6.339: NUMERICAL METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS
PROJECT ONE: FINITE DIFFERENCE METHODS

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In this project, we will utilize finite difference methods to solve the two-dimensional time-dependent Euler equations, a set of quasilinear hyperbolic partial differential equations, for the pressure field of a fluid flowing around a small perturbative bump.

The Euler equations mathematically represent the conservation of mass and momentum for a fluid and can be used to describe the flow of an inviscid fluid. They can be written as

$$\frac{D\mathbf{u}}{Dt} = -\nabla w + \mathbf{g} \tag{1a}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{1b}$$

Before tackling the questions, I will first introduce the finite difference operators that I use. A second-order finite-difference approximation for the first derivative of an arbitrary real-valued and sufficiently differentiable multivariable function $f = f(x)$ on a grid, as we saw in class, would be

$$\frac{\partial f(x_i)}{\partial x} = \frac{f(x_{i+1}) - f(x_{i-1})}{2\Delta x} + \mathcal{O}(\Delta x)^2 \tag{2}$$

where Δx is the spacing between two successive uniformly distributed grid points, while the second derivative can be approximated as

$$\frac{\partial^2 f(x_i)}{\partial x^2} = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{(\Delta x)^2} + \mathcal{O}(\Delta x)^2 \tag{3}$$

and together they can be used to discretize the governing Euler equations in space. Both finite difference operators (2) and (3) accurately approximate the derivatives up to a truncation error on the order of $(\Delta x)^2$, thus their classification as second-order operators.

It will also be useful at certain points to introduce the second-order forward difference operator

$$\frac{\partial f(x_i)}{\partial x} = \frac{-3f(x_i) + 4f(x_{i+1}) - f(x_{i+2})}{2\Delta x} + \mathcal{O}(\Delta x)^2 \tag{4}$$

along with the second-order backward difference operator

$$\frac{\partial f(x_i)}{\partial x} = \frac{3f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{2\Delta x} + \mathcal{O}(\Delta x)^2 \tag{5}$$

and the second-order mixed derivative operator assuming that f is now a multivariable function $f = f(x, y)$ with sufficiently smooth derivatives,

$$\frac{\partial^2 f_{i,j}}{\partial x \partial y} = \frac{f_{i+1,j+1} - f_{i+1,j-1} - f_{i-1,j+1} + f_{i-1,j-1}}{4\Delta x \Delta y} + \mathcal{O}(\Delta x^2, \Delta y^2) \tag{6}$$

where we have introduced the shorthand notation $f(x_i, y_j) \equiv f_{i,j}$ for brevity.

Question 1(a)—Derive the governing boundary condition for p' on boundary. The error in the boundary condition you derived should be the same as that in the linearized Euler equation.

The bottom ($y = 0$) and top ($y = H$) boundary conditions read

$$\nabla p \cdot \mathbf{n} = -\rho \left(\frac{\partial^2 F}{\partial t^2} + 2u \frac{\partial^2 F}{\partial t \partial x} + u^2 \frac{\partial^2 F}{\partial^2 x} \right) \tag{7}$$

where $F(x, t)$ describes the geometry for the lower and upper walls and $\mathbf{n}$ is the normal vector pointing into the flow field such that $\mathbf{n} = (F_x, -1)$ for the top wall and $\mathbf{n} = (-F_x, 1)$ for the bottom wall.

Expanding the gradient term in (7) we obtain

$$\nabla p = \nabla(p_0 + p') = \cancelto{0}{\nabla p_0} + \nabla p' = \nabla p'$$

where $\nabla p_0 = 0$ as p_0 represents the unperturbed pressure field and thus does not vary in time or space for our problem.

Question 1(b)—Derive a numerical scheme for the governing equation, using second-order finite-difference in space.

Taking a look at $p' = p'(x, y, t)$ first, which is the perturbed pressure field of the fluid flow, its governing equation can be written as

$$\frac{\partial p'}{\partial t} + \mathbf{u}_0 \cdot \nabla p' = q' \tag{8}$$

where $\mathbf{u}_0 = (u_0, v_0)$ is the unperturbed fluid flow velocity vector and

$$q' = \frac{Dp'}{Dt} \equiv \frac{\partial p'}{\partial t} + \mathbf{u}_0 \cdot \nabla p'$$

is the *material derivative* of p'. Expanding the dot product and gradient, the governing equation can be written as

$$\frac{\partial p'}{\partial t} = q' - u_0 \frac{\partial p'}{\partial x} - v_0 \frac{\partial p'}{\partial y} \tag{9}$$

where we haved moved the $\mathbf{u}_0 \nabla p'$ term to the right hand side. Now we can discretize the spatial derivatives using (2) to obtain

$$\boxed{\frac{dp'_{i,j}}{dt} = q'_{i,j} - u_0 \frac{p'_{i+1,j} - p'_{i-1,j}}{2\Delta x} - v_0 \frac{p'_{i,j+1} - p'_{i,j-1}}{2\Delta y}.} \tag{10}$$

The corresponding governing equation for q' is

$$\frac{\partial q'}{\partial t} + \mathbf{u}_0 \cdot \nabla q' = c_0^2 \nabla^2 p' \tag{11}$$

where c_0^2 is the unperturbed speed of sound in the fluid. Expanding the dot product and the Laplacian terms, we arrive at

$$\frac{\partial q'}{\partial t} = c_0^2 \left(\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} \right) - u_0 \frac{\partial q'}{\partial x} - v_0 \frac{\partial q'}{\partial y} \tag{12}$$

where again we have moved the $\mathbf{u}_0 \cdot \nabla q'$ term to the right hand side and which we can now discretize using (2) and (3) to obtain

$$\boxed{\begin{aligned} \frac{\partial q'_{i,j}}{\partial t} = c_0^2 &\left(\frac{p'_{i+1,j} - 2p'_{i,j} + p'_{i-1,j}}{(\Delta x)^2} + \frac{p'_{i,j+1} - 2p'_{i,j} + p'_{i,j-1}}{(\Delta y^2)} \right) \\ &\qquad - u_0 \frac{q'_{i+1,j} - q'_{i-1,j}}{2\Delta x} - v_0 \frac{q'_{i,j+1} - q'_{i,j-1}}{2\Delta y}. \end{aligned}} \tag{13}$$

Equations (10) and (13) each provide us with a set of $(N+1)(M+1)$ ordinary differential equations (ODEs) that may be integrated numerically to obtain the values of p' and q' at each grid point as a function of time, but we also require appropriate boundary conditions. Let us look at the boundary conditions for p' first. The left boundary condition reads $p = p_0$ at $x = 0$ but $p = p_0 + p'$ by definition so we must have that $p' = 0$ at $x = 0$ or

$$\boxed{p'_{0,j} = 0 \quad \text{for all } j} \tag{14}$$

The right boundary condition reads $\dfrac{dp'}{dx'} = 0$ which we can discretize using (2) to get

$$\frac{p'_{N+1,j} - p'_{N,j}}{2\Delta x} = 0 \tag{15}$$

or rather, that the boundary grid points must equal the ones to their left,

$$\boxed{p'_{N+1,j} = p'_{N,j} \quad \text{for all } j} \tag{16}$$

The bottom ($y = 0$) and top ($y = H$) boundary condition reads

$$\nabla p \cdot \mathbf{n} = -\rho \left(\frac{\partial^2 F}{\partial t^2} + 2u \frac{\partial^2 F}{\partial t \partial x} + u^2 \frac{\partial^2 F}{\partial^2 x} \right) \tag{17}$$

where $F(x,t)$ describes the geometry for the lower and upper walls and $\mathbf{n}$ is the normal vector pointing into the flow field such that $\mathbf{n} = (F_x, -1)$ for the top wall and $\mathbf{n} = (-F_x, 1)$ for the bottom wall.

Now taking a look at the boundary conditions for q', the condition on the left wall reads $q' = 0$ at $x = 0$ which may be discretized as

$$\boxed{q'_{0,j} = 0 \quad \text{for all } j} \tag{18}$$

Question 2(a)—On a bump described by function:

$$F(x) = 0.01 \left\{ 1 - \cos\left[\frac{8\pi}{L} \left(x - \frac{L}{4} \right) \right] \right\}^2$$

Plot the contour for p' at time $t = 2$ in your report (no figure generation in submitted code). Include the color-bar in the figure.

Question 3(a)—Estimate what the twist angle θ should be, so that the mesh aligns with the features of the solution.

We can manually measure the angle from figure 1. Choosing a vector $\mathbf{x}_1 = (1.74, 0)$ from the origin to the bottom left corner of the high pressure region to the left of the bump, and another vector $\mathbf{x}_2 = (1.14, 1.133)$ from $\mathbf{x}_1$ to along the high pressure region's straight left boundary, we can calculate the angle between the two vector using

$$\theta = \arccos \frac{\mathbf{x}_1 \cdot \mathbf{x}_2}{\|\mathbf{x}_1\| \, \|\mathbf{x}_2\|} \approx \frac{\pi}{4} \tag{19}$$

which results in an angle very close to 45°. Thus is makes sense to choose $\theta = \pi/4$ for the oblique mesh.

Question 3(b)—Derive the governing equation on this new mesh.

To do so we will make a coordinate transformation from Cartesian coordinates (x, y) to the oblique coordinate system, which we will denote by $(\tilde{x}, \tilde{y})$. The transformation is given by

$$\tilde{x} = x - y \tan\theta \tag{20a}$$
$$\tilde{y} = y \sec\theta \tag{20b}$$

which will transform the differential operators according the chain rule,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial x} + \frac{\partial f}{\partial \tilde{y}}\frac{\partial \tilde{y}}{\partial x} = \frac{\partial f}{\partial \tilde{x}}$$
$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial y} + \frac{\partial f}{\partial \tilde{y}}\frac{\partial \tilde{y}}{\partial y} = -\tan\theta\frac{\partial f}{\partial \tilde{x}} + \sec\theta\frac{\partial f}{\partial \tilde{y}}$$

so that they can be substituted by differential operators with respect to the oblique coordinate system

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial \tilde{x}} \tag{21a}$$
$$\frac{\partial}{\partial y} \to -\tan\theta\frac{\partial}{\partial \tilde{x}} + \sec\theta\frac{\partial}{\partial \tilde{y}} \tag{21b}$$

Substituting our new differential operators (21a) and (21b) into the governing equation for p' (9) we get

$$\frac{\partial p'}{\partial t} = q' - u_0\frac{\partial p'}{\partial \tilde{x}} - v_0\left(-\tan\theta\frac{\partial p'}{\partial \tilde{x}} + \sec\theta\frac{\partial p'}{\partial \tilde{y}}\right) \tag{22}$$

and doing the same for the governing equation for q' (12) we get

$$\frac{\partial q'}{\partial t} = c_0^2\left(\sec^2\theta\frac{\partial^2 p'}{\partial \tilde{x}^2} + \sec^2\theta\frac{\partial^2 p'}{\partial \tilde{y}^2} - 2\sec\theta\tan\theta\frac{\partial^2 p'}{\partial \tilde{x}\partial \tilde{y}}\right) - u_0\frac{\partial q'}{\partial \tilde{x}} - v_0\left(-\tan\theta\frac{\partial q'}{\partial \tilde{x}} + \sec\theta\frac{\partial p'}{\partial \tilde{y}}\right) \tag{23}$$

The boundary condition at the bottom wall now becomes more complicated involving multiple derivatives,

$$\tan\theta\frac{\partial p'}{\partial \tilde{x}} - \sec\theta\frac{\partial p'}{\partial \tilde{y}} = -\rho_0 u_0^2\frac{\partial^2 F}{\partial \tilde{x}^2}, \quad \frac{L}{4} < x < \frac{L}{2} \tag{24}$$

while the boundary condition at the top wall ($\tilde{y} = H$) is slightly simpler,

$$\tan\theta \frac{\partial p'}{\partial \tilde{x}} - \sec\theta \frac{\partial p'}{\partial \tilde{y}} = 0 \tag{25}$$

Question 3(c)—Derive a numerical scheme for the governing equation, using second-order finite-difference in space.

We will again take $v_0 = 0$, which will simplify the governing equations. Using the central difference finite-difference operator (2) for the first derivative we can discretize the governing equation for p' in the oblique coordinate system (22) as

$$\boxed{\frac{dp'_{i,j}}{dt} = q'_{i,j} - u_0 \frac{p'_{i+1,j} - p'_{i-1,j}}{2\Delta\tilde{x}}} \tag{26}$$

and results in exactly the same discretization used previously [see equation (10)]. Utilizing the finite-difference operator for the second derivative (3) and the finite-difference operator for the mixed second derivative we can also discretize the governing equation for q' in the oblique coordinate system as

$$\boxed{\begin{aligned}\frac{\partial q'_{i,j}}{\partial t} = c_0^2 \Bigg(&\sec^2\theta \frac{p'_{i+1,j} - 2p'_{i,j} + p'_{i-1,j}}{(\Delta\tilde{x})^2} + \sec^2\theta \frac{p'_{i,j+1} - 2p'_{i,j} + p'_{i,j-1}}{(\Delta\tilde{y}^2)} \\ &-2\sec\theta\tan\theta \frac{p'_{i+1,j+1} - p'_{i+1,j-1} - p'_{i-1,j+1} + p'_{i-1,j-1}}{4\Delta\tilde{x}\Delta\tilde{y}} \Bigg) - u_0 \frac{q'_{i+1,j} - q'_{i-1,j}}{2\Delta\tilde{x}}.\end{aligned}} \tag{27}$$

Using a second-order backward difference operator, the $\tilde{y} = H$ boundary condition can be written as a system of $N+1$ linear equations for the value of p', or more specifically, for the $N+1$ unknown values $p'_{i,M+1}$ along the $\tilde{y} = H$ boundary,

$$\frac{\tan\theta}{2\Delta\tilde{x}} p'_{i-1,M+1} + \frac{3\sec\theta}{2\Delta\tilde{y}} p'_{i,M+1} - \frac{\tan\theta}{2\Delta\tilde{x}} p'_{i+1,M+1} = \sec\theta \frac{4p'_{i,M} - p'_{i,M-1}}{2\Delta\tilde{y}} \tag{28}$$

and a similar process for the $\tilde{y} = 0$ boundary condition, except for the use of a second-order forward difference operator, produces another system of linear equations for the $N+1$ unknown values $p'_{i,1}$ along the $\tilde{y} = 0$ boundary,

$$\frac{\tan\theta}{2\Delta\tilde{x}} p'_{i-1,1} - \frac{3\sec\theta}{2\Delta\tilde{y}} p'_{i,1} - \frac{\tan\theta}{2\Delta\tilde{x}} p'_{i+1,1} = -\sec\theta \frac{4p'_{i,2} - p'_{i,3}}{2\Delta\tilde{y}} - \rho_0 u_0^2 \frac{F_{i+1} - 2F_i + F_{i-1}}{(\Delta\tilde{x})^2} \tag{29}$$

both of which may be easily solved in MATLAB due to their relatively small size.

Question 4—Same as question 2.

The highlighted/boxed equations from question (3b) together form the discretization we will use in the oblique coordinate system.