

MAT B41 — Techniques of the Calculus of Several Variables I

FALL 2023

This is a compilation of the notes from the MAT B41 lectures. The page and section references in parentheses occurring after definitions, theorems, other facts, and section titles refer to the textbook *Multivariable Calculus, 9th ed., Stewart, Clegg & Watson*. Certain graphs/figures are from this textbook, while others have been made using the GeoGebra Calculator Suite or the PGFPLOTS L^AT_EX package. Each of the facts (definitions, theorems, axioms, etc.) are numbered for cross-referencing purposes.

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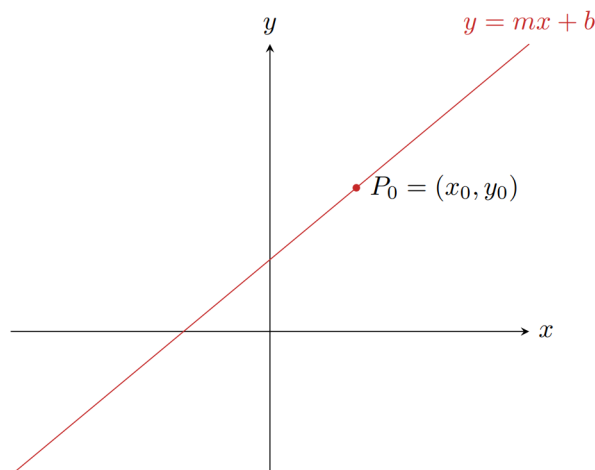
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Chapter 1 Geometry of Euclidean Space

1.1 Equations of Lines & Planes and Parametric Equations (§§10.1, 12.5)



In \mathbb{R}^2 , the equation of a line L is $y = mx + b$. Alternatively, in point-slope form, $y - y_0 = m(x - x_0)$ (given $P_0, P_1 \in L$ with $P_0 \neq P_1$ to compute m).

Definition 1.1 — Parametric Equations (p. 662)

Suppose that x and y are real-valued functions of t on an interval $I \subseteq \mathbb{R}$. That is, $x = f(t)$ and $y = g(t)$ with $t \in I$. These equations are called *parametric equations* with *parameter* t .

The set of points of x and y as t varies over I

$$\{(x, y) : x = f(t) \wedge y = g(t) \wedge t \in I\}$$

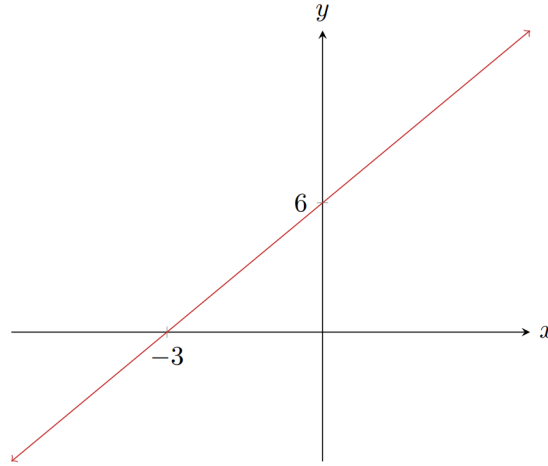
is the graph of the parametric equations of the *parametric curve*.

Example 1.2 (*Parametric Line*): Consider $x = t - 1$ and $y = 2t + 4$, where $t \in \mathbb{R}$.

- (a) Sketch the parametric curve.
- (b) Write the parametric curve in the form $y = f(x)$.
- (a) A table of values with some points as follows:

t	" $x(t)$ "	" $y(t)$ "
-1	-2	2
0	-1	4
1	0	6

Using these points, the curve is



(b) Since $x = t - 1 \iff t = x + 1$, we have

$$y = 2t + 4 = 2(x + 1) + 4 = 2x + 6$$

Therefore, $y = 2x + 6$.

Alternatively, we could proceed by noting that $y = 2t + 4 \iff t = \frac{1}{2}(y - 4)$, so

$$x + 1 = \frac{1}{2}(y - 4) \iff y = 2x + 6$$

Example 1.3: What curve/function in \mathbb{R}^2 is given by

$$x = \cos(t) \quad y = \sin(t) \quad t \in [0, 2\pi]$$

(in the form $y = f(x)$ or $f(x, y) = 0$)?

For all $t \in [0, 2\pi]$, $x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1$. Thus, the given curve is

$$f(x, y) = x^2 + y^2 - 1 = 0$$

1.2 More Lines in \mathbb{R}^3 (§12.5)

Definition 1.4 — Position Vector

A *position vector* represents a vector's components as a point with respect to the origin. In \mathbb{R}^3 , a position vector \mathbf{v} is denoted by $\mathbf{v} = \langle a, b, c \rangle$ for $a, b, c \in \mathbb{R}$.

Remark: The angled brackets for position vectors as in definition 1.4 are used to distinguish between ordered tuples representing points in space and vectors (e.g. $\langle a, b, c \rangle$ instead of (a, b, c) in \mathbb{R}^3).

Theorem 1.5 — Derivation of Line Equation in \mathbb{R}^3

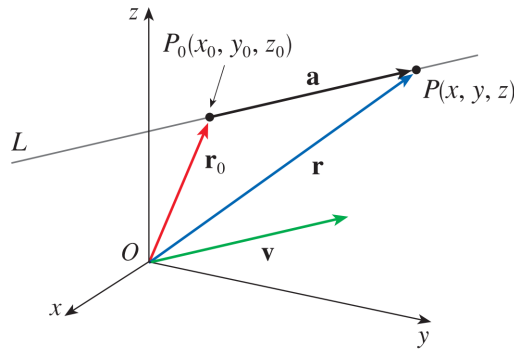
A line L in \mathbb{R}^3 may be determined by one of the following:

- Points $P_0, P \in L$ with $P_0 \neq P$
- A point $P_0 \in L$ and a *direction vector* — some $\mathbf{v} \in \mathbb{R}^3$ such that $\mathbf{v} \parallel L$

Given a point $P_0 = (x_0, y_0, z_0) \in L$ and a direction vector $\mathbf{v} = \langle a, b, c \rangle \in \mathbb{R}^3$ for L ,

$$x = x_0 + ta \quad y = y_0 + tb \quad z = z_0 + tc \quad t \in \mathbb{R}$$

are the parametric equations of L in \mathbb{R}^3 . Here, $t \in \mathbb{R}$ satisfies $\overrightarrow{P_0P} = t\mathbf{v}$ for some arbitrary $P = (x, y, z) \in L$.



PROOF: We will derive the line L given a point and a direction vector. Let $P_0 = (x_0, y_0, z_0) \in L$, \mathbf{v} be some direction vector for L , and $P = (x, y, z) \in L$ be an arbitrary point. Define position vectors \mathbf{r}_0 and \mathbf{r} from the origin to P_0 and P (respectively), and let $\mathbf{a} = \overrightarrow{P_0P}$.

We have $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}$. Note that $\mathbf{a} \parallel \mathbf{v}$, so $\mathbf{a} = t\mathbf{v}$ for some $t \in \mathbb{R}$. Therefore, $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$. Now let $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$, $\mathbf{r} = \langle x, y, z \rangle$, and $\mathbf{v} = \langle a, b, c \rangle$. It follows that

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} \iff \langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle \iff \langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

Therefore, $x = x_0 + ta$, $y = y_0 + tb$, and $z = z_0 + tc$ for $t \in \mathbb{R}$ are the parametric equations of L in \mathbb{R}^3 . ■

Definition 1.6 — Vector Equation of a Line (p. 865)

In the context of the derivation of the parametric equations of a line in \mathbb{R}^3 (the proof of theorem 1.5), the *vector equation* of L is

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

or equivalently,

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

Definition 1.7 — Symmetric Equations of a Line in \mathbb{R}^3

The *symmetric equations* of a line L in \mathbb{R}^3 is

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

where a , b , and c must be non-zero and $\langle x_0, y_0, z_0 \rangle$ and $\langle a, b, c \rangle$ are the same vectors as in the derivation of the parametric equations of L (the proof of theorem 1.5). If $a = 0$, $x = x_0$, and similarly $y = y_0$ and $z = z_0$ for the cases where $b = 0$ and $c = 0$ (respectively).

Remark: The symmetric equations in definition 1.7 follow from theorem 1.5, where $x = x_0 + ta \iff t = \frac{x - x_0}{a}$ for $a \neq 0$, and similarly $t = \frac{y - y_0}{b}$ for $b \neq 0$ and $t = \frac{z - z_0}{c}$ for $c \neq 0$.

1.3 Equations of Planes in \mathbb{R}^3

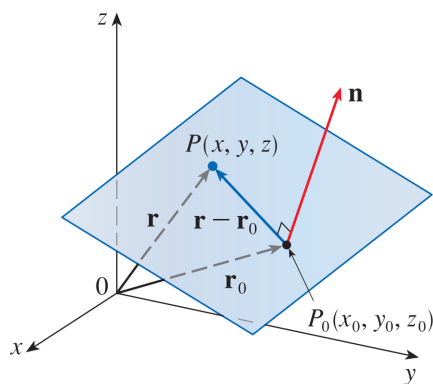
Theorem 1.8 — Derivation of Plane Equation in \mathbb{R}^3

The equation of a plane in \mathbb{R}^3 can be uniquely determined by one of the following:

- A point $P_0 = (x_0, y_0, z_0)$ in the plane and a *normal vector* to the plane
- A point $P_0 = (x_0, y_0, z_0)$ in the plane and a line L on the plane such that $P_0 \notin L$

Given a point $P_0 = (x_0, y_0, z_0)$ on the plane and a normal vector $\mathbf{n} = \langle a, b, c \rangle$ to the plane, the *scalar equation* of the plane is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$



PROOF: We will derive the plane's equation given a point and a normal vector. Let $P_0 = (x_0, y_0, z_0)$ be a point on the plane, \mathbf{n} be a normal vector to the plane, and $P = (x, y, z)$ be an arbitrary point on the plane. Suppose that $\mathbf{a} = \overrightarrow{P_0P}$ and \mathbf{r}_0 and \mathbf{r} are position vectors with respect to the points P_0 and P (respectively).

The set of all points P on the plane satisfy $\mathbf{n} \cdot \mathbf{a} = 0$ (as \mathbf{n} is normal to the plane). Also, $\mathbf{a} = \mathbf{r} - \mathbf{r}_0$, so $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$. Now let $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$, $\mathbf{r} = \langle x, y, z \rangle$, and $\mathbf{n} = \langle a, b, c \rangle$. It follows that

$$\begin{aligned} \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 &\iff \langle a, b, c \rangle \cdot (\langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle) = 0 \\ &\iff \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0 \\ &\iff a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \end{aligned}$$

Thus, $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ is the plane's scalar equation. ■

Remark: Unlike in linear algebra, “normal” and “orthogonal” are synonymous in this course, both representing perpendicularity (but *not* unit length).

Definition 1.9 — Vector Equation of a Plane (p. 868)

In the context of the derivation of the scalar equation of a plane in \mathbb{R}^3 (the proof of theorem 1.8), the *vector equation of the plane* is

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

Corollary 1.10 — Plane Alternative Scalar Equation

In the setup of the derivation of a plane's scalar equation in \mathbb{R}^3 (the proof of theorem 1.8), an equivalent form of the scalar equation of the plane is

$$ax + by + cz + d = 0$$

for $\mathbf{n} = \langle a, b, c \rangle$ and $d = -(ax_0 + by_0 + cz_0)$.

Remark: The equivalent form of a plane's scalar equation in corollary 1.10 is a result of expanding the scalar equation (as in theorem 1.8's proof) and collecting like terms.

Example 1.11: Find the scalar equation of a plane containing the points $P = (1, 1, -2)$, $Q = (0, 2, 1)$, and $R = (-1, -1, 0)$.

We first find a normal vector \mathbf{n} to the plane. Note that

$$\overrightarrow{QP} = \langle 1 - 0, 1 - 2, -2 - 1 \rangle = \langle 1, -1, -3 \rangle$$

$$\overrightarrow{QR} = \langle -1 - 0, -1 - 2, 0 - 1 \rangle = \langle -1, -3, -1 \rangle$$

Thus,

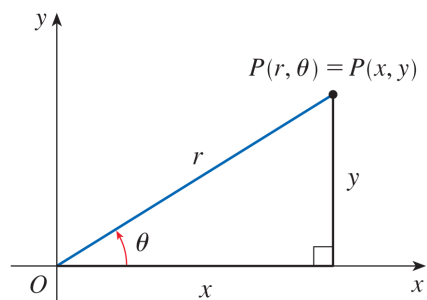
$$\mathbf{n} = \overrightarrow{QP} \times \overrightarrow{QR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -3 \\ -1 & -3 & -1 \end{vmatrix} = \mathbf{i}(1 - 9) - \mathbf{j}(-1 - 3) + \mathbf{k}(-3 - 1) = -8\mathbf{i} + 4\mathbf{j} - 4\mathbf{k} = \langle -8, 4, -4 \rangle$$

using cofactor expansion along the first row (note that the standard basis vectors \mathbf{i} , \mathbf{j} , and \mathbf{k} are treated just as any other entry in the matrix). Therefore, taking $P_0 = (1, 1, -2)$ (any point on the plane works here), the scalar equation of the desired plane is

$$-8(x - 1) + 4(y - 1) - 4(z - 2) = 0$$

by theorem 1.5.

1.4 Polar Coordinates (§10.3)



Definition 1.12 — Polar Coordinates (p. 686)

Let $(x, y) \in \mathbb{R}^2$. Each (x, y) can be represented using *polar coordinates* (r, θ) , where r is the *radial component* and θ is the *angular component*.

Let θ be the angle starting from the positive x -axis to the line segment between O and P . We have

$$x = r \cos \theta \quad y = r \sin \theta \quad \tan \theta = \frac{y}{x} \quad \text{provided } x \neq 0$$

where $r^2 = x^2 + y^2$.

Remark: If we restrict r and θ to $r > 0$ and $\theta \in [0, 2\pi)$, the polar representation is unique.

Example 1.13:

(a) Express $(1, -1)$ in polar coordinates such that $r > 0$ and $\theta \in [0, 2\pi)$.

(b) Convert $(2, \frac{3\pi}{2})$ to rectangular coordinates.

(a) We have

$$r^2 = 1^2 + (-1)^2 = 2 \implies r = \sqrt{2}$$

as $r > 0$ and

$$\tan \theta = \frac{-1}{1} \implies \theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$$

as $\theta \in [0, 2\pi)$.

(b) We have

$$x = 2 \cos\left(\frac{3\pi}{2}\right) = 2 \cdot 0 = 0$$

$$y = 2 \sin\left(\frac{3\pi}{2}\right) = 2 \cdot (-1) = -2$$

Therefore, $(x, y) = (0, -2)$.

Chapter 2 Functions

2.1 Functions of Two Variables (§14.1)

Definition 2.1 — Two-Variable Function (p. 934)

A real *two-variable function* $z = f(x, y)$ is a rule that assigns to each $(x, y) \in D$ exactly one $z \in \mathbb{R}$. Here,

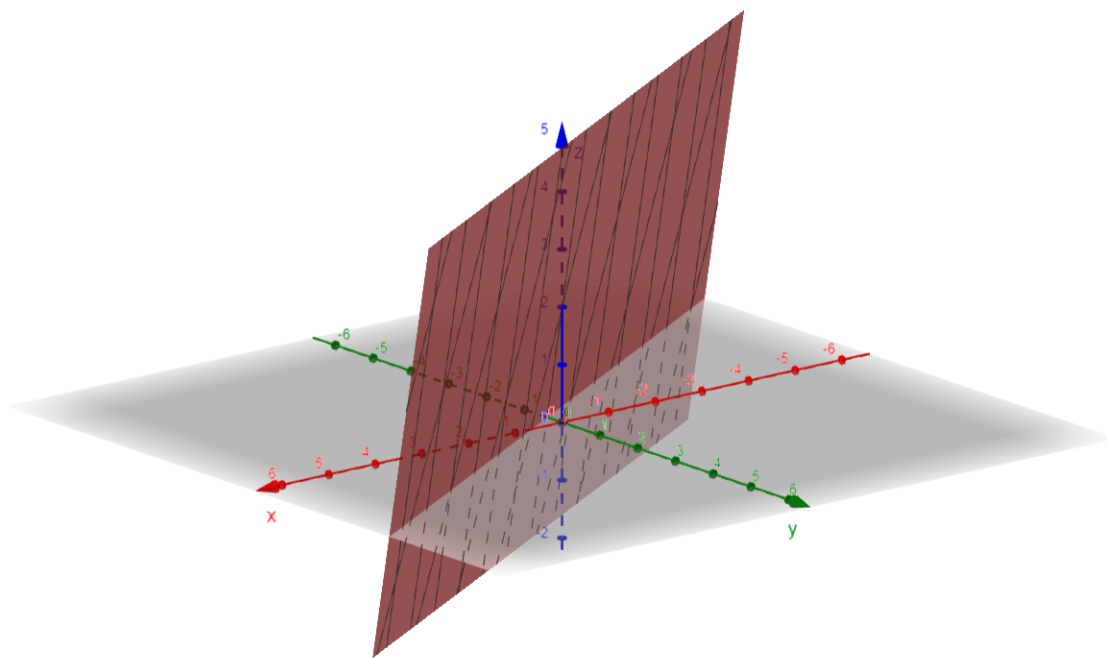
- $D = \{(x, y) \in \mathbb{R}^2 : z = f(x, y)\} = \text{dom}(f)$ is the *domain* of f
- The set $\{z \in \mathbb{R} : z = f(x, y) \text{ for some } (x, y) \in D\}$ is the *range* of f

Example 2.2: Find the domain D and the range of the following functions and graph them:

(a) $f(x, y) = -3x + 5y + 2$

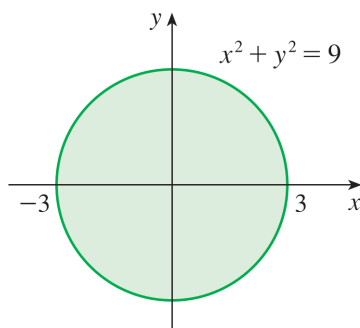
(b) $g(x, y) = \sqrt{9 - x^2 - y^2}$

- (a) $\text{dom}(f) = \{(x, y) \in \mathbb{R}^2 : f(x, y) = -3x + 5y + 2\} = \mathbb{R}^2$ as there are no x or y -values that make f undefined. For $k \in \mathbb{R}$, $-3x + 5y + 2 = k$ is the equation of a plane. Thus, $\text{range}(f) = \{z \in \mathbb{R} : z = f(x, y)\} = \mathbb{R}$. The graph is as follows:



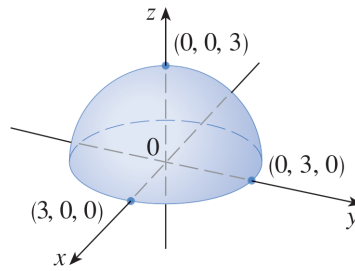
- (b) We have $\text{dom}(g) = \{(x, y) \in \mathbb{R}^2 : g(x, y) = \sqrt{9 - x^2 - y^2}\}$, where the function is defined if and only if

$$9 - x^2 - y^2 \geq 0 \iff 9 \geq x^2 + y^2$$



Thus, $\text{dom}(g) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}$. Also, $\text{range}(g) = \{z \in \mathbb{R} : z = f(x, y) \text{ for some } (x, y) \in \text{dom}(f)\} =$

$[0, 3]$ as $x^2 + y^2 \leq 9 \implies 9 - x^2 - y^2 \geq 0 \implies \sqrt{9 - x^2 - y^2} \geq 0$ and $x^2 + y^2 \geq 0 \implies 9 - x^2 - y^2 \leq 9 \implies \sqrt{9 - x^2 - y^2} \leq 3$. The graph is as follows:



Example 2.3: Consider $f(x, y) = \frac{(x^2 + 3y^2 - 9)(xy - 1)}{x}$. For what $(x, y) \in \mathbb{R}^2$ is $f(x, y)$ zero, undefined, positive, and negative? Illustrate these points.

Since f is a quotient,

$$\text{dom}(f) = \text{dom}(x^2 + 3y^2 - 9) \cap \text{dom}(xy - 1) \cap \text{dom}(x) \cap \{(x, y) \in \mathbb{R}^2 : x \neq 0\}$$

Note that $x^2 + 3y^2 - 9$, $xy - 1$, and x are polynomials in two variables, so we get

$$\text{dom}(f) = \mathbb{R}^2 \cap \mathbb{R}^2 \cap \mathbb{R}^2 \cap \{(x, y) \in \mathbb{R}^2 : x \neq 0\} = \{(x, y) \in \mathbb{R}^2 : x \neq 0\}$$

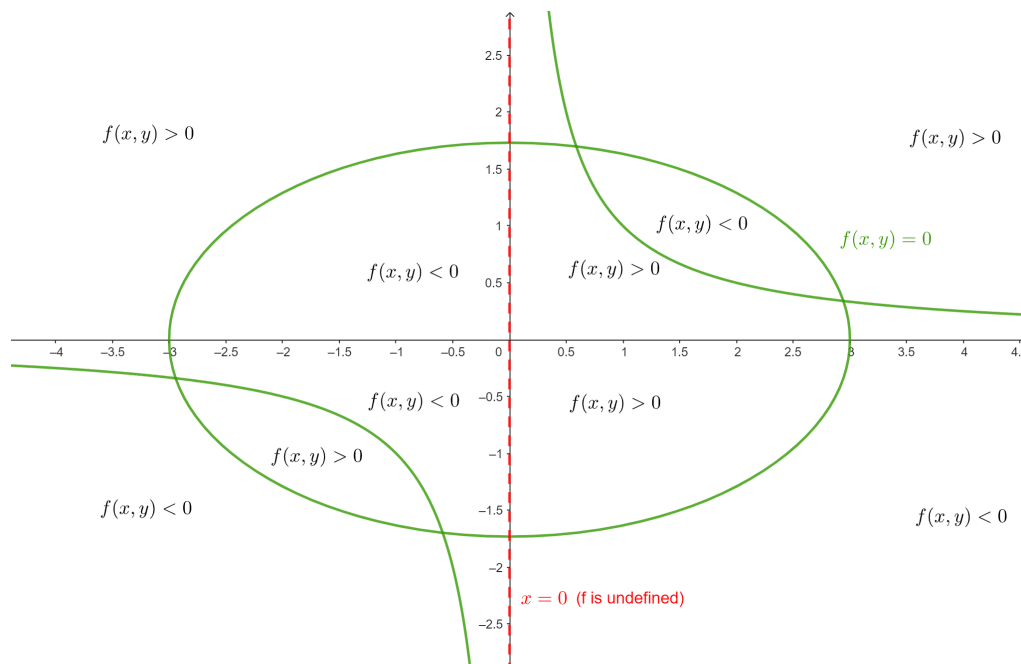
Thus, f is undefined for $\{(0, y) : y \in \mathbb{R}\}$ (i.e. the y -axis). Now observe that

$$f(x, y) = 0 \iff (x^2 + 3y^2 - 9)(xy - 1) = 0 \iff x^2 + 3y^2 - 9 = 0 \vee xy - 1 = 0$$

Here, $xy - 1 = 0 \iff y = \frac{1}{x}$ and $x^2 + 3y^2 - 9 = 0 \iff \frac{x^2}{9} + \frac{y^2}{3} = 1 \iff \left(\frac{x}{3}\right)^2 + \left(\frac{y}{\sqrt{3}}\right)^2 = 1$. Thus,

$$f(x, y) = 0 \iff \left(\frac{x}{3}\right)^2 + \left(\frac{y}{\sqrt{3}}\right)^2 = 1 \vee y = \frac{1}{x}$$

We determine where f is positive and negative by substituting appropriate sample points.



Observe that

$$\begin{aligned} f(-4, 2) &= \frac{117}{4} > 0 & f(-1, 1) &= -10 < 0 & f\left(1, \frac{1}{2}\right) &= \frac{13}{4} > 0 & f(2, 1) &= -1 < 0 & f(4, 2) &= \frac{91}{4} > 0 \\ f(-4, -2) &= -\frac{7}{4} < 0 & f(-2, -1) &= 4 > 0 & f\left(-1, -\frac{1}{2}\right) &= -\frac{19}{4} < 0 & f(1, -1) &= 22 > 0 & f(4, -1) &= -5 < 0 \end{aligned}$$

Remark: The intervals where a single-variable function is positive and negative can be determined by substituting sample points from each interval in the partition of \mathbb{R} formed by the function's roots and undefined points. The same information can be obtained for functions in two variables using sample points from each region in the partition of \mathbb{R}^2 formed by the function's roots and undefined points.

2.2 Graphs of Two-Variable Functions

Definition 2.4 — Graph of a Two-Variable Function (p. 937)

If $z = f(x, y)$ has domain D , then the *graph* (or *surface*) of f is

$$\{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D \wedge z = f(x, y)\} \subseteq \mathbb{R}^3$$

Example 2.5: Sketch the graph in \mathbb{R}^3 given by:

(a) $z + x + y - 1 = 0$

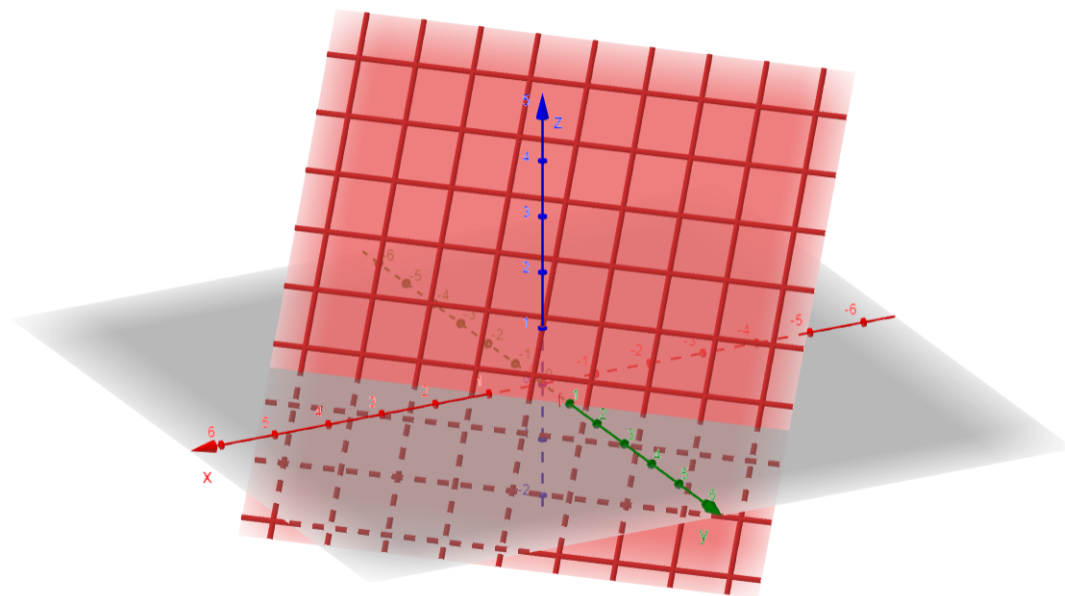
(b) $x^2 + y^2 - 4x + 2y + z^2 = 4$

(a) The given function is a plane. Note that $z + x + y - 1 = 0 \iff z = -x - y + 1$. Thus,

$$x = 0 \wedge y = 0 \implies z = 1 \implies P = (0, 0, 1) \in \text{graph}(f)$$

$$x = 0 \wedge z = 0 \implies y = 1 \implies Q = (0, 1, 0) \in \text{graph}(f)$$

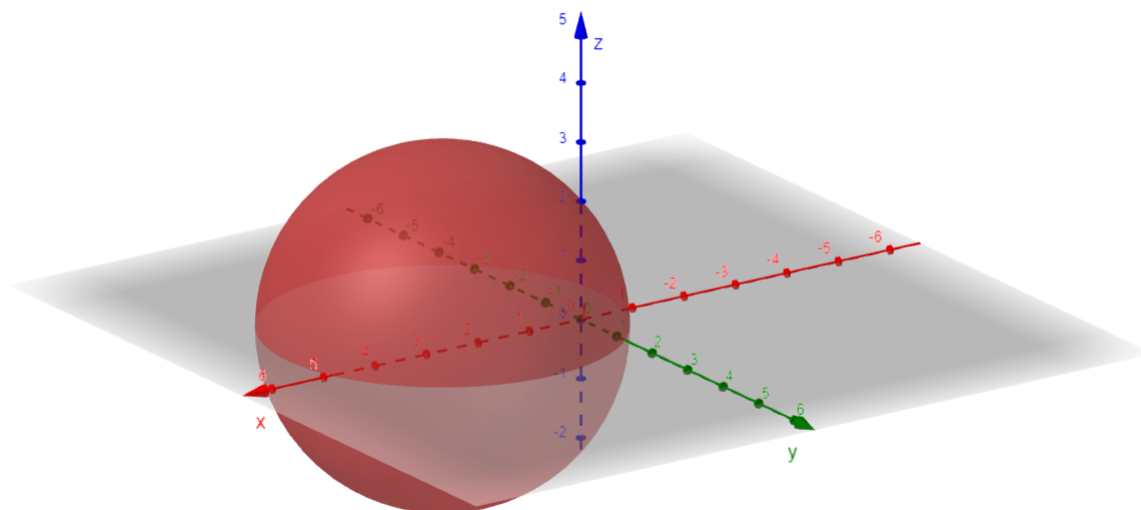
$$y = 0 \wedge z = 0 \implies x = 1 \implies R = (1, 0, 0) \in \text{graph}(f)$$



(b) Observe that

$$x^2 + y^2 - 4x + 2y + z^2 = 4 \iff (x^2 - 4x + 4) + (y^2 + 2y + 1) + z^2 = 4 + 4 + 1 \iff (x - 2)^2 + (y + 1)^2 + z^2 = 9$$

which is the equation of a sphere with radius 3, centered at $(2, -1, 0)$. The graph is as follows:



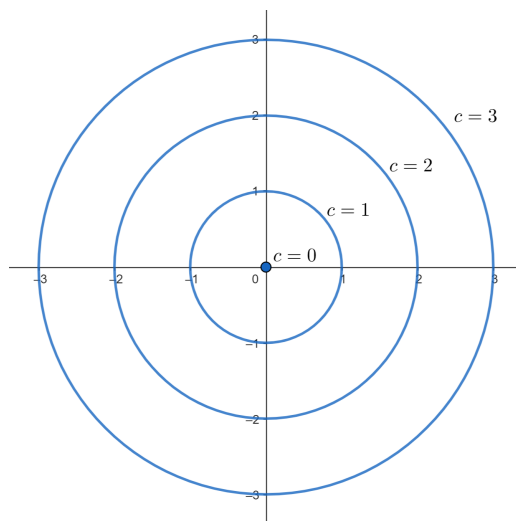
Definition 2.6 — Level Curve and Contour Map (p. 939)

Let $z = f(x, y)$ and $c \in \text{range}(f)$. The *level curve* (or *contour*) of f (for c) is the set of points $(x, y) \in \mathbb{R}^2$ that satisfy

$$f(x, y) = c$$

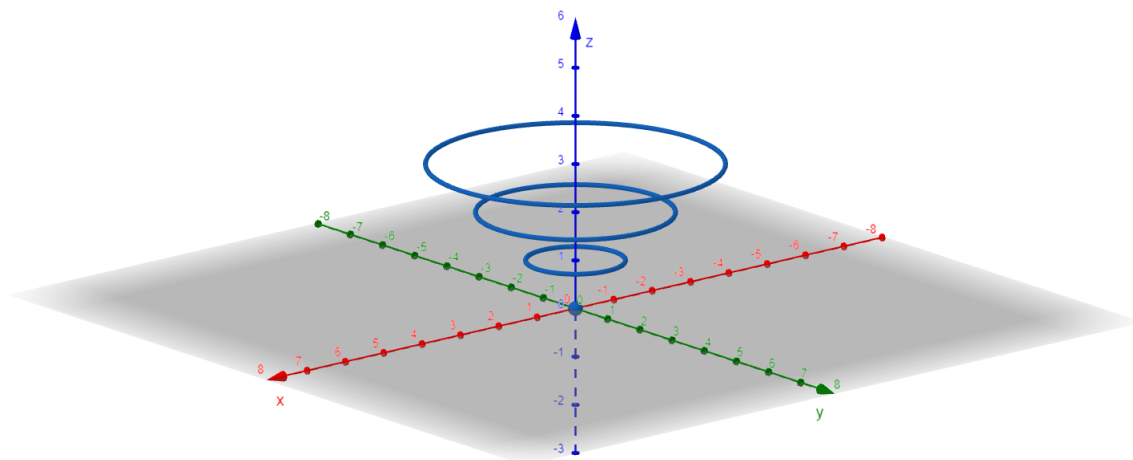
A collection of level curves is called a *contour diagram/map*.

Example 2.7: Consider the contour diagram

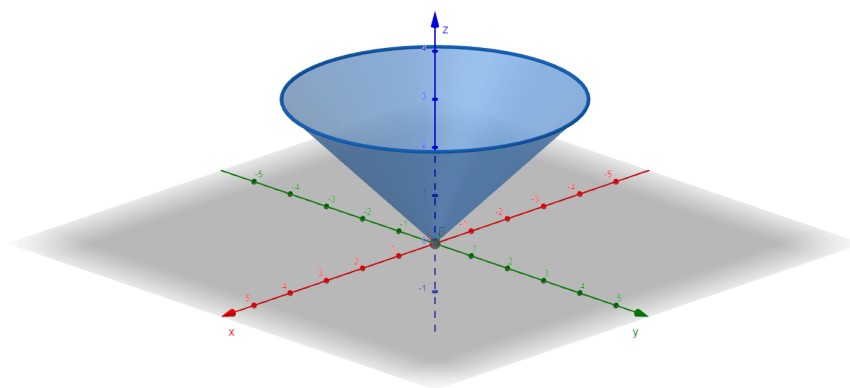


Provide a sketch of $z = f(x, y)$. Repeat this exercise for the same contour diagram, but with c values $c = 0$, $c = -1$, $c = -2$, and $c = -3$ in that order from the inner to outermost circle.

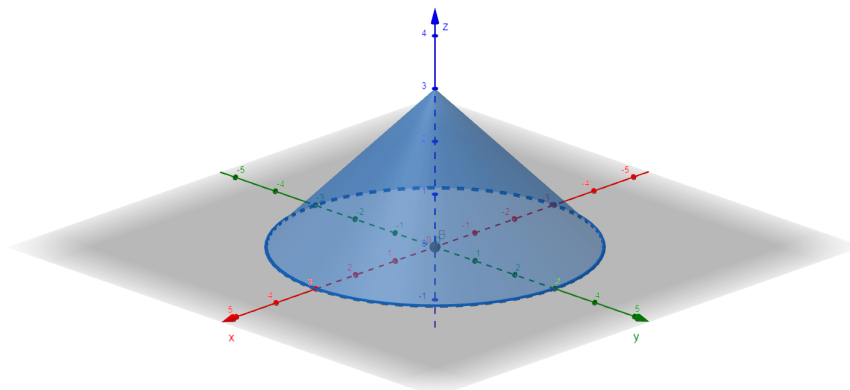
The given contours yield the graph



When the contours are interpolated, we obtain the following cone:



When the c values are reversed, we obtain the following cone:



Example 2.8: Draw the contour diagram of the graph for

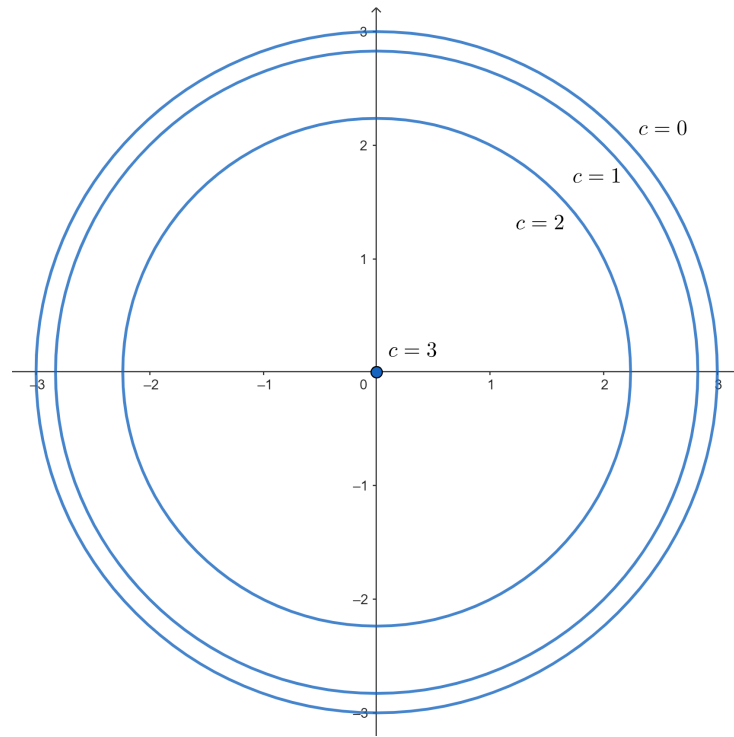
(a) $f(x, y) = \sqrt{9 - x^2 - y^2}$

(b) $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ for $a^2 \geq b^2 > 0$.

- (a) From example 2.2, $\text{dom}(f) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}$, so $\text{range}(f) = [0, 3]$ (as $0 \leq x^2 + y^2 \leq 9 \implies \sqrt{x^2 + y^2} \leq 3$). For $c = 0$, we have

$$\sqrt{9 - x^2 - y^2} = 0 \iff 9 - x^2 - y^2 = 0 \iff x^2 + y^2 = 9$$

Similarly, for $c = 1$, $c = 2$, and $c = 3$, we have $f(x, y) = 1 \iff x^2 + y^2 = 8$, $f(x, y) = 2 \iff x^2 + y^2 = 5$, and $f(x, y) = 3 \iff x^2 + y^2 = 0$ (respectively). This gives us



(b) Let $c \in \mathbb{R}^{\geq 0}$. We have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = c \iff b^2 \cdot x^2 + a^2 \cdot y^2 = a^2 b^2 \cdot c \iff (bx)^2 + (ay)^2 = (ab)^2 c$$

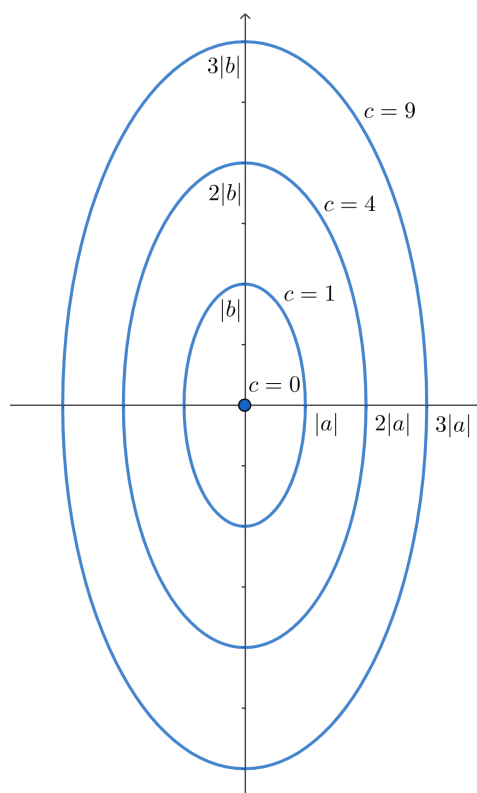
This is the equation of an ellipse centered at the origin. Setting $y = 0$ and solving for $|x|$ yields its radius along the x -axis, as follows:

$$(bx)^2 = (ab)^2 c - (ay)^2 = (ab)^2 c - (a \cdot 0)^2 = (ab)^2 c \implies x^2 = a^2 c \implies |x| = |a| \sqrt{c}$$

Similarly, setting $x = 0$ and solving for $|y|$ yields the ellipse's radius along the y -axis, as follows:

$$(ay)^2 = (ab)^2 c - (bx)^2 = (ab)^2 c - (b \cdot 0)^2 = (ab)^2 c \implies y^2 = b^2 c \implies |y| = |b| \sqrt{c}$$

Note that $c \geq 0$ in the previous calculations, so \sqrt{c} is defined. For $c = 0$, $c = 1$, $c = 4$, and $c = 9$, we thus have ellipses centered at the origin with x and y -axis radii 0 and 0, $|a|$ and $|b|$, $2|a|$ and $2|b|$, and $3|a|$ and $3|b|$. This gives us the contour diagram



2.3 Multivariable Functions

Definition 2.9 — Multivariable Function (p. 945)

Let $n \in \mathbb{Z}^+$ and $D \subseteq \mathbb{R}^n$. A *function of n variables* is a rule which assigns to each point $(x_1, \dots, x_n) \in D$ exactly one real number $z = f(x_1, \dots, x_n)$.

1. $D = \{(x_1, \dots, x_n) \in \mathbb{R}^n : z = f(x_1, \dots, x_n)\}$ is the *domain* of f
2. $\{z \in \mathbb{R} : z = f(x_1, \dots, x_n) \text{ for some } (x_1, \dots, x_n) \in D\}$ is the *range* (or *codomain*) of f
3. The set of points $\{(x_1, \dots, x_n, f(x_1, \dots, x_n)) \in \mathbb{R}^{n+1} : (x_1, \dots, x_n) \in D\}$ is the *graph* of f

Example 2.10: What the domain of $f(x, y, z) = \ln(z - y) + xy \sin(z)$?

Logarithms are defined on \mathbb{R}^+ , so we need $z - y > 0 \iff z > y$. xy is defined for all $x, y \in \mathbb{R}$, while $\sin(z)$ is defined for $-1 \leq z \leq 1$. Therefore,

$$\text{dom}(f) = \{(x, y, z) \in \mathbb{R}^3 : y < z \wedge -1 \leq z \leq 1\}$$

2.4 Limits (§14.2)

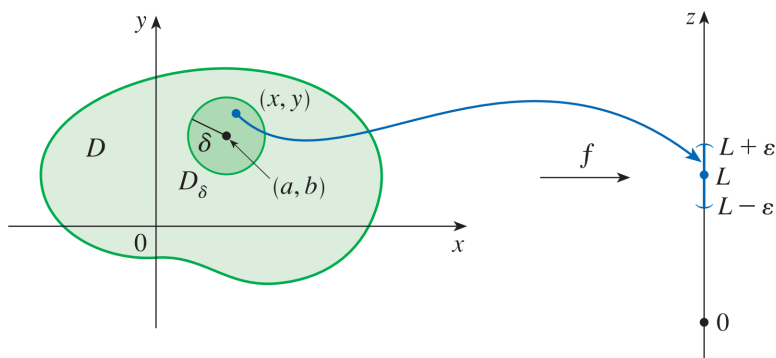
Let $a \in \mathbb{R}$ and $f: \mathbb{R} \rightarrow \mathbb{R}$. Recall that for single-variable functions, we say that the “limit of f as x approaches a ”, denoted by

$$\lim_{x \rightarrow a} f(x) = L$$

if and only if for all $\epsilon > 0$, there exists some $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$$

where $x \in \text{dom}(f)$ and $L \in \mathbb{R}$.



We now establish an analogous concept for multivariable functions.

Definition 2.11 — Two-Variable Function Limit (p. 952)

Let $(a, b) \in \mathbb{R}^2$ and $z = f(x, y)$. We say that the “limit of f as (x, y) approaches (a, b) ”, denoted by

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$$

if and only if for all $\epsilon > 0$, there exists some $\delta > 0$ such that

$$0 < \|(x, y) - (a, b)\| < \delta \implies |f(x, y) - L| < \epsilon$$

where $x \in D = \text{dom}(f)$ and $L \in \mathbb{R}$.

Remark: In definition 2.11, the *Euclidean norm* is used, which is given by

$$\|(x, y) - (a, b)\| = \sqrt{(x - a)^2 + (y - b)^2}$$

Example 2.12: Prove that $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x^2 + y^2}$ does not exist.

Let $f(x, y)$ be the given function and $D = \text{dom}(f)$. We want to choose a curve/path $C_1 \subseteq D$ for which $\lim_{\substack{(x,y) \rightarrow (a,b) \\ (x,y) \in C_1}} f(x, y) = L_1$ and another path $C_2 \subseteq D$ for which $\lim_{\substack{(x,y) \rightarrow (a,b) \\ (x,y) \in C_2}} f(x, y) = L_2$ such that $L_1 \neq L_2$.

Along the x -axis, $y = 0$, so we get

$$\lim_{(x,0) \rightarrow (0,0)} \frac{x^2}{x^2 + 0^2} = \lim_{x \rightarrow 0} \frac{x^2}{x^2} = \lim_{x \rightarrow 0} 1 = 1$$

for $x \neq 0$. Along the y -axis, $x = 0$, which gives us

$$\lim_{(0,y) \rightarrow (0,0)} \frac{0^2}{0^2 + y^2} = \lim_{y \rightarrow 0} 0 = 0$$

Since $0 \neq 1$, the limit depends on the path. Therefore, the limit does not exist.

Example 2.13: Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^4 + y^2}$ does not exist.

Along the x -axis, $y = 0$, so we get

$$\lim_{(x,0) \rightarrow (0,0)} \frac{x^2 \cdot 0}{x^4 + 0^2} = \lim_{x \rightarrow 0} 0 = 0$$

Along the curve $y = x^2$, we get

$$\lim_{(x,x^2) \rightarrow (0,0)} \frac{x^2 \cdot x^2}{x^4 + (x^2)^2} = \lim_{x \rightarrow 0} \frac{x^4}{2x^4} = \lim_{x \rightarrow 0} \frac{1}{2} = \frac{1}{2}$$

for $x \neq 0$. Since $0 \neq \frac{1}{2}$, the limit does not exist.

Example 2.14: Compute $\lim_{(x,y) \rightarrow (0,0)} \frac{e^{-(x^2+y^2)} - 1}{x^2 + y^2}$, if it exists.

Let $x = r \cos \theta$ and $y = r \sin \theta$ for $r \in \mathbb{R}^{\geq 0}$ and $\theta \in [0, 2\pi)$. As $(x, y) \rightarrow (0, 0)$,

$$r^2 = x^2 + y^2 \rightarrow 0 \iff r = \pm \sqrt{x^2 + y^2} \rightarrow 0$$

Since $r \geq 0$, $r \rightarrow 0^+$. Thus, by L'Hopital's Rule (for $\frac{0}{0}$ indeterminate forms), the limit becomes

$$\lim_{r \rightarrow 0^+} \frac{e^{-r^2} - 1}{r^2} = \lim_{r \rightarrow 0^+} \frac{e^{-r^2} \cdot (-2r)}{2r} = \lim_{r \rightarrow 0^+} -e^{-r^2} = -e^0 = -1$$

Example 2.15 (*One-Variable ϵ - δ*): Let $a, b, c \in \mathbb{R}$. Prove, using ϵ - δ , that $\lim_{x \rightarrow c} f(x)$ exists for $f(x) = ax + b$.

We want to show that for some $L \in \mathbb{R}$, for all $\epsilon > 0$, there exists some $\delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \epsilon$. Let $L = ac + b \in \mathbb{R}$ and $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{|a| + 1} > 0$, and suppose that $0 < |x - c| < \delta$. We want to show that $|f(x) - (ac + b)| < \epsilon$. Now

$$\begin{aligned} |f(x) - (ac + b)| &= |(ax + b) - (ac + b)| \\ &= |a(x - c)| \\ &= |a||x - c| && \text{(by the absolute value property } |AB| = |A||B| \text{)} \\ &< |a|\delta \\ &= |a| \cdot \frac{\epsilon}{|a| + 1} \\ &< \frac{|a| + 1}{|a| + 1} \cdot \epsilon && \text{(by increasing the numerator)} \\ &= \epsilon \end{aligned}$$

as desired.

Remark: The choice of $\delta = \frac{\epsilon}{|a| + 1}$ in example 2.15 instead of $\delta = \frac{\epsilon}{|a|}$ was made to handle the case where $|a| = 0$, while still allowing us to use the same reasoning as with $\frac{\epsilon}{|a|}$ with the inequalities.

Example 2.16 (*Two-Variable ϵ - δ*): Let $(a, b) \in \mathbb{R}^2$ and $A, B \in \mathbb{R}$. Prove, by definition (i.e. using ϵ - δ), that $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = Aa + Bb$ for $f(x, y) = Ax + By$.

We want to show that for all $\epsilon > 0$, there exists some $\delta > 0$ such that $0 < \|(x, y) - (a, b)\| < \delta \implies |f(x, y) - (Aa + Bb)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = \frac{\epsilon}{|A| + |B| + 1} > 0$, and suppose that $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$. We have

$$\begin{aligned} |f(x, y) - (Aa + Bb)| &= |(Ax + By) - (Aa + Bb)| \\ &= |A(x - a) + B(y - b)| \\ &\leq |A(x - a)| + |B(y - b)| && \text{(by the triangle inequality)} \\ &= |A||x - a| + |B||y - b| && \text{(by the absolute value multiplicative property)} \\ &= |A|\sqrt{(x - a)^2} + |B|\sqrt{(y - b)^2} && \text{(since } |\cdot| = \sqrt{\cdot^2} \text{)} \\ &\leq |A|\sqrt{(x - a)^2 + (y - b)^2} + |B|\sqrt{(y - b)^2 + (x - a)^2} && \text{(as the square root function is increasing and } \cdot^2 \geq 0 \text{)} \\ &= (|A| + |B|)\sqrt{(x - a)^2 + (y - b)^2} \end{aligned}$$

$$\begin{aligned}
&< (|A| + |B|)\delta \\
&= (|A| + |B|) \cdot \frac{\epsilon}{|A| + |B| + 1} \\
&< \frac{|A| + |B| + 1}{|A| + |B| + 1} \cdot \epsilon \\
&= \epsilon
\end{aligned}$$

as desired.

Example 2.17: Prove by definition that $\lim_{(x,y) \rightarrow (0,0)} \frac{x+y}{x^2+y^2+1} = 0$.

Let $\epsilon > 0$ be arbitrary. Choose $\delta = \frac{\epsilon}{2} > 0$, and suppose that $0 < \sqrt{x^2+y^2} < \delta$. We want to show that

$\left| \frac{x+y}{x^2+y^2+1} - 0 \right| < \epsilon$. Now

$$\begin{aligned}
\left| \frac{x+y}{x^2+y^2+1} - 0 \right| &= \left| \frac{x+y}{x^2+y^2+1} \right| && \text{(by the absolute value property } \left| \frac{A}{B} \right| = \frac{|A|}{|B|} \text{)} \\
&= \frac{|x+y|}{|x^2+y^2+1|} && \text{(since } x^2+y^2+1 > 0 \text{ for all } (x,y) \in \mathbb{R}^2 \text{)} \\
&= \frac{|x+y|}{x^2+y^2+1} && \text{(as } x^2+y^2+1 \geq 1 \text{ for all } (x,y) \in \mathbb{R}^2 \text{)} \\
&\leq |x+y| && \text{(by the triangle inequality)} \\
&\leq |x| + |y| && \text{(since } |\cdot| = \sqrt{\cdot^2} \text{)} \\
&= \sqrt{x^2} + \sqrt{y^2} && \text{(as } \cdot^2 \geq 0 \text{ and the square root function is increasing)} \\
&\leq \sqrt{x^2+y^2} + \sqrt{y^2+x^2} \\
&= 2\sqrt{x^2+y^2} \\
&< 2\delta \\
&= 2 \cdot \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

as desired.

Definition 2.18 — n -Variable Function Limit (p. 959)

Let $n \in \mathbb{Z}^+$, $\mathbf{a} \in \mathbb{R}^n$, and $f(\mathbf{x})$ be an n -variable function with domain $D \subseteq \mathbb{R}^n$. We say that

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} f(\mathbf{x}) = L$$

if for all $\epsilon > 0$, there exists some $\delta > 0$ such that

$$0 < \|\mathbf{x} - \mathbf{a}\| < \delta \implies |f(\mathbf{x}) - L| < \epsilon$$

where $\mathbf{x} \in D$ and $L \in \mathbb{R}$.

Theorem 2.19 — Limit Laws (p. 955)

Let $(a, b) \in \mathbb{R}^2$ and $f(x, y)$ and $g(x, y)$ be defined for all $(x, y) \neq (a, b)$ in a neighbourhood (a disk) around (a, b) .

If $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L_1 \in \mathbb{R}$ and $\lim_{(x,y) \rightarrow (a,b)} g(x, y) = L_2 \in \mathbb{R}$ exist then

1. $\lim_{(x,y) \rightarrow (a,b)} (f(x, y) \pm g(x, y)) = L_1 \pm L_2$
2. $\lim_{(x,y) \rightarrow (a,b)} cf(x, y) = cL_1$ for all $c \in \mathbb{R}$
3. $\lim_{(x,y) \rightarrow (a,b)} f(x, y)g(x, y) = L_1L_2$
4. $\lim_{(x,y) \rightarrow (a,b)} \frac{f(x, y)}{g(x, y)} = \frac{L_1}{L_2}$ (provided that $L_2 \neq 0$)

PROOF: Let $c \in \mathbb{R}$. Suppose that $f \rightarrow L_1$ and $g \rightarrow L_2$ as $(x, y) \rightarrow (a, b)$. To prove (1) and (2), we will show that $\lim_{(x,y) \rightarrow (a,b)} (f(x, y) + cg(x, y)) = L_1 + cL_2$ using the ϵ - δ definition. We want to show that for all $\epsilon > 0$, there exists some $\delta > 0$ such that

$$0 < \|(x, y) - (a, b)\| < \delta \implies |(f(x, y) + cg(x, y)) - (L_1 + cL_2)| < \epsilon$$

Let $\epsilon > 0$ be arbitrary. Since $f \rightarrow L_1$ as $(x, y) \rightarrow (a, b)$, there exists some $\delta_1 > 0$ such that

$$0 < \|(x, y) - (a, b)\| < \delta_1 \implies |f(x, y) - L_1| < \frac{\epsilon}{2}$$

Similarly, since $g \rightarrow L_2$ as $(x, y) \rightarrow (a, b)$, there exists some $\delta_2 > 0$ such that

$$0 < \|(x, y) - (a, b)\| < \delta_2 \implies |g(x, y) - L_2| < \frac{\epsilon}{2(|c| + 1)}$$

Choose $\delta = \min(\delta_1, \delta_2) > 0$, and suppose that $0 < \|(x, y) - (a, b)\| < \delta$. We want to show that $|f(x, y) + cg(x, y) - (L_1 + cL_2)| < \epsilon$. Now

$$\begin{aligned} |f(x, y) + cg(x, y) - (L_1 + cL_2)| &= |(f(x, y) - L_1) + c(g(x, y) - L_2)| \\ &\leq |f(x, y) - L_1| + |c(g(x, y) - L_2)| && \text{(by the triangle inequality)} \\ &= |f(x, y) - L_1| + |c||g(x, y) - L_2| && \text{(by an absolute value property)} \\ &< \frac{\epsilon}{2} + |c| \cdot \frac{\epsilon}{|c| + 1} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} && \text{(as } \frac{|c|}{|c| + 1} < 1) \\ &= \epsilon \end{aligned}$$

Example 2.20: Compute $\lim_{(x,y) \rightarrow (2,-1)} \frac{2x + 3y}{4x - 3y}$, if it exists.

Using limit laws, we obtain

$$\frac{\lim_{(x,y) \rightarrow (2,-1)} (2x + 3y)}{\lim_{(x,y) \rightarrow (2,-1)} (4x - 3y)} = \frac{2 \lim_{(x,y) \rightarrow (2,-1)} x + 3 \lim_{(x,y) \rightarrow (2,-1)} y}{4 \lim_{(x,y) \rightarrow (2,-1)} x - 3 \lim_{(x,y) \rightarrow (2,-1)} y} = \frac{1}{11}$$

Definition 2.21 — Two-Variable Rational Function

A *rational function of two variables* is a function of the form

$$\frac{p(x, y)}{q(x, y)}$$

where $p(x, y)$ and $q(x, y)$ are two-variable polynomials with $q(x, y) \neq 0$.

2.5 Continuity (§14.2)

Definition 2.22 — Two-Variable Function Continuity (p. 957)

Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function with domain $D \subseteq \mathbb{R}^2$ and $(a, b) \in \mathbb{R}^2$. f is *continuous at* (a, b) if and only if

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b)$$

That is, for all $\epsilon > 0$, there exists some $\delta > 0$ such that

$$\|(x, y) - (a, b)\| < \delta \implies |f(x, y) - f(a, b)| < \epsilon$$

f is *continuous on* D if and only if f is continuous at (x, y) for all $(x, y) \in D$.

Example 2.23: Prove that $f(x, y) = \begin{cases} \frac{2x+3y}{4x-3y} & \text{if } (x, y) \neq (2, -1) \\ 0 & \text{if } (x, y) = (2, -1) \end{cases}$ is discontinuous at $(2, -1)$.

From example 2.20, we have

$$\lim_{(x,y) \rightarrow (2,-1)} f(x, y) = \lim_{(x,y) \rightarrow (2,-1)} \frac{2x+3y}{4x-3y} = \frac{1}{11}$$

so the limit exists. By the definition of f , $f(2, -1) = 0$, so f is defined at $(2, -1)$. Since $\frac{1}{11} \neq 0$ (i.e. $\lim_{(x,y) \rightarrow (2,-1)} f(x, y) \neq f(2, -1)$), f is discontinuous at $(2, -1)$.

Theorem 2.24 — Continuity Properties (p. 957)

Let $f, g: \mathbb{R}^2 \rightarrow \mathbb{R}$ and $(a, b) \in \text{dom}(f) \cap \text{dom}(g)$. If f and g are continuous at (a, b) , then each of the following functions are continuous at (a, b) :

1. $f \pm g$
2. cf for all $c \in \mathbb{R}$
3. fg
4. $\frac{f}{g}$ (provided that $g(a, b) \neq 0$)

Theorem 2.25 — Continuity Composition (p. 958)

Let g be a two-variable function with domain $D \subseteq \mathbb{R}^2$ and range $R \subseteq \mathbb{R}$, and f be a single-variable function. Suppose that $(a, b) \in D$ and $z = g(a, b)$. If

- g is continuous at (a, b) and
- f is continuous at $z \in \text{dom}(f)$

then the two-variable composition $f \circ g$ is continuous at (a, b) .

Example 2.26: Determine the set of points at which the function $H(x, y) = \frac{xy}{1 + e^{x-y}}$ is continuous. Justify your answer.

H is a quotient.

- The numerator xy is a polynomial and thus continuous on its domain \mathbb{R}^2
- The denominator $1 + e^{x-y}$ is the sum of
 - 1, which is a polynomial and thus continuous on its domain \mathbb{R}^2
 - e^{x-y} , a composition of the polynomial $x - y$, which is continuous on its domain \mathbb{R}^2 , and the single-variable function e^t that is continuous on its domain \mathbb{R} . Thus, e^{x-y} is continuous on \mathbb{R}^2

- $1 + e^{x-y} \neq 0$ for all $(x, y) \in \mathbb{R}^2$ since $e^t > 0$ for all $t \in \mathbb{R}$

Therefore, H is continuous on the “common points of continuity”, namely

$$\mathbb{R}^2 \cap \mathbb{R}^2 - \left\{ (x, y) \in \mathbb{R}^2 : 1 + e^{x-y} = 0 \right\} = \mathbb{R}^2 \cap \mathbb{R}^2 - \emptyset = \mathbb{R}^2$$

Remark: In example 2.26, the fact that polynomials of two variables are continuous on their domains (\mathbb{R}^2) was proven in Q7 (c) of extra exercises 3.

Example 2.27 (§14.2 Q43): Determine the set of points for which $f(x, y) = \frac{1 + x^2 + y^2}{1 - x^2 - y^2}$ is discontinuous. Justify your answer.

f is a quotient whose numerator and denominator are both polynomials and thus continuous on \mathbb{R}^2 . Therefore, f is only discontinuous where its denominator is zero; that is,

$$1 - x^2 - y^2 = 0 \iff x^2 + y^2 = 1$$

so f is discontinuous on $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ (all points on the unit circle).

Example 2.28: Find $\lim_{(x,y) \rightarrow (-2,2)} e^{-xy} \cos(x+y)$, if it exists.

Using limit laws, we get

$$\left(\lim_{(x,y) \rightarrow (-2,2)} e^{-xy} \right) \left(\lim_{(x,y) \rightarrow (-2,2)} \cos(x+y) \right)$$

e^{-xy} is a composition of the polynomial $-xy$ (which is continuous on its domain \mathbb{R}^2) and the single-variable exponential e^t (which is continuous on its domain \mathbb{R}). Thus, e^{-xy} is continuous on \mathbb{R}^2 . In particular, it is continuous at $(-2, 2) \in \mathbb{R}^2$.

$\cos(x+y)$ is a composition of the polynomial $x+y$ (which is continuous on its domain \mathbb{R}^2) and the single-variable function $\cos(t)$ (which is continuous on its domain \mathbb{R}). Therefore, $\cos(x+y)$ is continuous on \mathbb{R}^2 and thus continuous at $(-2, 2) \in \mathbb{R}^2$.

By continuity at $(-2, 2)$, the limit becomes

$$e^{-(-2) \cdot 2} \cdot \cos(-2 + 2) = e^4 \cdot \cos(0) = e^4$$

Chapter 3 Differentiation

3.1 Partial Derivatives (§14.3)

Recall that for a single-variable function $f: \mathbb{R} \rightarrow \mathbb{R}$, the derivative is defined as

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

We now establish a similar concept for multivariable functions.

Definition 3.1 — Two-Variable Partial Derivatives

Let $z = f(x, y)$. The *partial derivative of f with respect to x* , denoted by

$$f_x \text{ or } \frac{\partial f}{\partial x} \text{ or } \frac{\partial z}{\partial x} \text{ or } D_1 f$$

is

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

Similarly, the *partial derivative of f with respect to y* , denoted by

$$f_y \text{ or } \frac{\partial f}{\partial y} \text{ or } \frac{\partial z}{\partial y} \text{ or } D_2 f$$

is

$$f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}$$

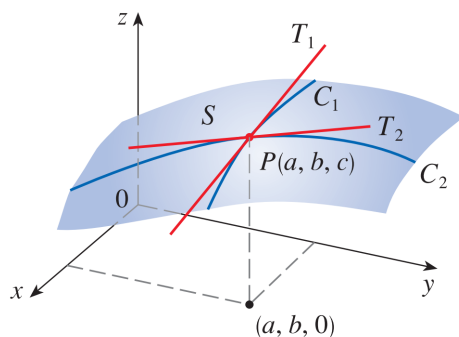
Remark: In the context of definition 3.1, suppose that $c = f(a, b)$ (for some $(a, b) \in \text{dom}(f)$),

$$C_1 = \{(x, b, f(x, b)) : (x, b) \in \text{dom}(f)\}$$

and

$$C_2 = \{(a, y, f(a, y)) : (a, y) \in \text{dom}(f)\}$$

Let T_1 and T_2 be the tangents to C_1 and C_2 (respectively) at (a, b) . The slopes of T_1 and T_2 are $f_x(a, b)$ and $f_y(a, b)$ (respectively), depicted in the following graph:



Example 3.2: Compute $f_x(0, 1)$ and $f_y(0, 1)$ by definition for $f(x, y) = x^2 - 3xy$.

By definition, $f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$, so

$$\begin{aligned} f_x(0, 1) &= \lim_{h \rightarrow 0} \frac{f(0+h, 1) - f(0, 1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(h^2 - 3h \cdot 1) - (0^2 - 3 \cdot 0 \cdot 1)}{h} \end{aligned}$$

(by the definition of f)

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{h^2 - 3h}{h} \\
&= \lim_{h \rightarrow 0} \frac{h(h - 3)}{h} \\
&= \lim_{h \rightarrow 0} (h - 3) \quad (\text{as } h \neq 0) \\
&= -3
\end{aligned}$$

Also, $f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y + h) - f(x, y)}{h}$, so

$$\begin{aligned}
f_y(0, 1) &= \lim_{h \rightarrow 0} \frac{f(0, 1 + h) - f(0, 1)}{h} \\
&= \lim_{h \rightarrow 0} \frac{(0^2 - 3 \cdot 0 \cdot (1 + h)) - 3 \cdot 0 \cdot (1 + h)}{h} \\
&= \lim_{h \rightarrow 0} \frac{0}{1 + h} \\
&= \lim_{h \rightarrow 0} 0 \\
&= 0
\end{aligned}$$

Example 3.3: Let $f(x, y) = xy^3 - x^2\sqrt{y}$ for $y \neq 0$. Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Treating y as a constant, we get

$$f_x(x, y) = y^3 - 2x\sqrt{y}$$

Similarly, treating x as a constant yields

$$f_y(x, y) = x \cdot 3y^2 - x^2 \cdot \frac{1}{2}y^{-1/2} = 3xy^2 - \frac{x^2}{2\sqrt{y}}$$

Example 3.4: Let $\frac{x^2}{2} + \frac{y^2}{4} + \frac{z^2}{3} = 1$. Compute f_x .

Implicitly differentiating with respect to x while treating y as a constant, we get

$$\begin{aligned}
\frac{2x}{2} + 0 + \frac{2z}{3} \cdot \frac{\partial z}{\partial x} &= 0 \iff \frac{2}{3}zf_x = -x \\
&\iff f_x = -\frac{3x}{2z}
\end{aligned}$$

Definition 3.5 — n -Variable Partial Derivatives (p. 966)

Let $z = f(x_1, \dots, x_n)$, where $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is a function. The *partial derivative of f with respect to x_i* is

$$\frac{\partial z}{\partial x_i} = \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_n)}{h}$$

This is denoted as

$$\frac{\partial z}{\partial x_i} \text{ or } \frac{\partial f}{\partial x_i} \text{ or } D_i f \text{ or } f_{x_i}$$

Definition 3.6 — Higher Order Partial Derivatives

Let $z = f(x, y)$. The second order partial derivatives of f are

$$\begin{aligned} f_{xx} &= \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) \\ f_{yy} &= \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) \\ (f_x)_y &= \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) \\ (f_y)_x &= \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) \end{aligned}$$

Example 3.7: Let $z = f(x, y) = xe^{-3y} + \sin(2x - 5y)$. Compute all second order partial derivatives of f .

Treating y as a constant, we get

$$f_x = e^{-3y} + \cos(2x - 5y) \cdot 2$$

Thus,

$$f_{xx} = \frac{\partial}{\partial x} (e^{-3y} + 2 \cos(2x - 5y)) = 0 - 2 \sin(2x - 5y) \cdot 2 = -4 \sin(2x - 5y)$$

Now treating x as a constant, we get

$$f_{xy} = \frac{\partial}{\partial y} (e^{-3y} + 2 \cos(2x - 5y)) = -3 \cdot e^{-3y} - 2 \sin(2x - 5y) \cdot (-5) = -3e^{-3y} + 10 \sin(2x - 5y)$$

Treating x as a constant yields

$$f_y = -3 \cdot xe^{-3y} + \cos(2x - 5y) \cdot (-5) = -3xe^{-3y} - 5 \cos(2x - 5y)$$

Thus,

$$f_{yy} = \frac{\partial}{\partial y} (-3xe^{-3y} - 5 \cos(2x - 5y)) = -3 \cdot (-3xe^{-3y}) + 5 \sin(2x - 5y) \cdot (-5) = 9e^{-3y} - 25 \sin(2x - 5y)$$

Treating y as a constant now yields

$$f_{yx} = \frac{\partial}{\partial x} (-3xe^{-3y} - 5 \cos(2x - 5y)) = -3e^{-3y} + 5 \sin(2x - 5y) \cdot 2 = -3e^{-3y} + 10 \sin(2x - 5y)$$

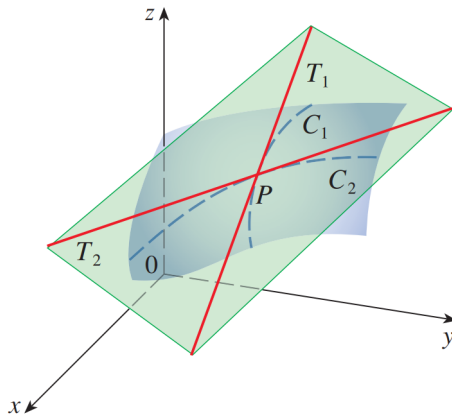
Theorem 3.8 — Clairaut

Let $z = f(x, y)$ and $(a, b) \in \text{dom}(f)$. Suppose there exists a disk $D \subseteq \text{dom}(f)$ such that $(a, b) \in D$. If f_{xy} and f_{yx} are continuous on D , then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Remark: The proof of theorem 3.8 can be found in Appendix F of the textbook.

3.2 Tangent Planes (§14.4)



Let (x_0, y_0, z_0) be a point on a surface $z = f(x, y)$, $\mathbf{u} = \langle 0, 1, f_y(x_0, y_0) \rangle$, and $\mathbf{v} = \langle 1, 0, f_x(x_0, y_0) \rangle$. In the above graph, T_1 and T_2 are the tangent lines to f at $P = (x_0, y_0, z_0)$, with position vectors \mathbf{u} and \mathbf{v} (respectively). Also, $C_1 = \{(x, y_0, f(x, y_0)) : (x, y_0) \in \text{dom}(f)\}$ and $C_2 = \{(x_0, y, f(x_0, y)) : (x_0, y) \in \text{dom}(f)\}$.

Note that \mathbf{u} and \mathbf{v} span a plane. It can be shown that $\mathbf{n} = \mathbf{u} \times \mathbf{v} = \langle f_x(x_0, y_0), f_y(x_0, y_0), -1 \rangle$.

Definition 3.9 — Tangent Plane Equation (p. 975)

Let $z = f(x, y)$ and $(x_0, y_0) \in \text{dom}(f)$. Suppose that f_x and f_y are continuous near (x_0, y_0) . The *tangent plane to f at (x_0, y_0, z_0)* (where $z_0 = f(x_0, y_0)$) has the equation

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

Remark: Definition 3.9 uses the equation of a plane in \mathbb{R}^3 , as in theorem 1.8.

Example 3.10: Find the tangent plane to $f(x, y) = x^3y^{-2}$ at $(1, 1, 1)$.

We have $f_x = 3y^{-2}x^2$ and $f_y = -2x^3y^{-3}$. These are both rational functions, continuous where $y \neq 0$. In particular, they are continuous at $(1, 1)$. We have $f_x(1, 1) = 3$ and $f_y(1, 1) = -2$, so the plane's equation (by definition 3.9) is

$$3(x - 1) - 2(y - 1) - (z - 1) = 0 \iff 3x - 2y - z = 0$$

Example 3.11: Find the point where the tangent plane to $f(x, y) = e^{x-y}$ at $P_0 = (1, 1, 1)$ intersects the z -axis.

We have $f_x = e^{x-y} \cdot 1 = e^{x-y}$ and $f_y = e^{x-y} \cdot (-1) = -e^{x-y}$. These are compositions of the polynomial $x - y$, which is continuous on its domain \mathbb{R}^2 , and a single-variable exponential $g(t) = \pm e^t$, which is continuous on its domain \mathbb{R} . Thus, the compositions are continuous on \mathbb{R}^2 . In particular, f_x and f_y are continuous at $(1, 1)$. Therefore, the tangent plane at P_0 is

$$e^{1-1}(x - 1) - e^{1-1}(y - 1) - (z - 1) = 0 \iff (x - 1) - (y - 1) - (z - 1) = 0 \iff x - y - z + 1 = 0$$

The tangent plane intersects the z -axis where $x = 0$ and $y = 0$, which yields

$$0 - 0 - z + 1 = 0 \iff z = 1$$

Therefore, the tangent plane intersects the z -axis at $(0, 0, 1)$.

Definition 3.12 — Differentiability

Let $z = f(x, y)$ and $(a, b) \in \mathbb{R}^2$. f is *differentiable at (a, b)* if and only if each of the following conditions hold:

- f_x and f_y both exist at (a, b)
- $$\lim_{(x,y) \rightarrow (a,b)} \frac{f(x, y) - (f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b))}{\|(x, y) - (a, b)\|} = 0$$

Remark: In definition 3.12, the term subtracted from $f(x, y)$ in the limit's numerator is the expression

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

obtained by isolating z in the equation for the tangent plane at (a, b) (as in definition 3.9).

Example 3.13: Let $f(x, y) = \begin{cases} \frac{2xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$. Is f differentiable at $(0, 0)$?

Consider $f_x(0, 0)$ and $f_y(0, 0)$. By definition,

$$f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(0 + h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \cdot \left(\frac{2h \cdot 0}{h^2 + 0^2} - 0 \right) = \lim_{h \rightarrow 0} \frac{1}{h} \cdot 0 = \lim_{h \rightarrow 0} 0 = 0$$

Also,

$$f_y(0, 0) = \lim_{h \rightarrow 0} \frac{f(0, 0 + h) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \cdot \left(\frac{2 \cdot 0 \cdot h}{0^2 + h^2} - 0 \right) = \lim_{h \rightarrow 0} \frac{1}{h} \cdot 0 = \lim_{h \rightarrow 0} 0 = 0$$

Now

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\frac{2xy}{x^2 + y^2} - (0 + 0 \cdot (x - 0) + 0 \cdot (y - 0))}{\sqrt{x^2 + y^2}} = \lim_{(x,y) \rightarrow (0,0)} \frac{\frac{2xy}{x^2 + y^2}}{\sqrt{x^2 + y^2}} = \lim_{(x,y) \rightarrow (0,0)} \frac{2xy}{(x^2 + y^2)^{3/2}}$$

Along the path $y = x$ for $x > 0$, the limit becomes

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2x^2}{(2x^2)^{3/2}} = \lim_{x \rightarrow 0^+} \frac{2x^2}{2^{3/2}x^3} = \lim_{x \rightarrow 0^+} \frac{1}{\sqrt{2}x} = \infty$$

so the limit does not exist. By definition, f is not differentiable at $(0, 0)$.

Theorem 3.14 — Major Differentiability Theorems (p. 977)

Let $z = f(x, y)$.

1. If f is differentiable at $(a, b) \in \text{dom}(f)$, then f is continuous at (a, b) .
2. If f_x and f_y both exist near (a, b) and f_x and f_y are continuous at (a, b) , then f is differentiable at (a, b) .

Example 3.15: Show that $f(x, y) = \begin{cases} \frac{2xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$ as in example 3.13 is not differentiable at $(0, 0)$ using

(1) in theorem 3.14.

Consider $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$. Along $y = x$, we have

$$\lim_{(x,x) \rightarrow (0,0)} \frac{2x \cdot x}{x^2 + x^2} = \lim_{x \rightarrow 0} \frac{2x^2}{2x^2} = \lim_{x \rightarrow 0} 1 = 1$$

Along $y = 0$, we get

$$\lim_{(x,0) \rightarrow (0,0)} \frac{2x \cdot 0}{x^2 + 0^2} = \lim_{x \rightarrow 0} 0 = 0$$

Since $0 \neq 1$, the limit does not exist, so f is not continuous at $(0, 0)$. By the contrapositive of (1) in theorem 3.14, f is not differentiable at $(0, 0)$.

Example 3.16: Come up with counterexamples to show that the converses of both parts in theorem 3.14 are false.

Consider $f(x, y) = (x + y)^{1/3}$. This is a composition of the polynomial $x + y$, which is continuous on its domain \mathbb{R}^2 , and the single-variable cube root function, which is continuous on its domain \mathbb{R} . Thus, f is continuous on \mathbb{R}^2 , including $(0, 0)$. Observe that

$$\begin{aligned} f_x(0, 0) &= \lim_{h \rightarrow 0} \frac{f(0 + h, 0) - f(0, 0)}{h} && \text{(by definition)} \\ &= \lim_{h \rightarrow 0} \frac{h^{1/3} - 0^{1/3}}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h^{2/3}} \end{aligned}$$

which does not exist. Thus, $f_x(0, 0)$ does not exist, so f is not differentiable at $(0, 0)$. Since f is continuous but not differentiable at $(0, 0)$, $f(x, y) = (x + y)^{1/3}$ is a counterexample for the converse of (1) in theorem 3.14.

Example 3.17: Let $f(x, y) = x^3y^{-2}$. Where is f differentiable? Justify your answer.

Treating y as a constant, we have $f_x = 3x^2y^{-2} = \frac{3x^2}{y^2}$. Similarly, $f_y = x^2 \cdot (-2y^{-3}) = -\frac{2x^2}{y^3}$ by treating x as a constant. f_x and f_y are rational functions and thus continuous on their domains. Now

$$\text{dom}(f_x) = \mathbb{R}^2 - \{(x, 0) : x \in \mathbb{R}\}$$

$$\text{dom}(f_y) = \mathbb{R}^2 - \{(x, 0) : x \in \mathbb{R}\}$$

Since f_x and f_y are rational functions, they are continuous on their domains. Therefore, f is differentiable on $\mathbb{R}^2 - \{(x, 0) : x \in \mathbb{R}\}$.

3.3 Vector-Valued Functions

Definition 3.18 — Vector-Valued Function

Let $m, n \in \mathbb{Z}^+$. A *vector-valued function* $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is an n -variable function whose range is a set of vectors in \mathbb{R}^m . That is, if $\underline{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, then

$$f(\underline{x}) = \langle f_1(\underline{x}), f_2(\underline{x}), \dots, f_m(\underline{x}) \rangle$$

for some functions f_1, f_2, \dots, f_m .

Example 3.19: One vector-valued function $f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is

$$f(x, y) = \left\langle xy, \frac{x}{y}, x + y \right\rangle$$

Definition 3.20 — Vector-Valued Derivative

Let $m, n \in \mathbb{Z}^+$ and $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a function so that $f(\underline{x}) = \langle f_1(\underline{x}), \dots, f_m(\underline{x}) \rangle$. The *derivative of f at $\underline{a} \in \mathbb{R}^n$* is the $m \times n$ matrix of partial derivatives

$$Df(\underline{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}_{m \times n}$$

Example 3.21: Find $Df(\underline{x})$ where $\underline{x} = (1, 1)$ and $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is given by $f(x, y) = x^3y^{-2}$.

From example 3.10, $f_x(1, 1) = 3$ and $f_y(1, 1) = -2$. Thus,

$$Df(1, 1) = \begin{bmatrix} \frac{\partial f}{\partial x} \Big|_{(1,1)} & \frac{\partial f}{\partial y} \Big|_{(1,1)} \end{bmatrix}_{1 \times 2} = \begin{bmatrix} 3 & -2 \end{bmatrix}$$

Example 3.22: Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be given by $f(x, y) = \left\langle xy, \frac{x}{y}, x + y \right\rangle$. Find $Df(2, 1)$.

We have $f_1(x, y) = xy$, $f_2 = \frac{x}{y}$, and $f_3(x, y) = x + y$, where each $f_i: \mathbb{R}^2 \rightarrow \mathbb{R}$. Thus, for $\underline{x} = (x, y)$,

$$Df(x, y) = \begin{bmatrix} \frac{\partial f_1}{\partial x}(\underline{x}) & \frac{\partial f_1}{\partial y}(\underline{x}) \\ \frac{\partial f_2}{\partial x}(\underline{x}) & \frac{\partial f_2}{\partial y}(\underline{x}) \\ \frac{\partial f_3}{\partial x}(\underline{x}) & \frac{\partial f_3}{\partial y}(\underline{x}) \end{bmatrix}_{3 \times 2} = \begin{bmatrix} y & x \\ \frac{1}{y} & -\frac{x}{y^2} \\ 1 & 1 \end{bmatrix} \implies Df(2, 1) = \begin{bmatrix} 1 & 2 \\ 1 & -2 \\ 1 & 1 \end{bmatrix}$$

Theorem 3.23 — Differentiation Rules

Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ and $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ be functions and $\underline{x} = (x, y) \in \text{dom}(f) \cap \text{dom}(g)$. If f and g are both differentiable at \underline{x} , then each of the following hold:

1. *Sum Rule:* $f + g$ is differentiable at \underline{x} with

$$D(f(\underline{x}) + g(\underline{x})) = Df(\underline{x}) + Dg(\underline{x})$$

2. *Constant Multiple Rule:* For all $c \in \mathbb{R}$, cf is differentiable at \underline{x} with

$$D(cf(\underline{x})) = cDf(\underline{x})$$

3. *Product Rule:* fg is differentiable at \underline{x} with

$$D(f(\underline{x})g(\underline{x})) = D(f(\underline{x}))g(\underline{x}) + f(\underline{x})D(g(\underline{x}))$$

4. *Quotient Rule:* $\frac{f}{g}$ is differentiable at \underline{x} , provided that $g(\underline{x}) \neq 0$, with

$$D\left(\frac{f(\underline{x})}{g(\underline{x})}\right) = \frac{D(f(\underline{x}))g(\underline{x}) - f(\underline{x})D(g(\underline{x}))}{(g(\underline{x}))^2}$$

Example 3.24: Let $h(x, y, z) = zxe^{xy}$ (so that $h: \mathbb{R}^3 \rightarrow \mathbb{R}$). Find $Dh(x, y, z) = Dh(\underline{x})$

(a) By the definition of the derivative.

(b) Using the product rule.

(a) By definition,

$$Dh(\underline{x}) = \begin{bmatrix} h_x(\underline{x}) & h_y(\underline{x}) & h_z(\underline{x}) \end{bmatrix}_{1 \times 3} = \begin{bmatrix} ze^{xy} + zxye^{xy} & zx^2e^{xy} & xe^{xy} \end{bmatrix}$$

(b) We have $h(\underline{x}) = g(\underline{x})f(\underline{x})$, where $g(\underline{x}) = zx$ ($g: \mathbb{R}^3 \rightarrow \mathbb{R}$) and $f(\underline{x}) = e^{xy}$ ($f: \mathbb{R}^3 \rightarrow \mathbb{R}$). Thus,

$$\begin{aligned} Dh(\underline{x}) &= D(g(\underline{x})f(\underline{x})) \\ &= D(g(\underline{x}))f(\underline{x}) + g(\underline{x})D(f(\underline{x})) && \text{(by the product rule)} \\ &= \begin{bmatrix} z & 0 & x \end{bmatrix}_{1 \times 3} e^{xy} + zx \begin{bmatrix} ye^{xy} & xe^{xy} & 0 \end{bmatrix}_{1 \times 3} \\ &= \begin{bmatrix} ze^{xy} + zxye^{xy} & zx^2e^{xy} & xe^{xy} \end{bmatrix} \end{aligned}$$

3.4 The Chain Rule (§14.5)

Theorem 3.25 — Chain Rule: Case I (p. 985)

Let $z = f(x, y) = f(x(t), y(t))$, where $x: \mathbb{R} \rightarrow \mathbb{R}$ and $y: \mathbb{R} \rightarrow \mathbb{R}$. If

- f is differentiable at (x, y) and
- $x(t)$ and $y(t)$ are differentiable at t

then z is differentiable and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

Example 3.26: Find $\frac{dw}{dt}$ for $w = \ln(x^2 + y^2 + z^2)$, where $x = \sin(t)$, $y = \cos(t)$, and $z = \tan(t)$,

(a) Using the multivariable chain rule (theorem 3.25).

(b) Using the single-variable chain rule (writing w as a function of t only).

(a) We have

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt} = \frac{2x}{x^2 + y^2 + z^2} \cdot \cos(t) + \frac{2y}{x^2 + y^2 + z^2} \cdot (-\sin(t)) + \frac{2z}{x^2 + y^2 + z^2} \cdot \sec^2(t)$$

Note that

$$x^2 + y^2 + z^2 = \sin^2(t) + \cos^2(t) + \tan^2(t) = 1 + \tan^2(t) = \sec^2(t)$$

so the original expression becomes

$$\frac{2 \sin(t) \cos(t)}{\sec^2(t)} - \frac{2 \cos(t) \sin(t)}{\sec^2(t)} + \frac{2 \tan(t) \sec^2(t)}{\sec^2(t)} = 2 \tan(t)$$

(b) As shown earlier, $x^2 + y^2 + z^2 = \sec^2(t)$. Thus, $w = \ln(\sec^2(t))$, so

$$\frac{dw}{dt} = \frac{1}{\sec^2(t)} \cdot 2 \sec(t) \cdot \sec(t) \tan(t) = 2 \tan(t)$$

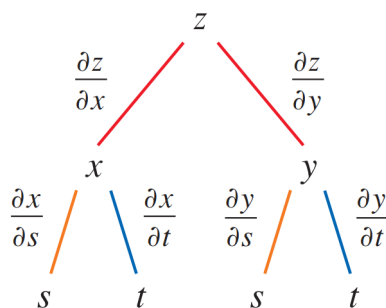
Theorem 3.27 — Chain Rule: Case II (p. 987)

Let $z = f(x, y) = f(x(s, t), y(s, t))$. If

- f is differentiable at (x, y) and
- x and y are differentiable at (s, t)

then

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}\end{aligned}$$



Example 3.28: Let $z = \tan^{-1}(xy)$, $x = st$, and $y = se^t$. Compute $\left. \frac{\partial z}{\partial s} \right|_{s=0}$.

We have

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s} = \frac{1}{1 + (xy)^2} \cdot y \cdot t + \frac{1}{1 + (xy)^2} \cdot x \cdot e^t$$

At $s = 0$, $x = 0 \cdot t = 0$ and $y = 0 \cdot e^t = 0$. Therefore,

$$\left. \frac{\partial z}{\partial s} \right|_{s=0} = \frac{1}{1 + 0^2} \cdot 0 \cdot t + \frac{1}{1 + 0^2} \cdot 0 \cdot e^t = 0$$

Theorem 3.29 — Chain Rule: General Case

Let $m, n, p \in \mathbb{Z}^+$, $g: \mathbb{R}^n \rightarrow \mathbb{R}^m$, and $f: \mathbb{R}^m \rightarrow \mathbb{R}^p$, where $g(\text{dom}(g)) \subseteq \text{dom}(f)$. Suppose that $\underline{x} \in \text{dom}(f)$. If

- g is differentiable at \underline{x} and
- f is differentiable at $g(\underline{x})$

then $f \circ g$ is differentiable at \underline{x} with

$$Df(g(\underline{x}))Dg(\underline{x})$$

Remark: In theorem 3.29, $Df(g(\underline{x}))$ is a $p \times m$ matrix and $Dg(\underline{x})$ is an $m \times n$ matrix. The multiplication performed is matrix multiplication.

Example 3.30: Let $g: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by $g(x, y) = \langle x + y^2, y + x^2 \rangle$ and $f(u, v) = \langle e^u, uv \rangle$.

(a) Compute $D(f \circ g)$ at $\underline{x} = (1, 2)$ using the chain rule (general case).

(b) Compute $D(f \circ g)$ at $(1, 2)$ directly (writing $f(g(x, y))$ as a vector-valued function of x and y only)

(a) This should be covered next lecture.

(b) We have $f(g(x, y)) = f(x + y^2, y + x^2) = \langle e^{x+y^2}, (x+y^2) \cdot (y+x^2) \rangle = \langle e^{x+y^2}, x^3 + y^3 + x^2y^2 + xy \rangle$. Letting

$f_1(x, y) = e^{x+y^2}$, $f_2(x, y) = x^3 + y^3 + x^2y^2 + xy$, and $\underline{x} = (x, y)$, we get

$$Df(g(\underline{x})) = \begin{bmatrix} \frac{\partial f_1}{\partial x}(\underline{x}) & \frac{\partial f_1}{\partial y}(\underline{x}) \\ \frac{\partial f_2}{\partial x}(\underline{x}) & \frac{\partial f_2}{\partial y}(\underline{x}) \end{bmatrix}_{2 \times 2} = \begin{bmatrix} e^{x+y^2} & e^{x+y^2} \cdot 2y \\ 3x^2 + 2xy^2 + y & 3y^2 + 2x^2y + x \end{bmatrix}$$

Thus,

$$Df(g(1, 2)) = \begin{bmatrix} e^{1+2^2} & e^{1+2^2} \cdot 2 \cdot 2 \\ 3 \cdot 1^2 + 2 \cdot 1 \cdot 2^2 + 2 & 3 \cdot 2^2 + 2 \cdot 1^2 \cdot 2 + 1 \end{bmatrix} = \begin{bmatrix} e^5 & 4e^5 \\ 13 & 17 \end{bmatrix}$$