Lecture 1

Schmid / Dr. Schmid (No Maam! :))

AERB 354

- Drop 1 minimal score HW
- midterm 30%, final 30%, HW 30%, participation 10%

This Friday : No class. → videos on Ecampus

Video - part I (8,19-1) Lecture 2

- Deterministic Experiments
  - follow physics Law
  - Reproducible

- Random Experiments
  - ex. communication system. a source generate a code 01100... transmit through channel. Thermal noise distorts signal received at receiver.

To describe a Random Experiment:

1. State the experiment
2. Specify the probability space $(S, \mathcal{F}, P)$

S → sample space

$\mathcal{F}$ → σ-algebra ← all possible events

P → probability assignment / measure

8,19-1 , 8,17-1

Experiments

1. 100 — 90 of \$1 — 9 of \$5 — 1 of \$50. pick a bill and note its value.

2. Toss a die twice. note the outcomes.

3. Pick an integer X between 0 and 9. Then pick another integer Y between 0 and X.

4. Pick a homogeneous stick of length L. Break it at random.

5. Toss a coin, until H is observed.

Sample spaces

1. $S = \{1, 5, 50\}$

2. $S = \{(1,1), (1,2), \dots, (1,6), (2,1), \dots, (2,6), \dots\dots, (6,6)\}$

3. $S = \{(0,0), (1,0), (1,1), \dots, (9,0), (9,1), \dots, (9,9)\}$

4. $S = \{\xi : \xi \in (0, L)\}$

5. $S = \{H, TH, TTH, TTTH, \dots\}$

Elementary Outcomes:

- mutually exclusive : it means if we perform an experiment and observe one outcome out of the sample space, the other outcomes cannot be observed.

Events / algebra

1. $A = \{$ a bill has either \$1 or \$50 $\}$

2. $B = \{$ both outcomes are even $\}$

3. $C = \{$ both outcomes are even and the value of Y is 2 units smaller than X $\}$

4. $D = \{$ the break occurs on the LHS $\}$

5. $E = \{$ H shows up in less than 10 tosses $\}$

+ Set Theory

  - Union $A \cup B$
  - Intersection $A \cap B$
  - Complement $A^c = \bar{A}$

Properties of Sets

- Commutative $A \cup B = B \cup A$ or $A \cap B = B \cap A$
- Associative $A \cap (B \cap C) = (A \cap B) \cap C$
- Distributive $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- De Morgan's Rule $(A \cap B)^C = A^C \cup B^C$

ex. Experiments ②

$A = \{$ both outcomes are even $\}$

$B = \{$ the sum of dots is $> 6 \}$

$$\Rightarrow \begin{cases} A = \{(2,2), (2,4), (2,6), (4,2), (4,4), (4,6), (6,2), (6,4), (6,6)\} \\ B = \{(1,6), (2,5), (2,6), \dots, (6,1), \dots, (6,6)\} \end{cases}$$

$$A \cap B = \{(2,6), (4,4), (4,6), (6,2), (6,4), (6,6)\}$$

$$(A \cup B)^C = A^C \cap B^n = \{(1,1), (1,2), \dots, (1,5), (2,1), (2,3), (3,1), (3,3), (4,1), (5,1)\}$$

Axioms of Probability

1, $0 \leq P(A) \leq 1$

2, $P(S) = 1$

3, A and B, and $\overbrace{A \cap B = \emptyset}^{\text{mutually exclusive}}$, then $P(A \cup B) = P(A) + P(B)$

3.a, $A_1, \dots, A_n$ are n events if $A_i \cap A_j = \emptyset$, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

Corollaries

1. $P(A^c) = 1 - P(A)$ since $\overbrace{A \cap A^c = \emptyset}^{\text{mutually exclusive}}$, $A \cup A^c = S$

2. $P(\emptyset) = 0$ since $P(S) = 1$, $\emptyset^c = S$

3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

4. $P\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} P(A_k) - \sum_{k<i} P(A_k A_i) + \sum P(A_k \cap A_i \cap A_j) - \dots$

$$\dots + (-1)^{n+1} P(A_1 \cap A_2 \cap \dots \cap A_n)$$

* 5. Union Bound : $P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$

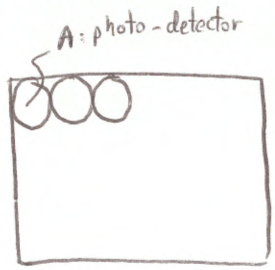
Video - Part II (Lecture 2 Cont'd)

$$(S, \mathcal{F}, P)$$

1. S is discrete. $\{a_1, \ldots, a_n\}$, $0 \leq P(a_i) \leq 1$, $\sum_{i=1}^{n} P(a_i) = 1$

2. S is countable, but infinite. $\{H, TH, TTH, \ldots\}$, $\sum_{i=1}^{+\infty} P(a_i) = 1$

Example CCD camera



Poisson process

$$P(1) = \lambda A \cdot e^{-\lambda A}$$

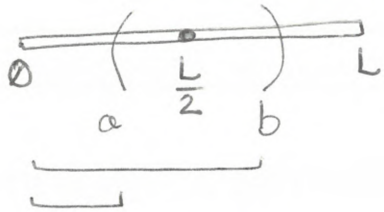
λ = intensity

$$P(0) = e^{-\lambda A}$$

$$P(k) = \frac{(\lambda A)^k}{k!} e^{-\lambda A} \qquad P(2) = \frac{(\lambda A)^2}{2!} e^{-\lambda A}$$

$$\sum_{k=0}^{+\infty} \frac{(\lambda A)^k}{k!} e^{-\lambda A} = e^{-\lambda A} \underbrace{\sum_{k=0}^{+\infty} \frac{(\lambda A)^k}{k!}}_{e^{\lambda A}} = 1$$

3. S is continuous.



$$(0,b) = \overbrace{(0,a) \cup [a,b)}^{\text{mutually exclusive}}$$

$$P(X \in (0,b)) = P(X \in (0,a) \cup X \in (a,b))$$

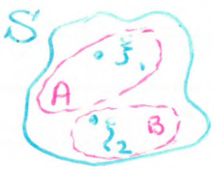
X → break occurs

$$= P(X \in (0,a)) + P(X \in (a,b))$$

$$\Rightarrow \boxed{P(X \in (a,b)) = P(X \in (0,b)) - P(X \in (0,a))}$$

Lecture 3

$\mathcal{S}$: sample space



- ξ_i's are Mutually Exclusive ~~Events~~ outcomes.

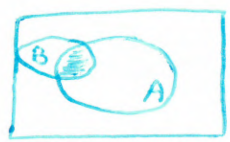
$\mathcal{F}$: σ-algebra

- the application of set theory on sets of outcomes

$\mathcal{P}$: Probability assignments

conditional probability

A, B are events



$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

If $A = B \Rightarrow P(B|A) = 1$

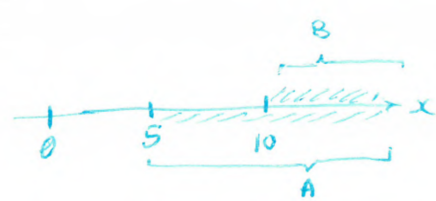
ex.

$$P(\text{Lifetime} \geq x) = e^{-x}, \quad x \geq 0$$

Events:

$A = \{\text{Lifetime} > 5\}$

$B = \{\text{Lifetime} > 10\}$



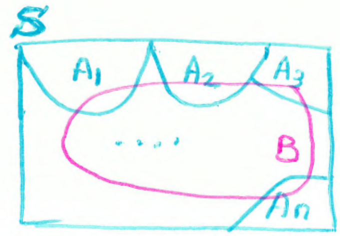
$P(A \cap B) = P(B) = e^{-10}$

$P(A \cup B) = P(A) = e^{-5}$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{e^{-10}}{e^{-5}} = e^{-5}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{e^{-10}}{e^{-10}} = 1$$

Theorem of Total Probability



$\{A_i\}_{i=1}^{n}$ are mutually exclusive.

$A_i \cap A_j = \emptyset$

$\bigcup_{i=1}^{n} A_i = S$ $\quad$ A_i's are portions of S.

$$B = B \cap S = B \cap \Big\{ \overbrace{\bigcup_{i=1}^{n} A_i}^{M.E.} \Big\} \xrightarrow[\text{Law}]{\text{Distributive}} \bigcup_{i=1}^{n} \overbrace{\{B \cap A_i\}}^{M.E.}$$

$$\Rightarrow P(B) = P\Big(\bigcup_{i=1}^{n} \overbrace{\{B \cap A_i\}}^{M.E.} \Big) = \sum_{i=1}^{n} P(B \cap A_i) = \sum_{i=1}^{n} P(B|A_i)\, P(A_i)$$

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ex.

Box 1: 99 R, 1 W $\qquad$ Box 2: 99 W, 1 R

P (to pick a W ball) = ?

Ball #:	1st	2nd		99th	100th
S: Box #	(1, r)	(2, r)	. . . . . .	(1, r)	(1, w)
	(2, w)	(2, w)	. . . . . .	(2, w)	(2, r)

Events: $A = \{\text{Box 1 is selected}\}$, $A^c = \{\text{Box 2 is selected}\}$

$B = \{\text{W ball is selected}\}$, $B^c = \{\text{R ball is selected}\}$

$$P(B) = P(B|A)\,P(A) + P(B|A^c)\,P(A^c)$$

$$\left.\begin{array}{l} P(A) = P(A^c) = \frac{1}{2} \\ P(B|A) = \frac{1}{100} \\ P(B|A^c) = \frac{99}{100} \end{array}\right\} \Rightarrow P(B) = \frac{1}{2}\frac{1}{100} + \frac{1}{2}\frac{99}{100} = \frac{1}{2} \; \blacksquare$$

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myself notes

→ Disjoint vs. Mutually Exclusive. Disjoint events never occur at the same time $A \cap B = \emptyset$.

M.E. events are another name for them. $P(A \cap B) = 0$.

$$\boxed{\text{Disjoint} \equiv \text{Mutually Exclusive}}$$

Lecture 4

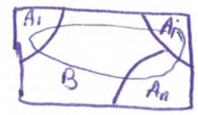
Given $P(B|A_j)$, $P(A_j)$
a priori

$P(A_i|B) = ?$
a posteriori

Review:

Thm. of Total Probability

$$P(B) = P(B|S) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$$



$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(B|A_i)P(A_i)}{P(B)}$$

$$= \frac{P(B|A_i)\,P(A_i)}{\sum_{i=1}^{n} P(B|A_i)\,P(A_i)} \quad \text{(Bayes' Theorem)}$$

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ex. Sample space

Box				
1	(1, r)	(1, r)	....	(1, w)
2	(2, w)	(2, w)	....	(2, r)

2 events:

$A = \{$Box 1 is selected$\}$

$B = \{$white ball is selected$\}$

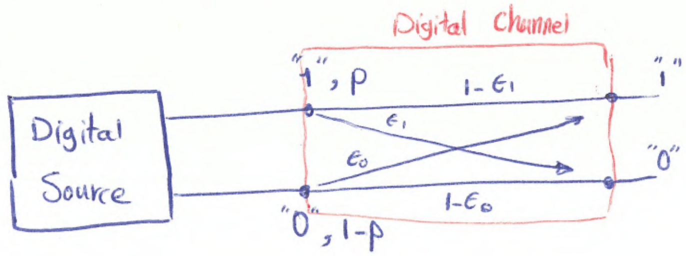
$$P(B) = P(B|A)\,P(A) + P(B|A^c)\,P(A^c)$$

$$= \frac{1}{100} \times \frac{1}{2} + \frac{99}{100} \times \frac{1}{2} = \frac{1}{2}$$

$P(A|B) = ? = P($ white ball came from box 1 $)$

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)} = \frac{\frac{1}{100} \times \frac{1}{2}}{\frac{1}{2}} = \frac{1}{100}$$

ex. (Binary Asymmetric Channel = BAC)



sample space S

		channel output 1	channel output 0
source output	1	(1, 1)	(1, 0)
	0	(0, 1)	(0, 0)

$A = \{ 1 \text{ is on the source output} \}$

$B = \{ 1 \text{ is on the channel output} \}$

$$P(A) = p \quad , \quad P(A^c) = 1 - p$$

Transition Probabilities in Channel $\rightarrow$ $P(B|A) = 1-\epsilon_1 \quad , \quad P(B|A^c) = \epsilon_0 \quad , \quad P(B^c|A^c) = 1-\epsilon_0 \; , \; P(B^c|A) = \epsilon_1$

Find $P(B)$, $P(B^c)$, $P(A|B)$, $P(A^c|B)$, $P(A^c|B^c)$, $P(A|B^c)$.

$$P(B) = P(B|A)P(A) + P(B|A^c)P(A^c)$$

$$= (1-\epsilon_1)\,p + \epsilon_0\,(1-p) =$$

$$P(B^c) = 1 - P(B)$$

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)} = \frac{(1-\epsilon_1)\,p}{(1-\epsilon_1)p + \epsilon_0(1-p)}$$

$$P(A^c|B) = 1 - P(A|B)$$

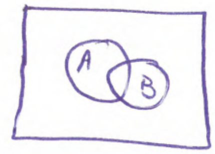
$$P(A^c|B^c) = \frac{P(B^c|A^c)\,P(A^c)}{P(B^c)} = \frac{(1-\epsilon_0)(1-p)}{1-P(B)}$$

$$P(A|B^c) = 1 - P(A^c|B^c)$$

Independence

A and B are independent iff

$$P(A \cap B) = P(A) \cdot P(B)$$



or

$$P(B|A) = P(B) = P(B|S)$$

Lecture 5

ex. $P(H) = a$ 2 tosses

$P(T) = 1 - a$ $S = \{(HH), (HT), (TH), (TT)\}$

$A = \{$ "H" is on the 1st trial $\} = \{(HH), (HT)\}$

$B = \{$ "H" is on the 2nd trial $\} = \{(TH), (HH)\}$

$$A \cap B = \{(HH)\}$$

Are A and B dependent?

$$P(A \cap B) = a^2$$

$$P(A) . P(B) = \left[a^2 + \underbrace{a(1-a)}_{a - a^2}\right] \times \left[\underbrace{(1-a)a}_{a - a^2} + a^2\right] = a^2$$

A and B are independent.

8,26 - 1 , 8/24 - 2

Defn. Events A, B, C are mutually independent if

1. They are pairwise independent: $\begin{cases} P(A\cap B) = P(A).P(B) \\ P(A\cap C) = P(A).P(C) \\ P(B\cap C) = P(B).P(C) \end{cases}$

2. $P(A\cap B\cap C) = P(A).P(B).P(C)$

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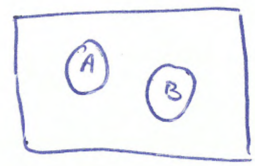
ex. $P(A\cap B) = p$ $\qquad P(A) = P(B) = P(C) = \frac{1}{5}$

$= P(A\cap C)$

$= P(B\cap C)$

, $P(A\cap B\cap C) = p$



A and B are mutually exclusive and they are dependent. if A occurs, B cannot

If $p = \frac{1}{25}$, the 4^{th} equation is not satisfied.

If $p = \frac{1}{125}$, the top 3 equations are not satisfied.

Sequence of Trials

$(A_1, A_2, \ldots, A_n)$ $\qquad P(A) = p$

Binary: $\underbrace{(A, A^c, \ldots, A, A)}$ $\qquad P(A^c) = 1-p$

k "A"s and $(n-k)$ "A^c"s.

$$P(A, A^c, \ldots, A, A) = p(1-p) \times \cdots \times p \times p$$

$$= p^k (1-p)^{n-k}$$

Sample Space of n-length tuples

$(A, \ldots, A, \ldots, A, A)_{1\times n}$ all A's $P(A \cdots A) = p^n$

$$n = \binom{n}{1} \begin{cases} \overbrace{A \cdots\cdots A}^{n-1} \ \overbrace{A^c}^{1} \\ A \cdots\cdots A^c \ A \\ \vdots \\ A^c \cdots\cdots A \ A \end{cases} \quad P(A \cdots A^c) = p^{n-1}(1-p)$$

$$\binom{n}{2} \begin{cases} \underbrace{A \cdots\cdots A}_{n-2} \ \underbrace{A^c A^c}_{2} \\ \vdots \\ A^c A^c A \cdots\cdots A \end{cases} \quad P(A \cdots A A^c A^c) = p^{n-2}(1-p)^2$$

$$\vdots$$

$$P(\underbrace{A \cdots A}_{k} \ \underbrace{A^c \cdots A^c}_{n-k}) = p^k \cdot (1-p)^{n-k}$$

$B = \{ k \ A\text{s are observed} \}$

$$P(B) = \binom{n}{k} p^k (1-p)^{n-k}$$

$C = \{ k^* \text{ or fewer } A\text{s are observed} \}$

$$P(C) = \sum_{k=0}^{k=k^*} \binom{n}{k} p^k (1-p)^{n-k}$$

8,26-2

Binomial Theorem

$$\sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} = (p + 1 - p)^n = 1$$

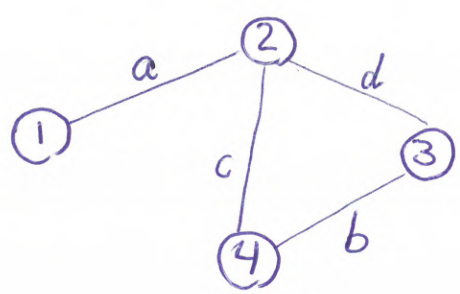
Stoch - Lecture 6

Binomial Probability Law

$$P(k \text{ hats in } n \text{ tosses}) = \binom{n}{k} p^k (1-p)^{n-k}$$

$p \equiv$ probability of a single "H"

ex.



each link may be in state "operate" or "does not operate".

$$P(\underbrace{\text{all nodes are connected}}_{B} \mid \underbrace{3 \text{ links operate}}_{A}) = ?$$

4-tuple Sample Space. $\|S\| = 16$ $\qquad a$ or a^c / $\bar{a}$

$$S = \{(a,b,c,d), (a^c, b, c, d), (a, b^c, c, d), \ldots, (a^c, b^c, c^c, d^c)\}$$

$P(a) = p$, $\quad P(a^c) = 1-p$

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

$B = \{(a,b,c,d), (a, b^c, c, d), (a, b, c^c, d), (a, b, c, d^c)\}$

$A = \{(a^c, b, c, d), (a, b^c, c, d), (a, b, c^c, d), (a, b, c, d^c)\}$

$A \cap B = \{ (a, b^c, c, d), (a, b, c^c, d), (a, b, c, d^c) \}$

$P(A \cap B) = p^3(1-p) + p^3(1-p) + p^3(1-p) = 3p^3(1-p)$

$P(A) = 4p^3(1-p)$

$\Rightarrow P(B|A) = \dfrac{3p^3(1-p)}{4p^3(1-p)} = \dfrac{3}{4}$

Binary Sequential Trials

$P(\text{"H" is observed on the } k^{th} \text{ toss})$

$= P((\underbrace{T, T, T, \ldots, T}_{\#(k-1)}, \underset{k^{th} \text{ toss}}{H}))$

$P(H) = p$

$P(T) = 1-p$

$= \underbrace{(1-p) \times (1-p) \times \cdots \times (1-p)}_{\#(k-1)} \times p = \underbrace{(1-p)^{k-1} p}_{\text{Geometric Law}}$

$S = \{ H, TH, \cancel{TTH}\ TTH, \ldots \}$

$$\sum_{k=1}^{+\infty} (1-p)^{k-1} p \underset{k-1 \triangleq \ell}{=} p \sum_{\ell=0}^{+\infty} (1-p)^{\ell} = p \cdot \frac{1}{1-(1-p)} = 1 \quad \blacksquare$$

$$P(\underbrace{\text{"H" appears } \underline{\text{after}} \ M^{th} \text{ trial}}_{B}) = ?$$

$$B = \{ \underbrace{TT \cdots T}_{M} H, \ \underbrace{TT \cdots T}_{M+1} H, \ \dots \}$$

$$P(B) = (1-p)^{M} \cdot p + (1-p)^{M+1} \cdot p + (1-p)^{M+2} \cdot p + \dots$$

$$= \sum_{k=M}^{+\infty} (1-p)^{k} p \underset{k-M=\ell}{=} p \sum_{\ell=0}^{+\infty} (1-p)^{\ell+M} = p(1-p)^{M} \frac{1}{1-(1-p)}$$

$$= (1-p)^{M} \quad \blacksquare$$

ex.

$S_1 = \{r, g, b\}$ $\quad P_r, P_g, 1-P_r-P_g$

$S_2 = \{1, 2, \dots, 6\}$ $\quad P_1 = P_2 = \dots = P_6 = \frac{1}{6}$

P(in n independent trials we observe n_r red ball, n_g green balls, $n_r - n_g - n_r$ blue balls)

$$= \frac{n!}{n_r!\, n_g!\, (n-n_r-n_g)!} P_r^{n_r} \cdot P_g^{n_g} \cdot (1-P_g-P_r)^{1-n_r-n_g}$$

Multinomial Law

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ex.1

n terminals

prob. of broadcasting is p

P(exactly one terminal is broadcasting)

Binomial

$$= \binom{n}{1} p (1-p)^{n-1}$$

where maximum happens?

$$\frac{dP(B)}{dp} = n(1-p)^{n-1} - n(n-1)p(1-p)^{n-2}$$

$$\frac{dP(B)}{dp} = 0 \Rightarrow n(1-p)^{n-2}[1-p-np+p] = 0$$

$$\Rightarrow \boxed{p^* = \frac{1}{n} \text{ (potential maximum)}}$$

you have to check 2nd derivative.

ex. 2

$P(t;+\infty)$

$= P(\text{lifetime} \geqslant t)$

$= e^{-(\lambda t)^2}, \quad t \geqslant 0$

we have 8 chips.

each chip has lifetime probability as described.

$P(\text{at least 2 chips are functioning after } 2/\lambda \text{ seconds}) = ?$

ex. 2

8 chips in a system.

$$P(\{\text{lifetime of a chip}\} > t) = e^{-(\lambda t)^2} \qquad t \geq 0$$

Find P(at least 2 chips are functioning after $\frac{2}{\lambda}$ seconds).

binomial

$$P(\text{lifetime} > \tfrac{2}{\lambda}) = e^{-4}$$

$$P(2 \cup 3 \cup 4 \cdots \cup 8) = 1 - P(0 \cup 1) = 1 - \binom{8}{0}(e^{-4})^0(1-e^{-4})^8 - \binom{8}{1}(e^{-4})^1(1-e^{-4})^7$$

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ex. 3 a die is tossed 5 times.

P(2x "1", 1x "4", 2 "5")

multinomial

$$= \frac{5!}{2!\,1!\,2!\,0!} \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^0 \left(\frac{1}{6}\right)^0 \left(\frac{1}{6}\right)^1 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^0$$

"1" "2" "3" "4" "5" "6"

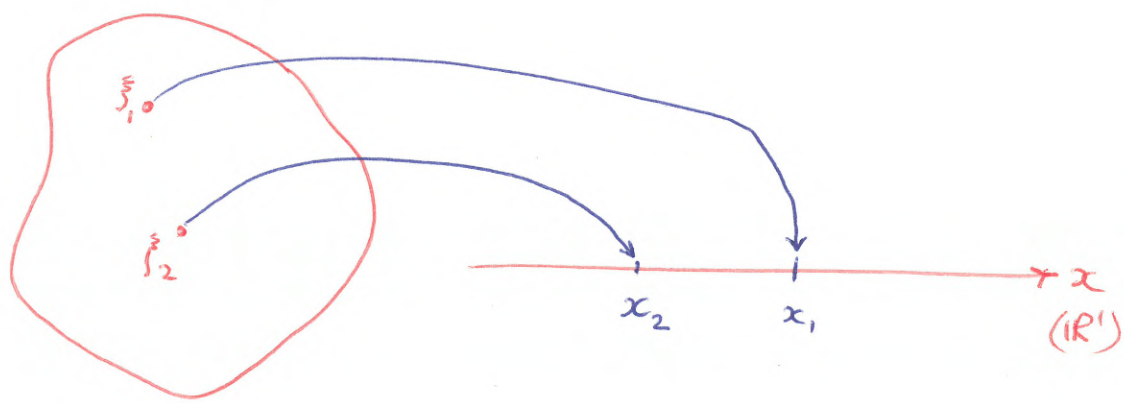
0! 0!

$$= \frac{5!}{2 \times 2} \left(\frac{1}{6}\right)^5$$

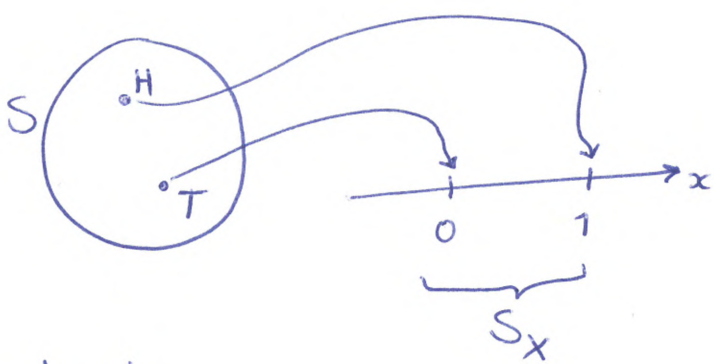
Random Variable

Defn. A real valued function $X(\xi)$ that assigns a number to every outcome $\xi \in S$ is called Random Variable (RV).

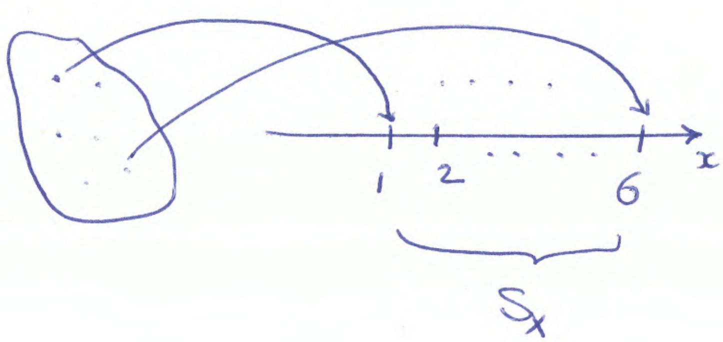
deterministic



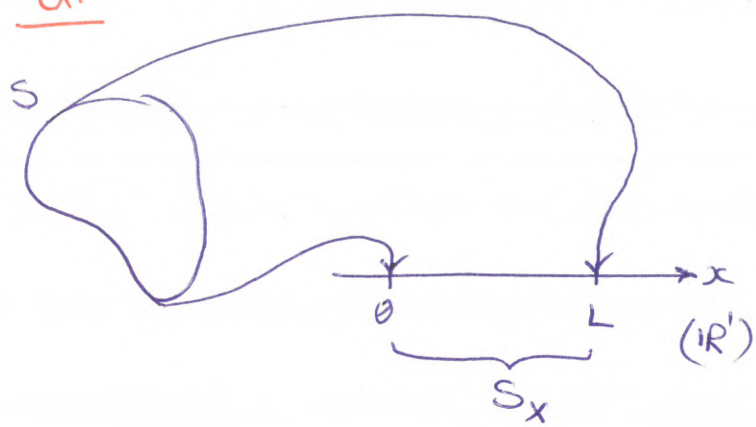
ex. coin toss

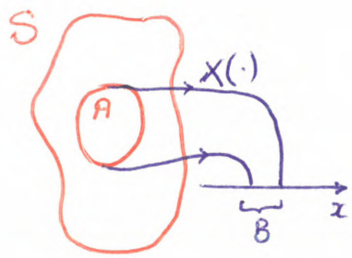


ex. die toss



ex. (break a homogeneous stick)





Defn: A and B are equivalent events if

$$A = \{ \xi : X(\xi) \in B \}$$

$$\Downarrow$$

$$P(A) = P(B)$$

Cumulative Distribution Function (cdf)

Defn: cdf of a random variable $X(\xi)$ is defined as

$$F_X(x) = P(\xi : X(\xi) \leq x) = P(X \leq x)$$

- properties of cdf:

1. $0 \leq F_X(x) \leq 1$

2. $\lim_{x \to +\infty} F_X(x) = P(\xi : X(\xi) \leq x) = 1$
$, x \to \infty$

3. $\lim_{x \to -\infty} F_X(x) = 0$

4. If $a < b$ then

$$F_X(a) \leq F_X(b)$$

5. $F_X(b) = F_X(a) + P_X(x \in (a, b]) \qquad a \leq b$

6. $F_X(x)$ is continuous from the right

$$\lim_{\epsilon \to 0} F_X(x+\epsilon) = F_X(x) = F_X(x^+) \qquad \epsilon > 0$$

7. If X is discrete

$$P(X = x) = F_X(x^+) - F_X(x^-)$$

If X is continuous

$$P(X = x) = 0$$

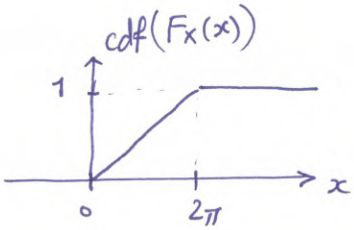
8. $P(X \geq x) = P(\xi : X(\xi) \geq x) = 1 - F_X(x)$

Stoch. Lect.

- note: we also have complex-valued R.V.. (out of scope of this course)

* $F_X(a) = P(X \leqslant a)$: Non-Decreasing Function

ex. 1



cdf of phase during asynchronous reception.

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \dfrac{x}{2\pi}, & 0 \leqslant x \leqslant 2\pi \\ 1, & x > 2\pi \end{cases}$$

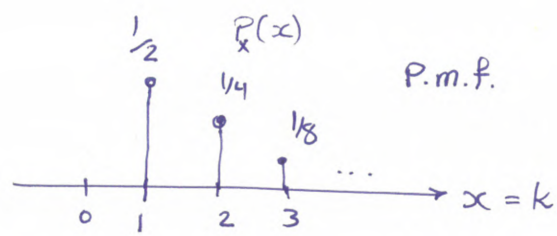
ex. 2 (Geometric Law)

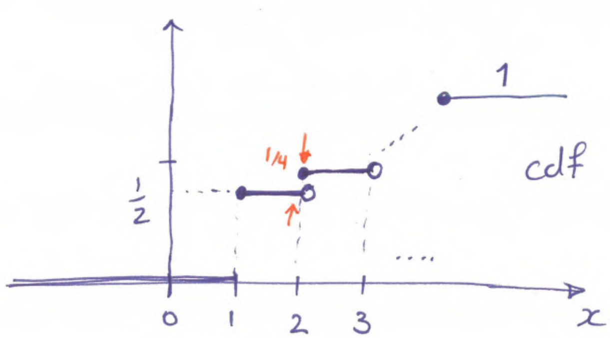
$S_X = \{1, 2, 3, 4, \dots\}$

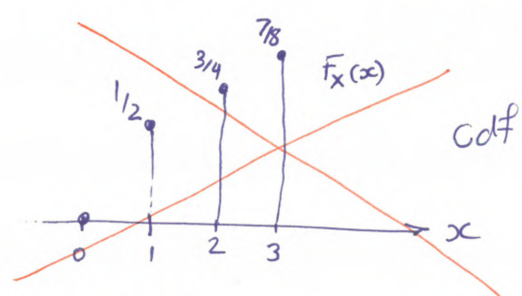
$P(\text{"1"}) = P(\text{"H"}) = p$

$P(X=k) = (1-p)^{k-1} \cdot p$

$p = \frac{1}{2} \rightarrow \quad P(x=k) = (\frac{1}{2})^k$







$\underbrace{\text{CDF}}_{\text{domain}}$ is always Continuous!

$$F_X(x) = \frac{1}{2}u(x-1) + \frac{1}{4}u(x-2) + \frac{1}{8}u(x-3) + \dots = \sum_{k=1}^{+\infty} (\tfrac{1}{2})^k u(x-k)$$

9,2,22 - 1

3 Types of cdf and R.V.'s

1. R.V. X is continuous if $F_X(x)$ is continuous.
2. R.V. X is discrete if $F_X(x)$ is piecewise constant.
3. R.V. X is of mixed type if $F_X(x)$ is piecewise continuous and also piecewise constant.

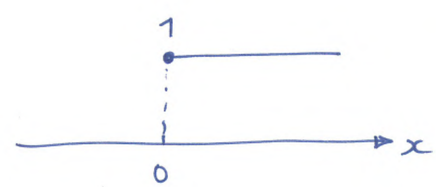
ex. 3

contact: on $\frac{1}{2}$, off $\frac{1}{2}$

If on, I in uniform $[0, 1]$

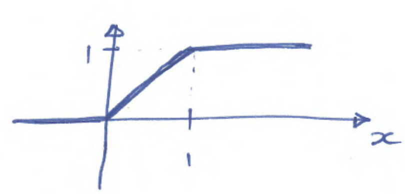
If off, $I = 0$

$F_X(x \mid \text{contact is off}) \rightarrow$



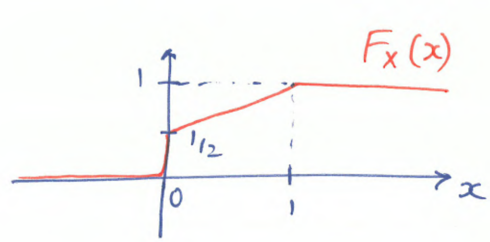
Thm. of Total Probability

$F_X(x \mid \text{contact is on}) \rightarrow$



$$F_X(x) = \frac{1}{2} F_X(x \mid \text{off}) + \frac{1}{2} F_X(x \mid \text{on})$$

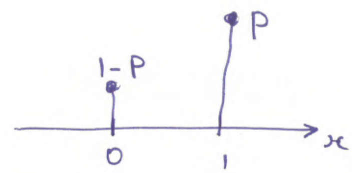
$$= \begin{cases} 0 & , x \le 0 \\ \frac{1}{2}(1+x) & , 0 \le x < 1 \\ 1 & , x \ge 1 \end{cases}$$



Discrete R.V.

1. R.V. X is Bernoulli (p)

$$X = \begin{cases} 1 & \text{with prob. } p \\ 0 & \text{with prob. } 1-p \end{cases}$$



2. $$\underbrace{\sum X_i}_{\text{iid Bernoulli}(p)} = Y \sim \text{Binomia}(n, p)$$

$$P(Y=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

3. Geometric R.V.

$$P(X=k) = (1-p)^{k-1} p \qquad k = 1, 2, 3, \dots$$

4. uniform over n outcomes

$$P(X=k) = \frac{1}{6} \qquad \text{if } n=6$$

5. Poisson $\quad S = \{0, 1, 2, \dots\} \quad np \rightsquigarrow \lambda$

$$P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

The probability density function (pdf)

X is continuous R.V.

$$f(x) = \frac{dF(x)}{dx}$$

pdf

if it exists

$$\lim_{\Delta \to 0} \frac{F(x+\Delta) - F(x)}{\Delta} \cdot \Delta = f_X(x)\, dx$$

rate of change of $F(x)$

Properties

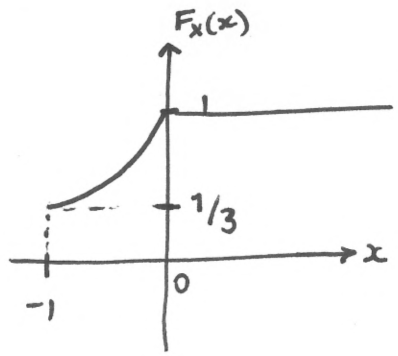
1. $f_X(x) \geq 0$

2. $P(a \leq X \leq b) = \int_a^b f_X(x)\, dx$

3. Let $a \to -\infty$: $P(X \leq b) = \int_{-\infty}^{b} f_X(x)\, dx = F_X(b)$

4. let $a \to -\infty$, $b \to +\infty$: $\int_{-\infty}^{+\infty} f_X(x)\, dx = P(-\infty \leq X < +\infty) = 1$

ex. (mixed type R.V.)



$$F_X(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{3} + \frac{2}{3}(x+1)^2 & -1 \leq x \leq 0 \\ 1 & x > 0 \end{cases}$$

The Dirac delta function

unit step function

Defn 1 $\delta(x) = \dfrac{du(x)}{dx}$

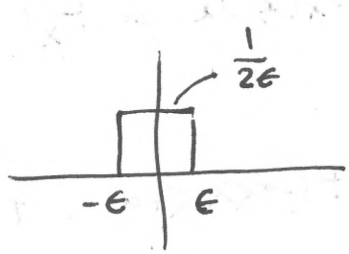
$$u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Defn 2
$$\begin{cases} \delta(x) = 0, \quad \text{if } x \neq 0 \\ \int_{-\infty}^{+\infty} \delta(x) = 1 = \int_{-\epsilon}^{+\epsilon} \delta(x) = 1 \end{cases}$$

Defn 3
$$\int_{-\infty}^{+\infty} \delta(x - x_0)\, g(x)\, dx = g(x_0)$$

Defn 4

$\lim_{\epsilon \to 0}$



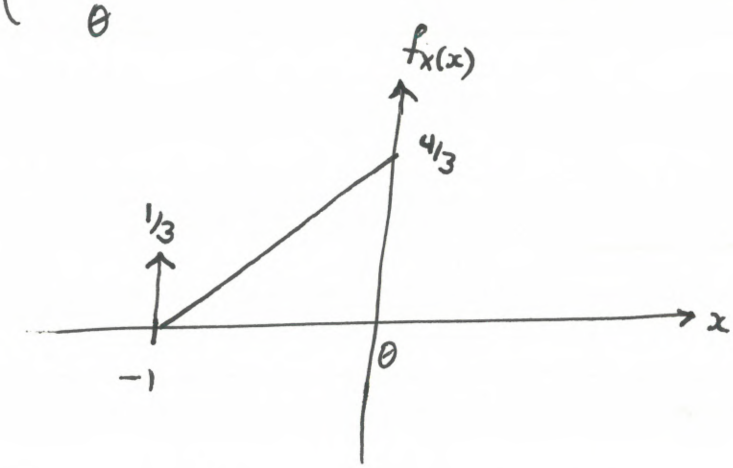
$= \delta(x)$

ex.

$$\Rightarrow \quad F_X(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{3}u(x) + \frac{2}{3}(x+1)^2 & -1 \leq x < 0 \\ 1 & x \geq 0 \end{cases}$$

Generalized pdf

$$f_X(x) = \begin{cases} 0 & \\ \frac{1}{3}\delta(x) + \frac{4}{3}(x+1) & -1 \leq x \leq 0 \\ 0 & \end{cases}$$



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ex. $X \sim$ geometric $\quad X \in \{1, 2, 3, \dots\}$

$p = \frac{1}{2}$

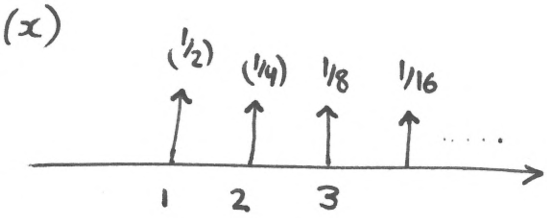
$$P_X(x = k) = \left(\frac{1}{2}\right)^k$$

$$F_X(x) = \frac{1}{2}u(x-1) + \frac{1}{4}u(x-2) + \frac{1}{8}u(x-3) + \dots + \frac{1}{2^k}u(x-k) + \cdots = \sum_{k=1}^{+\infty}\left(\frac{1}{2}\right)^k u(x-k)$$

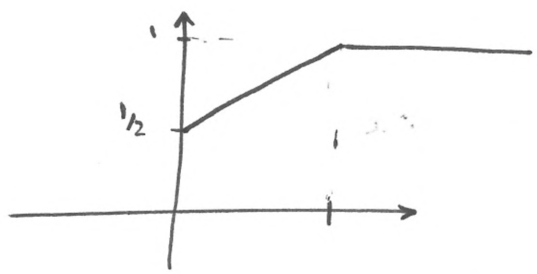
9/7/22 - 2

generalized pdf

$$f_X(x) = \frac{d\,F_X(x)}{dx} = \sum_{k=1}^{+\infty} \frac{1}{2^k}\,\delta(x-k)$$



<u>ex.</u> (monitor)

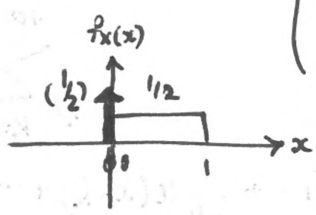


$$F_X(x) = \begin{cases} 0 & x<0 \\ \frac{1}{2}+\frac{x}{2} & 0 \le x < 1 \\ 1 & x > 1 \end{cases}$$

$\frac{1}{2}u(x) + \frac{x}{2}$

$$f_X(x) = \frac{dF_X(x)}{dx} = \begin{cases} 0 & x<0 \\ \frac{1}{2}\delta(x)+\frac{1}{2} & 0 \le x < 1 \\ 0 & x \geq 1 \end{cases}$$

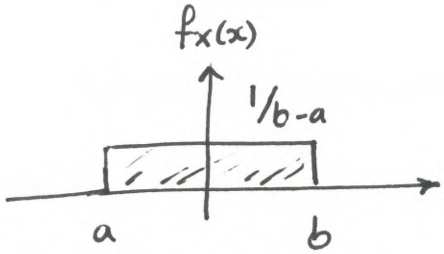
$$\int_{-\infty}^{+\infty} f_X(x) = 1$$



$$\int_{0^-}^{1} \frac{1}{2}\delta(x) + \frac{1}{2}\,dx = \frac{1}{2}+\frac{1}{2} = 1$$

1. X is uniform (a, b)

$X \sim U(a, b) = \text{unif}(a, b)$



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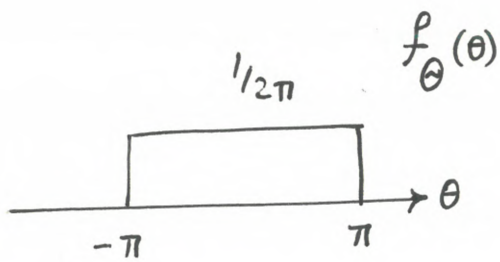
ex.

$\Theta \sim \text{unif}(-\pi, +\pi)$

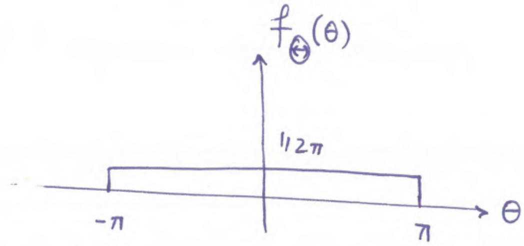


1. $P[\Theta \leq 0]$

2. $P[\Theta \leq \frac{\pi}{2}]$

3. $f_{\Theta}(x \mid \theta \leq 0)$

$\Theta \sim unif(-\pi, \pi)$



$$P(\Theta \leq 0) = \frac{1}{2}$$

$$P(\Theta \leq \pi/2) = \frac{3}{4}$$

$$f_\Theta(x \mid \Theta \leq 0) = ?$$

(conditional cdf)

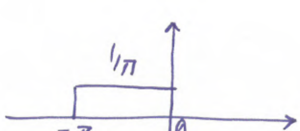
$$F_\Theta(x \mid \Theta \leq 0) = P(\Theta \leq x \mid \Theta \leq 0)$$

$$= \frac{P(\{\Theta \leq x\} \cap \{\Theta \leq 0\})}{P(\Theta \leq 0)} = \begin{cases} \dfrac{P(\Theta \leq x)}{1/2} & , -\pi < x < 0 \\[2ex] \dfrac{P(\Theta \leq 0)}{1/2} & , x \geq 0 \\[2ex] 0 & , x \leq -\pi \end{cases}$$

$$= \begin{cases} 1 + \dfrac{x}{\pi} & , -\pi < x < 0 \\[2ex] 1 & , x \geq 0 \\[2ex] 0 & , x \leq -\pi \end{cases}$$

(conditional pdf)

$$f_\Theta(x \mid \Theta \leq 0) = \frac{d}{dx} F_\Theta(x \mid \Theta \leq 0) = \begin{cases} \dfrac{1}{\pi} & , -\pi < x < 0 \\[2ex] 0 & , \text{else} \end{cases}$$



2. $X \sim$ exponential with parameter λ

$X \sim \text{expon}(\lambda)$

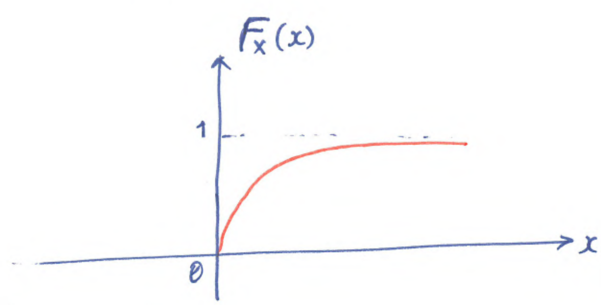
$$f_X(x) = \lambda e^{-\lambda x}, \qquad x \geqslant 0$$

$$= \lambda e^{-\lambda x} u(x)$$

* U is uniform $\xrightarrow{X = -\ln U}$ X is $\text{expon}(\lambda = 1)$

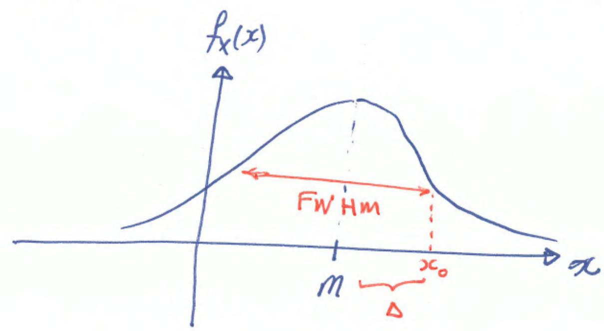
if $X \geqslant 0$:

$$F_X(x) = \int_0^x \lambda e^{-\lambda t} dt = -e^{-\lambda t}\Big|_0^x = 1 - e^{-\lambda x}$$



3. X is Gaussian with mean m and variance σ^2

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}$$



$$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\Delta^2}{2\sigma^2}} = \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}}$$

(Full Width Half Max)

FWHM

ln

$$\frac{\Delta^2}{2\sigma^2} = \ln 2$$

$$\Delta^2 = 2\sigma^2 \ln 2$$

$$\text{FWHM} = 2\Delta = 2\sigma\sqrt{2\ln 2}$$

$$\sigma \to \sqrt{\sigma^2} = \text{std}(X) \geq 0$$

Q-function

$$Q(y) = \int_y^{+\infty} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}}_{\mathcal{N}(0,1)} dt$$

tabulated for $y \geq 0$

- properties

1. $Q(0) = \frac{1}{2}$

2. $Q(z)$ is a decreasing function.

3. $Q(-z) = 1 - Q(z)$

4. $X \sim \mathcal{N}(m, \sigma^2) \to F_X(a) = 1 - Q\left(\frac{a-m}{\sigma}\right) = Q\left(\frac{m-a}{\sigma}\right)$

$$P(x>a) = \int_a^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}} dx$$

$$\frac{x-m}{\sigma} \triangleq t \Rightarrow \underline{x = \sigma t + m} \rightarrow$$

$$x > a$$
$$\sigma t + m > a$$
$$t > \frac{a-m}{\sigma}$$

$$= \int_{\frac{a-m}{\sigma}}^{+\infty} \frac{1}{\cancel{\sqrt{2\pi\sigma^2}}} e^{-t^2/2} (\cancel{\sigma}\, dt)$$

$$= Q\left(\frac{a-m}{\sigma}\right) = P\left(t > \frac{a-m}{\sigma}\right)$$

---

ex. $X \sim N(2, 10)$

$$P(X>4) = Q\left(\frac{4-2}{\sqrt{10}}\right) = Q\left(\frac{2}{\sqrt{10}}\right)$$

$$P(X \leq 6) = 1 - P(X>6) = 1 - Q\left(\frac{6-2}{\sqrt{10}}\right) = 1 - Q\left(\frac{4}{\sqrt{10}}\right)$$

$$P(X>-2) = Q\left(\frac{-2-2}{\sqrt{10}}\right) = Q\left(\frac{-4}{\sqrt{10}}\right) = 1 - Q\left(\frac{4}{\sqrt{10}}\right)$$

property 3

$$P(-2<X<7) = P(X>-2) - P(X>7)$$

$$= Q\left(\frac{-2-2}{\sqrt{10}}\right) - Q\left(\frac{7-2}{\sqrt{10}}\right)$$

$$= 1 - Q\left(\frac{4}{\sqrt{10}}\right) - Q\left(\frac{5}{\sqrt{10}}\right)$$

---

5. Laplace (2 sided exponential)

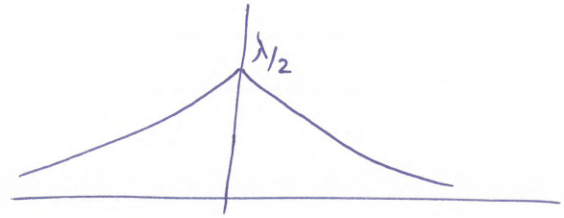
6. Cauchy

7. Gamma

(4) Laplace (double exponential)

- parameter λ

- $f_X(x) = \frac{\lambda}{2} e^{-\lambda |x|}$



* prior on distribution of natural images

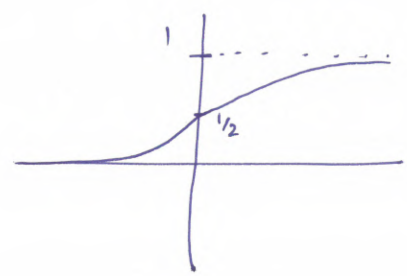
$(a<0)$

- $F_X(a) = \int_{-\infty}^{a} \frac{\lambda}{2} e^{+\lambda x}\, dx = \frac{1}{2} e^{\lambda x}\Big|_{-\infty}^{a} = \frac{1}{2} e^{\lambda a}$

$(a>0)$

- $F_X(a) = \frac{1}{2} + \int_{0}^{a} \frac{\lambda}{2} e^{-\lambda x}\, dx = \frac{1}{2} - \frac{1}{2} e^{-\lambda x}\Big|_{0}^{a} = 1 - \frac{1}{2} e^{-\lambda a}$

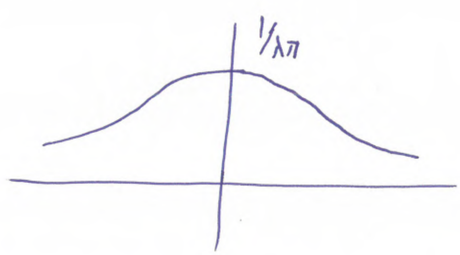
$$\Rightarrow \quad F_X(a) = \begin{cases} \frac{1}{2} e^{\lambda a} & , a<0 \\ 1 - \frac{1}{2} e^{-\lambda a} & , a>0 \end{cases}$$



(5) Cauchy (Heavy_tail R.V.)

- parameter λ

- $f_X(x) = \dfrac{\lambda/\pi}{\lambda^2 + x^2}$

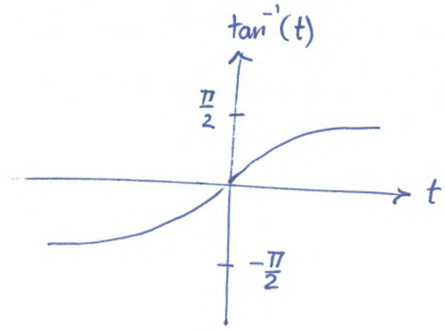


- $\begin{cases} X = \tan U \sim \text{Cauchy} \\ U \sim \text{Uniform} \end{cases}$

- $X = \dfrac{Y_1}{Y_2}$ $>$ indep. Gaussian RVs $\qquad X \sim$ Cauchy, $Y_i \sim$ Gaussians

- $$F_X(a) = \frac{\lambda}{\pi}\frac{1}{\lambda}\int_{-\infty}^{a} \frac{1}{\underset{1}{\cancel{\lambda^2}} + \left(\frac{x}{\lambda}\right)^2}\,\frac{dx}{\lambda}$$

$$= \frac{1}{\pi}\tan^{-1}\left(\frac{x}{\lambda}\right)\Big|_{-\infty}^{a} = \frac{1}{\pi}\left[\tan^{-1}\left(\frac{a}{\lambda}\right) + \frac{\pi}{2}\right]$$



(6) Gamma Family

- parameters : p (order), c (centrality (shift)), λ (scaling)

$$\begin{cases} X \sim \text{Gamma}(p) \\ f_X(x) = \dfrac{x^{p-1}\, e^{-x}}{\Gamma(p)}, \quad x \geqslant 0 \\ \Gamma(p) = \displaystyle\int_0^{+\infty} t^{p-1} e^{-t}\, dt \end{cases}$$

Gamma?

$\lambda f(\lambda(x-c))$

$$f_X(x;\lambda,c,p) = \frac{\lambda \cdot [\lambda(x-c)]^{p-1}\, e^{-\lambda(x-c)}}{\Gamma(p,\lambda,c)} \qquad x \geqslant c$$

☆ If $p = m$ (positive integer)

X is m-Erlang $=$ sum of m iid exponential RV's

☆ If $p = \frac{k}{2}$ and λ

X is chi_square RV with k real degree of freedom

$=$ sum of squared iid Gaussian RV each $N(0,1)$

$$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Poisson ?

$$P_X(0) = P_0 = (1-p)^n$$

$$p \to 0 \quad , \quad n \longrightarrow +\infty \quad \Rightarrow \quad \boxed{p \cdot n \triangleq \alpha}$$

$$P_0 = \left(1 - \frac{\alpha}{n}\right)^n \longrightarrow e^{-\alpha}$$

as $n \to +\infty$

$$\frac{P_{k+1}}{P_k} = \frac{\binom{n}{k+1} p^{k+1} (1-p)^{n-k-1}}{\binom{n}{k} p^k (1-p)^{n-k}} = \frac{p}{1-p} \frac{n-k}{k+1}$$

$$= \frac{\alpha/n}{1-\alpha/n} \frac{n-k}{k+1} \xrightarrow{n\to+\infty} \frac{\alpha}{k+1}$$

$$P_1 = P_0 \cdot \alpha = \alpha e^{-\alpha}$$

$$P_2 = P_1 \cdot \frac{\alpha}{2} = \frac{\alpha^2}{2} e^{-\alpha}$$

$$\vdots$$

$$P_k = \frac{\alpha^k}{k!} e^{-\alpha} \qquad k = 0, 1, 2, 3, \ldots$$

ex. rate = 10^9 bits/sec

$p = 10^{-9}$

$$p \cdot n = 1 = \alpha$$

$$P(X \geqslant 5) = 1 - \sum_{k=0}^{4} \frac{\alpha^k}{k!} e^{-\alpha}$$

Functions of Single R.V.

ex. $X_1, \ldots, X_n, \ldots$

$$|X_i|^2$$ ← some function of RV

---

ex. $X_1, X_2, \ldots, X_n, \ldots$

(sample mean) $S_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ ← //

(sample variance) $V_n = \frac{1}{n}\sum_{i=1}^{n} (X_i - S_n)^2$

---

ex. $X_1, \ldots, X_n$

$P_0(\underline{X})$

$P_1(\underline{X})$

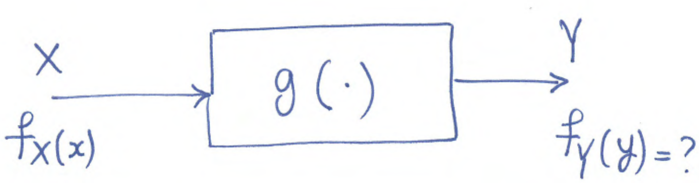
$$\log \frac{P_1(\underline{X})}{P_0(\underline{X})} \underset{H_0}{\overset{H_1}{\gtrless}} \underline{\text{Threshold}}$$

$X \sim$ known pdf $f_X(x)$

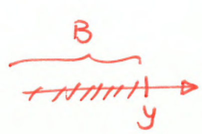
$Y = g(X)$

R.V. deterministic, known

Find pdf $f_Y(y)$.

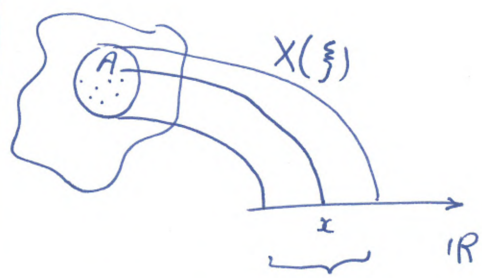


$A = \{x : g(x) \le y\}$ (aside) $\Longleftarrow$



A and B are equivalent.

aside:



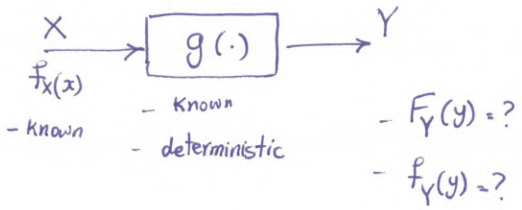
A and B are equivalent.

$P(A) = P(B)$

$A = \{\xi : X(\xi) \in B\}$

stoch. lect.

Transformation of a Single R.V.



Cookbook

1. Find 2 equivalent events.

$$B = (-\infty, y]$$

$$A = \{ x : g(x) \leq y \}$$

$$P(A) = P(B)$$

2. Write cdf of Y in terms of cdf of X

ignore mononocity of g just for a sec.

$$F_Y(y) = P(Y \leq y) = P(g(x) \leq y) = P(x \leq g^{-1}(y)) = F_X(g^{-1}(y))$$

3. $f_Y(y) = \dfrac{d}{dy} F_X(g^{-1}(y))$ → use Leibnitz rule.

→ derivative of a composite function

Find $F_X(g^{-1}(y))$ in closed form, then differentiate.

Aside:

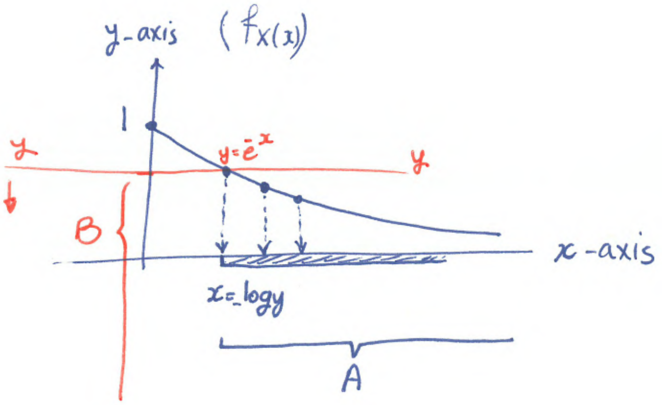
$$\frac{d}{d\xi} \int_{a(\xi)}^{b(\xi)} f(x, \xi)\, dx = \frac{\partial b(\xi)}{\partial \xi} f(b(\xi), \xi) - \frac{\partial a(\xi)}{\partial \xi} f(a(\xi), \xi) + \int_{a(\xi)}^{b(\xi)} \frac{\partial f}{\partial \xi}(x, \xi)\, dx$$

ex. $X \sim \exp(\lambda = 1)$

$$f_X(x) = \lambda e^{-\lambda x}, \qquad x \geq 0 \qquad \text{Find } f_Y(y).$$

$$g(x) = f_X(x) = \lambda e^{-\lambda x} \qquad x \geq 0$$

$$X \longrightarrow \boxed{g(\cdot)} \longrightarrow Y$$



correct ?

$$B = (-\infty, y], \qquad A = \{x : x > \log y\}$$

$$F_Y(y) = \int_{-\ln y}^{+\infty} e^{-x}\,dx = e^{-x}\Big|_{-\ln y}^{+\infty} = -e^{\ln y} = y$$

$$0 \leq y \leq 1$$

$$f_Y(y) = 1, \qquad 0 \leq y \leq 1 \qquad \text{(uniform R.V.)}$$

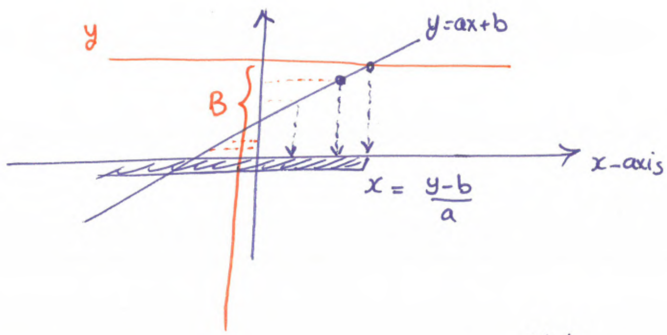
ex. (linear transformation)

$$Y = aX + b$$

$a > 0$ $\qquad$ $X \longrightarrow \boxed{aX + b} \longrightarrow Y$ $\qquad$ $f_Y(y) = ?$

$f_X(x)$

- known



$B = (-\infty, y]$

$A = \left\{ x : x < \frac{y-b}{a} \right\}$

1) $$F_Y(y) = P(Y \leq y) = \int_{-\infty}^{\frac{y-b}{a}} f_X(x)\, dx$$

2) $$F_Y(y) = P(Y \leq y) = P(aX + b \leq y) = P\left(X \leq \frac{y-b}{a}\right)$$

$$= \int_{-\infty}^{\frac{y-b}{a}} f_X(x)\, dx$$

Leibnitz

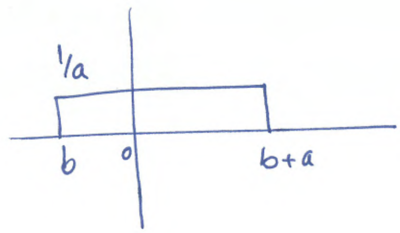
$$f_Y(y) = \frac{1}{a} \cdot f_X\left(\frac{y-b}{a}\right)$$

if $X \sim \text{Uniform}(0, 1)$

$$F_Y(y) = \int_0^{\frac{y-b}{a}} 1 \cdot dx \qquad \left(\text{if} \quad 0 < \frac{y-b}{a} < 1\right)$$

$$= \frac{y-b}{a} \qquad b < y < a+b$$

$$f_Y(y) = \frac{1}{a} \qquad b < y < a+b$$



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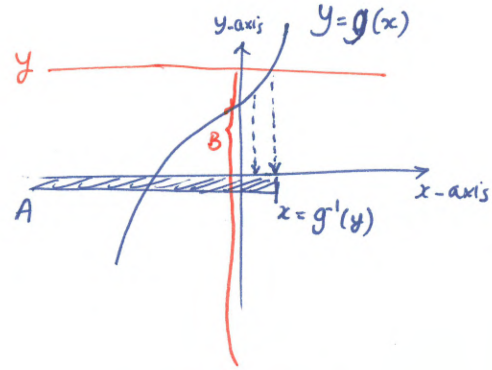
ex.3 (g is monotonic)

1. $g(\cdot)$ is monotonically increasing.

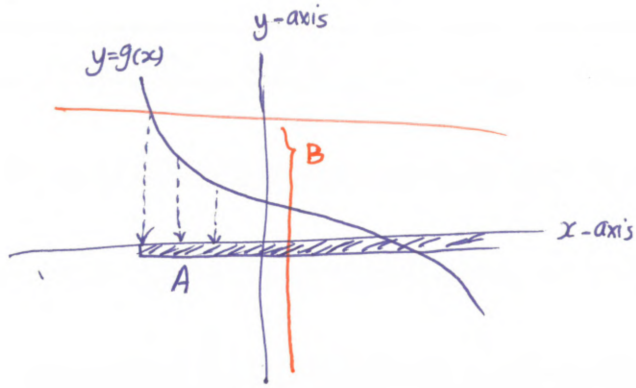


$B = (-\infty, y]$

$A = (-\infty, g^{-1}(y)]$

$$F_Y(y) = \int_{-\infty}^{g^{-1}(y)} f_X(x)\, dx \Rightarrow f_Y(y) = \frac{d}{dy} g^{-1}(y) \cdot f_X(g^{-1}(y))$$

2. g is monotonically decreasing.

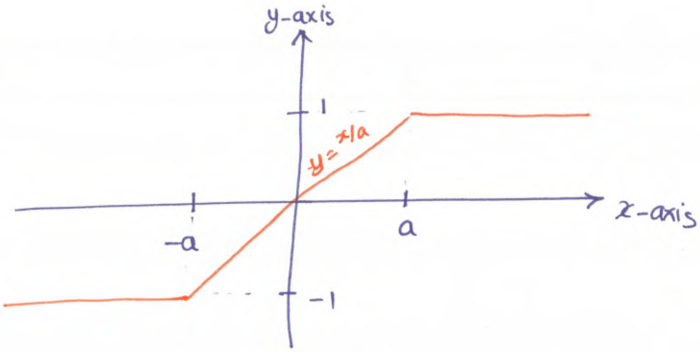


$$F_Y(y) = \int_{g^{-1}(y)}^{+\infty} f_X(x)\,dx \Rightarrow f_Y(y) = -\frac{dg^{-1}(y)}{dy} f_X\left(g^{-1}(y)\right)$$

(Both cases) For any g monotonic:

$$\underbrace{f_Y(y) = \left|\frac{dg^{-1}(y)}{y}\right| f_X\left(g^{-1}(y)\right)}$$

ex. ("Flat Spot" = soft threshold)

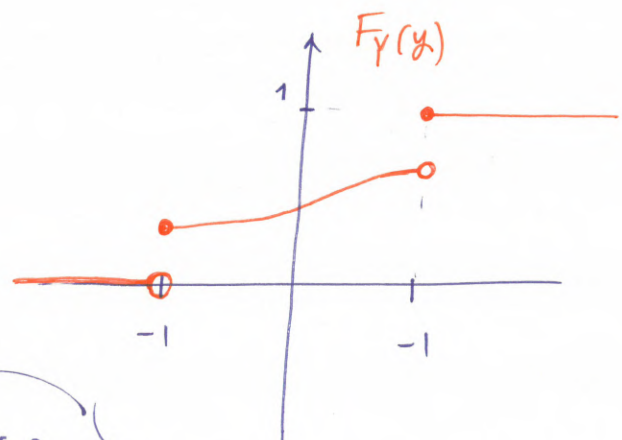


$$P(Y = -1) = \int_{-\infty}^{-a} f_X(x)\,dx = P(x \leq -a)$$

$$F_Y(y) = \begin{cases} 0 & , \; y < -1 \\ P(X \leq -a) & , \; y = -1 \\ \int_{-\infty}^{y \cdot a} f_X(x)\,dx & , \; -1 < y < 1 \\ & , \; y \geq 1 \end{cases}$$

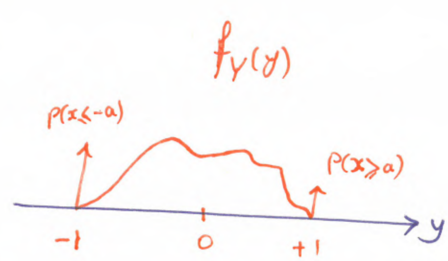
$$P(-1 < Y < 1) = \int_{-\infty}^{y \cdot a} f_X(x)\,dx =$$

$$P(y = 1) = \int_{a}^{+\infty} f_X(x)\,dx$$



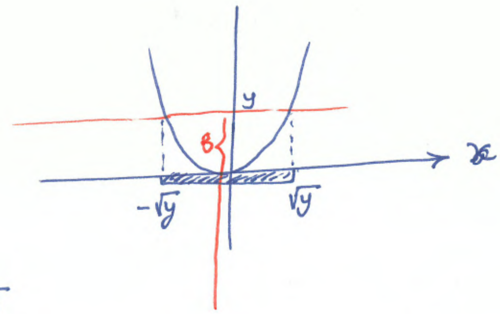
To Differentiate Discontinuities:
USE Generalized PDF.

$$f_Y(y) = \begin{cases} P(x \leq -a)\,\delta(y+1) + a f_X(ya) + P(x \geq a)\,\delta(y-1) & -1 \leq y \leq 1 \\ 0 & \text{else} \end{cases}$$



stoch. lect.

$Y = X^2$



$$F_Y(y) = P(Y \leqslant y) = \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,dx = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y})$$

i. $f_X(x) = e^{-x} \qquad x \geqslant 0$

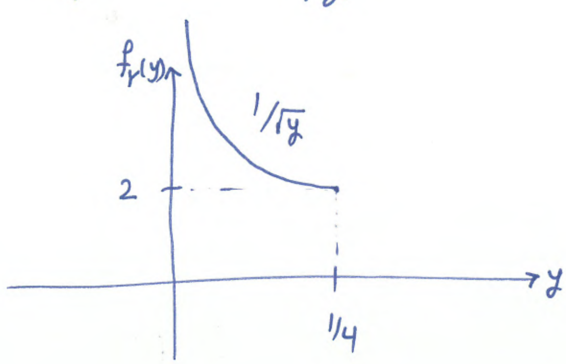
$F_Y(y) = 0 \quad y < 0$

$$F_Y(y) = \int_0^{\sqrt{y}} e^{-x}\,dx = 1 - e^{-\sqrt{y}} \qquad \Rightarrow f_Y(y) = -\frac{1}{2\sqrt{y}} e^{-\sqrt{y}}, \; y \geqslant 0$$

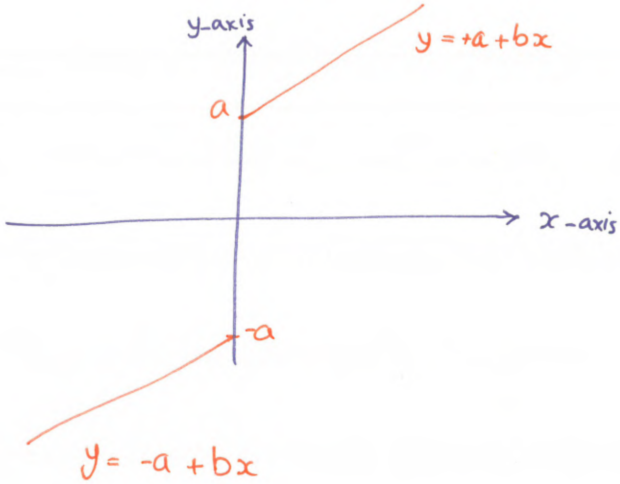
$y \geqslant 0$

ii. $f_X = 1 \quad -0.5 \leqslant x \leqslant 0.5 \quad \Rightarrow$ entire pdf of y will reside between $[-\frac{1}{4}, \frac{1}{4}]$

$$f_Y(y) = \frac{1}{2\sqrt{y}}(1) + \frac{1}{2\sqrt{y}}(1) = \frac{1}{\sqrt{y}} \qquad 0 < y \leqslant 1/4$$



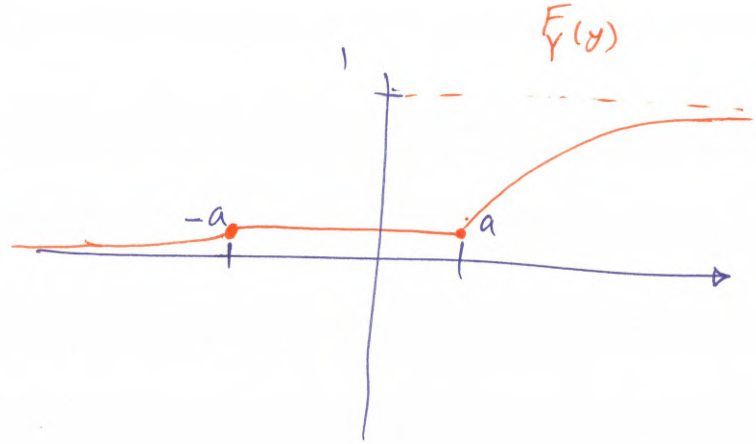
ex. 6 (Jump) $b>0$



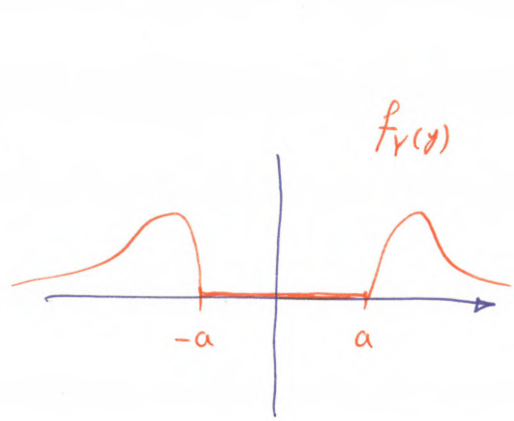
$y < -a$: $$F_Y(y) = \int_{-\infty}^{\frac{y+a}{b}} f_X(x)\, dx$$

$-a \leq y \leq a$: $$F_Y(y) = \int_{-\infty}^{0} f_X(x)\, dx$$

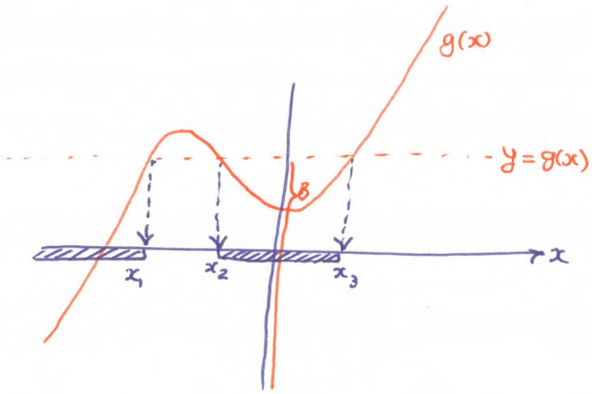
$y > a$: $$F_Y(y) = \int_{-\infty}^{\frac{y-a}{b}} F_X(x)$$



$$f_Y(y) = \begin{cases} \frac{1}{b} f_X\left(\frac{y+a}{b}\right) & , \quad y < a \\ 0 & , \quad -a \leq y \leq a \\ \frac{1}{b} f_X\left(\frac{y-a}{b}\right) & , \quad y > a \end{cases}$$



General Transformation of a Single R.V.



x_1, x_2, x_3 : $g^{-1}(y)$

roots of equation. $\boxed{g^{-1}(y) = x}$

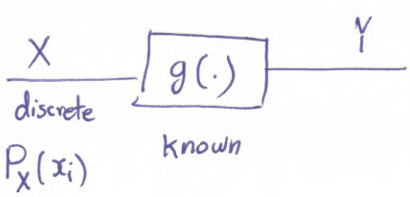
$$F_Y(y) = \int_{-\infty}^{x_1(y)} f_X(x)\,dx + \int_{x_2(y)}^{x_3(y)} f_X(x)\,dx$$

$$f_Y(y) = \frac{dx_1(y)}{dy} f_X(x_1(y)) + \frac{dx_3(y)}{dy} f_X(x_3(y)) - \frac{dx_2(y)}{dy} f_X(x_2(y))$$

Questions to reflect:

1. $f_Y(y)$, and $g(\cdot)$ are known. Can we find $f_X(x)$?

2.



what will be $P_Y(y)$?
(pmf)

Functions of a R.V.

Definition: $E[X] = m_X = \bar{X} = \langle X \rangle$ Expectation $\equiv$ mean

$\xrightarrow{\text{continuous}}$ $= \int_{-\infty}^{+\infty} x \cdot f_X(x)\, dx$

$\xrightarrow{\text{discrete}}$ $= \sum_i x_i \cdot P_X(x_i)$

properties:

1. Linearity $E[aX+b] = a\,E[X] + b$

2. If $x = c$ and $f_X(x)$ is symmetric around c,
   even

   then c is expeted value of R.V. X : $E[X] = c$.

✪ properties are not valid unless $E[|X|] < \infty$

ex.

$$f_X(x) = \frac{1}{\pi(1+x^2)} \quad \text{(Cauchy)}$$

$$E[|X|] = \int_{-\infty}^{+\infty} |x| \cdot \frac{1}{\pi(1+x^2)}\, dx = \lim_{\substack{a \to +\infty \\ b \to -\infty}} \int_{b}^{+a} |x| \cdot \frac{1}{\pi(1+x^2)}\, dx$$

$$= \lim_{\substack{a \to +\infty \\ b \to -\infty}} \left\{ \underbrace{\int_0^a \frac{x}{\pi(1+x^2)}\,dx}_{\text{change of variable}} + \underbrace{\int_b^0 \frac{-x}{\pi(1+x^2)}\,dx}_{\frac{1}{2\pi}\ln(1+b^2)} \right.$$

$$\underbrace{\int_0^a \frac{1}{\pi(1+x^2)} \frac{dx^2}{2}}$$

$$\underbrace{\frac{1}{2\pi} \ln(1+x^2) \Big|_0^a}$$

$$\frac{1}{2\pi} \ln(1+a^2)$$

$$= \lim_{\substack{a \to +\infty \\ b \to -\infty}} \frac{1}{2\pi} \ln(1+a^2).(1+b^2) \longrightarrow +\infty \qquad \text{No convergence!}$$

ex.1 $X \sim \text{Binomial}(n,p)$

$$X = \sum_{i=1}^{n} Y_i \quad , \quad Y_i \sim \text{Bernoulli}(p) \rightarrow E[Y_i] = p$$

$$E[X] = E\left[\sum_{i=1}^{n} Y_i\right] = \sum_{i=1}^{n} \overbrace{E[Y_i]}^{p} = n.p$$

linearity

ex.2 $X \sim \exp(\lambda)$

$$E[X] = \int_0^{+\infty} dx\, x \cdot \lambda e^{-\lambda x} \overset{\text{by parts}}{=\!=\!=} -x e^{-\lambda x}\Big|_0^{+\infty} + \int_0^{+\infty} e^{-\lambda x} dx$$

$$= \frac{e^{-\lambda x}}{\lambda}\Big|_0^{+\infty} = \frac{1}{\lambda}$$

ex.3 $X \sim \mathcal{L}aplace(\lambda)$

$$f_X(x) = \frac{\lambda}{2} e^{-\lambda|x|}$$

even symmetric around zero $\Rightarrow$ $E[X] = 0$.
(property 2.)

Expected Value of a Transformed R.V.

show that $\underset{f_Y(y)}{E}[Y] = \underset{f_X(x)}{E}[g(X)]$ (LOTUS)

$\boxed{\text{proof for } g(\cdot) \text{ monotonic!}}$

$$\underset{f_Y(y)}{E}[Y] = \int_{-\infty}^{+\infty} dy \left|\frac{dg(y)}{dy}\right| f_X(g^{-1}(y)) = \cancel{\overset{+}{\underset{-}{\text{or}}}} \int g(x) \left|\frac{dx}{dy}\right| f_X(x) \left|\frac{dg(x)}{dx}\right| dx$$

$$dy = dg(x) = \frac{dg(x)}{dx} \cdot dx$$

$$= \int_{-\infty}^{+\infty} g(x) f_X(x)\, dx$$

for example: $E[e^X] = \int_{-\infty}^{+\infty} e^X f_{X}(x)\, dx$

stoch. lect.

1. $g(X) = X$

$$\underset{f_Y}{E}[\overbrace{g(X)}^{Y}] = \underset{f_X}{E}[g(X)] = E[X]$$

2. $g(X) = X^k$ $\qquad k = 0, 1, 2, \dots$ $\qquad$ $\boxed{\text{Non-Central Moments}}$

$$m_k = \underset{f_X}{E}[g(X)] = \underset{f_X}{E}[X^k] = \underbrace{\int_{-\infty}^{+\infty} x^k \cdot f_X(x)\, dx}_{\text{continuous}} = \underbrace{\sum_i x_i^k \cdot P_X(x_i)}_{\text{discrete}}$$

$\rightarrow$ k^{th} non-central moment

$$m_0 = E[X^0] = 1$$

$$m_1 = E[X^1] = m_X$$

$$m_2 = E[X^2] = \text{average power in } X$$

3. $g(X) = (X - E[X])^k$, $\qquad k = 0, 1, 2, \dots$ $\qquad$ $\boxed{\text{Central Moments}}$

$$E[(X - E[X])^k] = \mu_k$$

$\rightarrow$ k^{th} central moment

$$\mu_0 = E[(X - E[X])^0] = 1$$

$$\mu_1 = E[X - E[X]] = 0$$

Variance or σ_X^2 or $Var(X)$ $\equiv \mu_2 = E[(X - E[X])^2] = E[X^2 - 2XE[X] + E^2[X]] = E[X^2] - E^2[X]$

$\mu_3 = E[(X - m_X)^3] \equiv$ measure of skewness of $f_X(x)$

$\mu_4 = E[(X - m_X)^4] \equiv$ Kurtoisis $\equiv$ measure of Gaussianity

$$\mu_k = \int_{-\infty}^{+\infty} dx \, (x - \underbrace{E[X]}_{m_X})^k f_X(x)$$

ex.1 [k^{th} non-central moment for Bernoulli(p)]

$$m_k = E[X^k]$$

$$= 1^k \cdot p + 0^k \cdot (1-p)$$

ex.2 $X \sim \mathcal{N}(a, b^2)$

prove $Var(X) = b^2$

$$Var(X) = \int_{-\infty}^{+\infty} (x-a)^2 \frac{e^{-\frac{(x-a)^2}{2b^2}}}{\sqrt{2\pi b^2}} dx = \int_{-\infty}^{+\infty} dy \; y^2 \cdot \frac{e^{-y^2/2b^2}}{\sqrt{2\pi b^2}}$$

$$= \frac{1}{\sqrt{2\pi b^2}} \frac{d}{d(\underbrace{1/2b^2}_{\xi^2})} \int_{-\infty}^{+\infty} \frac{e^{-y^2/2b^2} dy}{1}$$

5. $g(X) = e^{j\nu X}$ → $\boxed{\text{Characteristic Function of Random Variable } X}$, $j^2 = -1$

$$E[g(X)] = E[e^{j\nu X}] = \int_{-\infty}^{+\infty} e^{j\nu X} f_X(x)\, dx = M_X(j\nu)$$

$$= \Phi_X(j\omega)$$

Given $M_X(j\nu)$ we can write Fourier Transform:

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} M_X(j\nu)\, e^{-j\nu x}\, d\nu$$

properties:

1. $M_X(j\nu)$ is continuous in ν.
2. $M_X(j0) = 1$
3. $|M_X(j\nu)| \leq 1$
4. $M_X(j\nu)$ is non-negative definite.
5. $M_X(j\nu)$ generates non-central moments.

$$m_k = \int_{-\infty}^{+\infty} x^k f_X(x)\, dx$$

$$m_k = \frac{d}{d(j\nu)^k} M_X(j\nu) \Big|_{\nu=0}$$

Bochner's Theorem

$A(j\nu)$ is characteristic function if:

1. $A(j\nu)$ is continuous.
2. $A(j0) = 1$
3. $A(j\nu)$ is non-negative definite.

ex.1 $X \sim \exp(\lambda)$

$$M_X(j\nu) = E\left[e^{j\nu X}\right] = \int_0^{+\infty} e^{j\nu x} \cdot \lambda e^{-\lambda x}\, dx$$

$e^{-(\lambda - j\nu)x}$

$$= -\lambda \left. \frac{e^{-(\lambda - j\nu)}}{\lambda - j\nu} \right|_0^{+\infty} = \frac{\lambda}{\lambda - j\nu}$$

(we knew $E[X] = \frac{1}{\lambda}$)

$$m_1 = \frac{d}{dj\nu} \left. \frac{\lambda}{\lambda - j\nu} \right|_{\nu=0} = \frac{\lambda}{(\lambda - j\nu)^2} = \frac{1}{\lambda} = E[X]$$

$$m_2 = \frac{d^2}{d(j\nu)^2}(\dots) = \left. \frac{+2\lambda}{(\lambda - j\nu)^3} \right|_{\nu=0} = \frac{2}{\lambda^2} = E[X^2]$$

$$\mathrm{Var}(X) = -\left(\frac{1}{\lambda}\right)^2 + \frac{2}{\lambda^2} = \frac{1}{\lambda^2}$$

stoch. lect.

$$\Phi_X(s) = E[e^{sX}]$$

$$= \int_{-\infty}^{+\infty} e^{sx} f_X(x)\,dx$$

$$M_X(j\nu) = E[e^{j\nu X}]$$

ex. $X \sim \text{Poisson}(\lambda)$

$$P(X=n) = \frac{\lambda^n}{n!} e^{-\lambda}$$

$$M_X(j\nu) = E[e^{j\nu X}] = \sum_{n=0}^{+\infty} e^{j\nu n} \frac{\lambda^n}{n!} e^{-\lambda} = e^{-\lambda} \sum_{n=0}^{+\infty} \frac{(e^{j\nu}\lambda)^n}{n!}$$

$$= e^{-\lambda} \cdot e^{\lambda e^{j\nu}}$$

$$E[X] = \frac{d}{d(j\nu)} M_X(j\nu)\Big|_{\nu=0} = e^{-\lambda} e^{\lambda e^{j\nu}} \lambda \cdot e^{j\nu}\Big|_{\nu=0} = \lambda$$

$$E[X^2] = \frac{d^2}{d(j\nu)^2} M_X(j\nu)\Big|_{\nu=0} = e^{-\lambda} e^{\lambda e^{j\nu}} (\lambda e^{j\nu})^2 + e^{-\lambda} e^{\lambda e^{j\nu}} \lambda e^{j\nu}\Big|_{\nu=0} = \lambda^2 + \lambda$$

$$Var[X] = E[X^2] - E^2[X] = \lambda^2 + \lambda - \lambda^2 = \lambda$$

ex. $X \sim \text{Bernoulli}(p)$

$$M_X(jv) = E[e^{jvX}] = e^{jv0} \cdot (1-p) + e^{jv1} \cdot p = \underline{1 - p + e^{jv}p}$$

ex. $X \sim \mathcal{N}(0, 1)$

$$M_X(jv) = E[e^{jvX}] = \int_{-\infty}^{+\infty} \underbrace{e^{jvx} \frac{1}{\sqrt{2\pi}} e^{-j\frac{x^2}{2}}}\, dx$$

trying to make it another Gaussian

$$\underbrace{e^{-\frac{1}{2}(x^2 - 2jvx + (jv)^2 - (jv)^2)}}_{e^{-\frac{1}{2}(x-jv)^2} \cdot e^{\frac{+(jv)^2}{2}}}$$

$$= \frac{e^{(jv)^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x-jv)^2} dx = 1$$

$$= e^{-v^2/2}$$

$$Y = a + bX \qquad X \sim N(0,1) \qquad \Rightarrow Y = N(a, b^2)$$

$$M_Y(jv) = \underset{f_Y}{E}\left[e^{jvY}\right] = \underset{f_X}{E}\left[e^{jv(a+bX)}\right] = e^{jva}\, E_{f_X}\left[e^{jvbx}\right]$$

$$vb = u \quad \Rightarrow \quad E\left[e^{juX}\right] = e^{\frac{(ju)^2}{2}}$$

$$\Rightarrow M_Y(jv) = e^{jva}\, e^{\frac{(jvb)^2}{2}} = e^{-\frac{1}{2}b^2v^2} \cdot e^{jva}$$

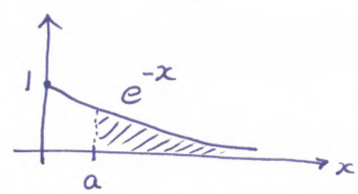
$$v^2_{\text{coeff.}} \equiv -\frac{1}{2}b^2$$

$$v^1_{\text{coeff.}} \equiv ja$$

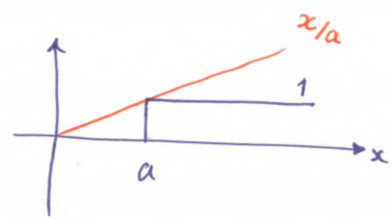
Markoff and Chebyshev Inequalities

Markov?!

1. R.V. X with $x \in [0, +\infty)$



$$P(X>a) = \int_a^{+\infty} f_X(x)\,dx$$



$$= \int_0^{+\infty} f_X(x)\, u(x-a)\,dx$$

$$\leq \frac{1}{a}\int_0^{+\infty} x\, f_X(x)\,dx = \frac{E[X]}{a}$$

ex. $X \sim \text{geometric}(p)$

$$P(X=k) = (1-p)^{k-1} \cdot p$$

$$E(X) = 1/p$$

$$P(X \geq \theta) \leq \frac{1}{10p}$$ markoff

2. R.V. X with mean m_X and variance σ_X^2

$$P(X > m_X + \epsilon)$$

$$P(X \leq m_X - \epsilon)$$

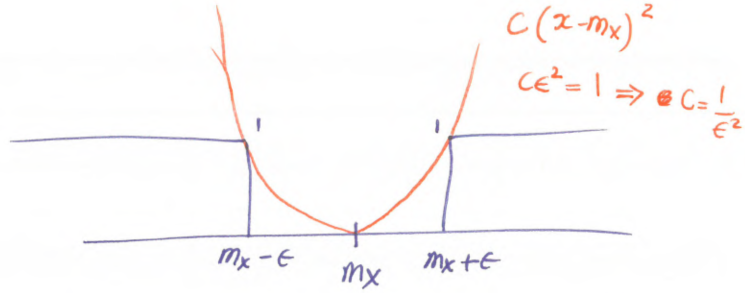
$$P(|X - m_X| > \epsilon) = P(\{X > m_X + \epsilon\} \cup \{X < m_X - \epsilon\})$$

$$= P(X > m_X + \epsilon) + P(X < m_X - \epsilon)$$

$$= \int_{m_X+\epsilon}^{+\infty} f_X(x)\,dx + \int_{-\infty}^{m_X-\epsilon} f_X(x)\,dx$$

$$= \int_{-\infty}^{+\infty} u(x-(m_X+\epsilon)) + u(-(x-(m_X+\epsilon))\, f_X(x)\,dx$$

$$\leq \underbrace{\int_{-\infty}^{+\infty} \frac{(x-m_X)^2}{\epsilon^2} f_X(x)\, dx}_{\frac{\sigma_X^2}{\epsilon^2}}$$



$$\Rightarrow \boxed{P\left(|X-m_X| > \epsilon\right) \leq \frac{\sigma_X^2}{\epsilon^2}}$$

chebyshev's inequality

stoch. lect.

$X: m_X, \sigma_X^2$

$$P(|X - m_X| > \epsilon) \leqslant \frac{\sigma_X^2}{\epsilon^2}$$

ex.1

$$X \sim \exp(\overset{\overset{2}{\shortparallel}}{\lambda})$$

$$E(X) = \frac{1}{2}$$

$$Var(X) = \frac{1}{4}$$

$$P(|X - \tfrac{1}{2}| > \epsilon) \leqslant \frac{1}{4\epsilon^2}$$

ex.2

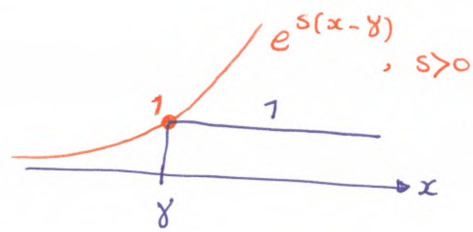
$$X \sim \mathcal{N}(m_X, \sigma_X^2)$$

$$P(|X - m_X| > \epsilon) \leqslant \frac{\sigma_X^2}{\epsilon^2}$$

Chernoff Bound (Tightest Exponential Bound)

- X has mean m_X

$$P(X - m_X > \epsilon) = P(X > \underbrace{m_X + \epsilon}_{\gamma}) = \int_{\gamma}^{+\infty} f_X(x)\,dx = \int_{-\infty}^{+\infty} u(x-\gamma) f_X(x)\,dx$$



$$\leqslant \int_{-\infty}^{+\infty} \underbrace{e^{s(x-\gamma)}}_{e^{sx} \cdot e^{-s\gamma}} f_X(x)\,dx$$

Moment Generating Function

convex in s

$$\Phi_X(s) = E[e^{sX}] = \int_{-\infty}^{+\infty} e^{sX} f_X(x)\,dx$$

$\varphi_X(s) = \ln \Phi_X(s)$ Log-Moment Generating Function

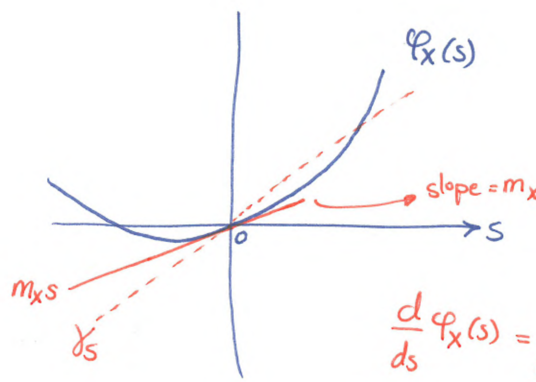
convex in s

$$\Rightarrow \; < \; \bigcirc = \underbrace{e^{-s\gamma} \cdot e^{\varphi_X(s)}}_{\text{Chernoff Bound}}$$

$\begin{pmatrix} s \geq 0 \\ \gamma > m_X \end{pmatrix}$

find minimum to tighten bound.
(w.r.t. s)

$$\underset{s}{\arg\min}\; e^{-s\gamma + \varphi_X(s)} = \underset{s}{\arg\min}\; -s\gamma + \varphi_X(s) = \underset{s}{\arg\max}\; s\gamma - \varphi_X(s)$$



$$\frac{d}{ds}\left\{ s\gamma - \varphi_X(s) \right\}\Big|_{s^*} = 0$$

$$\gamma = \varphi_X'(s^*)$$

$$\frac{d}{ds}\varphi_X(s) = \frac{d}{ds}\ln E[e^{sX}] = \frac{E[xe^{sx}]}{E[e^{sx}]}\Bigg|_{s=0} = E(X) = m_X$$

Solve for s^* which is a function of γ.

we need γs be above $m_X s \Rightarrow \gamma > m_X$.

ex. $X \sim \exp(2)$

$m_X = 2$

$$P(X - \tfrac{1}{2} > \epsilon) = P(X > \underbrace{\tfrac{1}{2} + \epsilon}_{\gamma})$$

MGF: $$\Phi_X(s) = E[e^{sX}] = \int_0^{+\infty} e^{sx}\, 2e^{-2x}\, dx = \frac{2e^{-(2-s)x}}{-(2-s)} = \frac{2}{2-s}$$

log-MGF: $$\varphi_X(s) = \ln \Phi_X(s) = \ln \frac{2}{2-s} = \ln 2 - \ln(2-s)$$

$$\frac{d}{ds}\left(s\gamma - \ln 2 + \ln(2-s)\right)\Big|_{s^*} = 0$$

$$\gamma - \frac{1}{2-s} = 0 \Rightarrow \boxed{s^* = 2 - \frac{1}{\gamma}}$$

$$P(X > \gamma) \leq \underbrace{e^{-\left((2-\frac{1}{\gamma})\gamma - \ln 2 + \ln(2 - 2 + \frac{1}{\gamma})\right)}}_{\underbrace{e^{-(2\gamma - 1 - \ln 2 - \ln\gamma)}}_{e^{-2\gamma + 1 + \ln 2 + \ln\gamma}}}$$

---

ex. $X \sim \mathcal{N}(0, \sigma_X^2)$

$$P(X > \gamma) \leq e^{-\max_{s \geq 0} \{s\gamma - \varphi_X(s)\}}$$

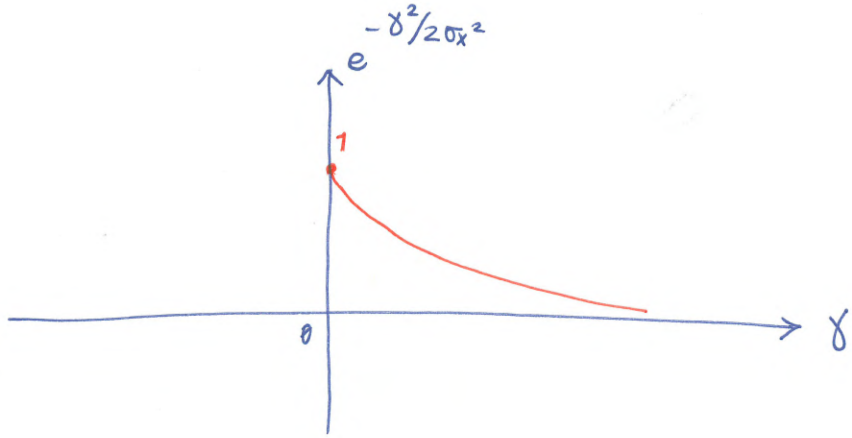
$$\Phi_X(s) = e^{\frac{s^2 \sigma_X^2}{2}}$$

$\varphi_X(s) = s^2 \sigma_X^2 / 2$

$$\frac{d}{ds}\left(s\gamma - \frac{s^2\sigma_x^2}{2}\right)\Bigg|_{s^*} = 0$$

$$\gamma - \cancel{2}\, s\sigma_x^2 = 0 \quad \Rightarrow \quad \boxed{s^* = \frac{\gamma}{\sigma_x^2}}$$

$$\Rightarrow \quad P(X > \gamma) \leq \underbrace{\underbrace{e^{-\left(\frac{\gamma}{\sigma_x^2}\cdot\gamma - \frac{\frac{\gamma^2}{\sigma_x^4}\cdot\sigma_x^2}{2}\right)}}_{e^{-\frac{\gamma^2}{\sigma_x^2} + \frac{1}{2}\frac{\gamma^2}{\sigma_x^2}}}}_{e^{-\frac{\gamma^2}{2\sigma_x^2}}}$$



ex. $X \sim \text{Poisson}(\lambda)$

$E[X] = \lambda$ the $\gamma > \lambda$

$$P(X \geq \gamma) = e^{-\max\limits_{s \geq 0} \{ s\gamma - \varphi_X(s) \}}$$

$$\Phi_X(s) = e^{-\lambda} \cdot e^{\lambda e^s}$$

$$\varphi_X(s) = -\lambda + \lambda e^s$$

$$\frac{d}{ds} \{ s\gamma + \lambda - \lambda e^s \} = 0$$

$$\gamma - \lambda e^s = 0$$

$$\boxed{\ln \frac{\gamma}{\lambda} = s^*}$$

$$\underset{\gamma > \lambda}{P(X > \gamma)} \leq \underbrace{e^{-\left(\ln \frac{\gamma}{\lambda} \cdot \gamma + \lambda - \overbrace{\lambda e^{\ln \frac{\gamma}{\lambda}}}^{-\gamma} \right)}}_{e^{-\left(\underbrace{\gamma \ln \frac{\gamma}{\lambda} + \lambda - \gamma}_{} \right)}}$$

this is called a (weired / named) distance!

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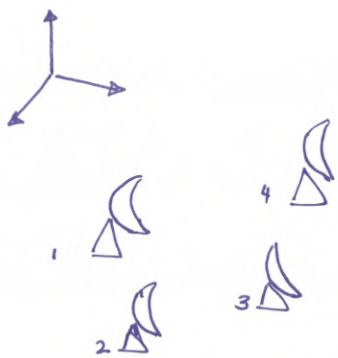
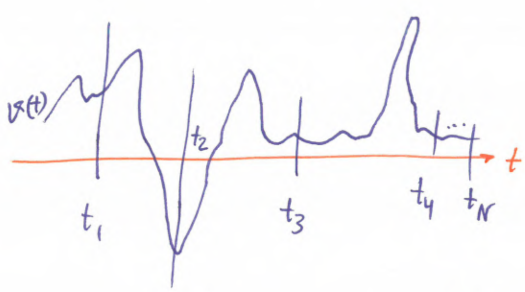
Multiple (vector) Random Variables

ex. 1



- geometry is known
- $s(t)\, e^{j2\pi t f_0}$
- $$\underline{r}(t) = \begin{bmatrix} r_1(t) \\ r_2(t) \\ r_3(t) \\ r_4(t) \end{bmatrix} = s(t)\, e^{j2\pi f_0 t} \cdot \begin{bmatrix} 1 \\ e^{+j\omega_0 \tau_1} \\ e^{j\omega_0 \tau_2} \\ e^{j\omega_0 \tau_3} \end{bmatrix} + \underset{\text{noise}}{n(t)}$$

ex. 2

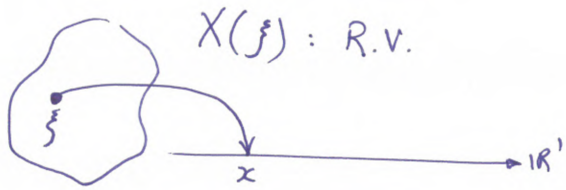


$$\underline{r} = \begin{bmatrix} v(t_1) \\ v(t_2) \\ \vdots \\ v(t_N) \end{bmatrix} + \underbrace{\begin{bmatrix} n(t_1) \\ n(t_2) \\ \vdots \\ n(t_N) \end{bmatrix}}_{\text{colored noise}}$$

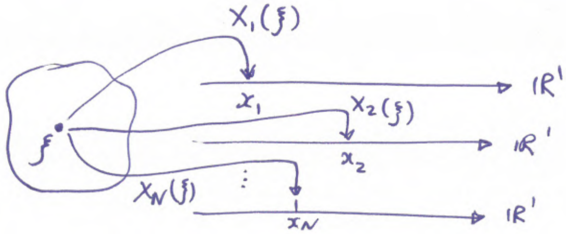
Def'n:

$X(\xi)$: R.V.

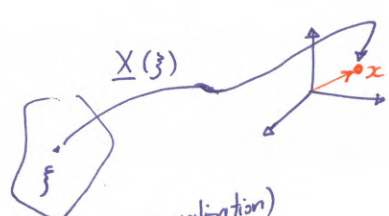
single



multiple (vector)

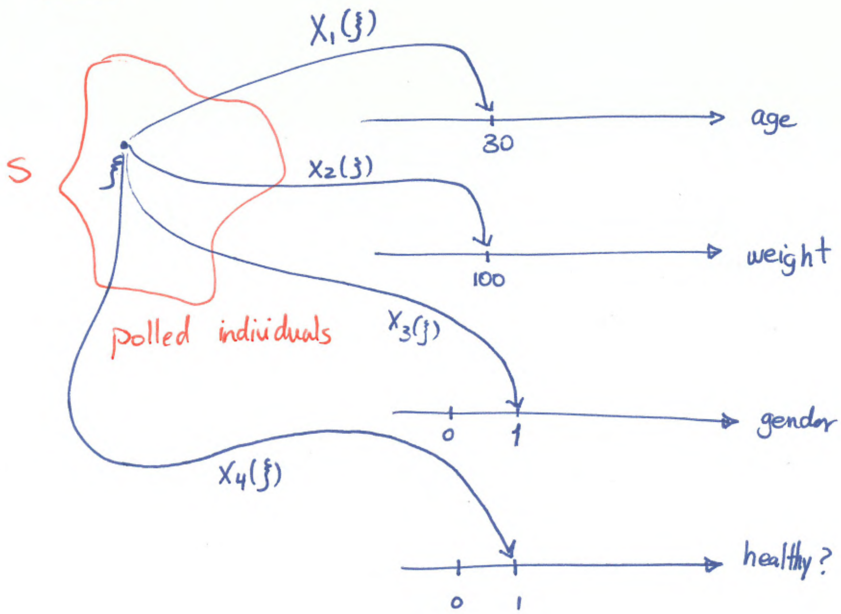


$\Rightarrow$



(another visualization)

ex.



Event
$A = \{$ a healthy woman of age below 30 and weight between $110 < w < 140 \}$

$$A = \{ \xi : X_1(\xi) < 30, \; X_2(\xi) > 110, \; X_2(\xi) < 140, \; X_3(\xi) = 1, \; X_3(\xi) = 1 \}$$

ex. Random Process

$$X(t) = A \cos(\omega_0 t + \varphi)$$

RV's

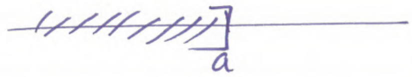
$$\begin{bmatrix} X(t_1) \\ \vdots \\ X(t_N) \end{bmatrix} \qquad \underline{x} = \begin{bmatrix} 5 \\ 2 \\ 4 \\ \vdots \\ 1 \end{bmatrix}$$

ex. Non-Product Type Events

$$C = \{ (x,y) : \; x^2 + y^2 = 25 \}$$

$$D = \{ (x_1, \cdots, x_N) : \; \max\{ x_1, x_2, \cdots x_N \} \leq \alpha \}$$

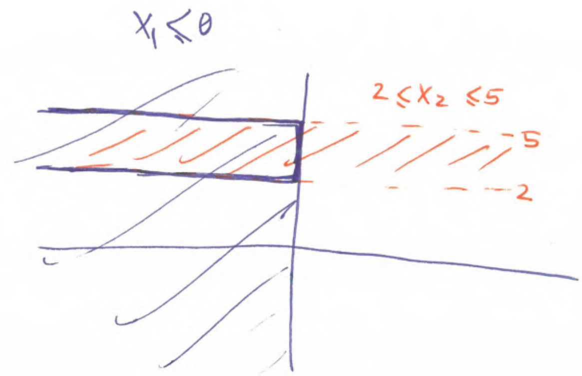
Building Blocks

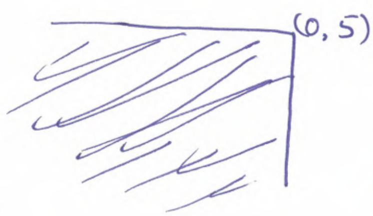
1D 

2D 

ex.

(2D) $A = \{ X_1 \leqslant 0,\ 2 \leqslant X_2 \leqslant 5 \}$





F

G

goal $\equiv$ F − G

$$F_{\underline{X}}(\underline{x}) = P(X_1 \leqslant x_1,\ X_2 \leqslant x_2,\ \ldots,\ X_N \leqslant x_N) \equiv \text{Joint Cumulative Distribution Function}$$

also written as $F_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n)$

properties :

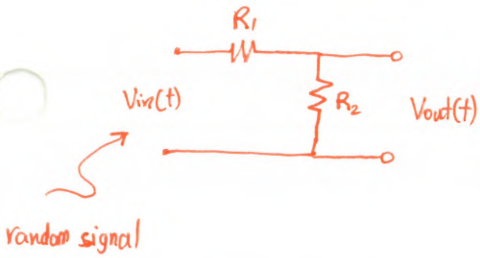
1. $$\lim_{\substack{x_1 \to +\infty \\ x_2 \to +\infty}} \underbrace{F_{X_1, X_2}(x_1, x_2)}_{P(X_1 \leqslant x_1,\ X_2 \leqslant x_2)} = 1 .$$

2. $$\lim_{\substack{x_1 \to -\infty \\ x_2 \to -\infty}} F_{X_1, X_2}(x_1, x_2) = 0 .$$

3. $$\lim_{x_1 \to +\infty} F_{X_1, X_2}(x_1, x_2) = F_{X_2}(x_2)$$

(marginal cdf)

stoch. lect.



$t = t_0$

$$\underline{r}(t_0) = \begin{bmatrix} r_1(t_0) \\ r_2(t_0) \end{bmatrix} = \begin{bmatrix} V_{in}(t) \\ \frac{R_2}{R_1+R_2} V_{in}(t) \end{bmatrix} + \begin{bmatrix} n_1(t_0) \\ n_2(t_0) \end{bmatrix}$$

Joint CDF (D=2)

$$F_{X_1,X_2}(a,b) = P(X_1 \leq a,\ X_2 \leq b)$$

1. $$\lim_{\substack{a\to+\infty \\ b\to+\infty}} F_{X_1,X_2}(a,b) = \lim_{\substack{a\to+\infty \\ b\to+\infty}} P\Big(\underbrace{\{\xi : X(\xi \leq a\}}_{S} \cap \underbrace{\{\xi : X_2(\xi) \leq b\}}_{S}\Big) = 1$$

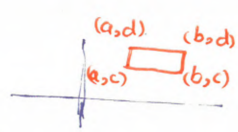
2. $$\lim_{\substack{a\to-\infty \\ b\to-\infty}} F_{X_1,X_2}(a,b) = \lim_{\substack{a\to-\infty \\ b\to-\infty}} P\Big(\underbrace{\{\xi : X_1(\xi) \leq a\}}_{\emptyset} \cap \underbrace{\{\xi : X_2(\xi) \leq b\}}_{\emptyset}\Big) = 0$$

3. $$\lim_{b\to-\infty} F_{X_1,X_2}(a,b) = \lim_{b\to-\infty} P\Big(\{\xi : X_1(\xi) \leq a\} \cap \underbrace{\{\xi : X_2(\xi) \leq b\}}_{\emptyset}\Big) = 0$$

$\boxed{\text{marginal CDF}}$

4. $$\lim_{a\to+\infty} F_{X_1,X_2}(a,b) = \lim_{a\to+\infty} P\Big(\underbrace{\{\xi : X_1(\xi) \leq a\}}_{S} \cap \{\xi : X_2(\xi) \leq b\}\Big) = P(\{\xi : X_2(\xi) \leq b\}) = F_{X_2}(b)$$

$\boxed{\text{For all ordered pairs } a \leq b \text{ and } c \leq d}$



5. $$P(a < X_1 \leq b,\ c < X_2 \leq d) = F_{X_1,X_2}(b,d) - F_{X_1,X_2}(b,c) - F_{X_1,X_2}(a,d) + F_{X_1,X_2}(a,c)$$

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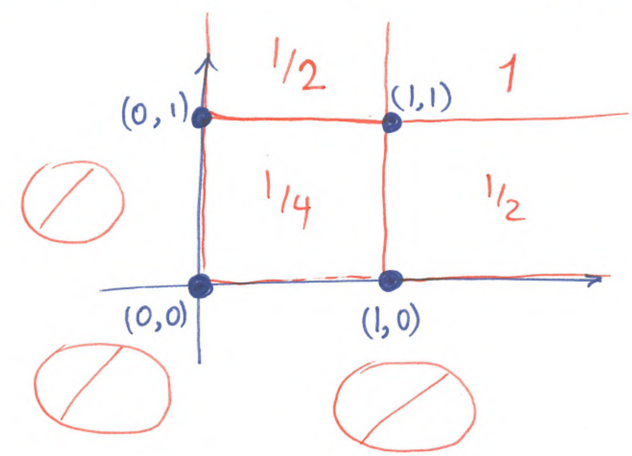
Theorem. $G_{X_1,X_2}(a,b)$ is a valid joint cdf if:

① It is non-decreasing from left to right and from bottom to top.

② For any ordered choice of $a \leq b$ and $c \leq d$

$$P(a < X_1 \leq b,\ c < X_2 \leq d) \geq 0.$$

ex. (2 independent Bernoulli$\left(\frac{1}{2}\right)$)



1. If $-\infty < a < +\infty$, $b < 0$

$$F_{X_1,X_2}(a,b) = 0$$

If $a < 0$, $b > 0$ $F_{X_1,X_2}(a,b) = 0$

2. If $0 \leq a < 1$, $0 \leq b < 1$, $F_{X_1,X_2}(0,b) = 1/4$

3. If $a \geq 1$, $0 \leq b < 1$, $F_{X_1,X_2}(a,b) = 1/2$

4. If $0 \leq a < 1$, $b \geq 1$, $F_{X_1,X_2}(a,b) = 1/2$

5. If $a \geq 1$, $b \geq 1$, $F_{X_1,X_2}(a,b) = 1$

stoch. lect.

$$2D \rightarrow F_{X_1,X_2}(x_1,x_2) = P(X_1 \leqslant x_1,\ X_2 \leqslant x_2)$$

Joint PDF of X_1 and X_2

$$f_{X_1,X_2}(x_1,x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F_{X_1,X_2}(x_1,x_2)$$

$$F_{X_1,X_2}(x_1,x_2) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} f_{X_1,X_2}(\alpha,\beta)\, d\alpha\, d\beta$$

- properties :

① $f_{X_1,X_2}(x_1,x_2) \geqslant 0$

② $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X_1,X_2}(\alpha,\beta)\, d\alpha\, d\beta = 1$

③ marginal pdf

$$f_{X_1}(x_1) = \int_{-\infty}^{+\infty} f_{X_1,X_2}(x_1,\alpha)\, d\alpha$$

$$f_{X_2}(x_2) = \int_{-\infty}^{+\infty} f_{X_1,X_2}(\alpha,x_2)\, d\alpha$$

note^{I}:

$$f_{X_1,X_2}(x_1,x_2) \underset{\cancel{\Longleftarrow}}{\Longrightarrow} f_{X_1}(x_1)\ ,\ f_{X_2}(x_2)$$

note^{II}:

$$F_{X_1}(x_1) = F_{X_1,X_2}(x_1,\infty)$$

$$= \int_{-\infty}^{x_1} \underbrace{\boxed{\int_{-\infty}^{+\infty} f_{X_1,X_2}(\alpha,\beta)\, d\beta}}_{f_{X_1}(\alpha)} d\alpha$$

Independence of R.V.'s

A and B, two events, are independent iff:

$$P(A \cap B) = P(A, B) = P(A) \cdot P(B)$$

X and Y independent iff: $P(X \in A,\ Y \in B) = P(X \in A) \cdot P(Y \in B)$

$A = \{X \leq x\}$

,

$B = \{Y \leq y\}$ then

$$\underline{F_{XY}(x, y)} = P(X \leq x,\ Y \leq y) = P(X \leq x) \cdot P(Y \leq y)$$

$$= \underline{F_X(x) \cdot F_Y(y)}$$

→ Differentiate $F_{X,Y}(x, y)$ w.r. to x and y

$$f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y)$$

For X and Y discrete R.V.'s

$$P(X = x_i,\ Y = y_i) = P(X = x_i) \cdot P(Y = y_i)$$

Note: Assume $h(\cdot)$ and $g(\cdot)$ are two known deterministic transformations.

X and Y are independent.

assume $Z = h(X)$, $W = g(Y)$

Then Z and W are independent.

$$\boxed{X \perp\!\!\!\perp Y \xrightarrow[g(\cdot)]{h(\cdot),} Z \perp\!\!\!\perp W}$$

proof:

A and $\tilde{A}$ are equivalent:

$$P(X \in A) = P(Z \in \tilde{A})$$

B and $\tilde{B}$ are equivalent:

$$P(Y \in B) = P(W \in \tilde{B})$$

$$P(Z \in \tilde{A}, W \in \tilde{B}) = P(X \in A, Y \in B)$$

$$= P(X \in A) \cdot P(Y \in B)$$

Conditional cdf and pdf

A and B are two events, over the same S

$$P(A|B) = \frac{P(A, B)}{P(B)_{(\neq 0)}}$$

$$A = \{\xi : X_1(\xi) \leq x_1\}$$

$$B = \{\xi : X_2(\xi) \in I\}$$

$$\underbrace{P(X_1 \leq x \mid X_2 \in I)}_{F_{X_1}(x \mid X_2 \in I)} = \frac{P(X_1 \leq x_1,\ X_2 \in I)}{P(X_2 \in I)} = \frac{\int_{-\infty}^{x_1} \int_{\beta \in I} f_{X_1, X_2}(\alpha, \beta)\, d\beta\, d\alpha}{\int_{\beta \in I} f_{X_2}(\beta)\, d\beta}$$

① choose

$$I = (-\infty,\ x_2)$$

$$F_{X_1}(x_1 \mid X_2 \leq x_2) = P(X_1 \leq x_1 \mid X_2 \leq x_2) = \frac{P(X_1 \leq x_1,\ X_2 \leq x_2)}{P(X_2 \leq x_2)}$$

$$= \frac{F_{X_1, X_2}(x_1, x_2)}{F_{X_2}(x_2)}$$

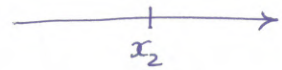
$$= \frac{\int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f_{X_1, X_2}(\alpha, \beta)\, d\beta\, d\alpha}{\int_{-\infty}^{x_2} f_{X_2}(\beta)\, d\beta}$$

The pdf of X_1, given $X_2 \leq x_2$:

$$f_{X_1}(x_1 \mid X_2 \leq x_2) = \frac{d}{dx_1} F_{X_1}(x_1 \mid X_2 \leq x_2)$$

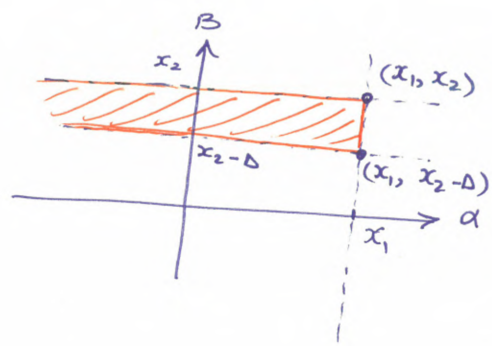
$$= \frac{\int_{-\infty}^{x_2} f_{X_1, X_2}(x_1, \beta)\, d\beta}{\int_{-\infty}^{x_2} f_{X_2}(\beta)\, d\beta}$$

② choose $I = \{x_2\}$



$$F_{X_1}(x_1 \mid X_2 \in (x_2 - \Delta, x_2])$$

$$= \frac{P(X_1 \leq x_1, \; X_2 \in (x_2 - \Delta, x_2])}{P(X_2 \in (x_2 - \Delta, x_2])}$$



$$= \frac{F_{X_1,X_2}(x_1, x_2) - F_{X_1,X_2}(x_1, x_2 - \Delta)}{F_{X_2}(x_2) - F_{X_2}(x_2 - \Delta)} \times \frac{\Delta}{\Delta}$$

as $\Delta \longrightarrow 0$

$$= \frac{\frac{d}{dx_2} F_{X_1,X_2}(x_1, x_2)}{\frac{d}{dx_2} F_{X_2}(x_2)}$$

differentiate both sides w.r. to x_1:

$$\underbrace{f_{X_1}(x_1 \mid X_2 = x_2)}_{f_{X_1|X_2}(x_1|x_2)} = \frac{d}{dx_1} F_{X_1}(x_1 \mid X_2 = x_2)$$

$$= \frac{f_{X_1,X_2}(x_1, x_2)}{f_{X_2}(x_2)}$$

Conditional Expectation

$$\left(E[Y \mid X = x] = \int_{-\infty}^{+\infty} y \cdot f_{Y|X}(y|x)\, dy \right) \times f_X(x)$$

integrate

$$\int_{-\infty}^{+\infty} E[Y|X=x]\, f_X(x)\, dx = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y\, \underbrace{f_{Y|X}(y|x)\, dy\; f_X(x)}_{f_{XY}(x,y)}\, dx$$

$$= \int_{-\infty}^{+\infty} y \int_{-\infty}^{+\infty} f_{XY}(x,y)\, dx \quad dy$$

$$= \int_{-\infty}^{+\infty} y\, f_Y(y)\, dy$$

$$E_{f_X}\left[E_{f_{Y|X}}[Y|X] \right] = E[Y]$$

iterated expectation:

$$\Rightarrow \quad \boxed{E[Y] = E\left[E[Y|X] \right]}$$

note: $E[Y|X]$ is a function of x.

ex.

$$Z = \sum_{i}^{N} X_i \quad \text{iid } \exp(\cdot\lambda)$$

$N \sim \text{Poisson}(\gamma)$

$$E[Z]$$

$$E[Z|N=n] = E\left[\sum_{i=1}^{n} X_i\right]$$

$E[\cdot]$: a linear operator

$$= \sum_{i=1}^{n} E[X_i] = \frac{n}{\lambda}$$

$$E[E[Z|N]] = \sum_{n=0}^{+\infty} \frac{n}{\lambda} \cdot \frac{\gamma^n}{n!} e^{-\gamma} = E\left[\frac{N}{\lambda}\right] = \frac{\gamma}{\lambda}$$

Entropy and Conditional Entropy

entropy: X is discrete $x_1, \dots, x_n$ with probabilites $p_1, \dots, p_n$.

$$-\log p_i = \log \frac{1}{p_i}$$

$$\sum_{i=1}^{n} p_i(-\log p_i) = E\left[\log \frac{1}{P(x)}\right] \triangleq H(X)$$

cond. entropy: $y_1, \dots, y_n$

$$-E\left[\log(P(Y|X=x_j))\right]_{x_1,\dots,x_m} = -\sum_{i=1}^{n} P(y_i|X=x_j) \log P(y_i|X=x_j)$$

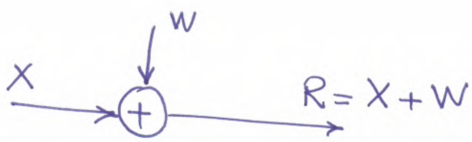
to reach the conditional entropy we should evaluate Expectation of above w.r.t. x_j.

$$-\sum_{j=1}^{m}\sum_{i=1}^{n} \underbrace{P(X=x_j)\,P(Y=y_i \mid X=x_j)}_{P(X=x_j,\ Y=y_i)} \log P(Y=y_i \mid X=x_j)$$

$$H(Y|X) = E\big[E[\log P(Y|X)]\big]$$

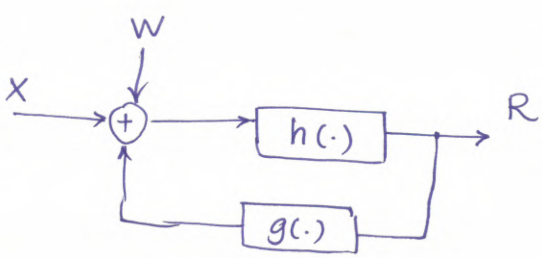
Transformations of Two R.V.'s

ex.



$$f_{X,W}(x, w)$$

ex.



ex. Binary Detection

$H_1 :\quad \underline{R} \sim f_{\underline{R}|H_1}(\underline{r} \mid H_1)$

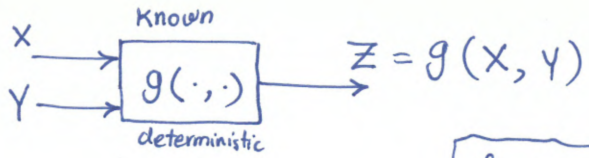
(pure noise) $H_0 :\quad \underline{R} \sim f_{\underline{R}|H_0}(\underline{r} \mid H_0)$

a R.V. as well

Likelihood Ratio : $\dfrac{f_{\underline{R}|H_1}(\underline{r} \mid H_1)}{f_{\underline{R}|H_0}(\underline{r} \mid H_0)} \triangleq \Lambda$

$\Lambda \overset{H_1}{\gtrless} \text{Threshold}$

1. A single function of two R.V.'s



$$\boxed{f_Z(z) = ?}$$

$f_{X,Y}(x,y)$: known

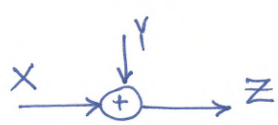
1- Identify equivalent events

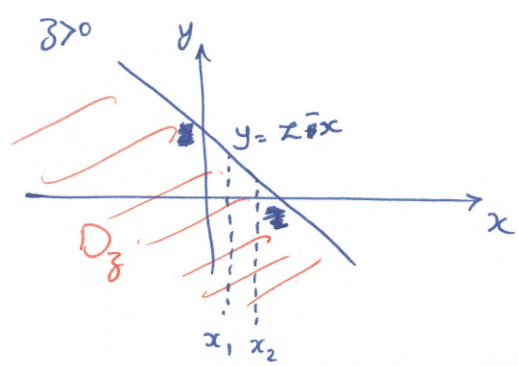
$$A = (-\infty, z]$$

$$D_z = \{(x,y) : g(x,y) \leq z\}$$

2- $$F_Z(z) = \iint_{D_z} f_{X,Y}(x,y)\,dx\,dy$$

ex.1 $Z = X + Y$





1. $A = [-\infty, z]$

$D_z = \{(x,y) : x+y \leq z\}$

2. $$F_Z(z) = P(Z \leq z) = P(X+Y \leq z) = \int_{-\infty}^{z-x}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx\,dy$$

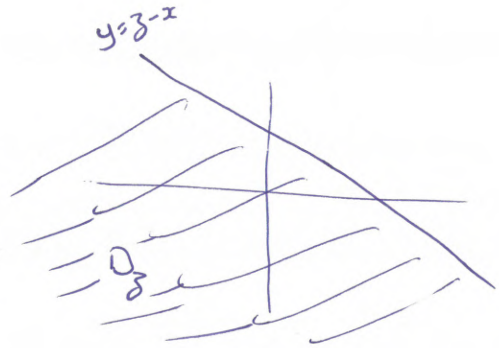
they may be better to change orders.

$f_Z(z) = ?$ $$\frac{d}{dz}F_Z(z) = \int_{-\infty}^{+\infty} f_{X,Y}(x, z-x)\,dx$$

Midterm . Oct. 26th

ex. 1 $Z = X + Y$

$f_{X,Y}(x,y)$ is known.



Collection of (x,y) which satisfy $x+y \leq z$

$$F_Z(z) = \iint_{D_z} f_{X,Y}(x,y)\,dy\,dx$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{z-x} f_{X,Y}(x,y)\,dy\,dx$$

"x" "y"

$$f_Z(z) = \int_{-\infty}^{+\infty} f_{X,Y}(x,\overset{z-x}{y})\,dx$$

Independence and Convolution

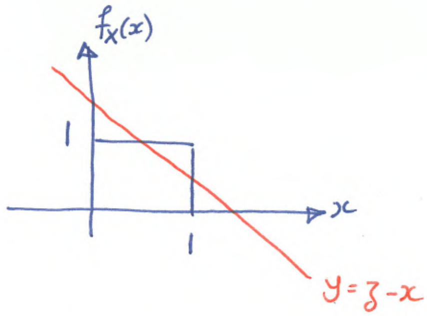
$Z = X+Y$, $X \perp\!\!\!\perp Y$

This means $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(x) \cdot f_Y(z-x)\,dx = f_X(z) * f_Y(z)$$

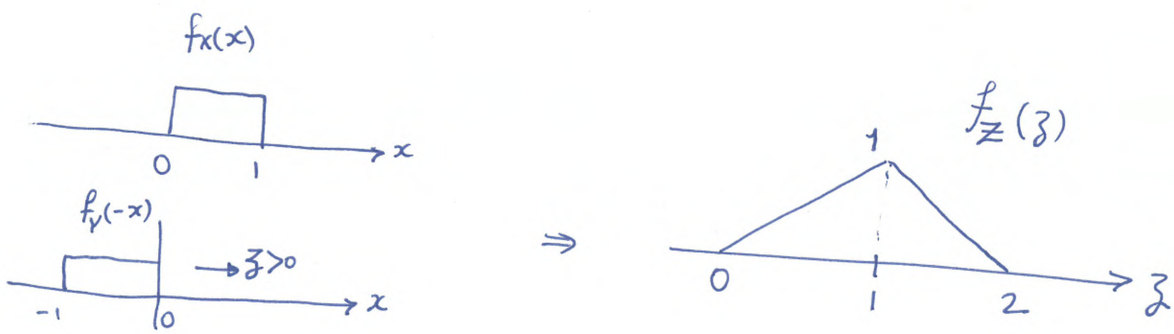
Assume X and Y are iid $U(0,1)$

$$Z = X + Y$$



main solution : find collection of x, y which $x + y \leq Z$ then integrate ... then differentiate ...

easier solution : independence $\Rightarrow$ convolution



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ex. 2 $\quad Z = \dfrac{X}{Y}$

$f_{XY}(x, y)$ is known.

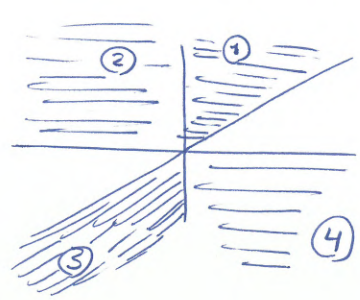
Find $f_Z(z)$. at first we need $P(Z \leq z)$

$$\begin{cases} z > 0 \xrightarrow{\text{only } ①, ③} \\ z \leq 0 \end{cases}$$

$y = \dfrac{x}{z}$

$\dfrac{x}{y} \leq z$

$y \geq \dfrac{x}{z}$



on ② and ④ inequality already satisfied.

$$F_Z(z) = \iint\limits_{(x,y) \in D_z} f_{X,Y}(x,y)\, dy\, dx$$

$$= \int_0^{+\infty}\int_{\frac{x}{z}}^{+\infty} dy\, dx\, f_{X,Y}(x,y) + \int_{-\infty}^{0}\int_0^{+\infty} dy\, dx\, f_{X,Y}(x,y)$$
① ②

$$+ \int_{-\infty}^{0}\int_{-\infty}^{x/z} dy\, dx\, f_{XY}(x,y) + \int_0^{+\infty}\int_{-\infty}^{0} dy\, dx\, f_{XY}(x,y)$$
③ ④

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \int_0^{+\infty} dx \left(\frac{x}{z^2}\right) f_{XY}\left(x, \frac{x}{z}\right) + 0 + \int_{-\infty}^{0} dx \left(\frac{-x}{z^2}\right) f_{XY}\left(x, \frac{x}{z}\right) + 0$$
① ② ③ ④

task: derive as well for $z \leq 0$.

---

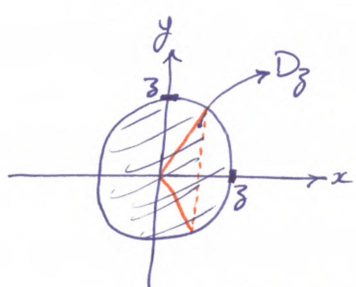
ex.3 $\quad Z = \sqrt{X^2 + Y^2}$

$f_{XY}(x,y)$ is known.

$z \geq 0$

$F_Z(z) = P(Z \leq z)$

$D_z = \{ (x,y) : \underbrace{\sqrt{x^2+y^2}}_{Z} \leq z \}$

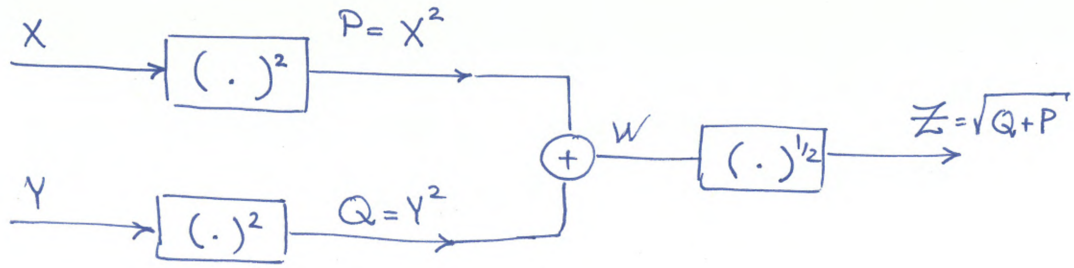


$$F_Z(z) = \int_{-z}^{z}\int_{-\sqrt{z^2-x^2}}^{\sqrt{z^2-x^2}} f_{XY}(x,y)\, dy\, dx$$

Polar coordinates will be more of interest!

10/10/22 - 2

Let X and Y are iid $\mathcal{N}(0,1)$.



$$M_P(jv) = E_{f_P}\left[e^{jvP}\right] = E_{f_X}\left[e^{jvX^2}\right]$$

$$= \int_{-\infty}^{+\infty} \frac{e^{jvx^2} \, e^{-x^2/2}}{\sqrt{2\pi}} dx \times \frac{\sqrt{1-j2v}}{\sqrt{1-2jv}} = 1$$

$$e^{-\frac{1}{2}(1-2jv)x^2}$$

$$\frac{1}{\sigma^2} \triangleq 1 - 2jv$$

$$= \frac{1}{\sqrt{-2jv+1}}$$

as well: $$M_Q(jv) = \frac{1}{\sqrt{1-2jv}}$$

$$W = P + Q \Rightarrow M_W(jv) = E_{f_W}\left[e^{jvW}\right] = E_{f_{P,Q}}\left[e^{jvP} \cdot e^{jvQ}\right]$$

$$= E_{f_P \cdot f_Q}\left[e^{jvP} \cdot e^{jvQ}\right] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{jvP} e^{jvQ} f_P(p) \cdot f_Q(q) \, dp \, dq$$

$$= \int_{-\infty}^{+\infty} e^{jvp} f_P(p)\,dp \int_{-\infty}^{+\infty} e^{jvq} f_Q(q)\,dq\}$$

$$= M_P(jv) \cdot M_Q(jv)$$

$$\boxed{M_W(jv) = \frac{1}{1-j2v}}$$

$W \sim \exp\left(\frac{1}{2}\right)$

$$Z = W^{1/2}$$

$$F_Z(z) = P(Z \leq z) = P(W^{1/2} \leq z) = P(W \leq z^2)$$

$$= \int_{0}^{z^2} \frac{1}{2} e^{-\frac{1}{2}w}\,dw$$

$$f_Z(z) = \cancel{2} z \frac{1}{\cancel{2}} e^{-z^2/2} \quad , \quad z \geq 0$$

stoch. lect.

exercise

X, Y, Z are independent. $F_X(x)$, $F_Y(y)$, $F_Z(z)$

(a)

$$P(|X|<5,\ Y>2,\ Z^2 \geq 2)$$

mutual independence of X, Y, Z :

$$P(-5<X<5) \cdot [1 - P(Y \leq 2)] \cdot P(\{Z \leq -\sqrt{2}\} \cup \{Z \geq +\sqrt{2}\})$$

$$= [F_X(5^-) - F_X(-5^+)][1 - F_Y(2^+)][F_Z(-\sqrt{2}^+) + 1 - F_Z(\sqrt{2}^-)]$$

(b)

$$P(X>5,\ Y<0,\ Z=1)$$

we don't know if Z is continuous or discrete.

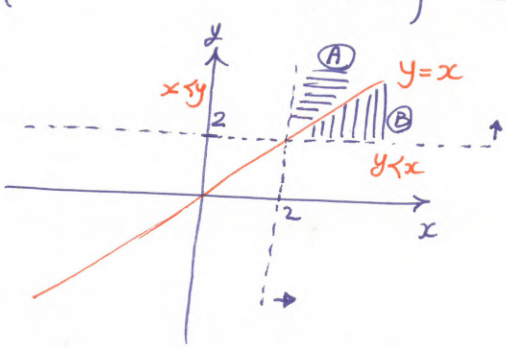
$$= \{1 - F_X(5^+)\} \cdot F_Y(0^-) \cdot \{F_Z(1^+) - F_Z(1^-)\}$$

(c)

$$P(\min(X,Y,Z) > 2)$$

$$\min(X,Y) > 2$$

$\{(x,y) : \min(x,y) > 2\}$ → first find where both are equal. (a line on x-y plane)



Ⓐ: x is min., $x>2$	Ⓑ: y is min., $y>2$

$$\Rightarrow P(\min(X,Y) > 2) = P(X>2,\ Y>2)$$

$\boxed{\text{more difficult to do:} \quad P(\min(X,Y) < 2) = ?}$

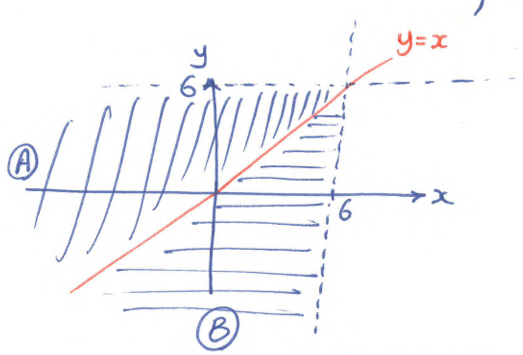
and therefore to generalize: $P(\min(X,Y,Z) > 2) = P(X>2, Y>2, Z>2)$

$$= \{1 - F_X(2^+)\}\{1 - F_Y(2^+)\}\{1 - F_Z(2^+)\}$$

d) $P(\max(X,Y,Z) < 6)$

first easier one: $P(\max(X,Y) < 6) = ?$ $\Rightarrow P(X<6, Y<6)$

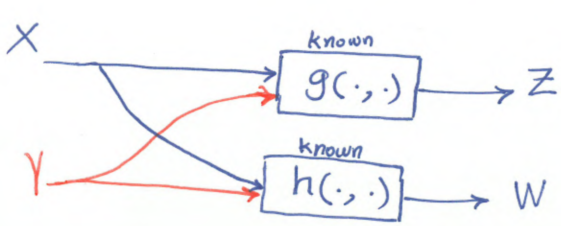
$\{(x,y): \max(x,y) < 6\}$



$\Rightarrow P(\max(X,Y,Z) < 6) = P(X<6, Y<6, Z<6)$

$$= F_X(6^-) \cdot F_Y(6^-) \cdot F_Z(6^-)$$

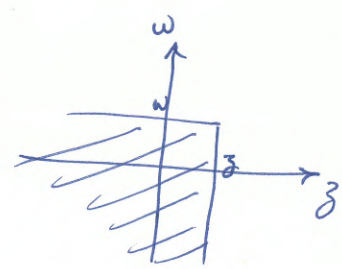
Two Transformations of Two R.V.'s



$f_{X,Y}(x,y)$ known

Find $f_{Z,W}(z,w)$

$A = \{Z \leq z, W \leq w\}$



$$D_{z,w} = \{(x,y) : g(x,y) \leq z,\ h(x,y) \leq w\}$$

$$F_{Z,W}(z,w) = \iint\limits_{(x,y) \in D_{z,w}} f_{X,Y}(x,y)\,dx\,dy$$

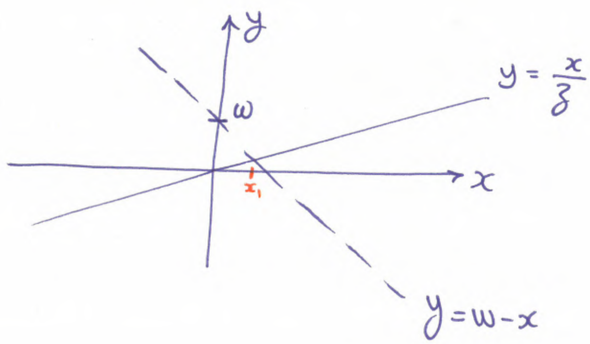
---

ex.

$$Z = \frac{X}{Y} \qquad W = X + Y$$

$z > 0,\ w > 0$

$$\left\{(x,y) : \frac{x}{y} \leq z,\ x+y \leq w\right\}$$



$$\frac{x}{z} = w - x$$

$$x_1 = \frac{zw}{1+z}$$

$$F_{Z,W}(z,w) = \int_{-\infty}^{0} \left\{ \int_{-\infty}^{x/z} f_{X,Y}(x,y)\,dy + \int_{0}^{w-x} f_{XY}(x,y)\,dy \right\} dx$$

$$+ \ \cdots\cdots$$

Joint PDF

$Z = g(X, Y)$ and $W = h(X, Y)$

$f_{XY}(x, y)$ is given.

proposition: To find $f_{Z,W}(z, w)$ in terms of $f_{XY}(x, y)$

1. Solve $g(x, y) = z$ and $h(x, y) = w$

   Assume n solutions (x_i, y_i), $i = 1, \dots, n$

   $x_i(z, w)$ $\quad$ $y_i(z, w)$

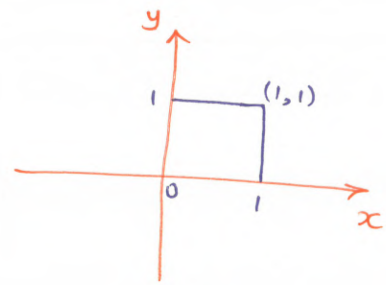
2. Write

$$f_{Z,W}(z, w) = \sum_{i=1}^{n} \frac{f_{X,Y}(x_i, y_i)}{|J(x_i, y_i)|} \quad \left| \frac{dx\, dy}{dz\, dw} \right| dy\, dw$$

$$J(x, y) = \det \begin{vmatrix} \partial z/\partial x & \partial w/\partial x \\ \partial z/\partial y & \partial w/\partial y \end{vmatrix}$$

ex 4.2 $\quad X \perp\!\!\!\perp Y \sim \text{Uniform}(0, 1)$

(a) $P\left(X^2 \leq \frac{1}{2},\ |Y-1| < \frac{1}{2} \right)$



indep.
$$\overset{\text{indep.}}{=} P\left(-\frac{1}{\sqrt{2}} < X < \frac{1}{\sqrt{2}}\right) \cdot P\left(\frac{1}{2} < Y < \frac{3}{2}\right)$$

$$= \int_0^{1/\sqrt{2}} 1\, dx \times \frac{1}{2} = \frac{1}{2\sqrt{2}}$$

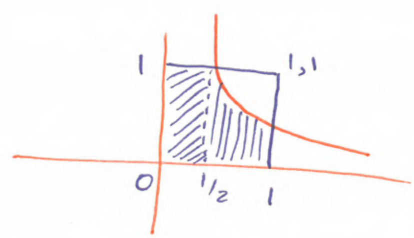
(b) $P\left(\frac{X}{2} < 1,\ Y > 0 \right)$

$$= P(X < 2)\ P(Y < 0)$$

$$= 1$$

(c) $P\left(XY < \frac{1}{2} \right)$

(x,y) : $\quad y < \frac{1}{2x}$



$$= \frac{1}{2} + \int_{1/2}^{1} \int_0^{\frac{1}{2x}} 1\, dy\, dx$$

$$= \frac{1}{2} + \int_{1/2}^{1} \frac{1}{2x}\, dx = \frac{1}{2} + \frac{1}{2} \ln x \Big|_{1/2}^{1} = \frac{1}{2} + \frac{1}{2}\ln 2$$

(d) $P(\min\{X, Y\} > \frac{1}{3})$

$= P(X > \frac{1}{3}, Y > \frac{1}{3})$

$= \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$

$Z = g(X, Y)$

$W = h(X, Y)$

- $f_{XY}(x,y)$ is known
- $f_{ZW}(z,w) = ?$

1. For every (z, w) solve:

$$z = g(x,y)$$
$$w = h(x,y$$

assume $(x_i, y_i) = (x_i(z,w), y_i(z,w))$

$i = 1, \ldots, n$

2. Write

$$f_{ZW}(z,w) = \sum_{i=1}^{n} \frac{f_{XY}(x_i, y_i)}{|J(x_i, y_i)|}$$

$$J(x,y) = \det \begin{bmatrix} \partial z/\partial x & \partial z/\partial y \\ \partial w/\partial x & \partial w/\partial y \end{bmatrix}$$

ex.1 (Rotation)

$$Z = X\cos\varphi + Y\sin\varphi$$

$$W = -X\sin\varphi + Y\cos\varphi$$

$$\begin{bmatrix} Z \\ W \end{bmatrix} = \overbrace{\begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}}^{Q:\ \text{orthogonal matrix}} \begin{bmatrix} X \\ Y \end{bmatrix}$$

$\rightarrow \begin{cases} QQ^T = I \\ \det(Q) = 1 \end{cases}$?

because of orthogonality? (or just $\sin^2 + \cos^2 = 1$.)

$$\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}^{-1}}_{\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}} \begin{bmatrix} y \\ x \end{bmatrix}$$

$$f_{ZW}(z,w) = f_{XY}(z\cos\varphi - w\sin\varphi,\ y\sin\varphi + w\cos\varphi)$$

$$|J| = \det(Q^T) = 1$$

---

ex.2 (Polar Transformation)

$$R = \sqrt{X^2 + Y^2} \qquad (r \geq 0)$$

$$\varphi = \tan^{-1}\left(\frac{Y}{X}\right) \qquad (-\pi \leq \varphi \leq \pi)$$

$x = r\cos\varphi$

$y = r\sin\varphi$

$|J|^{-1}$ $\quad = r$

$$f_{R,\varphi}(r,\varphi) = f_{XY}(r\cos\varphi,\ r\sin\varphi)\left|\frac{dx\,dy}{dr\,d\varphi}\right| dr\,d\varphi$$

$$\left|\frac{dx\,dy}{dr\,d\varphi}\right| = \left|\det\begin{bmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi}\end{bmatrix}\right| = \left|\det\begin{bmatrix}\cos\varphi & -r\sin\varphi\\ \sin\varphi & r\cos\varphi\end{bmatrix}\right|$$

$$= \left|r\cos^2\varphi + r\sin^2\varphi\right| = |r| \overset{r\geq 0}{=} r$$

Auxiliary Variable

ex. Let $Z = \dfrac{X}{X+Y} = \dfrac{1}{1+\frac{Y}{X}}$

introduce $W = Y$

(z, w)

$$\begin{cases} z = \dfrac{x}{x+y} \\ w = y \end{cases} \Rightarrow \begin{cases} x = \dfrac{zw}{1-z} \\ y = w \end{cases}$$

$$f_{ZW}(z,w) = f_{XY}\left(\frac{zw}{1-z},\ w\right)\cdot\left|\det\begin{bmatrix}\frac{w}{1-z}+\frac{zw}{(1-z)^2} & \frac{z}{1-z}\\ 0 & 1\end{bmatrix}\right|$$

$$f_{ZW}(z,w) = f_{XY}\left(\frac{zw}{1-z}, w\right) \left|\frac{w}{(1-z)^2}\right|$$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_{XY}\left(\frac{zw}{1-z}, w\right) \frac{|w|}{(1-z)^2}\, dw$$

Functions of Multiple Random Variables

1. $Z = g(X, Y)$ from $f_{XY}(x,y)$ can find $f_Z(z)$

$$\underset{z \sim f_Z(z)}{E}(Z) = \int_{-\infty}^{+\infty} z\, f_Z(z)\, dz = \underset{(x,y)\sim f_{XY}}{E}[g(X,Y)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y)\, f_{XY}(x,y)\, dx\, dy$$

ex. 1

$Z = -\log f_{XY}(x,y)$

$$E[Z] = E_{f_{XY}}[-\log f_{XY}(X,Y)]$$

$$= -\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y) \log f_{XY}(x,y)\, dx\, dy$$

Joint Differential Entropy

ex.2 $\quad X$ and Y are iid $\exp(\lambda)$

$$M = \frac{1}{2}(X+Y) \qquad \text{Sample (mean) estimate}$$

$$E[M] = \frac{1}{2}E[X] + \frac{1}{2}E[Y] = \frac{1}{2}\frac{1}{\lambda} + \frac{1}{2}\frac{1}{\lambda} = \frac{1}{\lambda}$$

linearity of $E[\cdot]$

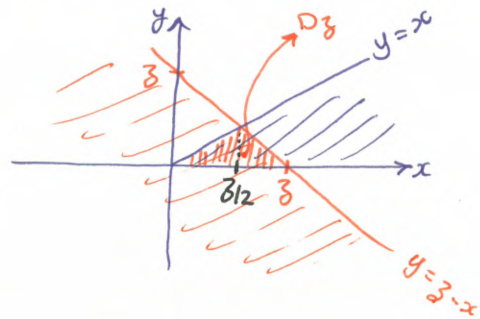
$\Rightarrow$ Estimate is Unbiased as $E[M]$ equals $E\left[\frac{X+Y}{2}\right]$.

ex 4.3

X, Y are R.V.'s. $f_{XY}(x,y) = \begin{cases} 2e^{-(x+y)} & 0 \leq y \leq x < +\infty \\ 0 & \text{o.w.} \end{cases}$

$Z = X + Y \qquad f_Z(z) = ?$

! we cannot do convolution, due to dependency in $0 \leq y \leq x < +\infty$



$D_z = \{(x,y) : x + y = z\}$

z is positive.

$$F_Z(z) = \iint_{(x,y)\in D_z} f_{XY}(x,y)\,dx\,dy = \int_0^{z/2} dx \int_0^{x} dy\; 2e^{-(x+y)} + \int_{z/2}^{z} dx \int_0^{z-x} dy\; 2e^{-(x+y)}$$

---

ex.

X, Y are iid $\mathrm{Exp}(\lambda)$

$$M = \frac{1}{2}(X+Y)$$

$$E[M] = \frac{1}{\lambda}$$

unbiased estimator.

$$V = \frac{1}{2}\left[(X-M)^2 + (Y-M)^2\right] = \frac{1}{2}\left[\left(\frac{X-Y}{2}\right)^2 + \left(\frac{Y-X}{2}\right)^2\right]$$

$$= \frac{1}{4}(X-Y)^2$$

$$E[V] = \frac{1}{2} E[X^2 - 2XY + Y^2]$$

$$= \frac{1}{4} E[X^2] - \frac{1}{2} E[XY] + \frac{1}{4} E[Y^2]$$

$$= \frac{1}{2\lambda^2} + \frac{1}{2\lambda^2} - \frac{1}{2} E[XY]$$

Covariance

$$\boxed{\mathrm{Cov}(X,Y) = E[(X - E[X])(Y - E[Y])]}$$

$$= E[XY] - E[E[X]Y] - E[X E[Y]] + E[X]E[Y]$$

$$= E[XY] - E[X]E[Y]$$

Correlation Coefficient (normalized covariance)

$$\rho_{X,Y} = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{Var}(X) \cdot \mathrm{Var}(Y)}}$$

$$|\rho_{X,Y}| \leq 1 \implies [\mathrm{Cov}(X,Y)]^2 \leq \mathrm{Var}(X) \cdot \mathrm{Var}(Y)$$

Def'n. X and Y are uncorrelated if $\operatorname{cov}(X,Y)=0$.

$$\Longrightarrow \quad E[XY] = E[X]\cdot E[Y]$$

Def'n. X and Y are orthogonal if $E[XY]=0$.

$$\operatorname{Cov}(X,Y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\, f_{XY}(x,y)\,dx\,dy \;-\; E[X]E[Y]$$

X and Y are discrete $P_{XY}(x,y)$

$$\operatorname{cov}(X,Y) = \sum_x \sum_y P_{XY}(x,y)\,(x-\underbrace{m_X}_{E[X]})\,(y-\underbrace{m_Y}_{E[Y]})$$

★ If $X \perp\!\!\!\perp Y \Rightarrow \operatorname{cov}(X,Y)=0$.

$\not\Leftarrow$

but Gaussians satisfy this inverse!!!

$$V = \frac{1}{2}(X-M)^2 + \frac{1}{2}(Y-M)^2$$

$$E[V] = \frac{1}{4}\left[\frac{2}{\lambda^2}+\frac{2}{\lambda^2}\right] - \frac{1}{2}E[XY]$$

$$E[XY] = \int_0^{+\infty}\int_0^{+\infty} xy \; \lambda^2 e^{-\lambda x} e^{\lambda y} \, dx\, dy$$

$$= \int_0^{+\infty} \lambda x e^{-\lambda x} dx \int_0^{+\infty} \lambda y e^{-\lambda y} dy$$

$$= E[X] \, E[Y] = \frac{1}{\lambda^2}$$

$$\Rightarrow E[V] = \frac{1}{\lambda^2} - \frac{1}{2\lambda^2} = \frac{1}{2\lambda^2} \quad \text{(Biased Estimator) of } var(X)$$

to make it unbiased: $V_{unbiased} = \frac{1}{n-1} \times V_{biased}$?

Joint Moments

Def'n The $(k, \ell)^{th}$ joint moment for two RV's X and Y is defined as

$$m_{k,\ell} = E[X^k Y^\ell]$$

noncentered joint moment

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^k y^\ell f_{XY}(x,y) \, dx\, dy$$

Central Joint Moments

$$\mu_{k,\ell} = E[(X - m_X)^k (Y - m_Y)^\ell]$$

$\rightarrow \quad \mu_{0,0} = 1$

$\rightarrow \quad \mu_{1,0} = 0$

$\rightarrow \quad \mu_{2,0} = Var(X) \qquad \mu_{0,2} = Var(Y)$

$\rightarrow \quad \mu_{1,1} = cov(X, Y)$

5. Joint Characteristic Function

$$\Phi_{X,Y}(ju, jv) = \underset{f_{X,Y}}{E}\left[e^{juX + jvY}\right]$$

or M

$$= \iint_{-\infty}^{+\infty} e^{jux + jvy} \; f_{XY}(x,y)\; dx\, dy$$

fourier transform of char. function gives us pdf

$$f_{XY}(x,y) = \frac{1}{2\pi}\iint_{-\infty}^{+\infty} \Phi_{XY}(ju, jv)\; e^{-jvx}\, e^{-jvy}\; du\, dv$$

I. $$\Phi_{XY}(ju, jv)\Big|_{v=0} = \Phi_X(ju)$$

$$\Phi_{XY}(ju, jv)\Big|_{u=0} = \Phi_Y(jv)$$

II. $Z = X + Y$ $\quad f_X(x),\ f_Y(y)$ are known.

$X \perp\!\!\!\perp Y$

$$f_Z(z) = f_X(z) * f_Y(z) = \int_{-\infty}^{+\infty} f_X(t)\, f_Y(z-t)\, dt$$

$$M_Z(jv) \triangleq \underset{f_Z}{E}\left[e^{jvz}\right] = \underset{f_{XY}}{E}\left[e^{jv(X+Y)}\right]$$

$$= \iint_{-\infty}^{+\infty} e^{jux + jvy}\; f_X(x)\, f_Y(y)\; dx\, dy$$

$$= \int_{-\infty}^{+\infty} M_X(ju)\, e^{jvy} f_Y(y)\, dy = M_X(ju) . M_Y(jv)$$

III. $(k, \ell)^{th}$ Joint Moment

$$m_{k,\ell} = E[X^k Y^\ell]$$

$$m_{k,\ell} = \frac{\partial^k}{\partial (ju)^k} \frac{\partial^\ell}{\partial (jv)^\ell} \Phi_{XY}(ju, jv) \Big|_{\substack{u=0 \\ v=0}}$$

Ex 4.4

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} Z \\ W \end{bmatrix} = \begin{bmatrix} aZ + bW \\ cZ + dW \end{bmatrix}$$

$$ad \neq bc$$

Express $\Phi_{XY}(ju, jv)$ in terms of $\varphi_{Z,W}(ju, jw)$

$$\Phi_{XY}(ju, jv) = E_{f_{XY}}\left[e^{juX + jvY} \right]$$

$$= E_{f_{XY}}\left[e^{ju(aZ+bW) + jv(cZ+dW)} \right]$$

$$= E_{f_{ZW}}\left[e^{j(ua+cv)Z + j(ub+vd)W} \right]$$

$$= \Phi_{ZW}(j(ua+vc), j(ub+vd))$$

$$E[XY] = E[(aZ+bW)(cZ+dW)]$$

$$= E[acZ^2 + (bc+ad)ZW + bd\,W^2]$$

$$= ac\,\underbrace{E[Z^2]}_{\text{2nd moment}} + (bc+ad)\,\underbrace{E[ZW]}_{\text{joint moment}} + bd\,\underbrace{E[W^2]}_{\text{2nd moment}}$$

Ex 4.7

$$M_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

X_i's iid

$f_X(x)$

(sample mean)

$$\Phi_{M_n}(j\nu) = E_{f_{M_n}}[e^{j\nu M_n}]$$

$$\Phi_{M_n}(j\nu) = E_{f_{\underline{X}} = f_{X_1}\cdots f_{X_n}}\left[e^{j\nu \frac{1}{n}\sum_{i=1}^{n} x_i}\right]$$

$$= \int_{-\infty}^{+\infty}\cdots\int e^{j\frac{\nu}{n}x_1 + j\frac{\nu}{n}x_2 + \cdots + j\frac{\nu}{n}x_n} f_{X_1}(x_1)\times f_{X_2}(x_2)\times\cdots\times f_{X_n}(x_n)\,dx_1\cdots dx_n$$

$$= \prod_{i=1}^{n}\left\{\underbrace{\int_{-\infty}^{+\infty} e^{j\frac{\nu}{n}x_i} f_{X_i}(x_i)\,dx_i}_{M_{X_i}(j\frac{\nu}{n})}\right\}$$

$$\overset{X_i\text{'s iid}}{=\!=} \left[M_{X}(j\tfrac{\nu}{n})\right]^n$$

Aside:

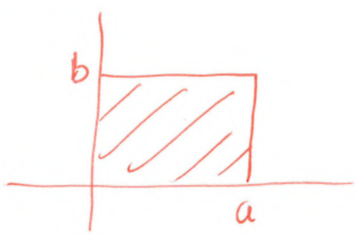
2D (Spatial Fourier Transform)

$g(x, y)$

$$G(u, v) = \iint_{-\infty}^{+\infty} g(x, y)\, e^{-j2\pi ux}\, e^{-j2\pi vy}\, dx\, dy$$

$$g(x, y) = \iint_{-\infty}^{+\infty} G(u, v)\, e^{j2\pi ux}\, e^{j2\pi vy}\, du\, dv$$

ex.1



$$G(u, v) = \iint_{\text{"}x\text{" } \text{"}y\text{"}} 1 \cdot e^{-j2\pi ux}\, e^{-j2\pi vy}\, dx\, dy$$

$$= \left(\int e^{-j2\pi ux}\, dx\right)\left(\int e^{-j2\pi vyu}\, dy\right)$$

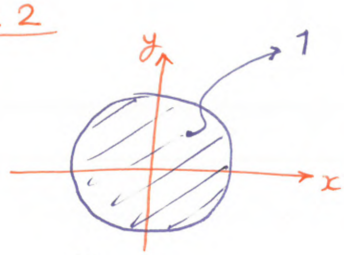
$$= \left.\frac{e^{-j2\pi xu}}{(-j2\pi u)}\right|_0^a \left.\frac{e^{-j2\pi vy}}{-j2\pi v}\right|_0^b = \left(\frac{1 - e^{-j2\pi au}}{2j\pi u}\right)\left(\frac{1 - e^{-j2\pi bv}}{j2\pi v}\right)$$

$$= e^{-j\pi a u}\left(\frac{e^{j\pi a u} - e^{-j\pi a u}}{j2\pi u}\right) \times e^{-j\pi b v}\left(\frac{e^{j\pi b v} - e^{-j\pi b v}}{j2\pi v}\right)$$

$$= e^{-j\pi a u - j\pi b v} \quad \frac{a}{a}\frac{\sin(\pi a u)}{\pi u} \quad \frac{b}{b}\frac{\sin(\pi b v)}{\pi v}$$

$a \operatorname{sinc}(au)$ $\quad$ $b \operatorname{sinc}(bv)$

ex. 2



$g(x,y)$

$$G(u,v) = \iint_{(x,y):\, x^2+y^2 \leq 1} 1 \cdot e^{-j2\pi xu - j2\pi yv} \, dx\, dy$$

change of variables (coordinates): $x = r\cos\varphi$, $y = r\sin\varphi$

$$\underbrace{\left|\frac{dx\,dy}{dr\,d\varphi}\right|}_{\left|\det\begin{bmatrix}\cos\varphi & \sin\varphi \\ -r\sin\varphi & r\cos\varphi\end{bmatrix}\right| \;=\; |r|} dr\,d\varphi$$

$$\Rightarrow \int_0^1\int_0^{2\pi} |r|\, e^{-j2\pi r\cos\varphi\, u - j2\pi r\sin\varphi\, v} \, d\varphi\, dr$$

$\boxed{\begin{aligned} u &= \rho\cos\theta \\ v &= \rho\sin\theta \end{aligned}}$

$$= \iint r\, e^{\overbrace{-j2\pi r\rho\cos\varphi\cos\theta - j2\pi r\rho\sin\varphi\sin\theta}^{-j2\pi r\rho\cos(\varphi-\theta)}} \, d\varphi\, dr$$

$$= \int_0^1 r\, 2\pi\, \overbrace{J_0(2\pi r\rho)}^{\text{zero-order Bessel function}} \, dr$$

$$= \frac{\overbrace{J_1(2\pi\rho)}^{\text{first-order Bessel function}}}{\rho}$$

(Airy function / response of a telescope) ex. $X_1, \dots, X_n$ n RV's (not iid necessarily)

$a_1, \dots, a_n$

$$Y = \sum_{i=1}^{n} a_i X_i \qquad (\text{ex. BLUE})$$

$E[Y] = ?$, $Var[Y] = ?$ in terms of $E[X_i] = m_i$ and $Cov[X_i, X_j]$.

$$E[Y] = E\Big[\sum_{i=1}^{n} \underbrace{a_i}_{\text{known}} X_i\Big] = \sum a_i E[X_i] = \sum a_i m_i$$

$$Var[Y] = E\big[(Y - E[Y])^2\big] = E\Big[\big(\sum a_i (X_i - m_i)\big)^2\Big]$$

$$= E\Big[\big(\sum_i a_i (X_i - m_i)\big)\big(\sum_j a_j (X_j - m_j)\big)\Big]$$

$$= E\Big[\sum_i \sum_j a_i (X_i - m_i)(X_j - m_j) a_j\Big]$$

$$= \sum_i \sum_j a_i a_j E\big[(X_i - m_i)(X_j - m_j)\big]$$

$$= \sum_i \sum_j a_i a_j Cov(X_i, X_j)$$

$$\underline{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Let's Vectorize!

$$\underline{m} = \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix}$$

$$\Rightarrow \quad Y = a^T X \qquad \text{(inner product)}$$

$$\underline{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$$

Variances

Covariances

$$K_{n \times n} = \begin{bmatrix} \mathrm{Cov}(X_1, X_1) & \cdots & \mathrm{Cov}(X_1, X_n) \\ \vdots & \ddots & \vdots \\ \mathrm{Cov}(X_n, X_1) & \cdots & \mathrm{Cov}(X_n, X_n) \end{bmatrix}$$

Covariances

- symmetric matrix
- $K^T = K$

$$\Rightarrow \quad \begin{aligned} \mathrm{Var}(Y) &= \underline{a}^T K \underline{a} \\ E[Y] &= \underline{a}^T \underline{m} \end{aligned}$$

If $\mathrm{cov}(X_i, X_j) = 0$ for $i \neq j \Rightarrow$ K will be Diagonal!

The Characteristic Function of Y

$$M_Y(j\nu) = E[e^{j\nu Y}]$$

$$= E\left[e^{j\nu \sum_{i=1}^{n} a_i X_i}\right]$$

★ X_i are indep.

$$= \prod_i E\left[e^{j\nu a_i X_i}\right] =$$

$$= \prod_i M_{X_i}(e^{j\nu a_i})$$

Ex 4.6

$$\underline{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} \qquad E[X_i] = m$$

$$K = \begin{bmatrix} \sigma^2 & \rho\sigma^2 & & \emptyset \\ \rho\sigma^2 & \ddots & \ddots & \\ & \ddots & \ddots & \rho\sigma^2 \\ \emptyset & & \rho\sigma^2 & \sigma^2 \end{bmatrix} \qquad S_n = \sum_{i=1}^{n} X_i$$

$|\rho| \leq 1$ correlation coefficient

$E[S_n] = ?$ $\quad Var[S_n] = ?$

$$E[S_n] = E\left[\sum X_i\right] = \sum E[X_i] = \sum m = \boxed{n.m}$$

$$Var[S_n] = \sum_{i=1} \sum_{j=1} cov(X_i, X_j) = \boxed{n\sigma^2 + 2(n-1)\rho\sigma^2}$$

stoch. lect.

Jointly Gaussian Random Variables

$X_1, \dots, X_n \quad , \quad E[\,|X_i|^2\,] < \infty \qquad (*)$

X_i : real-valued $\qquad E[\,X^T X\,] < \infty$

Defn. 1 (Linear Sum Form)

$X_1, \dots, X_n$ satisfying $(*)$ are jointly Gaussian iff

$$Y = \sum_{i=1}^{n} a_i X_i = a^T X \quad \text{is Gaussian RV.}$$

for any choice of finite $\{a_i\}$

$$E[Y] = E[a^T X] = a^T m_X$$

$$\sigma_Y^2 = Var[Y] = a^T \underbrace{K_X}_{} a$$

- $n \times n$
- symmetric
- non-negative definite : $\sigma_Y^2 \geq 0$, K_X has non-negative eigenvalues

→ If Y is Gaussian with mean m_Y and variance σ_Y^2

$$M_Y(j\nu) = e^{j\nu m_Y - \frac{\nu^2}{2}\sigma_Y^2}$$

$$= e^{j\nu a^T m_X - \frac{1}{2}\nu a^T K_X a \nu^T}$$

$$= M_X(j v a)$$

set $v = 1$ and $a = v$ (just a change of variable $v^{new} = a v^{old}$)

The joint characteristic function of $X_1, \dots, X_n$ is

$$M_X(jv) = e^{j v^T m_X - \frac{1}{2} v^T K_X v} \qquad (**)$$

Defn 2. (characteristic Function Form)

Vector $\underline{X}$ with characteristic function $(**)$ is a Jointy Gaussian vector.

---

properties of jointly gaussian:

I. Jointy Gaussian RVs are completely characterized by their mean m_X and covariance matrix K_X.

II. If X_i and X_j in vector $\underline{X}$ such that $\mathrm{Cov}(X_i, X_j) = 0$, for $i \neq j$ (uncorrelated), $X_1, \dots, X_n$ are INDEPENDENT!

$$K_X = \begin{bmatrix} \sigma_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_n^2 \end{bmatrix}$$

Then $M_X(jv) = \prod_{i=1}^{n} e^{j v_i m_{X_i} - \frac{1}{2} v_i^2 \sigma_i^2}$

$$= \prod_{i=1}^{n} M_{X_i}(jv_i)$$

III. A : subset of a jointly Gaussian vector $X = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$

$$M_{\underline{X}}(jv_1, 0, jv_3, 0, \ldots)$$

is another jointy Gaussian vector.

IV. Any linear transformation A, of a jointly Gaussian Vector $\underline{X}$ is a jointly Gaussian vector, as well.

$$Z = \underset{\substack{\downarrow \\ \text{constant coefficients}}}{A_{m\times n}} X_{n\times 1}$$

Use Defn 1 for jointly Gaussian:

$$(b_{m\times 1}) \rightarrow \quad Y = b^T . Z = \underbrace{b^T_{1\times m} A_{m\times n}}_{d^T_{1\times n}} X_{n\times 1} = d^T \underset{\substack{\downarrow \\ \text{Jointy Gaussian}}}{X}$$

$$\underset{\text{(is Jointly Gaussian)}}{Z} \Leftarrow Y \text{ is Jointy Gaussian} \Leftarrow$$

$$m_Z = E[Z] = E[AX] = Am_X$$

$$K_Z = E[(Z - m_Z)(Z - m_Z)^T] = E[A(X - m_X)(X - m_X)A^T] =$$

$$= A\ E[(X-m_X)(X-m_X)^T]\ A^T$$

$$= A K_X A^T$$

- Linear Algebra to Decorrelate a Jointy Gaussian

$$K_X\ Q = Q\ \Lambda_X$$

Λ_X: $n \times n$ matrix of eigenvalues (Diagonal)

$$\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

K_X:
- $n \times n$
- symmetric
- non-negative definite

Q: matrix of eigenvectors of K_X (orthogonal matrix) ($Q^{-1} = Q^T$)

$$\begin{bmatrix} | & | & | & & | \\ q_1 & q_2 & q_3 & \cdots & q_n \\ | & | & | & & | \end{bmatrix}$$

set $A = Q^T$, then

$$Z = Q^T X \quad \Rightarrow \quad Z \text{ has uncorrelated components.}$$

stoch. lect.

set $B = K_{X_{n\times n}}$

$$K_X Q = Q \Lambda_X$$

$$Q = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix}_{n\times n} \qquad \Lambda_X = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}_{n\times n}$$

→ Try $\underline{Y} = Q^T \underline{X}$ to decorrelate RV's in X.

constant coefficients

$$E[\underline{Y}] = Q^T \underline{m_X}$$

covariance matrix

$$K_Y = E[\underbrace{(Y - m_Y)}_{n\times 1}(Y - m_Y)^T]$$

$$= E[Q^T(X - m_X)(X - m_X)^T Q]$$

$$= Q^T \underbrace{E[(X - m_X)(X - m_X)^T]}_{K_X} Q$$

$$= Q^T Q \Lambda_X$$

$$= \Lambda_X$$ → diagonal covariance matrix for K_Y, shows that $\underline{Y}$ components are decorrelated.

because it is Gaussian, they are independent as well.

eigenvalues of K_X will be variances of $\underline{Y}$.

new transformation $\underset{\sim}{Z}_{n\times 1} = \Lambda_X^{-1/2} \underline{Y} = \Lambda_X^{1/2} Q^T \underline{X}$

$K_Z = I$ $\quad$ Z is <u>whitenned</u> version of X.

Jointly Gaussian PDF

$X_i \sim$ indep Gaussian $(0, \sigma_i^2)$ $\quad$ $\underline{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$

$i = 1 \dots n$

$$f_{\underline{X}}(\underline{x}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-\frac{1}{2\sigma_i^2}(x_i - 0)^2}$$

$$\underset{n\times n}{K_{\underline{X}}} = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{bmatrix}$$

$$K_X^{-1} = \begin{bmatrix} \frac{1}{\sigma_1^2} & & \\ & \ddots & \\ 0 & & \frac{1}{\sigma_n^2} \end{bmatrix}$$

$$\det K_X = \prod_{i=1}^{n} \sigma_i^2$$

$$f_{\underline{X}}(\underline{x}) = \frac{1}{\sqrt{(2\pi)^n \det(K_X)}} e^{-\frac{1}{2}\underline{x}^T K_X^{-1} \underline{x}}$$

$$\underline{Y} = A_{n\times n}\underline{X}_{n\times 1} + \underline{m}_{X\, n\times 1} \rightsquigarrow \underline{y} = A\underline{x} + m_Y$$

$$x = A^{-1}(y - m_Y)$$

$$f_{\underline{Y}}(y) = f_{\underline{X}}(A^{-1}(y - m_Y)) \cdot \frac{1}{|J(y)|}$$

$$\left|\frac{dy}{dx}\right| = \left|\det\begin{bmatrix} \frac{\partial y_1}{\partial x_1} & & \frac{\partial y_n}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_1}{\partial x_n} & & \frac{\partial y_n}{\partial x_n} \end{bmatrix}\right|$$

$$f_Y(y) = \frac{1}{\sqrt{(2\pi)^n \det(K_X)}} \cdot \frac{1}{\det(A)} e^{-\frac{1}{2}\underbrace{(A^{-1}(y-m_Y))^T K_X^{-1}(A^{-1}(y-m_Y))}_{(y-m_Y)^T \underbrace{A^{-T} K_X^{-1} A^{-1}}_{(A K_X A^T)^{-1}} (y-m_Y)}}$$

$$\det(A) = \sqrt{\det(A) . \det(A^T)}$$

becaus all matrices are square:

$$\det(K_X)\det(A)\det(A^T) = \det(K_X A A^T) = \det(\underbrace{A K_X A^T}_{K_Y})$$

$$f_{\underline{Y}}(y) = \frac{1}{\sqrt{(2\pi)^n \det(K_Y)}} e^{-\frac{1}{2}(y - m_Y)^T K_Y^{-1}(y - m_Y)}$$

$$Y \sim N(m_Y, K_Y)$$

---

ex.

$n = 2$

$$E[Y] = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \qquad K_Y = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_1\sigma_2\rho & \sigma_2^2 \end{bmatrix}$$

$$\mathrm{cov}(Y_1, Y_2) = \sigma_1\sigma_2\rho$$

$$K_Y^{-1} = \frac{1}{\sigma_1^2 \sigma_2^2 (1-\rho)} \begin{bmatrix} \sigma_2^2 & -\sigma_1\sigma_2\rho \\ -\sigma_1\sigma_2\rho & \sigma_1^2 \end{bmatrix}$$

$$f_Y(\underline{y}) = \frac{1}{\sqrt{(2\pi)^2 \sigma_1^2 \sigma_2^2 (1-\rho)}} \, e^{-\frac{1}{2}(\underline{y}-\underline{m}_Y)^T K_Y (\underline{y}-\underline{m}_Y)}$$

ex.

$$K_X = \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}$$

$$K_X Q = Q \Lambda_X$$

$$\det\left(\begin{bmatrix} 5-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}\right) = (5-\lambda)(1-\lambda) - 1 = 0$$

$$\lambda^2 - 6\lambda + 4 = 0 \Rightarrow \{ 3 \pm \sqrt{5}$$

$$\det(K_X - \lambda I) = 0$$

$$\Lambda_Y = \begin{bmatrix} 3+\sqrt{5} & 0 \\ 0 & 3-\sqrt{5} \end{bmatrix}$$

$$K_X q_1 = \lambda q_1$$

$$\begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix} = (3+\sqrt{5}) \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix}$$

$$5q_{11} + q_{12} = (3+\sqrt{5}) q_{11}$$

$$q_{11} + q_{12} = (3+\sqrt{5}) q_{12}$$

stoch. lect.

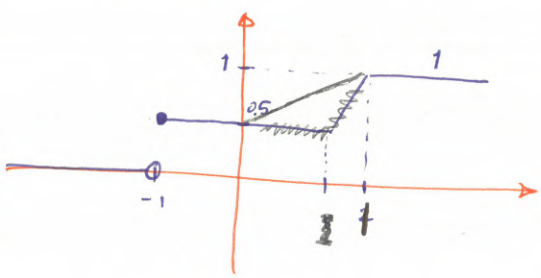
$F_X(x) = P(X \leq x)$ continuous from right. (no matter of being mixed, continuous or discrete)

$P(a < X \leq b) = F_X(b) - F_X(a)$

P.4

$$\begin{cases} 0 & x < -1 \\ 0.5 & -1 \leq x \leq 0 \\ \frac{1+x}{2} & 0 < x \leq 1 \\ 1 & x > 1 \end{cases}$$

(a)



mixed R.V.

(b)

$P(X \leq -1) = 0.5$

$P(X = -1) = F_X(-1^+) - F_X(-1^-) = 0.5$

$P(X < 0.5) = 0.75$

$P(0.5 < X < 0.5) = 0.75 - 0.5 = 0.25$

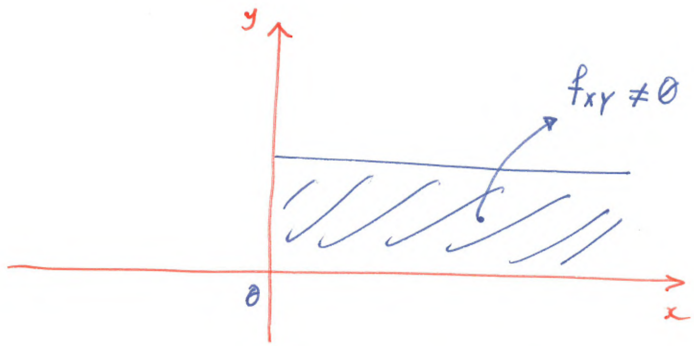
$P(X > -1) = 1 - P(X \leq -1) = 1 - 0.5 = 0.5$

$P(X \leq 2) = 1$

$P(X > 3) = 1 - F_X(3) = 1 - 1 = 0$

P.3

$$f_{XY}(x,y) = \begin{cases} C y e^{-xy} & , \quad x \geqslant 0, \quad 0 \leqslant y \leqslant 1 \\ 0 & , \quad \text{o.w.} \end{cases}$$



a)

$$C \int_0^1 y \underbrace{\int_0^{+\infty} e^{-xy}\,dx}_{\left.\frac{1}{-y} e^{-xy}\right|_{0=x}^{+\infty=x}}\,dy = C\int_0^1 y\left[0 + \frac{1}{y}\right]dy = \boxed{C = 1}$$

(b)

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx = \int_0^{+\infty} y e^{-xy}\,dx = y\,\frac{1}{y} = \begin{cases} 1 & 0 \leqslant y \leqslant 1 \\ 0 & \text{o.w.} \end{cases}$$

(c)

$$f_X(x) = \int_0^1 y e^{-xy}\,dy$$

by parts

$(\int u\,dv = uv - \int v\,du)$

$$= -\frac{y}{x} e^{-xy}\Big|_0^1 + \int_0^1 \frac{e^{-xy}}{x}\,dy$$

$$= \frac{1}{x}e^{-x} - \frac{e^{-xy}}{x^2}\Big|_0^1 = -\frac{e^{-x}}{x} - \frac{e^{-x}}{x^2} + \frac{1}{x^2}, \quad x \geqslant 0$$

$$E[|X|] \rightarrow \infty$$

$$E[|X|] = E[X] = \int_0^{+\infty} x\left(\frac{1}{x^2} - \frac{e^{-x}}{x} - \frac{e^{-x}}{x^2}\right) dx$$

does NOT converge!

(d)

$P(X < 1)$ (using joint pdf) solve

$$= \int_0^1 \int_0^1 y e^{-xy} \, dx \, dy$$

(e) $f_{X|Y}(x \mid y=2)$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{f_{XY}(x,2)}{f_Y(2)} = 0$$

(joint pdf is zero @ $y=2$)

(f) are X and Y indpendent? NO! $X \not\perp\!\!\!\perp B$

$$f_{XY}(x,y) \neq f_X(x)\, f_Y(y)$$

(g) $E[XY] = ?$

Correlated?

$$\mathrm{cov}(X,Y) = E[XY] - E[X].E[Y]$$
$$= E[(X - E(X))(Y - E(Y))]$$

P.2

$$f_X(x) = p a e^{-ax} + (1-p) b e^{-bx} \qquad x \geqslant 0$$

$0 \leqslant P \leqslant 1$

$a, b > 0$

inverse Fourier Transform of pdf $\equiv$ Characteristic Function

(a) $$M_X(jv) = E[e^{jvX}] = pa\int_0^{+\infty} e^{jvx} e^{-ax}\,dx + \int (1-p) b\, e^{+jvx} e^{-bx}\,dx$$

$$= pa \left.\frac{e^{-(a-jv)x}}{-(a-jv)}\right|_0^{+\infty} +$$

$$= \frac{pa}{a-jv} + \frac{(1-p)b}{b-jv}$$

(b) $$E[X] = \left.\frac{d}{d(jv)} M_X(jv)\right|_{v=0} = \left. -\frac{pa}{(a-jv)^2} + \frac{(1-p)b}{(b-jv)^2}\right|_{v=0} = \frac{p}{a} + \frac{1-p}{b}$$

(c) $$Var[X] = E[X^2] - E^2[X] = \left.\frac{d^2}{d(jv)^2} M_X(jv)\right|_{v=0} - E^2[X]$$

P.7 $\quad X \sim \text{Poisson}\left(\frac{3}{4}\right)$

$\lambda = \frac{3}{4}$

$$P\Big(\big|X - \underbrace{E[X]}_{\frac{3}{4}}\big| > \overbrace{\tfrac{1}{4}}^{\epsilon}\Big) \leq \frac{\overbrace{Var[X]}^{3/4}}{\underbrace{1/16}_{\epsilon^2}}$$

$$P\left(\left|X - \tfrac{3}{4}\right| > \tfrac{1}{4}\right) = P\left(\left\{X - \tfrac{3}{4} > \tfrac{1}{4}\right\} \cup \left\{X - \tfrac{3}{4} < -\tfrac{1}{4}\right\}\right)$$

mutually exclusive

$$= P(X > 1) + P\left(X < \tfrac{1}{2}\right)$$

$$= 1 - \underbrace{P(X \leq 1)}_{} + P(X = 0)$$

$$= 1 - \overbrace{\cancel{P(X=0)} - P(X=1)} + \cancel{P(X=0)}$$

$$= 1 - P(X=1)$$

$$= 1 - \frac{3}{4}e^{-3/4} = 1 - 0.3543$$

P.8 $\longrightarrow$ nice problem $\rightarrow$ check it for midterm.

P.10

$X \sim \text{unif}(0, a)$, n observations

$Y = \max\{X_1, \ldots, X_n\}$

(a) $P(Y \leq y) = F_Y(y)$

$$= P[\max(X_1, \ldots, X_n) \leq y]$$

this is the right way for max. then can write in terms of products.

all X_i's smaller than y

$$= \left(P[X_i < y]\right)^n$$

$$= \left(\frac{1}{a}\int_0^a d\alpha\right)^n$$

$$= \left(\frac{y}{a}\right)^n \qquad 0 \leq y \leq a$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{n y^{n-1}}{a^n}, \quad 0 \leq y \leq a$$

$$E[Y] = \int_0^a \frac{n y^{n-1}}{a^n} y \, dy = \frac{n}{n+1} \frac{a^{n+1}}{a^n} = \frac{n}{n+1} a$$

as $n \to \infty$ $E[Y] \to a$

unbiased estimator

P.12

$$X \quad , \quad F_X(x)$$

$$E[F_X(x)] = \int F_X(x) \frac{dF_X(x)}{\cancel{dx}} \cancel{dx} = \left.\frac{F_X^2(x)}{2}\right|_{-\infty}^{+\infty} = \frac{1}{2}$$

$$Var[F_X(x)] = ?$$

stoch. lect.

- Sums of Random Variables and Weak Law of Large Numbers

$$X_1, X_2, \dots, X_n, \dots \quad , \quad E[X_i] = m_i$$

probability $\xrightarrow{\text{WLLN}}$ statistics

$$S_n = \sum_{i=1}^{n} X_i$$

$$E[S_n] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} m_i \underset{\text{mean}}{\overset{\text{equal}}{=\!=}} n \cdot m_X$$

$$\text{var}[S_n] = E\left[(S_n - E[S_n])^2\right] = E\left[\sum_{i=1}^{n}(X_i - m_i)\sum_{j=1}^{n}(X_j - m_j)\right]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \underbrace{E\left[(X_i - m_i)(X_j - m_j)\right]}_{\text{Cov}(X_i, X_j)}$$

if X_i and X_j are pairwise uncorrelated $\text{cov}(X_i, X_j) = 0$ for $i \neq j$

$$= \sum_{i=1}^{n} \sigma_i^2 \underset{\text{variance}}{\overset{\text{equal}}{=\!=}} n\,\sigma_X^2$$

→ normalizing S_n

$$S_n^* \triangleq \frac{S_n - E[S_n]}{n}$$

$$S_n^* \xrightarrow[\text{in what sense?}]{?} 0$$

$$E[S_n^*] = 0$$

Let's measure deviation of S_n from its mean, by means of probability:

$$P(|S_n^* - 0| > \epsilon) \xrightarrow[\text{as } n \to +\infty]{} 0$$ (if it goes to zero, we can answer that question)

$$\Rightarrow \quad S_n^* \xrightarrow[\text{as } n\to+\infty]{P} 0$$

P → convergence in probability

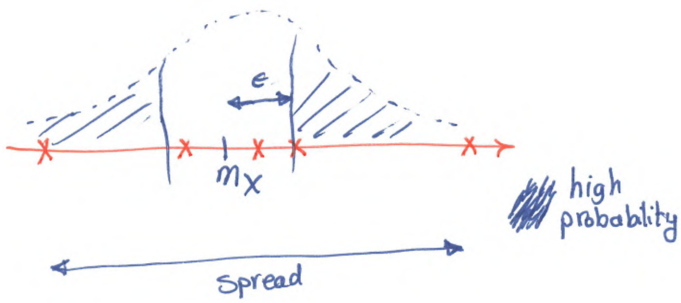
11, 2, 22 - 1

$$\mathrm{Var}(S_n^*) = E\left[(S_n^*)^2\right] = \frac{1}{n}\,\mathrm{var}(S_n) \underset{\substack{\text{uncorrelated} \\ \text{and equal variance}}}{=} \frac{n\sigma_X^2}{n^2} = \frac{\sigma_X^2}{n}$$

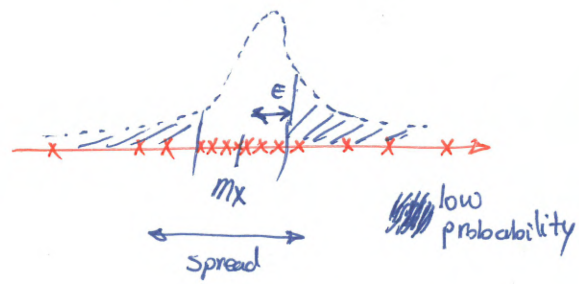
$E[S_n^*] = 0$

$$X_i \sim iid\,,\ m_X,\ \sigma_X^2 \quad \Rightarrow \quad \frac{S_n}{n} \xrightarrow[\text{as } n \to +\infty]{P} m_X$$

n is small



n is large



: amount of probability of divergence from mean

X_i are iid with mean $E[X_i] = m_X$ and $\mathrm{Var}(X_i) = \sigma_X^2$

chebyshev's inequality

$$P\left(\left|\frac{S_n}{n} - m_X\right| > \epsilon\right) \leq \frac{\sigma_X^2}{n}$$

$$\frac{S_n}{n} \xrightarrow[\text{as } n \to +\infty]{P} m_X$$

a variant of Weak Law of Large Numbers (WLLN).

Theorem (markoff)

If $\{X_i\}$ is such that $\lim\limits_{n \to +\infty} \frac{1}{n^2} var\left(\sum_{i=1}^{n} X_i\right) = 0$,

then $S_n^* \xrightarrow[\text{as } n \to +\infty]{P} 0$.

Theorem (Chebyshev)

If $\{X_i\}$ are pairwise uncorrelated and

$$\lim_{n \to +\infty} \frac{1}{n^2} \sum_{i=1}^{n} \sigma_i^2 = 0, \text{ then}$$

$$S_n^* \xrightarrow{P} 0 .$$

Theorem (Bernoulli)

Let $k(n)$ be the number of occurances of event A in n trials, and the probability $P(A)$ is p. Then $Z_n \triangleq \frac{k(n)}{n}$

$$P(|Z_n - p| > \epsilon) \xrightarrow[\text{as } n \to +\infty]{} 0$$

$$X_i = \begin{cases} 1 & \text{if } A \\ 0 & \text{if } A^c \end{cases}$$

* Use Chebyshev Theorem to prove the statement.

$$Z_n = \frac{\sum_{i=1}^{n} X_i}{n}$$

$$E[Z_n] = p$$

$$Var[Z_n] = \frac{np(1-p)}{n^2} = \frac{p(1-p)}{n}$$

as $n \to +\infty \Rightarrow Var[Z_n] \to 0$ so $Z_n \xrightarrow[\text{as } n\to+\infty]{P} p$

Stoch. lect.

X_i are iid.

$\underbrace{\quad}_{m_X \quad \sigma_X^2}$

converge in probability

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow[\text{as } n\to+\infty]{P} m_X \equiv P\left(\left|\frac{1}{n}\sum X_i - m_X\right| > \epsilon\right) \xrightarrow[\text{as } n\to+\infty]{} 0$$

chebyshev: $\leq \dfrac{\sigma_X^2}{n}$

goal of today:

(sample cdf $\longrightarrow$ true cdf) by introducing for Poisson

ex. (Poisson Form)

$Y_1, \dots, Y_n, \dots$ independent (not necessarily iid)

$Y_i \leq y$

$$X_i = \begin{cases} 1 & , \text{if } Y_i \leq y \\ 0 & , \text{else} \end{cases}$$

$E[X_i] = 1 \times P(X_i = 1) + 0 \times P(X_i = 0)$

$= P(Y_i \leq y) = F_{Y_i}(y)$ cdf of Y_i

$$\frac{1}{n}\sum_{i=1}^{n} X_i : \boxed{\text{Estimator of } F_Y(y)} \equiv \hat{F}_Y(y)$$

$Var(X_i) = E[X_i^2] - E^2[X_i] = 1^2 \cdot P(X_i = 1) - F_{Y_i}^2(y) = F_{Y_i}(y)\,(1 - F_{Y_i}(y)) < 1$ we know

Apply Chebyshev form of WLLN:

$$\lim_{n\to+\infty} \frac{1}{n^2}\sum_{i=1}^{n} \sigma_i^2 = 0 \qquad (*)$$

since $Var(X_i) < 1$ then $(*)$ is satisfied.

$$P\left(\left|\overbrace{\frac{1}{n}\sum_{i=1}^{n} X_i}^{\hat{F}_Y(y)} - \frac{1}{n}\sum_{i=1}^{n} F_{Y_i}(y)\right| > \epsilon\right) \xrightarrow[\text{as } n\to+\infty]{} 0$$

$$\frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\sum_{i=1}^{n} F_{Y_i}(y) \xrightarrow[\text{as } n\to+\infty]{P} 0$$

Theorem (Khinchin form)

If $\{X_i\}$ are iid

with $E[X_i] < \infty$, then $\frac{1}{n}\left(\sum X_i - \sum E[X_i]\right) \xrightarrow[\text{as } n\to+\infty]{P} 0$

$$E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = m_X \qquad \text{unbiased estimate}$$

iid $\qquad m_X$

$$M_n = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad \text{(sample mean)}$$

M_n is consistent estimate of m_X if $M_n \xrightarrow[\text{as } n\to+\infty]{P} m_X$

Central Limit Theorem (CLT)

$\{X_i\}$: iid with mean m_X and variance σ_X^2

$S_n = \sum_{i=1}^{n} X_i \qquad E[S_n] = n\, m_X$

$\qquad Var(S_n) = n\,\sigma_X^2$

Thm. CLT

Let S_n be sum of iid RV's $X_1, \dots, X_n, \dots$ each with $E[X_i] = m_X$ and $var(X_i) = \sigma_X^2$.

Define a new sequence of RV's

$$S_n^* = \frac{S_n - E[S_n]}{\sigma_{S_n}}$$

Then the cdf of S_n^* converges to cdf of a RV Z with cdf of Gaussian $\mathcal{N}(0,1)$.

$$(**) \qquad F_{S_n^*}(\alpha) \xrightarrow[\text{as } n\to+\infty]{} \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt$$

(**) is the same as

$$S_n^* \xrightarrow[\text{as } n \to +\infty]{d} Z$$

Convergence in distribution

$Z \sim \mathcal{N}(0,1)$

Theorem (Continuity)

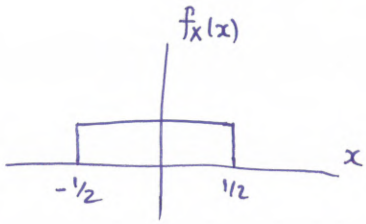
$$X_n \xrightarrow{d} X$$

sequence of RV's.

$$\Phi_{X_n}(jv) \longrightarrow \Phi_X(jv)$$

ex.

$$X_i \sim U\left(-\tfrac{1}{2}, \tfrac{1}{2}\right)$$



$$\begin{cases} E[X] = 0 \\ Var(X) = \frac{1}{12} \end{cases}$$

$$S_n = \sum_{i=1}^{n} X_i$$

$$S_n^* = \frac{S_n - \cancel{E[S_n]}^{\,0}}{\sqrt{Var[S_n]}} = \frac{S_n}{\sqrt{\frac{n}{12}}}$$

$Var(S_n^*) = 1$
$E[S_n^*] = 0$

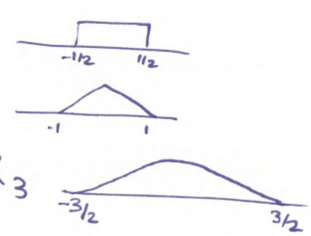
$$\begin{cases} E[S_n] = 0 \\ Var[S_n] = n \cdot \frac{1}{12} \end{cases}$$

aside:

$S_1 = X_1$

$S_2 = X_1 + X_2$

$S_3 = X_1 + X_2 + X_3$



The CLT

$$S_n = \sum_{i=1}^{n} X_i \qquad \text{iid}$$

$$S_n^* = \frac{S_n - E[S_n]}{\sigma_{S_n}} \xrightarrow[\text{as } n \to +\infty]{\substack{\text{convergence} \\ \text{in distribution}}} Z \sim \mathcal{N}(0, 1)$$

$$F_{S_n^*}(\alpha) \xrightarrow[n \to +\infty]{} F_Z(\alpha)$$

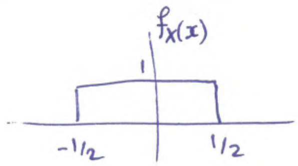
Continuity Theorem

For $\underbrace{X_n}_{\text{sequence of RVs}} \longrightarrow X$, it is necessary and sufficient that

$$M_{X_n}(jv) \xrightarrow[\text{as } n \to +\infty]{} M_X(jv)$$

ex.

$X_i \sim$ iid $\text{unif}\left(-\frac{1}{2}, \frac{1}{2}\right)$



$$S_n = \sum_{i=1}^{n} X_i \qquad E[X_i] = 0$$

$$Var[X_i] = E[X_i^2] = \frac{1}{12}$$

$$E[S_n] = 0$$

$$Var[S_n] = n\, Var[X_i] = \frac{n}{12}$$

no dependence on n

$$\underline{S_n^* = \sum X_i \sqrt{\frac{12}{n}}} \qquad \underline{E[S_n^*] = 0}, \quad \underline{Var[S_n^*] = 1}$$

Apply Continuity Theory to show $S_n^* \xrightarrow{dist.} Z \sim \mathcal{N}(0,1)$

$$M_{S_n^*}(jv) = E\left[e^{jv\sqrt{\frac{12}{n}}\sum x_i}\right]$$

x_i's iid

$$= \left(E\left[e^{jv\sqrt{\frac{12}{n}}x_1}\right]\right)^n$$

$$= \left(\underbrace{\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{jv\sqrt{\frac{12}{n}}x}\,dx}_{\left.\frac{e^{jv\sqrt{\frac{12}{n}}x}}{jv\sqrt{\frac{12}{n}}}\right|_{-1/2}^{1/2}}\right)^n$$

$$= \left(\frac{e^{jv\sqrt{\frac{12}{n}}\frac{1}{2}} - e^{-jv\sqrt{\frac{12}{n}}\frac{1}{2}}}{jv\sqrt{\frac{12}{n}}}\right)^n = \left(\frac{1}{v\sqrt{\frac{3}{n}}}\sin\left(v\sqrt{\frac{3}{n}}\right)\right)^n$$

taylor: $\boxed{\sin(x) = x - \frac{x^3}{3!}} + \frac{x^5}{5!} - \dots \Rightarrow$

$$\cong \left(\frac{v\sqrt{\frac{3}{n}} - \frac{1}{6}v\sqrt{\frac{3}{n}}\,v^2\cdot\frac{3}{n}}{v\sqrt{\frac{3}{n}}}\right)^n$$

$$\boxed{\text{aside} \quad \lim_{n\to+\infty}\left(1-\frac{a}{n}\right)^n = e^{-a}}$$

$$= \left(1 - \frac{1}{2}\frac{v^2}{n}\right)^n$$

as $n \to +\infty$ converges $e^{-\frac{1}{2}v^2}$

Done!

Characteristic function of Gaussian with $\mu=0$, $\sigma^2=1$

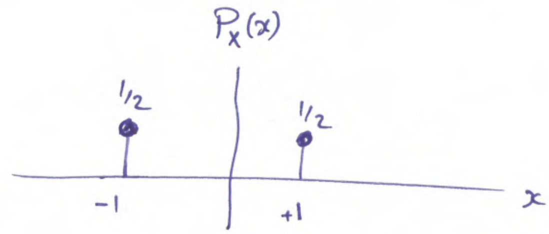
Thus by continuity Theorem:

$$S_n^* \xrightarrow[\text{as } n \to +\infty]{\text{dist.}} Z \sim \mathcal{N}(0,1)$$

---

ex.2 (binary R.V.)

$$X_i = \begin{cases} 1, & \text{with prob } \frac{1}{2} \\ -1, & \text{with prob } 1/2 \end{cases}$$
iid



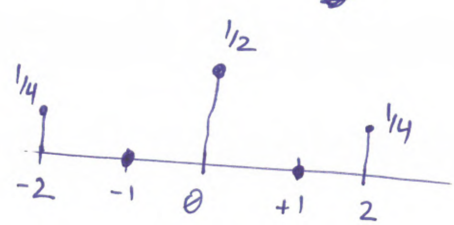
$$S_n = \sum_{i=1}^{n} X_i$$

$$E[X_i] = 0 \qquad E[S_n] = 0$$

$$Var[X_i] = E[X_i^2] = 1 \qquad Var[S_n] = n \qquad S_n^* = \frac{S_n}{\sqrt{n}}$$

by continuity Thm.:

$n=2$ PMF X_1+X_2



$$M_{S_n^*}(jv) = E\left[e^{jv\frac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i}\right]$$

iid

$$= \left(E\left[e^{jv\frac{1}{\sqrt{n}}x_1}\right]\right)^n$$

$$= \left[e^{\frac{jv}{\sqrt{n}}}\frac{1}{2} + e^{-jv\frac{1}{\sqrt{n}}}\frac{1}{2}\right]^n$$

aside

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \cdots}$$

$$= \left[\cos(v/\sqrt{n})\right]^n \simeq \left(1 - \frac{v^2}{2n}\right)^n \xrightarrow{n \to +\infty} e^{-v^2/2}$$

Gaussian $\mathcal{N}(0,1)$

Done!

$$\Rightarrow S_n^* \xrightarrow[\text{as } n \to +\infty]{d} Z \sim \mathcal{N}(0,1)$$

ex. 2

$P_{\text{bit flip}} = 0.15$

Bernoulli Trials

$n = 100$

$\underbrace{\sum_{i=1}^{100}}_{\text{counting bit flips}} X_i \rightarrow$ flip happens of ith trial

$$P\Big(\underbrace{\sum_{i=1}^{100} X_i}_{S_n} \leq 20\Big)$$

$E[X_i] = p = 0.15 \qquad Var[X_i] = p(1-p) = 0.15 \times 0.85$

$E[S_n] = np = 15$

$Var[S_n] = np(1-p) = 12.75$

$$\underbrace{P\left(\frac{\sum X_i - 15}{\sqrt{12.75}} \leq \frac{20-15}{\sqrt{12.75}}\right)}_{} \approx N(0,1)$$

$$\approx 1 - Q\left(\frac{5}{\sqrt{12.75}}\right)$$

[stoch. lect.]

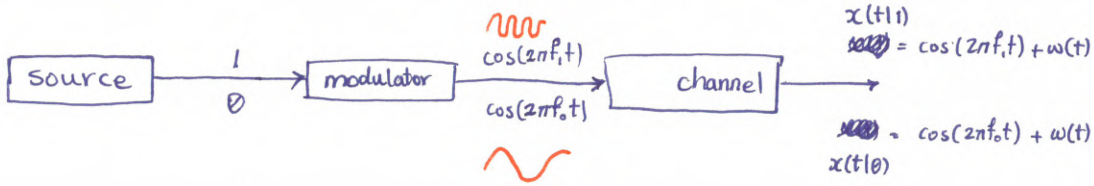
Basics of $\boxed{\text{Classical}}$ Detection

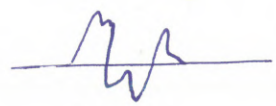
i.e., we are dealing with sample data only.

- Inferential Statistics
  - Hypothesis Testing
    - maybe wrong or right.
  - Estimation Theory
    - always wrong
    - how much close to true answer?

$\tau < t < \tau + T$

ex. 1





ex. 2 (radar, incoherent detection)

$s(t) = c(t) \cdot \sin(2\pi f_0 t)$

H_1: $r(t) = A\, c(t-\tau) \cdot \sin(2\pi \underset{\downarrow\, f_0 \pm f_D}{f_1} (t-\tau) + \varphi)$

H_0: $r(t) = w(t)$

$r(t), \quad \tilde{\tau} < t < \tilde{\tau} + T$

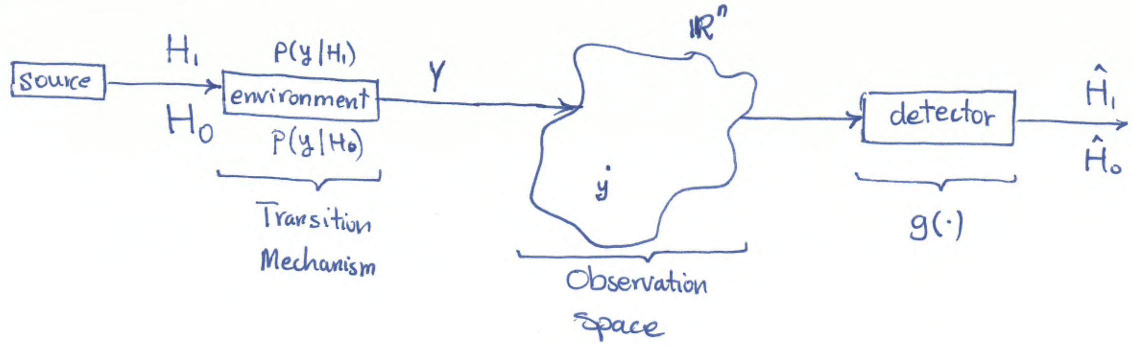
ex. 3 (sonar, radio astronomy)

H_1: $r(t) = \underbrace{s(t)}_{\text{random process}} + w(t)$

H_0: $r(t) = w(t)$

Given $r(t)$

mathematical framework



4 outputs:

decide H_1, and H_1 is true
decide H_0, and H_0 is true } correct decision

" H_1, " H_0 " "
" H_0, " H_1 " " } error in decision

ex.

	X = 0	X = 1	X = 2	X = 3
H_1	0.0	$\boxed{0.1}$	$\boxed{0.3}$	$\boxed{0.6}$
H_0	$\boxed{0.4}$	0.3	0.2	0.1

$\leftarrow P_{X|H_1}$.

is this way of decision good? $\longrightarrow$ we need a measure!

$\boxed{\text{Conditional Prob. of Error.}}$

$$P_{\substack{\text{False}\\\text{Alarm}}} = P(H_1|H_0) = P(\text{decide } H_1 \mid H_0 \text{ is true})$$

$$P_{\text{miss}} = P(H_0|H_1) = P(\text{decide } H_0 \mid H_1 \text{ is true})$$

$P_{FA} = P(H_1|H_0) = 0.3 + 0.2 + 0.1 = 0.6$

$P_{\text{miss}} = P(H_0|H_1) = 0$.

prior probability $\begin{cases} P(H_0) = \pi_0 \\ P(H_1) = \pi_1 = 1 - \pi_0 \end{cases}$

$\boxed{\text{Total Probability of Error}}$

$$P(\text{Error}) = P_{\substack{\text{False}\\\text{Alarm}}} \cdot \pi_0 + P_{\text{miss}} \cdot \pi_1$$

- maximum likelihood (ML) will minimize the total probability of error.

$P_{X|H_1}(x|H_1) > P_{X|H_0}(x|H_0)$, then we decide H_1.

$P_{X|H_1}(x|H_1) < P_{X|H_0}(x|H_0)$, then we decide H_0.

$\xrightarrow[\text{compact}]{\text{more}}$ $$P_{X|H_1}(x|H_1) \underset{H_0}{\overset{H_1}{\gtrless}} P_{X|H_0}(x|H_0)$$

revisit

Stoch. lect.

Recap: Maximum Likelihood Decision Rule

$P_{X|H_1}(x)$

$P_{X|H_0}(x)$

$X = x$

$$P(\text{error}) = \underbrace{P_{FA}}_{\text{False Alarm}} \pi_0 + P_{miss}\,\pi_1$$

$$= P(\text{decide } H_1 \mid H_0 \text{ is true})\,\pi_0 + P(\text{decide } H_0 \mid H_1 \text{ is true})\,\pi_1$$

If $\pi_1 = \pi_0$ or they are unknown.

$$P_{X|H_1}(x) \underset{\text{decide } H_0}{\overset{\text{decide } H_1}{\gtrless}} P_{X|H_0}(x) \longrightarrow \frac{P_{X|H_1}(x)}{P_{X|H_0}(x)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

Likelihood Ratio

$$\xrightarrow[\substack{\text{monotonic} \\ \text{(no change} \\ \text{on direction)}}]{\ln(\cdot)} \quad \ell(x) = \ln P_{X|H_1}(x) - \ln P_{X|H_0}(x) \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

---

ex. 2

H_1 : $X \sim$ Poisson (6)

H_0 : $X \sim$ Poisson (2)

$$P_{X|H_i}(x) = \frac{\lambda_i^x}{x!} e^{-\lambda_i}$$

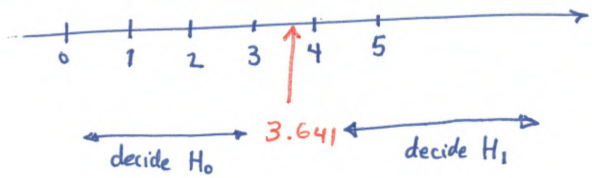
ML rule is

$$\ell(x) = x\ln 6 - \ln x! - 6 - (x\ln 2 - \ln x! - 2)$$

$$= x\ln 3 - 4$$

$$x \underset{H_0}{\overset{H_1}{\gtrless}} \frac{4}{\ln 3} \simeq \underbrace{3.641}_{\text{Threshold}}$$

in general, x is a R.V. X.



$$P_{FA} = P(\text{decide } H_1 \mid H_0 \text{ is true}) = \sum_{x > \text{Threshold}} P_{X|H_0}(x) = \sum_{x=4}^{+\infty} \frac{2^x}{x!} e^{-2}$$

$$= 1 - \sum_{x=0}^{3} \frac{2^x}{x!} e^{-2}$$

$$P_{miss} = P(\text{decide } H_0 \mid H_1 \text{ is true}) = \sum_{x \lesssim \text{Threshold}} P_{X|H_1}(x) = \sum_{x=0}^{3} \frac{6^x}{x!} e^{-6}$$

$$P_{error} = \frac{1}{2} P_{FA} + \frac{1}{2} P_{miss}$$

MAP test

$$P(\text{error}) = P_{FA} \underbrace{\pi_0}_{P(H_0)} + P_{miss} \underbrace{\pi_1}_{P(H_1)} \qquad (\pi_0 + \pi_1 = 1)$$

$$\underbrace{P(H_1 \mid X=x)}_{\frac{P(H_1, X=x)}{\cancel{P(X=x)}}} \underset{H_2}{\overset{H_1}{\gtrless}} P(H_0 \mid X=x)$$

$$\frac{P(H_0, X=x)}{\cancel{P(X=x)}}$$

don't have access to it

Use Bayes Rule

$$P(X=x \mid H_1) \cdot P(H_1) \underset{H_0}{\overset{H_1}{\gtrless}} P(X=x \mid H_0) P(H_0)$$

$$\Lambda(x) = \frac{P_{X|H_1}(x)}{P_{X|H_0}(x)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{\pi_1}$$

a R.V.

take ln $\longrightarrow$ $$\ell(x) = \ln P_{X|H_1}(x) - \ln P_{X|H_0}(x) \underset{H_0}{\overset{H_1}{\gtrless}} \ln\left(\frac{\pi_0}{\pi_1}\right)$$

log-likelihood ratio

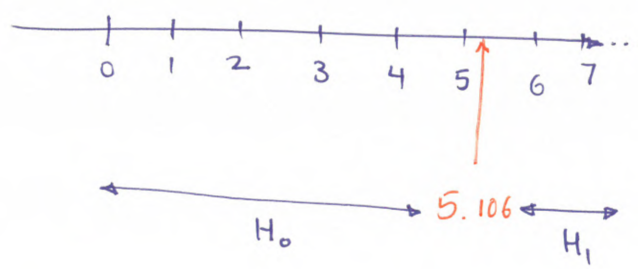
ex.2 revisit

Assume $\pi_0 = 5\pi_1$ $\qquad \pi_1 + \pi_0 = 1 \longrightarrow \pi_1 = \frac{1}{6}, \ \pi_0 = \frac{5}{6}$

$$\ell(x) = x\ln 3 - 4 \underset{H_0}{\overset{H_1}{\gtrless}} \ln 5$$

$$x \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\ln 5 + 4}{\ln 3}$$

a R.V.



$$P_{FA} = P(\text{decide } H_1 \mid \underbrace{H_0 \text{ is true}}) = \sum_{x=6}^{+\infty} \frac{2^x}{x!} e^{-2}$$

use H_0 for calculating prob's

$$P(\text{error}) = \frac{5}{6} P_{FA} + \frac{1}{6} P_{miss}$$

$$P_{miss} = P(\text{decide } H_0 \mid H_1 \text{ is true}) = \sum_{x=0}^{5} \frac{6^x}{x!} e^{-6}$$

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Continuous RV's

$H_1: \quad X \sim f_{X|H_1}(x)$

$H_0: \quad X \sim f_{X|H_0}(x)$

$$\Lambda(x) = \frac{f_{X|H_1}(x)}{f_{X|H_0}(x)} \underset{H_0}{\overset{H_1}{\gtrless}} 1 \quad \text{ML}$$

$$\text{MAP} \quad \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{\pi_1}$$

ex.3 (location testing)

noise $\mathcal{N}(0, \sigma^2)$

$H_1: \quad X = m_1 + W \longrightarrow X \sim \mathcal{N}(m_1, \sigma^2)$

$H_0: \quad X = m_0 + W \longrightarrow X \sim \mathcal{N}(m_0, \sigma^2)$

$m_0 < m_1$

ex.3 (location testing)

$H_1: \quad X \sim \mathcal{N}(m_1, \sigma^2)$

$H_0: \quad X \sim \mathcal{N}(m_0, \sigma^2)$

$$f_{X|H_1}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

(assumption: $m_1 > m_0$)

The MAP test:

π_1, π_0 are given.

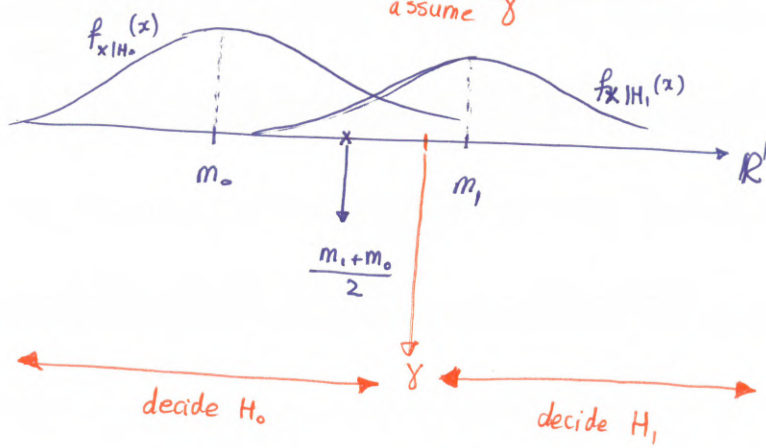
$$\ell(x) = \ln f_{X|H_1}(x) - \ln f_{X|H_0}(x)$$

$$= -\frac{1}{2\sigma^2}(x-m_1)^2 - \cancel{\frac{1}{2}\ln(2\pi\sigma^2)} - \left\{-\frac{1}{2\sigma^2}(x-m_0)^2 - \cancel{\frac{1}{2}\ln(2\pi\sigma^2)}\right\} \underset{H_0}{\overset{H_1}{\gtrless}} \ln\frac{\pi_0}{\pi_1}$$

$$= \frac{xm_1}{\sigma^2} - \frac{xm_0}{\sigma^2} - \frac{m_1^2}{2\sigma^2} + \frac{m_0^2}{2\sigma^2} \underset{H_0}{\overset{H_1}{\gtrless}} \ln\frac{\pi_0}{\pi_1}$$

$$\frac{x(m_1-m_0)}{\sigma^2} \underset{H_0}{\overset{H_1}{\gtrless}} \ln\frac{\pi}{\pi_0} + \frac{m_1^2 - m_0^2}{2\sigma^2}$$

$$x \gtrless \underbrace{\frac{\sigma^2 \ln \pi_1/\pi_0}{m_1 - m_0} + \frac{m_1+m_0}{2}}_{\text{assume } \gamma}$$



$$P_{FA}(\underbrace{\pi_1, \pi_0}_{\gamma}) = P(H_1 | H_0) = \int_{\gamma}^{+\infty} f_{X|H_0}(x)\, dx$$

$$= \int_{\gamma}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-m_0)^2} dx$$

$$= Q\left(\frac{\gamma - m_0}{\sigma}\right)$$

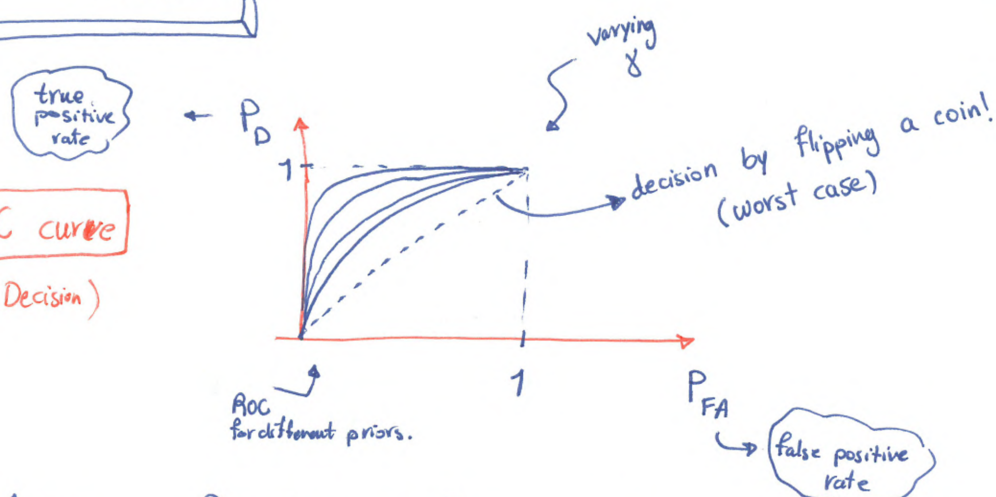
$$P_{miss}(\gamma) = P(H_0 | H_1) = \int_{-\infty}^{\gamma} f_{X|H_1}(x)\, dx = 1 - \int \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\pi\sigma^2}(x-m_1)^2} dx$$

$$= 1 - Q\left(\frac{\gamma - m_0}{\sigma}\right)$$

$$\boxed{P_{Detection}(\gamma) = 1 - P_{miss}(\gamma)}$$

ROC curve

(Binary Decision)



total probability of error: $P(\text{error}) = \pi_0 P_{FA} + \pi_1 P_{miss}$

---

ex.4

H_1 : small car $\quad X \sim N(0, a^2)$

H_0 : truck $\quad X \sim N(0, b^2)$

$b^2 > a^2 > 0$

π_0 and π_1 are known.

MAP test

$$\ell(x) = \ln f_{X|H_1}(x) - \ln f_{X|H_0}(x) = -\frac{x^2}{2a^2} - \frac{1}{2}\ln(2\pi a^2) + \frac{x^2}{2b^2} + \frac{1}{2}\ln(2\pi b^2)$$

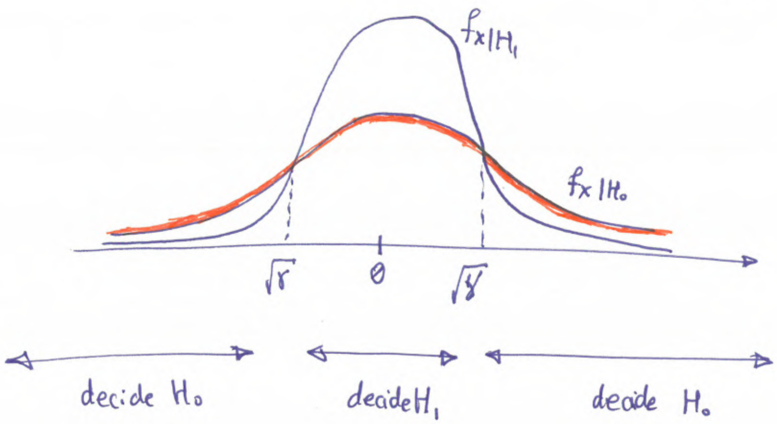
$$= x^2 \frac{1}{2}\left(\frac{1}{b^2} - \frac{1}{a^2}\right) \underset{H_0}{\overset{H_1}{\gtrless}} \ln\frac{\pi_0}{\pi_1} - \frac{1}{2}\ln\frac{b^2}{a^2}$$

$$x^2 \underset{H_0}{\overset{H_1}{\lessgtr}} \frac{\ln \frac{\pi_0}{\pi_1} - \frac{1}{2}\ln \frac{b^2}{a^2}}{\frac{1}{2}\left(\frac{1}{b^2} - \frac{1}{a^2}\right)} \quad \gamma$$

$$|x| \underset{H_0}{\overset{H_1}{\lessgtr}} \sqrt{\frac{\ln \pi_0/\pi_1 - \frac{1}{2} b^2/a^2}{\frac{1}{2}(1/b^2 - 1/a^2)}}$$

Decide H_1 if $\sqrt{\gamma} \leqslant x \leqslant \sqrt{\gamma}$

Decide H_0 if $x > \sqrt{\gamma}$ or $x < -\sqrt{\gamma}$



$$P_{FA}(\pi_0, \pi_1) = P(H_1 | H_0)$$

$$= \int_{-\sqrt{\gamma}}^{\sqrt{\gamma}} f_{X|H_0}(x)\, dx = 1 - 2Q\left(\frac{\sqrt{\gamma}}{b}\right)$$

$$P_{miss} = P(H_0 | H_1)$$

$$= \int_{-\infty}^{-\sqrt{\gamma}} f_{X|H_1}(x)\, dx + \int_{\sqrt{\gamma}}^{+\infty} f_{X|H_1}(x)\, dx = 2Q(\sqrt{\gamma}/a)$$

ex.5 $\quad X \sim N(\theta, \sigma^2)$

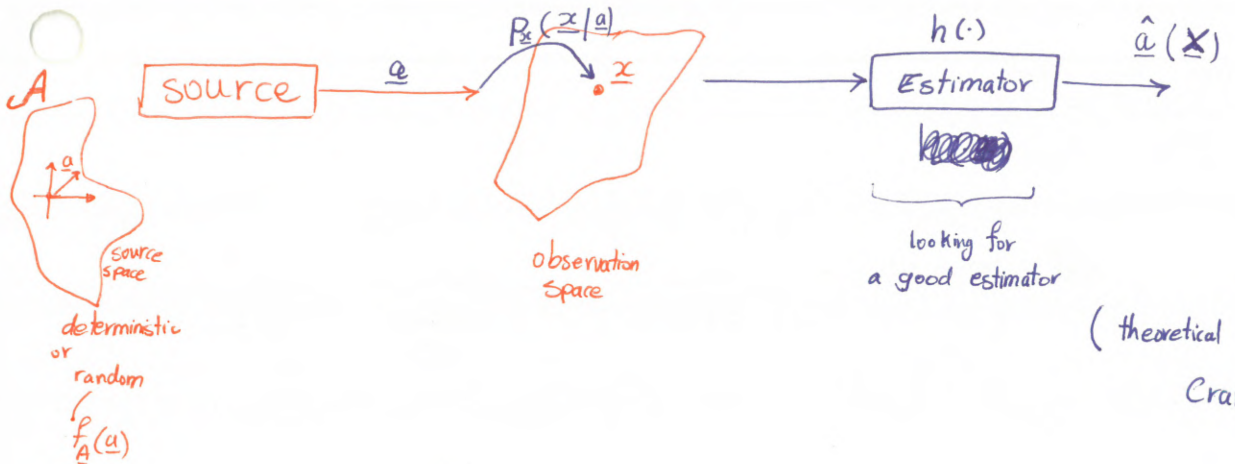
H_1 : $\quad Y = e^X$

H_0 : $\quad Y = X^2$

π_1, π_0 are known.

Run the MAP decision rule.

Introduction to Estimation Theory



(theoretical goodness:
Cramer-Rao Bound)

3 levels of complexity

① Estimation of parameters in the presence of noise

$$x(t) = A\,\overbrace{m(t)\cos(2\pi f_0 t + \varphi)}^{\text{unknown}} + w(t)$$

$0 < t < T$

Gaussian noise (known)

② Estimation of parameters in the presence of noise and unwanted parameters

eg. radar : $x(t) = C\,\underbrace{m(t-\tau)}\cos(2\pi(f \pm f_D)(t-\tau) + \varphi)$

parameters: f_D, τ

unwanted parameters: C, φ

③ Sonar / Astronomy

$$x(t) = S(t : \underbrace{m(t),\ k(t,s)}_{\text{unknown}}) + w(t) + I(t)$$

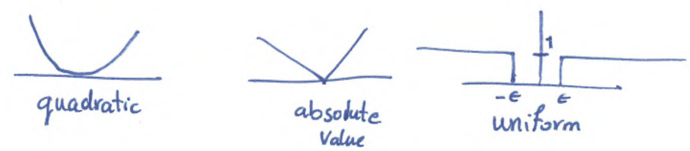
good estimator:
- unbiased
- minimum variance

$\rightarrow$ MVUE

MVUE's are hard to have!

Solution: define COST function.

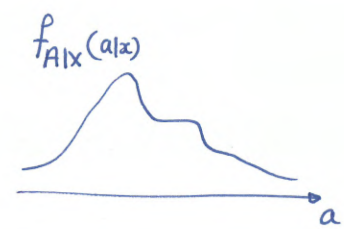
$c \triangleq a - \hat{a}$ differen c's:



MAP (Maximum A Posteriori) Estimate

$\mathcal{A}$ is random space.

A is a Random Variable with known pdf $f_A(a)$



$$f_{A|X}(a|\mathbf{x})$$

$$\hat{a}_{MAP}(x) = \underset{a \in \mathcal{A}}{\arg\max}\, f_{A|X}(a|x)$$

we get help from Bayes:

$$f_{A|X}(a|x) = \frac{f_{X|A}(x|a)\, f_A(a)}{f_X(x)}$$

to find maximum:

$$\left.\frac{\partial f_{A|X}(a|x)}{\partial a}\right|_{a = \hat{a}_{MAP}(x)} = 0$$

(necessary condition)

$$\ln f_{A|X}(a|x) = \ln f_{X|A}(x|a) + \ln f_A(a) - \underbrace{\ln f_X(x)}$$

doesn't matter for derivate w.r.t. a.

$$\frac{\partial}{\partial a}(\cdot) = 0$$

ex.

$$R = A + W \sim N(0, \sigma_W^2)$$

$A \sim N(0, \sigma_A^2)$, indep.

Given $R = r$, find $\hat{a}_{MAP}(r)$.

$$f_{R|A}(r|a) = N(a, \sigma_W^2)$$

$$f_A(a) = N(0, \sigma_A^2)$$

$$\frac{\partial}{\partial a}\left(\ln f_{R|A}(r|a) + \ln f_A(a)\right) = \frac{\partial}{\partial a}\left(-\frac{1}{2\sigma_W^2}(r-a)^2 - \frac{1}{2\sigma_A^2}a^2\right)$$

+ terms not functions of "a"

$$\Rightarrow \quad -\frac{1}{2\sigma_W^2}2(r-a) - \frac{2}{2\sigma_A^2}a = 0$$

$$\frac{r}{\sigma_W^2} = \left(\frac{1}{\sigma_A^2} - \frac{1}{\sigma_W^2}\right)a$$

$$\Rightarrow \boxed{\hat{a}_{MAP} = \frac{1/\sigma_W^2}{1/\sigma_A^2 - 1/\sigma_W^2}\, r}$$

is it biased?

$$E[\hat{a}_{MAP}(R)] = \frac{1/\sigma_W^2}{1/\sigma_A^2 - 1/\sigma_W^2} E[R] = 0 = E[A]$$

$$R \sim N(0, \sigma_A^2 + \sigma_W^2)$$

unbiased

how much variance?

$$Var(\hat{a}_{MAP}) = E\left[\left(\frac{\cancel{1}\sigma_W^2}{1/\sigma_A^2 - 1/\sigma_W^2} R\right)^2\right] - 0$$

$$= \frac{\sigma_A^4}{(\sigma_W^2 - \sigma_A^2)^2} \times \sigma_A^2 + \sigma_W^2$$

There is variance for the estimator. $\Leftarrow$ $Var(\hat{a}_{MAP})$ small $\Leftarrow$ if σ_A^2 small

stoch. lect.

Maximum Likelihood Estimation

A: parameter space is deterministic (No A-priori knowledge about parameters)

- assumption : uniform prior $\left(\frac{1}{A}\right)$ $\longrightarrow$ if unbounded set A, this assumption is wrong.

$$\underbrace{\ln f_{X|A}(x|a)}_{\text{log-likelihood function}} + \ln \underbrace{f_A(a)}_{\text{a number only!}}$$

we can ignore it at the time of differentiation.

$$\frac{\partial}{\partial a}\left\{ \cdot \right\}\Bigg|_{a=\hat{a}_{ML}(x)} = 0$$

or

$$\hat{a}_{ML}(x) = \underset{a \in A}{\arg\max} \ln f_X(x;a)$$

---

ex.

$Y_1, \dots, Y_n$ are iid $\exp(a)$

$$f_{Y_i}(y_i; a) = a e^{-ay_i}, \quad y_i \geq 0$$

Find $\hat{a}_{ML}(y)$.

$$\ln \prod_{i=1}^{n} f_{Y_i}(y_i; a) = \sum_{i=1}^{n} \ln a e^{-ay_i} = n \ln a - a \sum_{i=1}^{n} y_i$$

$$\Rightarrow \frac{\partial}{\partial a}\left(n \ln a - a \sum y_i\right) = 0 \Rightarrow \boxed{\hat{a}_{ML} = \left(\frac{\sum y_i}{n}\right)^{-1}}$$

$\left(\hat{a}_{ML}(y)\right)^{-1} =$ sample mean

$$Z = \sum_{i=1}^{n} y_i \quad , \quad y_i \sim \text{iid } \exp(a)$$

new R.V

$\Rightarrow$ Z will have Erlang distribution.

$$\hat{a}_{ML} = \frac{n}{Z}$$

$$M_Z(j\nu) = E\left[e^{j\nu Z}\right] = E\left[e^{j\nu \sum Y_i}\right] = \left\{E\left[e^{j\nu Y_1}\right]\right\}^n = \left\{\frac{1}{1 - j\frac{\nu}{a}}\right\}^n$$

wikipedia $\Rightarrow$ $f_Z(z) = \dfrac{a e^{-az} (az)^{n-1}}{(n-1)!} \quad , \quad z \geq 0$

$$E_{f_Z}\left[\hat{a}_{ML}(Y_i)\right] = E_f\left[\frac{n}{Z}\right]$$

$$= \int_0 \frac{an\,(za)^{n-2}}{(n-2)!\,(n-1)}\, a e^{-az}\, dz = \frac{an}{n-1} \quad \boxed{\text{biased}}$$

as $n \to +\infty$

$\hat{a}_{ML}(Y)$ is asymptotically unbiased.

Consistency

$$P\left(\left|\frac{n}{\sum Y_i} - \frac{n}{n-1} a\right| > \epsilon\right) \leq \frac{\mathrm{Var}\left(\frac{n}{\sum Y_i}\right)}{\epsilon^2}$$

$\to 0$ as $n \to +\infty$

$$\mathrm{var}\left(\frac{n}{\sum Y_i}\right) = E\left[\left(\frac{n}{Z}\right)^2\right] - \left(\frac{an}{n-1}\right)^2$$

$$E\left[\left(\frac{n}{Z}\right)^2\right] = n^2 \int_0^{+\infty} \frac{(az)^{n-1}}{z^2} \frac{a e^{-az}}{(n-1)!} \, dz$$

$$= \frac{n^2 a^2}{(n-1)(n-2)} \underbrace{\frac{(za)^{n-3} \, a e^{-az}}{(n-3)!}}_{(n-2)\ \text{Erlang}}$$

$$E\left[\frac{n^2}{Z^2}\right] = \frac{n^2 a^2}{(n-1)(n-2)}$$

$$\Rightarrow \mathrm{Var}(.) = \frac{n^2 a^2 (n-1) - a^2 n^2 (n-2)}{(n-1)^2 (n-2)} = \frac{3 a^2 n^2}{(n-1)^2 (n-2)} \xrightarrow[n \to +\infty]{\text{as}} 0$$

$$\Rightarrow P(.) \to 0 \Rightarrow \text{consistent estimate}$$

best bound on Estimate (non-defeatable) $\to$ Cramer-Rao

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ex. 5 Binary Hypothesis Testing $X \sim N(0, \sigma^2)$

$H_1: \quad Y = e^X$

$H_0: \quad Y = X^2$

~~$P(Y = e^X$~~

H_1

$$P(Y \leq y) = P(e^X \leq y) = P(X \leq \ln y)$$

$$= \int_{-\infty}^{\ln y} N(0, \sigma^2)\, dx = \frac{1}{y} f_X(\ln y)$$

$$= \frac{1}{y\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(\ln y - \theta)^2}, \quad y \geq 0$$

H_0

$$P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y})$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\, dx$$

$$f_{Y|H_0}(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y}) = \frac{1}{\sqrt{y}} f_X(\sqrt{y}) = \frac{1}{\sqrt{y}\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(\sqrt{y} - \theta)^2}$$

The MAP Test

$$\frac{f_{Y|H_1}(y)}{f_{Y|H_0}(y)} \underset{H_0}{\overset{H_0}{\gtrless}} \frac{\pi_0}{\pi_1} \longrightarrow -\ln(y) - \frac{1}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}(\ln y)^2$$

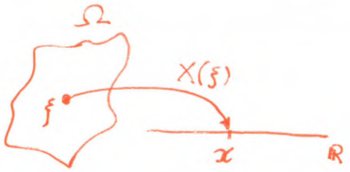
$$+ \frac{1}{2}\ln y + \frac{1}{2\sigma^2} y \underset{H_0}{\overset{H_1}{\gtrless}} \ln\frac{\pi_0}{\pi_1}$$

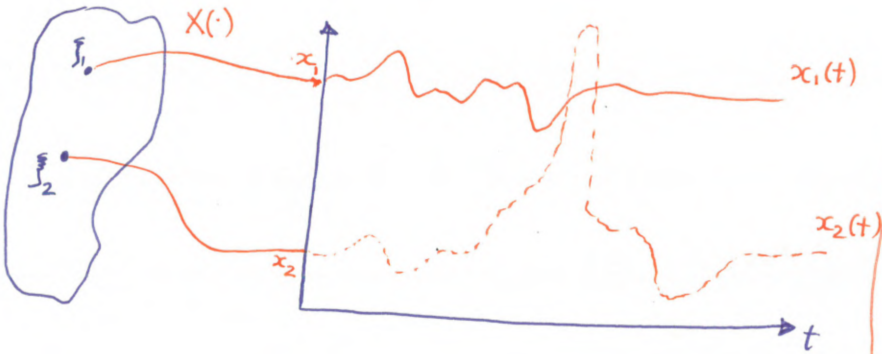
$$\Rightarrow -\frac{1}{2}\ln y - \frac{1}{2\sigma^2}(\ln y)^2 + \frac{y}{2\sigma^2} \underset{H_0}{\overset{H_1}{\gtrless}} \ln\frac{\pi_0}{\pi_1} + \frac{1}{2}\ln(2\pi\sigma^2)$$

stoch. lect.

Recap. $(\Omega, \mathcal{F}, \mathcal{P})$

RVs





time or index space

Terminology

- when $t = t_0$ is fixed
  R.P. $\longrightarrow$ R.V
- when $\xi = \xi_1$ is fixed
  R.P. $\longrightarrow$ sample path
- when t and ξ are varying
  R.P. $\longrightarrow$ ensemble

$$X(\cdot, t) : \underbrace{\Omega}_{\text{sample space}} \times \underbrace{T}_{\text{index space}} \longrightarrow \underbrace{S}_{\text{state space}}$$

ex. index spaces

$T = \{t_1, t_2, \ldots, t_n, \ldots\}$ discrete - parameter Random Process (R.P.)

$T = \{t \in \mathbb{R}_1^+\}$ Continuous - parameter R.P.

$T = \left\{ t = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \in \mathbb{R}^2 \right\}$ Spatial parameter

$T = \left\{ t = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \in \mathbb{R}_+^3 \right\}$

Complete Characterization of a Random Process.

$$\overbrace{P(X(t_1) < x_1, \dots, X(t_n) < x_n)}^{F_{X(t_1), X(t_2), \dots, X(t_n)}(x_1, x_2, \dots, x_n)}$$ for any finite n, and any choice of $t_1, \dots, t_n$ in T, and for any values of $x_1, \dots, x_n$ in S.

3 ways to specify a R.P.

1. By rule of construction
2. By parametrization $\quad s(t) = \underline{\underline{A}} \cos(w_0 t + \underline{\underline{\Theta}})$

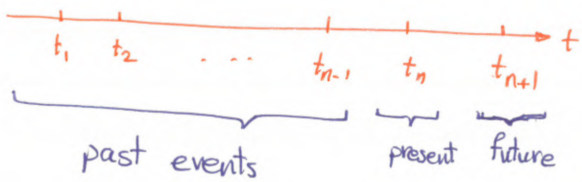
   R.V.'s with $f_{A,\Theta}(a, \theta)$
3. By transformation

   (typically linear) transformation applied to a Gaussian R.P.

ex. (rule of construction) Markov Random Process

- 2 types of index (parameter) space T
- 2 types of state space S

index space		state space: discrete	state space: continuous
	discrete	markov chain	discrete parameter markov process
	continuous	continuous parameter markov chain	continuous parameter markov process



Def'n: $X(t)$ is Markov Process if

$$P(X(t_{n+1}) \leq x_{n+1} \mid \underbrace{X(t_n) \leq x_n, \ldots, X(t_1) \leq x_1}_{})$$

$$= P(X(t_{n+1}) \leq x_{n+1} \mid \underbrace{X(t_n) \leq x_n}_{})$$

Consider joint CDF of $(n+1)$ samples

$$P(X(t_{n+1}) \leq x_{n+1}, X(t_n) \leq x_n, \ldots, X(t_1) \leq x_1)$$

$$= P(X(t_{n+1}) \leq x_n \mid X(t_n) \leq x_n, \ldots, X(t_1) \leq x_1)$$

$$\times \underbrace{\boxed{P(X(t_n) \leq x_n, \ldots, X(t_1) \leq x_1)}}_{}$$

$$P(X(t_n) \leq x_n \mid X(t_{n-1}) \leq x_{n-1}, \ldots, X(t_1) \leq x_1)$$

$$\times \; P(X(t_{n-1}) \leq x_{n-1}, \ldots, X(t_1) \leq x_1)$$

$$= P(X(t_1) \leq x_1) \prod_{i=2}^{n} P(X(t_{i+1}) \leq x_{i+1} \mid X(t_i) \leq x_i, \ldots, X(t_1) \leq x_1)$$

$$= P(X(t_1) \leq x_1) \prod_{i=2}^{n} P(X(t_{i+1}) \leq x_{i+1} \mid X(t_i) \leq x_i)$$

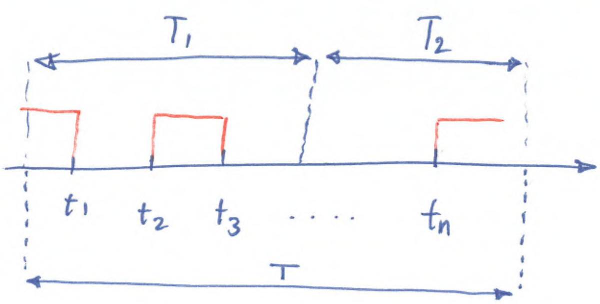
To completely specify a Markov Process.

1) Type of a Markov Process

2) Conditional Probability of sample at t_{i+1}, given sample at t_i

3) $P(X(t_i) \leq x_1)$ (marginal)

ex. 2 (rule of construction) Random Telegraph Wave (RTW)



* average intensity λ

* 2-state R.P. $S=\{0, 1\}$

Rules of Construction

1. $P(X(t)=0) = P(X(t)=1) = \frac{1}{2}$

2. At every t_i R.P. $X(t)$ changes state $0 \rightarrow 1$ or $1 \rightarrow 0$

3. # of events over T

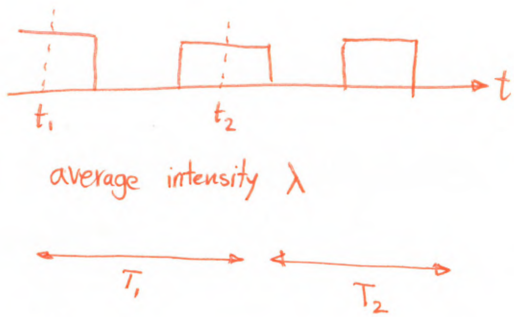
$P(\text{\# events over } T \text{ is } k) = \frac{(\lambda T)^k}{k!} e^{-\lambda T}$ Poisson

4. # of events over disjoint T_1 and T_2

are independents R.V.'s.

stoch. lect.

ex. 2 (RTW = rule based)



2 disjoint intervals

$X(t)$ is a binary state.

1. $P(X(t)=1) = P(X(t)=0) = \frac{1}{2}$

2. $X(t)$ changes state $0 \to 1$ or $1 \to 0$ at every event

3. # events over T is Poisson.

$$P(k \text{ events over } T) = \frac{(\lambda T)^k}{k!} e^{-\lambda T}$$

4. # events over disjoint intervals are independent random variables.

$$P(X(t_2)=1 \mid X(t_1)=1)$$

$$= P(\#\text{events over } T \text{ is even})$$

$$= \sum_{k \text{ is even}} \frac{(\lambda T)^k}{k!} e^{-\lambda T}$$

solve by a trick

$$= \sum_{0}^{+\infty} \left[\frac{1+(-1)^k}{2}\right] \frac{(\lambda T)^k}{k!} e^{-\lambda T}$$

$$\frac{(-\lambda T)^k}{k!}$$

$$= e^{-\lambda T}\left(\frac{1}{2} e^{\lambda T} + \frac{1}{2} e^{-\lambda T}\right)$$

$$= \frac{1}{2}\left(1 + e^{-2\lambda T}\right)$$

$$P(X(t_2)=1 \mid X(t_1)=0) = \frac{1}{2}\left(1-e^{-2\lambda T}\right)$$

$$P(X(t_1)=1,\ X(t_2)=1,\ X(t_3)=1)$$

t_1 t_2 t_3 ↘ disjoint!

[redacted]

no dependency.

$$= P(X(t_3)=1 \mid \cancel{X(t_1)=1},\ X(t_2)=1)\cdot \underbrace{P(X_2(t)=1 \mid X(t_1)=1)}_{\frac{1}{2}\left(1+e^{-2\lambda(t_2-t_1)}\right)} \cdot \underbrace{P(X(t_1)=1)}_{\frac{1}{2}}$$

$$= \left(\frac{1}{2}\right)^3 \left(1+e^{-2\lambda(t_2-t_1)}\right)\left(1+e^{-2\lambda(t_3-t_2)}\right)$$

Defn. A random process $X(\xi;t)$ is $\boxed{\text{strict stationary}}$ of order 1, if

$$F_{X(t+\tau)}(x) = F_{X(t)}(x)$$

$X(\xi;t)$ is strict stationary of order 2, if

$$F_{X(t_1+\tau),\,X(t_2+\tau)}(x_1,x_2) = F_{X(t_1),\,X(t_2)}(x_1,x_2)$$

$\boxed{\star}$ A Random Process is strict sense stationary (SSS) if it is SSS for all orders.

---

ex. RTW is a SSS R.P.

$$P(X(t_1)=1, \ldots, X(t_n)=1)$$

$$= P(X(t_1)=1) \prod_{i=2}^{n} P(X(t_i)=1 \mid X(t_{i-1})=1$$

$$= \left(\frac{1}{2}\right)^n \prod_{i=2}^{n} \left(1 + e^{-2\lambda(t_i \overset{+\tau}{} - t_{i-1} \overset{-\tau}{})}\right) \rightarrow$$ so it is SSS for all order.

Partial Characterization of a Random Proces

$$X(\xi, t)$$

1. $m_X(t) = E[X(\xi, t)]$

mean function

ex: (parametrization)

$$X(t) = \underbrace{A \cos(2\pi f_0 t + \theta)}_{f_{A,\Theta}(a,\theta)}$$

$$m_X(t) = E[A \cos(2\pi f_0 t + \theta)]$$

$$= \int_0^{2\pi}\int_{-\infty}^{+\infty} a \cos(2\pi f_0 t + \theta)\, f_{A,\Theta}(a,\theta)\, dA\, d\theta$$

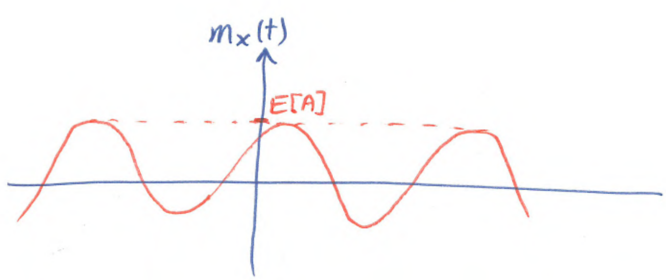
$$\xrightarrow{A \perp\!\!\!\perp \Theta} = \underbrace{\int a \, f_A(a)\,da}_{E[A]} \underbrace{\int_0^{2\pi} \cos(2\pi f_0 t + \theta)\, f_\Theta(\theta)\, d\theta}$$

$$\xrightarrow{\Theta \sim \text{Uniform}(0,2\pi)} = E[A] \underbrace{\int_0^{2\pi} \frac{\cos(2\pi f_0 t + \theta)}{2\pi}\, d\theta}_{0} = 0$$

---

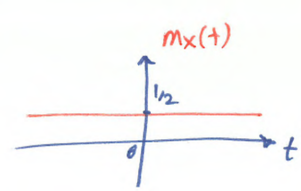
if $A \sim f_A(a)$ and Θ is deterministic

$$m_X(t) = E[A] \cdot \cos(2\pi f_0 t + \theta)$$



ex.2 (RTW)

$$E[X(t)] = 1 \cdot P(X(t)=1) + 0 \cdot P(X(t)=0)$$
$$= \frac{1}{2}$$



2. Autocorrelation Function

$$R_{XX}(t, u) = E[X(t) \cdot X(u)]$$

real-valued

- if $X(t)$ is complex-valued $\longrightarrow$ $R_{XX}(t,u) = E[X(t)\, X^*(u)]$

stoch. lect.

$X(\xi, t)$

$\Omega \times T \longrightarrow S$

to characterize a R.P.:

1. The mean function $m_X(t) = E_{f(X(t))}[X(t)]$

2. The correlation function

auto cross

If $X(t)$ is a real-valued R.P. $R_{XX}(t,u) = E[X(t)X(u)]$

complex-valued $R_{XX}(t,u) = E[X(t)X^*(u)]$

3. $X(t)$ and $Y(t)$ are two R.P.'s $R_{XY}(t,u) = E[X(t) \cdot Y^*(u)]$

ex.

$$X(t) = A\cos(2\pi f_0 t + \Theta)$$

assume $A \perp\!\!\!\perp \Theta$, $\Theta \sim \mathrm{Unif}(0, 2\pi)$

$$R_{XX}(t,u) = E[X(t)X(u)] = E[A^2\cos(2\pi f_0 t + \Theta)\cos(2\pi f_0 u + \Theta)]$$

$$\overset{A \perp\!\!\!\perp \Theta}{=} E[A^2]\, E\left[\frac{1}{2}\cos(2\pi f_0(t+u) + 2\Theta) + \frac{1}{2}\cos(2\pi f_0(t-u))\right]$$

$$= \frac{1}{2}E[A^2]\cos(2\pi f_0\underbrace{(t-u)}_{\tau})$$

$R_{XX}(t,u) = R_{XX}(\underbrace{t-u}_{\tau}) = R_{XX}(\tau) \longrightarrow$
- is a function of τ only. ①
- is a continuous function. ②
- is even. ③
- $R_{XX}(0) \geq |R_{XX}(t)|$ ④

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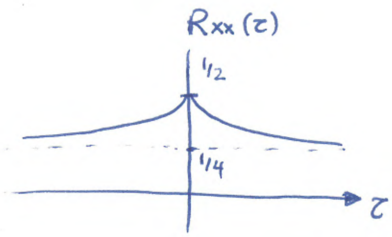
ex. (RTW) → Random Telegraph Wave

$$R_{XX}(t,u) = E[X(t)X(u)] = 1.1.\ P(X(t)=1,\ X(u)=1)$$
$$+ 1.0.\ P(1,\ 0)$$
$$+ 0\ 1\ \ P(0,\ 1)$$
$$+ 0.0.\ P(0,\ 0)$$

$t > u$

$u > t \rightarrow \frac{1}{4}\left(1 + e^{-2\lambda \overbrace{|u-t|}^{\tau}}\right)$



1. function of difference of time $R_{XX}(t-u) = R_{XX}(\tau)$

2. $R_{XX}(\tau)$ is continuous

3. $R_{XX}(\tau)$ is an even function

4. $R_{XX}(\tau) \leq R_{XX}(0)$

3rd way of characterization

like Covariance Function

$$K_{XX}(t,u) = E[(X(t) - m_X(t))(X(u) - m_X(u))]$$

real-valued

$\boxed{\text{WSS}}$ $X(t)$ is a wide sense stationary (WSS) if

① $E[X(t)] = m_X(t) = \text{const} < \infty$

② $R_{XX}(t, u) = R_{XX}(t-u, 0) = R_{XX}(\tau, 0)$

For a WSS process:

Power Spectral Density (PSD)

$$S_X(f) = \int_{-\infty}^{+\infty} R_{XX}(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

$$R_{XX}(\tau) = \int_{-\infty}^{+\infty} S_X(f)\, e^{j2\pi f\tau}\, df$$

ex.

$X(t) = \cos(2\pi f_0 t + \theta)$

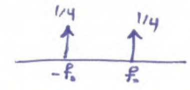
$\theta \sim \text{Unif}(0, 2\pi)$

$$\left.\begin{aligned} &E[X(t)] = m_X(t) = 0 = \text{const} \\ &R_{XX}(t,u) = \frac{1}{2}\cos(2\pi f_0 \underbrace{(t-u)}_{\tau}) \end{aligned}\right\} \Rightarrow X(t) \text{ is WSS.}$$

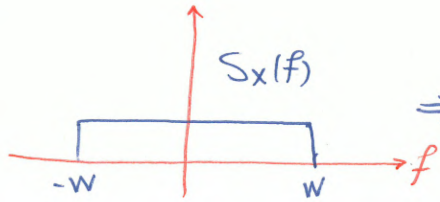
Thus PSD is given as

$$S_X(f) = \frac{1}{2}\int_{-\infty}^{+\infty} \left[\frac{1}{2} e^{j2\pi f_0\tau} + \frac{1}{2} e^{-j2\pi f_0\tau}\right] e^{-j2\pi f\tau}\, d\tau$$

$$= \frac{1}{4}\delta(f-f_0) + \frac{1}{4}\delta(f+f_0)$$



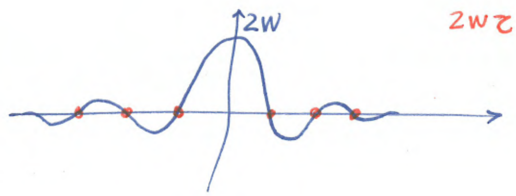
ex.



$\Rightarrow$ $x(t)$ is WSS.

$$R_{xx}(\tau) = \int S_x(f) \ldots = \int_{-W}^{W} 1 \; e^{+j2\pi f\tau}\, df = \frac{e^{j2\pi f W} - e^{-j2\pi f W}}{j2\pi\tau}$$

$$= \frac{1}{\pi\tau}\sin(2\pi f W) \times \frac{2W}{2W} = \boxed{2W \,\mathrm{sinc}\,(2W\tau)}$$



$2W\tau = \pi k$

Properties of PSD

① $S_x(f) \geq 0$

② real-valued

③ Even function of f.

$\boxed{\text{stoch. lect.}}$

$X(t)$: WSS

- $E[X(t)] = m_X(t) = \text{const}$
- $E[X(t)X(\cancel{u})] = R_{XX}(\tau)$ $\quad t+\tau$

PSD

$$\underbrace{S_X(f)} = \int_{-\infty}^{+\infty} \underbrace{R_{XX}(\tau)} e^{j2\pi f\tau} d\tau \qquad [\text{Watts}/\text{Hertz}]$$

- ≥ 0
- even
- real-valued

- even
- real-valued
- continuous
- $|R_{XX}(\tau)| \leq R_{XX}(0)$

(f_1, f_2)

$$\int_{f_1}^{f_2} S_X(f)\, df$$

$$P = \int_{-\infty}^{+\infty} S_X(f)\, df$$

average power

$E[|X(t)|^2]$ = average power

If $X(t)$ is WSS :

$$E[\underbrace{|X(t)|^2}_{X(t)X^*(t)}] = R_{XX}(0)$$

$$R_{XX}(\tau)\Big|_{\tau=0} = \int_{-\infty}^{+\infty} S_{XX}(f)\, e^{j2\pi f\tau} df \Big|_{\tau=0}$$

Parseval's Theorem

$$P = E[\,|X(t)|^2\,] = \int_{-\infty}^{+\infty} S_X(f)\, df$$

Linear Filters

- for LTI systems

$$R(t) = X(t) + W(t)$$

received signal (additive noisy)

$$R(t) \longrightarrow \boxed{\text{Linear Filter}} \longrightarrow \hat{X}(t) = Y(t)$$

$$\underset{\text{WSS}}{X(t)} \longrightarrow \overset{h(t)}{\boxed{\text{LTI}}} \longrightarrow Y(t) = X(t) * h(t) = \int_{-\infty}^{+\infty} h(\lambda)\, x(t-\lambda)\, d\lambda$$

- an LTI system completely charachterized by impulse response ($h(t)$)

$\boxed{\text{aside}}\ \boxed{\text{I}}$

$X_1(t) \to Y_1(t)$

$X_2(t) \to Y_2(t)$

$aX_1(t) + bX_2(t) \to aY_1(t) + bY_2(t)$

(linear system)

$\boxed{\text{aside}}\ \boxed{\text{II}}$

$x(t) \to y(t)$

$x(t-\tau) \to y(t-\tau)$

(time-invariant system)

$$H(f) = \int_{-\infty}^{+\infty} h(t)\, e^{-j2\pi f t}\, dt$$

$X(t)$: m_X, $R_X(\tau)$, W_{SS}

Stability analysis

- BIBO criterion (Bounded Input Bounded Output)

$$E[\,|X(t)|^2] < \infty$$

$$\left|E[\,|Y(t)|^2]\right| = \left|E\Big[\underbrace{\int_{-\infty}^{+\infty} h(\alpha)\, x(t-\alpha)\, d\alpha \int_{-\infty}^{+\infty} h^*(\beta)\, x^*(t-\beta)\, d\beta}_{\iint_{-\infty}^{+\infty} h(\alpha)\, h^*(\beta)\; x(t-\alpha)\, x^*(t-\beta)\, d\alpha\, d\beta}\Big]\right|$$

$$= \left|\iint h(\alpha)\, h^*(\beta)\; \underbrace{E\big[X(t-\alpha)\, X^*(t-\beta)\big]}_{R_{XX}(\alpha-\beta)}\, d\alpha\, d\beta\right|$$

$[X(t)$ is WSS$]$

$$= \left|\iint h(\alpha)\, h^*(\beta)\; R_{XX}(\alpha-\beta)\, d\alpha\, d\beta\right|$$

$$\leq \iint |h(\alpha)|\,|h^*(\beta)|\;|R_{XX}(\alpha-\beta)|\, d\alpha\, d\beta$$

$$\leq R_{XX}(0) \iint |h(\alpha)|\,|h^*(\beta)|\, d\alpha\, d\beta = R_{XX}(0) \left\{\int_{-\infty}^{+\infty} |h(\alpha)|\, d\alpha\right\}^2$$

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$$\int_{-\infty}^{+\infty} |h(\alpha)|\, d\alpha < \infty$$

$$\boxed{H(0) < \infty}$$ BIBO

Show $Y(t)$ is WSS.

1. $E[Y(t)] = \text{const}$

2. $R_{YY}(\tau)$

① $\Rightarrow$ $E[Y(t)] = E\left[\int_{-\infty}^{+\infty} h(\alpha)\, X(t-\alpha)\, d\alpha\right] = \int_{-\infty}^{+\infty} h(\alpha) \underbrace{E[X(t-\alpha)]}_{\text{const}} d\alpha =$

$$= m_X \underbrace{\int_{-\infty}^{+\infty} h(\alpha)\, d\alpha \; e^{-j2\pi f\alpha}\Big|_{\alpha=0}}_{H(0)}$$

$$\Rightarrow \boxed{m_Y = m_X \cdot H(0)}$$

② $\Rightarrow$ $R_{YY}(t, t+\tau) = E[Y(t)\, Y^*(t+\tau)] = E\left[\int_{-\infty}^{+\infty} h(\alpha) x(t-\alpha) d\alpha \int_{-\infty}^{+\infty} h^*(\beta) x(t+\tau-\beta) d\beta\right]$

$$= \iint_{-\infty}^{+\infty} h(\alpha) \underbrace{E[X(t-\alpha) X^*(t+\tau-\beta)]}_{R_{XX}(\tau-\beta+\alpha)} h^*(\beta)\, d\alpha d\beta = R_{YY}(\tau)$$

α, β are marginalized out.

$$S_Y(f) = \int_{-\infty}^{+\infty} R_{YY}(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

$$= \underset{``\tau"}{\int_{-\infty}^{+\infty}} \underset{``\beta"\ ``\alpha"}{\iint_{-\infty}^{+\infty}} h(\alpha) R_{XX}(\underbrace{t-\beta+\alpha}_{\lambda}) h(\beta)\, e^{-j2\pi f\tau}\, d\alpha\, d\beta\, \underset{d\lambda}{d\tau}$$

$\lambda+\beta-\alpha$

$$= \underset{\lambda\ \beta\ \alpha}{\iiint} h(\alpha) R_{XX}(\lambda) h^*(\beta)\, e^{-j2\pi f\lambda}\, e^{+j2\pi f\alpha}\, e^{-j2\pi f\beta}\, d\alpha\, d\beta\, d\lambda$$

$$= H(f)\, H^*(f)\, S_X(f)$$

$$\boxed{S_Y(f) = S_X(f) \cdot |H(f)|^2}$$

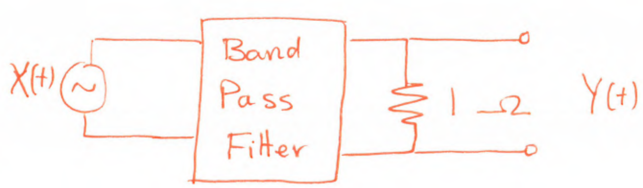
ex.

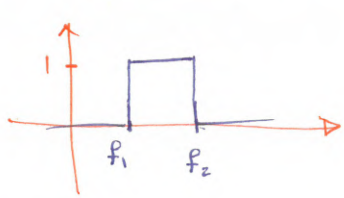
$X(t)$

WSS

$E[X(t)] = 0$

$S_X(f)$





$$P_Y = \int_{-\infty}^{+\infty} S_Y(f)\, df$$

$S_Y(f) = ?$

$S_Y(f) = S_X(f) \cdot |H(f)|^2$

$$P_Y = \int_{f_1}^{f_2} S_X(f)\, df$$

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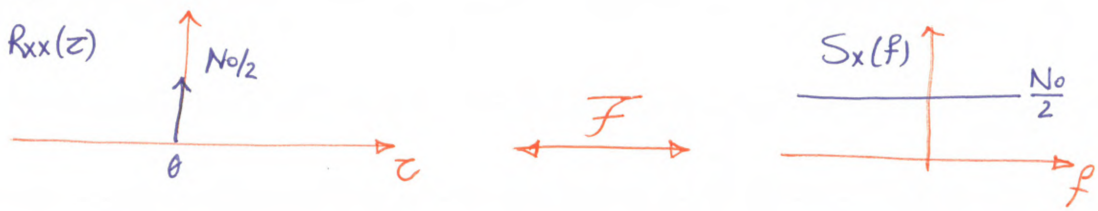
Def'n: $X(t)$ is a white noise R.P. if: (is also WSS)

- $E[X(t)] = 0$
- $E[X(t)X(u)] = R_{XX}(\underbrace{t-u}_{\tau}, 0) = R_{XX}(\tau) = \frac{N_0}{2}\delta(t)$



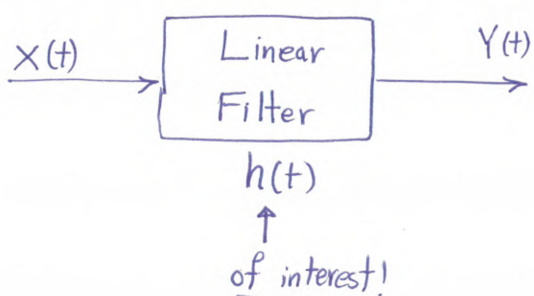
The Wiener Filter

random observed signal: $X(t)$

$$X(t) = Z(t) + W(t)$$

- known mean values ($E[\cdot] = 0$)
- $S_Z^{(t)}, S_{ZW}^{(t)}, S_W^{(t)}$

all are WSS.



$Y(t) = \hat{Z}(t) = h(t) * X(t)$

→ we need a criterion to measure how good the estimate $\hat{z}(t)$ is:

$$MMSE(t) = E\left[\left| z(t) - \hat{z}(t) \right|^2 \right]$$

Find $\min\limits_{h(t)\,:\, h(t) \text{ is LTI}} MMSE(t)$.

Calculus of Variations

$$h(t) = h_o(t) + \epsilon \, h_\epsilon(t)$$

variation

$$\min MMSE(\epsilon) \rightsquigarrow \frac{\partial}{\partial \epsilon} MMSE \Big|_{\epsilon = 0} = 0$$

Assume RP's are real-valued.

$$MMSE = E\left[z^2(t) - 2\, z(t)\, \hat{z}(t) + \hat{z}(t) \right]$$

$$, \quad \hat{z}(t) = \int_{-\infty}^{+\infty} h(\alpha)\, X(t-\alpha)\, d\alpha$$

$h_o(\alpha) + \epsilon h_\epsilon(\alpha)$

with β

$$\longrightarrow \; = E\left[z^2(t) - 2 z(t) \int_{-\infty}^{+\infty} h(\alpha) X(t-\alpha) d\alpha + \iint_{-\infty}^{+\infty} h(\alpha) h(\beta) X(t-\alpha) X(t-\beta)\, d\beta\, d\alpha \right]$$

$$\frac{\partial}{\partial \epsilon} MMSE = E\Big[-\cancel{2}\, Z(t) \int_{-\infty}^{+\infty} h_\epsilon(\alpha)\, X(t-\alpha)\, d\alpha$$

$$+ \cancel{2} \iint_{-\infty}^{+\infty} h_\epsilon(\alpha) \left[h_o(\beta) + \epsilon h_\epsilon(\beta) \right] X(t-\alpha) X(t-\beta)\, d\alpha\, d\beta \Big]$$

all are WSS.

$$= -\int_{-\infty}^{+\infty} \Big(h_\epsilon(\alpha) \underbrace{E[Z(t)\, X(t-\alpha)]}_{R_{XZ}(\alpha)} - h_\epsilon(\alpha) \int_{-\infty}^{+\infty} h_o(\beta)\, R_{XX}(\alpha-\beta)\, d\beta \Big)\, d\alpha$$

$$= \int_{-\infty}^{+\infty} h_\epsilon(\alpha) \left\{ R_{XZ}(\alpha) - \int_{-\infty}^{+\infty} h_o(\beta)\, R_{XX}(\alpha-\beta)\, d\beta \right\} d\alpha$$

$$\boxed{R_{XZ}(\alpha) = \int_{-\infty}^{+\infty} h_o(\beta)\, R_{XX}(\alpha-\beta)\, d\beta}$$

Wiener - Hopf Equation

$$S_{XZ}(f) = H_o(f) \cdot S_X(f)$$

$$\boxed{H_o(f) = \frac{S_{XZ}(f)}{S_X(f)}}$$

Non - Causal!

If $Z(t)$ and $W(t)$ are uncorrelated:

$$R_{XX}(\tau) = R_{ZZ}(\tau) + R_{WW}(\tau)$$

$$\Rightarrow \quad S_X(f) = S_Z(f) + S_W(f)$$

$$\Longrightarrow \quad H_o(f) = \frac{S_Z(f)}{S_Z(f) + S_W(f)}$$

If $W(t)$ is white: $\quad H_o(f) = \dfrac{S_Z(f)}{S_Z(f) + \frac{N_o}{2}}$ (still Non-Causal!)