6.339: NUMERICAL METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS
PROJECT TWO: FINITE VOLUME METHODS

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In this project, we will utilize finite volume methods to study dense traffic flow and traffic jams modeled as shockwaves. We model traffic in each lane by a scalar hyperbolic conservation law, following what is known as the Lighthill-Whitman-Richards model.

We use a scalar hyperbolic conservation law to model traffic density $\rho^{(\ell)}(x,t)$ for n lanes indexed by $\ell = 1, 2, \ldots, n$

$$\frac{\partial \rho^{(\ell)}}{\partial t} + \frac{\partial(\rho^{(\ell)} v^{(\ell)})}{\partial x} = s \tag{1}$$

where $v^{(\ell)}(x,t)$ is the average velocity of the cars. This, however, provides us with only one equation for two unknowns and thus we specify the velocity by

$$v(\rho) = v_{\max}\left(1 - \frac{\rho^2}{\rho_{\max}^2}\right) \tag{2}$$

giving us a traffic flux of

$$f(\rho) = \rho v = v_{\max}\left(\rho - \frac{\rho^3}{\rho_{\max}^2}\right) \tag{3}$$

The source term

$$s^{(\ell)} = \alpha \sum_{\substack{|k-\ell|=1 \\ 1 \le k,\ell \le n}} \rho^{(k)} - \rho^{(\ell)} \tag{4}$$

models the density of traffic that is switching lanes from neighboring lanes. α is the fraction of drivers that change lanes.

We will split up our one-dimensional grid into a number of cells indexed by $i = 1, 2, \ldots, N$. We will index the edges of the cell i by $i - \frac{1}{2}$ for the left boundary of the cell, and by $i + \frac{1}{2}$ for the right boundary of the cell. So we can think of i as indexing the cell centers.

To derive a first-order conservative finite-volume scheme for a single lane, we will consider the volume averages of the traffic density $\rho(x,t)$ at two different times. The volume average of the traffic density at cell i, $\rho_i = \rho(x_i, t)$, at a time t_1 over $x \in \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]$ must exist by the mean value theorem and is given by

$$\bar{\rho}_i(t_1) = \frac{1}{\Delta x_i} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \rho(x,t_1)\, dx$$

and an identical expression can be written for the volume average at a later time $t_2 > t_1$. Now, integrating the scalar conservation law in time from $t = t_1$ to $t = t_2$ we can write

$$\int_{t_1}^{t_2} \frac{\partial \bar{\rho}}{\partial t}\, dt + \int_{t_1}^{t_2} \frac{\partial(\bar{\rho} v)}{\partial x}\, dt = 0$$

where the first integral can be evaluated using the second fundamental theorem of calculus, sometimes referred to as the Newton–Leibniz axiom, and rearranged to obtain $\bar{\rho}_i$ at a later time

$$\bar{\rho}(x,t_2) = \bar{\rho}(x,t_1) - \int_{t_1}^{t_2} \frac{\partial(\bar{\rho}_i v_i)}{\partial x}\, dt$$

We can now calculate $\rho_i(t_2)$ as

$$\begin{aligned}
\bar{\rho}_i(t_2) &= \frac{1}{\Delta x_i} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left[\rho(x,t_1) - \int_{t_1}^{t_2} \frac{\partial(\rho v)}{\partial x}\, dt\right] dx \\
&= \frac{1}{\Delta x_i} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \rho(x,t_1)\, dx - \frac{1}{\Delta x_i} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{t_1}^{t_2} \frac{\partial(\rho v)}{\partial x}\, dt\, dx \\
&= \bar{\rho}_i(t_1) - \frac{1}{\Delta x_i} \int_{t_1}^{t_2} \left[\rho(x_{i+\frac{1}{2}},t) v(x_{i+\frac{1}{2}},t) - \rho(x_{i-\frac{1}{2}},t) v(x_{i-\frac{1}{2}},t)\right] dt \\
&= \bar{\rho}_i(t_1) - \frac{1}{\Delta x_i} \left[\int_{t_1}^{t_2} F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right] dt
\end{aligned}$$

which can be rearranged to write

$$\bar{\rho}_i(t_2) - \bar{\rho}_i(t_1) = \frac{d}{dt} \int_{t_1}^{t_2} \rho_i(t)\, dt = \int_{t_1}^{t_2} \left(-\frac{F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}}{\Delta x_i}\right) dt$$

where the integrands inside the two integrals must be the same so that

$$\frac{d\bar{\rho}_i}{dt} = -\frac{F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}}{\Delta x_i}$$

and if we approximate the time derivate by a first-order forward difference finite difference operator $\dot{\bar{\rho}}_i = (\bar{\rho}_i^{n+1} - \bar{\rho}_i^n)/\Delta t$ and further rearrange, we obtain

$$\bar{\rho}_i^{n+1} = \bar{\rho}_i^n - \frac{\Delta t}{\Delta x_i} \left(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right) \tag{5}$$