Problem 1:

To compute the intersection, let's solve the system of 2 linear equations with 3 unknowns:

$$\begin{cases} (a) & x + 2y - z = 0 \\ (b) & 3x - 3y + z = 0 \end{cases}$$

This translates to the following augmented matrix: $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 3 & -3 & 1 & 0 \end{array}\right]$

Let's find the reduced row echelon form:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 3 & -3 & 1 & 0 \end{array}\right] \xrightarrow{R_2 : R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -9 & 4 & 0 \end{array}\right] \xrightarrow{R_2 : \frac{-R_2}{9}} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & \frac{-4}{9} & 0 \end{array}\right] \ldots$$

$$\ldots \xrightarrow{R_1 : R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & \frac{-1}{9} & 0 \\ 0 & 1 & \frac{-4}{9} & 0 \end{array}\right] \#\text{rref}$$

The rref corresponds to the system: $\begin{cases} x - \frac{1}{9}z = 0 \\ y - \frac{4}{9}z = 0 \end{cases}$

and z is a free variable. Let's put $z = \alpha$ for some $\alpha \in \mathbb{R}$.

Therefore the solution to the system is:

$$x = \frac{1}{9}\alpha \ , \ y = \frac{4}{9}\alpha \ , \ z = \alpha. \quad (\forall \alpha \in \mathbb{R})$$

Since the 2 planes are not identical, there intersection is a line that passes through the set of vectors

$$\left\{ \alpha \begin{bmatrix} 1/9 \\ 4/9 \\ 1 \end{bmatrix} \right\} (\forall \alpha \in \mathbb{R}).$$

Problem 2:

- Since matrix A has only one special solution, the dimension of the nullspace of A is 1, and the rank of A is $n - \dim(N(A))$ where n is the number of columns of A.

Therefore $\text{rank}(A) = 4-1 = 3$.

The nullspace of A is given by the set of combinations αs where $\alpha \in \mathbb{R}$ and s is the special solution to $Ax=0$.

Therefore the complete solution to $Ax=0$ is $x = \alpha s = \alpha \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$.

- Since $\text{rank}(A) = 3$ and $\dim(N(A)) = 1$, the row reduced form of A must have 3 pivot columns and 1 free column.

The free column in $\text{rref}(A)$ is $-1 \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ and its position is in column 3 because the free variable in $s = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$ is the third variable s_3.

$$R = RREF(A) = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This can be verified as $\underbrace{\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}}_{s} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{0}}$

- Matrix A has $m=3$ rows and $n=4$ columns. Also we have $\text{rank}(A) = m$ and $\text{rank}(A) < n$. Therefore there is infinitely many solutions to the system $Ax=b$ and hence it can be solved for every vector b.

Also to note that $b \in C(A)$ and $b \in \mathbb{R}^3$ while $C(A)$ is actually all $\mathbb{R}^3$.

Problem 3:

To prove that S is linearly independent if and only if the set S' is linearly independent, let's show that:

① S is linearly independent $\Longrightarrow$ S' is linearly independent;

② S' is linearly independent $\Longrightarrow$ S is linearly independent.

① Suppose $S = \{u_1, u_2, \dots u_n\}$ is linearly independent.

$$c_1 u_1 + c_2 (u_1 + u_2) + \dots + c_{n-1}(u_1 + u_2 + \dots + u_{n-1}) + c_n (u_1 + u_2 + \dots + u_n) = 0$$

$$\Rightarrow (c_1 + c_2 + \dots + c_{n-1} + c_n) u_1 + (c_2 + \dots + c_{n-1} + c_n) u_2 + \dots$$
$$\dots + (c_{n-1} + c_n) u_{n-1} + c_n u_n = 0$$

$$\Rightarrow a u_1 + b u_2 + \dots + c u_{n-1} + d u_n = 0$$

with $a = c_1 + \dots + c_n$, $b = c_2 + \dots + c_n$, $c = c_{n-1} + c_n$, $d = c_n$

$\Rightarrow a = b = \dots = c = d = 0$, since $u_1, u_2, \dots, u_n$ are linearly independent.

$\Rightarrow c_1 = c_2 = \dots = c_{n-1} = c_n = 0$

$\Rightarrow u_1, u_1 + u_2, \dots, u_1 + \dots + u_n$ are linearly independent.

Hence $S' = \{u_1, \sum_{i=1}^{2} u_i, \dots, \sum_{i=1}^{n} u_i\}$ is linearly independent.

② Suppose $S' = \{u_1, \sum_{i=1}^{2} u_i, \dots, \sum_{i=1}^{n} u_n\}$ is linearly independent.

Therefore $c_1 u_1 + c_2 (u_1 + u_2) + \dots + c_n (u_1 + u_2 + \dots + u_n) = 0$

$$\Rightarrow c_1 = c_2 = \dots = c_n = 0$$

Also $c_1 u_1 + c_2 (u_1 + u_2) + \dots + c_n (u_1 + u_2 + \dots + u_n) = 0$

$\Rightarrow (c_1 + c_2 + \dots + c_n) u_1 + (c_2 + c_3 + \dots + c_n) u_2 + \dots + c_n u_n = 0$

$\Rightarrow c_1 + c_2 + \dots + c_n = 0$, $c_2 + c_3 + \dots + c_n = 0, \dots$, $c_n = 0$

because $c_1 = c_2 = \dots = c_n = 0$

$\Rightarrow u_1, u_2, \dots, u_n$ are linearly independent.

Hence $S = \{u_1, u_2, \dots, u_n\}$ is linearly independent.

\# From ① & ②, we conclude that S is linearly independent $\Longleftrightarrow$ S' is linearly independent.

Problem 4:

- We know that in a $m \times n$ Matrix, the maximum number of pivots is m. Since C is a 3×4 matrix, there is is at least 1 ("4-3") free column. Therefore $Cx=0$ has a non-zero solution at least. Let k be this solution. Therefore $k \in N(C)$ and $k \neq \vec{0}$.

$$k \in N(C) \Rightarrow Ck = 0$$
$$\Rightarrow B(Ck) = 0$$
$$\Rightarrow (BC)k = 0$$
$$\Rightarrow k \in N(BC)$$
$$\Rightarrow k \in N(A) \quad (\text{because } \forall v \in \mathbb{R}^4 : Av = B(Cv))$$

Therefore $N(A)$ contains at least a non zero vector. This also means that the dimension of $C(A)$ is $\leqslant 3$ since $\dim(N(A)) + \dim(C(A)) = n$ and A is 4×4 matrix.

Let's assume $C(A)$ has dimension 3, then 3 independent vectors in $\mathbb{R}^4$ form the basis of $C(A)$. These 3 vectors in $\mathbb{R}^4$ do not span the whole $\mathbb{R}^4$ but rather a 3 dimensional subspace in $\mathbb{R}^4$. Therefore there are vectors in $\mathbb{R}^4$ which are not in $C(A)$