EE 513: Stochastic Systems Theory

Fall 2022 Schmid

Homework Assignment 5

Distributed: Friday, October 28, 2022

Due: Wednesday, November 9, 2022

Reference Material:

1. class notes
2. Leon-Garcia'08, Sec. 6.1 – Sec. 6.4, Ch. 7

The following exercises are for self study. The solutions will be provided.

Exercise 5.1 (Leon-Garcia'94, problem 17, page 319)

A fair die is tossed independently 100 times. Let N_i be the number showing on the top face on the *i*th toss. Use the Chebychev inequality to lower bound the probability that $\sum_{i=1}^{100} N_i$ is between 300 and 400.

Exercise 5.2 (Leon-Garcia'94, problem 29, page 320)

A binary communication channel introduces bit errors independently with probability p=0.15; that is, p is the probability any bit is received in error. Use the central limit theorem to estimate the probability that there are 20 or fewer bits in error in 100-bit transmissions.

Below are required problems. Three problems will be selected at random and graded.

Problem 5.1

Let $\{X_n\}$ be a sequence of independent and identically distributed, continuous random variables. The common probability density for these random variables is uniform on the interval (-a, a), and their common variance equals one. Let

$$S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \,.$$

Let $f_{S_n}(s)$ denote the probability density of S_n.

a. Determine (analytically) $f_{S_n}(s)$ for n = 2 and n = 3.

b. What does the central limit theorem imply about $f_{S_n}(s)$ as n tends towards infinity?

c. Write and implement a computer program to determine $f_{S_n}(s)$ for any n.

Hint: the characteristic function of S_n can be readily expressed in terms of the characteristic function of *X*. Fourier transforms can then be used to calculate the probability density of S_n. Use Matlab to draw a graph of the density of S_n for n = 1, 2,…,10. Discuss your results.

Problem 5.2 (L-G'94, Problem 81, p. 327)

Let X_n be the sequence of i.i.d. distributed outputs of an information source. At time n, the source produces symbols according to the following probabilities

Symbol	Probability	Codeword
A	1/2	0
B	1/4	10
C	1/8	110
D	1/16	1110
E	1/16	1111

a. The self-information of the input at time *n* is defined by the random variable $Y_n = -\log P[X_n]$. Thus, for example, if the output is C, the self-information is $-\log_2 1/8 = 3$. Find the mean and variance of Y_n. Note that the expected value of the self-information is equal to the entropy of X.

b. Consider the sequence of arithmetic average of the self-information:

$$S_n = \frac{1}{n}\sum_{k=1}^{n} Y_k$$

Does the weak law of large numbers apply to S_n? <u>Answers without a proof will not be accepted.</u>

Problem 5.3

Let $X_1, X_2, \cdots, X_n$ be a sequence of independent, identically distributed random variables for which the mean *m* and variance σ^2 are unknown. The *sample mean* and the *sample variance* are defined by

$$M_n = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad V_n = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - M_n)^2,$$

respectively. $X_1, X_2, \cdots, X_n$ can be regarded as the result of repeated, independent performances of a random experiment in which a random variable *X* is measured to estimate its mean and variance via the sample mean and sample variance, respectively.

a. Find the mean and variance of M_n. Is it a biased/unbiased and consistent/not consistent estimate?
b. Demonstrate that $E[V_n] = \sigma^2$, which means that the sample variance is an unbiased estimate of the variance.
c. Determine the expected value of a modified sample variance in which the factor 1/(*n*-1) is replaced by 1/*n*. Note that this modified sample variance <u>is a biased estimate of the variance</u>.

Problem 5.4

The resistors R_1, R_2, R_3, R_4 are independent random variables and each is uniform in the interval (450, 550). Using the central limit theorem, find $P(1900 \le R_1 + R_2 + R_3 + R_4 \le 2100)$.

Problem 5.5

Let S is Gaussian random variable with mean zero and variance 1. Define R as R = S + W, where W is another Gaussian random variable (independent of S) with mean zero and variance 1.

(a) Show that S and R are jointly Gaussian.
(b) Find their covariance matrix.
(c) Find an orthogonal transformation (matrix **Q**) that makes S and R uncorrelated. Solve this part numerically.

Bonus Problem 5.6 (try, but you do not have to solve it) (Gubner'06, Problem 13, p. 378)

The digital signal processing chip in a wireless communication receiver generates the n-dimensional Gaussian vector X with mean zero and positive definite covariance matrix C. It then computes the vector $Y = C^{-1/2}X$. (Since $C^{-1/2}$ is invertible, there is no loss of information in applying such a transformation). Finally, the decision statistic $V = ||Y||^2 = \sum_{k=1}^{n} Y_k^2$ is computed.

(a) Find the multivariate density of Y.

(b) Find the density of Y_k^2 for $\boldsymbol{k = 1 \ldots n}$.

(c) Find the density of V.