CpE 520: HW #2

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Question 1

Part (a): Inverse of $\mathbf{y_{n\times 1} = A_{n\times m}x_{m\times 1}}$ where n¡m and rank=n

In this case there are more unknowns than equations so the system is underdetermined. Because the matrix A is full rank with $rank = n$, AA^T will be an $n \times n$ non-singular square matrix. Therefore, there will be an inverse for it like below:

$$(AA^T)(AA^T)^{-1} = I \tag{*}$$

As it is clear from the above equation, we may use $A^T(AA^T)^{-1}$ as a right inverse for matrix A. We first assume that the inverse we are seeking for has an equation like $\mathbf{x_{m\times 1} = R_{m\times n}y_{n\times 1}}$. Finding $R_{m\times n}$ will be the goal. We propose $R_{m\times n} = A^T(AA^T)^{-1}$ and through the equations below we will check if it will result in our main equation:

$$\boxed{\mathbf{x= A^T(AA^T)^{-1}y}}$$

$$\begin{aligned}
&\mathbf{x= A^T(AA^T)^{-1}y}\\
\xrightarrow{\times A}\ &\mathbf{Ax = AA^T(AA^T)^{-1}y}\\
\xrightarrow{\text{using } *}\ &\mathbf{Ax = y}
\end{aligned}$$

Part (b): Inverse of $\mathbf{y_{n\times 1} = A_{n\times m}x_{m\times 1}}$ where n¿m and rank=m

In this case there are more equations than unknowns, so the system is overdetermined. Because the matrix A is full rank with $rank = m$, A^TA will be an $m \times m$ non-singular square matrix. Therefore, there will be an inverse for it like below:

$$(A^TA)^{-1}(A^TA) = I \tag{**}$$

As it is clear from the above equation, we may use $(A^TA)^{-1}A^T$ as a left inverse for matrix A.

$$\begin{aligned}
&\mathbf{Ax= y}\\
\xrightarrow{\times A^T}\ &\mathbf{A^TAx= A^Ty}\\
\xrightarrow[\text{using } *]{\times (A^TA)^{-1}}\ &\mathbf{x= (A^TA)^{-1}A^Ty}
\end{aligned}$$

$$\boxed{\mathbf{x= (A^TA)^{-1}A^Ty}}$$

Question 2

Part (a): Minimizing $\tilde{\mathbf{x}} = \min_{\mathbf{x}}\{\|\mathbf{x}\|_2^2 + \frac{1}{2}\lambda^T(\mathbf{y} - \mathbf{Ax})\}$ to find the solution of $\mathbf{y_{n\times 1}} = \mathbf{A_{n\times m}x_{m\times 1}}$ where n¡m and rank=n

(Underdetermined System)
We define the function $J(x, \lambda)$ as:

$$J(x,\lambda) = \|x\|_2^2 + \frac{1}{2}\lambda^T(y - Ax)$$

now we find its partial derivatives and let them be zero:

$$\frac{\partial J}{\partial x} = 2x - \frac{1}{2}A^T\lambda = 0 \to x = \frac{1}{4}A^T\lambda \tag{1}$$
$$\frac{\partial J}{\partial \lambda} = \frac{1}{2}(y - Ax) = 0 \to y = Ax \tag{2}$$

by putting equation (1) into (2):

$$y = \frac{1}{4}AA^T\lambda$$

we know that AA^T is invertible so we can multiply both sides by $(AA^T)^{-1}$:

$$\lambda = 4(AA^T)^{-1}y$$

we use this λ to replace it in equation (1):

$$x = \frac{1}{4}A^T(4(AA^T)^{-1}y)$$

$$\boxed{x = A^T(AA^T)^{-1}y}$$

Part (b-1): Minimizing $\tilde{\mathbf{x}} = \min_{\mathbf{x}}\{(\mathbf{y} - \mathbf{Ax})^2\}$ to find the solution of $\mathbf{y_{n\times 1}} = \mathbf{A_{n\times m}x_{m\times 1}}$ where n¿m and rank=m

(Overdetermined System)
We define the function $J(x)$ as:

$$\begin{aligned} J(x) = (y - Ax)^2 &= \|y - Ax\|_2^2 \\ &= (y - Ax)^T(y - Ax) \\ &= y^Ty - 2y^TAx + x^TA^TAx \end{aligned}$$

now we find its partial derivatives and let them be zero:

$$\begin{aligned} \frac{\partial J}{\partial x} &= -2A^Ty + 2A^TAx = 0 \\ A^TAx &= A^Ty \end{aligned}$$

we know that A^TA is invertible so we can multiply both sides by $(A^TA)^{-1}$:

$$\boxed{x = (A^TA)^{-1}A^Ty}$$

Part(b-2) : Minimizing $\tilde{\mathbf{x}} = \min_{\mathbf{x}}\{(\mathbf{y} - \mathbf{A}\mathbf{x})^2 + \frac{1}{2}\lambda\|\mathbf{x}\|_2^2\}$ to find the solution of $\mathbf{y}_{n\times 1} = \mathbf{A}_{n\times m}\mathbf{x}_{m\times 1}$ where n¿m and rank=m

(Overdetermined System)
We define the function $J(x)$ as:

$$\begin{aligned} J(x) = (y - Ax)^2 + \frac{1}{2}\lambda\|x\|_2^2 &= \|y - Ax\|_2^2 + \frac{1}{2}\lambda\|x\|_2^2 \\ &= (y - Ax)^T(y - Ax) + \frac{1}{2}\lambda x^T x \\ &= y^T y - 2y^T Ax + x^T A^T Ax + \frac{1}{2}\lambda x^T x \end{aligned}$$

now we find its partial derivatives and let them be zero:

$$\begin{aligned} \frac{\partial J}{\partial x} = -2A^T y + 2A^T Ax + \lambda x = 0 \\ (A^T A + \frac{1}{2}\lambda I)x = A^T y \end{aligned}$$

$$\boxed{x = (A^T A + \frac{1}{2}\lambda I)^{-1} A^T y}$$