12.818: INTRODUCTION TO ATMOSPHERIC DATA AND LARGE-SCALE DYNAMICS

PROJECT SEVEN: THE QUASI-GEOSTROPHIC VORTICITY EQUATION

ALI RAMADHAN

In this project we will study synoptic systems by employing the *quasi-geostrophic vorticity equation*, which can be expressed mathematically in pressure coordinates as

$$\frac{D_g}{Dt}(\zeta_g + f) = -f(\nabla \cdot \boldsymbol{v}) = f\frac{\partial \omega}{\partial p} = \frac{f}{\varrho}\frac{\partial(\varrho w)}{\partial z}$$

where

$$\frac{D_g}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{v}_g \cdot \nabla$$

is the geostrophic derivative operator, $\zeta_g + f$ is the absolute vorticity, f is the planetary vorticity, ζ_g is the geostrophic relative vorticity evaluated on an isobaric surface, ω is the vertical velocity in pressure coordinates, w is the vertical velocity in height coordinates, $\boldsymbol{v}$ is the wind velocity field, and $\boldsymbol{u}_g$ is the geostrophic wind velocity field.

Flows in *quasi-geostrophic motion* have the Coriolis force and pressure gradient forces *almost* balancing each other, with interia providing the residual effect, as opposed to geostrophic balance where the Coriolis and pressure gradient forces are exactly in balance.

The quasi-geostrophic equation exhibits some useful properties. One is that it states that the sum of the relative, planetary, and stretching vorticities must be conserved following geostrophic motions. Another is that it allows for the determination of the geostrophic wind velocity and temperature fields from knowledge of the geopotential height only. Furthermore, if the time evolution of the geopotential height field is also known, vertical motions in the atmosphere may be inferred.

Note that the *absolute vorticity* $\boldsymbol{\omega}$ is the curl of the absolute velocity, $\boldsymbol{\omega} = \nabla \cdot \boldsymbol{u}_a$, while the *relative vorticity* $\boldsymbol{\omega}$ is the curl of the relative velocity, $\boldsymbol{\omega} = \nabla \cdot \boldsymbol{u}$. Furthermore, in meteorology, we are generally interested in the vertical components of the absolute vorticity, denoted η, and the vertical component of the relative vorticity, denoted ζ, where $\eta = \hat{\boldsymbol{z}} \cdot \boldsymbol{\omega}_a$ and $\zeta = \hat{\boldsymbol{z}} \cdot \boldsymbol{\omega}$. The difference between the absolute and relative vorticity is called the *planetary vorticity*, and denoted f where $f = \hat{\boldsymbol{z}} \cdot (\nabla \times \boldsymbol{u}_e) = 2\Omega \sin\phi$ which happens to be the Coriolis parameter and thus $\eta = \zeta + f$.

Note that positive vorticity is associated with low pressure systems or cyclonic storms in the Northerm Hemisphere, while negative vorticity is associated with high pressure systems. In the Southern Hemisphere, negative vorticity is associated with cyclonic storms.

1 Quasi-geostrophic scaling

In this section we will estimate the scales associated with a synoptic system and compare them to the predicted quasi-geostrophic scaling. To identify synoptic systems we plot the mean sea level pressure over North America on November 2, 2017 (0Z) in figure 1. Unfortunately the weather looks relatively boring around Boston, MA, however, there is a low pressure system to the west over the central United States centered over the states of Colorado, Nebraska, and Kansas, as well as a high pressure system to the east over the Northern Atlantic Ocean.

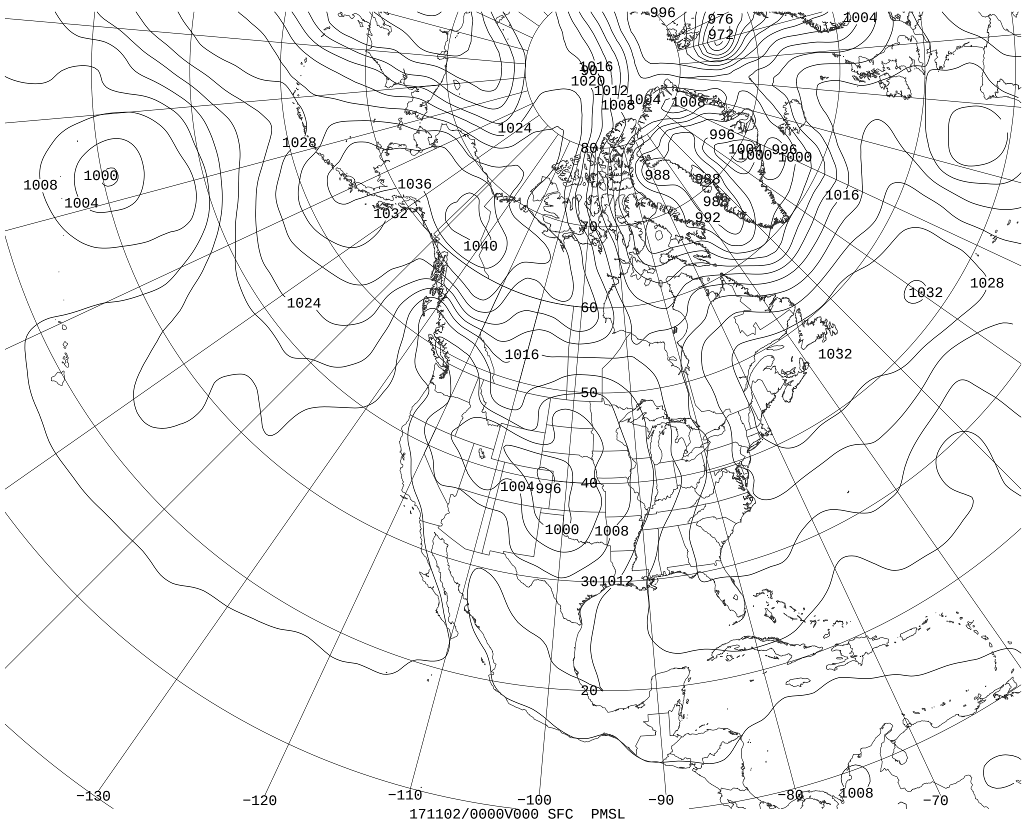

Figure 1: Contour plot of the mean sea level pressure over North America on November 2, 2017 (0Z).

Figure 2 shows a meridional cross-section of the potential temperature (K) and observed wind velocity (normal to the cross-section) through Boston, MA from 25 °N to 70 °N, while figure 3 shows a zonal cross-section from 120 °W to 20 °E, both on November 2, 2017 (0Z).

We will be more interested in the zonal cross-section (figure 3) as it passes through the low and high pressure systems we identified in figure 1. In particular we will focus on the low-pressure system.

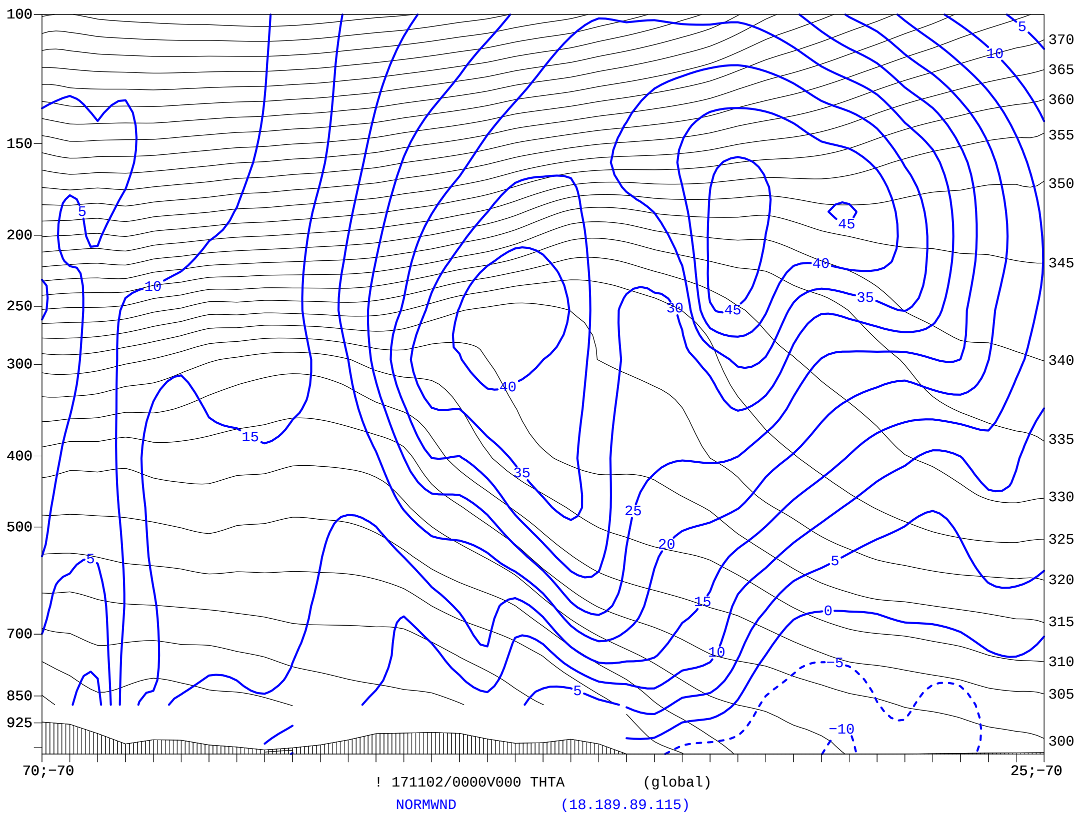

Figure 2: Meridional cross-section of the potential temperature (K) and observed wind velocity (normal to the cross-section) from 25 °N to 70 °N along the 70 °W meridian (chosen to pass through Boston, MA) on November 2, 2017 (0Z).

We can estimate the troposhperic scale height $H = \frac{RT_0}{g}$ where T_0 represents a mean tropospheric temperature in the region of the system. By inspection of the potential temperature contours over the low-pressure system in figure 3 we see that $T_0 = 290\,\text{K}$ is a reasonable estimate of the surface temperature as potential temperature and air temperature coincide at mean sea level pressure. The actual air temperature may be higher due to the low-pressure system being near the Rocky Mountains which are significantly above sea level. Using $R = 287.058\,\text{J/kg K}$ this gives us a tropospheric scale height of $H = 8.5\,\text{km}$

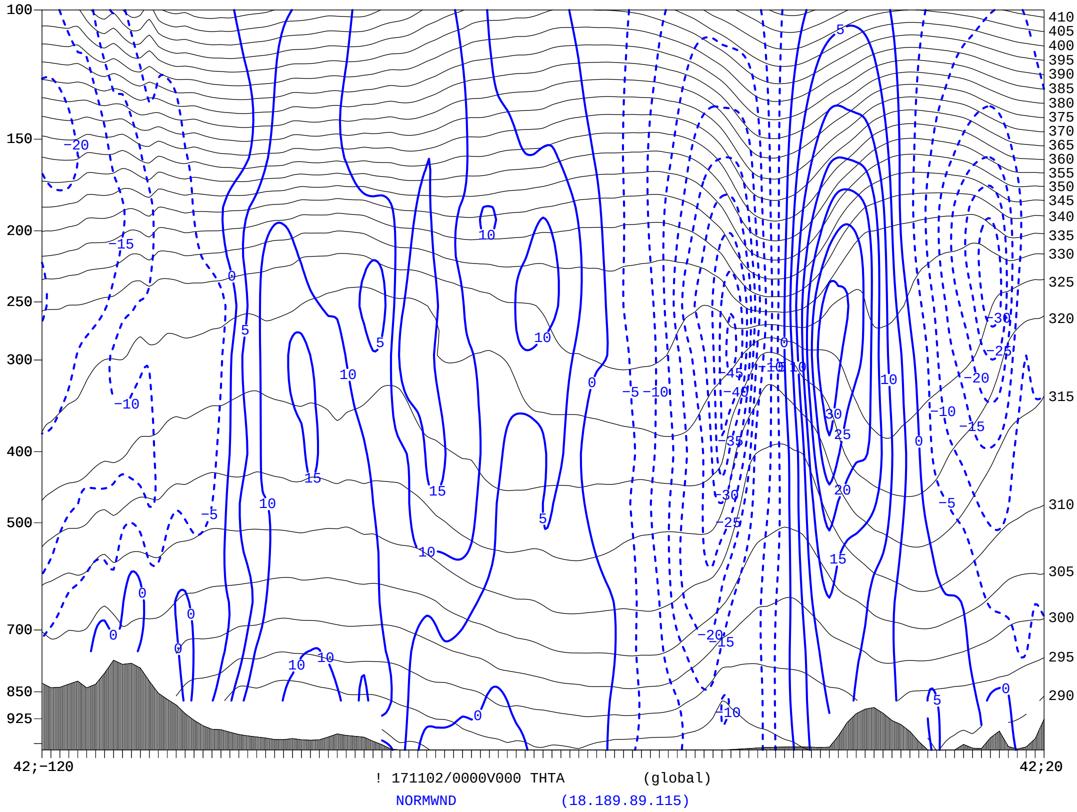

Figure 3: Zonal cross-section of the potential temperature (K) and observed wind velocity (normal to the cross-section) from 120 °W to 20 °E along the 42 °N meridian (chosen to pass through Boston, MA) on November 2, 2017 (0Z).

which seems reasonable for the mid-latitudes.

We can estimate the buoyancy frequency $N^2 = \dfrac{g}{\vartheta_0}\dfrac{\partial\vartheta}{\partial z}$ where ϑ_0 is a reference mean potential temperature. We will take ϑ_0 to be the mean potential temperature over the low-pressure system.

2 The Quasi-geostrophic vorticity equation

We will now evaluate the different terms in the quasi-geostrophic vorticity equation. To start off, we plot the relative vorticity ζ_g at 500 hPa over Eastern Asia in figure 4 and we will eventually focus on the synoptic system bordering Mongolia and China.

Note that GEMPACK plots the negative of the vorticity and so

$$-\boldsymbol{u} \cdot \nabla(\zeta_g + f) = f \nabla \cdot \boldsymbol{u}$$

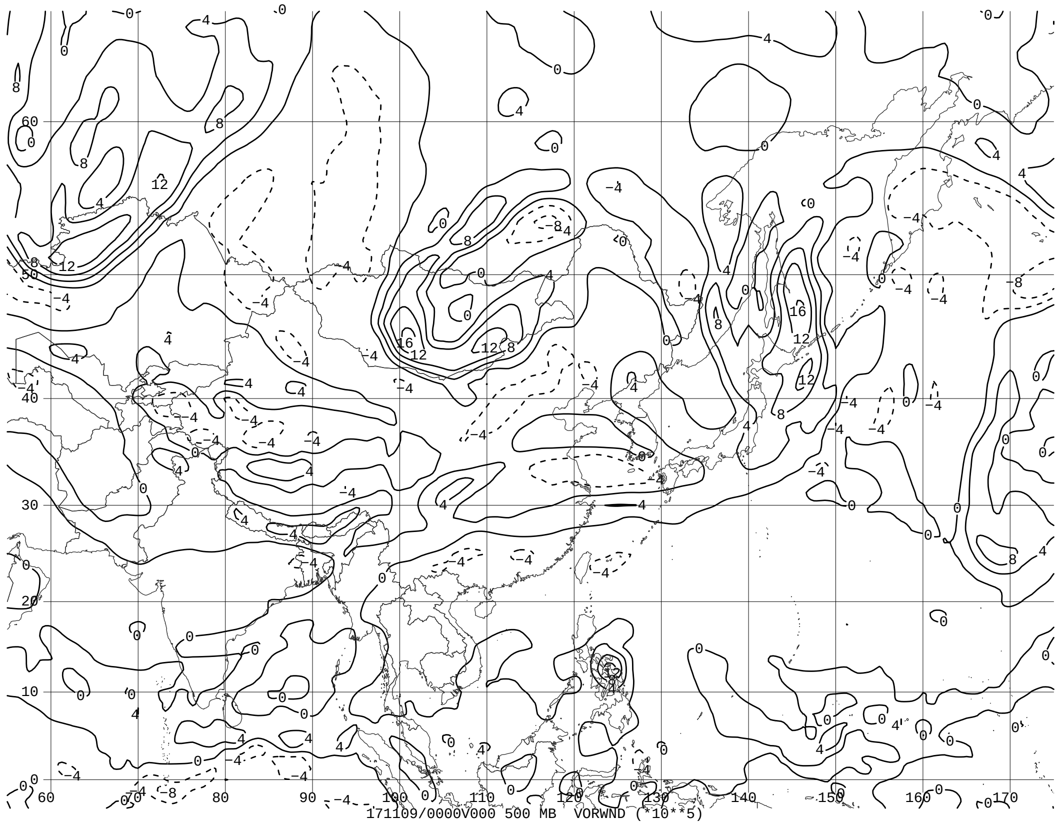

Figure 4: Contour plot of the vorticity ($\times 10^5$) field calculated from the observed wind velocity field at 500 hPa over Eastern Asia on November 9, 2017 (0Z).

3 The Brunt–Väisälä frequency

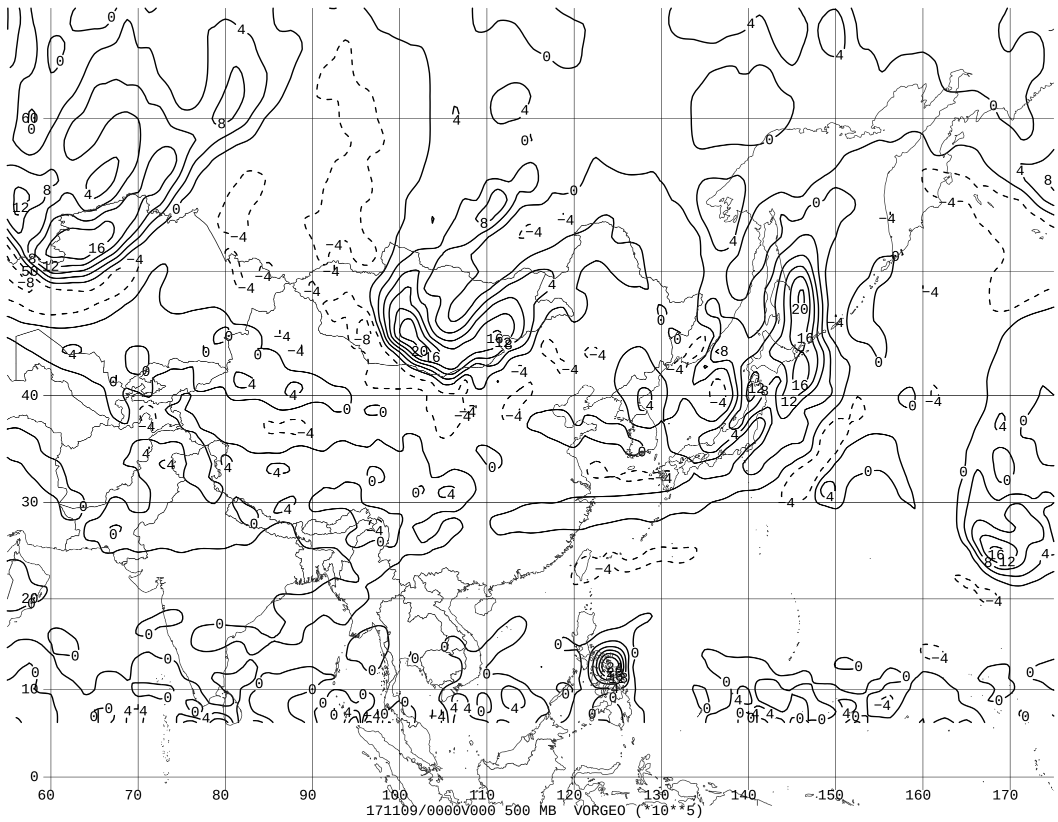

Figure 5: Contour plot of the geostrophic vorticity ($\times 10^5$) field (that is, calculated from the geostrophic wind velocity field) at 500 hPa over Eastern Asia on November 9, 2017 (0Z).

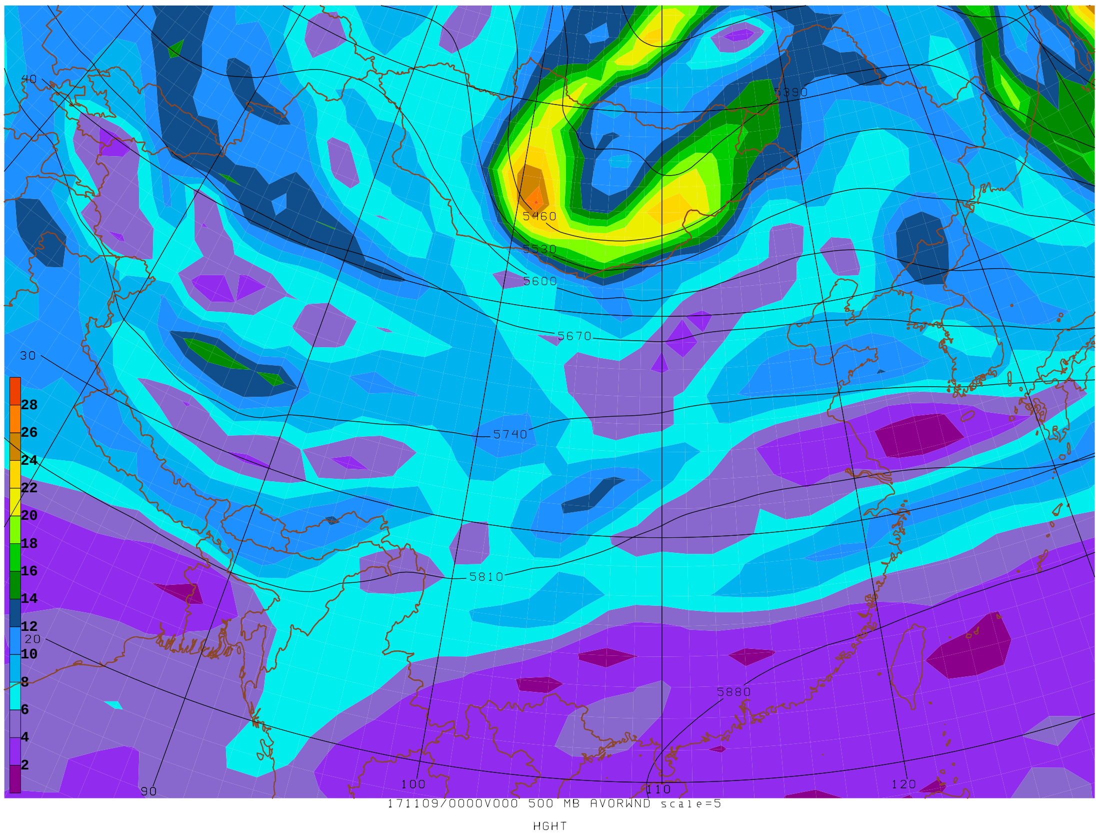

Figure 6: Contour plot of the absolute vorticity $\zeta_g + f$ ($\times 10^5$) and geopotential height field at 500 hPa over China on November 9, 2017 (0Z).

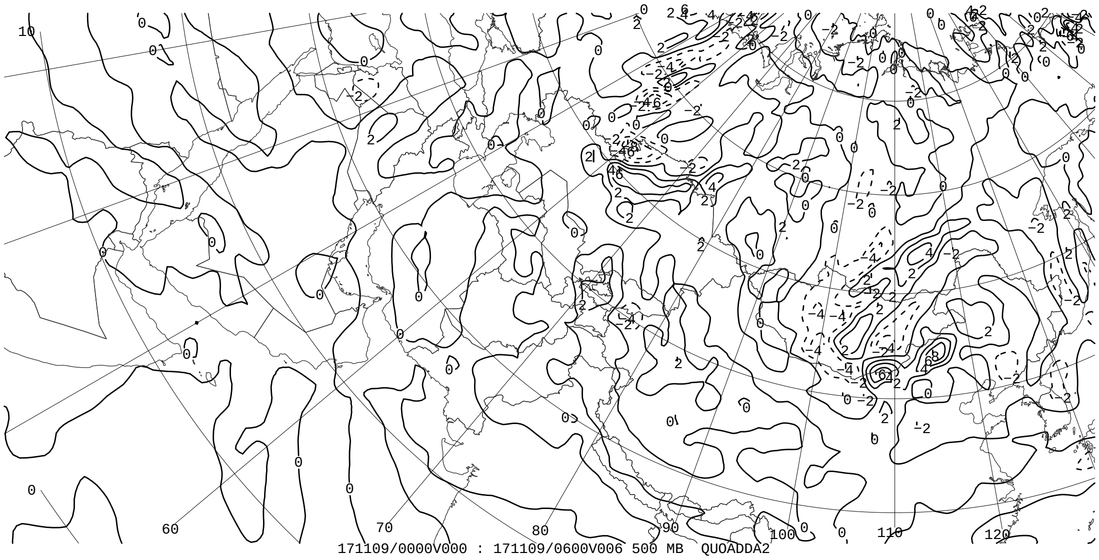

Figure 7: Contour plot of the horizontal advection of absolute vorticity $\boldsymbol{u}_g \cdot \nabla(\zeta_g + f)$ ($\times 10^5$) and geopotential height field at 500 hPa over Eastern Asia on November 9, 2017 (0Z).

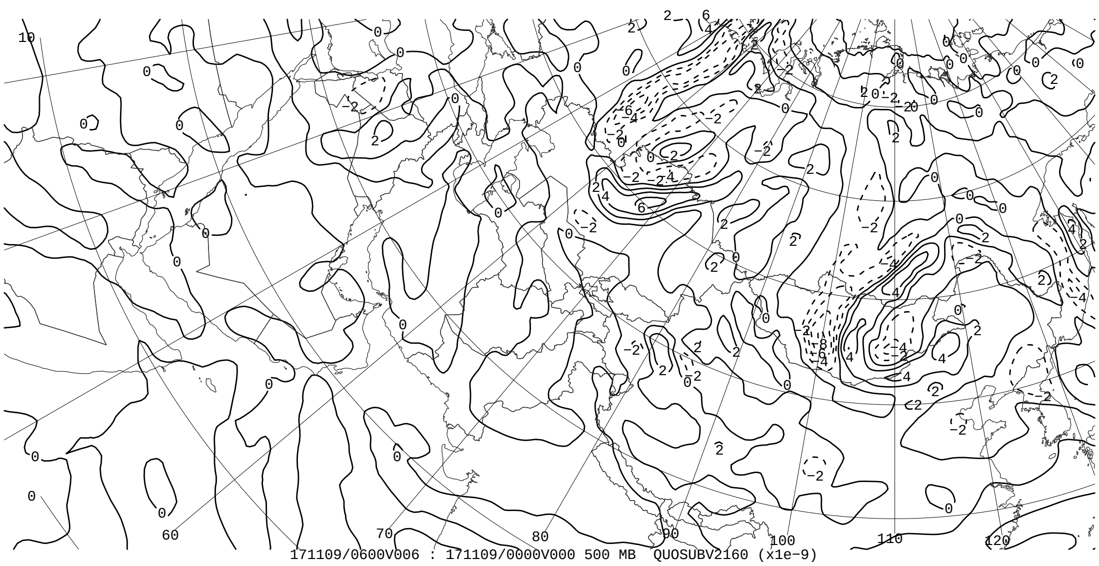

Figure 8: Contour plot of the rate of change of relative vorticity $\partial\zeta/\partial t$ ($\times 10^9$) and geopotential height field at 500 hPa over Eastern Asia on November 9, 2017 (0Z).

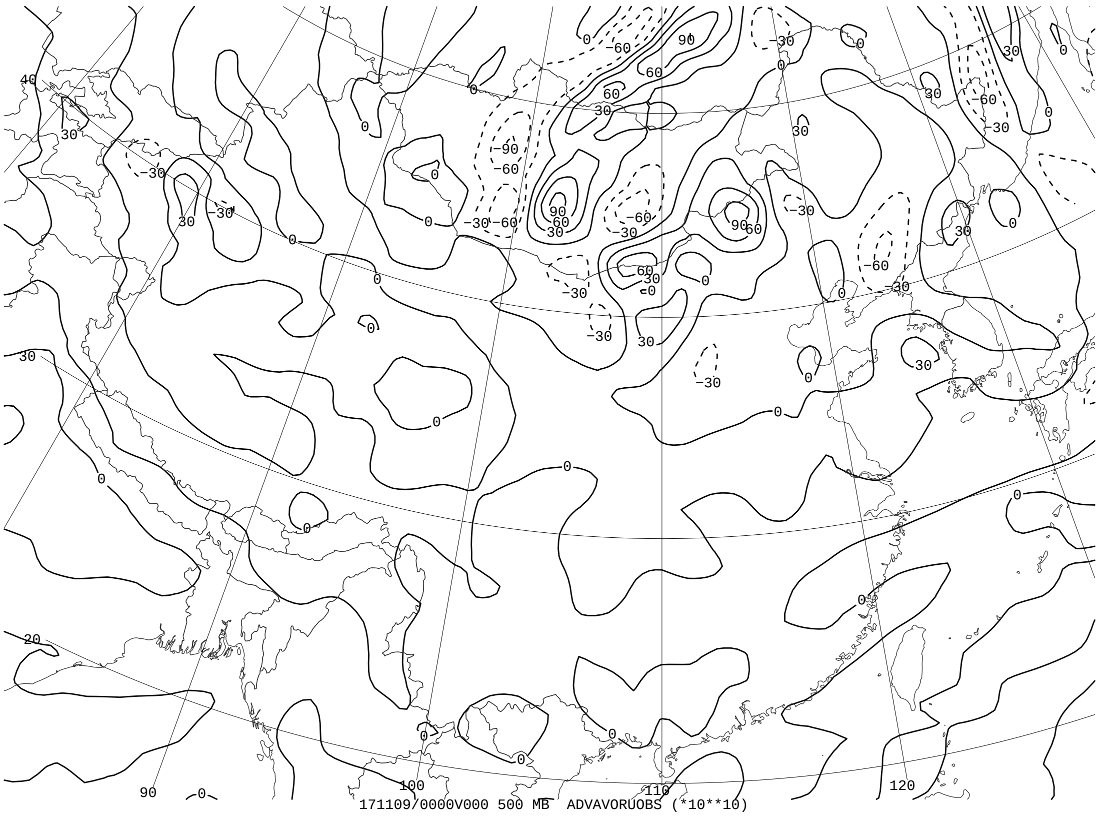

Figure 9: Contour plot of the advection of absolute vorticity ($\times 10^{10}$) at 500 hPa over Eastern Asia on November 9, 2017 (0Z).