6.339: NUMERICAL METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS
PROJECT ONE: FINITE DIFFERENCE METHODS

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In this project, we will utilize finite difference methods to solve the two-dimensional time-dependent Euler equations, a set of quasilinear hyperbolic partial differential equations, for the pressure field of a fluid flowing around a small perturbative bump.

The Euler equations mathematically represent the conservation of mass and momentum for a fluid and can be used to describe the flow of an inviscid fluid. They can be written as

$$\frac{D\mathbf{u}}{Dt} = -\nabla w + \mathbf{g} \tag{1a}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{1b}$$

> *Question 1(a)—Derive the governing boundary condition for p' on boundary. The error in the boundary condition you derived should be the same as that in the linearized Euler equation.*

The bottom ($y = 0$) and top ($y = H$) boundary conditions read

$$\nabla p \cdot \mathbf{n} = -\rho \left(\frac{\partial^2 F}{\partial t^2} + 2u \frac{\partial^2 F}{\partial t \partial x} + u^2 \frac{\partial^2 F}{\partial^2 x} \right) \tag{2}$$

where $F(x, t)$ describes the geometry for the lower and upper walls and $\mathbf{n}$ is the normal vector pointing into the flow field such that $\mathbf{n} = (F_x, -1)$ for the top wall and $\mathbf{n} = (-F_x, 1)$ for the bottom wall.

Expanding the gradient term in (2) we obtain

$$\nabla p = \nabla(p_0 + p') = \cancelto{0}{\nabla p_0} + \nabla p' = \nabla p' \tag{3}$$

where $\nabla p_0 = 0$ as it represents the unperturbed pressure field and thus does not vary in time or space.

Question 1(b)—Derive a numerical scheme for the governing equation, using second-order finite-difference in space.

A second-order finite-difference approximation for the first derivative of an arbitrary real-valued and twice-differentiable function f on a grid would be

$$\frac{\partial f(x_i)}{\partial x} \approx \frac{f(x_{i+1}) - f(x_{i-1})}{2\Delta x} \tag{4}$$

while the second derivative can be approximated as

$$\frac{\partial^2 f(x_i)}{\partial x^2} \approx \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{(\Delta x)^2} \tag{5}$$

and together they can be used to discretize the governing Euler equations in space.

Taking a look at $p' = p'(x, y, t)$ first, which is the perturbed pressure field of the fluid flow, its governing equation can be written as

$$\frac{\partial p'}{\partial t} + \mathbf{u}_0 \cdot \nabla p' = q' \tag{6}$$

where $\mathbf{u}_0 = (u_0, v_0)$ is the unperturbed fluid flow velocity vector and

$$q' = \frac{Dp'}{Dt} \equiv \frac{\partial p'}{\partial t} + \mathbf{u}_0 \cdot \nabla p'$$

is the *material derivative* of p'. Expanding the dot product and gradient, (7) can be written as

$$\frac{\partial p'}{\partial t} = q' - u_0 \frac{\partial p'}{\partial x} - v_0 \frac{\partial p'}{\partial y} \tag{7}$$

where we haved moved the $\mathbf{u}_0 \nabla p'$ term to the right hand side. Now we can discretize the spatial derivatives using (4) to obtain

$$\boxed{\frac{dp'_{i,j}}{dt} = q'_{i,j} - u_0 \frac{p'_{i+1,j} - p'_{i-1,j}}{2\Delta x} - v_0 \frac{p'_{i,j+1} - p'_{i,j-1}}{2\Delta y}.} \tag{8}$$

The corresponding governing equation for q' is

$$\frac{\partial q'}{\partial t} + \mathbf{u}_0 \cdot \nabla q' = c_0^2 \nabla^2 p' \tag{9}$$

where c_0^2 is the unperturbed speed of sound in the fluid. Expanding the dot product and the Laplacian terms, we arrive at

$$\frac{\partial q'}{\partial t} = c_0^2 \left(\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} \right) - u_0 \frac{\partial q'}{\partial x} - v_0 \frac{\partial q'}{\partial y} \tag{10}$$

where again we have moved the $\mathbf{u}_0 \cdot \nabla q'$ term to the right hand side and which we can now discretize using (4) and (5) to obtain

$$\boxed{\begin{aligned} \frac{\partial q'_{i,j}}{\partial t} = c_0^2 &\left(\frac{p'_{i+1,j} - 2p'_{i,j} + p'_{i-1,j}}{(\Delta x)^2} + \frac{p'_{i,j+1} - 2p'_{i,j} + p'_{i,j-1}}{(\Delta y^2)} \right) \\ &\qquad - u_0 \frac{q'_{i+1,j} - q'_{i-1,j}}{2\Delta x} - v_0 \frac{q'_{i,j+1} - q'_{i,j-1}}{2\Delta y}. \end{aligned}} \tag{11}$$

Equations (8) and (11) each provide us with a set of $(N+1)(M+1)$ ordinary differential equations (ODEs) that may be integrated numerically to obtain the values of p' and q' at each grid point as a function of time, but we also require appropriate boundary conditions. Let us look at the boundary conditions for p' first. The left boundary condition reads $p = p_0$ at $x = 0$ but $p = p_0 + p'$ by definition so we must have that $p' = 0$ at $x = 0$ or

$$\boxed{p'_{0,j} = 0 \quad \text{for all } j} \tag{12}$$

The right boundary condition reads $\dfrac{dp'}{dx'} = 0$ which we can discretize using (4) to get

$$\frac{p'_{N+1,j} - p'_{N,j}}{2\Delta x} = 0 \tag{13}$$

or rather, that the boundary grid points must equal the ones to their left,

$$\boxed{p'_{N+1,j} = p'_{N,j} \quad \text{for all } j} \tag{14}$$

The bottom ($y = 0$) and top ($y = H$) boundary condition reads

$$\nabla p \cdot \mathbf{n} = -\rho \left(\frac{\partial^2 F}{\partial t^2} + 2u \frac{\partial^2 F}{\partial t \partial x} + u^2 \frac{\partial^2 F}{\partial^2 x} \right) \tag{15}$$

where $F(x, t)$ describes the geometry for the lower and upper walls and $\mathbf{n}$ is the normal vector pointing into the flow field such that $\mathbf{n} = (F_x, -1)$ for the top wall and $\mathbf{n} = (-F_x, 1)$ for the bottom wall.

Now taking a look at the boundary conditions for q', the condition on the left wall reads $q' = 0$ at $x = 0$ which may be discretized as

$$\boxed{q'_{0,j} = 0 \quad \text{for all } j} \tag{16}$$

Question 2(a)—On a bump described by function:

$$F(x) = 0.01 \left\{ 1 - \cos\left[\frac{8\pi}{L} \left(x - \frac{L}{4} \right) \right] \right\}^2$$

Plot the contour for p' at time $t = 2$ in your report (no figure generation in submitted code). Include the color-bar in the figure.

Question 3(a)—Estimate what the twist angle θ should be, so that the mesh aligns with the features of the solution.

Question 3(b)—Derive the governing equation on this new mesh.

To do so we will make a coordinate transformation from Cartesian coordinates (x, y) to the oblique coordinate system, which we will denote by $(\tilde{x}, \tilde{y})$. The transformation is given by

$$\tilde{x} = x - y \tan\theta \tag{17a}$$
$$\tilde{y} = y \sec\theta \tag{17b}$$

which will transform the differential operators according the chain rule,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \tilde{x}} \frac{\partial \tilde{x}}{\partial x} + \frac{\partial f}{\partial \tilde{y}} \frac{\partial \tilde{y}}{\partial x} = \frac{\partial f}{\partial \tilde{x}}$$
$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial \tilde{x}} \frac{\partial \tilde{x}}{\partial y} + \frac{\partial f}{\partial \tilde{y}} \frac{\partial \tilde{y}}{\partial y} = -\tan\theta \frac{\partial f}{\partial \tilde{x}} + \sec\theta \frac{\partial f}{\partial \tilde{y}}$$

so that they can be substituted by differential operators with respect to the oblique coordinate system

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial \tilde{x}} \tag{18a}$$
$$\frac{\partial}{\partial y} \to -\tan\theta \frac{\partial}{\partial \tilde{x}} + \sec\theta \frac{\partial}{\partial \tilde{y}} \tag{18b}$$

Question 3(c)—Derive a numerical scheme for the governing equation, using second-order finite-difference in space.

Same as question 2.