$, $q' \in \mathcal{C}$
4. If $q \in \mathcal{C}$, there exists some $q^+ \in \mathcal{C}$ with $q^+ > q$

Remark: The idea of a cut (as in definition 1.31) is to construct an "interval" of rational numbers (only) before defining the real numbers (and intervals). That is, cuts construct

$$(-\infty, a) \cap \mathbb{Q}$$

for $a \in \mathbb{R}$ without relying on $\mathbb{R}$.

Definition 1.32 — $\mathbb{R}$ as a Set of Cuts

The field $\mathbb{R}$ is defined by

$$\mathbb{R} = \{\mathcal{C} : \mathcal{C} \text{ is a cut}\}$$

Let $\mathcal{C}_1, \mathcal{C}_2 \in \mathbb{R}$. The field has addition defined as

$$\mathcal{C}_1 + \mathcal{C}_2 = \{x + y : x \in \mathcal{C}_1 \wedge y \in \mathcal{C}_2\}$$

ordering defined as

$$\mathcal{C}_1 \leq \mathcal{C}_2 \iff \mathcal{C}_1 \subseteq \mathcal{C}_2$$

the additive identity defined as the set of all negative rationals $\hat{0} = \mathbb{Q}^-$, and multiplication defined as

1. $\mathcal{C}_1 \cdot \mathcal{C}_2 = \{q \in \mathbb{Q} : q < xy \text{ for some } x \in \mathcal{C}_1 \text{ and } y \in \mathcal{C}_2 \text{ with } x > 0 \wedge y > 0\}$ if $\mathcal{C}_1 > \hat{0}$ and $\mathcal{C}_2 > \hat{0}$
2. $-\big(\mathcal{C}_1 \cdot (-\mathcal{C}_2)\big)$ if $\mathcal{C}_1 > \hat{0}$ and $\mathcal{C}_2 < \hat{0}$
3. $-\big((-\mathcal{C}_1) \cdot \mathcal{C}_2\big)$ if $\mathcal{C}_1 < \hat{0}$ and $\mathcal{C}_2 > \hat{0}$
4. $(-\mathcal{C}_1) \cdot (-\mathcal{C}_2)$ if $\mathcal{C}_1 < \hat{0}$ and $\mathcal{C}_2 < \hat{0}$
5. $\hat{0}$ if $\mathcal{C}_1 = \hat{0}$ or $\mathcal{C}_2 = \hat{0}$

Remark: The set of cuts $\mathbb{R}$ (as in definition 1.32) is the set of real numbers, whose existence is asserted by theorem 1.25.

Theorem 1.33 — Least Upper Bound Property

Let $\mathcal{A}$ be a cut. For any bounded above collection of cuts $S = \{\mathcal{C} : \mathcal{C} \subset \mathcal{A}\}$ satisfies

$$\sup(S) = \bigcup_{\mathcal{C} \in S} \mathcal{C}$$

That is, the supremum exists.

Remark: Using theorem 1.33, each $a \in \mathbb{R}$ can now be uniquely identified as the supremum of a cut $\mathcal{C}$.

Chapter 2 Sequences

2.1 Cardinality

Definition 2.1 — Injective, Surjective, and Bijective

Let A and B be sets and $f\colon A \to B$ ($f(A) \subseteq B$).

We say that f is *one-to-one* (or *injective*) if $f(x) = f(y) \implies x = y$.

We say that f is *onto* (or *surjective*) if $f(A) = B$.

f is *bijective* if it is one-to-one and onto.

Example 2.2: There is no bijection from $\{1, 2\}$ to $\{1, 2, 3\}$. Similarly, there is no bijection from $\{1, 2\}$ to $\{3\}$.

Definition 2.3 — Cardinality Equality

Let A and B be sets. We say that A and B have the *same cardinality* if there exists a bijection $f\colon A \to B$. The cardinality of A is denoted by $\operatorname{card}(A)$ or $|A|$.

Example 2.4: Do $\mathbb{Z}$ and $\mathbb{N}$ have the same cardinality?

Consider $f\colon \mathbb{N} \to \mathbb{Z}$ given by

$$f(n) = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even} \\ -\dfrac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

f is a bijection that maps the even natural numbers to the positive integers and the odd natural numbers to the non-positive integers. Therefore, $|\mathbb{N}| = |\mathbb{Z}|$.

2.2 Sequences

Definition 2.5 — Sequence

Let A be a set. A *sequence* is a function $f\colon \mathbb{N} \to A$. We denote the outputs of sequences by

$$\{f(1), f(2), f(3), \ldots, f(n), \ldots\} \equiv \{f_1, f_2, f_3, \ldots, f_n, \ldots\}$$

A *sequence of real numbers* is a collection

$$(a_1, a_2, a_3, \ldots, a_n, \ldots)$$

where each $a_i \in \mathbb{R}$. We will use (a_n) to denote a sequence of real numbers.

Example 2.6: Some sequences are $\{1, 2, 3, \ldots\}$ and $\{2, 1, 3, 4, 5, \ldots\}$.

Definition 2.7 — Bounded Sequence

A sequence (a_n) is *bounded* if there exists some $M \in \mathbb{R}^+$ such that for all $n \in \mathbb{N}$,

$$|a_n| < M$$

The sequence is *bounded below* if there exists some $r \in \mathbb{R}$ such that for all $n \in \mathbb{N}$,

$$a_n > r$$

Similarly, the sequence is *bounded above* if there exists some $s \in \mathbb{R}$ such that for all $n \in \mathbb{N}$,

$$a_n < s$$

Definition 2.8 — Convergence

A sequence (a_n) *converges* to $\alpha \in \mathbb{R}$, denoted by

$$\lim_{n\to\infty} a_n = \alpha \text{ or } a_n \to \alpha$$

if for all $\epsilon > 0$, there exists some $N > 0$ such that

$$n > N \implies |a_n - \alpha| < \epsilon$$

Proposition 2.9 — Properties of Convergent Sequences

The following properties hold for any convergent sequence:

1. The limit of the sequence is unique
2. The sequence is bounded

PROOF: Let (a_n) be a convergent sequence. For (1), assume that the sequence has limits $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 \neq \alpha_2$ (for the sake of contradiction). Without loss of generality, suppose that $\alpha_2 > \alpha_1$. By definition, for all $\epsilon_1 > 0$, there exists some $N_1 > 0$ such that $n > N_1 \implies |a_n - \alpha_1| < \epsilon_1$. Similarly, for all $\epsilon_2 > 0$, there exists some $N_2 > 0$ such that $n > N_2 \implies |a_n - \alpha_2| < \epsilon_2$. Take $\epsilon = \dfrac{\alpha_2 - \alpha_1}{2}$ and $N = \max(N_1, N_2)$. We have

$$\alpha_1 + \epsilon < \alpha_2 - \epsilon$$

so for $n > N$ and $\epsilon_1 = \epsilon_2 = \epsilon$, both $|a_n - \alpha_1| < \epsilon$ and $|a_n - \alpha_2| < \epsilon$ cannot be satisfied. This is because $|a_n - \alpha_1| < \epsilon \implies a_n - \alpha_1 < \epsilon \implies a_n < \alpha_1 + \epsilon$, while $|a_n - \alpha_2| < \epsilon \implies a_n - \alpha_2 > -\epsilon \implies a_n > \alpha_2 - \epsilon$. By contradiction, the limit must be unique.

To prove (2), suppose that $(a_n) \to \alpha \in \mathbb{R}$. Let $\epsilon = 1$ with $|a_n - \alpha| < \epsilon$ when $n > N \in \mathbb{N}$. If $1 \le n \le N$,

$$\min\{a_1, \dots, a_N\} \le a_n \le \max\{a_1, \dots, a_N\}$$

For $n > N$, since $|a_n - \alpha| < \epsilon = 1$, we have

$$a_n - \alpha < 1 \implies a_n < \alpha + 1$$

and

$$a_n - \alpha > -1 \implies a_n > \alpha - 1$$

so $\alpha - 1 < a_n < \alpha + 1$. In either case, (a_n) is bounded both above and below, so (a_n) is bounded. ∎

Example 2.10: Observe that $a_n = (-1)^n$ is bounded, but has no limit. This shows that the converse of (2) in proposition 2.9 does not hold in general.

Example 2.11: Consider the sequence given by $a_n = \dfrac{1}{n}(-1)^n$. We will show that $\frac{1}{n}(-1)^n \to 0$.

Let $\epsilon > 0$. We want to find some $N > 0$ such that for all $n > N$,

$$|a_n - 0| < \epsilon \iff \left|\frac{1}{n}(-1)^n\right| < \epsilon \iff \frac{1}{n} < \epsilon \iff n > \frac{1}{\epsilon}$$

Now choose $N = \dfrac{1}{\epsilon}$. As shown above, this choice guarantees that $n > N \implies |a_n - 0| < \epsilon$.

Example 2.12: Consider the sequence given by $a_n = \dfrac{1}{n^2}$. We will show that $a_n \to 0$.

Let $\epsilon > 0$. We will find some $N > 0$ such that for all $n > N$,

$$|a_n - 0| < \epsilon \iff \left|\frac{1}{n^2} - 0\right| < \epsilon \iff \frac{1}{n^2} < \epsilon \iff n^2 > \frac{1}{\epsilon} \iff n > \frac{1}{\sqrt{\epsilon}}$$

Choose $N = \dfrac{1}{\sqrt{\epsilon}}$. As shown above, this choice of N guarantees that $n > N \implies |a_n - 0| < \epsilon$.

Example 2.13 *($\mathbb{N}$ Is Not Bounded Above)*: Consider the sequence given by $a_n = n$. Assume that $a_n \to L \in \mathbb{R}$. For all $\epsilon > 0$, there exists some $N > 0$ such that $n > N \implies |a_n - L| < \epsilon$. Take $\epsilon = 1$ and N_1 be its corresponding value of N. For all $n > N_1$, we have

$$|a_n - L| < \epsilon \iff |n - L| < 1 \iff -1 < n - L < 1 \iff L - 1 < n < L + 1$$

so for all $n > N_1$, $n < L + 1$, which contradicts the fact that $a_n \to L$. By contradiction, a_n does not converge to any $L \in \mathbb{R}$.

Proposition 2.14 — Limit Properties

Let (a_n) and (b_n) be convergent sequences that converge to $a \in \mathbb{R}$ and $b \in \mathbb{R}$ (respectively). The following properties hold:

1. $(a_n + b_n) \to a + b$
2. For all $c \in \mathbb{R}$, $c \cdot a_n \to c \cdot a$
3. $a_n \cdot b_n \to a \cdot b$

Proof: We will prove properties (1) and (3).

By definition, for all $\epsilon > 0$, there exists some $N_1 > 0$ such that $n > N_1 \implies |a_n - a| < \epsilon$. Similarly, for all $\epsilon > 0$, there exists some $N_2 > 0$ such that $n > N_2 \implies |b_n - b| < \epsilon$.

To prove (1), let $\epsilon > 0$. For all $n > \max(N_1, N_2)$, we have

$$\big|(a_n + b_n) - (a + b)\big| = \big|(a_n - a) + (b_n) - b\big| \leq |a_n - a| + |b_n - b| < \epsilon + \epsilon = 2\epsilon$$

using the triangle inequality. Thus, if $N > 0$ corresponds to $\frac{\epsilon}{2}$, we have

$$\big|(a_n + b_n) - (a + b)\big| < 2 \cdot \frac{\epsilon}{2} = \epsilon$$

To prove (3), note that

$$\begin{aligned}
|a_n \cdot b_n - a \cdot b| &= |a_n b_n - a_n b + a_n b - ab| \\
&= \big|a_n(b_n - b) + b(a_n - a)\big| \\
&\leq \big|a_n(b_n - b)\big| + \big|b(a_n - a)\big| && \text{(by the triangle inequality)} \\
&= |a_n||b_n - b| + |b||a_n - a| \\
&< |a_n|\,\epsilon + |b|\,\epsilon \\
&= \epsilon\big(|a_n| + |b|\big) \\
&< \epsilon\big(M_1 + |b|\big) && \text{(for some } M_1 \in \mathbb{R}\text{, by the boundedness of } (a_n))
\end{aligned}$$

Take $N > 0$ to be a value corresponding to $\dfrac{\epsilon}{M_1 + |b|}$. We thus have

$$n > N \implies |a_n \cdot b_n - a \cdot b| < \frac{\epsilon}{\big(M_1 + |b|\big)} \cdot \big(M_1 + |b|\big) = \epsilon$$

■

Definition 2.15 — Cauchy Sequence

A sequence (a_n) is *Cauchy* if for all $\epsilon > 0$, there exists some $N > 0$ such that

$$n > N \land m > N \implies |a_n - a_m| < \epsilon$$

Remark: Cauchy sequences are used to prove convergence without finding limits.

Proposition 2.16 — Boundedness of Cauchy Sequences

Every Cauchy sequence is bounded.

PROOF: Let $\epsilon = 1$. For some $N > 0$, we have $|a_n - a_m| < 1$ for all $n, m > N$ by definition. Fix $m = m_0 > N$, and suppose that $n \geq N$. We have

$$|a_n - a_{m_0}| < 1$$

so a_n is bounded whenever $n \geq N$. If $1 \leq n < N$, we let $t = \min\left\{a_1, \ldots, a_{\lfloor N \rfloor}\right\}$ and $T = \max\left\{a_1, \ldots, a_{\lfloor N \rfloor}\right\}$ so that

$$t \leq a_n \leq T$$

In either case, $\min(t, a_{m_0} - 1)$ and $\max(T, a_{m_0} + 1)$ are lower and upper bounds for (a_n) (respectively). Therefore, (a_n) is bounded. ■

Example 2.17: Consider the sequence given by $a_n = \dfrac{1}{n}$. We will show that (a_n) is Cauchy.

Let $\epsilon > 0$. Note that

$$\begin{aligned} |a_n - a_m| &= \left|\frac{1}{n} - \frac{1}{m}\right| \\ &\leq \left|\frac{1}{n}\right| + \left|-\frac{1}{m}\right| && \text{(by the triangle inequality)} \\ &= \frac{1}{n} + \frac{1}{m} \end{aligned}$$

Choose $N = \dfrac{2}{\epsilon}$. If $n > N$ and $m > N$, then

$$|a_n - a_m| \leq \frac{1}{n} + \frac{1}{m} < \frac{1}{2/\epsilon} + \frac{1}{2/\epsilon} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Theorem 2.18 — Convergence of Cauchy Sequences

A sequence is Cauchy if and only if it is convergent.

PROOF: Let (a_n) be a sequence.

($\Longrightarrow$) Suppose that $a_n \to L \in \mathbb{R}$. We have

$$\begin{aligned} |a_n - a_m| &= \big|(a_n - L) - (a_m - L)\big| \\ &\leq |a_n - L| + \big|-(a_m - L)\big| && \text{(by the triangle inequality)} \\ &= |a_n - L| + |a_m - L| \end{aligned}$$

There exists some $N > 0$ such that if $n > N$ and $m > N$, then $|a_n - L| < \frac{\epsilon}{2}$ and $|a_m - L| < \frac{\epsilon}{2}$, so

$$|a_n - L| + |a_m - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus, if a sequence converges, then it is Cauchy.

($\Longleftarrow$) Now suppose that (a_n) is Cauchy. We will show that (a_n) converges to the supremum of its "eventual" lower bounds. More formally, consider the set

$$S = \{x \in \mathbb{R} : n \geq k \implies a_i > x \text{ for some } k \in \mathbb{N}\}$$

Intuitively, k is the "eventual" index after which (a_n) is bounded below by x. Since Cauchy sequences are bounded, (a_n) has a lower bound, so $S \neq \emptyset$. Furthermore, (a_n) is bounded above as it is Cauchy. We will now show that $\lim\limits_{n\to\infty} a_n = \sup(S)$.

Suppose that for some $\epsilon > 0$, $|a_n - a_m| < \epsilon$ whenever $n > N$ and $m > N$ (for some $N > 0$). We fix m and let n vary. Note that

$$|a_n - a_m| < \epsilon \iff a_m - \epsilon < a_n < a_m + \epsilon$$

Thus, $a_m - \epsilon \in S$ (as it is a lower bound for the sequence when $n > N$), which yields $a_m - \epsilon \leq \sup(S)$. Also, $a_m + \epsilon \notin S$ (as it is not a lower bound for the sequence when $n > N$), so $\sup(S) \leq a_m + \epsilon$. It follows that

$$a_m - \epsilon \leq \sup(S) \leq a_m + \epsilon \implies \big|a_m - \sup(S)\big| \leq \epsilon$$

For any $\epsilon' > 0$, let $\epsilon = \frac{\epsilon'}{2}$. We have shown that for all $\epsilon' > 0$,

$$m > N \implies \left|a_m - \sup(S)\right| \leq \epsilon = \frac{\epsilon'}{2} < \epsilon'$$

Therefore, $\lim_{m\to\infty} a_m = \sup(S)$. ∎

Remark: The idea of "eventual" lower bounds used in the proof for theorem 2.18 alludes to the idea of a lim inf, as covered later in definition 2.25.

Lemma 2.19 — Cauchy by Exponential Bound

Let (a_n) be a sequence. If $|a_{n+1} - a_n| \leq \frac{1}{2^n}$ for all $n \in \mathbb{N}$, then (a_n) is Cauchy.

Proof: Fix $\epsilon > 0$, and let $n > m > N$ (where $N > 0$). We have

$$\begin{aligned}
|a_n - a_m| &= \left|(a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \ldots + (a_{m+1} - a_m)\right| \\
&\leq |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \ldots + |a_{m+1} - a_m| && \text{(by the triangle inequality)} \\
&\leq \frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \ldots + \frac{1}{2^m} \\
&= \frac{1}{2^m}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1-m}}\right) \\
&= \frac{1}{2^m} \cdot \frac{1 - (1/2)^{(n-1-m)+1}}{1 - (1/2)} && \text{(by the finite geometric series sum formula)} \\
&= \frac{1}{2^m} \cdot 2 \cdot \left(1 - \left(\frac{1}{2}\right)^{n-m}\right) \\
&= \frac{1}{2^{m-1}} - \frac{1}{2^{m-1}} \cdot \frac{1}{2^{n-m}} \\
&= \frac{1}{2^{m-1}} - \frac{1}{2^{n-1}} \\
&\leq \frac{c}{2^{N-1}}
\end{aligned}$$

The last inequality holds for some $c \in \mathbb{R}^+$ as $n > m > N \implies \frac{1}{2^{n-1}} < \frac{1}{2^{m-1}} < \frac{1}{2^{N-1}}$. Thus, for $N = \log_2\left(\frac{2c}{\epsilon}\right)$, we have

$$|a_n - a_m| \leq \frac{c}{2^{N-1}} = \frac{c}{2^{\log_2(2c/\epsilon)-1}} = \frac{2c}{2c/\epsilon} = \epsilon$$

∎

Definition 2.20 — Subesequence

Let (a_n) be a sequence. (b_k) is a *subsequence* of (a_n) if for all $k \in \mathbb{N}$,

$$b_k = a_{i_k}$$

where each $i_k \in \mathbb{N}$ and $i_1 < i_2 < i_3 < \ldots$.

Example 2.21: Consider $(a_n) = (-1)^n$. We have $(a_n) = \{-1, 1, -1, 1, -1, 1, \ldots\}$, so for $i_k = 2k$,

$$a_{i_k} = a_{2k} = (-1)^{2k} = 1$$

Thus, $a_{i_k} \to 1$. Similarly, for $j_k = 2k + 1$,

$$a_{j_k} = a_{2k+1} = (-1)^{2k+1} = -1$$

so $a_{j_k} \to -1$.

Theorem 2.22 — Bolzano–Weierstrass

Any bounded sequence has a *convergent* subsequence.

PROOF: Suppose that the sequence (a_n) is bounded with $|a_n| \leq M \in \mathbb{R}^{\geq 0}$ for all $n \in \mathbb{N}$. There are infinitely many terms of the sequence in the interval $[-M, M]$, so there must be infinitely many terms in $[-M, 0]$ or $[0, M]$. Without loss of generality, suppose that $[0, M]$ has infinitely many terms, and choose a_{i_1} to be one such term. Again, there must be infinitely many terms in $\left[0, \frac{M}{2}\right]$ or $\left[\frac{M}{2}, M\right]$. Without loss of generality, suppose that $\left[0, \frac{M}{2}\right]$ has infinitely many terms, and take a_{i_2} to be one such term with $i_2 > i_1$. Continuing this process, we obtain the subsequence $\left(a_{i_k}\right)$ with

$$\left|a_{i_{k+1}} - a_{i_k}\right| \leq \frac{M}{2^{k-1}}$$

for all $k \in \mathbb{N}$, as $a_{i_k}, a_{i_{k+1}} \in \left[0, \frac{M}{2^{k-1}}\right]$ (without loss of generality). By lemma 2.19, $\left(a_{i_k}\right)$ is Cauchy and thus converges. ■

Proposition 2.23 — Bounded Monotone Convergence

If a sequence $A = (a_n)$ is increasing and bounded above, then $a_n \to \sup(A)$.

Similarly, if the sequence is decreasing and bounded below, then $a_n \to \inf(A)$.

2.3 Limit Superior and Limit Inferior

Definition 2.24 — Supremum and Infimum of Unbounded Sets

Let $A \subseteq \mathbb{R}$. If A is not bounded above, we define $\sup(A) = \infty$. Similarly, we define $\inf(A) = -\infty$ if A is not bounded below.

Definition 2.25 — Limit Superior and Limit Inferior

Let (a_n) be a sequence and $A_k = \{a_k, a_{k+1}, a_{k+2}, \ldots\}$ for each $k \in \mathbb{N}$. The *limit superior* of (a_n) is defined as

$$\limsup a_n = \lim_{k\to\infty} \sup(A_k)$$

Similarly, the *limit inferior* of (a_n) is defined as

$$\liminf a_n = \lim_{k\to\infty} \inf(A_k)$$

Remark: In definition 2.25, the values of $\sup(A_k)$ form a decreasing sequence as the least upper bound of fewer terms in (a_n) is computed as $k \to \infty$. Similarly, the values of $\inf(A_k)$ form an increasing sequence as the greatest lower bound of fewer terms is computed as $k \to \infty$.

By proposition 2.23, $\limsup a_n$ and $\liminf a_n$ exist (they could be finite or infinite).

Example 2.26: Compute the $\limsup$ and $\liminf$ of the sequence given by $a_n = n \cdot \sin\left(n \cdot \frac{\pi}{2}\right)$.

By the periodicity of sin (as evaluated at integer multiples of $\frac{\pi}{2}$), $a_n = n \cdot \sin\left(n \cdot \frac{\pi}{2}\right)$ alternates between 0, n, 0, and $-n$. Thus, $\limsup a_n = \infty$ and $\liminf a_n = -\infty$.

Example 2.27: Show that if $\liminf a_n = \limsup a_n \in \mathbb{R}$ for some sequence (a_n), then (a_n) converges.

Let $A_k = \{a_k, a_{k+1}, a_{k+2}, \ldots\}$, where $k \in \mathbb{N}$. For all $a_n \in A_k$ (i.e. $n \geq k$), we have $\inf(A_k) \leq a_n \leq \sup(A_k)$. As $k \to \infty$, $n \to \infty$ for $n \geq k$. Thus,

$$\begin{aligned}
\lim_{k\to\infty} \inf(A_k) \leq \lim_{k\to\infty} a_n \leq \lim_{k\to\infty} \sup(A_k) \implies & \lim_{k\to\infty} \inf(A_k) \leq \lim_{n\to\infty} a_n \leq \lim_{k\to\infty} \sup(A_k) \\
\implies & \liminf a_n \leq \lim_{n\to\infty} a_n \leq \limsup a_n \\
\implies & \liminf a_n \leq \lim_{n\to\infty} a_n \leq \liminf a_n \qquad (\text{as } \liminf a_n = \limsup a_n)
\end{aligned}$$

$$\implies \qquad \lim_{n\to\infty} a_n = \liminf a_n$$

Since $\liminf a_n \in \mathbb{R}$, (a_n) converges.

Proposition 2.28 — Subsequence lim sup and lim inf Inequality

For any subsequence $\left(a_{n_k}\right)$ of a sequence (a_n), we have

$$\liminf a_n \leq \liminf a_{n_k} \leq \limsup a_{n_k} \leq \limsup a_n$$

Furthermore, there exists a subsequence $\left(b_{n_k}\right)$ of (a_n) such that

$$\lim_{k\to\infty} b_{n_k} = \limsup a_n$$

and another subsequence $\left(c_{n_k}\right)$ of (a_n) such that

$$\lim_{k\to\infty} c_{n_k} = \liminf a_n$$

Example 2.29: Show that $a_n = n^{1/n} \to 1$ as $n \to \infty$.

Let $x_n = a_n - 1$ for all $n \in \mathbb{N}$. We have

$$\begin{aligned}
x_n = n^{1/n} - 1 \iff n^{1/n} &= x_n + 1 \\
\implies \quad n &= (x_n+1)^n \\
&= 1 + nx_n + \frac{n(n-1)}{2}x_n^2 + \ldots + x_n^n && \text{(by the binomial theorem)} \\
&\geq \frac{n(n-1)}{2}x_n^2 \\
\implies \quad x_n^2 &\leq n \cdot \frac{2}{n(n-1)} && \text{(assuming that } n \neq 1\text{)} \\
&= \frac{2}{n-1} \\
\implies \quad x_n &\leq \sqrt{\frac{2}{n-1}} \to 0 && \text{as } n \to \infty
\end{aligned}$$

Since $x_n = a_n - 1 \to 0$, $a_n = n^{1/n} \to 1$ as $n \to \infty$.

Chapter 3 Series of Numbers

3.1 Convergence of Series

Definition 3.1 — Series

Let (a_n) be a sequence. We define the *series* of (a_n) to be the *infinite sum*

$$S = a_1 + a_2 + a_3 + \dots$$

Formally, the *partial sums* of (a_n) are given by

$$S_n = \sum_{i=1}^{n} a_i$$

where $n \in \mathbb{N}$. We have

$$S = \lim_{n\to\infty} S_n$$

Example 3.2: Consider the sequence given by $a_n = 1$ for all $n \in \mathbb{N}$. We have $S = a_1 + a_2 + a_3 + \dots = 1+1+1+\dots = \infty$.

Definition 3.3 — Series Convergence and Divergence

Let (a_n) be a sequence and $S = a_1 + a_2 + a_3 + \dots$ be its series. The series *converges* if S is finite and *diverges* if S is infinite (or does not exist).

Example 3.4: Consider the sequence given by $a_n = \dfrac{1}{2^n}$. We have

$$\begin{aligned}
S = a_1 + a_2 + \dots + a_n + \dots &= \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n} + \dots \\
&= \lim_{n\to\infty} \left(\frac{1}{2} + \dots + \frac{1}{2^n} \right) \\
&= \frac{1}{2} \lim_{n\to\infty} \left(1 + \dots + \frac{1}{2^{n-1}} \right) \\
&= \frac{1}{2} \cdot \frac{1-(1/2)^n}{1-1/2} \\
&= \frac{1}{2} \cdot \frac{1-0}{1/2} && \left(\text{as } \tfrac{1}{2^n} \to 0 \text{ as } n \to \infty\right) \\
&= 1
\end{aligned}$$

Example 3.5: Consider the sequence given by $a_n = (-1)^n$. We may obtain

$$\sum_{i=1}^{\infty} a_i = -1+1-1+1-1+1+\dots = (-1+1)+(-1+1)+(-1+1)+\dots = 0+0+0+\dots = 0$$

or

$$\sum_{i=1}^{\infty} a_i = -1+1-1+1-1+\dots = -1+(1-1)+(1-1)\dots = -1+0+0+\dots = -1$$

We have $0 \neq -1$, so the series does not converge.

Remark: Example 3.5 incorrectly utilizes associativity with addition, even though the property may not hold for certain infinite sums. This is more of an intuitive reasoning why the series does not converge, rather than a rigorous justification.

Example 3.6: Consider the sequence given by $a_n = (-1)^n$, as in example 3.5. We have $S_1 = -1$, $S_2 = -1+1 = 0$,

$S_3 = -1+1-1 = -1$, and $S_4 = -1+1-1+1 = 0$. In general,

$$S_n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Thus, the series does not converge.

Proposition 3.7 — Vanishing Condition

Let (a_n) be a sequence. If the series of (a_n) converges, then $a_n \to 0$ as $n \to \infty$.

Proof: Since the series of (a_n) converges, the sequence of its partial sums (S_n) converges, so (S_n) is a Cauchy sequence. Thus, for all $\epsilon > 0$, there exists some $N > 0$ such that for all $m > N$ and $n > N$, $|S_m - S_n| < \epsilon$. Taking $m = n+1 > N$, we obtain

$$|S_{n+1} - S_n| < \epsilon \iff |a_{n+1}| < \epsilon \iff |a_{n+1} - 0| < \epsilon$$

Thus, for all $n > N+1$, $|a_n - 0| < \epsilon$, so $\lim_{n\to\infty} a_n = 0$. ∎

Remark: The converse of proposition 3.7 does not hold in general.

Theorem 3.8 — Cauchy Criterion for Series

Let (a_n) be a sequence and (S_n) be its partial sums. The series of (a_n) converges if and only if for all $\epsilon > 0$, there exists some $N > 0$ such that for all $m > N$ and $n > N$,

$$|S_m - S_n| < \epsilon$$

Proposition 3.9 — Non-Negative Sequence Bounded Series

Suppose that (a_n) is a sequence with $a_n \geq 0$ for all $n \in \mathbb{N}$. Let (S_n) be its partial sums. The series of (a_n) converges if and only if (S_n) is bounded.

Proof: For all $n \in \mathbb{N}$,

$$S_{n+1} = a_1 + \ldots + a_n + a_{n+1} = S_n + a_{n+1} \geq S_n$$

as $a_{n+1} \geq 0$. Thus, (S_n) is increasing, so it converges if and only if it is bounded by proposition 2.23. ∎

3.2 Series Convergence Tests

Theorem 3.10 — Cauchy Condensation Test

Let (a_n) be a sequence with $a_1 \geq a_2 \geq a_3 \geq \ldots \geq 0$. The series of (a_n) converges if and only if the series

$$\sum_{k=1}^{\infty} 2^k \cdot a_{2^k}$$

converges.

Proof: Let $S = \sum_{n=1}^{\infty} a_n$ and $S' = \sum_{n=0}^{\infty} 2^n \cdot a_{2^n}$. We have

$$\begin{aligned} S &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + \ldots + a_{15} + a_{16} + \ldots \\ &\leq a_1 + a_2 + a_2 + a_4 + a_4 + a_4 + a_4 + a_8 + a_8 + \ldots + a_8 + a_{16} + \ldots = S' \end{aligned}$$

since $a_3 \leq a_2$, $a_5 \leq a_4$, $a_6 \leq a_4$, and so on. Also,

$$\begin{aligned} S &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + \ldots + a_{15} + a_{16} + \ldots \\ &\geq a_2 + a_4 + a_4 + a_8 + a_8 + a_8 + a_8 + a_{16} + \ldots + a_{16} + a_{32} + \ldots \\ &= \frac{1}{2}\left(2a_2 + 2^2 a_{2^2} + 2^3 a_{2^3} + \ldots\right) \end{aligned}$$

$$= \frac{1}{2}(S' - a_1)$$

The inequality holds since $a_1 \geq a_2$, $a_2 \geq a_4$, $a_3 \geq a_4$, $a_5 \geq 8$, and so on. We have shown that

$$S' \geq S \geq \frac{1}{2}(S' - a_1)$$

so S is bounded if and only if S' is bounded. Furthermore, S and S' are monotonic, so they converge if and only if they are bounded (by proposition 2.23). Thus, S converges if and only if S' converges. ■

Example 3.11 *(Harmonic Series Divergence)*: The harmonic series $S = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$ diverges.

Proof: Let $a_n = \frac{1}{n}$. We have $a_n > 0$ for all $n \in \mathbb{N}$ and

$$\sum_{k=1}^{\infty} 2^k \cdot a_{2^k} = 1 + 2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} + 8 \cdot \frac{1}{8} + \ldots = 1 + 1 + 1 + 1 + \ldots = \infty$$

Thus, the series diverges (by theorem 3.10). ■

Proposition 3.12 — Comparison Test

Let (a_n) be a sequence and $\sum_{k=1}^{\infty} b_k$ be a convergent series. If $|a_n| \leq b_n$ for all $n \in \mathbb{N}$, then the series of (a_n) converges.

Proof: Since the series of (b_n) is convergent, it is Cauchy. Thus, for all $\epsilon > 0$, there exists some $N > 0$ such that if $m, n \in \mathbb{N}$ satisfy $n \geq m > N$, then $\left|\sum_{k=m}^{n} b_k\right| < \epsilon$. Note that $b_n \geq |a_n|$ for all $n \in \mathbb{N}$, so each $b_n \geq 0$ and thus $\left|\sum_{k=m}^{n} b_k\right| = \sum_{k=m}^{n} b_k$. We now obtain

$$\sum_{k=m}^{n} a_k \leq \sum_{k=m}^{n} |a_k| \leq \sum_{k=m}^{n} b_k = \left|\sum_{k=m}^{n} b_k\right| < \epsilon$$

Thus, the series of (a_n) is Cauchy, so it converges. ■

Remark: In proposition 3.12, the fact that $|a_n| \leq b_n$ for all $n \in \mathbb{N}$ ensures that $b_n \geq 0$ for all $n \in \mathbb{N}$.

Theorem 3.13 — Ratio Test

Let (a_n) be a sequence. If

$$\limsup_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$$

then the series of (a_n) converges.

Proof: Let $\beta \in \mathbb{R}$ with $\beta < 1$. Suppose that for all $\epsilon > 0$ such that $\gamma = \beta + \epsilon < 1$, there exists some $N \in \mathbb{N}$ such that

$$n > N \implies \left|\frac{a_{n+1}}{a_n}\right| \leq \gamma$$

We have $|a_{n+1}| \leq \gamma \cdot |a_n|$, so

$$\begin{aligned}
|a_{N+1}| &\leq \gamma |a_N| \\
|a_{N+2}| &\leq \gamma |a_{N+1}| \leq \gamma \cdot \gamma |a_N| = \gamma^2 |a_N| \\
&\vdots \\
|a_{N+k}| &\leq \gamma^k |A_N|
\end{aligned}$$

for all $k \in \mathbb{N}$. Thus, by the triangle inequality,

$$|a_N + a_{N+1} + \ldots + a_{N+k}| \leq |a_N| + |a_{N+1}| + \ldots + |a_{N+k}| \leq |a_N| \cdot (1 + \gamma + \ldots + \gamma^k)$$

The partial sums $1 + \gamma + \ldots + \gamma^k$ form a geometric series. Since $\gamma < 1$, this series converges. Therefore, (a_n) converges by proposition 3.12. ■

Remark: Both theorem 3.16 and theorem 3.13 do not require computations involving partial sums. Note that the lim sups must be strictly less than 1 in both tests for the conclusions to follow (equality cannot hold).

Example 3.14: Consider the sequence given by $a_n = \dfrac{1}{n!}$. For all $n \in \mathbb{N}$, $a_n \geq 0$, so

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{a_{n+1}}{a_n} = \frac{1/(n+1)!}{1/n!} = \frac{1}{n+1}$$

Thus, $\limsup\limits_{n\to\infty} a_n = 0 < 1$, so the series converges by theorem 3.13.

Example 3.15: Consider the sequence given by $a_n = \dfrac{2^n}{n!}$. For all $n \in \mathbb{N}$, $a_n \geq 0$, so

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}/(n+1)!}{2^n/n!} = \frac{n! \cdot 2^n \cdot 2}{n! \cdot (n+1) \cdot 2^n} = \frac{2}{n+1} \to 0$$

as $n \to \infty$. Thus, $\limsup\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$, so the series converges (by theorem 3.13).

Theorem 3.16 — Root Test

Suppose that (a_n) is a sequence. If

$$\limsup_{n\to\infty} |a_n|^{1/n} < 1$$

then the series of (a_n) converges.

Example 3.17 *(Ratio Test Inconclusivity for p-Series)*: Consider the sequence given by $a_n = \dfrac{1}{n^r}$, where $r \in \mathbb{R}^+$. For all $n \in \mathbb{N}$, we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1/(n+1)^r}{1/n^r} = \frac{n^r}{(n+1)^r} = \frac{1}{\left(\dfrac{n+1}{n}\right)^r} = \frac{1}{(1+1/n)^r} \to \frac{1}{1^r} = 1$$

as $n \to \infty$ since $\frac{1}{n} \to 0$. Thus, $\limsup\left|\frac{a_{n+1}}{a_n}\right| = 1$, so the ratio test is inconclusive for the sequence given by $a_n = \frac{1}{n^r}$.

Example 3.18 *(p-Series Convergence)*: Consider the sequence given by $a_n = \dfrac{1}{n^r}$, as in example 3.17. For all $n \in \mathbb{N}$, $a_n = \frac{1}{n^r} \geq 0$ and $a_n \geq a_{n+1}$. Thus, $\sum\limits_{n=1}^{\infty} a_n$ converges if and only if $\sum\limits_{n=1}^{\infty} 2^n \cdot a_{2^n}$ converges by theorem 3.10.

$$\sum_{n=1}^{\infty} 2^n \cdot a_{2^n} = \sum_{n=1}^{\infty} 2^n \cdot \frac{1}{(2^n)^r} = \sum_{n=1}^{\infty} \frac{2^n}{2^{nr}} = \sum_{n=1}^{\infty} 2^{n(1-r)} = \sum_{n=1}^{\infty} \left(2^{1-r}\right)^n$$

This is a geometric series, which converges if and only if

$$2^{1-r} < 1 \iff 1 - r < 0 \iff r > 1$$

Therefore, the sequence given by $a_n = \frac{1}{n^r}$ converges if and only if $r > 1$.

Theorem 3.19 — Abel's Convergence Test

Let (a_n) and (b_n) be sequences and (S_n) be the partial sums of (a_n). $\sum\limits_{n=1}^{\infty} a_n \cdot b_n$ converges if each of the following conditions hold:

1. $b_n \leq \ldots \leq b_2 \leq b_1$
2. $\lim\limits_{n\to\infty} b_n = 0$
3. (S_n) is bounded

Remark: In theorem 3.19, (a_n) is usually a sequence of coefficients for some "main" sequence (b_n).

Example 3.20 *(Harmonic Series Divergence by Abel's Test)*: Consider the sequence given by $a_n = \frac{1}{n}$. The corresponding series is

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} 1 \cdot \frac{1}{n}$$

Let $a_n = 1$ and $b_n = \frac{1}{n}$. Note that (b_n) satisfies the requirements in theorem 3.19, but the partial sums of (a_n) are not bounded. Thus, (a_n) diverges by theorem 3.19.

Example 3.21: Consider the series $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n}$. Let $a_n = (-1)^{n+1}$ and $b_n = \frac{1}{n}$. For all $n \in \mathbb{N}$,

$$a_{n+1} = \frac{1}{n+1} \leq \frac{1}{n} = a_n$$

Furthermore,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

Also, if (S_n) are the partial sums of (a_n), then

$$\begin{aligned} S_n &= \begin{cases} 1 - 1 + 1 - 1 + \ldots + 1 - 1 & \text{if } n \text{ is even} \\ 1 - 1 + 1 - 1 + \ldots + 1 & \text{if } n \text{ is odd} \end{cases} \\ &= \begin{cases} 0 + 0 + \ldots + 0 & \text{if } n \text{ is even} \\ 0 + 0 + \ldots + 0 + 1 & \text{if } n \text{ is odd} \end{cases} \\ &= \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

Thus, (S_n) is bounded. By theorem 3.19, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ converges.

Proposition 3.22 — Alternating Series Test

Let (b_n) be a sequence. If $b_1 \geq b_2 \geq b_3 \geq \ldots$ and $b_n \to 0$ as $n \to \infty$, then $S = \sum\limits_{n=1}^{\infty} (-1)^n b_n$ converges. Furthermore, if (S_n) are its partial sums, then $|S - S_n| \leq b_{n+1}$ for all $n \in \mathbb{N}$.

Proof: Let $a_n = (-1)^n$. The partial sums of this sequence are either -1 or 0, so they are bounded. Also, (b_n) is decreasing and $b_n \to 0$ as $n \to \infty$, so by theorem 3.19, $\sum\limits_{n=1}^{\infty} a_n b_n = \sum\limits_{n=1}^{\infty} (-1)^n b_n = S$ converges.

Now for all $k \in \mathbb{N}$, we have

$$|S - S_k| = \left| \sum_{n=k+1}^{\infty} (-1)^n b_n \right| = \left| (-1)^{k+1} b_{k+1} + (-1)^{k+2} b_{k+2} + \ldots \right| = |b_{k+1} - b_{k+2} + b_{k+3} - \ldots|$$

Observe that since (b_n) is decreasing,

$$b_{k+2} - b_{k+3} + b_{k+4} - b_{k+5} + \ldots = (b_{k+2} - b_{k+3}) + (b_{k+4} - b_{k+5}) + \ldots \leq (b_{k+2} - b_{k+4}) + (b_{k+4} - b_{k+6}) + \ldots = b_{k+2}$$

Furthermore, since $b_{k+2} \geq b_{k+3}$, $b_{k+4} \geq b_{k+5}$, and so on, $b_{k+2} - b_{k+3} + \ldots \geq 0$ using the pairing of terms above. Also, $b_{k+2} - b_{k+3} + \ldots \leq b_{k+2} \leq b_{k+1}$, so $0 \leq b_{k+2} - b_{k+3} + \ldots \leq b_{k+1}$. Thus,

$$|b_{k+1} - b_{k+2} + b_{k+3} - \ldots| = \big|b_{k+1} - (b_{k+2} - b_{k+3} + \ldots)\big| \leq b_{k+1}$$

Therefore, for all $k \in \mathbb{N}$,

$$|S - S_k| \leq b_{k+1}$$

■

Example 3.23 *(Irrationality of e)*: Let $e = 1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \ldots$. Consider the sequence given by $a_n = \dfrac{1}{n!}$ and its partial sums (A_n). For all $n \in \mathbb{N}$,

$$\begin{aligned}
|e - A_n| &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \frac{1}{(n+4)!} \cdots \\
&= \frac{1}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \frac{1}{(n+2)(n+3)(n+4)} + \ldots\right) \\
&< \frac{1}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)^2} + \frac{1}{(n+2)^3} + \ldots\right) \\
&= \frac{1}{(n+1)!} \cdot \frac{1}{1 - 1/(n+2)} && \text{(by the geometric series formula; note that } \left|\tfrac{1}{n+2}\right| < 1\text{)} \\
&= \frac{1}{(n+1)!} \cdot \frac{n+2}{n+2-1} \\
&= \frac{1}{(n+1)!} \cdot \frac{n+2}{n+1} \\
&\leq \frac{1}{(n+1)!} \cdot \frac{n+1}{n} && \text{(as } n^2 + 2n \leq n^2 + 2n + 1 \implies n(n+2) \leq (n+1)^2 \implies \tfrac{n+2}{n+1} \leq \tfrac{n+1}{n}\text{)} \\
&= \frac{1}{n \cdot n!}
\end{aligned}$$

Note that $e > A_n$, so $|e - A_n| = e - A_n$. Now assume that $e \in \mathbb{Q}$. Let $e = \dfrac{p}{q}$ for some $p, q \in \mathbb{N}$. As determined previously,

$$0 < e - A_q = \big|e - A_q\big| < \frac{1}{q} \cdot \frac{1}{q!} \implies 0 < q! \cdot e - q! \cdot \left(1 + \frac{1}{2!} + \ldots + \frac{1}{q!}\right) < \frac{1}{q}$$

Here, $q! \cdot e \in \mathbb{N}$ (by the assumption) and $q! \cdot \left(1 + \frac{1}{2!} + \ldots + \frac{1}{q!}\right) \in \mathbb{N}$, so the above quantity is an integer strictly between 0 and $\frac{1}{q}$. Since $\frac{1}{q} \leq 1$ for all $q \in \mathbb{N}$, it must be an integer strictly between 0 and 1, a contradiction. Therefore, e must be irrational.

Chapter 4 Topology of $\mathbb{R}$

4.1 Open and Closed Sets

Definition 4.1 — Neighbourhood

A *neighbourhood* of $x \in \mathbb{R}$ is an interval of the form

$$N_x(\epsilon) = (x - \epsilon, x + \epsilon)$$

where $\epsilon \in \mathbb{R}^+$.

Definition 4.2 — Open Set

A subset $S \subseteq \mathbb{R}$ is said to be *open* if for all $x \in S$, there exists some neighbourhood $N_x(\epsilon_x) \subseteq S$ (where $\epsilon_x > 0$).

Example 4.3: Consider the set $[0, 1]$. Observe that

$$N_0(\epsilon) = (-\epsilon, \epsilon) \nsubseteq [0, 1]$$

for $\epsilon > 0$. Thus, $[0, 1]$ is not open.

Now consider $(0, 1)$. For all $x \in (0, 1)$, note that $x - \epsilon > 0 \iff x > \epsilon$ and $x + \epsilon < 1 \iff \epsilon < 1 - x$. Thus, taking $\epsilon < \min(x, 1 - x)$ yields $N_x(\epsilon) \subseteq (0, 1)$ for all $x \in (0, 1)$, so $(0, 1)$ is open.

Proposition 4.4 — Preservation of Open Sets Under Unions and Finite Intersections

Any union of open sets is open.

The *finite* intersection of open sets is open.

Proof: Consider a collection O_i of open sets, where $i \in I$ (for some index set I). We will show that their union is open. Let $x \in \bigcup_{i \in I} O_i$. We have $x \in O_j$ for some $j \in I$. Since O_j is open, there exists some $\epsilon > 0$ such that

$$(x - \epsilon, x + \epsilon) \subseteq O_j \subseteq \bigcup_{i \in I} O_i$$

Therefore, $\bigcup_{i \in I} O_i$ is open by definition.

Now let $O_1, \ldots, O_n$ be open sets. We will show that their intersection is open. Suppose that $x \in \bigcap_{i=1}^{n} O_i$. We have

$$x \in O_i \implies (x - \epsilon_i, x + \epsilon_i) \subseteq O_i$$

for each i, where each $\epsilon_i > 0$. Choose $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\}$. For all i, we have $(x - \epsilon, x + \epsilon) \subseteq O_i$. Therefore,

$$(x - \epsilon, x + \epsilon) \subseteq \bigcap_{i=1}^{n} O_i$$

so $\bigcap_{i=1}^{n} O_i$ is open. ∎

Remark: In proposition 4.4, the preservation of open sets under union holds for both finite and infinite unions, including countably infinite ones. The preservation of open sets under intersection does not hold in general for infinite intersections.

Example 4.5: Consider the open sets $O_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ for $n \in \mathbb{N}$. We have

$$\bigcap_{n=1}^{\infty} O_n = \{0\}$$

which is not open. This shows that an infinite intersection of open sets is not necessarily open.

Definition 4.6 — Closed Set

A subset $S \subseteq \mathbb{R}$ is said to be *closed* if

$$S^{\mathsf{c}} = \mathbb{R} - S$$

is open.

Example 4.7 *(Closed Sets not Preserved Under Infinite Unions)*: Consider the closed sets $C_n = \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$. We have

$$\bigcup_{n=1}^{\infty} C_n = (0, 1)$$

which is not closed.

Remark: Some sets can be neither open nor closed.

Proposition 4.8 — Characterization of Open Sets in $\mathbb{R}$

If $O \subseteq \mathbb{R}$ is open, then

$$O = \bigcup_{i=1}^{\infty} I_i$$

where the I_i are pairwise disjoint open intervals. There may be either finitely many or countably infinite intervals in the union.

Proof: Let $x, y \in O$. Define $\sim$ as $x \sim y$ if and only if all $r \in \mathbb{R}$ between x and y (inclusive) satisfy $r \in O$.

Note that for any $x \in O$, the only number between x and x is x itself. This is in O, so $x \sim x$. Also, for any $x, y \in O$ with $x \sim y$, all $r \in \mathbb{R}$ between x and y are in O. Thus, all $r \in \mathbb{R}$ between y and x satisfy $r \in O$, so $y \sim x$. Finally, if $x, y, z \in O$ satisfy $x \sim y$ and $y \sim z$, then all $r \in \mathbb{R}$ that are between x and y or between y and z are in O. Assuming that $x \leq y \leq z$, we have $r \in O$ for all r between x and z (the other cases follow with some additional reasoning), so $x \sim z$. Therefore, $\sim$ is an equivalence relation.

The quotient set $O/\!\sim$ contains pairwise disjoint equivalence classes. Consider an equivalence class containing x and y. $x \sim y$ if and only if all $r \in \mathbb{R}$ between x and y are in O. Thus, $x \sim r$ and $y \sim r$ for each such r (as all numbers between x and r are in O, and all numbers between r and y are in O). It follows that all $r \in \mathbb{R}$ between x and y are in the same equivalence class as x and y, so the equivalence class must be an interval I.

For any $x \in I$, there exists some $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq O$ (as $I \subseteq O$ and O is open). Since $x \sim y$ for all $y \in O$, $x \sim s$ for all $s \in (x - \epsilon, x + \epsilon)$ (as $(x - \epsilon, x + \epsilon) \subseteq O$). Each such s is in the same equivalence class as x, so $(x - \epsilon, x + \epsilon) \subseteq I$ and thus I is open.

Since there exists a rational number between any two distinct real numbers, each equivalence class I has a distinct rational number. There are countably many rational numbers, so there are countably many such equivalence classes I. We have thus shown that O is a union of countably many pairwise disjoint open intervals; more formally,

$$O = \bigcup_{i=1}^{\infty} I_i$$

for some pairwise disjoint open intervals I_i. ■

Proposition 4.9 — Closed Sets and Cauchy Limit Points

Suppose that $S \subseteq \mathbb{R}$. S is closed if and only if for every Cauchy sequence $(x_n) \subseteq S$ with $x_n \to x \in \mathbb{R}$ (as $n \to \infty$), $x \in S$.

Proof: Let $S \subseteq \mathbb{R}$.

($\Longrightarrow$) Suppose that S is closed and $(x_n) \subseteq S$ be Cauchy. Since (x_n) is Cauchy, $x_n \to x \in \mathbb{R}$. We will show that $x \in S$. Assume that $x \notin S$ so that $x \in S^{\mathsf{c}}$, which is open. Thus, for some $\epsilon > 0$, $(x - \epsilon, x + \epsilon) \subseteq S^{\mathsf{c}}$, so $|x_n - x| < \epsilon \implies x_n \in S^{\mathsf{c}} \implies x_n \notin S$

for all n. It follows that (x_n) cannot converge to x, which is a contradiction. Therefore, for every Cauchy sequences $(x_n) \subseteq S$, there exists some $x \in S$ such that $x_n \to x \in S$.

$(\Longleftarrow)$ Now suppose that for every Cauchy sequence in S converges to an element of S. Assume, for the sake of contradiction, that S is not closed. It follows that S^{c} is not open, so for some $t \in S^{\mathsf{c}}$, $N_t(\epsilon) \not\subseteq S^{\mathsf{c}}$ for all $\epsilon > 0$. For every $k \in \mathbb{N}$, we can thus choose a point $a_k \in N_t\left(\frac{1}{k}\right)$, which yields a sequence (a_k) such that $a_k \to t \notin S$. This is a contradiction (as (a_k) is Cauchy by convergence). Therefore, S must be closed by contradiction. ■

4.2 Properties of Open and Closed Sets

Definition 4.10 — Accumulation Point

Let $S \subseteq \mathbb{R}$. $s \in \mathbb{R}$ is an *accumulation point* of S if every neighbourhood $N_s(\epsilon)$ (where $\epsilon > 0$) contains infinitely many distinct elements in S.

The *set of accumulation points* of S is denoted by $\operatorname{Acc}(S)$.

Proposition 4.11 — Accumulation Points as Limits

$s \in \mathbb{R}$ is an accumulation point of $S \subseteq \mathbb{R}$ if and only if there exists a non-constant sequence $(x_n) \subseteq S$ (i.e. $x_n \neq x'_n$ for some distinct $n, n' \in \mathbb{N}$) such that $x_n \to S$.

Example 4.12: Consider $S = (0, 1]$. Note that the sequence given by $x_n = \frac{1}{n}$ satisfies $x_n \in S$ for all $n \in \mathbb{N}$ and $x_n \to 0$. Since (x_n) is non-constant, 0 is an accumulation point of S (by proposition 4.11).

Example 4.13: Let $S = (0, 1] \cup \{2\}$. Note that $2 \notin \operatorname{Acc}(S)$ since the only sequence $(x_n) \subseteq S$ with $x_n \to 2$ is the constant sequence given by $x_n = 2$.

Example 4.14: It can be shown that

- $\operatorname{Acc}(S) = \emptyset$ for any finite set $S \subseteq \mathbb{R}$
- $\operatorname{Acc}(\mathbb{N}) = \operatorname{Acc}(\mathbb{Z}) = \emptyset$
- $\operatorname{Acc}(\mathbb{Q}) = \mathbb{R}$ (this is equivalent to the *density* of $\mathbb{Q}$ in $\mathbb{R}$)

Definition 4.15 — Boundary Point

Suppose that $S \subseteq \mathbb{R}$. $s \in \mathbb{R}$ is a *boundary point* of S if every neighbourhood $N_s(\epsilon)$ (where $\epsilon > 0$) contains both elements of S and S^{c}.

The *set of boundary points* of S is denoted by ∂S.

Remark: In definition 4.15, $N_x(\epsilon)$ is only required to have *at least one* element from each of S and S^{c}, not necessarily infinitely many from both.

Example 4.16: It can be shown that

- $\partial S = S$ for any finite set $S \subseteq \mathbb{R}$
- $\partial \mathbb{N} = \mathbb{N}$
- $\partial \mathbb{Z} = \mathbb{Z}$
- $\partial \mathbb{Q} = \mathbb{R}$
- $\partial \mathbb{R} = \emptyset$

Definition 4.17 — Isolated Point

Let $S \subseteq \mathbb{R}$. $s \in S$ is an *isolated point* of S if there exists some neighbourhood $N_s(\epsilon)$ (where $\epsilon > 0$) such that

$$N_s(\epsilon) \cap S = \{s\}$$

The *set of isolated points* of S is denoted by $\text{Iso}(S)$.

Remark: In definition 4.17, $N_x(\epsilon) \cap S = \{x\}$ denotes that every number in $N_x(\epsilon)$ other than x is in S^c.

Example 4.18: Consider $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. For each $n \in \mathbb{N}$ with $n \neq 1$, $\frac{1}{n+1} < \frac{1}{n} < \frac{1}{n-1}$. Thus, for $\epsilon = \min\left(\frac{1}{n} - \frac{1}{n+1}, \frac{1}{n-1} - \frac{1}{n}\right) > 0$,

$$N_x(\epsilon) \cap S = \left\{\frac{1}{n}\right\}$$

If $n = 1$, we have $\left(1 - \frac{1}{2}, 1 + \frac{1}{2}\right) \cap S = \{1\}$ as for $n \neq 1$, $n > 2 \implies \frac{1}{n} < \frac{1}{2}$. Combining the two cases, $\text{Iso}(S) = S$.

Definition 4.19 — Interior Point

Suppose that $S \subseteq \mathbb{R}$. $s \in S$ is an *interior point* of S if there exists some neighbourhood $N_s(\epsilon)$ (where $\epsilon > 0$) such that

$$N_s(\epsilon) \subseteq S$$

The *set of interior points* of S is denoted by $\text{Int}(S) = \mathring{S}$.

Example 4.20: It can be shown that

- $\mathring{\mathbb{N}} = \emptyset$
- $\mathring{\mathbb{Q}} = \emptyset$
- $\mathring{\mathbb{R}} = \mathbb{R}$

Proposition 4.21 — Set Relations Between Accumulation, Isolation, Boundary, and Interior Points

For all sets $S \subseteq \mathbb{R}$, we have:

1. $\text{Acc}(S) \cap \text{Iso}(S) = \emptyset$
2. $\text{Iso}(S) \subseteq \partial S$
3. $\partial S = \partial(S^c)$
4. $S - \text{Iso}(S) \subseteq \text{Acc}(S)$
5. $S \subseteq \partial(S) \cup \mathring{S}$
6. $\mathring{S} \cap \partial S = \emptyset$
7. $\partial S - \text{Iso}(S) \subseteq \text{Acc}(A)$

Proof: We will prove (5). Suppose that $S \subseteq \mathbb{R}$ and $x \in S$. Assume that $x \notin \mathring{S}$. By definition, for every neighbourhood $N_x(\epsilon)$ (where $\epsilon > 0$),

$$N_x(\epsilon) \cap S^c \neq \emptyset$$

Since $x \in N_x(\epsilon)$,

$$N_x(\epsilon) \cap S \neq \emptyset$$

so by definition, $x \in \partial S$. ■

Corollary 4.22 — Bolzano–Weierstrass for Accumulation Points

Let $S \subseteq \mathbb{R}$ be an infinite and bounded set. It follows that $\text{Acc}(S) \neq \emptyset$.

PROOF: We can extract a bounded sequence $(a_n) \subseteq S$. By theorem 2.22, there exists a subsequence (a_{n_k}) of (a_n) that converges to some $l \in \mathbb{R}$. Since each $a_{n_k} \in S$ and for all $\epsilon > 0$, there exists some $N > 0$ such that

$$n_k > N \implies |a_{n_k} - l| < \epsilon$$

there are arbitrarily many elements of S in $N_l(\epsilon)$ for any $\epsilon > 0$. By definition, $l \in \text{Acc}(S)$, so $\text{Acc}(S) \neq \emptyset$. ∎

Proposition 4.23 — Closed Set Convergent Subsqeuence

Let $S \subseteq \mathbb{R}$ be a non-empty, closed, and bounded set. For any sequence $(a_n) \subseteq S$, there exists a convergent subsequence (a_{n_k}) of (a_n) such that $a_{n_k} \to l \in S$.

PROOF: Since S is bounded, any sequence $(a_n) \subseteq S$ is bounded and thus has a convergent subsequence (a_{n_k}) by theorem 2.22. Note that S is closed and (a_{n_k}) is Cauchy (by convergence), so $a_{n_k} \to l$ for some $l \in S$ by proposition 4.9. ∎

4.3 Compact Sets

Definition 4.24 — Compactness

A set $S \subseteq \mathbb{R}$ is said to be *compact* if and only if every sequence in S has a subsequence which converges to an element of S.

Example 4.25: Consider $S = [0, 1]$. Since $S \neq \emptyset$, S is closed, and S is bounded, every sequence in S has a subsequence which converges to an element of S by proposition 4.23. Thus, S is compact by definition.

Example 4.26: Consider $S = (0, 1)$. The sequence given by $a_n = \dfrac{1}{n}$ satisfies $a_n \in S$ for all $n \in \mathbb{N}$ and $a_n = \frac{1}{n} \to 0 \notin S$. Thus, every subsequence of (a_n) converges to $0 \notin S$, so S is not compact by definition.

Example 4.27: The set $[0, \infty)$ is not compact as the sequence given by $a_n = n$ satisfies $(a_n) \subseteq [0, \infty)$, but does not converge. Thus, no subsequence of (a_n) can converge to an element of $[0, \infty)$.

Example 4.28: $\{1\}$ is compact.

Theorem 4.29 — Heine–Borel

A set $S \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.

PROOF:

($\Longrightarrow$) If S is compact, then for all sequences $(x_n) \subseteq S$ with $x_n \to x$, $x \in S$ (by definition). Thus, S is closed. Now assume, for the sake of contradiction, that S is not bounded. There exists a sequence $(x_n) \subseteq S$ such that $|x_n| \to \infty$, so any subsequence (x_{n_k}) of (x_n) cannot converge to an element of S. This contradicts the fact that S is compact, so S must also be bounded.

($\Longleftarrow$) Suppose that S is closed and bounded. For any sequence $(x_n) \subseteq S$, (x_n) is bounded. By theorem 2.22, there exists a subsequence (x_{n_k}) of (x_n) that converges. Since S is closed, $x_{n_k} \to x \in S$. Therefore, S is compact by definition. ∎

Definition 4.30 — Open Covering

A collection of open sets $\{O_i : i \in I\}$ (where I is some index set) is said to be an *open covering* of $S \subseteq \mathbb{R}$ if

$$\bigcup_{i \in I} O_i \supseteq S$$

Definition 4.31 — Open Subcovering

Let $\mathcal{C} = \{O_i : i \in I\}$ be an open covering of $S \subseteq \mathbb{R}$. An open covering of S that is a subset of $\mathcal{C}$ is said to be an *open subcovering* of $\mathcal{C}$.

Example 4.32: Consider the open sets of the form $O_i = (i-1, i+1)$ for $i \in \mathbb{R}$. $\{O_i : i \in [0,1]\}$ is an open covering of $[0,1]$ as each $i \in [0,1]$ is in O_i, so $\bigcup_{i \in [0,1]} \supseteq [0,1]$.

Theorem 4.33 — Compactness and Finite Subcoverings

A set $S \subseteq \mathbb{R}$ is compact if and only if every open covering of S has a finite subcovering.

Proof:

($\Longrightarrow$) Exercise.

($\Longleftarrow$) Suppose that for all open coverings of S, there exists a finite subcovering. Note that $\{O_n : n \in \mathbb{N}\}$, where each $O_n = (-n, n)$, is an open covering as each $(-n, n)$ is an open set and $\bigcup_{n \in \mathbb{N}} O_n = \mathbb{R} \supseteq S$. There must exist a finite subcovering satisfying

$$(-k, k) \supseteq \bigcup_{n \leq k} O_n \supseteq S$$

where $k \in \mathbb{N}$ is the largest index of any element in the subcovering. Thus, S is bounded. Now assume, for contradiction, that S is not closed. For some sequence $(x_n) \subseteq S$, $x_n \to l \notin S$. Consider the open sets of the form $O_n = (-\infty, l) \cup \left(l + \frac{1}{n}, \infty\right)$, which satisfy

$$\bigcup_{n \in \mathbb{N}} O_n = (-\infty, l) \cup (l, \infty)$$

■

Remark: Since each open covering is a collection of open sets $\{O_i : i \in I\}$, the cardinality of the subcovering mentioned in theorem 4.33 refers to the number of open sets in the subcovering being finite (rather than the cardinality of each open set in the subcovering being finite).

Example 4.34: Consider the interval $(0,1)$ and the open sets of the form $O_n = \left(\frac{1}{n}, 1\right)$ for $n \in \mathbb{N}$. For all $s \in (0,1)$, there exists some $n \in \mathbb{N}$ such that $\frac{1}{n} < s$ (by the Archimedean property, as $\frac{1}{n} < s \iff 1 < ns$). Thus, $s \in \left(\frac{1}{n}, 1\right)$ for some n, so $(0,1) \subseteq \bigcup_{n \in \mathbb{N}} O_n$. Therefore, $\left\{\left(\frac{1}{n}, 1\right) : n \in \mathbb{N}\right\}$ is an open covering of $(0,1)$.

Now each $O_n = \left(\frac{1}{n}, 1\right) \subseteq (0,1)$ (as $\frac{1}{n} > 0$), so we have

$$\bigcup_{n \in \mathbb{N}} O_n = (0,1)$$

Additionally, any finite subset of $\{O_n : n \in \mathbb{N}\}$ is not a subcovering as for all $k \in \mathbb{N}$,

$$\bigcup_{n \leq k} O_n = \left(\frac{1}{k}, 1\right) \not\supseteq (0,1)$$

since $\frac{1}{k} > 0$ for all k.

Proposition 4.35 — Intersection of Compact Nested Sets

Let K_i be non-empty compact sets for all $i \in \mathbb{N}$ such that $K_1 \supseteq K_2 \supseteq K_3 \supseteq \ldots$. It follows that

$$\bigcap_{i=1}^{\infty} K_i \neq \emptyset$$

Proof: Let (x_n) be a sequence with $x_n \in K_n$ for each $n \in \mathbb{N}$. Since K_1 is compact, there exists a convergent subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \to x \in K_1$. Now each $x_n \in K_n$, so $x_{n_m} \in K_m \supseteq K_{m+1} \supseteq K_{m+2} \supseteq \ldots$ for all $m \in \mathbb{N}$ with $m \geq n_k$ (as $n_m \geq m$). Thus, $(x_{n_k})_{k=m}^{\infty} \subseteq K_m$ for all m. ■

Remark: Note that proposition 4.35 does not hold for closed sets; compactness is required.

Example 4.36: Consider the closed sets of the form $C_n = [n, \infty)$ for $n \in \mathbb{N}$. Note that $C_1 \supseteq C_2 \supseteq C_3 \supseteq \ldots$ and $\bigcup\limits_{n=1}^{\infty} = \emptyset$. This is a counterexample demonstrating that proposition 4.35 does not hold for closed sets in general.

4.4 The Cantor Set

Definition 4.37 — Cantor Set

Let $C_0 = [0, 1]$ and C_{j+1} be the set obtained by taking the union of intervals C_j and removing the middle third from each interval in the union (e.g. $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$ and $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$). The *Cantor set* is defined as

$$C = \bigcap_{j=0}^{\infty} C_j$$

Proposition 4.38 — Cantor Set Properties

The following properties hold for the Cantor set C:

- $C \neq \emptyset$
- C is uncountable (i.e. $|C| = |\mathbb{R}|$)
- C has length 0
- C is compact

4.5 Connected and Disconnected Sets

Definition 4.39 — Disconnected Set

A set $S \subseteq \mathbb{R}$ is *disconnected* if and only if there exist open sets U and V such that $U \cap S \neq \emptyset$, $V \cap S \neq \emptyset$, $(U \cap S) \cap (V \cap S) = \emptyset$, and

$$S = (U \cap S) \cup (V \cap S)$$

If no such U and V exist, S is said to be *connected*.

Example 4.40: Show that $T = (-1, 0) \cup (0, 1)$ is disconnected.

Consider $U = (-1, 0)$ and $V = (0, 1)$. These are open sets with $U \cap T \neq \emptyset$, $V \cap T \neq \emptyset$, $(U \cap T) \cap (V \cap T) = \emptyset$, and $T = (U \cap T) \cup (V \cap T)$. By definition, T is disconnected.

Example 4.41: Show that $X = [-1, 1]$ is connected.

Assume, for contradiction, that X is disconnected with some open sets U and V satisfying the definition of disconnected sets. Let $a \in U$ and $b \in V$ with $a < b$ (without loss of generality). We have $[a, b] \subseteq [-1, 1]$. Consider

$$A = \left\{x \in [a, b] : [a, x] \subseteq U\right\}$$

Since $A \neq \emptyset$ (e.g. $a \in A$) and A is bounded, $\alpha = \sup(A) \in \mathbb{R}$. Now $a \leq \alpha \leq b$, so $\alpha \in [-1, 1]$.

If $\alpha \in U$, for some $\epsilon > 0$, $\left[a, \alpha + \frac{\epsilon}{2}\right] \subseteq U$ (as U is open). This contradicts the fact that $\alpha = \sup(A)$ (as $\alpha < \alpha + \frac{\epsilon}{2} \in U$). If $\alpha \in V$, $\left[\alpha - \frac{\epsilon}{2}, \alpha\right] \subseteq V$ for some $\epsilon > 0$ (as V is open). Since $\alpha - \frac{\epsilon}{2}$ is an upper bound for U, this contradicts the fact that $\alpha = \sup(A)$.

In either case, X must be connected by contradiction.

Theorem 4.42 — Connectedness of Intervals

$S \subseteq \mathbb{R}$ is connected if and only if S is an interval.

PROOF:

($\Longrightarrow$) If S is not an interval, then there exist $a, b \in S$ with $a < b$ such that some $t \notin S$, $a < t < b$. For $U = \{x \in \mathbb{R} : x < t\}$ and $V = \{x \in \mathbb{R} : x > t\}$, $U \cap S \neq \emptyset$ (as $a \in U \cap S$) and $V \cap S \neq \emptyset$ (as $b \in V \cap S$). Also, $(U \cap S) \cap (V \cap S) = \emptyset$ (since $U \cap V = \emptyset$) and $S = (U \cap S) \cup (V \cap S)$. By definition, S is disconnected, so any connected set must be an interval (by the contrapositive of the proven statement).

($\Longleftarrow$) By similar reasoning as in example 4.41, it can be shown that any interval is connected. ■

Definition 4.43 — Totally Disconnected

A set $S \subseteq \mathbb{R}$ is said to be *totally disconnected* if for all $x, y \in S$ with $x \neq y$, there exist disjoint open sets U and V such that $x \in U$, $y \in V$, and

$$S = (S \cap U) \cup (S \cap V)$$

Proposition 4.44 — Cantor Set Disconnectedness

The Cantor set C is totally disconnected.

PROOF: Let $x, y \in C$ with $x \neq y$ and $\delta = |x - y| > 0$. There exists some $j \in \mathbb{N}$ such that $\dfrac{1}{3^j} < \delta$. Now each interval in C_j has length $\frac{1}{3^j}$, so x and y belong to different intervals in C_j.

Choose some $t \in \mathbb{R}$ such that $x < t < y$ and $t \notin C_j$. It can be shown that $U = (-\infty, t)$ and $V = (t, \infty)$ satisfy the definition of disconnectedness for C. ■

4.6 Perfect Sets

Definition 4.45 — Perfect Set

A set $A \subseteq \mathbb{R}$ is said to be *perfect* if A is closed and for all $x \in A$, $x \in \mathrm{Acc}(A)$.

Example 4.46: $\{1\}$ is not perfect.

Example 4.47: $[0, 1]$ is perfect.

Example 4.48 *(Cantor Set Perfectness)*: The Cantor set C is perfect. We have

$$C = \bigcap_{j=0}^{\infty} C_j$$

where each C_j is closed. Thus, C is closed. Now for any $x \in C$, we will construct a sequence $(x_k) \subseteq C$ such that $x_k \neq x$ for all $k \in \mathbb{N}$ and $x_k \to x$. For each k, we pick x_k to be an endpoint of C_k containing x that is not x. Thus,

$$|x - x_k| \leq \frac{1}{3^k}$$

for all k, so $x_k \to x$.

Theorem 4.49 — Uncountability of Non-Empty Perfect Sets

If $A \subseteq \mathbb{R}$ is non-empty, then A is uncountable.

PROOF: Suppose that A is not uncountable. Since A contains accumulation points, it cannot be finite as it contains a non-trivial (i.e. non-constant) sequence. Thus, we can let $A = \{a_1, a_2, a_3, \ldots\}$. For $U = (a_1 - 1, a_1 + 1)$, we have $|U \cap A| = \infty$.

Now let U_2 be an open set such that $\overline{U_2} \subseteq U$, $U_2 \cap A \neq \emptyset$, and $a_1 \notin \overline{U_2}$. Generally, given an open set U_j such that $U_j \cap A \neq \emptyset$ and $a_j \notin \overline{U_j}$, we can construct an open set U_{j+1} such that $\overline{U_{j+1}} \subseteq U_j$ and $a_j \notin \overline{U_{j+1}}$. Consider

$$S = \bigcap_{j=1}^{\infty} \left(\overline{U_j} \cap A\right)$$

Since the $\overline{U_j} \cap A$ are nested compact sets, $S \neq \emptyset$. Now for all j, $a_j \notin \overline{U_{j+1}} \implies a_j \notin S$, which contradicts the fact that $a_j \in A$. Therefore, A must be uncountable by contradiction. ■

Chapter 5 Continuity

5.1 Limits

Definition 5.1 — Limit

Let $E \subseteq \mathbb{R}$, $f\colon E \to \mathbb{R}$ be a function, $P \in \operatorname{Acc}(E)$, and $L \in \mathbb{R}$. We say that

$$\lim_{E \ni x \to P} f(x) = L$$

if and only if for all $\epsilon > 0$, there exists some $\delta > 0$ such that for all $x \in E$,

$$0 < |x - P| < \delta \implies \big|f(x) - L\big| < \epsilon$$

Proposition 5.2 — Uniqueness of Limits

Let $f\colon E \to \mathbb{R}$ be a function with $E \subseteq \mathbb{R}$ and $P \in \operatorname{Acc}(E)$. If $\lim_{x \to P} f(x) = L$ and $\lim_{x \to P} f(x) = M$, then $L = M$.

Proof: Assume, for contradiction, that $L \neq M$. Let $\epsilon = \dfrac{|L - M|}{4} > 0$. By definition, there exists some $\delta_1 > 0$ such that for $x \in E$,

$$0 < |x - P| < \delta_1 \implies \big|f(x) - L\big| < \epsilon$$

Similarly, there exists some $\delta_2 > 0$ such that for $x \in E$,

$$0 < |x - P| < \delta_2 \implies \big|f(x) - M\big| < \epsilon$$

Take $\delta = \min(\delta_1, \delta_2)$. Suppose that $x \in E$ and $0 < |x - P| < \delta$. Using the triangle inequality, we have

$$|L - M| = \big|L - f(x) + f(x) - M\big| \leq \big|L - f(x)\big| + \big|f(x) - M\big| < \epsilon + \epsilon = 2\epsilon = 2 \cdot \frac{|L - M|}{4} = \frac{|L - M|}{2}$$

which is a contradiction. Therefore, $L = M$ by contradiction. ∎

Theorem 5.3 — Properties pf Limits

Suppose that $f\colon E \to \mathbb{R}$ and $g\colon E \to \mathbb{R}$ are functions with $E \subseteq \mathbb{R}$ and $P \in \operatorname{Acc}(E)$. If $\lim_{x \to P} f(x) = L$ and $\lim_{x \to P} g(x) = M$, then

1. $\lim_{x \to P} (f \pm g)(x) = L \pm M$
2. $\lim_{x \to P} (f \cdot g)(x) = L \cdot M$
3. $\lim_{x \to P} \dfrac{f}{g}(x) = \dfrac{L}{M}$ provided that $M \neq 0$

Proof: We will prove (2). Let $\epsilon > 0$ so that $\epsilon < 1$. By definition, there exists some $\delta_1 > 0$ such that for $x \in E$,

$$0 < |x - P| < \delta_1 \implies \big|f(x) - L\big| < \frac{\epsilon}{2\big(|M| + 1\big)}$$

Similarly, there exists some $\delta_2 > 0$ such that for $x \in E$,

$$0 < |x - P| < \delta_2 \implies \big|g(x) - M\big| < \frac{\epsilon}{2\big(|L| + 1\big)}$$

Take $\delta = \min(\delta_1, \delta_2)$. Suppose that $x \in E$ and $0 < |x - P| < \delta$. We have

$$\begin{aligned}
&\big|f(x)g(x) - LM\big| \\
&= \big|f(x)g(x) - Lg(x) + Lg(x) - LM\big| \\
&\leq \big|f(x)g(x) - Lg(x)\big| + \big|Lg(x) - LM\big| && \text{(by the triangle inequality)}
\end{aligned}$$

$$
\begin{aligned}
&= \left|g(x)\right|\left|f(x)-L\right| + |L|\left|g(x)-M\right| \\
&< \left(|M|+1\right)\cdot\left|f(x)-L\right| + |L|\left|g(x)-M\right| && \left(\text{since } \left|g(x)-M\right| < \epsilon < 1 \implies \left|g(x)\right| + |-M| < 1 \implies \left|g(x)\right| < |M|+1\right) \\
&< \left(|M|+1\right)\cdot\left|f(x)-L\right| + \left(|L|+1\right)\left|g(x)-M\right| \\
&< \left(|M|+1\right)\cdot\frac{\epsilon}{2\left(|M|+1\right)} + \left(|L|+1\right)\cdot\frac{\epsilon}{2\left(|L|+1\right)} \\
&< \frac{\epsilon}{2}+\frac{\epsilon}{2} && \left(\text{as } 0 < |M|+1 < 2\left(|M|+1\right) \implies \tfrac{|M|+1}{2\left(|M|+1\right)} < 1\text{; similar reasoning holds for } L\right) \\
&= \epsilon
\end{aligned}
$$

■

Proposition 5.4 — Sequential Function Limit

Let $f\colon E \to \mathbb{R}$ be a function with $E \subseteq \mathbb{R}$ and $P \in \operatorname{Acc}(E)$. It follows that

$$\lim_{x\to P} f(x) = L$$

if and only if for all sequences $\left(a_j\right) \subseteq E \setminus \{P\}$,

$$a_j \to P \implies \lim_{j\to\infty} f(a_j) = L$$

Proof:

($\Longleftarrow$) Suppose that $\lim\limits_{x\to P} \neq L$. By definition, there exists some $\epsilon > 0$ such that for all $\delta > 0$,

$$0 < |x-P| < \delta \;\not\Longrightarrow\; \left|f(x)-L\right| < \epsilon$$

for some $x \in E$. That is, there exists some $x \in E$ such that $0 < |x-P| < \delta$ but $\left|f(x)-L\right| \geq \epsilon$. For $\delta_j = \dfrac{1}{j}$ ($j \in \mathbb{N}$), there thus exists a sequence $\left(x_j\right) \subseteq E \setminus \{P\}$ such that $0 < \left|x_j - P\right| < \frac{1}{j}$ for all j but $\left|f(x)-L\right| \geq \epsilon$. We have shown that $x_j \to P$ but $f(x_j) \not\to L$.

($\Longrightarrow$) Now suppose that there exists a sequence $\left(x_j\right) \subseteq E \setminus \{P\}$ such that $x_j \to P$ but $f(x_j) \not\to L$ (this is the negation of the hypothesis). By definition, there exists some $\epsilon > 0$ such that for infinitely many j, $\left|f(x_j)-L\right| \geq \epsilon$. Thus, for all $\delta > 0$ and sufficiently large j, $0 < \left|x_j - P\right| < \delta$ but

$$\left|f(x_j)-L\right| \geq \epsilon$$

for infinitely many j. Therefore, $\lim\limits_{x\to P} f(x) \neq L$. ■

5.2 Continuity

Definition 5.5 — Continuity

Suppose that $f\colon E \to \mathbb{R}$ is a function with $E \subseteq \mathbb{R}$ and $P \in E \cap \operatorname{Acc}(E)$. f is *continuous* at P if

$$\lim_{x\to P} f(x) = f(P)$$

Definition 5.6 — Inverse Image

Let $f\colon E \to \mathbb{R}$ be a function with $E \subseteq \mathbb{R}$ and $W \subseteq \mathbb{R}$. The *inverse image of W under f* is

$$f^{-1}(W) = \left\{x \in E : f(x) \in W\right\}$$

Theorem 5.7 — Continuity and Inverse Images

Suppose that $f\colon E \to \mathbb{R}$ is a function with $E \subseteq \mathbb{R}$. f is continuous if and only if for all open sets U, there exists an open set A such that $f^{-1}(U) = A \cap E$.

PROOF:

($\Longrightarrow$) Suppose that f is continuous. Let U be an open set and $P \in f^{-1}(U)$. We have $f(P) \in U$, so there exists some $\epsilon > 0$ such that $\big(f(P)-\epsilon, f(P)+\epsilon\big) \subseteq U$ as U is open. Since f is continuous at P, there exists some $\delta > 0$ such that

$$x \in E \cap \big((P-\delta, P) \cup (P, P+\delta)\big) \implies f(x) \in \big(f(P)-\epsilon, f(P)+\epsilon\big) \subseteq U$$

($\Longleftarrow$) Now suppose that for all open sets U, $f^{-1}(U) = A \cap E$ for some open set A. For all $P \in E \cap \operatorname{Acc}(E)$ and every $\epsilon > 0$, $U_\epsilon = \big(f(P)-\epsilon, f(P)+\epsilon\big)$ is open. Thus, $f^{-1}(U_\epsilon) = E \cap A$ for some open set A, so if $P \in A$, then $(P-\delta, P+\epsilon) \subseteq A$ for some $\delta > 0$ (as A is open). Therefore,

$$E \cap (P-\delta, P+\delta) \subseteq E \cap A \implies f\big(E \cap (P-\delta, P+\delta)\big) \subseteq f(E \cap A) \subseteq U_\epsilon$$

so f is continuous at any $P \in E \cap \operatorname{Acc}(E)$ by definition. ■

Theorem 5.8 — Preservation of Compactness by Continuity

Let $f\colon E \to \mathbb{R}$ be a continuous function with domain $E \subseteq \mathbb{R}$. If $C \subseteq \mathbb{R}$ is compact, then $f(C)$ is also compact.

PROOF: Consider an open covering $\{O_i\}$ of $f(C)$. By definition, each O_i is open and $\bigcup_i O_i \supseteq f(C)$. We will show that finitely many of these O_i cover $f(C)$. For each i, let $f^{-1}(O_i) = \widetilde{O_i} \cap C$. Each $\widetilde{O_i}$ is open, and for any $p \in C$,

$$f(p) \in f(C) \implies f(p) \in O_j$$

for some j. Thus,

$$p \in f^{-1}(O_j) \implies p \in \widetilde{O_j} \cap C \implies p \in \widetilde{O_j}$$

Thus, $\left\{\widetilde{O_j}\right\}$ is an open covering of C. Since C is compact, there exists a finite subcovering $\left\{\widetilde{O_{j_1}}, \ldots, \widetilde{O_{j_n}}\right\}$ of $\left\{\widetilde{O_j}\right\}$. By definition,

$$\bigcup_{k=1}^{n} O_{j_k} \supseteq C$$

Furthermore, since $f^{-1}(O_i) = \widetilde{O_i} \cap C$ for each i, we have $O_i = f(\widetilde{O_i} \cap C)$ and thus

$$\bigcup_{k=1}^{n} f(\widetilde{O_{j_k}} \cap C) = f(C)$$

We have shown that

$$\bigcup_{k=1}^{n} O_{j_k} \supseteq \bigcup_{k=1}^{n} f(\widetilde{O_{j_k}} \cap C) = f(C)$$

so every open covering $\{O_i\}$ of $f(C)$ has a finite subcovering $\{O_{j_k}\}$. Therefore, $f(C)$ is compact. ■

Remark: In general, continuous functions do *not* necessarily preserve open or closed sets, but they preserve compact sets.

Corollary 5.9 — Boundedness by Continuity and Compactness

If $f\colon C \to \mathbb{R}$ is a continuous function, where $C \subseteq \mathbb{R}$ is compact, then $f(C)$ is bounded.

Corollary 5.10 — Extreme Value Theorem

Suppose that $f\colon C \to \mathbb{R}$ is a continuous function, where $C \subseteq \mathbb{R}$ is compact. It follows that

$$\sup\big(f(C)\big) \in f(C) \qquad\qquad \inf\big(f(C)\big) \in f(C)$$

That is, f attains some minimum and maximum values on C.

5.3 Continuity and Connectedness

Proposition 5.11 — Continuity and Connectedness

Let $f\colon L \to \mathbb{R}$ be a continuous function, where $L \subseteq \mathbb{R}$ is connected. It follows that $f(L)$ is connected.

Proof: We will prove the contrapositive of the statement. Suppose that $f(L)$ is not connected. By definition, there exist some sets $U \subseteq \mathbb{R}$ and $V \subseteq \mathbb{R}$ that satisfy each of the following conditions:

1. U and V are open
2. $U \cap f(L) \neq \emptyset$ and $V \cap f(L) \neq \emptyset$
3. $\big(U \cap f(L)\big) \cap \big(V \cap f(L)\big) = \emptyset$
4. $\big(U \cap f(L)\big) \cup \big(V \cap f(L)\big) = f(L)$

Now consider $f^{-1}\big(U \cap f(L)\big)$ and $f^{-1}\big(V \cap f(L)\big)$. We have $f^{-1}\big(U \cap f(L)\big) \cap f^{-1}\big(V \cap f(L)\big) = \emptyset$. Also, $f^{-1}\big(U \cap f(L)\big)$ and $f^{-1}\big(V \cap f(L)\big)$ are open by the continuity of f. Finally, $f^{-1}\big(U \cap f(L)\big) \cup f^{-1}\big(V \cap f(L)\big) = L$, so L is disconnected by definition.

Justifying each of the above claims was left as an exercise. ■

Remark: In the context of proposition 5.11, L and $f(L)$ are intervals when working in $\mathbb{R}$ by theorem 4.42.

Corollary 5.12 — Intermediate Value Theorem

Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function, where $a, b \in \mathbb{R}$ with $a \leq b$. For all $y \in \mathbb{R}$ between $f(a)$ and $f(b)$ (inclusive), there exists some $x \in [a,b]$ such that

$$f(x) = y$$

Definition 5.13 — Uniform Continuity

Let $f\colon A \to \mathbb{R}$, where $A \subseteq \mathbb{R}$. f is *uniformly continuous* on E if for every $\epsilon > 0$, there exists some $\delta > 0$ such that for all $x_0 \in E$,

$$0 < |x - x_0| < \delta \implies \big|f(x) - f(x_0)\big| < \epsilon$$

Remark: In definition 5.13, δ may depend on ϵ (but not x_0 or x).

Example 5.14: Consider $f(x) = x^2$. We have $f\colon \mathbb{R} \to \mathbb{R}$. For any $\epsilon > 0$ and $x_0 \in \mathbb{R}$, choose $\delta = \mathit{value}$. If $|x - x_0| < \delta$,

$$\big|f(x) - f(x_0)\big| = \left|x^2 - x_0^2\right| = |x - x_0||x + x_0| \approx |x - x_0||x_0|$$

as $x \approx x_0$ for values of x sufficiently close to x_0. Now

$$|x - x_0||x_0| < \epsilon \iff |x - x_0| < \frac{\epsilon}{|x_0|} \to 0$$

as $x_0 \to \infty$. Thus, f is not uniformly continuous on $\mathbb{R}$.

(This is not a rigorous argument; a more precise proof can be found in the textbook.)

Theorem 5.15 — Uniform Continuity and Compactness

Let $f\colon C \to \mathbb{R}$ be a continuous function, where $C \subseteq \mathbb{R}$ is compact. It follows that f is uniformly continuous.

Proof: Fix $\epsilon > 0$. For any $x_i \in C$, choose some $\delta_i > 0$ such that

$$|x - x_i| < \delta_0 \implies \big|f(x) - f(x_0)\big| < \frac{\epsilon}{2}$$

by the continuity of f. Consider $N_{x_i} = \left(x - \frac{\delta_i}{2}, x + \frac{\delta_i}{2}\right)$. Observe that $\{N_{x_i} : x_i \in C\}$ is an open cover of C. Since C is compact, there exists a corresponding finite subcovering $\{N_{x_0}, \ldots, N_{x_n}\}$. Now choose $\delta = \min\{\delta_0, \ldots, \delta_n\}$.

To be completed next lecture. ■

Corollary 5.16 — Fixed Point

Let $f\colon [0,1] \to [0,1]$ be a continuous function. There exists some $x_0 \in [0,1]$ such that $f(x_0) = x_0$.

PROOF: Consider the function $h(x) = f(x) - x$. We have $h(0) = f(0) - 0 \geq 0$ and $h(1) = f(1) - 1 \leq 0$. Since h is continuous on $[0,1]$, by corollary 5.12, there exists some $x_0 \in [0,1]$ such that $h(x_0) = 0$. For this x_0, we have

$$h(x_0) = f(x_0) - x_0 = 0 \implies f(x_0) = x_0$$ ■

Remark: Corollary 5.16 follows from corollary 5.12 (Intermediate Value Theorem).

5.4 Continuity and Monotonicity

Definition 5.17 — One-Sided Limits

Let f be a function and $x_0, L \in \mathbb{R}$. We say that

$$\lim_{x \to x_0^+} f(x) = L$$

if and only if for all $\epsilon > 0$, there exists some $\delta > 0$ such that for all $x \in E$,

$$x \in (x_0, x_0 + \delta) \implies \left|f(x) - L\right| < \epsilon$$

This is known as the *right limit*. Similarly, we say that

$$\lim_{x \to x_0^-} f(x) = L$$

if and only if for all $\epsilon > 0$, there exists some $\delta > 0$ such that for all $x \in E$,

$$x \in (x_0 - \delta, x_0) \implies \left|f(x) - L\right| < \epsilon$$

This is known as the *left limit*.

Definition 5.18 — Discontinuity Classifications

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. For any $x_0 \in \mathbb{R}$, we have three possible cases with respect to continuity:

1. f is continuous at x_0
2. $\lim\limits_{x \to x_0^-} f(x)$ and $\lim\limits_{x \to x_0^+} f(x)$ exist, but one of the following conditions holds:
   (a) $\lim\limits_{x \to x_0^-} f(x) \neq \lim\limits_{x \to x_0^+} f(x)$
   (b) $\lim\limits_{x \to x_0^-} f(x) = \lim\limits_{x \to x_0^+} f(x)$ exists but $\lim\limits_{x \to x_0} f(x) \neq f(x_0)$

   In each of these cases, f is said to have a *discontinuity of the first type* at x_0
3. One of $\lim\limits_{x \to x_0^-} f(x)$ and $\lim\limits_{x \to x_0^+} f(x)$ does not exist; f is said to have a *discontinuity of the second type* at x_0

Example 5.19 *(Dirichlet Function)*: The Dirichlet function is given by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

This function is discontinuous everywhere.

Proposition 5.20 — Monotonicity and Discontinuity Classification

Let $f\colon (a,b) \to \mathbb{R}$ be a monotonic function, where $a, b \in \mathbb{R}$ with $a < b$. Any discontinuities of f must be of the first kind.

Proof: Without loss of generality, assume that f is increasing. Suppose that f is discontinuous at $x_0 \in (a,b)$. We will show that $\lim_{x \to x_0^- f(x)}$ exists. Note that for $\delta > 0$, $(x_0 - \delta, x_0)$ is bounded, and we have $f(x_0 - \delta) \leq f(x) \leq f(x_0)$ for all $x \in (a,b)$ (as f is increasing). Thus, $M = \sup\Big(f\big((a, x_0)\big)\Big) \in \mathbb{R}$.

To be completed next lecture. ■

Proposition 5.21 — Monotonicity and Countable Discontinuities

Let $f\colon (a,b) \to \mathbb{R}$ be a monotonic function, where $a, b \in \mathbb{R}$ with $a < b$. It follows that f has at most countably many discontinuities.

Chapter 6 Derivatives

6.1 Derivatives

Definition 6.1 — Derivative

Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a function. f is *differentiable* at $x_0 \in \mathbb{R}$ if

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. The above quantity is known as the *derivative* of f at x_0, denoted by $f'(x_0)$.

Proposition 6.2 — Differentiability and Continuity

If f is differentiable at $x_0 \in \mathbb{R}$, then f is continuous at x_0.

Proof: We have

$$\lim_{x\to x_0} \big(f(x) - f(x_0)\big) = \lim_{x\to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0}\right) \cdot \lim_{x\to x_0} (x - x_0) = f'(x_0) \cdot 0 = 0$$

so $\lim_{x\to x_0} f(x) = f(x_0)$. ■

Remark: The converse of proposition 6.2 does *not* hold in general. $f(x) = |x|$ at $x_0 = 0$ is a counterexample.

Theorem 6.3 — Differentiation Rules

Suppose that f and g are differentiable functions. Each of the following hold:

1. $f + g$ is differentiable with $(f + g)'(x) = f'(x) + g'(x)$
2. λf is differentiable for all $\lambda \in \mathbb{R}$, with $(\lambda f)'(x) = \lambda f'(x)$
3. $f \cdot g$ is differentiable with $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$
4. $\dfrac{f}{g}$ is differentiable with $\left(\dfrac{f}{g}\right)'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$
5. $f \circ g$ is differentiable provided that f is differentiable at $g(x)$, with $(f \circ g)'(x) = f'\big(g(x)\big) \cdot g'(x)$

Proof: We will prove the chain rule. Let p be a differentiable function and $r \in \mathbb{R}$. Note that for $\delta > 0$, if $0 < |x - r| < \delta$, then $x = r + h$ for some $h \in \mathbb{R}$ with $-\delta < h < \delta$. Thus,

$$\begin{aligned}
\lim_{x\to r} \frac{p(x) - p(r)}{x - r} &= \lim_{h\to 0} \frac{p(r+h) - p(r)}{h} \\
&= p'(r) \\
\implies \lim_{h\to 0} \left(\frac{p(r+h) - p(r)}{h} - p'(r)\right) &= 0 \\
\implies \lim_{h\to 0} \frac{p(r+h) - p(r) - h \cdot p'(r)}{h} &= 0
\end{aligned}$$

Letting $V(h) = \dfrac{p(r+h) - p(r) - h \cdot p'(r)}{h}$, we have

$$h \cdot V(h) = p(r+h) - p(r) - h \cdot p'(r) \implies p(r+h) = p(r) + h \cdot p'(r) + h \cdot V(h)$$

Using this fact, if $0 < |x - x_0| < \delta$, then

$$\begin{aligned}
f(x + h_1) &= f(x) + h_1 \cdot f'(x) + h_1 \cdot V_1(h_1) \\
g(x + h_2) &= g(x) + h_2 \cdot g'(x) + h_2 \cdot V_2(h_2)
\end{aligned}$$

for some functions V_1 and V_2 such that $\lim_{h_1\to 0} V_1(h_1) = 0$ and $\lim_{h_2\to 0} V_2(h_2) = 0$. It follows that

$$f\big(g(x_0 + h_2)\big) = f\Big(g(x_0) + h_2 \cdot \big(g'(x_0) + V_2(h_2)\big)\Big)$$

$$= f\big(g(x_0)\big) + h_2 \cdot \big(g'(x_0) + V_2(h_2)\big) \cdot f'\big(g(x_0)\big) + h_2 \cdot \big(g'(x_0) + V_2(h_2)\big) \cdot V_1\Big(h_2 \cdot \big(g'(x_0) + V_2(h_2)\big)\Big)$$

Thus,

$$\begin{aligned}
& f\big(g(x_0+h_2)\big) - f\big(g(x_0)\big) = h_2 \cdot \big(g'(x_0) + V_2(h_2)\big) \cdot f'\big(g(x_0)\big) + h_2 \cdot \big(g'(x_0) + V_2(h_2)\big) \cdot V_1\Big(h_2 \cdot \big(g'(x_0) + V_2(h_2)\big)\Big) \\
\implies \quad & \frac{f\big(g(x_0+h_2)\big) - f\big(g(x_0)\big)}{h_2} = \big(g'(x_0) + V_2(h_2)\big) \cdot f'\big(g(x_0)\big) + \big(g'(x_0) + V_2(h_2)\big) \cdot V_1\Big(h_2 \cdot \big(g'(x_0) + V_2(h_2)\big)\Big) \\
\implies \lim_{h_2 \to 0} & \frac{f\big(g(x_0+h_2)\big) - f\big(g(x_0)\big)}{h_2} = \big(g'(x_0) + 0\big) \cdot f'\big(g(x_0)\big) + \big(g'(x_0) + 0\big) \cdot 0 \\
& = f'\big(g(x_0)\big) \cdot g'(x_0)
\end{aligned}$$

■

Theorem 6.4 — Fermat's Theorem

If $f\colon (a,b) \to \mathbb{R}$ is differentiable (where $a, b \in \mathbb{R}$ with $a < b$) and $x_0 \in (a,b)$ is a local extremum, then $f'(x_0) = 0$.

Theorem 6.5 — Darboux's Theorem

Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable and $a, b \in \mathbb{R}$ with $a < b$ and $f'(a) < f'(b)$. If $s \in \mathbb{R}$ satisfies $f'(a) < s < f'(b)$, then there exists some $x_0 \in (a,b)$ such that

$$f'(x_0) = s$$

Proof: Suppose that $s \in \mathbb{R}$ satisfies $f'(a) < s < f'(b)$. Consider the function $g(x) = f(x) - sx$. We have

$$g'(x) = f'(x) - s$$

so $g'(a) = f'(a) - s < 0$ (as $f'(a) < s$) and $g'(b) = f'(a) - s > 0$ (as $f'(b) > s$). Since $f(x)$ and sx are differentiable, so is g. Thus, g must have a local minimum at some $x_0 \in (a,b)$ by continuity. By theorem 6.4,

$$g'(x_0) = 0 \implies f'(x_0) - s = 0 \implies f'(x_0) = s$$

■

Lemma 6.6 — Rolle's Theorem

Suppose that f is a function and $a, b \in \mathbb{R}$ with $a < b$. If f is continuous on $[a,b]$ and differentiable on (a,b), and $f(a) = f(b) = 0$, then there exists some $x_0 \in (a,b)$ such that

$$f'(x_0) = 0$$

Theorem 6.7 — Mean Value Theorem

Suppose that f is a function and $a, b \in \mathbb{R}$ with $a < b$. If f is continuous on $[a,b]$ and differentiable on (a,b), then there exists some $x_0 \in (a,b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(x_0)$$

Proof: Consider the function $g(x) = f(x) - \left(\dfrac{f(b) - f(a)}{b - a} \cdot (x - a) + f(a)\right)$. Now $g(a) = g(b) = 0$, so by lemma 6.6, there exists some $x_0 \in (a,b)$ such that

$$g'(x_0) = 0 \implies f'(x_0) - \frac{f(b) - f(a)}{b - a} = 0 \implies f'(x_0) = \frac{f(b) - f(a)}{b - a}$$

■

Example 6.8: Consider the function

$$f(x) = \begin{cases} x^{3/2} \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

We have

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{x^{3/2}\sin\left(\frac{1}{x}\right)}{x}=\lim_{x\to 0}\sqrt{x}\sin\left(\frac{1}{x}\right)=0$$

Thus, $f'(0)=0$. For $x>0$,

$$f'(x)=\frac{3}{2}\sqrt{x}\cdot\sin\left(\frac{1}{x}\right)-\frac{x^{3/2}}{x^2}\cdot\cos\left(\frac{1}{x}\right)=\frac{3}{2}\sqrt{x}\sin\left(\frac{1}{x}\right)-\frac{1}{\sqrt{x}}\cos\left(\frac{1}{x}\right)$$