12.818: INTRODUCTION TO ATMOSPHERIC DATA AND LARGE-SCALE DYNAMICS

PROJECT SEVEN: THE QUASI-GEOSTROPHIC VORTICITY EQUATION

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In this project we will study synoptic systems by employing the *quasi-geostrophic vorticity equation*, which can be expressed mathematically in pressure coordinates as

$$\frac{D_g}{Dt}(\zeta_g + f) = -f(\nabla \cdot \boldsymbol{v}) = f\frac{\partial \omega}{\partial p} = \frac{f}{\varrho}\frac{\partial(\varrho w)}{\partial z}$$

where

$$\frac{D_g}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{v}_g \cdot \nabla$$

is the geostrophic derivative operator, $\zeta_g + f$ is the absolute vorticity, f is the planetary vorticity, ζ_g is the geostrophic relative vorticity evaluated on an isobaric surface, ω is the vertical velocity in pressure coordinates, w is the vertical velocity in height coordinates, $\boldsymbol{v}$ is the wind velocity field, and $\boldsymbol{u}_g$ is the geostrophic wind velocity field.

Flows in *quasi-geostrophic motion* have the Coriolis force and pressure gradient forces *almost* balancing each other, with interia providing the residual effect, as opposed to geostrophic balance where the Coriolis and pressure gradient forces are exactly in balance.

The quasi-geostrophic equation exhibits some useful properties. One is that it states that the sum of the relative, planetary, and stretching vorticities must be conserved following geostrophic motions. Another is that it allows for the determination of the geostrophic wind velocity and temperature fields from knowledge of the geopotential height only. Furthermore, if the time evolution of the geopotential height field is also known, vertical motions in the atmosphere may be inferred.

Note that the *absolute vorticity* $\boldsymbol{\omega}$ is the curl of the absolute velocity, $\boldsymbol{\omega} = \nabla \cdot \boldsymbol{u}_a$, while the *relative vorticity* $\boldsymbol{\omega}$ is the curl of the relative velocity, $\boldsymbol{\omega} = \nabla \cdot \boldsymbol{u}$. Furthermore, in meteorology, we are generally interested in the vertical components of the absolute vorticity, denoted η, and the vertical component of the relative vorticity, denoted ζ, where $\eta = \hat{\boldsymbol{z}} \cdot \boldsymbol{\omega}_a$ and $\zeta = \hat{\boldsymbol{z}} \cdot \boldsymbol{\omega}$. The difference between the absolute and relative vorticity is called the *planetary vorticity*, and denoted f where $f = \hat{\boldsymbol{z}} \cdot (\nabla \times \boldsymbol{u}_e) = 2\Omega \sin\phi$ which happens to be the Coriolis parameter and thus $\eta = \zeta + f$.

Note that positive vorticity is associated with low pressure systems or cyclonic storms in the Northerm Hemisphere, while negative vorticity is associated with high pressure systems. In the Southern Hemisphere, negative vorticity is associated with cyclonic storms.

1 Quasi-geostrophic scaling

In this section we will estimate the scales associated with a synoptic system and compare them to the predicted quasi-geostrophic scaling. To identify synoptic systems we plot the mean sea level pressure over North America on November 2, 2017 (0Z) in figure 1. Unfortunately the weather looks relatively boring around Boston, MA, however, there is a low pressure system to the west over the central United States centered over the states of Colorado, Nebraska, and Kansas, as well as a high pressure system to the east over the Northern Atlantic Ocean.

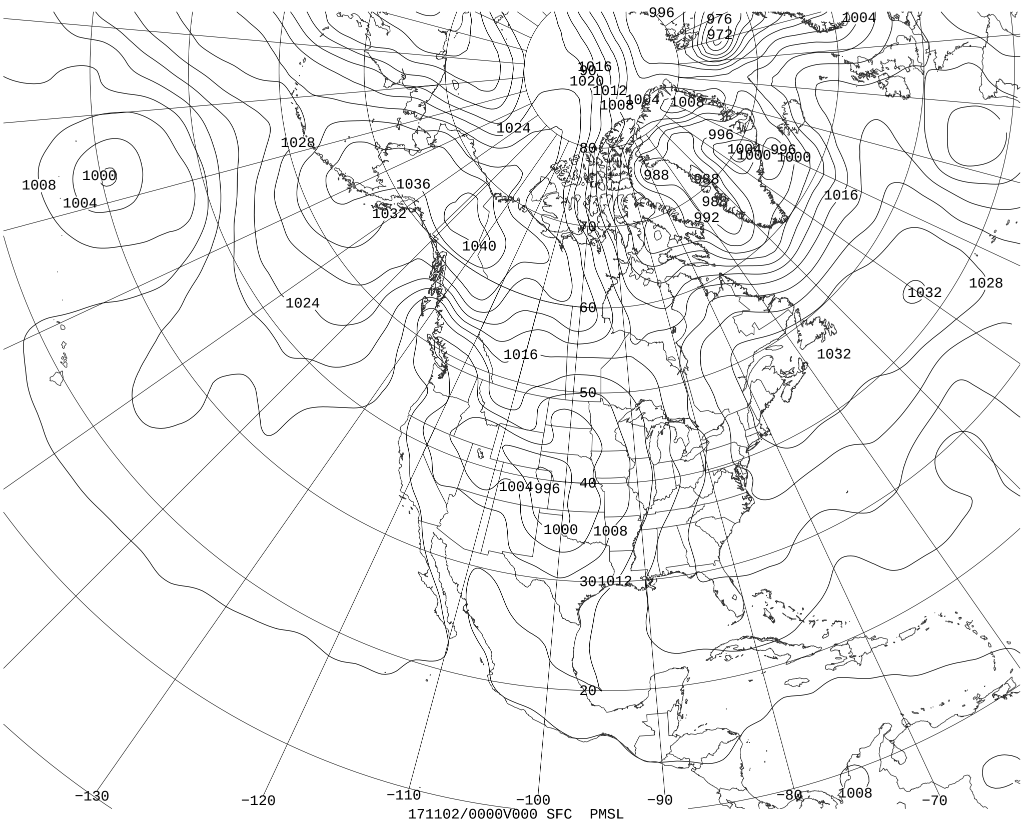

Figure 1: Contour plot of the mean sea level pressure over North America on November 2, 2017 (0Z).

Figure 2 shows a meridional cross-section of the potential temperature (K) and observed wind velocity (normal to the cross-section) through Boston, MA from 25 °N to 70 °N, while figure 3 shows a zonal cross-section from 120 °W to 20 °E, both on November 2, 2017 (0Z).

We will be more interested in the zonal cross-section (figure 3) as it passes through the low and high pressure systems we identified in figure 1. In particular we will focus on the low-pressure system.

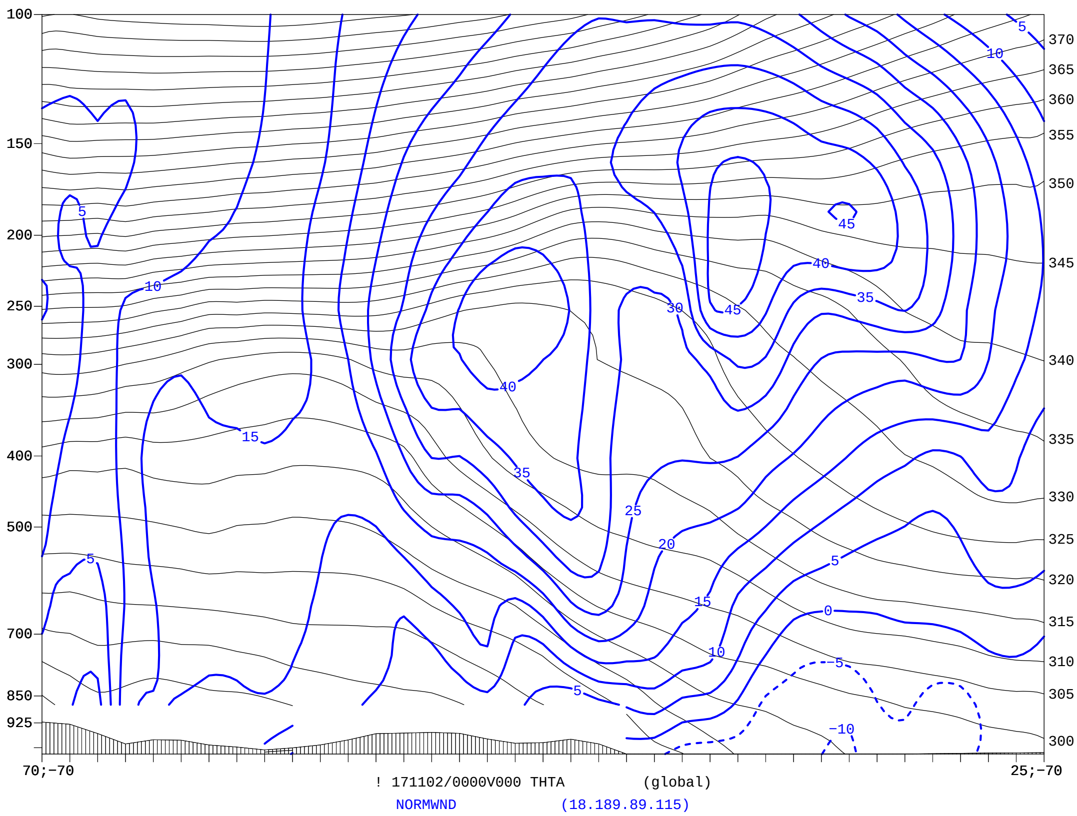

Figure 2: Meridional cross-section of the potential temperature (K) and observed wind velocity (normal to the cross-section) from 25 °N to 70 °N along the 70 °W meridian (chosen to pass through Boston, MA) on November 2, 2017 (0Z).

We can estimate the troposhperic scale height $H = \frac{RT_0}{g}$ where T_0 represents a mean tropospheric temperature in the region of the system. By inspection of the potential temperature contours over the low-pressure system in figure 3 we see that $T_0 = 290\,\mathrm{K}$ is a reasonable estimate of the surface temperature as potential temperature and air temperature coincide at mean sea level pressure. The actual air temperature may be higher due to the low-pressure system being near the Rocky Mountains which are significantly above sea level. Using $R = 287.058\,\mathrm{J/kg\,K}$ this gives us a tropospheric scale height of $H = 8.5\,\mathrm{km}$

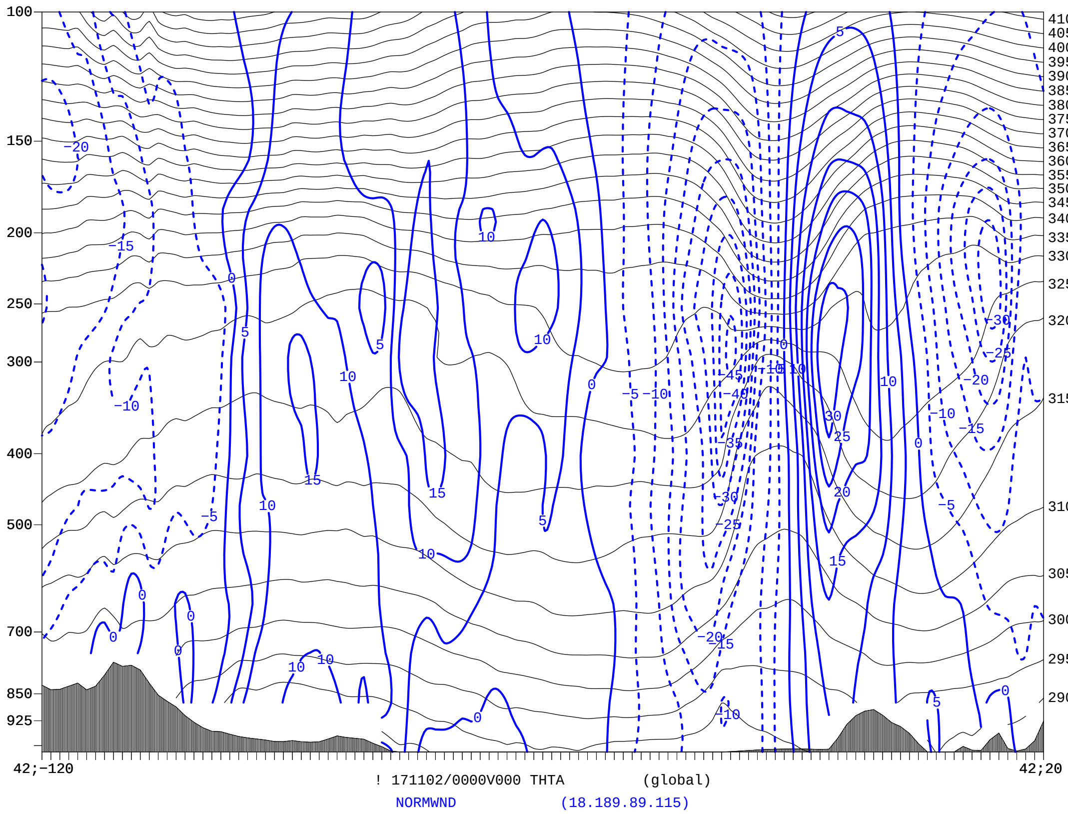

Figure 3: Zonal cross-section of the potential temperature (K) and observed wind velocity (normal to the cross-section) from 120 °W to 20 °E along the 42 °N meridian (chosen to pass through Boston, MA) on November 2, 2017 (0Z).

which seems reasonable for the mid-latitudes.

We can estimate the buoyancy frequency $N^2 = \dfrac{g}{\vartheta_0}\dfrac{\partial\vartheta}{\partial z}$ where ϑ_0 is a reference mean potential temperature. Taking the troposphere to lie between 1000 hPa and 200 hPa, the potential temperature increases from 290 K to 340 K so the mean potential temperature over the troposphere is $\vartheta_0 = 315\,\mathrm{K}$ over a height of approximately 12 km and so $\partial\vartheta/\partial z \approx 50\,\mathrm{K}/12\,000\,\mathrm{m} = 0.0042$ we get a buoyancy frequency of $N = 0.01\,\mathrm{s}^{-1}$ for the troposphere. For the layer bounded by 500 hPa and 300 hPa; $\vartheta_0 = 312.5\,\mathrm{K}$, $\partial\vartheta/\partial z \approx 10\,\mathrm{K}/4300\,\mathrm{m} = 0.0042$, and so $N \approx 0.008\,\mathrm{s}^{-1}$, just slightly lower.

The Rossby radius of deformation is $L_D = \dfrac{NH}{f} \approx 850\,\mathrm{km}$ where we took $N \approx 0.01\,\mathrm{s}^{-1}$,

$H \approx 8.5\,\text{km}$, and $f \approx 1 \times 10^{-4}\,\text{rad/s}$.

The Richardson number is defined as $\text{Ri} = N^2 / \left(\frac{\partial u}{\partial z}\right)^2$. Taking $N \approx 0.01\,\text{s}^{-1}$ for the troposphere, we will estimate $\partial u/\partial z \approx (25\,\text{m/s})/12\,000\,\text{m} = 0.004\,\text{s}^{-1}$ where we took $u(200\,\text{hPa}) \approx 25\,\text{m/s}$. Then $\text{Ri} \approx 6.25$ for the troposphere. For the layer bounded by 500 hPa and 300 hPa we can use the fact that $u(300\,\text{hPa}) \approx 38\,\text{m/s}$ and $u(500\,\text{hPa}) \approx 21\,\text{m/s}$ so that $\partial u/\partial z \approx (27\,\text{m/s})/4300\,\text{m} = 0.006\,\text{s}^{-1}$ and thus $\text{Ri} \approx 1.6$.

Looking at the zonal cross-section (figure 3) I would estimate that the low-pressure system over Colorado has a length scale of approximately 3200 km from one wind velocity peak to the next (the 15 m/s contours), while the high-pressure system over the North Atlantic has a length scale of approximately 2040 km from the $-45\,\text{m/s}$ contour to the 30 m/s contour. This are obviously quite larger than the Rossby radius of deformation estimate of 850 km that we came up with. Two reason for this discrepancy may be that we are using the wrong definition for the length scale of a synoptic system and that we are measuring this length scale at the wrong height.

2 The Quasi-geostrophic vorticity equation

We will now evaluate the different terms in the quasi-geostrophic vorticity equation. To start off, we plot the relative vorticity ζ_g at 500 hPa over Eastern Asia in figure 4 and we will eventually focus on the synoptic system bordering Mongolia and China.

It is a little difficult to see without colored contouring but there is a vorticity peak south of the Mongolia-China border peaking at $16 \times 10^{-5}\,\text{s}^{-1}$. There are two other similar vorticity peaks, one north of Japan and another on the top-left corner of the map. The troughs are muted in comparison and some prominent troughs tend to lie between the peaks.

As the planetary vorticity f coincides with the Coriolis parameter which is on the order of $10^{-4}\,\text{s}^{-1}$, the relative vorticity seems to be roughly on the same order, especially at the vorticity peaks. The vorticity troughs have a smaller magnitude at around $-4 \times 10^{-5}\,\text{s}^{-1}$, but they are negative while $f > 0$ for the Northern Hemisphere.

Figure 5 shows a contour plot of the absolute vorticity $\zeta_g + f$ and geopotential height field at 500 hPa over China. The projection has changed and so we now focus on the vorticity peak above Mongolia. I would expect the largest advection of absolute vorticity around regions with the highest vorticity gradients, that is, along the fronts (colored green in figure 5) in the direction of the geostrophic flow as the horizontal advection term is given by $\boldsymbol{u}_g \cdot \nabla(\zeta_g + f)$.

Plotting the observed horizontal advection of absolute vorticity in figure 6, it's a little difficult to see but the advection is indeed the greatest surrounding the vorticity peak/front with some horizontal advection around the vorticity troughs. The magnitude of the field peaks at above $9 \times 10^{-9}\,\text{s}^{-2}$ over the peaks and below $-9 \times 10^{-9}\,\text{s}^{-2}$ over the troughs.

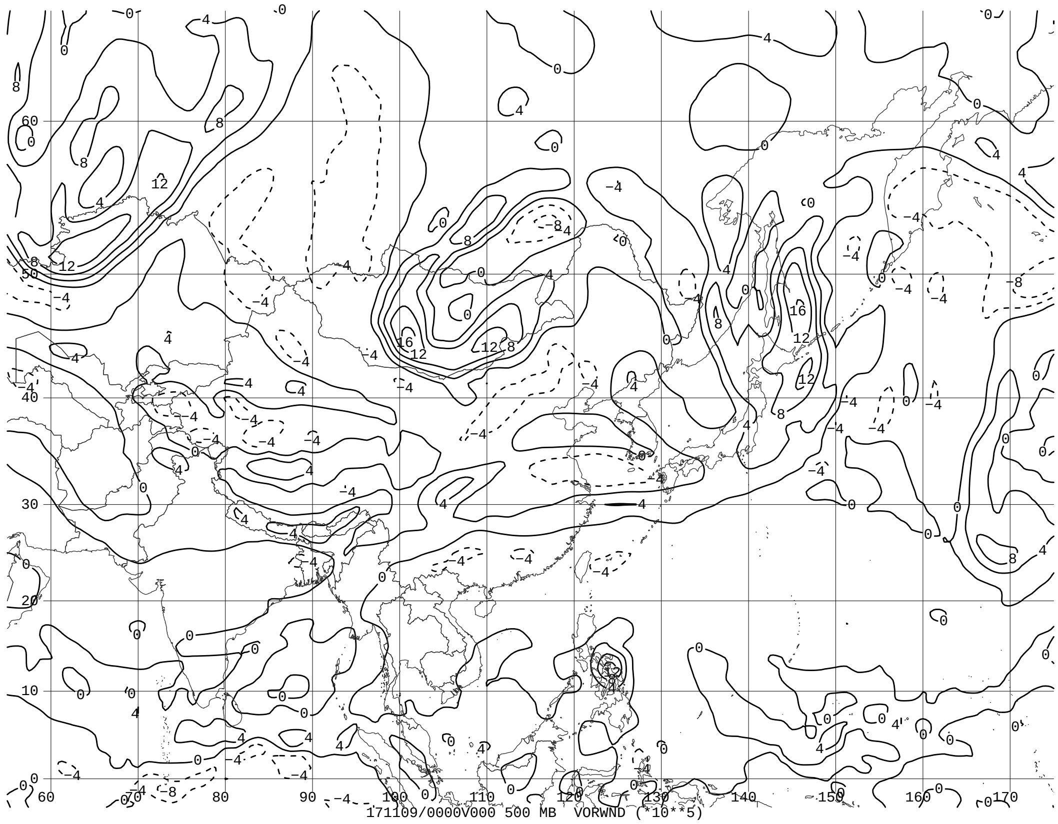

Figure 4: Contour plot of the vorticity ($\times 10^5\,\mathrm{s}^{-1}$) field calculated from the observed wind velocity field at 500 hPa over Eastern Asia on November 9, 2017 (0Z).

In figure 7 we plot the surface pressure to check if it is consistent with the pattern of upper level divergence. It agrees very well as there is a clear high-pressure system over Mongolia below which we saw a vorticity front and some associated horizontal advection of absolute vorticity.

Figure 8 shows a contour plot of the time derivative of the relative vorticity $\partial\zeta/\partial t$ over Eastern Asia computed as a finite different from the relative vorticity at 6Z and 0Z. We see its magnitude peaks around vorticity peaks at values up to $6 \times 10^{-9}\,\mathrm{s}^{-2}$ and around vorticity troughs with values up to $-8 \times 10^{-9}\,\mathrm{s}^{-2}$.

The magnitudes of the $\partial\zeta/\partial t$ and $\boldsymbol{u}_g \cdot \nabla(\zeta_g + f)$ terms is very similar with the $\boldsymbol{u}_g \cdot \nabla(\zeta_g + f)$ term being slightly larger. Thus we would expect the magnitude of the residual field to be approximately the difference between the $\partial\zeta/\partial t$ and $\boldsymbol{u}_g \cdot \nabla(\zeta_g + f)$ fields, or roughly $< 3 \times 10^{-9}\,\mathrm{s}^{-2}$.

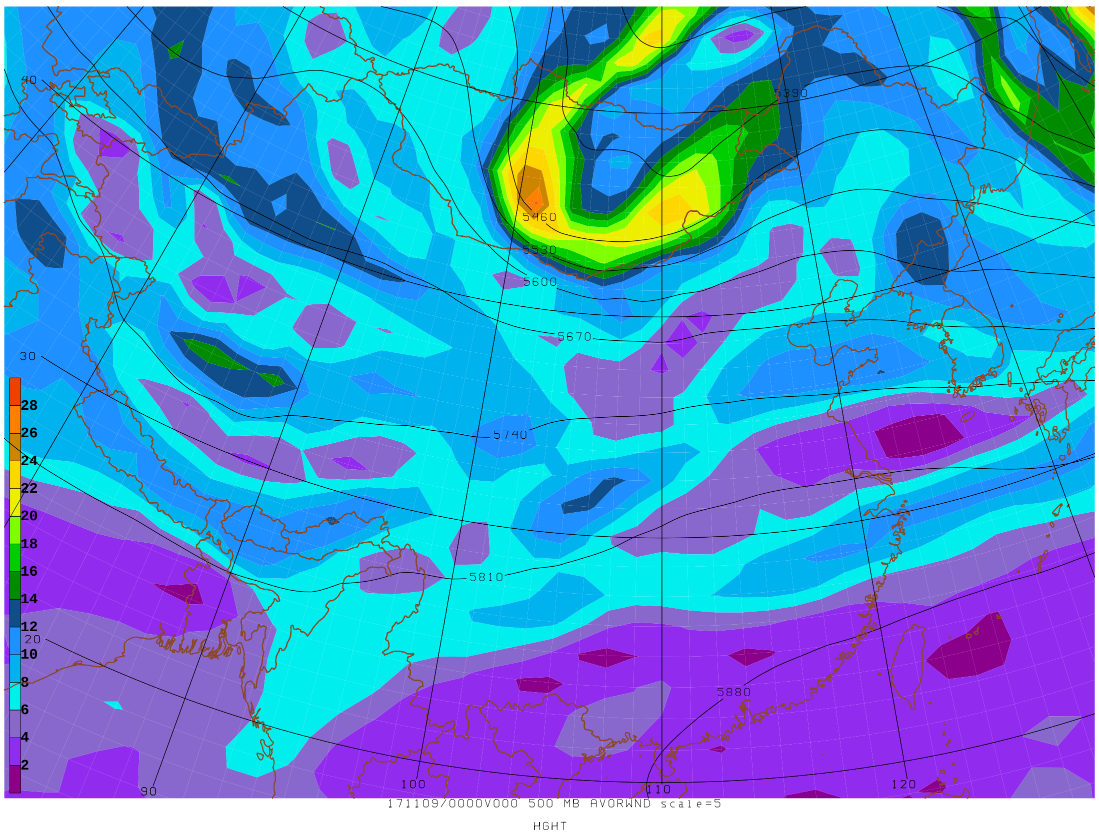

Figure 5: Contour plot of the absolute vorticity $\zeta_g + f$ ($\times 10^5\,\mathrm{s}^{-1}$) and geopotential height field at 500 hPa over China on November 9, 2017 (0Z).

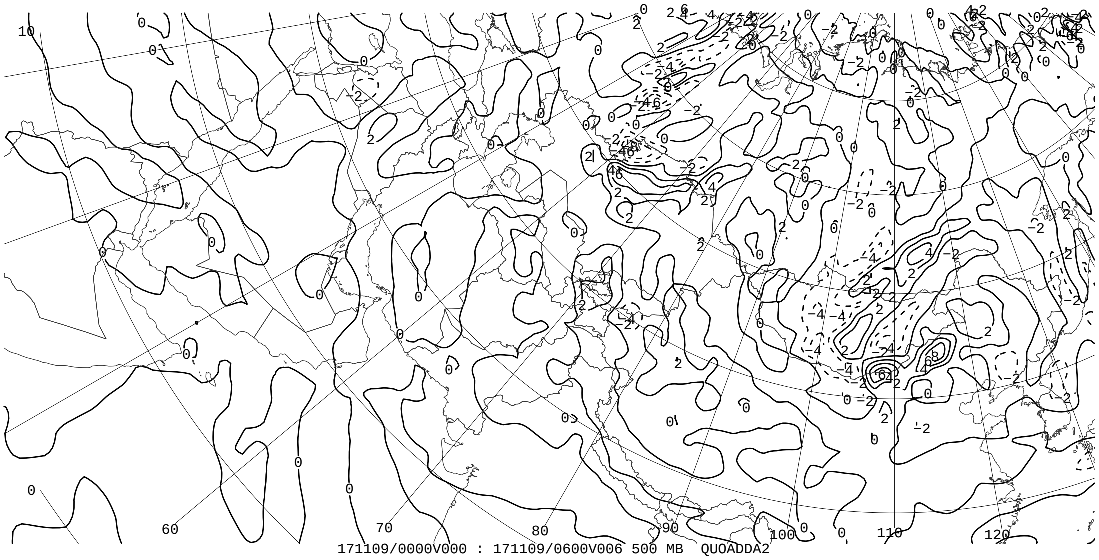

Figure 6: Contour plot of the horizontal advection of absolute vorticity $\boldsymbol{u}_g \cdot \nabla(\zeta_g + f)$ ($\times 10^{10}\,\mathrm{s}^{-2}$) and geopotential height field at 500 hPa over Eastern Asia on November 9, 2017 (0Z).

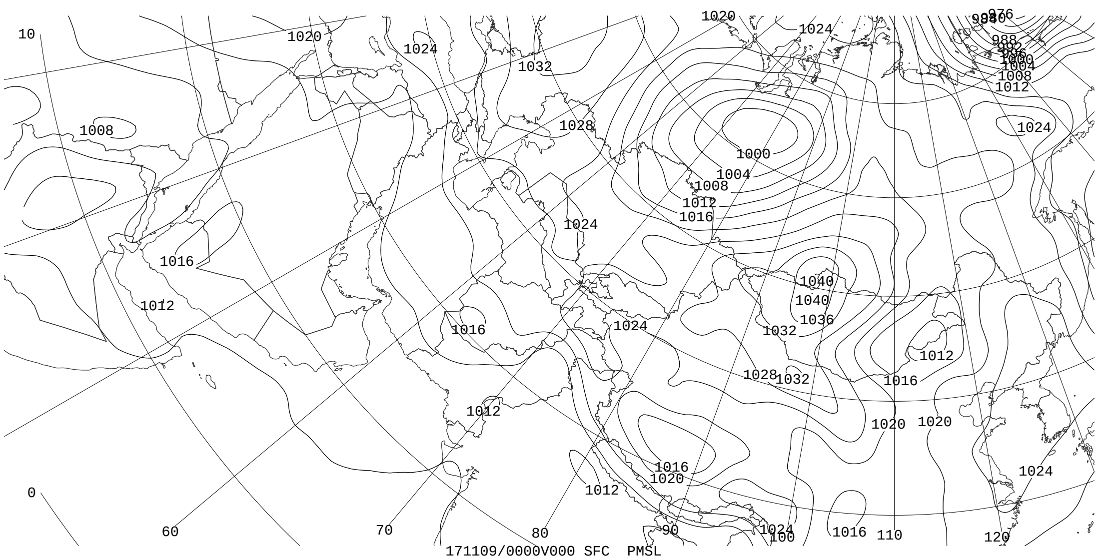

Figure 7: Contour plot of the mean sea level pressure over Eastern Asia on November 9, 2017 (0Z).

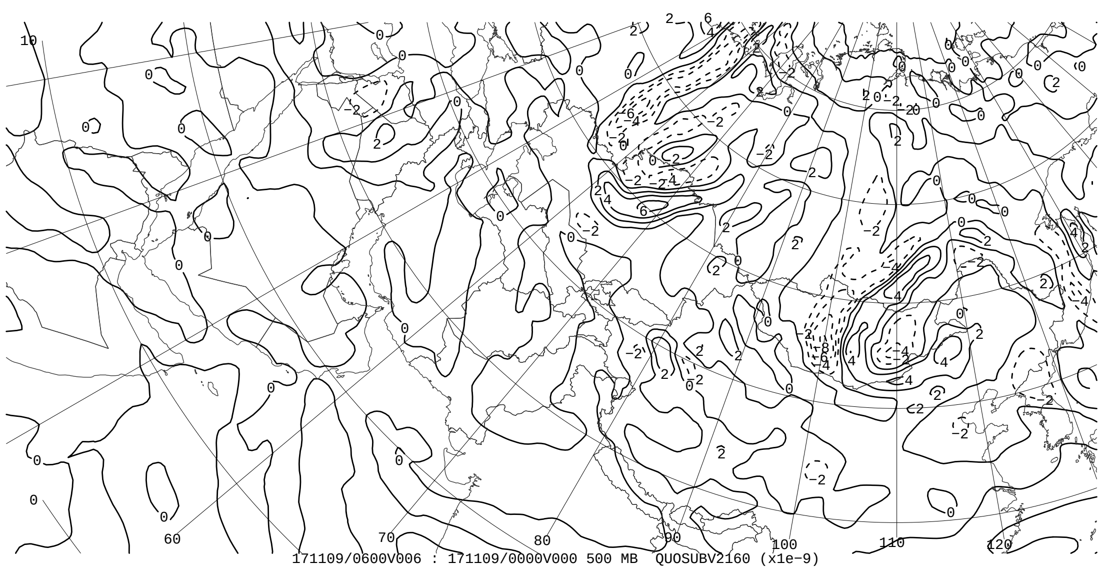

Figure 8: Contour plot of the rate of change of relative vorticity $\partial\zeta/\partial t$ ($\times 1 \times 10^9\,\mathrm{s}^{-2}$) and geopotential height field at 500 hPa over Eastern Asia on November 9, 2017 (0Z).