EE 513: Stochastic Systems Theory
Fall 2022 **Schmid**

Final (take home)

Distributed on December 15, 2022 (at 9 am)
Due is on December 16, 2022 (at 9 am)

Do your own work. The test is take home and thus open book and notes.

Name: Ali Zafari

Pledge: "I have neither given nor received unauthorized aid on this examination."

Signed:

Problem	Points	Points received
1	13	
2	15	
3	12	
4	15	
5	15	
6	15	
7	15	
Bonus	10	
Total		

1 a $\quad X_i \sim \text{Pois}(\lambda) \Rightarrow p(X_i) = \dfrac{\lambda^{X_i} e^{-\lambda}}{X_i!}$

$$\mathcal{L} = p(X_1)\,p(X_2)\cdots p(X_n) = \prod_{i=1}^{n} p(X_i) = \frac{\lambda^{\sum X_i}\, e^{-n\lambda}}{X_1!\, X_2! \cdots X_n!}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Rightarrow \sum X_i \, \lambda^{\sum X_i - 1} e^{-n\lambda} - n e^{-n\lambda} \lambda^{\sum X_i} = 0$$

$$\lambda^{\sum X_i} e^{-n\lambda}\left(\frac{\sum X_i}{\lambda} - n\right) = 0$$

$$\Rightarrow \boxed{\lambda_{MLE} = \frac{\sum X_i}{n}}$$

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1 b $\quad \text{bias} \triangleq E[\lambda_{MLE}] - \lambda = E\left[\frac{\sum X_i}{n}\right] - \lambda = \frac{n\lambda}{n} - \lambda = 0$

Variance : $\quad Var(\lambda_{MLE}) = Var\left(\frac{\sum X_i}{n}\right) = \frac{1}{n^2}\, n \times \overbrace{Var(X_i)}^{\lambda} = \frac{\lambda}{n}$

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② Bayesian decision rule:

(assumming $r \in [0,5]$)

$$\ln \frac{P(R=r \mid H_1)}{P(R=r \mid H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \ln \frac{\pi_0}{\pi_1}$$

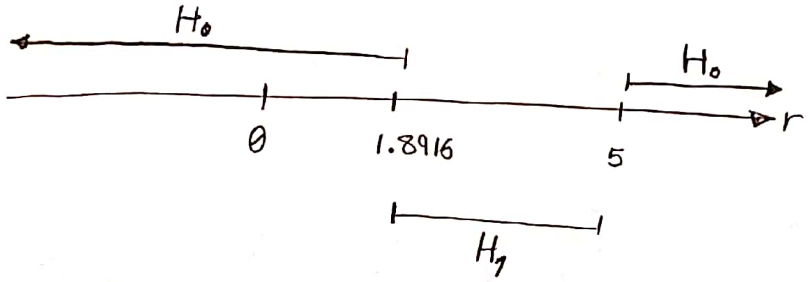
$$\ln \frac{\sqrt{2\pi}}{5} + \frac{r^2}{2} \underset{H_0}{\overset{H_1}{\gtrless}} \ln 3$$

$$r^2 \underset{H_0}{\overset{H_1}{\gtrless}} 3.5782$$

$0 \leqslant r \leqslant 5$

$$r \underset{H_0}{\overset{H_1}{\gtrless}} 1.8916$$

Decision boundaries:



$$P_{FA} = P(H_1 \mid H_0) = \int_{1.8916}^{5} P(r \mid H_0)\, dr = \int_{1.8916}^{5} \frac{1}{\sqrt{2\pi}} e^{-\frac{r^2}{2}} dr = Q(1.8916) - Q(5) = 0.0293$$

$$P_{miss} = P(H_0 \mid H_1) = \int_{-\infty}^{1.8916} P(r \mid H_1)\, dr + \int_{5}^{\infty} P(r \mid H_1)\, dr = \frac{1}{5} \times (1.8916 - 0) + 0 = 0.3632$$

$$P_{error} = \pi_0 P_{FA} + \pi_1 P_{miss} = \frac{3}{4} \times 0.0293 + \frac{1}{4} \times 0.3632 = 0.1128$$

3

$$m_X = \frac{1}{3} + 0 - \frac{1}{3} = 0$$

$$\sigma_X^2 = E[X^2] - \overset{0}{\cancel{E[X]^2}} = \frac{1}{3} + 0 + \frac{1}{3} = \frac{2}{3}$$

$$E[M_n] = \frac{n\,E[X]}{n} = 0$$

$$Var[M_n] = \frac{1}{n^2}\, n\,\sigma_X^2 = \frac{\sigma_X^2}{n} = \frac{2}{3n}$$

to prove WLLN, we start by Chebyshev's inequality:

$$P\left(\left|M_n - E[M_n]\right| > \epsilon\right) \leq \frac{Var[M_n]}{\epsilon^2}$$

$$P\left(\left|M_n - 0\right| > \epsilon\right) \leq \frac{2}{3n\epsilon^2}$$

it can be seen from equation above that as $n \to \infty$, the sample mean (M_n) will converge to $m_X = 0$ in probability.

$$M_n \xrightarrow[\text{as } n \to \infty]{\text{in Prob.}} 0$$

(4)

$\lambda = 15 \frac{\text{message}}{\text{second}} \times 60 = 900 \frac{\text{message}}{\text{minute}}$

$k \sim Pois(900)$

$k_n \triangleq k_1 + k_2 + \cdots + k_{900}$

$, k_i \sim Pois(1)$

$\Rightarrow k_n \sim Pois(900)$

aside

Sum of independent Poisson random variables is a Poisson random variable.

proof:

$X_1 \sim Pois(\lambda_1)$

$X_2 \sim Pois(\lambda_2)$ $\quad X_1 \perp\!\!\!\perp X_2$

$Y \triangleq X_1 + X_2$

$\Phi_Y(jv) = \Phi_{X_1}(jv) \cdot \Phi_{X_2}(jv)$

$= e^{\lambda_1(e^{jv}-1)} \cdot e^{\lambda_2(e^{jv}-1)}$

$= e^{(\lambda_1+\lambda_2)(e^{jv}-1)}$

$\Rightarrow Y \sim Pois(\lambda_1 + \lambda_2)$

from CLT:

$$\frac{k_n - E[k_n]}{\sigma_{k_n}} \sim \mathcal{N}(0,1)$$

$$E[k_n] = \sum_{1}^{900} E[k_i] = 900 \times 1$$

$$\sigma_{k_n}^2 = \sum_{1}^{900} \sigma_{k_i}^2 = 900 \Longrightarrow \sigma_{k_n} = 30$$

$$\Rightarrow \frac{k_n - 900}{30} \sim \mathcal{N}(0,1)$$

$$\Rightarrow P(k_n > 950) = P\left(\frac{k_n - 900}{30} > \underbrace{\frac{950-900}{30}}_{5/3}\right) \stackrel{CLT}{=\!=} Q\left(\frac{5}{3}\right) = 0.0478$$

5 $\boxed{a}$

$$E[X(t)] \overset{A \perp\!\!\!\perp \theta}{=} E_A[A]\, E_\theta[\cos(2\pi f_c t + \theta)] = 2 \times 0 = 0$$

$$R_{XX}(t,u) = E_{A,\theta}[A\cos(2\pi f_c t + \theta) \times A\cos(2\pi f_c u + \theta)]$$

$$= E_A[A^2]\left\{\underbrace{E_\theta\left[\tfrac{1}{2}\cos(2\pi f_c(t+u) + 2\theta)\right]}_{0} + E_\theta\left[\tfrac{1}{2}\cos(2\pi f_c(t-u))\right]\right\}$$

$$= \left(\tfrac{1}{2}\times 16 + \tfrac{1}{2}\times 0\right)\left(\tfrac{1}{2}\cos(2\pi f_c(t-u))\right)$$

$$= 4\cos(2\pi f_c(t-u))$$

$$\xrightarrow{\tau \triangleq t-u} \quad R_{XX}(\tau) = 4\cos 2\pi f_c \tau$$

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5 $\boxed{b}$ I. $E[X(t)] = 0$ is constant and bounded ($<\infty$)

II. $R_{XX}(t,u)$ is a function of $t-u$ ($\triangleq \tau$)

$\Rightarrow$ Therefore $X(t)$ is WSS random process.

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5 c

Frequency response of whole LTI system is:

$$G(f) = \frac{Z(f)}{X(f)} = \frac{X(f) - H(f)X(f)}{X(f)} = 1 - H(f) = \frac{f^2}{1+f^2}$$

PSD of the output will be:

$$S_Z(f) = S_X(f) \, |G(f)|^2$$

PSD of the input is the Fourier Transform of autocorrelation function of it:

$$S_X(f) = \mathcal{F}\{R_{XX}(\tau)\} = \mathcal{F}\left\{4 \frac{e^{j2\pi f_c \tau} + e^{-j2\pi f_c \tau}}{2}\right\}$$

$$= 2\,\delta(f - f_c) + 2\,\delta(f + f_c)$$

$$\xrightarrow{f_c=1} S_Z(f) = \left[2\delta(f-1) + 2\delta(f+1)\right] \left|\frac{f^2}{1+f^2}\right|^2$$

$$= 2\left(\frac{(+1)^2}{1+1^2}\right)^2 \delta(f-1) + 2\left(\frac{(-1)^2}{1+(-1)^2}\right)^2 \delta(f+1)$$

$$= \frac{1}{2}\delta(f-1) + \frac{1}{2}\delta(f+1)$$

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5 d

$$\underbrace{E\left[|Z(t)|^2\right]}_{\text{average power}} = R_{ZZ}(0) = \mathcal{F}^{-1}\left\{\frac{1}{2}\delta(f-1) + \frac{1}{2}\delta(f+1)\right\}\Big|_{\tau=0} = \cos 2\pi\tau\Big|_{\tau=0} = 1$$

6 a $\quad Z = X + 2Y$

$$P(Z \leqslant z) = F_Z(z) = \iint_{D_z} f_{XY}(x,y)\,dy\,dx$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{\frac{z-x}{2}} f_{XY}(x,y)\,dy\,dx$$

$$f_Z(z) = \frac{d\,F_Z(z)}{dz} = \int_{-\infty}^{+\infty} \frac{1}{2} f_{XY}\left(x, \frac{z-x}{2}\right) dx$$

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6 b

$$f_{ZW}(z,w) = f_{XY}\left(\overbrace{\frac{2w+z}{5}}^{x}, \overbrace{\frac{2z-w}{5}}^{y}\right) \times \frac{1}{|J(x,y)|}$$

$$|J(x,y)| = \left| \det \begin{bmatrix} \partial z/\partial x & \partial z/\partial y \\ \partial w/\partial x & \partial w/\partial y \end{bmatrix} \right| = \left| \det \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \right| = |-5| = 5$$

7 a

I. X is Gaussian.

II. Since X and Y are jointly Gaussian, any linear combination of them is also Gaussian. As a result $X+aY$ is Gaussian.

III. linear combination of X and $X+aY$, i.e. $AX + B(X+aY)$, can be written as linear combination of X and Y, i.e. $(A+B)X + (Ba)Y$. Therefore X and $X+aY$ are $\boxed{\text{Jointly Gaussian}}$.

As a result, uncorrelatedness is same as independence for X and $X+aY$.

$$Cov(X, X+aY) = E[(X - \underbrace{E[X]}_{0})(X+aY - \underbrace{E[X+aY]}_{0})]$$

$$= E[X^2] + a\,E[XY]$$

We need to find $E[X^2]$ and $E[XY]$:

$\rightarrow \sigma_X^2 = E[X^2] - \underbrace{E^2[X]}_{0} \Rightarrow \boxed{E[X^2] = 16}$

$\rightarrow \rho = \dfrac{Cov(X,Y)}{\sqrt{\sigma_X^2\,\sigma_Y^2}} = \dfrac{E[XY] - \overbrace{E[X]}^{0}\overbrace{E[Y]}^{0}}{4\times 3} \Rightarrow \boxed{E[XY] = 12\rho}$

$\Rightarrow Cov(X, X+aY) = 0 \Rightarrow 16 + 12a\rho = 0 \Rightarrow \boxed{\rho = \dfrac{-4}{3a}}$

7 b

I. To have $2X + dY$ and $(X - dY)^2$ independent, we know that $g(2X + dY)$ and $f((X-dY)^2)$ must also be independent for any choice of g and f.

II. Therefore $2X + dY$ and $X - dY$ must be independent.

III. linear combination of $2X + dY$ and $X - dY$, i.e. $A(2X+dY) + B(X-dY)$, can be written as $(2A+B)X + (Ad - Bd)Y$ which is a linear combination of X and Y, and hence $\boxed{\text{Jointly Gaussian}}$.

As a result, uncorrelatedness is same as independence for $2X+dY$ and $X-dY$.

$$Cov(2X+dY, X-dY) = E[(2X+dY - \cancel{E[2X+dY]}^{0})(X-dY - \cancel{E[X-dY]}^{0})]$$

$$= E[2X^2 - dXY - d^2Y^2]$$

from part (a)

$$= 2 \times 16 - d(12\rho) - d^2 \times 9 = 0$$

$$\Rightarrow 9d^2 + 12\rho d - 32 = 0$$

$$\Rightarrow d = \frac{-12\rho \pm \sqrt{144\rho^2 + 1152}}{18}$$

a few examples of d

if $\rho = 0 \rightarrow d = \pm 1.8856$

if $\rho = 0.5 \rightarrow d = 1.58$, $d = -2.24$

if $\rho = 1 \rightarrow d = 1.33$, $d = -2.67$

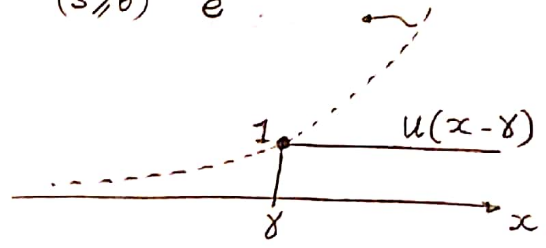
CamScanner

Bonus Problem a

$$P(X > \overbrace{E[X] + \epsilon}^{\triangleq \gamma}) = P(X > \gamma) = \int_{\gamma}^{+\infty} f_X(x)\, dx = \int_{-\infty}^{+\infty} u(x-\gamma)\, f_X(x)\, dx$$

$(\epsilon > 0)$ $\qquad$ $(\gamma > E[X])$

replacing step function by $(s \geq 0)$ $e^{s(x-\gamma)}$ an exponential, we will find an upperbound for the probability:



$$P(X > \gamma) \leqslant \int_{-\infty}^{+\infty} e^{s(x-\gamma)} f_X(x)\, dx = \underbrace{\boxed{\int_{-\infty}^{+\infty} e^{sX} f_X(x) dx}}_{\text{Moment Generating Function (MGF)}}^{\Phi_X(s)} e^{-s\gamma}$$

we already know MGF is defined as:

$$\Phi_X(s) = E[e^{sX}] = \int_{-\infty}^{+\infty} e^{sX} f_X(x)\, dx$$

we now define log-MGF:

$$\varphi_X(s) \triangleq \ln \Phi_X(s)$$

$$\Rightarrow P(X > \gamma) \leqslant e^{\varphi_X(s)} \cdot e^{-s\gamma} \qquad (\underline{s \geq 0},\ \underline{\gamma > E[X]})$$

$e^{-s\gamma + \varphi_X(s)}$ is the Chernoff Bound.

Now we try to find the tightest Chernoff Bound:

$$\Rightarrow s^* = \underset{s \geq 0}{\arg\min}\ e^{-s\gamma + \varphi_X(s)} = \underset{s \geq 0}{\arg\min}\ -s\gamma + \varphi_X(s) = \underset{s \geq 0}{\arg\max}\ s\gamma - \varphi_X(s)$$

Bonus Problem 6

$$X \sim \mathcal{N}(5, 4) \quad \Rightarrow \quad \Phi_X(s) = e^{5s + 2s^2}$$

$$\Rightarrow \quad \varphi_X(s) = 5s + 2s^2$$

to find s^* which maximizes $s\gamma - \varphi_X(s)$, we take derivate of it:

$$\frac{d}{ds}\left\{ s\gamma - 5s - 2s^2 \right\} = 0 \quad \Rightarrow \quad \gamma - 5 - 4s = 0 \quad \Rightarrow \quad \boxed{s^* = \frac{\gamma - 5}{4}}$$

So the Chernoff Bound will be:

$$P(X > \gamma) \leq e^{-\gamma\left(\frac{-5+\gamma}{4}\right) + 5\left(\frac{-5+\gamma}{4}\right) + 2\left(\frac{-5+\gamma}{4}\right)^2} = e^{-\frac{(\gamma-5)^2}{8}}$$

For Gaussian we have access to the actual value of probabilities:

$$P(X > \gamma) = P(X - E[X] > \epsilon) = P(X - 5 > \epsilon) = P\left(\frac{X-5}{2} > \frac{\epsilon}{2}\right) = Q\left(\frac{\epsilon}{2}\right)$$

So to compare, we need to check values of $Q\left(\frac{\epsilon}{2}\right)$

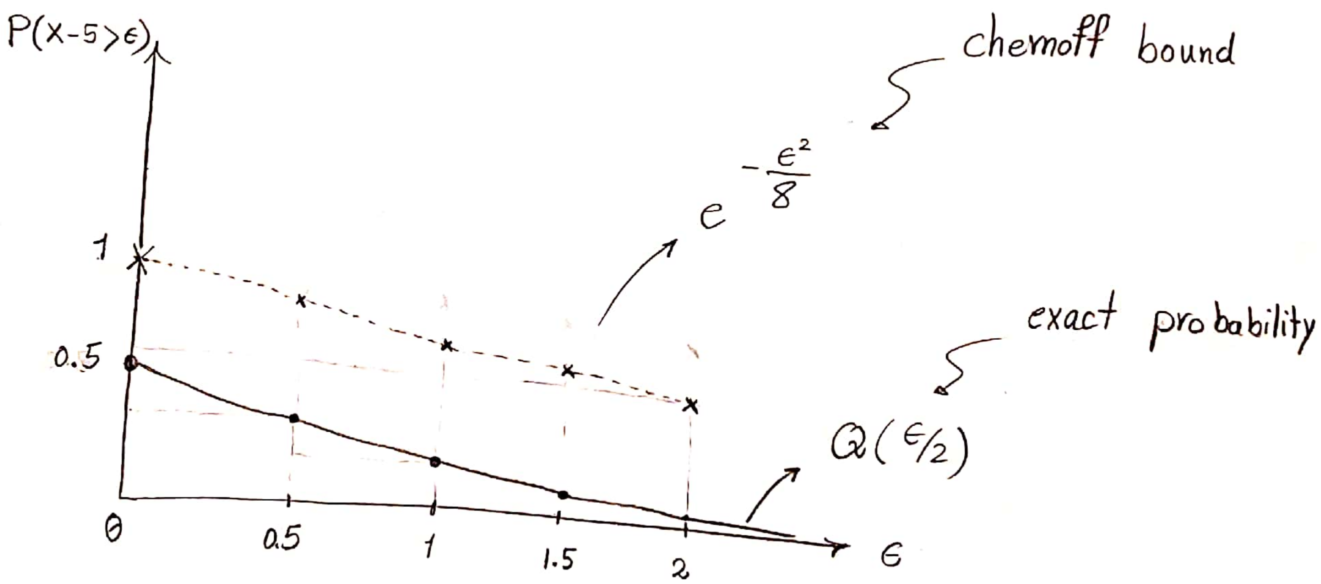
and $e^{(\gamma-5)^2/8} = e^{\epsilon^2/8}$.

Chernoff Bound : $P(X-5>\epsilon) \leqslant e^{-\epsilon^2/8}$

$(\gamma = 5+\epsilon)$

Actual Value : $P(X-5>\epsilon) = Q\left(\frac{\epsilon}{2}\right)$

$\boxed{\text{Bonus Problem}}\boxed{c}$



On Larger scale →