Homework 5

EE 513 — Stochastic Systems Theory

Name: Ali Zafari **Student Number:** 800350381 **Fall 2022**

Problem 5.1

((a)) First, since the variance of X_i is one, then $\frac{(2a)^2}{12} = 1$ and therefore $a = \sqrt{3}$.

To find a general form for the distribution of S_n we define a random variable $Y_i = \frac{X_i}{\sqrt{n}}$. And therefore $S_n = \sum Y_i$.

As all X_i's are iid, so are Y_i's. Therefore the distribution of S_n is the convolution of distribution of Y_i's.

$$f_Y(y) = \begin{cases} \frac{\sqrt{n}}{2a} & \frac{-a}{\sqrt{n}} < y < \frac{a}{\sqrt{n}} \\ 0 & o.w. \end{cases}$$

- S_2

$$f_{S_2}(s) = f_Y(y) * f_Y(y) = \begin{cases} \frac{1}{6}(s + \sqrt{6}) & -\sqrt{6} < s < 0 \\ \frac{1}{6}(-s + \sqrt{6}) & 0 < s < \sqrt{6} \\ 0 & o.w. \end{cases}$$

- S_3

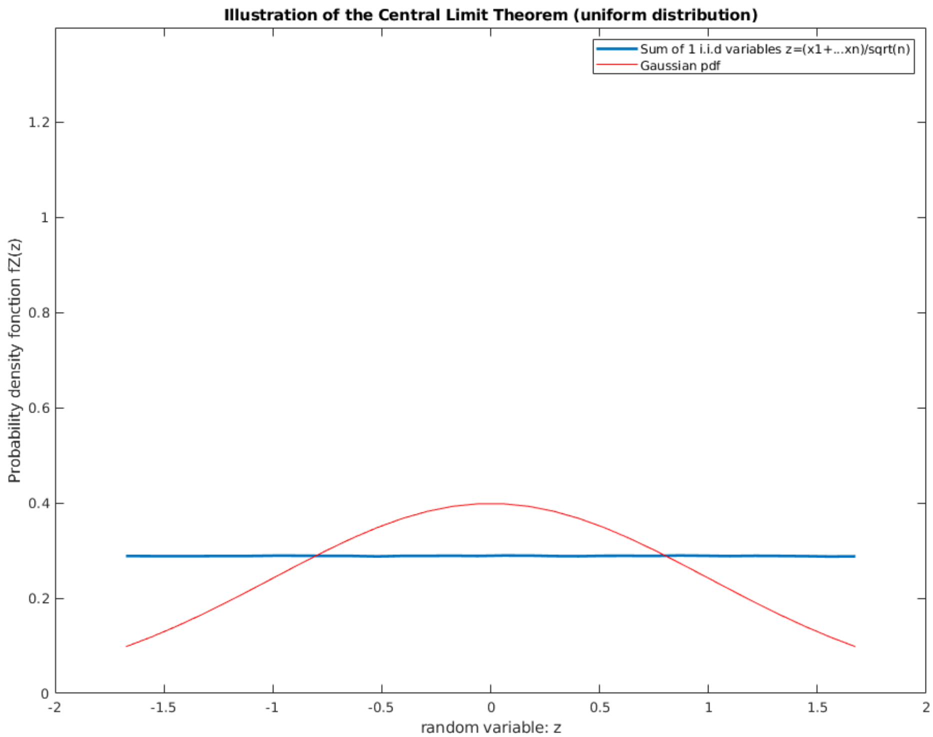
$$f_{S_3}(s) = f_Y(y) * f_Y(y) * f_Y(y) = \begin{cases} \frac{(s+3)^2}{16} & -3 < s < -1 \\ \frac{-s^2+3}{8} & -1 < s < 1 \\ \frac{(-s+3)^2}{16} & 1 < s < 3 \\ 0 & o.w. \end{cases}$$

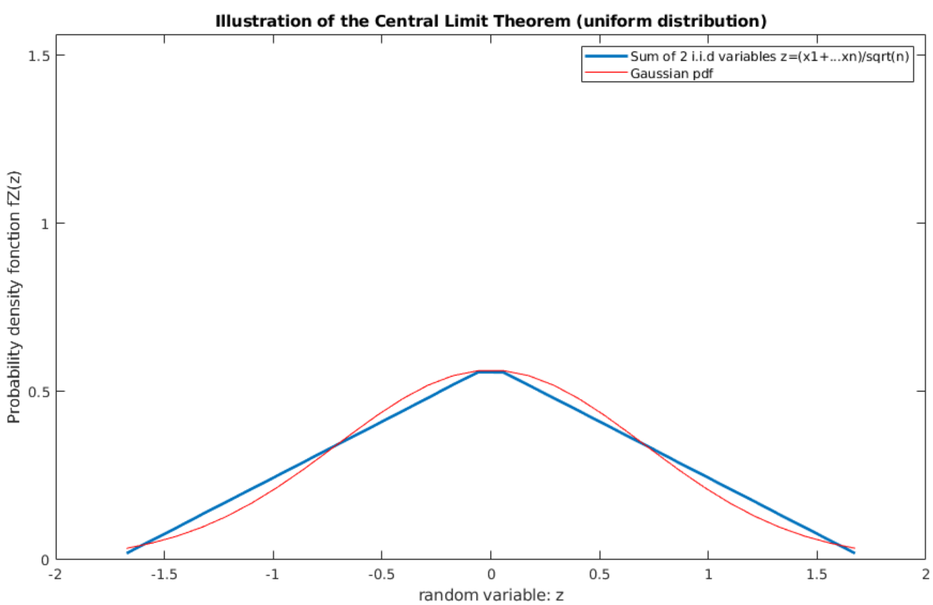
((b)) By applying CLT, we have $\frac{S_n - \mathbb{E}[S_n]}{\sqrt{Var(S_n)}} \sim \mathcal{N}(0, 1)$.

$$\mathbb{E}[S_n] = 0$$
$$Var(S_n) = \frac{1}{n} \times \sum Var(X_i) = 1$$

Therefore: $S_n \sim \mathcal{N}(0, 1)$ as $n \to +\infty$.

((c)) Following images showing pdf of sample mean compared with a Gaussian at each step.





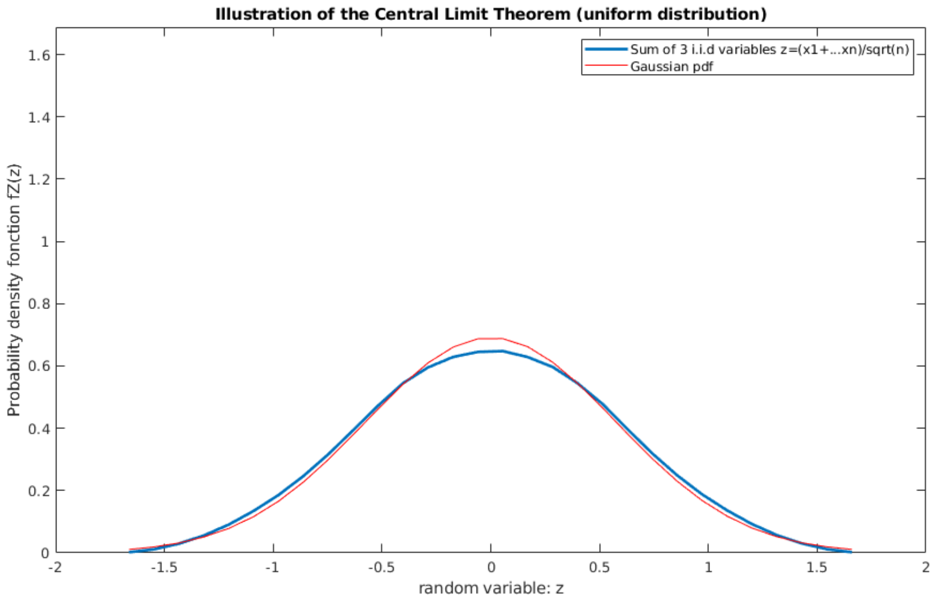


Illustration of the Central Limit Theorem (uniform distribution)
Sum of 3 i.i.d variables z=(x1+...xn)/sqrt(n)
Gaussian pdf
Probability density fonction fZ(z)
random variable: z

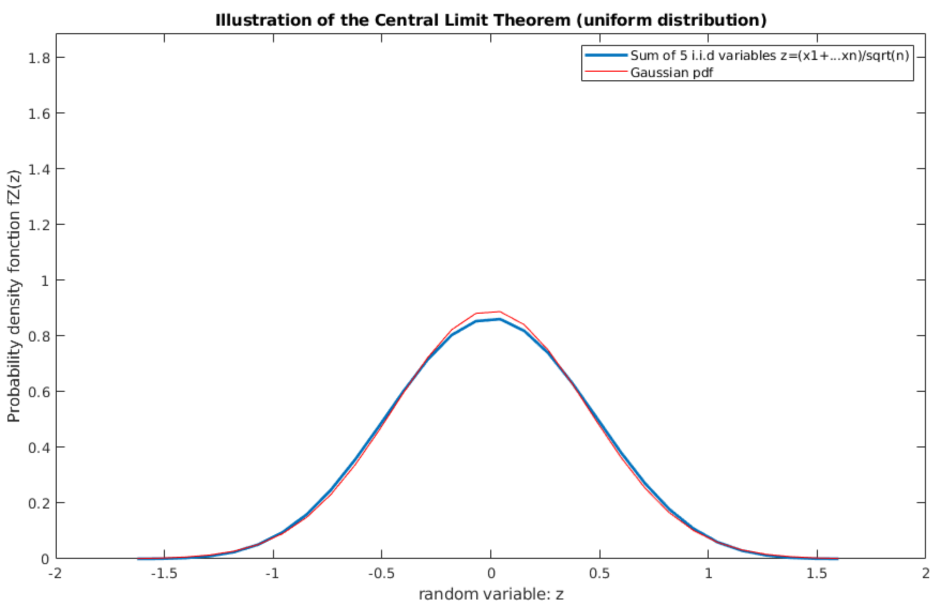
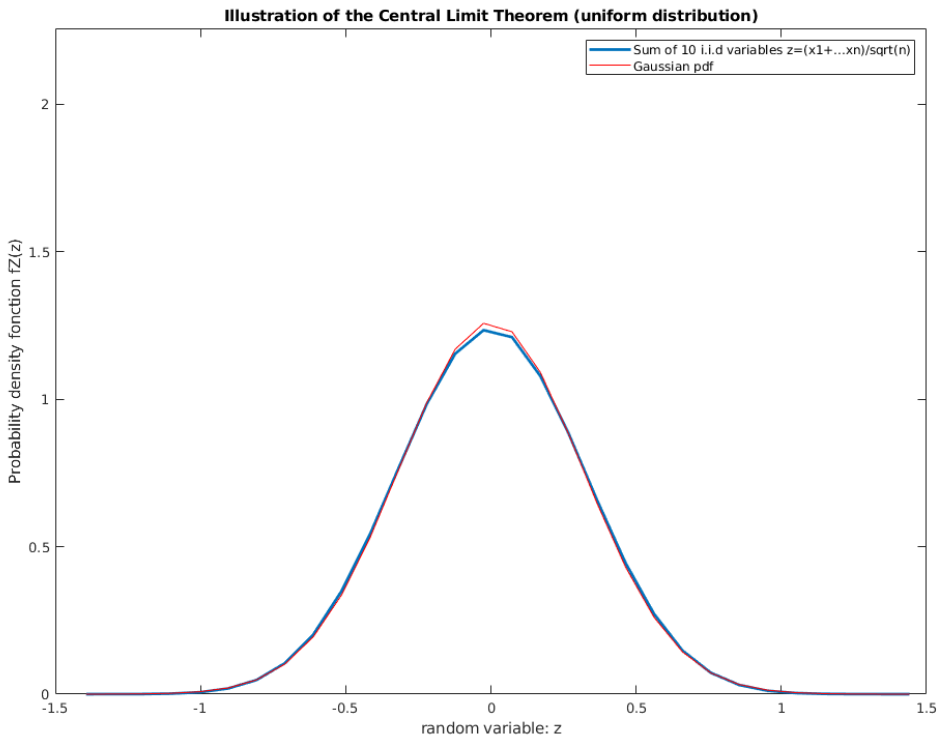


Illustration of the Central Limit Theorem (uniform distribution)
Sum of 5 i.i.d variables z=(x1+...xn)/sqrt(n)
Gaussian pdf
Probability density fonction fZ(z)
random variable: z



Problem 5.2

((a)) For $Y = -\log(p(X))$:

$$\begin{aligned}
\mathbb{E}_{y\sim P_Y(y)}[Y] &= \mathbb{E}_{x\sim P_X(x)}[-\log(p(x))] \\
&= -\sum_{x\in\mathcal{X}} p(x)\log(p(x)) \\
&= \frac{1}{2}\times 1 + \frac{1}{4}\times 2 + \frac{1}{8}\times 3 + \frac{1}{16}\times 4 + \frac{1}{16}\times 4 \\
&= 1.875
\end{aligned}$$

to find variance, we need $E_{y\sim P_Y(y)}[Y^2] = 4.625 - 1.875^2 = 1.109375$:

$$\begin{aligned}
\mathbb{E}_{y\sim P_Y(y)}[Y^2] &= \mathbb{E}_{x\sim P_X(x)}[\log^2(p(x))] \\
&= \sum_{x\in\mathcal{X}} p(x)\log^2(p(x)) \\
&= \frac{1}{2}\times 1 + \frac{1}{4}\times 4 + \frac{1}{8}\times 9 + \frac{1}{16}\times 16 + \frac{1}{16}\times 16 \\
&= 4.625
\end{aligned}$$

hence: $Var(Y) = \mathbb{E}[Y^2] - \mathbb{E}^2[Y] = 4.625 - 1.875^2 = 1.109375$ ✓

((b)) First we calculate the mean and variance of the arithmetic average of self-information (X_i's are iid and Y_i's are iid as well):

$$\mathbb{E}[S_n] = n \times \frac{1}{n}\mathbb{E}[Y_1] = 1.875$$

$$Var(S_n) = n \times \frac{1}{n^2}Var(Y_1) = \frac{1.109375}{n}$$

Weak Law of Large Numbers (WLLN): as $n \to +\infty$ the difference between average self-information S_n and its mean will converge **in probability** to zero.
It can be proved by Chebyshev's inequality:

$$P(|S_n - \mathbb{E}[S_n]| > \epsilon) \leq \frac{Var(S_n)}{\epsilon^2} = \frac{1.109375}{n\epsilon^2}$$ ✓

hence, if $n \to +\infty$ then $P(|S_n - 1.875| > \epsilon) \to 0$.

Problem 5.3

((a))

$$\mathbb{E}[M_n] = n \times \frac{1}{n}\mathbb{E}[X_i] = m$$

$$Var(M_n) = n \times \frac{1}{n^2}Var(X_i) = \frac{\sigma^2}{n}$$

Since the expected valued of sample mean is equal to the mean, it is an **unbiased** estimator of mean of the random variable X.

To check if the estimator of mean is consistent, we should see if it converges in probability to the true mean of the random variable X or not. We exploit the *Chebyshev*'s inequality to check this criterion.

$$P(|M_n - m| > \epsilon) \leq \frac{Var(M_n)}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2}$$

as we can see, when $n \to +\infty$ then $P(|M_n - m| > \epsilon) \to 0$. So the estimator M_n is **consistent**.

((b))

$$
\begin{aligned}
\mathbb{E}[V_n] &= \frac{1}{n-1}\sum_i \mathbb{E}[(X_i - M_n)^2] \\
&= \frac{1}{n-1}\sum_i \mathbb{E}[X_i^2 - 2X_iM_n + M_n^2] \\
&= \frac{1}{n-1}\sum_i \mathbb{E}[X_i^2] - 2\mathbb{E}[X_iM_n] + \mathbb{E}[M_n^2] \\
&= \frac{1}{n-1}\sum_i \mathbb{E}[X_i^2] - 2\mathbb{E}[X_i\frac{X_1 + \cdots + X_i + \cdots + X_n}{n}] + \mathbb{E}[M_n^2] \\
&= \frac{1}{n-1}\sum_i \mathbb{E}[X_i^2] - \frac{2}{n}(\mathbb{E}[X_i^2] + (n-1)\mathbb{E}[X_iX_{j_{(j\neq i)}}]) + \mathbb{E}[M_n^2] \\
&= \frac{1}{n-1}\sum_i \mathbb{E}[X_i^2] - \frac{2}{n}(\mathbb{E}[X_i^2] + (n-1)\mathbb{E}[X_i]\mathbb{E}[X_{j_{(j\neq i)}}]) + \mathbb{E}[M_n^2] \\
&= \frac{1}{n-1}\sum_i \sigma^2 + m^2 - \frac{2}{n}(\sigma^2 + m^2 + (n-1)m^2) + \frac{\sigma^2}{n} + m^2 \\
&= \frac{1}{n-1}\sum_i \frac{n\sigma^2 + nm^2 - 2\sigma^2 - 2m^2 + -2nm^2 + 2m^2 + \sigma^2 + nm^2}{n} \\
&= \frac{1}{n-1}\sum_i \frac{(n-1)\sigma^2}{n} \\
&= \frac{1}{n-1}n\frac{(n-1)\sigma^2}{n} \\
&= \sigma^2
\end{aligned}
$$

((c))

$$
\begin{aligned}
\mathbb{E}[V_n^{biased}] &= \frac{1}{n}\sum_i \mathbb{E}[(X_i - M_n)^2] \\
&= \ldots \text{Following the same steps above} \ldots \\
&= \frac{1}{n}\sum_i \frac{(n-1)\sigma^2}{n} \\
&= \frac{1}{n}n\frac{(n-1)\sigma^2}{n} \\
&= \frac{(n-1)}{n}\sigma^2
\end{aligned}
$$

Problem 5.4

First, calculating the mean and variance of the sum of resistors, i.e. $S_4 = \sum_{i=1}^{4} R_i$:

$$\mathbb{E}[S_4] = 4 \times \mathbb{E}[R_1] = 4 \times 500 = 2000$$
$$Var(S_4) = 4 \times Var(R_1) = 4 \times \frac{100^2}{12} = \frac{10000}{3}$$

By applying CLT, we have $\frac{S_4 - \mathbb{E}[S_4]}{\sqrt{Var(S_4)}} \sim \mathcal{N}(0, 1)$. Therefore:

$$\begin{aligned} P(1900 \le S_4 \le 2100) &= P(\frac{1900 - 2000}{100/\sqrt{3}} \le \frac{S_4 - 2000}{100/\sqrt{3}} \le \frac{2100 - 2000}{100/\sqrt{3}}) \\ &= Q(-\sqrt{3}) - Q(\sqrt{3}) \\ &= 0.9167 \end{aligned}$$

Problem 5.5

((a)) For S and R to be jointly Gaussian, we should check that if their linear combination follows a Guassian distribution or not. To summarize:

$$\begin{aligned} S &\sim \mathcal{N}(0, 1) \\ W &\sim \mathcal{N}(0, 1) \\ R &\sim \mathcal{N}(0, 2) \end{aligned}$$

Let's assume random variable $P = aS + bR = (a + b)S + bW$. We calculate the characteristic function of P.

$$\begin{aligned} \mathbb{M}_P[jv] &= \mathbb{E}[e^{jv((a+b)S+bW)}] \\ &= e^{\frac{-v^2(a+b)^2}{2}} . e^{\frac{-v^2(b)^2}{2}} \\ &= e^{\frac{-v^2((a+b)^2+b^2)}{2}} \end{aligned}$$

Thus random variable P is Gaussian with $\mathcal{N}(0, (a+b)^2 + b^2)$. Then S and R are jointly Gaussian.

((b)) By knowing that $\mathbb{E}[SR] - \mathbb{E}[R]\mathbb{E}[S] = \mathbb{E}[S^2 + SW] = 1 + 0 = 1$,

$$K = \begin{bmatrix} Var(S) & \mathbb{E}[SR] - \mathbb{E}[R]\mathbb{E}[S] \\ \mathbb{E}[SR] - \mathbb{E}[R]\mathbb{E}[S] & Var(R) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

((c)) The orthogonal transformation will be the eigenvectors of the covariance matrix. By using *Matlab* we find the eigenvectors and eigenvalues of covariance matrix.

$$Q = \begin{bmatrix} -0.8507 & 0.5257 \\ 0.5257 & 0.8507 \end{bmatrix}$$

By using this transformation, the joint Gaussian will have the covariance matrix containing the eigenvalues of the original covariance matrix:

$$\Lambda = \begin{bmatrix} 0.3820 & 0 \\ 0 & 2.6180 \end{bmatrix}$$

Problem 5.5

((a)) Since, elements of vector Y is linear combination of elements in X, distribution of Y will be jointly Gaussian as well. Therefore by knowing the mean and covariance matrix of Y, its distribution will be fully characterized.

$$\mathbb{E}[Y] = \mathbb{E}[C^{-1/2}X] = 0$$
$$\mathbb{E}[YY^T] = \mathbb{E}[C^{-1/2}XX^TC^{-1/2^T}] = C^{-1/2}CC^{-1/2^T} = C^{-1/2}C^{1/2}C^{1/2^T}C^{-1/2^T} = I_{n\times n}$$

Therefore $Y \sim \mathcal{N}(0, I_{n\times n})$.

((b)) Let's define $Q \triangleq Y_k^2$, then we start with its CDF to find its PDF:

$$P(Y_k^2 \leq q) = P(-\sqrt{q} \leq Y_k \leq \sqrt{q}) = \int_{-\sqrt{q}}^{\sqrt{q}} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

Using Leibniz integral rule, then we have a χ^2 distribution with 1 degree of freedom, as was expected from a square of normal random variable:

$$f_{Y_k^2}(y_k^2) = f_Q(q) = \frac{1}{2\sqrt{q}} \frac{1}{\sqrt{2\pi}} e^{-\frac{q}{2}} + \frac{1}{2\sqrt{q}} \frac{1}{\sqrt{2\pi}} e^{-\frac{q}{2}} = \frac{1}{\sqrt{2\pi q}} e^{-\frac{q}{2}}$$

((c)) The sum of squares of n independent Gaussian random variables (which is exactly the same as $V = \sum_{k=1}^{n} Y_k^2$), follows a χ^2 distribution with n degrees of freedom.
