Problem 1

Let $\alpha \in \mathbb{R}$ and $x_1, x_2, \ldots$ be a convergent sequence in the sublevel set $S_\alpha = \{x \in \mathrm{dom} f \mid f(x) \leq \alpha\}$, such that $\lim_{i\to\infty} x_i = x$.
The sublevel set is closed if $x \in s_\alpha$. To show that, two conditions should be satisfied:

1. $\underline{x \in \mathrm{dom} f}$

   Since $\mathrm{dom} f$ is a closed set, the limit point of any convergent sequence in it, is a member of it, *i.e.*, $x \in \mathrm{dom} f$.

2. $\underline{f(x) \leq \alpha}$

   Since we showed $x \in \mathrm{dom} f$:

$$\forall x_i \quad f(x_i) \leq \alpha \Rightarrow \lim_{i\to\infty} f(x_i) \leq \lim_{i\to\infty} \alpha \xRightarrow[x \in \mathrm{dom} f]{\text{continuous } f,} f(\lim_{i\to\infty} x_i) \leq \alpha \Rightarrow f(x) \leq \alpha$$

as α was chosen arbitrarily, any sublevel set of f is a closed set, so f is a closed function.

Problem 2

f continuous, $\mathbf{dom} f$ open

$\boxed{\Rightarrow}$ $\{x_i\} \in \mathbf{dom} f, \lim_{i\to\infty} x_i = x \in \mathbf{bd}\,\mathbf{dom} f, \lim_{i\to\infty} f(x_i) = \infty \Rightarrow f$ closed

Let $\alpha \in \mathbb{R}$ and $x_1, x_2, \ldots$ be a convergent sequence in the sublevel set $S_\alpha = \{x \in \mathbf{dom} f \mid f(x) \le \alpha\}$ such that $\lim_{i\to\infty} x_i = x$. The sublevel set is closed if $x \in S_\alpha$. To show that, two conditions should be satisfied:
The limit point of the sequence (x) can exist in only two cases:

1. $x \in \mathbf{int}\,\mathbf{dom} f$
   By hypothesis $\mathbf{int}\,\mathbf{dom} f = \mathbf{dom} f$, so $x \in \mathbf{dom} f$. Then by continuity of f we have $\lim_{i\to\infty} f(x_i) \le \lim_{i\to\infty} \alpha \Rightarrow f(\lim_{i\to\infty} x_i) \le \alpha \Rightarrow f(x) \le \alpha$. Therefore f is closed in this case.
2. $x \in \mathbf{bd}\,\mathbf{dom} f$
   In this case, if $\lim_{i\to\infty} f(x_i) \ne \infty$, it contradicts with $\mathbf{dom} f$ being open.

Therefore, f is closed, with the hypothesis.

$\boxed{\Leftarrow}$ f closed $\Rightarrow \{x_i\} \in \mathbf{dom} f, \lim_{i\to\infty} x_i = x \in \mathbf{bd}\,\mathbf{dom} f, \lim_{i\to\infty} f(x_i) = \infty$

(proof by contradiction) Suppose $x_1, x_2, \ldots$ be a convergent sequence in the $\mathbf{dom} f$ with the limit point $\lim_{i\to\infty} x_i = x \in \mathbf{bd}\,\mathbf{dom} f$ such that $\lim_{i\to\infty} f(x_i) = B < \infty$. Then considering the sublevel set $S_B = \{x \in \mathbf{dom} f \mid f(x) \le B\}$, since the function is closed, $x \in S_B$. By definition of S_B, then, $x \in \mathbf{dom} f$ as well. This is in contradiction with $\mathbf{dom} f$ being an open function since we assumed x to be on the boundary of $\mathbf{dom} f$. Therefore $\lim_{i\to\infty} f(x_i)$ cannot be less than ∞.

Problem 3 [BV 2.1]

Induction on k:

1. **Basis step.** for k=2 $\theta_1 x_1 + \theta_2 x_2 \in C$ where $x_1, x_2 \in C$ and $\theta_1, \theta_2 \geq 0$ and $\theta_1 + \theta_2 = 1$.

2. **Inductive hypothesis.** assume that it holds also for $k-1$, *i.e.*, $\theta_1 x_1 + \cdots + \theta_{k-1} x_{k-1} \in C$ where $x_1, \dots, x_{k-1} \in C$ and $\theta_1, \dots, \theta_{k-1} \geq 0$ and $\theta_1 + \cdots + \theta_{k-1} = 1$.

3. **Inductive step.** Let $x_1, \dots, x_k \in C$ and $\theta_1, \dots, \theta_k \geq 0$ and $\theta_1 + \cdots + \theta_k = 1$, and also w.l.o.g. suppose that $\theta_k \neq 1$ (There must exists at least one $\theta_i \neq 1 \;\; i \in \{1, \dots, k\}$):

$$\begin{aligned}
\theta_1 x_1 + \cdots + \theta_{k-1} x_{k-1} + \theta_k x_k &= (1-\theta_k)\underbrace{(\frac{\theta_1}{1-\theta_k} x_1 + \cdots + \frac{\theta_{k-1}}{1-\theta_k} x_{k-1})}_{:= x^* \in C \text{ (\textbf{inductive hypothesis})}} + \theta_k x_k \\
&= \underbrace{(1-\theta_k) x^* + \theta_k x_k}_{\in C \text{ (\textbf{basis step})}}
\end{aligned}$$

   where the second equality follows from the **inductive hypothesis** and the third follows from **basis step**.

Therefore for any k, $\theta_1 x_1 + \cdots + \theta_k x_k \in C$, given $x_1, \dots, x_k \in C$ and $\theta_1, \dots, \theta_k \geq 0$ and $\theta_1 + \cdots + \theta_k = 1$.

Problem 4 [BV 2.2]

(a) $\boxed{\Rightarrow}$ $C \cap L$ is convex for any line $L \Rightarrow C$ is convex

Let distinct points $x, y \in C$ and let the line L pass through points x and y. Since $C \cap L$ is convex then $\theta x + (1-\theta)y \in C \cap L \quad \forall \theta \in [0,1]$ and as a result $\theta x + (1-\theta)y \in C \ \ \forall \theta \in [0,1]$.

$\boxed{\Leftarrow}$ C is convex $\Rightarrow C \cap L$ is convex for any line L

Any line L is a convex set by definition. Intersection of any two convex sets is a convex set (♠). Since C is convex, $C \cap L$ will be a convex set.

(b) $\boxed{\Rightarrow}$ set $A \cap L$ is affine for any line $L \Rightarrow$ set A is affine

Let distinct points $x, y \in A$ and let the line L pass through points x and y. Since $A \cap L$ is affine then $\theta x + (1-\theta)y \in A \cap L \quad \forall \theta \in \mathbb{R}$ and as a result $\theta x + (1-\theta)y \in A \ \ \forall \theta \in \mathbb{R}$.

$\boxed{\Leftarrow}$ set A is affine $\Rightarrow A \cap L$ is affine for any line L

Any line L is an affine set by definition. Intersection of any two affine sets is an affine set (♣). Since A is affine, $A \cap L$ will be an affine set.

Remark (♠) Intersection of two convex sets is a convex set.

proof. Suppose C_1 and C_2 be two convex sets. Let $x, y \in C_1 \cap C_2$. Since $x, y \in C_1$, by convexity of C_1 we have $\theta x + (1-\theta)y \in C_1 \ \ \forall \theta \in [0,1]$. Also, since $x, y \in C_2$, by convexity of C_2 we have $\lambda x + (1-\lambda)y \in C_2 \ \ \forall \lambda \in [0,1]$. As a result, $\gamma x + (1-\gamma)y \in C_1 \cap C_2 \ \ \forall \gamma \in [0,1]$. Therefore $C_1 \cap C_2$ is convex.

Remark (♣) Intersection of two affine sets is an affine set.

proof. Same as proof above, except for affine sets.