6.339: NUMERICAL METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS
PROJECT TWO: FINITE VOLUME METHODS

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In this project, we will utilize finite volume methods to study dense traffic flow and traffic jams modeled as shockwaves. We model traffic in each lane by a scalar hyperbolic conservation law, following what is known as the Lighthill-Whitman-Richards model.

We use a scalar hyperbolic conservation law to model traffic density $\rho^{(\ell)}(x,t)$ for n lanes indexed by $\ell = 1, 2, \dots, n$

$$\frac{\partial \rho^{(\ell)}}{\partial t} + \frac{\partial(\rho^{(\ell)} v^{(\ell)})}{\partial x} = s \tag{1}$$

where $v^{(\ell)}(x,t)$ is the average velocity of the cars. This, however, provides us with only one equation for two unknowns and thus we specify the velocity by

$$v(\rho) = v_{\text{max}} \left(1 - \frac{\rho^2}{\rho_{\text{max}}^2} \right) \tag{2}$$

giving us a traffic flux of

$$f(\rho) = \rho v = v_{\text{max}} \left(\rho - \frac{\rho^3}{\rho_{\text{max}}^2} \right) \tag{3}$$

The source term

$$s^{(\ell)} = \sum_{\substack{|k-\ell|=1 \\ 1 \le k,\ell \le n}} \alpha \left(\rho^{(k)} - \rho^{(\ell)} \right) \tag{4}$$

models the density of traffic that is switching lanes from neighboring lanes. α is the fraction of drivers that change lanes.

We will split up our one-dimensional grid into a number of cells indexed by $i = 1, 2, \dots, N$. We will index the edges of the cell i by $i - \frac{1}{2}$ for the left boundary of the cell, and by $i + \frac{1}{2}$ for the right boundary of the cell. So we can think of i as indexing the cell centers.

To derive a first-order conservative finite-volume scheme for a single lane, we will consider the volume averages of the traffic density $\rho(x,t)$ at two different times. The volume average of the traffic density at cell i, $\rho_i = \rho(x_i, t)$, at a time t_1 over $x \in \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]$ must exist by the mean value theorem and is given by

$$\bar{\rho}_i(t_1) = \frac{1}{\Delta x_i} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \rho(x, t_1)\; dx$$

and an identical expression can be written for the volume average at a later time $t_2 > t_1$. Now, integrating the scalar conservation law in time from $t = t_1$ to $t = t_2$ we can write

$$\int_{t_1}^{t_2} \frac{\partial \bar{\rho}}{\partial t}\; dt + \int_{t_1}^{t_2} \frac{\partial(\bar{\rho} v)}{\partial x}\; dt = 0$$

where the first integral can be evaluated using the second fundamental theorem of calculus, sometimes referred to as the Newton–Leibniz axiom, and rearranged to obtain $\bar{\rho}_i$ at a later time

$$\bar{\rho}(x, t_2) = \bar{\rho}(x, t_1) - \int_{t_1}^{t_2} \frac{\partial(\bar{\rho}_i v_i)}{\partial x}\; dt$$

We can now calculate $\rho_i(t_2)$ as

$$\begin{aligned}
\bar{\rho}_i(t_2) &= \frac{1}{\Delta x_i} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left[\rho(x, t_1) - \int_{t_1}^{t_2} \frac{\partial(\rho v)}{\partial x}\; dt\right]\; dx \\
&= \frac{1}{\Delta x_i} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \rho(x, t_1)\; dx - \frac{1}{\Delta x_i} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{t_1}^{t_2} \frac{\partial(\rho v)}{\partial x}\; dt\; dx \\
&= \bar{\rho}_i(t_1) - \frac{1}{\Delta x_i} \int_{t_1}^{t_2} \left[\rho(x_{i+\frac{1}{2}}, t) v(x_{i+\frac{1}{2}}, t) - \rho(x_{i-\frac{1}{2}}, t) v(x_{i-\frac{1}{2}}, t)\right]\; dt \\
&= \bar{\rho}_i(t_1) - \frac{1}{\Delta x_i} \left[\int_{t_1}^{t_2} F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right]\; dt
\end{aligned}$$

which can be rearranged to write

$$\bar{\rho}_i(t_2) - \bar{\rho}_i(t_1) = \frac{d}{dt} \int_{t_1}^{t_2} \rho_i(t)\; dt = \int_{t_1}^{t_2} \left(-\frac{F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}}{\Delta x_i}\right)\; dt$$

where the integrands inside the two integrals must be the same so that

$$\frac{d\bar{\rho}_i}{dt} = -\frac{F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}}{\Delta x_i}$$

and if we approximate the time derivate by a first-order forward difference finite difference operator $\dot{\bar{\rho}}_i = (\bar{\rho}_i^{n+1} - \bar{\rho}_i^n)/\Delta t$ and further rearrange, we obtain

$$\bar{\rho}_i^{n+1} = \bar{\rho}_i^n - \frac{\Delta t}{\Delta x_i}\left(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right) \tag{5}$$

> *Question 1(a)—For the numerical flux function use Godunov's scheme in which the flux is the exact solution to the Riemann problem at the interface between two volumes, i.e.,*

For the Gudonov scheme we will show that we can solve the Riemann problem exactly and without the use of a brute force search. We first notice that the flux function (3) is a cubic function, increasing monotonically until it attains a maximum value of $\rho_{\text{max}}/\sqrt{3}$ and then decreases monotonically. The maximum was found by setting the derivative of (3) to zero and solving for the value of ρ that maximizes $f(\rho)$:

$$\frac{df}{d\rho} = v_{\text{max}}\left(1 - \frac{3\rho^2}{\rho_{\text{max}}^2}\right) = 0 \quad \Longrightarrow \quad \rho = \frac{\rho_{\text{max}}}{\sqrt{3}}$$

Focusing on the case when $\rho_i < \rho_{i+1}$ first, we are interested in finding the minimum of $f(\rho)$. If

$$\min_{\rho\in[\rho_i,\rho_{i+1}]} f(\rho) = \begin{cases} f(\rho_i), & \text{if } \rho_i < \rho_{i+1} \leq \frac{\rho_{\text{max}}}{\sqrt{3}} \\ f(\rho_{i+1}), & \text{if } \frac{\rho_{\text{max}}}{\sqrt{3}} \leq \rho_i < \rho_{i+1} \\ \min\left\{f(\rho_i), f(\rho_{i+1})\right\}, & \text{if } \rho_i < \frac{\rho_{\text{max}}}{\sqrt{3}} < \rho_{i+1} \end{cases} \tag{6}$$

$$\max_{\rho\in[\rho_i,\rho_{i+1}]} f(\rho) = \begin{cases} f(\rho_i), & \text{if } \rho_{i+1} < \rho_i \leq \frac{\rho_{\text{max}}}{\sqrt{3}} \\ f(\rho_{i+1}), & \text{if } \frac{\rho_{\text{max}}}{\sqrt{3}} \leq \rho_{i+1} < \rho_i \\ f\left(\frac{\rho_{\text{max}}}{\sqrt{3}}\right), & \text{if } \rho_{i+1} < \frac{\rho_{\text{max}}}{\sqrt{3}} < \rho_i \end{cases} \tag{7}$$

Question 1(b)—Look at the problem of a traffic accident causing a lane ($x \in [0,10]$) to be blocked at time $t = 0$ and solve the continuity equation (1).

Use the following problem parameters: $\rho_{\max} = 1$, $v_{\max} = 1$. An accident happened at $t = 0$ at $x = 5$ and is cleared at $t = 1$. Due to the accident, the lane is completely blocked or the velocity is zero at $x = 5$. The initial condition is $\rho(x, t = 0) = \rho_0$. The boundary conditions, if applicable, are $\rho(0, t) = \rho_0$, $\rho(10, t) = \rho_0$. Consider two conditions: i. Light traffic: $\rho_0 = 0.2\rho_{\max}$, ii. Traffic jam: $\rho_0 = 0.8\rho_{\max}$.

Solve the PDE from $t = 0$ to $t = 2$. When are the specified boundary conditions not applicable? Why? How do you modify the set of boundary conditions? In what conditions can we prescribe a density on the left side of the boundary? In what conditions can we prescribe a density on the right side of the boundary? Why? Describe what happens to ρ and v as time evolves due to the blockage. For the traffic jam conditions, how is this related to the domino effect?

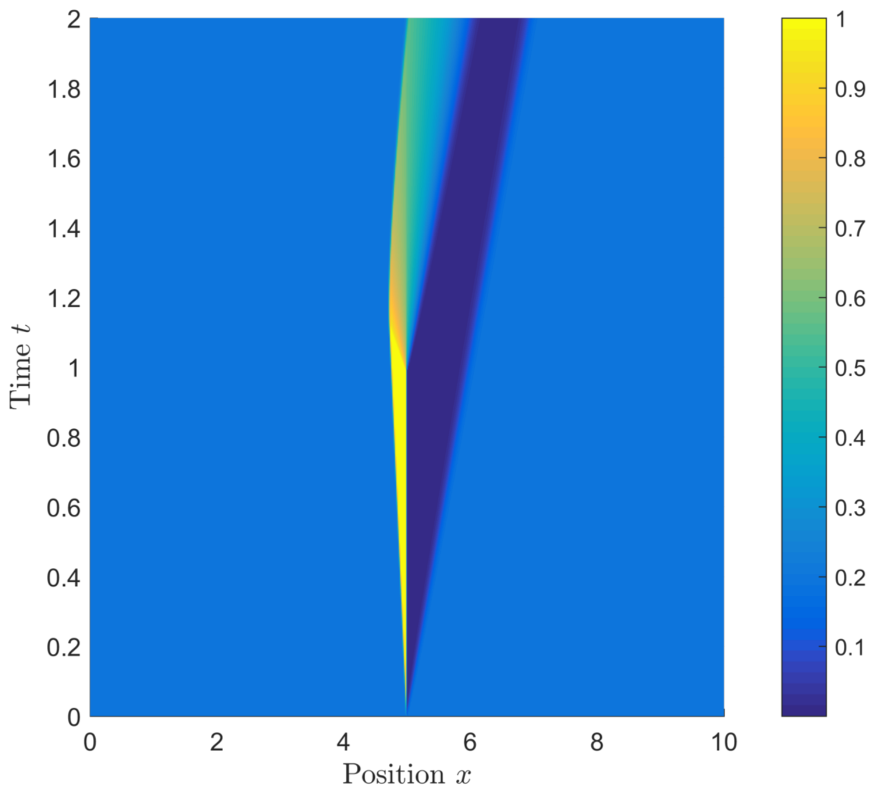


Figure 1: Heatmap of $\rho(x, t)$ for light traffic: $\rho(0, t) = 0.2\rho_{\max}$.

The minmod scheme $\phi(r) = \max\{0, \min\{r, 1\}\}$. Superbee $\phi(r) = \max\{0, \min\{2r, 1\}, \min\{r, 2\}\}$.

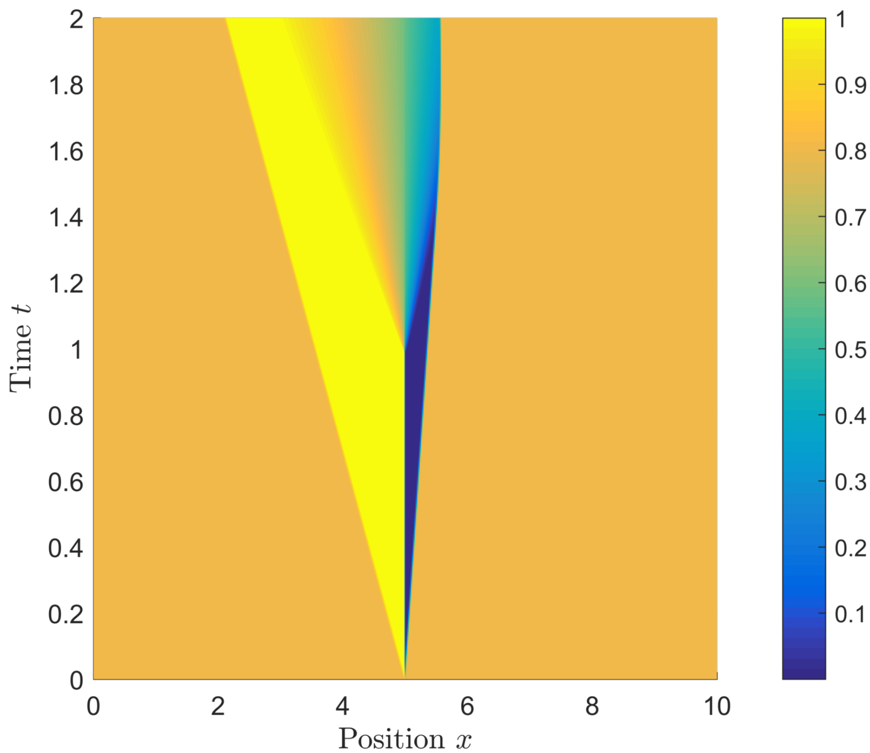
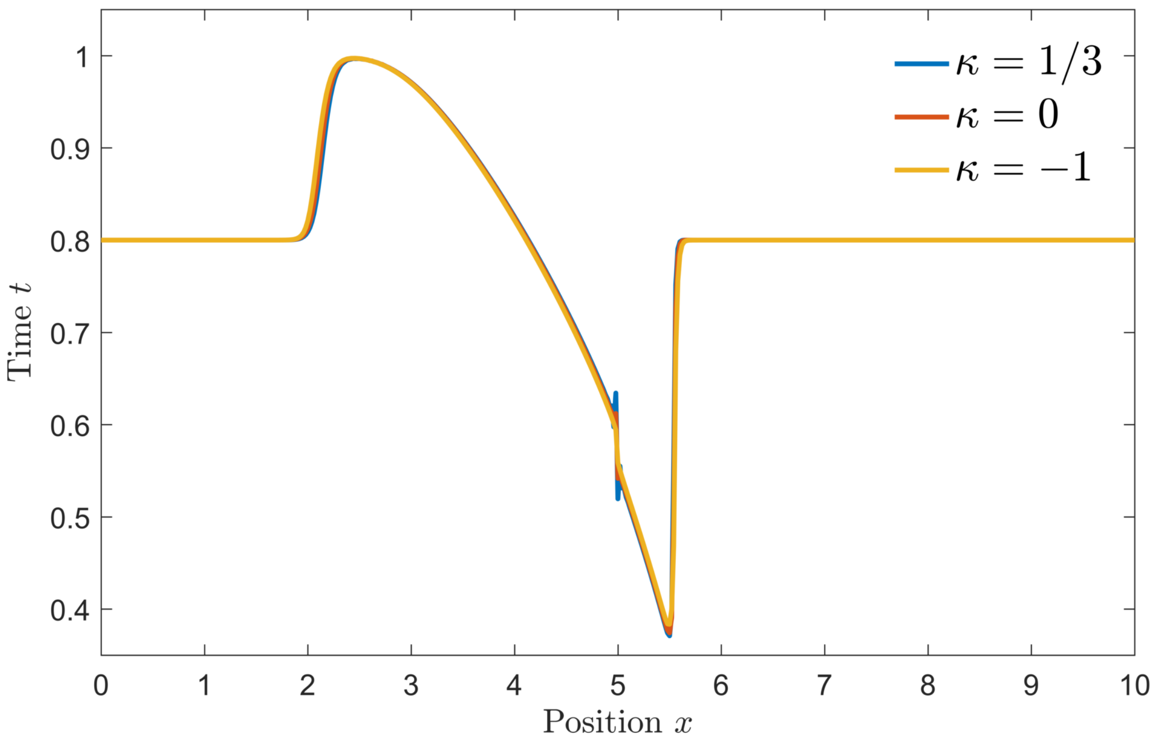


Figure 2: Heatmap of $\rho(x,t)$ for heavy traffic: $\rho(0,t) = 0.8\rho_{\max}$.

van Leer $\phi(r) = \dfrac{r + |r|}{1 + |r|}$.



$\kappa = 1/3$
$\kappa = 0$
$\kappa = -1$
Time t
Position x
1
0.9
0.8
0.7
0.6
0.5
0.4
0
1
2
3
4
5
6
7
8
9
10

Figure 3:

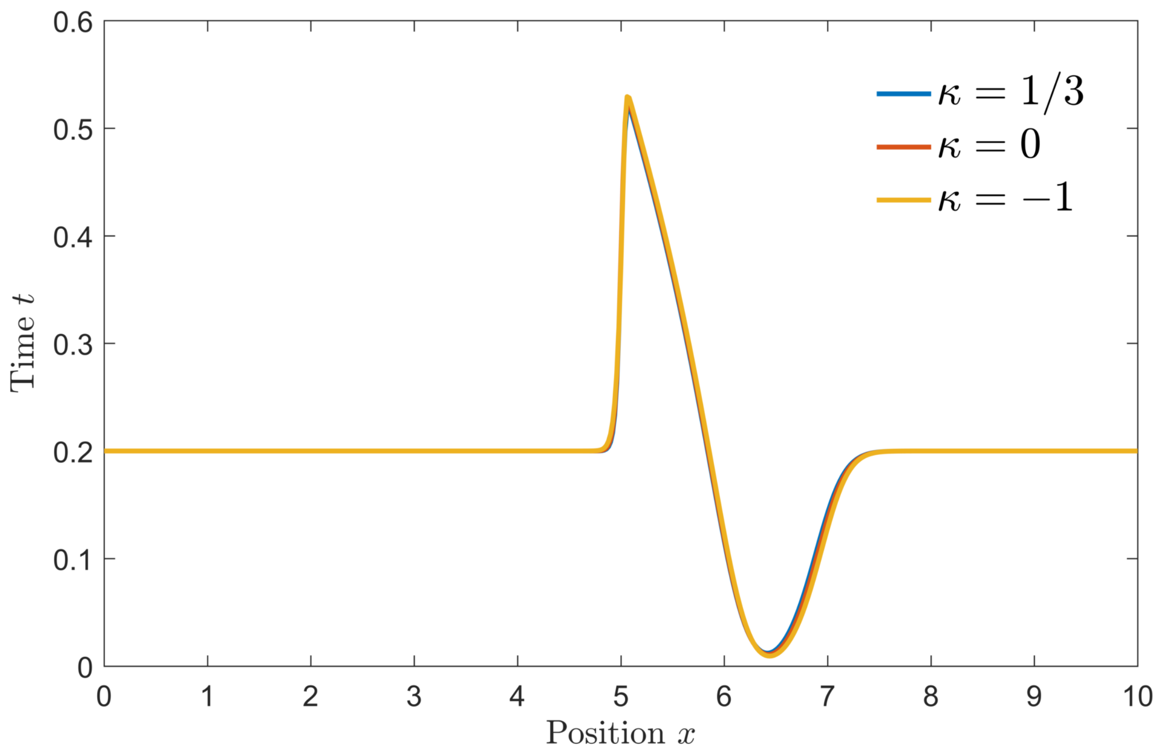

$\kappa = 1/3$
$\kappa = 0$
$\kappa = -1$
0.6
0.5
0.4
0.3
0.2
0.1
0
Time t
0
1
2
3
4
5
6
7
8
9
10
Position x

Figure 4:

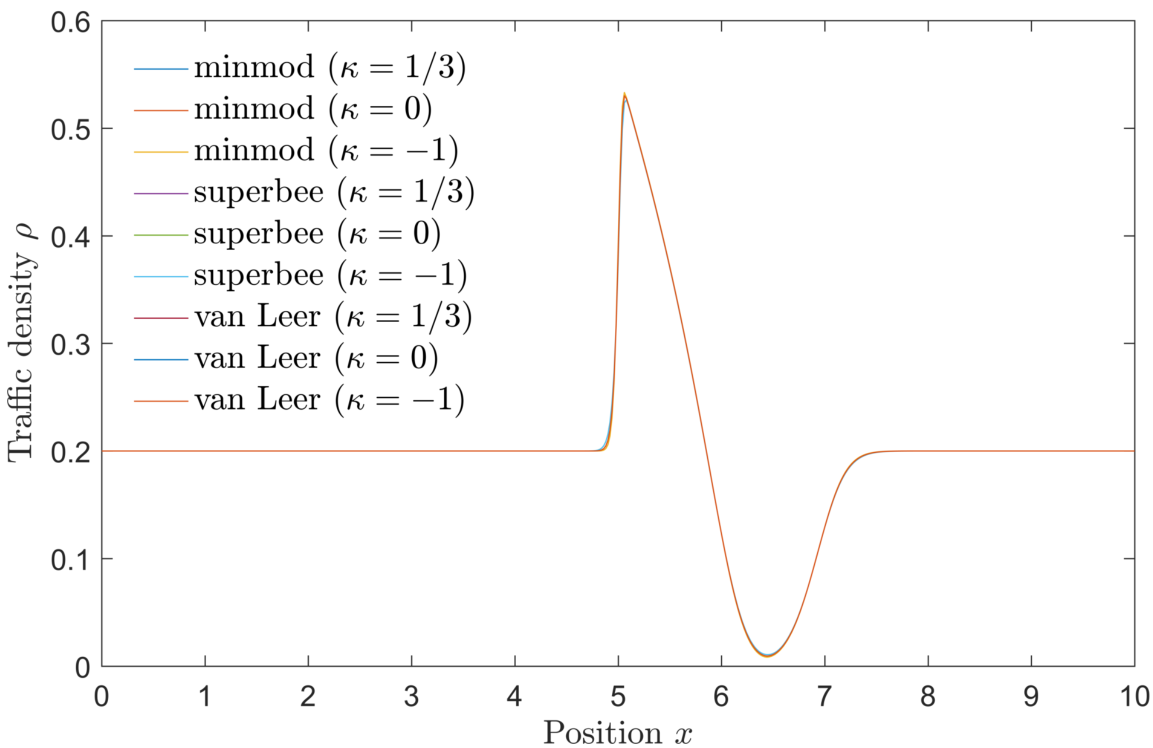

0.6
0.5
0.4
0.3
0.2
0.1
0
0 1 2 3 4 5 6 7 8 9 10
Traffic density ρ
Position x
minmod ($\kappa = 1/3$)
minmod ($\kappa = 0$)
minmod ($\kappa = -1$)
superbee ($\kappa = 1/3$)
superbee ($\kappa = 0$)
superbee ($\kappa = -1$)
van Leer ($\kappa = 1/3$)
van Leer ($\kappa = 0$)
van Leer ($\kappa = -1$)

Figure 5: