Problem 1.2 Device Sizing.

(a) False. Design B is violating the constraint (by having total area greater than 50), therefore it can not be considered as a solution of the problem.

(b) False. Both designs A and C are satisfying the constraint of the problem and offer the same value for the objective function (power consumption), therefore none of them is preferrable over the other.

(c) True. Design A is offering smaller value for the objective function (power) than design D and still satisfying the area constraint, therefore design D cannot be optimal.

Problem 12.1 Some famous inequalities.

(a) Let $a = (a_1, \ldots, a_n)^T, b = (b_1, \ldots, b_n)^T \in \mathbb{R}^n$, and nonnegative function g be defined as $g(t) = \|a + tb\|_2^2$ where $t \in \mathbb{R}$.

To find where the minimum of g occurs:

$$\frac{d}{dt}g(t) = 0 \to \frac{d}{dt}[(a_1 + tb_1)^2 + \cdots + (a_n + tb_n)^2] = 0 \to 2b^T(a + tb) = 0 \to t = -\frac{b^T a}{b^T b}$$

Using the minimum value of g:

$$\begin{aligned}
0 \leq \min_t g(t) &= g(-\frac{b^T a}{b^T b}) = \|a - \frac{b^T a}{b^T b} b\|_2^2 \\
&= \langle a - \frac{b^T a}{b^T b} b, a - \frac{b^T a}{b^T b} b\rangle \\
&= \langle a, a\rangle - \frac{b^T a}{b^T b}\langle a, b\rangle - \frac{b^T a}{b^T b}\langle b, a\rangle + \left(\frac{b^T a}{b^T b}\right)^2 \langle b, b\rangle \\
&= a^T a - \frac{(b^T a)^2}{b^T b} - \frac{(b^T a)^2}{b^T b} + \frac{(b^T a)^2}{b^T b} \\
&= a^T a - \frac{(b^T a)^2}{b^T b} \\
&= \|a\|_2^2 - \frac{(b^T a)^2}{\|b\|_2^2},
\end{aligned}$$

therefore we have $|b^T a| \leq \|a\|_2 \|b\|_2$.

(b) Given $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$, let $\mathbb{1} = (1, \dots, 1)^T \in \mathbb{R}^n$ and $y = (|x_1|, \dots, |x_n|)^T$. Using Cauchy-Schwarz inequality:

$$
\begin{aligned}
|y^T \mathbb{1}| &\leq \|y\|_2 \|\mathbb{1}\|_2 \\
\underbrace{\sum_{k=1}^{n} |x_k|}_{\|x\|_1} &\leq (\|x\|_2)(\sqrt{n})
\end{aligned}
$$

(c) Given $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n_{++}$, let $y = \frac{1}{\sqrt{n}}(\sqrt{x_1}, \dots, \sqrt{x_n})^T$ and $z = \frac{1}{\sqrt{n}}(\sqrt{1/x_1}, \dots, \sqrt{1/x_n})^T$. Using Cauchy-Schwarz inequality:

$$
\begin{aligned}
|y^T z| &\leq \|y\|_2 \|z\|_2 \\
1 &\leq \Big(\sqrt{\frac{1}{n}\sum_{k=1}^{n} x_k}\Big)\Big(\sqrt{\frac{1}{n}\sum_{k=1}^{n} \frac{1}{x_k}}\Big) \\
1^2 &\leq \Big(\frac{1}{n}\sum_{k=1}^{n} x_k\Big)\Big(\frac{1}{n}\sum_{k=1}^{n} \frac{1}{x_k}\Big) \\
\Big(\frac{1}{n}\sum_{k=1}^{n} \frac{1}{x_k}\Big)^{-1} &\leq \Big(\frac{1}{n}\sum_{k=1}^{n} x_k\Big)
\end{aligned}
$$