# EE 513: Stochastic Systems Theory

Fall 2022 Schmid

Homework Assignment 4
Distributed: Monday, October 10, 2022
Due: Monday, October 24, 2022

**Note: Test 1 is tentatively scheduled on October 26, Wednesday.**

**Coverage: everything in class notes, Leon-Garcia through Ch. 6, and homework assignments through (and including) HW Set 4.**

**Rules: do your own work.**

**Reference Material:**

1. class notes
2. Leon-Garcia'08, Chapters 3 – 6.

**Three of the following problems will be selected at random and graded.**

**Problem 4.1**

Compare the Chebyshev inequality and the exact probability for the event $\{|X - m| \geq c\}$ as a function of $c$ for

(a) $X$ is a uniform random variable in the interval $[-\pi/2, \pi/2]$;
(b) $X$ is Laplacian random variable with parameter 2;
(c) $X$ is a Gaussian random variable with mean zero and variance $\sigma^2 = 4$.

**Problem 4.2**

Let $X$ be a discrete random variable with the following probability mass function

$$X = \begin{cases} -2, & p(-2) = 1/8 \\ -1, & p(-1) = 3/8 \\ 1, & p(1) = 3/8 \\ 2, & p(2) = 1/8 \end{cases}$$

(a) Find the characteristic function of the random variable, $\Phi_X(jv) = E[\exp(jvX)]$.
(b) Find the entropy of the random variable.

*Note* that the entropy of a discrete random variable is defined as $H(X) = E\left[log_2\left(\frac{1}{p(X)}\right)\right]$. Entropy is known as a measure of uncertainty in a random variable. Lossless compression is impossible beyond the entropy limit (measured in bits per symbol).

(c) Set the probability of each of the four outcomes to 1/4. Find $H(X)$ and compare to the result in part (b).

**Problem 4.3**

Let $X_1$ and $X_2$ be statistically independent random variables, each of which is uniformly distributed over the interval (0,2). Define random variables $Y_1$ and $Y_2$ according to:

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$

Also, let A be the event $\{\omega : |Y_2(\omega)| \le 1\}$.

(a) Determine $\Pr(A \mid Y_1 = 1)$.

(b) Determine $\Pr(Y_1 \ge 0 \mid A)$.

(c) Determine the conditional p.d.f. $f_{Y_2|Y_1}(y_2|y_1)$ of $Y_2$ given that $Y_1 = y_1$.

**Problem 4.4 (Leon-Garcia'94, problem 9, page 257)**

Let *X* and *Y* denote the amplitudes of noise signals at two antennas. Their joint probability density function is given by

$$f_{X,Y}(x,y) = \begin{cases} axe^{-ax^2/2} bye^{-by^2/2}, & x > 0, \quad y > 0, \quad a > 0, \quad b > 0 \\ 0, & otherwise \end{cases}$$

(a) Determine the joint cumulative distribution function of *X* and *Y*.

(b) Determine the probability that *X* exceeds *Y*.

(c) Determine the probability densities of *X* and *Y*.

**Problem 4.5**

Let $X$ and $Y$ be random variables with the following joint density function

$$f_{X,Y}(x,y) = \begin{cases} C, & x^2 + y^2 \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Evaluate the constant $C$.

(b) Determine the marginal density function for $X$.

(c) Determine the conditional density function for $X$ given that $Y = \sqrt{3}/2$.

(d) Are $X$ and $Y$ uncorrelated? Why or why not?

(e) Are $X$ and $Y$ independent? Why or why not?

(f) Now let $R$ and $\Theta$ be random variables given by the transformation of $X$ and $Y$ to polar coordinates, i.e., $R = (X^2 + Y^2)^{1/2}$ and $\Theta = \tan^{-1}(Y/X)$.

(g) Determine the marginal probability density function for $R$ and $\Theta$.

(h) Are $R$ and $\Theta$ independent? Why or why not?

**Problem 4.6 (Set k = 3)**

Let $X_1, X_2, \cdots, X_k$ be k statistically independent, identically distributed, continuous random variables, all having the same cumulative distribution function $F_X(x)$. Define the random variable

$$Z = \min(X_1, X_2, \cdots, X_k) \text{ and } W = \max(X_1, X_2, \cdots, X_k).$$

(a) Show that the joint cumulative distribution function for $(Z, W)$ is given by:

$$F_{Z,W}(z,w) = \{F_X(w)\}^k - \{F_X(w) - F_X(z)\}^k u(v-u),$$

where $u(\,.\,)$ is the unit-step function (that is, $u(t) = 0$ for $t < 0$ and $u(t) = 1$ for $t \geq 0$).

(b) Determine the marginal cumulative distribution functions, $F_Z(z)$ and $F_W(w)$, of $Z$ and $W$, respectively.

**Problem 4.7**

Let $X$ and $Y$ be statistically independent random variables with probability densities $f_X(x)$ and $f_Y(y)$, respectively.

(a) Let $Z = X + Y$. Demonstrate that the probability density of $Z$ is the convolution of the densities of $X$ and $Y$, and demonstrate that the characteristic function of $Z$ is the product of the characteristic functions of $X$ and $Y$.

(b) Let $W = X - Y$. Demonstrate that the probability density of $W$ is the correlation of the densities of $X$ and $Y$. Determine the relationship between the characteristic function of $W$ and those of $X$ and $Y$.

(c) Two random variables, $X$ and $Y$, are said to be *equal in distribution* if their probability densities are equal; that is, $f_X(\alpha) = f_Y(\alpha)$. The notation "$X \overset{d}{=} Y$" is used to indicate that $X$ and $Y$ are equal in distribution. Suppose that $X$ and $Y$ are independent random variables such that $Y$ is Gaussian distributed with mean m and variance $\sigma^2$, and $X \overset{d}{=} Y$. Use the results of parts (a) and (b) to determine the probability density of $X$-$Y$. This result shows that even though $X \overset{d}{=} Y$ holds, it is not true that $X - Y \overset{d}{=} 0$, so the symbol "$\overset{d}{=}$" needs to be interpreted with care.

**Problem 4.8**

Let $Y = X_1 + X_2 + \cdots + X_n$, where $X_i$ are mutually independent random variables.

(a) Suppose that $X_i$ is normally distributed $N(m_i, \sigma_i^2)$ for each $i$. Show that $Y$ is also normally distributed with mean $\sum_{i=1}^{n} m_i$ and variance $\sum_{i=1}^{n} \sigma_i^2$.

(b) Suppose that $X_i$ is Poisson distributed with parameter $\lambda_i$ for each $i$; that is,

$$\Pr(X_i = k) = \frac{\lambda_i^k}{k!} e^{-\lambda_i}.$$

Show that $Y$ is also Poisson distributed with parameter $\sum_{i=1}^{n} \lambda_i$.

(c) Suppose that $X_i$ is Cauchy distributed with parameters $a_i$ and $b_i$ for each $i$; that is,

$$f_{X_i}(x) = \frac{1}{\pi b_i}\left(1 + \frac{(x - a_i)^2}{b_i^2}\right)^{-1}.$$

Show that $Y$ is also Cauchy distributed with parameters $a_Y = \sum_{i=1}^{n} a_i$ and $b_i = \sum_{i=1}^{n} b_i$.

(d) Suppose that $X_i$ is chi-square with $N_i$ degrees of freedom for each $i$; that is,

$$f_{X_i}(x) = \frac{x^{\left(\frac{N_i}{2}\right)-1}}{2^{\left(\frac{N_i}{2}\right)}\sigma^{N_i}\Gamma\left(\frac{N_i}{2}\right)} e^{-\frac{x}{2\sigma^2}} u(x),$$

where $u(x) = 1$, for $x \geq 0$, and $u(x) = 0$, otherwise. Show that $Y$ is also chi-square with $N_Y = \sum_{i=1}^{n} N_i$ degrees of freedom.

Note: probability density functions that are preserved under summation are called *reproducing densities*. So, the normal, Poisson, Cauchy, and chi-square densities are all examples of reproducing densities.