CpE 520: HW #1

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Problem 1

Part One: X_1

$$X_1 = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

Rank=2:
X_1's column space has a dimension of 2. (The third column is a linear combination of the first and second columns)

SVD:
Singular value decomposition is like:

$$\boxed{X_{1_{2\times 3}} = U_{1_{2\times 2}} \Sigma_{1_{2\times 3}} V^T_{1_{3\times 3}}}$$

We know that $X_1^T X_1 = V \Sigma^T \Sigma V^T$ so:

$$X_1^T X_1 = \begin{bmatrix} 5 & 4 & 7 \\ 4 & 5 & 5 \\ 7 & 5 & 10 \end{bmatrix}$$

$\Sigma^T \Sigma$ is actually a diagonal matrix of eigenvalues of $X_1^T X_1$ which are also equal to the squared singular values of X_1:

$$det(X_1^T X_1 - \lambda I) = -\lambda(\lambda^2 - 20\lambda + 35) = 0$$
$$\lambda_1 = 18.0623$$
$$\lambda_2 = 1.9377$$
$$\lambda_3 = 0$$

Now we find the eigenvector corresponding to each eigenvalue using the row elimination to find the null space for $(X_1^T X_1 - \lambda_i I) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$:

- $\boldsymbol{\lambda_1 = 18.0623}$

$$\begin{bmatrix} 1 & 0 & -0.7207 \\ 0 & 1 & -0.6035 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

by choosing $x_3 = 1$ we have:

$$v_1 = \begin{bmatrix} 0.7207 \\ 0.6035 \\ 1 \end{bmatrix}$$

- $\boldsymbol{\lambda_2 = 1.9377}$

$$\begin{bmatrix} 1 & 0 & -0.2168 \\ 0 & 1 & 1.916 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

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by choosing $x_3 = 1$ we have:

$$v_2 = \begin{bmatrix} 0.2168 \\ -1.916 \\ 1 \end{bmatrix}$$

- $\boldsymbol{\lambda_3 = 0}$

$$\begin{bmatrix} 1 & 0 & 1.6667 \\ 0 & 1 & -0.3333 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

by choosing $x_3 = 1$ we have:

$$v_3 = \begin{bmatrix} -1.6667 \\ 0.3333 \\ 1 \end{bmatrix}$$

then we normalize each eigenvector to make the V matrix orthogonal:

$$V = \begin{bmatrix} 0.5251 & 0.0998 & -0.8452 \\ 0.4397 & -0.8821 & 0.169 \\ 0.7286 & 0.4604 & 0.5071 \end{bmatrix}$$

To find U matrix we use the equation $X_1 V = U\Sigma$ and this equation in vector form could be used more effectively as $u_i = \frac{1}{\sigma_i} X_1 v_i$ (except for the $\sigma_i = 0$), noting that normalizing again is necessary for u_i:

$$u_1 = \begin{bmatrix} 0.8649 \\ 0.5019 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 0.5019 \\ -0.8649 \end{bmatrix}$$

Finally:

$$X_1 = \begin{bmatrix} 0.8649 & 0.5019 \\ 0.5019 & -0.8649 \end{bmatrix} \begin{bmatrix} 4.25 & 0 & 0 \\ 0 & 1.392 & 0 \end{bmatrix} \begin{bmatrix} 0.5251 & 0.4397 & 0.7286 \\ 0.0998 & -0.8821 & 0.4604 \\ -0.8452 & 0.169 & 0.5071 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

Part Two: X_2

$$X_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 3 & 1 \end{bmatrix}$$

Rank = :
X_2 is the transpose of X_1 so its rank is equivalently 2.

SVD:

$$\boxed{X_{2_{3\times 2}} = U_{2_{3\times 3}} \Sigma_{2_{3\times 2}} V^T_{2_{2\times 2}}}$$

To find the svd of X_2 we could use the same procedure of **Part One** but it is obvious that X_2 is the transpose of X_1, so we can do some calculations:

$$X_1 = U_1 \Sigma_1 V_1^T$$
$$X_2 = U_2 \Sigma_2 V_2^T$$

$$\begin{aligned} X_2 = X_1^T &= (U_1 \Sigma_1 V_1^T)^T \\ &= V_1 \Sigma_1^T U_1^T \end{aligned}$$

$$X_2 = \begin{bmatrix} 0.5251 & 0.0998 & -0.8452 \\ 0.4397 & -0.8821 & 0.169 \\ 0.7286 & 0.4604 & 0.5071 \end{bmatrix} \begin{bmatrix} 4.25 & 0 \\ 0 & 1.392 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.8649 & 0.5019 \\ 0.5019 & -0.8649 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 3 & 1 \end{bmatrix}$$

Part Three: X_3

$$X_3 = \begin{bmatrix} 1 & 3 & 2 & 5 \\ 2 & 5 & 3 & 9 \\ 2 & 1 & -1 & 5 \\ 3 & 2 & -1 & 8 \\ 1 & 1 & 0 & 3 \end{bmatrix}$$

Rank = 2:
X_3's column space consist of the 1^{st} and 3^{rd} columns only. (2^{nd} and 4^{th} columns are linear combinations of them)

SVD:

$$\boxed{X_{3_{5\times4}} = U_{3_{5\times5}} \Sigma_{3_{5\times4}} V_{3_{4\times4}}^T}$$

By using the same algorithm as in **Part One** we can calculate SVD for X_3. We know that $X_3^T X_3 = V \Sigma^T \Sigma V^T$ so:

$$X_3^T X_3 = \begin{bmatrix} 19 & 22 & 3 & 60 \\ 22 & 40 & 18 & 84 \\ 3 & 18 & 15 & 24 \\ 60 & 84 & 24 & 204 \end{bmatrix}$$

$\Sigma^T \Sigma$ is actually a diagonal matrix of eigenvalues of $X_3^T X_3$ which are also equal to the squared singular values of X_3:

$$det(X_3^T X_3 - \lambda I) = \lambda^2(\lambda^2 - 278\lambda + 4692) = 0$$
$$\lambda_1 = 259.9504$$
$$\lambda_2 = 18.0496$$
$$\lambda_3 = 0$$
$$\lambda_4 = 0$$

Now we find the eigenvector corresponding to each eigenvalue using the row elimination to find the null space for $(X_3^T X_3 - \lambda_i I) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$:

- $\boldsymbol{\lambda_1 = 259.9504}$

$$\begin{bmatrix}1 & 0 & 0 & -0.2892\\ 0 & 1 & 0 & -0.4217\\ 0 & 0 & 1 & -0.1325\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = 0$$

by choosing $x_4 = 1$ we have:

$$v_1 = \begin{bmatrix}0.2892\\ 0.4217\\ 0.1325\\ 1\end{bmatrix}$$

- $\boldsymbol{\lambda_2 = 18.0496}$

$$\begin{bmatrix}1 & 0 & 0 & -1.6331\\ 0 & 1 & 0 & 2.2661\\ 0 & 0 & 1 & 3.8992\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = 0$$

by choosing $x_4 = 1$ we have:

$$v_2 = \begin{bmatrix}1.6331\\ -2.2661\\ -3.8992\\ 1\end{bmatrix}$$

- $\boldsymbol{\lambda_{3,4} = 0}$

For distinct eigenvalues of a symmetric real-valued matrix we could be sure about orthogonality of their corresponding eigenvectors. So to find two orthogonal eigenvectors for two eigenvalues $\lambda_3 = 0$ and $\lambda_3 = 0$ their orthogonality should be under consideration.

$$\begin{bmatrix}1 & 0 & -1 & 2\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = 0$$

we now assume to eigenvectors:

$$v3 = \begin{bmatrix}x_3 - 2x_4\\ -x_3 - x_4\\ x_3\\ x_4\end{bmatrix} v4 = \begin{bmatrix}x'_3 - 2x'_4\\ -x'_3 - x'_4\\ x'_3\\ x'_4\end{bmatrix}$$

their dot product must be zero:

$$v_3^T v_4 = 3x_3x'_3 + 6x_4x'_4 - x_3x'_4 - x_4x'_3 = 0 \tag{*}$$

by choosing $x_3 = 0$ and $x_4 = 1$ we have:

$$v_3 = \begin{bmatrix}-2\\ -1\\ 0\\ 1\end{bmatrix}$$

now we are forced to choose x'_3 and x'_4 accordingly to hold the equation (*) true. So we must choose $x'_3 = 6$ and $x'_4 = 1$

$$v_4 = \begin{bmatrix} 4 \\ -7 \\ 6 \\ 1 \end{bmatrix}$$

then we normalize each eigenvector to make the V matrix orthogonal:

$$V = \begin{bmatrix} 0.2557 & 0.333 & -0.8165 & 0.3961 \\ 0.3728 & -0.4625 & -0.4082 & -0.6931 \\ 0.1172 & -0.7958 & 0 & 0.5941 \\ 0.8842 & 0.2042 & 0.4082 & 0.099 \end{bmatrix}$$

To find U matrix we use the equation $X_3X_3^T = U\Sigma\Sigma^TU^T$ and follow the same procedure that were used to find V, here only the final solutions for vectors of U are mentioned:

$$u_1 = \begin{bmatrix} 0.374 \\ 0.6627 \\ 0.3218 \\ 0.5253 \\ 0.2035 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} -0.3826 \\ -0.517 \\ 0.4756 \\ 0.5893 \\ 0.1137 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} 0.8018 \\ -0.5345 \\ 0 \\ 0 \\ 0.2673 \end{bmatrix}$$

$$u_4 = \begin{bmatrix} -0.0344 \\ -0.0516 \\ -0.8087 \\ 0.585 \\ 0 \end{bmatrix}$$

$$u_5 = \begin{bmatrix} 0.8435 \\ -0.5336 \\ 0.0351 \\ 0.0511 \\ 0 \end{bmatrix}$$

Finally:

$$X_3 = \begin{bmatrix} 0.374 & -0.3826 & 0.8018 & -0.0344 & 0.8435 \\ 0.6627 & -0.517 & -0.5345 & -0.0516 & -0.5336 \\ 0.3218 & 0.4756 & 0 & -0.8087 & 0.0352 \\ 0.5253 & 0.5893 & 0 & 0.585 & 0.0511 \\ 0.2035 & 0.1137 & 0.2673 & 0 & 0 \end{bmatrix} \begin{bmatrix} 16.123 & 0 & 0 & 0 \\ 0 & 4.2485 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} 0.2557 & 0.3729 & 0.1172 & 0.8842 \\ 0.3333 & -0.4625 & -0.7958 & 0.2041 \\ -0.8165 & -0.4082 & 0 & 0.4082 \\ 0.3961 & -0.6931 & 0.5941 & 0.099 \end{bmatrix} \approx \begin{bmatrix} 1 & 3 & 2 & 5 \\ 2 & 5 & 3 & 9 \\ 2 & 1 & -1 & 5 \\ 3 & 2 & -1 & 8 \\ 1 & 1 & 0 & 3 \end{bmatrix}$$

Problem 2

Part One: $SVD\{X_1X_2\}$

$$\begin{aligned} X1X2 &= (U_1\Sigma_1V_1^T)(U_2\Sigma_2V_2^T) \\ &= (U_1\Sigma_1V_1^T)(V_1\Sigma_1^TU_1^T) \\ &= (U_1\Sigma_1)(V_1^TV_1)(\Sigma_1^TU_1^T) \\ &= U_1(\Sigma_1\Sigma_1^T)U_1^T \end{aligned}$$

$$X_1X_2 = \begin{bmatrix} 0.8649 & 0.5019 \\ 0.5019 & -0.8649 \end{bmatrix} \begin{bmatrix} 18.0623 & 0 \\ 0 & 1.9377 \end{bmatrix} \begin{bmatrix} 0.8649 & 0.5019 \\ 0.5019 & -0.8649 \end{bmatrix}$$

Part Two: $SVD\{X_2X_1\}$

$$\begin{aligned} X2X1 &= (U_2\Sigma_2V_2^T)(U_1\Sigma_1V_1^T) \\ &= (V_1\Sigma_1^TU_1^T)(U_1\Sigma_1V_1^T) \\ &= (V_1\Sigma_1^T)(U_1^TU_1)(\Sigma_1V_1^T) \\ &= V_1(\Sigma_1^T\Sigma_1)V_1^T \end{aligned}$$

$$X_2X_1 = \begin{bmatrix} 0.5251 & 0.0998 & -0.8452 \\ 0.4397 & -0.8821 & 0.169 \\ 0.7286 & 0.4604 & 0.5071 \end{bmatrix} \begin{bmatrix} 18.0623 & 0 & 0 \\ 0 & 1.9377 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.5251 & 0.4397 & 0.7286 \\ 0.0998 & -0.8821 & 0.4604 \\ -0.8452 & 0.169 & 0.5071 \end{bmatrix}$$

Part Three: $SVD\{X_3X_3^T\}$

$$\begin{aligned} X_3X_3^T &= (U_3\Sigma_3V_3^T)(U_3\Sigma_3V_3^T)^T \\ &= (U_3\Sigma_3V_3^T)(V_3\Sigma_3^TU_3^T) \\ &= (U_3\Sigma_3)(V_3^TV_3)(\Sigma_3^TU_3^T) \\ &= U_3(\Sigma_3\Sigma_3^T)U_3^T \end{aligned}$$

$$\begin{aligned} X_3X_3^T = &\begin{bmatrix} 0.374 & -0.3826 & 0.8018 & -0.0344 & 0.8435 \\ 0.6627 & -0.517 & -0.5345 & -0.0516 & -0.5336 \\ 0.3218 & 0.4756 & 0 & -0.8087 & 0.0352 \\ 0.5253 & 0.5893 & 0 & 0.585 & 0.0511 \\ 0.2035 & 0.1137 & 0.2673 & 0 & 0 \end{bmatrix} \begin{bmatrix} 259.9504 & 0 & 0 & 0 & 0 \\ 0 & 18.0496 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &\times \begin{bmatrix} 0.374 & 0.6627 & 0.3218 & 0.5253 & 0.2035 \\ -0.3826 & -0.517 & 0.4756 & 0.5893 & 0.1137 \\ 0.8018 & -0.5345 & 0 & 0 & 0.2673 \\ -0.0344 & -0.0516 & -0.8087 & 0.585 & 0 \\ 0.8435 & -0.5336 & 0.0352 & 0.0511 & 0 \end{bmatrix} \end{aligned}$$

Part Four: $SVD\{X_3^T X_3\}$

$$\begin{aligned} X_3^T X_3 &= (U_3 \Sigma_3 V_3^T)^T (U_3 \Sigma_3 V_3^T) \\ &= (V_3 \Sigma_3^T U_3^T)(U_3 \Sigma_3 V_3^T) \\ &= (V_3 \Sigma_3^T)(U_3^T U_3)(\Sigma_3 V_3^T) \\ &= V_3 (\Sigma_3^T \Sigma_3) V_3^T \end{aligned}$$

$$\begin{aligned} X_3^T X_3 = &\begin{bmatrix} 0.2557 & 0.3333 & -0.8165 & 0.3961 \\ 0.3729 & -0.4625 & -0.4082 & -0.6931 \\ 0.1172 & -0.4082 & 0 & 0.5941 \\ 0.8842 & -0.6931 & 0.4082 & 0.099 \end{bmatrix} \begin{bmatrix} 259.9504 & 0 & 0 & 0 \\ 0 & 18.0496 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \times &\begin{bmatrix} 0.2557 & 0.3729 & 0.1172 & 0.8842 \\ 0.3333 & -0.4625 & -0.7958 & 0.2041 \\ -0.8165 & -0.4082 & 0 & 0.4082 \\ 0.3961 & -0.6931 & 0.5941 & 0.099 \end{bmatrix} \end{aligned}$$

Problem 3

Part One: X_1^{-1}

X_1 will have a pseudo-inverse as:

$$
\begin{aligned}
X_1^{-1} &= (U_1 \Sigma_1 V_1^T)^{-1} \\
&= V_1 \Sigma_1 U_1^{-1} \\
&= V_1 \Sigma_1^{-1} U_1^T
\end{aligned}
$$

$$
X_1^{-1} = \begin{bmatrix} 0.5251 & 0.0998 & -0.8452 \\ 0.4397 & -0.8821 & 0.169 \\ 0.7286 & 0.4604 & 0.5071 \end{bmatrix} \begin{bmatrix} 0.2353 & 0 \\ 0 & 0.7184 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.8649 & 0.5019 \\ 0.5019 & -0.8649 \end{bmatrix}
$$

this inverse only works as $X_1 X_1^{-1} = I$ because X_1 is a rectangular matrix.

Part Two: X_2^{-1}

X_2 will have a pseudo-inverse as:

$$
\begin{aligned}
X_2^{-1} &= (U_2 \Sigma_2 V_2^T)^{-1} \\
&= (V_1 \Sigma_1^T U_1^T)^{-1} \\
&= U_1 (\Sigma_1^T)^{-1} V_1^T
\end{aligned}
$$

$$
X_2^{-1} = \begin{bmatrix} 0.8649 & 0.5019 \\ 0.5019 & -0.8649 \end{bmatrix} \begin{bmatrix} 0.2353 & 0 & 0 \\ 0 & 0.7184 & 0 \end{bmatrix} \begin{bmatrix} 0.5251 & 0.4397 & 0.7286 \\ 0.0998 & -0.8821 & 0.4604 \\ -0.8452 & 0.169 & 0.5071 \end{bmatrix}
$$

this inverse only works as $X_2^{-1} X_2 = I$ because X_2 is a rectangular matrix.

Part Three: $(X_1 X_2)^{-1}$

From **Part One** of **Problem 2** we know that $X_1 X_2 = U_1 (\Sigma_1 \Sigma_1^T) U_1^T$ so:

$$
\begin{aligned}
(X_1 X_2)^{-1} &= (U_1 (\Sigma_1 \Sigma_1^T) U_1^T)^{-1} \\
&= (U_1^T)^{-1} (\Sigma_1 \Sigma_1^T)^{-1} U_1^{-1} \\
&= U_1 (\Sigma_1 \Sigma_1^T)^{-1} U_1^T
\end{aligned}
$$

$$
(X_1 X_2)^{-1} = \begin{bmatrix} 0.8649 & 0.5019 \\ 0.5019 & -0.8649 \end{bmatrix} \begin{bmatrix} 0.0554 & 0 \\ 0 & 0.5161 \end{bmatrix} \begin{bmatrix} 0.8649 & 0.5019 \\ 0.5019 & -0.8649 \end{bmatrix}
$$

$X_1 X_2$ is a square matrix with no zero in its singular values. So it has an inverse as written above.

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Part Four: $(X_2X_1)^{-1}$

X_2X_1 has zero in its singular values on the diagonal elements of its Σ so it won't have an inverse.

Part Five: X_3^{-1}

X_3 has zero in its singular values on the diagonal elements of its Σ so it won't have an inverse.