6.339: NUMERICAL METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS
PROJECT ONE: FINITE DIFFERENCE METHODS

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Question 1(a)—Derive the governing boundary condition for p' on boundary. The error in the boundary condition you derived should be the same as that in the linearized Euler equation.

Question 1(b)—Derive a numerical scheme for the governing equation, using second-order finite-difference in space.

A second-order finite-difference approximation for the first derivative of an arbitrary real-valued and twice-differentiable function f on a grid would be

$$\frac{\partial f(x_i)}{\partial x} \approx \frac{f(x_{i+1}) - f(x_{i-1})}{2\Delta x} \tag{1}$$

while the second derivative can be approximated as

$$\frac{\partial^2 f(x_i)}{\partial x^2} \approx \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{(\Delta x)^2} \tag{2}$$

and together they can be used to discretize the governing Euler equation in space.

Taking a look at $p' = p'(x, y, t)$ first, which is the perturbed pressure field of the fluid flow, its governing equation can be written as

$$\frac{\partial p'}{\partial t} + \mathbf{u}_0 \cdot \nabla p' = q' \tag{3}$$

where $\mathbf{u}_0 = (u_0, v_0)$ is the unperturbed fluid flow velocity vector and

$$q' = \frac{Dp'}{Dt} \equiv \frac{\partial p'}{\partial t} + \mathbf{u}_0 \cdot \nabla p'$$

is the *material derivative* of p'. Expanding the dot product and gradient, (4) can be written as

$$\frac{\partial p'}{\partial t} = q' - u_0 \frac{\partial p'}{\partial x} - v_0 \frac{\partial p'}{\partial y} \tag{4}$$

where we haved moved the $\mathbf{u}_0 \nabla p'$ term to the right hand side. Now we can discretize the spatial derivatives using (1) to obtain

$$\frac{dp'_{i,j}}{dt} = q'_{i,j} - u_0 \frac{p'_{i+1,j} - p'_{i-1,j}}{2\Delta x} - v_0 \frac{p'_{i,j+1} - p'_{i,j-1}}{2\Delta y}. \tag{5}$$

The corresponding governing equation for q' is

$$\frac{\partial q'}{\partial t} + \mathbf{u}_0 \cdot \nabla q' = c_0^2 \nabla^2 p' \tag{6}$$

where c_0^2 is the unperturbed speed of sound in the fluid. Expanding the dot product and the Laplacian terms, we arrive at

$$\frac{\partial q'}{\partial t} = c_0^2 \left(\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} \right) - u_0 \frac{\partial q'}{\partial x} - v_0 \frac{\partial q'}{\partial y} \tag{7}$$

where again we have moved the $\mathbf{u}_0 \cdot \nabla q'$ term to the right hand side and which we can now discretize using (1) and (2) to obtain

$$\begin{aligned} \frac{\partial q'_{i,j}}{\partial t} = c_0^2 \left(\frac{p'_{i+1,j} - 2p'_{i,j} + p'_{i-1,j}}{(\Delta x)^2} + \frac{p'_{i,j+1} - 2p'_{i,j} + p'_{i,j-1}}{\partial(\Delta y^2)} \right) \\ - u_0 \frac{q'_{i+1,j} - q'_{i-1,j}}{2\Delta x} - v_0 \frac{q'_{i,j+1} - q'_{i,j-1}}{2\Delta y}. \end{aligned} \tag{8}$$

Equations (5) and (8) each provide us with a set of $(N+1)(M+1)$ ordinary differential equations (ODEs) that may be integrated numerically to obtain the values of p' and q' at each grid point as a function of time, but we also require appropriate boundary conditions. Let us look at the boundary conditions for p' first. The left boundary condition reads $p = p_0$ at $x = 0$ but $p = p_0 + p'$ by definition so we must have that $p' = 0$ at $x = 0$ or

$$p'_{0,j} = 0 \quad \text{for all } j \tag{9}$$

The right boundary condition reads $\dfrac{dp'}{dx'} = 0$ which we can discretize using (1) to get

$$\frac{p'_{N+1,j} - p'_{N,j}}{2\Delta x} = 0 \quad \text{or} \quad p'_{N+1,j} = p'_{N,j} \quad \text{for all } j \tag{10}$$

The bottom ($y = 0$) and top ($y = H$) boundary condition reads

$$\nabla p \cdot \mathbf{n} = -\rho \left(\frac{\partial^2 F}{\partial t^2} + 2u \frac{\partial^2 F}{\partial t \partial x} + u^2 \frac{\partial^2 F}{\partial^2 x} \right) \tag{11}$$

where $F(x, t)$ describes the geometry for the lower and upper walls and $\mathbf{n}$ is the normal vector pointing into the flow field such that $\mathbf{n} = (F_x, -1)$ for the top wall and $\mathbf{n} = (-F_x, 1)$ for the bottom wall.

Now taking a look at the boundary conditions for q', the condition on the left wall reads $q' = 0$ at $x = 0$ which may be discretized as

$$q'_{0,j} = 0 \quad \text{for all } j \tag{12}$$