Homework 2
EE 513 — Stochastic Systems Theory

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Problem 2.1

40/40
Good work.

((a))

$$p_b : \text{Probability of Board working}$$
$$p : \text{Probability of Chip working}$$

$$\begin{aligned} p_b &= P(\text{at least 7 chips out of 8 chips working}) \\ &= \binom{8}{7} p^7(1-p) + \binom{8}{8} p^8 \\ &= -8p^8 + 8p^7 + p^8 \\ &= 8p^7 - 7p^8 \end{aligned}$$

((b))

$$\begin{aligned} P(\text{At least ONE board working}) &\geq 99.9\% \\ 1 - \binom{n}{0} p_b^0 (1-p_b)^n &\geq 0.999 \\ 0.001 &\geq (1-p_b)^n \\ -3 &\geq n \log(1-p_b) \\ \frac{-3}{\log(1-p_b)} &\leq n \\ \frac{-3}{\log(1-8p^7-7p^8)} &\leq n \end{aligned}$$

Problem 2.2

((a)) Area under the Laplacian pdf must be one:

$$
\begin{aligned}
\int_{-\infty}^{+\infty} f_X(x)dx &= 1\\
\int_{-\infty}^{+\infty} ae^{-b|x|}dx &= 1\\
a[\quad \int_{-\infty}^{0} e^{bx}dx + \int_{0}^{+\infty} e^{-bx}dx \quad] &= 1\\
a[\quad \frac{1}{b}e^{bx}\Big|_{-\infty}^{0} \quad + \quad \frac{1}{-b}e^{-bx}\Big|_{0}^{+\infty} \quad] &= 1 \qquad [\text{only if } b \geq 0]\\
a[\quad \frac{1}{b}(1-0) \quad + \quad \frac{1}{-b}(0-1) \quad] &= 1\\
\frac{2a}{b} &= 1\\
b &= 2a
\end{aligned}
$$

if $a = \frac{1}{2}$ then $b = 1$, then the cdf will be:

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{2}e^{-|t|}dt$$

$$
= \begin{cases}
\frac{1}{2}e^{t}\Big|_{-\infty}^{x} = \frac{1}{2}e^{x} & , \quad x \leq 0\\
\\
\frac{1}{2}e^{t}\Big|_{-\infty}^{0} + \frac{-1}{2}e^{-t}\Big|_{x}^{0} = 1 - \frac{1}{2}e^{-x} & , \quad x > 0
\end{cases}
$$

$$
= \begin{cases}
\frac{1}{2}e^{x} & , \quad x \leq 0\\
1 - \frac{1}{2}e^{-x} & , \quad x > 0
\end{cases}
$$

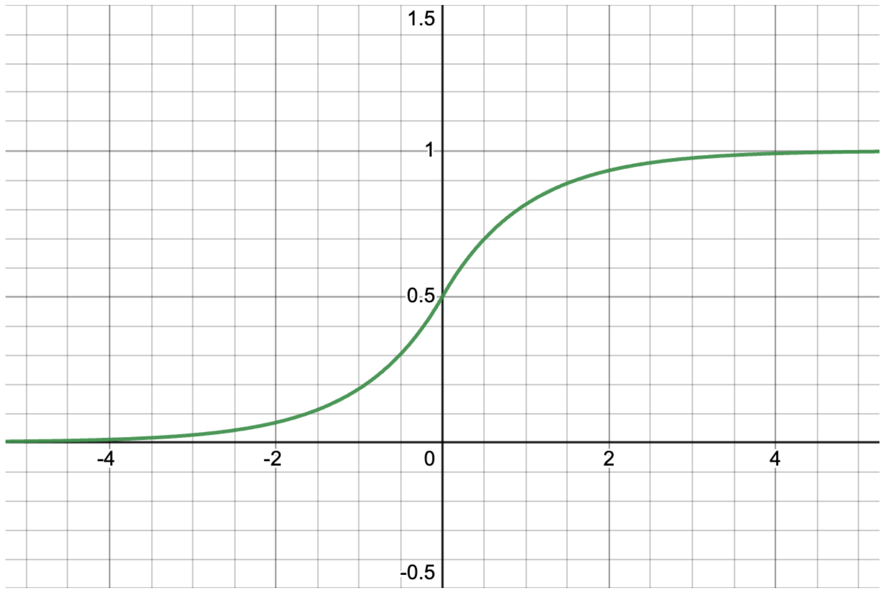


Figure 1: CDF $F_X(x)$

((b)) To find relationship between a and b:

$$\int_{-\infty}^{+\infty} f_X(x)dx = 1$$

$$\int_{-\infty}^{+\infty} \frac{a}{b^2 + x^2} = 1$$

$$\frac{a}{b}\int_{-\infty}^{+\infty} \frac{\frac{1}{b}}{1 + (\frac{x}{b})^2} = 1$$

$$\frac{a}{b}\arctan(\frac{x}{b})\Big|_{-\infty}^{+\infty} = 1$$

$$\frac{a}{b}(+\frac{\pi}{2} - (-\frac{\pi}{2})) = 1$$

$$\frac{a}{b}\pi = 1$$

$$b = \pi a$$

if $a = \frac{1}{2}$ then $b = \pi$, then the cdf will be:

$$\begin{aligned}
F_X(x) &= \int_{-\infty}^{x} \frac{1/2}{(\frac{\pi}{2})^2 + t^2} dt \\
&= \frac{1}{\pi} \int_{-\infty}^{x} \frac{\frac{1}{\pi/2}}{1 + (\frac{t}{\pi/2})^2} dt \\
&= \frac{1}{\pi} \arctan(\frac{x}{\pi}/2) \Big|_{-\infty}^{x} \\
&= \frac{1}{\pi} [\arctan(\frac{x}{\pi/2}) + \frac{\pi}{2}]
\end{aligned}$$

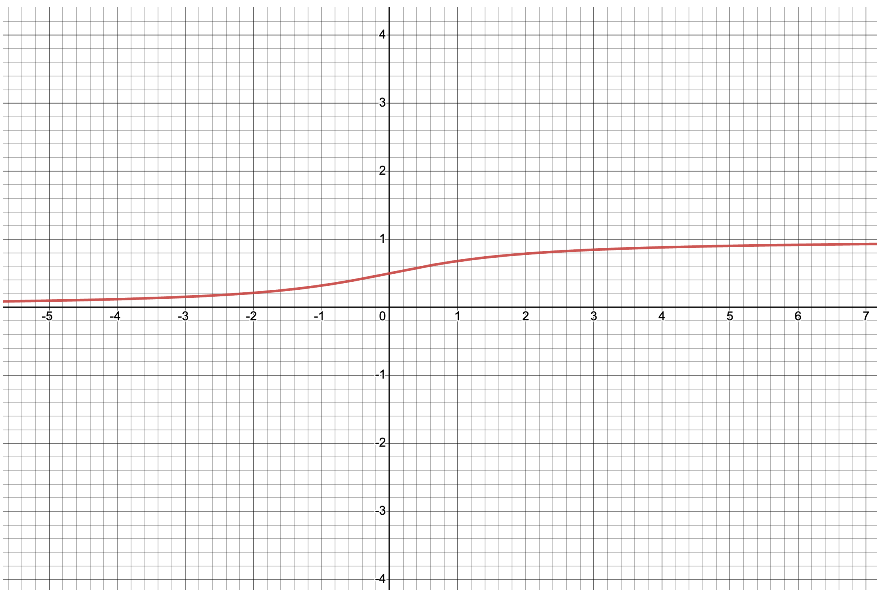


Figure 2: CDF $F_X(x)$

Problem 2.3

$$X \sim \mathcal{N}(5, 16)$$

let us define Y as $Y \triangleq \frac{X-5}{4}$, then

$$Y \sim \mathcal{N}(0, 1)$$

hence by using the change of variable ($X = 4Y + 5$) we will be able to calculate the probabilities based on the Q-function which is defined for the Standard Normal distribution ($\mathcal{N}(0, 1)$).

$$Q(y) = P[Y > y] = \int_y^\infty \frac{1}{\sqrt{2\pi}} \exp(-\frac{u^2}{2}) du$$

((a)) To calculate probabilities in this part we assumed that $Q(-x) = 1 - Q(x)$,which will be proved in section ((d)) of this problem.

$$\mathbf{P[X > 4]} = P[4Y + 5 > 4] = P[Y > \frac{-1}{4}] = Q(\frac{-1}{4}) = 0.5987$$

Closed or open intervals for the Gaussian distributed random variable has no impact on the final probability value:

$$\mathbf{P[X \geq 7]} = P[4Y + 5 \geq 7] = P[Y \geq \frac{1}{2}] = P[Y = \frac{1}{2}] + P[Y > \frac{1}{2}] = 0 + Q(\frac{1}{2}) = 0.3085$$

$$\begin{aligned}\mathbf{P[2 < X < 7]} &= P[X > 2] - P[X \geq 7] = P[X > 2] - P[X > 7] \\ &= P[4Y + 5 > 2] - P[4Y + 5 > 7] = P[Y > \frac{-3}{4}] - P[Y > \frac{2}{4}] \\ &= Q(\frac{-3}{4}) - Q(\frac{2}{4}) = 0.4648\end{aligned}$$

$$\begin{aligned}\mathbf{P[6 \leq X \leq 8]} &= P[X \geq 6] - P[X > 8] = P[X > 6] - P[X > 8] \\ &= P[4Y + 5 > 6] - P[4Y + 5 > 8] = P[Y > \frac{1}{4}] - P[Y > \frac{3}{4}] \\ &= Q(\frac{1}{4}) - Q(\frac{3}{4}) = 0.1747\end{aligned}$$

((b))

$$\begin{aligned}
P[X < a] &= 0.8869 \\
1 - P[X > a] &= 0.8869 \\
P[X > a] &= 0.1131 \\
P[4Y + 5 > a] &= 0.1131 \\
P[Y > \frac{a-5}{4}] &= 0.1131 \\
Q(\frac{a-5}{4}) &= 0.1131 \qquad \text{(Q-function Table)} \\
\frac{a-5}{4} &= 1.2102 \\
a &= 9.8408
\end{aligned}$$

((c))

$$\begin{aligned}
P[13 < X \leq c] &= 0.0123 \\
P[X > 13] - P[X > c] &= 0.0123 \\
P[4Y + 5 > 13] - P[4Y + 5 > c] &= 0.0123 \\
P[Y > 2] - P[Y > \frac{c-5}{4}] &= 0.0123 \\
Q(2) - Q(\frac{c-5}{4}) &= 0.0123 \qquad \text{(Q-function Table)} \\
0.0228 - Q(\frac{c-5}{4}) &= 0.0123 \\
\frac{c-5}{4} &= Q^{-1}(0.0105) \qquad \text{(Q-function Table)} \\
c &= 14.2391
\end{aligned}$$

((d)) Before deriving the asked equation, we note that for a standard Gaussian random variable:

$$\begin{aligned}
1 &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp(-\frac{u^2}{2}) du \\
1 &= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp(-\frac{u^2}{2}) du + \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-\frac{u^2}{2}) du \qquad (\dagger) \\
1 &= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp(-\frac{u^2}{2}) du + Q(x)
\end{aligned}$$

Now we can derive $Q(-x)$:

$$\begin{aligned}
Q(-x) &= \int_{-x}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-\frac{u^2}{2}) du \qquad \text{(change of variable: } v = -u) \\
&= \int_{x}^{-\infty} \frac{-1}{\sqrt{2\pi}} \exp(-\frac{v^2}{2}) dv \\
&= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp(-\frac{v^2}{2}) dv \qquad \text{(using (†))} \\
&= 1 - \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-\frac{u^2}{2}) du \qquad \checkmark \\
&= 1 - Q(x)
\end{aligned}$$

Problem 2.4

((a)) Relative frequencies of X and Y:

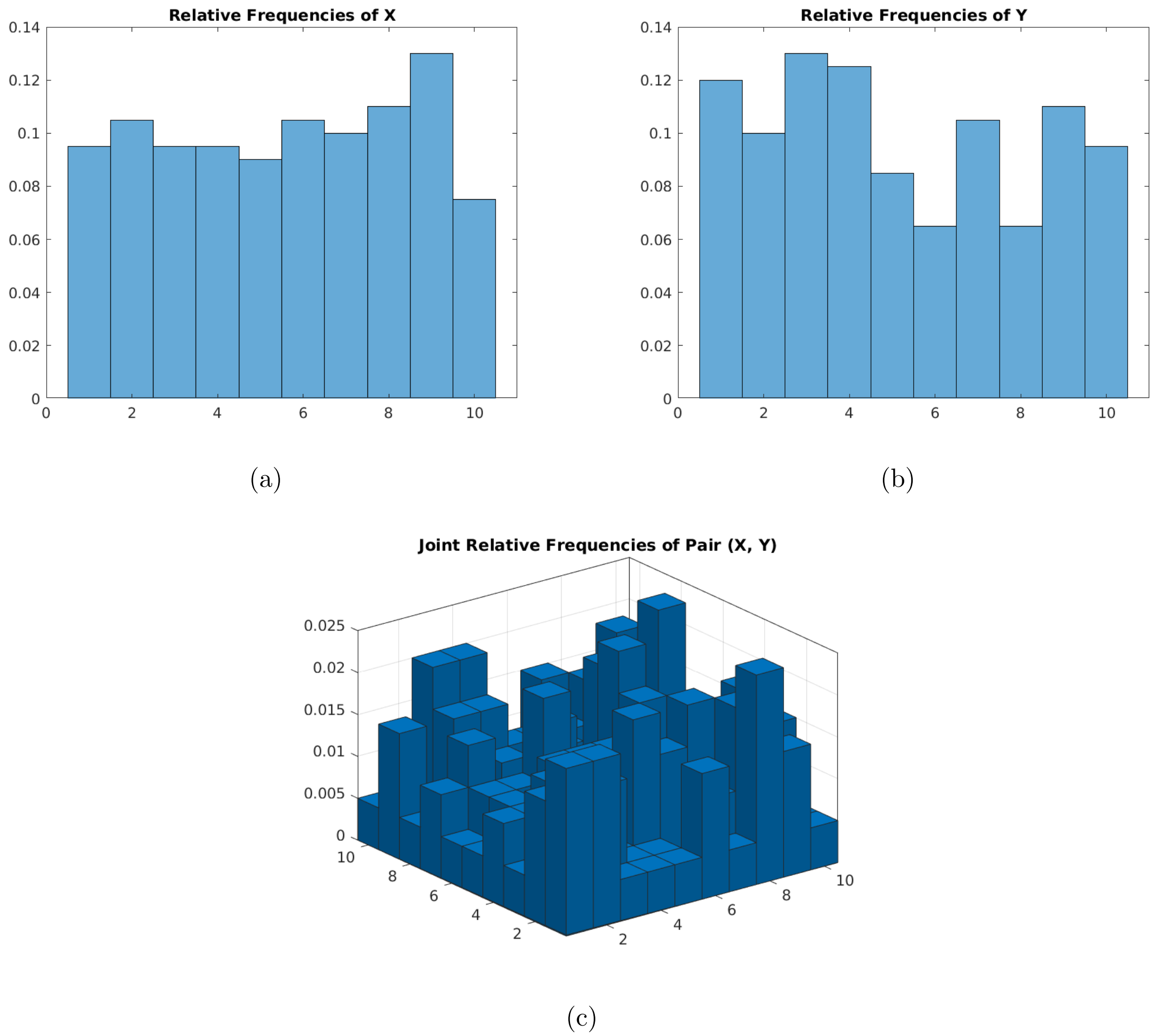
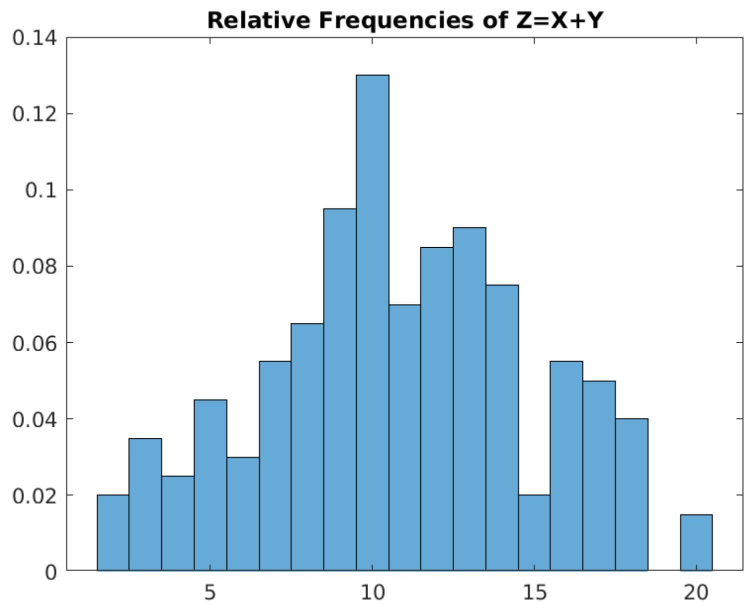


Figure 3: 200 pairs of (X_i, Y_i) drawn uniformly from $\{1, 2, \ldots, 10\}$.

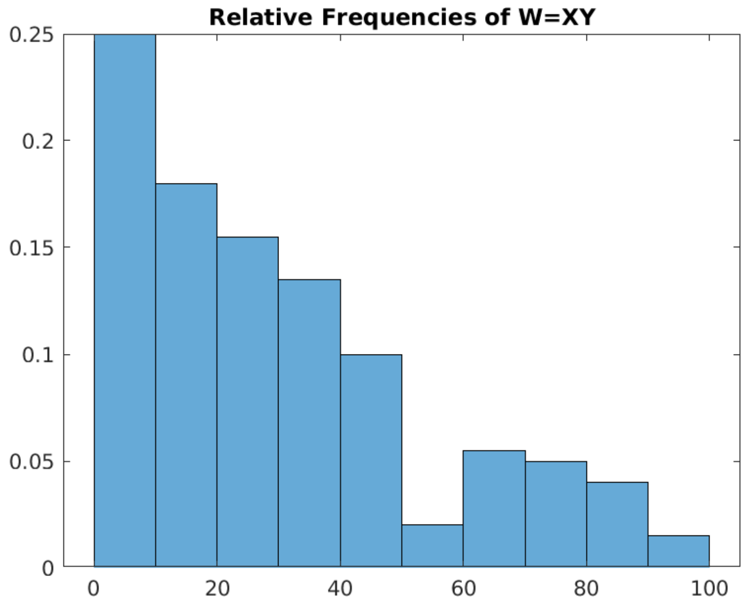
((b)) Z which is sum of two uniform random variables, has a probability distribution which is the convolution of a uniform by itself. Therefore the distribution of Z will have the shape of a Triangular function:



(a)

Figure 4: 200 samples of $Z = X + Y$.

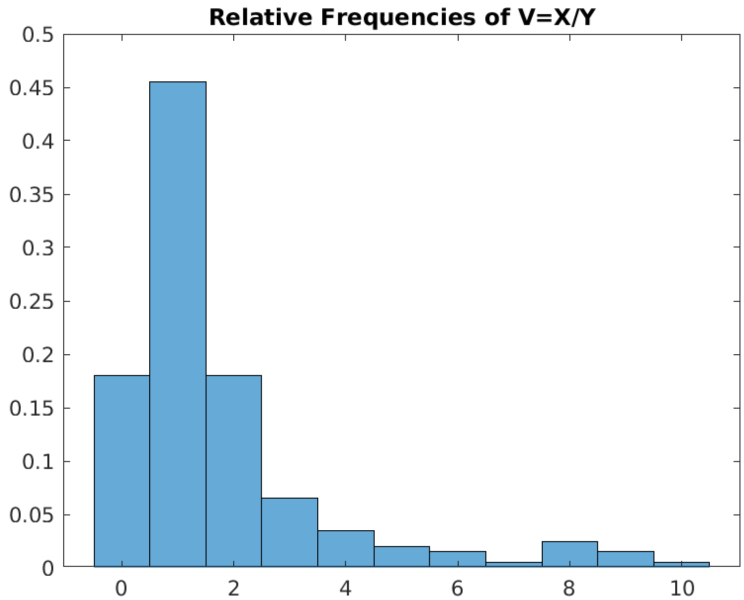
((c)) Relative frequencies of $W = XY$. It looks like a Poisson distribution, $Pois(\lambda)$ with relatively small λ :



(a)

Figure 5: 200 samples of $W = XY$.

((d)) Relative frequencies of $V = X/Y$. It looks like a Poisson distribution, $Pois(\lambda)$ with relatively bigger λ:

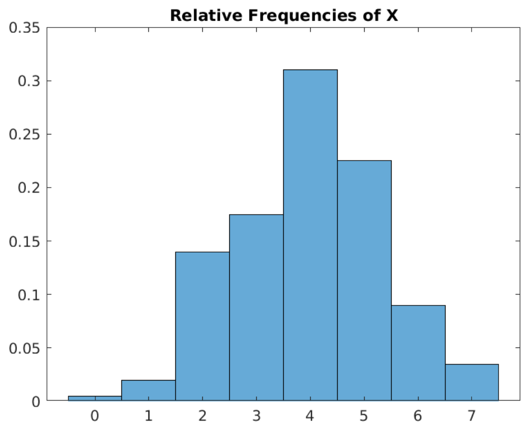


(a)

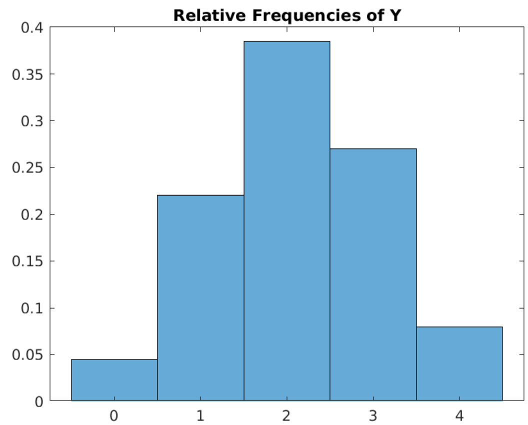
Figure 6: 200 samples of $V = X/Y$.

Problem 2.5

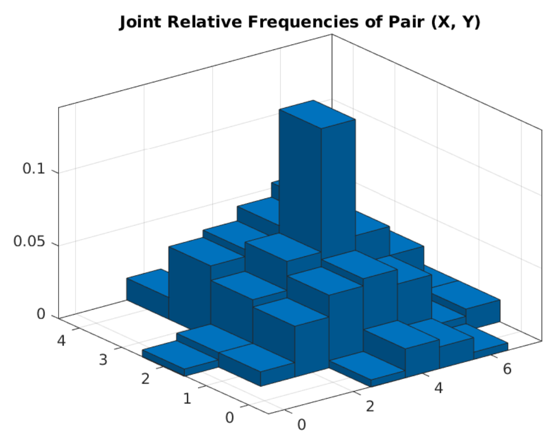
((a)) Relative frequencies of X and Y:



(a)



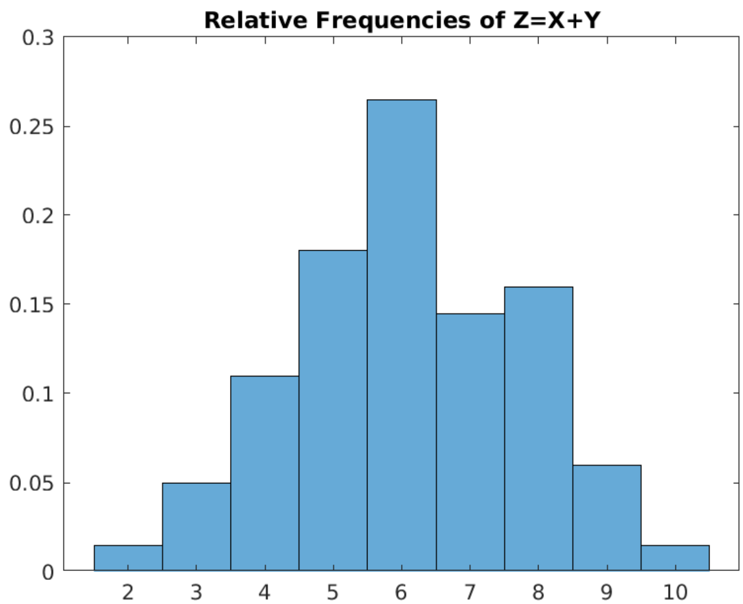
(b)



(c)

Figure 7: 200 pairs of (X_i, Y_i) drawn from $Binomial(8, 0.5)$ and $Binomial(4, 0.5)$, respectively.

((b)) Relative frequencies of $Z = X + Y$. Sum of these two binomial distributions will be also a binomial distribution with parameters $n = 8+4$, $p = 0.5$. The distribution with the expected value of 6 is discernible from the plot below:



(a)

Figure 8: 200 samples of $Z = X + Y$.

Problem 2.6

((a)) Relative frequencies of X and Y:

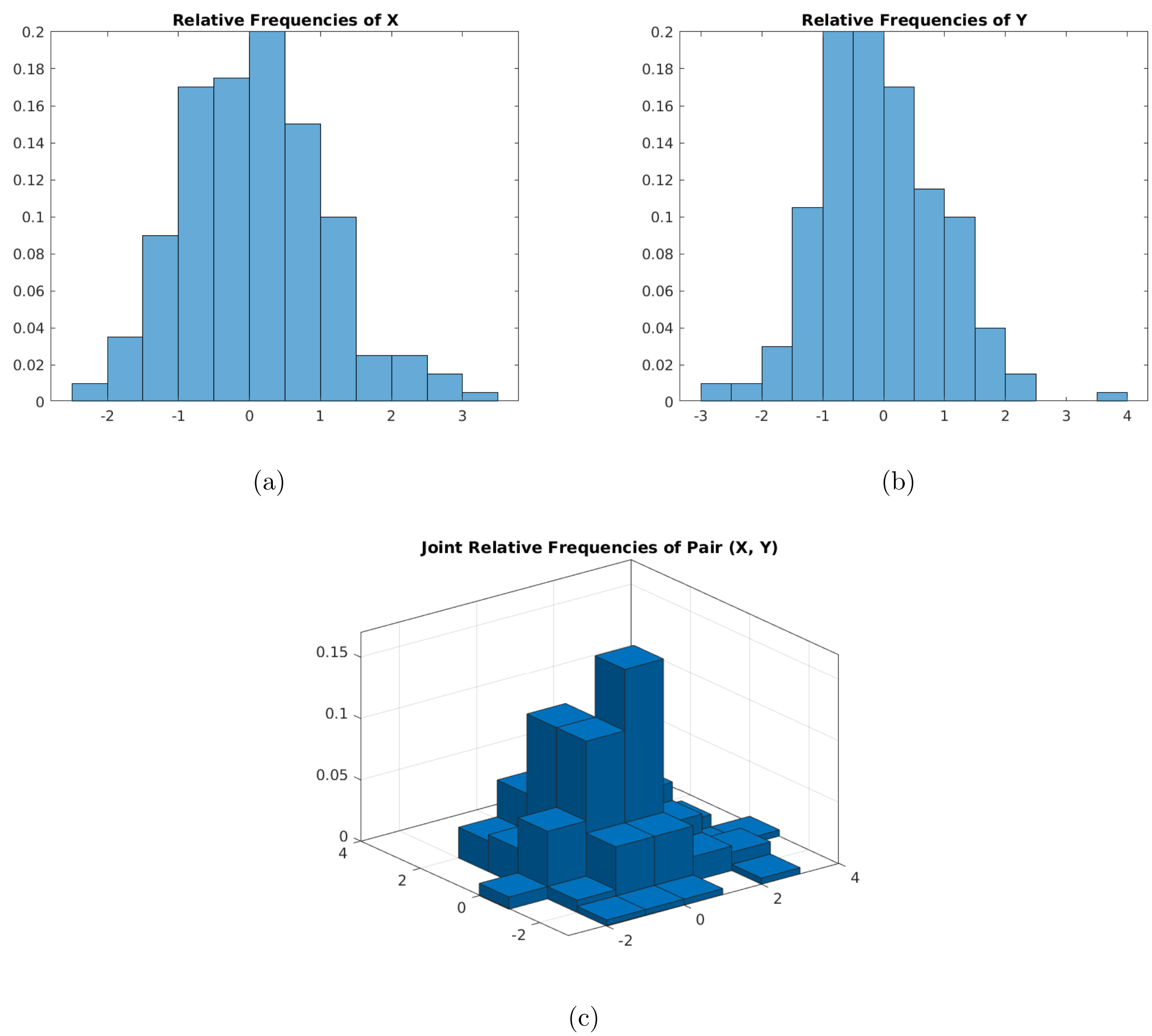
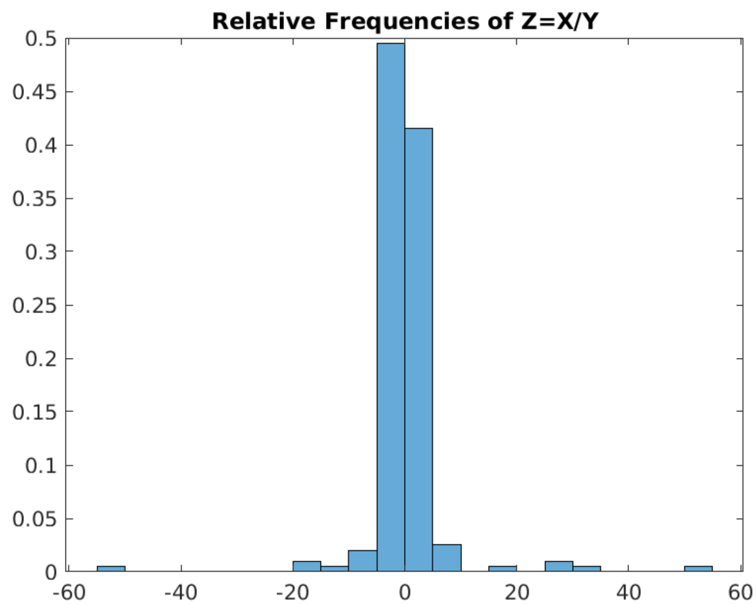


Figure 9: 200 pairs of (X_i, Y_i) drawn from $\mathcal{N}(0, 1)$.

((b)) Relative frequencies of $Z = X/Y$. The ratio of these two independent normally distributed random variable will follow Cauchy distribution as can be seen from the plot below:



(a)

Figure 10: 200 samples of $Z = X/Y$.

Problem 2.7

Relative frequencies of $Z = -ln(X)$, in which X is drawn from $\mathcal{U}(0, 1)$. The resulting Z will follow an Exponential distribution, i.e. $Exp(\lambda)$, with parameter $\lambda = 1$:

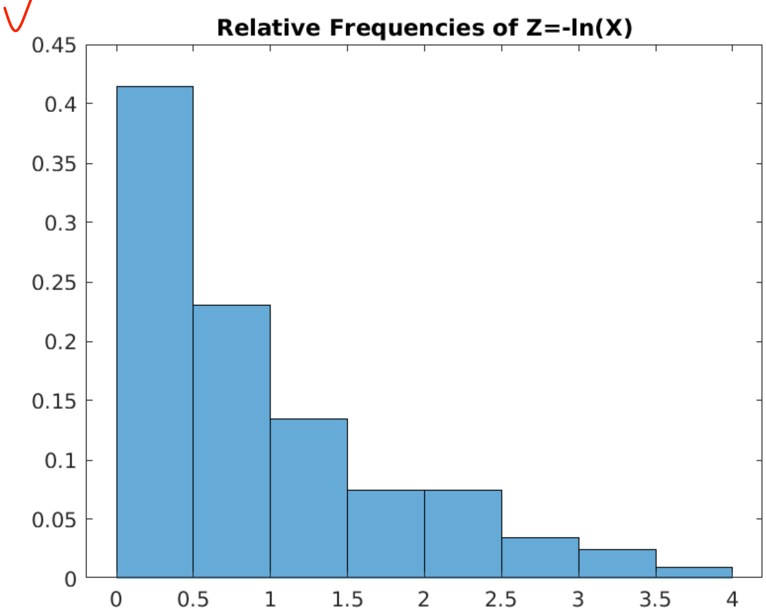


Figure 11: 200 samples of $Z = -ln(X)$.

Problem 2.8

((a)) The mean and variance of Binomial distribution:

$$E(X) = n.p = 8 \times 0.5 = 4$$
$$Var(X) = n.p.(1-p) = 8 \times 0.5 \times 0.5 = 2$$

((b))

$$M_{200} = 3.9750 \qquad \Rightarrow \qquad |Error_{mean}| = 0.025$$
$$V_{200} = 1.9039 \qquad \Rightarrow \qquad |Error_{var}| = 0.0961$$

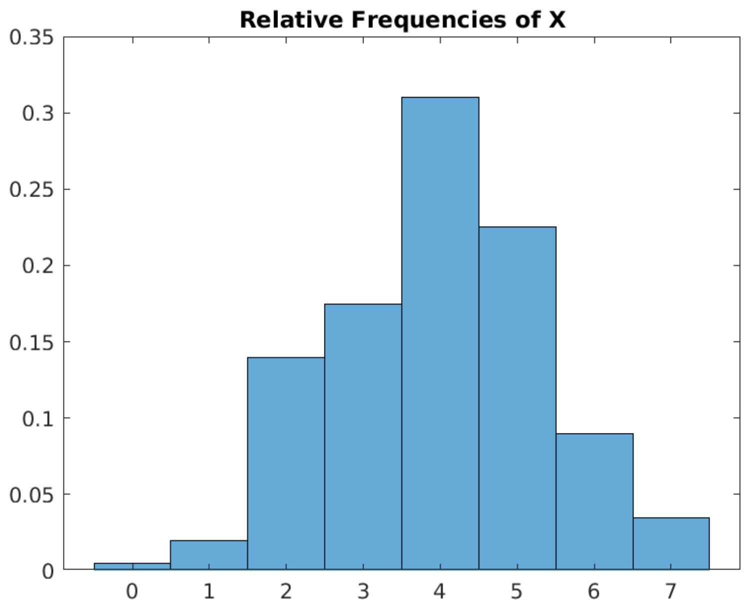


Figure 12: 200 samples of X.

Problem 2.9

((a)) The mean and variance of Y:

$$E(Y) = E(2X+5) = 2E(X) + 5 = 5$$
$$Var(Y) = Var(2X+5) = 4Var(X) = 4$$

((b))

$$M_{200} = 5.1483 \quad \Rightarrow \quad |Error_{mean}| = 0.1483$$
$$V_{200} = 3.9714 \quad \Rightarrow \quad |Error_{var}| = 0.0286$$

Relative frequencies of $Y = 2X + 5$:

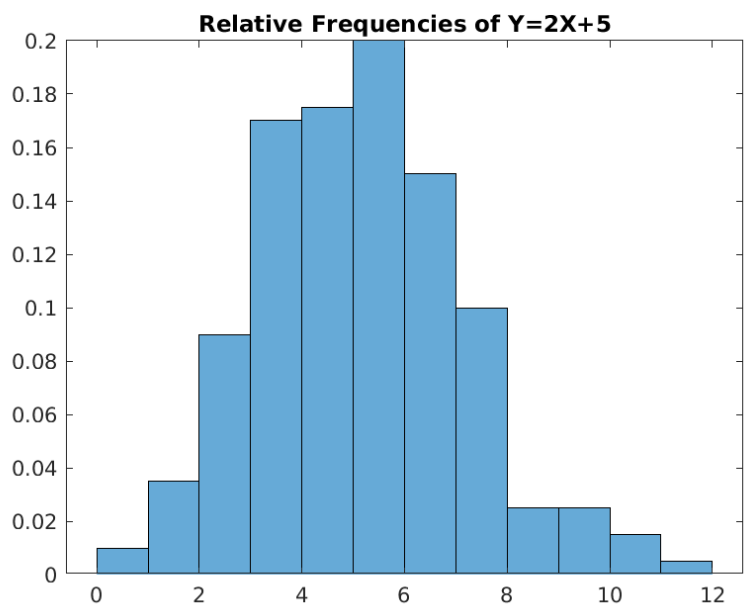


Figure 13: 200 samples of $Y = 2X + 5$.