Homework 4

EE 513 - Stochastic Systems Theory

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30/30

4.1 a

$$X \sim \text{Uniform}\left(-\pi/2, +\pi/2\right)$$

$$\begin{cases} m = E[X] = 0 \\ \sigma^2 = Var[X] = E[X^2] - E^2[X] = E[X^2] = \displaystyle\int_{-\pi/2}^{\pi/2} \frac{1}{\pi} x^2\, dx = \frac{x^3}{3}\Big|_{-\pi/2}^{\pi/2} = \frac{\pi^2}{12} \end{cases}$$

$$\rightarrow P(|X-m| \geq c) = P(|X| \geq c) = ?$$

exact:

$$P(|X| \geq c) = P(\{X \leq -c\} \cup \{X \geq c\}) = P(X \leq -c) + P(X \geq c)$$

$$\overset{\text{symmetric}}{\underset{0 \leq c \leq \frac{\pi}{2}}{=\!=}} 2P(X \geq c) = 2\int_c^{\pi/2} \frac{1}{\pi}\, dx = \boxed{1 - \frac{2}{\pi} c}$$

Chebyshev's:

$$P(|X| \geq c) \leq \frac{\sigma^2}{c^2} = \frac{\pi^2}{12c^2}$$

4.1 b

$$X \sim Lap.(2) \rightsquigarrow f_{X}(x) = e^{-2|x|}$$

$$\begin{cases} m = E[X] = 0 \\ \sigma^2 = Var[X] = 1 \end{cases}$$

$$\rightarrow P(|X-m| \geq c) = P(|X| \geq c) = ?$$

exact:

$$P(|X| \geq c) \overset{\text{symmetric}}{\underset{c \geq 0}{=\!=}} 2P(X \geq c) = 2\int_c^{+\infty} e^{-2x}\, dx = -e^{-2x}\Big]_c^{+\infty} = \boxed{e^{-2c}}$$

Chebyshev's: $P(|X| \geq c) \leq \frac{\sigma^2}{c^2} = \frac{1}{c^2}$

4.1 c

$$X \sim N(\underset{m}{0}, \underset{\sigma^2}{4})$$

exact: $P(|X| \geq c) \underset{c \geq 0}{\overset{\text{symmetric}}{=\!=}} 2\int_c^{+\infty} \frac{1}{\sqrt{2\pi 4}} e^{-\frac{x^2}{8}} dx =$

Chebyshev's: $P(|X| \geq c) \leq \frac{4}{c^2}$

4.2 a

$$\Phi_X(jv) = E[e^{jvX}] = \sum_{i=1}^{4} P_X(x_i)\, e^{jvx_i}$$

$$= \frac{1}{8} e^{j2v} + \frac{3}{8} e^{jv} + \frac{3}{8} e^{-jv} + \frac{1}{8} e^{-j2v}$$

$$= \frac{1}{4} \cos 2v + \frac{3}{4} \cos v$$

✓

4.2 b

$$H(X) = E[-\log_2 P_X(x)] = -\frac{1}{8}\log \frac{1}{8} - \frac{3}{8}\log \frac{3}{8} - \frac{3}{8}\log \frac{3}{8} - \frac{1}{8}\log \frac{1}{8}$$

$= 1.8113$ [bits/symbol] ✓

4.2 c

$H(X) = E[-\log_2 P_X^{new}(x)] = -4 \times \frac{1}{4} \log(1/4) = 2$ ✓ [bits/symbol] ✓

$H_{uniform}^{(x)} > H_{P_X}(x) \Rightarrow$ Uniform needs more bits $\Rightarrow$ Higher Uncertainty.

4.3 a

$$P(A \mid Y_1 = 1) = P(|Y_2| \leq 1 \mid Y_1 = 1) = P(-1 \leq Y_2 \leq +1 \mid Y_1 = 1)$$

$$= \int_{-1}^{+1} f_{Y_2|Y_1}(y_2 \mid 1)\, dy_2 = \int_{-1}^{+1} \frac{f_{Y_1 Y_2}(1, y_2)}{f_{Y_1}(1)}\, dy_2$$

→ So, we need joint PDF of (Y_1, Y_2) and Marginal PDF of Y_1 at $y_1 = 1$.

I. Joint PDF Y_1, Y_2

$$\begin{cases} Y_1 = X_1 + X_2 \\ Y_2 = X_1 - X_2 \end{cases} \Rightarrow \begin{cases} X_1 = \frac{1}{2} Y_1 + \frac{1}{2} Y_2 \\ X_2 = \frac{1}{2} Y_1 - \frac{1}{2} Y_2 \end{cases}$$

$$\underset{\text{(uniform)}}{f_{X_1 X_2}(x_1, x_2)} = \begin{cases} 1/4 & , 0 \leq x_1 \leq 2 \quad , 0 \leq x_2 \leq 2 \\ 0 & , \quad \text{o.w.} \end{cases}$$

$$f_{Y_1 Y_2}(y_1, y_2) = \frac{f_{X_1 X_2}\left(\frac{1}{2}Y_1 + \frac{1}{2}Y_2 \,,\, \frac{1}{2}Y_1 - \frac{1}{2}Y_2\right)}{\left| \det \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \right|} = \frac{1/4}{2} = \frac{1}{8}$$

II. Marginal @ $Y_1 = 1$

$$\text{if } Y_1 = 1 \Rightarrow X_1 + X_2 = 1 \Rightarrow 0 \leq X_1, X_2 \leq 1 \xrightarrow{Y_2 = X_1 - X_2} -1 \leq Y_2 \leq 1$$

$$f_{Y_1}(y_1)\Big|_{y_1=1} = \int_{Y_2 : \{y_2 \mid y_1 = 1\}} f_{Y_1 Y_2}(y_1, y_2)\, dy_2 = \int_{-1}^{+1} \frac{1}{8}\, dy_2 = \frac{1}{4}$$

→ Finally we have: $\Longrightarrow$ $P(A \mid Y_1 = 1) = \int_{-1}^{+1} \frac{1/8}{1/4}\, dy_2 = \boxed{1}$

4.3 b

$$P(Y_1 \geqslant 0 | A) = P(Y_1 \geqslant 0 \mid |Y_2| \leqslant 1) \overset{\text{independent}}{=\!=} P(Y_1 \geqslant 0) = 1$$

Events $Y_1 \geqslant 0$ and $|Y_2| \leqslant 1$ are independent, because no matter of what will be the value of Y_2, Y_1 is always non-negative.

4.3 c

$$f_{Y_2|Y_1}(y_2|y_1) = \frac{f_{Y_1,Y_2}(y_1,y_2)}{f_{Y_1}(y_1)} = \frac{1/8}{f_{Y_1}(y_1)}$$

$$f_{Y_1}(y_1) = \begin{cases} y_1/4 , & 0 \leqslant y_1 \leqslant 2 , \ -y_1 \leqslant y_2 \leqslant y_1 \\ \frac{4-y_1}{4} , & 2 \leqslant y_1 \leqslant 4 , \ y_1-4 \leqslant y_2 \leqslant 4-y_1 \end{cases}$$

$$\Rightarrow f_{Y_2|Y_1}(y_2|y_1) = \begin{cases} \frac{1}{2y_1} , & 0 \leqslant y_1 \leqslant 2 , \ -y_1 \leqslant y_2 \leqslant y_1 \\ \frac{1}{2(4-y_1)} , & 2 \leqslant y_1 \leqslant 4 , \ 4-y \leqslant y_2 \leqslant 4-y_1 \end{cases}$$

4.4 a

$$F_{XY}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{XY}(u,v)\,du\,dv$$

$$= \int_{0}^{y}\int_{0}^{x} \left(au e^{-\frac{au^2}{2}}\right)\left(bv e^{-\frac{bv^2}{2}}\right) du\,dv$$

$$= ab \int_{0}^{x} u e^{-\frac{au^2}{2}} du \int_{0}^{y} v e^{-\frac{bv^2}{2}} dv$$

$$= \left(1-e^{-ax^2/2}\right)\left(1-e^{-by^2/2}\right)$$ ✓

4.4 b

$$P(X>Y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{u} f_{XY}(u,v)\,dv\,du$$

$$= \int_{0}^{+\infty} au e^{-au^2/2}\left(\int_{0}^{u} bv e^{-bv^2/2} dv\right) du$$

$$= \int_{0}^{+\infty} au e^{-au^2/2} du - \int_{0}^{+\infty} au e^{-(a+b)u^2/2} du$$

$$= -e^{-au^2/2}\Big|_{0}^{+\infty} - \frac{a}{a+b}\left(-e^{-(a+b)u^2/2}\right)\Big|_{0}^{+\infty}$$

$$= \frac{b}{a+b}$$ ✓

4.4 $\boxed{c}$

$$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dy$$

$$= \int_0^{+\infty} ax\,e^{-ax^2/2}\,by\,e^{-by^2/2}\,dy$$

$$= \begin{cases} ax\,e^{-ax^2/2} & x > 0 \\ 0 & \text{o.w.} \end{cases} \quad \checkmark$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx$$

$$= \begin{cases} by\,e^{-by^2/2} & y > 0 \\ 0 & \text{o.w.} \end{cases} \quad \checkmark$$

4.5 a

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dx\, dy = 1$$

$$\int_{-1}^{+1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} C \quad dy\, dx = 1$$

$$\int_{-1}^{+1} 2C\sqrt{1-x^2}\, dx = 1$$

$$2C\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x)\right]_{-1}^{+1} = 1$$

$$2C \times \frac{\pi}{2} = 1$$

$$\boxed{C = \frac{1}{\pi}}$$

4.5 b

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\, dy = \frac{2}{\pi}\sqrt{1-x^2} \qquad -1 \leqslant x \leqslant +1$$

$$f_Y(x) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi}\, dx = \frac{2}{\pi}\sqrt{1-y^2} \qquad -1 \leqslant y \leqslant +1$$

4.5 c

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{1/\pi}{\frac{2}{\pi}\sqrt{1-y^2}} = \frac{1}{2\sqrt{1-y^2}} \qquad -1 \leq y \leq 1, \quad -\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2}$$

$$f_{X|Y}\left(x, y=\frac{\sqrt{3}}{2}\right) = \frac{1}{2\sqrt{1/4}} = 1 \qquad -\frac{1}{2} \leq x \leq \frac{1}{2}$$

4.5 d

uncorrelated only if $Cov(X,Y) \stackrel{?}{=} 0$

$Cov(X,Y) = 0$, is the same as $E[XY] = E[X] \cdot E[Y]$

$$E[XY] = \int_{-1}^{+1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi} xy \, dx \, dy = \int_{-1}^{1} y \times 0 \, dy = 0$$

$$E[X] = \int_{-1}^{+1} x \frac{2}{\pi} \sqrt{1-x^2} \, dx = 0$$

$$E[Y] = 0$$

$\Rightarrow$ since $E[XY] = E[X]E[Y]$, then X and Y are Uncorrelated.

4.5 e

$$f_{XY}(x,y) = \frac{1}{\pi} \stackrel{?}{=} f_X(x) f_Y(y) = \frac{4}{\pi^2} \sqrt{1-x^2} \sqrt{1-y^2}$$

$\Rightarrow$ Since $f_{XY}(x,y) \neq f_X(x) f_Y(y)$, the X and Y are dependent.

4.5 $\boxed{f, g}$

$$\begin{cases} R = (X^2 + Y^2)^{1/2} \\ \Theta = \tan^{-1}(Y/X) \end{cases}$$

$$F_R(r) = P(R \leqslant r) = P(\sqrt{X^2+Y^2} \leqslant r) = \begin{cases} \pi r^2 (\frac{1}{\pi}) & 0 \leqslant r \leqslant 1 \\ 1 & r > 1 \end{cases}$$

$$\Rightarrow f_R(r) = \frac{d}{dr} F_R(r) = \begin{cases} 2r & 0 \leqslant r \leqslant 1 \\ 0 & \text{o.w.} \end{cases}$$

$$F_\Theta(\theta) = P(\Theta \leqslant \theta) = P(\tan^{-1}(Y/X) \leqslant \theta) = \begin{cases} 0 & \theta < -\pi/2 \\ \frac{\theta + \pi/2}{\pi} & -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2} \\ 1 & \theta > \pi/2 \end{cases}$$

$$f_\Theta(\theta) = \frac{d}{d\theta} F_\Theta(\theta) = \begin{cases} 1/\pi & -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2} \\ 0 & \text{o.w.} \end{cases}$$

4.5 $\boxed{h}$ to check r and θ dependency we calculate their jont pdf.

$$\begin{cases} R = \sqrt{X^2+Y^2} = g(X, Y) \\ \Theta = \tan^{-1}(Y/X) = h(X, Y) \end{cases}$$

inverse functions

$X \geqslant 0$: $X = R\cos\Theta$ $\quad Y = R\sin\Theta$

$X < 0$: $X = -R\cos\Theta$ $\quad Y = -R\sin\Theta$

$$f_{R,\Theta}(r,\theta) = \sum' \frac{f_{X,Y}(x_i, y_i)}{\left| \det \begin{vmatrix} \frac{\partial r}{\partial x_i} & \frac{\partial r}{\partial y_i} \\ \frac{\partial \theta}{\partial x_i} & \frac{\partial \theta}{\partial y_i} \end{vmatrix} \right|} = \sum_{\substack{x_i > 0 \\ \text{and} \\ x_i < 0}} f_{XY}(x_i, y_i) \det \begin{vmatrix} \frac{\partial x_i}{\partial r} & \frac{\partial x_i}{\partial \theta} \\ \frac{\partial y_i}{\partial r} & \frac{\partial y_i}{\partial \theta} \end{vmatrix}$$

$$= \frac{1}{\pi} \left\| \begin{matrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{matrix} \right\| + \frac{1}{\pi} \left\| \begin{matrix} -\cos\theta & r\sin\theta \\ -\sin\theta & -r\cos\theta \end{matrix} \right\|$$

$$= \frac{r}{\pi} + \frac{r}{\pi} = \frac{2r}{\pi} \qquad 0 \leq r \leq 1 \ , \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

Since $f_{R,\Theta}(r,\theta) = f_R(r) \cdot f_\Theta(\theta)$, then R and Θ are independant

4.6 a

(k=3)

$$Z = \min\{X_1, X_2, X_3\}$$

$$W = \max\{X_1, X_2, X_3\}$$

$$F_{Z,W}(z,\omega) = Pr(Z \leqslant z, W \leqslant \omega)$$

$$= \begin{cases} Pr(W \leqslant \omega) & , \omega < z \quad \longrightarrow \text{all } X_1, X_2, X_3 \text{ equal} \\ Pr(W \leqslant \omega) - Pr(Z > z, W \leqslant \omega) & , \omega \geqslant z \end{cases}$$

$$= \begin{cases} [F_X(\omega)]^3 & , \omega < z \\ [F_X(\omega)]^3 - [F_X(\omega) - F_X(z)]^3 & , \omega > z \end{cases}$$

$$= [F_X(\omega)]^3 - [F_X(\omega) - F_X(z)]^3 u(\omega - z)$$

4.6 b

$$F_Z(z) = P_Z(Z \leqslant z) = 1 - P(Z > z) = 1 - [1 - F_X(z)]^3$$

$$F_W(\omega) = P_W(W \leqslant \omega) = [F_X(\omega)]^3$$

4.7 a

$$F_Z(z) = \int_{-\infty}^{+\infty} \int_{-\infty}^{y=z-x} \underbrace{f_{XY}(x,y)}_{f_X(x)\, f_Y(y)} \, dy \, dx$$

$$= \int_{-\infty}^{+\infty} f_X(x) \int_{-\infty}^{z-x} f_Y(y) \, dy \, dx$$

convolution

$$f_Z(z) = \frac{d}{dz} F_Z(z) \underset{\text{integral}}{\overset{\text{Liebnits}}{=}} \int_{-\infty}^{+\infty} f_X(x) \, f_Y(z-x) \, dx = f_X(z) * f_Y(z)$$

$$\Phi_Z[j\nu] = E[e^{j\nu z}] = \int_{-\infty}^{+\infty} f_Z(z) \, e^{j\nu z} dz = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_X(x) f_Y(z-x) e^{j\nu z} \, dx \, dz$$

$$= \int_{-\infty}^{+\infty} f(x) \, e^{+j\nu x} dx \int_{-\infty}^{+\infty} f_Y(z-x) \, e^{j\nu(z-x)} \, d(z-x)$$

$$= \Phi_X(j\nu) \cdot \Phi_Y(j\nu)$$ ✓

4.7 b

$$F_W(w) = \int_{-\infty}^{+\infty} \int_{-\infty}^{w+y} \underbrace{f_{XY}(x,y)}_{f_X(x)\, f_Y(y)} \, dx \, dy$$

$$= \int_{-\infty}^{+\infty} f_Y(y) \int_{-\infty}^{w+y} f_X(x) \, dx \, dy$$

$$f_W(w) = \frac{dF_W(w)}{dw} \overset{(\text{Liebnits})}{=} \int_{-\infty}^{+\infty} f_X(w+y) \, f_Y(y) \, dy$$

correlation

$$\Phi_W(ju) = E[e^{ju}] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_X(w+y)\, f_Y(y)\, e^{juw}\, dy\, dw$$

$$= \underbrace{\int_{-\infty}^{+\infty} f_Y(y)\, e^{-juw}\, dy}\int_{-\infty}^{+\infty} f_X(w+y)\, e^{ju(w+y)}\, d(w+y)$$

$$= \Phi_X(ju)\ \Phi_Y(-ju)$$ ✓

4.7 $\boxed{c}$

$$\Phi_W(ju) = \Phi_X(ju)\ \Phi_Y(ju) \xrightarrow{\text{Gaussian}} \left(e^{jmu - \frac{\sigma^2}{2}u^2}\right)\left(e^{-jmu - \frac{\sigma^2}{2}u^2}\right)$$

$$= e^{-\sigma^2 u^2}$$ ✓

$$\Rightarrow \quad f_W(w) = \frac{1}{\sqrt{2\pi(2\sigma^2)}}\, e^{\frac{w^2}{4\sigma^2}}, \quad -\infty < w < +\infty$$

So the subtraction of $X - Y$ has a Gaussian distribution of $N(0, 2\sigma^2)$. ✓

4.8 a

we use the characteristic function of Y:

$$\Phi_Y(j\nu) = \prod_i \Phi_{X_i}(j\nu) = e^{j\nu\left(\sum_i m_i\right) - \nu^2\left(\frac{\sum \sigma_i^2}{2}\right)}$$

$$\Rightarrow f_Y(y) = \frac{1}{\sqrt{2\pi \sum_i \sigma_i^2}} e^{\frac{(y-\sum m_i)^2}{2\sum \sigma_i^2}} \quad -\infty \leq y \leq +\infty$$

4.8 b

$$\Phi_Z(j\nu) = \prod_i e^{\lambda_i(\nu-1)} = e^{\left[\sum_i \lambda_i\right](\nu-1)}$$

$$P_Y(k) = \frac{\sum_i \lambda_i}{k!} e^{-\left[\sum_i \lambda_i\right]} \qquad k=0,1,2,\ldots$$

4.8 c

$$W = Y - \sum a_i = \sum (X_i - a_i) = \sum X_i^{new}$$

$$\Phi_W(j\nu) = \prod_i e^{-b_i|\nu|} = e^{-(\sum b_i)|w|}$$

$$f_W = \frac{\sum_i b_i / \pi}{w^2 + \left(\sum_i b_i\right)^2} \Rightarrow f_Y(y) = \frac{1}{\pi(\sum b_i)}\left[1 + \frac{(y-\sum a_i)^2}{(\sum b_i)^2}\right]^{-1}$$

$$-\infty < y < +\infty$$

4.8 $\boxed{d}$

$$W = \frac{Y}{\sigma^2} = \sum \frac{X_i}{\sigma^2} = \sum X_i^{new}$$

$$\Phi_W(jv) = \prod_i \left(\frac{1}{1-j2v}\right)^{N_i/2} = \left(\frac{1}{1-j2v}\right)^{\frac{\sum_i N_i}{2}}$$

$$f_W(w) = \frac{w^{\frac{\sum N_i}{2}-1}}{2^{\frac{\sum N_i}{2}} \Gamma\left(\sum_i N_i\right)} e^{-w/2} u(w)$$

$$f_Y(y) = \sigma^2 f_W(w) = \frac{y^{\frac{\sum N_i}{2}-1} e^{-\frac{y}{2\sigma^2}}}{2^{\frac{\sum N_i}{2}} \sigma^{\sum N_i} \Gamma\left(\sum N_i\right)} u(y)$$

$$, \quad -\infty < y < +\infty$$