2.097J / 6.339J / 16.920J
Numerical Methods for Partial Differential Equations
Massachusetts Institute of Technology - Fall 2017

Project for Boundary Element Methods

Issued: November 1st 2017 *Due: November 17th 2017*

Instructions

This problem set concerns the implementation of a quadrature rule, followed by the implementation of a Nyström boundary element method. In addition to answering the questions, please provide some helpful plots. For the quadrature questions, provide plots of the convergence curves (in n) where appropriate. Similarly for the Nyström method, provide a convergence plot of the error as n increases, using your maximum value of n to provide the reference solution which you can take to be exact. Also provide a plot of the solution σ around the boundary.

Problem 1 - Quadrature (50 pts)

This question concerns the trapezoidal rule. The aim is to prove the error associated with this rule, then implement the rule in Matlab to approximate some integrals, and hopefully observe this error in practice. Finally we shall approximate the integral of a periodic function for which we observe a curious phenomenon, namely that the trapezoidal rule is spectrally accurate for periodic functions.

Consider the definite integral

$$I[f] = \int_a^b f(x)\mathrm{d}x.$$

In class we proved the error in approximating I via the composite midpoint rule on n intervals is

$$I[f] - \frac{(b-a)}{n}\sum_{i=1}^{n} f\left(a + \frac{b-a}{n}\left(i - \frac{1}{2}\right)\right) = \frac{(b-a)^3}{24n^2}f''(\eta) \tag{1}$$

for η some point in $[a, b]$. In order to prove this, we made use of the Taylor series about the midpoint of the interval $c = (a + b)/2$,

$$f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}, \tag{2}$$

for some ξ between x and c. The final term is called "Lagrange's remainder term" - a derivation of this is provided in the appendix to Lecture 2's handwritten notes.

We noted from (1) that the error depends on the second derivative of f, hence the midpoint rule is exact when integrating polynomials of degree less than or equal to 1. Another rule which is exact for polynomials of degree less than or equal to 1 is the trapezoidal rule.

The trapezoidal rule on a single interval is

$$\int_a^b f(x)\mathrm{d}x \approx \frac{b-a}{2}(f(a)+f(b)).$$

This is equivalent to approximating the area under $f(x)$ by a trapezoid. It can also be seen as the average of the left and right Riemann sums. To improve the accuracy, one can subdivide $[a,b]$ into n intervals and use the rule on each interval. This leads to the composite trapezoidal rule:

$$\int_a^b f(x)\mathrm{d}x \approx \frac{b-a}{n}\left[\frac{f(a)+f(b)}{2} + \sum_{i=1}^{n-1} f\left(a + i\frac{b-a}{n}\right)\right] \tag{3}$$

(a) (20 pts) Prove that the error in approximating I via the composite trapezoidal rule is

$$-\frac{(b-a)^3}{12n^2}f''(\eta)$$

for some $\eta \in [a,b]$.

[Hint: Consider one interval in the summation in (3), $[x_i, x_{i+1}]$ and let $c = (x_i + x_{i+1})/2$ be the midpoint of this interval. Perform integration by parts on the following

$$\int_{x_i}^{x_{i+1}} (x-c)f'(x)\mathrm{d}x$$

and you should observe an integral form of the error. Then Taylor expand f about c and carry out the integration.]

(b) (20 pts) Write a Matlab script to implement the composite trapezoidal rule (3). Then use it to approximate the following integrals for number of intervals $n = 1,\ldots,100$. For each calculate the absolute error between the exact integral (which can be calculated by hand) and the approximated value, and plot the error as a function of n on a loglog plot. State how fast the error decays for each ($\mathcal{O}(1/n)$, $\mathcal{O}(1/n^2)$, $\mathcal{O}(1/n^3),\ldots$), and explain why this is the case if you can.

(i)

$$I = \int_2^{10} 3x\mathrm{d}x.$$

(ii)

$$I = \int_0^{\pi} \sin(x)\mathrm{d}x.$$

(iii)

$$I = \int_0^1 \mathrm{e}^{2\cos(2\pi x)} \mathrm{d}x.$$

Exact:besseli(0,2) in Matlab.

(iv)

$$I = \int_0^{2\pi} |\cos(x)| \mathrm{d}x$$

(c) (10 pts) You might have noticed that the error for example (iii) above converges spectrally. That is, the error decreases exponentially with increasing numbers of points, which happens when the trapezoidal rule is applied to some periodic functions. There are other periodic functions in the above set of examples, why doesn't the error converge spectrally for those other examples? (For more details on this, see the article below).

The Exponentially Convergent Trapezoidal Rule, Trefethen and Weiderman
`http://epubs.siam.org/doi/pdf/10.1137/130932132`

Problem 2 - Nyström method for Laplace's equation (50 pts)

For this problem, the boundary is the unit circle (all points x, y such that $\sqrt{x^2 + y^2} = 1$). The boundary condition on the circle is of Neumann type, specifically, the normal derivative of the potential (with the normal pointing out of the circle) is

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma} = \frac{1}{3 + 2\cos\theta + \cos 2\theta}$$

where $\theta \in [-\pi, \pi]$, $x = \cos(\theta)$, and $y = \sin(\theta)$

For the 2-D Laplace's equation, the monopole integral formulation of the interior Neumann problem is given by

$$\frac{\partial u_\Gamma(\vec{x})}{\partial n_{\vec{x}}} = +\pi\sigma(\vec{x}) - \int_\Gamma^{PV} \frac{(\vec{x} - \vec{x}')^T n_{\vec{x}}}{\|\vec{x} - \vec{x}'\|^2} \sigma(\vec{x}') d\Gamma' \qquad \vec{x} \in \Gamma$$

and for the exterior Neumann problem, the monopole integral formulation is given by

$$\frac{\partial u_\Gamma(\vec{x})}{\partial n_{\vec{x}}} = -\pi\sigma(\vec{x}) - \int_\Gamma^{PV} \frac{(\vec{x} - \vec{x}')^T n_{\vec{x}}}{\|\vec{x} - \vec{x}'\|^2} \sigma(\vec{x}') d\Gamma' \qquad \vec{x} \in \Gamma.$$

a) (20 pts) Use a Nyström method to solve the exterior Neumann problem, using equally spaced quadrature points on the circle and equal quadrature weights. What do you notice about the matrix elements?

b) (15 pts) When using the Nyström method with equally spaced points to solve the exterior Neumann problem, how quickly does the method converge. That is, how do the solution errors decrease with increasing numbers of points?

c) (15 pts) Now try solving the *interior* Neumann problem with the equally spaced point Nyström method. Is the interior problem easy to solve? What goes wrong? Does a solution exist?

d) (Optional and challenging) Now try solving the *exterior* Neumann problem with the Nyström method, but change the boundary from a circle to an ellipse. How will you select where to put the points?

Appendix

Exact solution for verification of numerical solution in 2(a), 2(b):
Let

$$f(\theta) := \frac{1}{3 + 2\cos\theta + \cos 2\theta}.$$

Then the exact solution to the exterior Neumann problem stated above is given by

$$\sigma(\theta) = -\frac{1}{\pi}\left(\frac{I}{2} + f(\theta)\right), \quad \theta \in [-\pi, \pi],$$

where

$$I := -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\mathrm{d}\theta.$$