Homework 6

EE 513 — Stochastic Systems Theory

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Problem 6.1

For $x \in \{0, 1, \ldots, 5\}$:

$$H_1 : \qquad p(X = x|H_1) = \binom{5}{x}(\frac{2}{3})^x(\frac{1}{3})^{5-x}$$

$$H_\circ : \qquad p(X = x|H_\circ) = \binom{5}{x}(\frac{1}{2})^x(\frac{1}{2})^{5-x}$$

ML Decision Rule:

$$\log \frac{p(X = x|H_1)}{p(X = x|H_\circ)} \underset{H_\circ}{\overset{H_1}{\gtrless}} 0$$

$$\log \frac{2^{5+x}}{3^5} \underset{H_\circ}{\overset{H_1}{\gtrless}} 0$$

$$x \underset{H_\circ}{\overset{H_1}{\gtrless}} 5\frac{\log 3}{\log 2} - 5$$

$$x \underset{H_\circ}{\overset{H_1}{\gtrless}} 2.92$$ ✓

$$P_{FalseAlarm} = P(H_1|H_\circ) = \sum_{x \in \{3,4,5\}} p(X = x|H_\circ) = 0.5$$ ✓

$$P_{Miss} = P(H_\circ|H_1) = \sum_{x \in \{0,1,2\}} p(X = x|H_1) = 0.2098$$ ✓

$$P_{Error} = \frac{1}{2}P_{FalseAlarm} + \frac{1}{2}P_{Miss} = 0.3549$$ ✓

MAP Decision Rule:

$$\log \frac{p(X = x|H_1)}{p(X = x|H_\circ)} \underset{H_\circ}{\overset{H_1}{\gtrless}} \log \frac{\pi_0}{\pi_1}$$

$$\log \frac{2^{5+x}}{3^5} \underset{H_\circ}{\overset{H_1}{\gtrless}} \log \frac{0.8}{0.2}$$ ✓

$$x \underset{H_\circ}{\overset{H_1}{\gtrless}} 4.92$$ ✓

$$P_{FalseAlarm} = P(H_1|H_\circ) = \sum_{x\in\{5\}} p(X=x|H_\circ) = \frac{1}{2^5} = 0.03125$$

$$P_{Miss} = P(H_\circ|H_1) = \sum_{x\in\{0,1,2,3,4\}} p(X=x|H_1) = 1-(\frac{2}{3})^5 = 0.8683$$

$$P_{Error} = 0.8P_{FalseAlarm} + 0.2P_{Miss} = 0.1987$$

Problem 6.2

For $y \in \{1,2,3,\ldots\}$:

$$\begin{aligned} H_1: &\quad p(Y=y|H_1) = 0.5(1-0.5)^{y-1} \\ H_\circ: &\quad p(Y=y|H_\circ) = 0.2(1-0.2)^{y-1} \end{aligned}$$

((a)) ML decision rule:

$$\begin{aligned} \log\frac{p(Y=y|H_1)}{p(Y=y|H_\circ)} &\underset{H_\circ}{\overset{H_1}{\gtrless}} 0 \\ y &\underset{H_\circ}{\overset{H_1}{\gtrless}} 1+\frac{\log(2/5)}{\log(5/8)} \\ y &\underset{H_\circ}{\overset{H_1}{\gtrless}} 2.94 \end{aligned}$$

There is a flip in the unequality when dicidung by a negative number.

((b)) For the ML decision rule:

$$P_{FalseAlarm} = P(H_1|H_\circ) = \sum_{y\in\{3,4,\ldots\}} p(Y=y|H_\circ) = 0.64$$

$$P_{Miss} = P(H_\circ|H_1) = \sum_{y\in\{1,2\}} p(Y=y|H_1) = 0.75$$

((c)) MAP decision rule:

$$\begin{aligned} \log\frac{p(Y=y|H_1)}{p(Y=y|H_\circ)} &\underset{H_\circ}{\overset{H_1}{\gtrless}} \log\frac{\pi_\circ}{\pi_1} \\ \log\frac{p(Y=y|H_1)}{p(Y=y|H_\circ)} &\underset{H_\circ}{\overset{H_1}{\gtrless}} \log\frac{2/3}{1/3} \\ y &\underset{H_\circ}{\overset{H_1}{\gtrless}} 2+\frac{\log(2/5)}{\log(5/8)} \\ y &\underset{H_\circ}{\overset{H_1}{\gtrless}} 3.94 \end{aligned}$$

Doublecheck the result; check the dircetion of the inequality.

For the MAP decision rule:

$$P_{FalseAlarm} = P(H_1|H_\circ) = \sum_{y\in\{4,5,\dots\}} p(Y=y|H_\circ) = 0.512$$

$$P_{Miss} = P(H_\circ|H_1) = \sum_{y\in\{1,2,3\}} p(Y=y|H_1) = 0.875$$

((d)) ML decision rule:

$$P_{Error} = \frac{2}{3}P_{FalseAlarm} + \frac{1}{3}P_{Miss} = 0.6766$$

MAP decision rule:

$$P_{Error} = \frac{2}{3}P_{FalseAlarm} + \frac{1}{3}P_{Miss} = 0.633$$

Problem 6.3

((a)) Since S and N are independent the distribution of Y will be:

$$H_1 : f_Y(y) = f_N(y) * f_S(y) = \int_0^y e^{-t}e^{-(y-t)dt} = ye^{-y}u(y)$$
$$H_\circ : f_Y(y) = f_N(y) = e^{-y}u(y)$$

The likelihood ratio will be:

$$\Lambda(y) = y \qquad , y \geq 0$$

((b)) By assuming a threshold of 1 (equal priors), we can calculate the total probability of error:

$$P_{FalseAlarm} = P(H_1|H_\circ) = \int_1^{+\infty} e^{-y}dy = e^{-1}$$

$$P_{Miss} = P(H_\circ|H_1) = \int_0^1 ye^{-y}dy = 1 - 2e^{-1}$$

Therefore:

$$P_{Error} = \frac{1}{2}P_{FalseAlarm} + \frac{1}{2}P_{Miss} = \frac{1}{2}(1-e^{-1})$$

Problem 6.4

We first derive the likelihood function. It is a function of both true parameters μ and σ:

$$\mathcal{L} = \sum_i \log f(v_i) = -\frac{n}{2} \log 2\pi\sigma^2 + \sum_1^n \frac{(v_i - \mu)^2}{2\sigma^2}$$

now the derivatives with respect to both parameters should be found and simultaneously set to zero, to find the maximum likelihood estimates for both of them (derivation wrt to σ^2 will make no difference for variance estimation):

$$\frac{\partial \mathcal{L}}{\partial \mu} = \sum_1^n \frac{v_i - \mu}{\sigma^2} = 0 \longrightarrow \hat{\mu} = \frac{1}{n} \sum_1^n v_i$$

$$\frac{\partial \mathcal{L}}{\partial \sigma^2} = -\frac{n}{2\sigma^2} - \sum_1^n \frac{(v_i - \hat{\mu})^2}{2\sigma^4} = 0 \longrightarrow \hat{\sigma^2} = \frac{1}{n} \sum_1^n (v_i - \hat{\mu})$$

Then we evaluate the expectations for both of the estimators:

$$\mathbb{E}[\hat{\mu}] = \frac{1}{n} \sum_1^n \mathbb{E}[v_i] = \mu \qquad \text{(UNBIASED ESTIMATOR)}$$

$$\begin{aligned} \mathbb{E}[\hat{\sigma^2}] &= \frac{1}{n} \mathbb{E}[\sum_1^n (v_i - \hat{\mu})] \\ &= \sigma^2 + \mu^2 + \frac{-\sigma^2 - \mu^2 - n\mu^2 + \mu^2}{n} \\ &= \frac{n-1}{n} \sigma^2 \qquad \text{(BIASED ESTIMATOR)} \end{aligned}$$

Problem 6.5

MAP estimate will maximize the posterior which is equivalent to maximizing the product of likelihood and prior.

$$\frac{\partial}{\partial a} \sum_k \log f(A|R_k) = 0 \longrightarrow \frac{\partial}{\partial a} \sum_k \log f(R_k|A) f(A) = 0$$

$$\frac{\partial}{\partial a} \left(\frac{-1}{\sigma_N^2} \sum (r_k - a)^2 + \frac{-1}{2\sigma_A^2} a^2 + \ldots \text{not dependent on } a \ldots \right) = 0$$

$$\hat{a} = \frac{\sigma_A^2}{K\sigma_A^2 + \sigma_N^2} \sum r_k$$
