Homework 7

EE 513 — Stochastic Systems Theory

Name: Ali Zafari **Student Number:** 800350381 **Fall 2022**

Problem 7.1

((a))

$$
\begin{aligned}
\mathbb{E}_\Theta[X(t)] &= \int_{-\pi}^{\pi} \frac{1}{2\pi} 2\sin(2\pi(1000)t + \theta)d\theta \\
&= -\frac{1}{\pi}\cos(2\pi(1000)t + \pi) + \frac{1}{\pi}\cos(2\pi(1000)t - \pi) \\
&= -\frac{1}{\pi}[\cos 2\pi(1000)t\cos\pi - sin\pi\sin 2\pi(1000)t] + \cos 2\pi(1000)t\cos\pi + sin\pi\sin 2\pi(1000)t] \\
&= 0
\end{aligned}
$$

((b))

$$
\begin{aligned}
R_{XX}(t,u) &= \mathbb{E}_\Theta[X(t)X(u)] \\
&= \mathbb{E}_\Theta[2\sin(2\pi(1000)t + \theta) \times 2\sin(2\pi(1000)u + \theta)] \\
&= 2\mathbb{E}_\Theta[\cos(2\pi(1000)(t-u))] - 2\mathbb{E}_\Theta[\cos(2\pi(1000)(t+u) + 2\theta)] \\
&= 2\cos(2\pi(1000)(t-u))
\end{aligned}
$$

((c)) $X(t)$ IS wide sense stationary (WSS), since it satisfies following 2 conditions:

- $\mathbb{E}_\Theta[X(t)]$ is constant with respect to time.
- Autocorrelation function ($R_{XX}(t,u)$) can be written as a function of difference in time, i.e. $R_{XX}(t,u) = R_{XX}(t-u) = R_{XX}(\tau) = 2\cos(2\pi(1000)\tau)$.

Then the power spectral density will be the Fourier transform of autocorrelation function:

$$
\begin{aligned}
S_X(f) &= \int_{-\infty}^{+\infty} R_{XX}(\tau)e^{-j2\pi f\tau}d\tau \\
&= \int_{-\infty}^{+\infty} \frac{e^{j2\pi 1000\tau} + e^{-j2\pi 1000\tau}}{2} e^{-j2\pi f\tau}d\tau \\
&= \delta(f-1000) + \delta(f+1000)
\end{aligned}
$$

Problem 7.2

((a)) For a WSS process, the autocorrelation function is the inverse Fourier transform of its power spectral density:

$$S_W(f) \xrightarrow{\mathcal{F}^{-1}} R_W(\tau) = \frac{N_0}{2}\delta(\tau) = 0.1\delta(\tau)$$

((b)) For an LTI system we have:

$$m_{\widetilde{W}} = m_W H(0) = 0 \times 2 = 0$$

((c))

$$\begin{aligned}\widetilde{W}(t) = W(t) * h(t) \xrightarrow{\mathcal{F}} S_{\widetilde{W}}(f) &= S_W(f)|H(f)|^2 \\ &= \frac{N_0}{2} \times 4rect(\frac{f}{20}) \\ &= 0.4rect(\frac{f}{20})\end{aligned}$$

((d))

$$S_{\widetilde{W}}(f) = 0.4rect(\frac{f}{20}) \xrightarrow{\mathcal{F}^{-1}} R_{\widetilde{W}}(\tau) = 8sinc(20\tau)$$
$$\mathbb{E}[\widetilde{W}^2(t)] = R_{\widetilde{W}}(0) = 8sinc(20 \times 0) = 8$$

((e)) A Gaussian process at any specific time is a Gaussian random variable. Since the filter is LTI, the output will also be a Gaussian process and by considering it at time t_1 it will be a Gaussian random variable, which is characterized only by its mean and variance:

$$\begin{aligned}\mathbb{E}[\widetilde{W}(t_1)] &= \int_{-\infty}^{\infty} h(x)W(t_1 - x)dx \\ &= \int_{-\infty}^{\infty} h(x)\mathbb{E}[W(t_1 - x)]dx \\ &= 0\end{aligned}$$

$$\begin{aligned}Var[\widetilde{W}(t_1)] &= \mathbb{E}[\widetilde{W}^2(t_1)] - \mathbb{E}^2[\widetilde{W}(t_1)] \\ &= 8 - 0\end{aligned}$$

therefore the the output at time t_1 is a Gaussian random variable: $\widetilde{W}(t_1) \sim \mathcal{N}(0, 8)$

((f)) Using Q-function it is easy to calculate:

$$P(\widetilde{W}(t_1) > 3) = Q(\frac{3}{\sqrt{8}})$$

Problem 7.3

Signal to Noise Ratio at the output can be written as:

$$SNR = \frac{P_Y}{P_{N_{output}}}$$

where power is the area under the curve of power spectral density for each targetted signal.

$$\begin{aligned} P_{N_{output}} &= \int_{-\infty}^{+\infty} \frac{N_0}{2}|H(f)|^2 df \\ &= 0.001 \times (25) \times (2000) + 2 \times 0.001 \times (4) \times (1000) \\ &= 58 \quad [\text{Watts}] \end{aligned}$$

$$\begin{aligned} P_Y &= \int_{-\infty}^{+\infty} \frac{6000}{3000} rect(\frac{f}{3000})|H(f)|^2 df \\ &= 50 \times (2000) + 2 \times 8 \times (500) \\ &= 108,000 \quad [\text{Watts}] \end{aligned}$$

Therefore to have the SNR in decibels, we have:

$$SNR_{dB} = 10 \log \frac{P_Y}{P_{N_{output}}} = 10 \log \frac{108000}{58} = 32.7$$

Problem 7.4

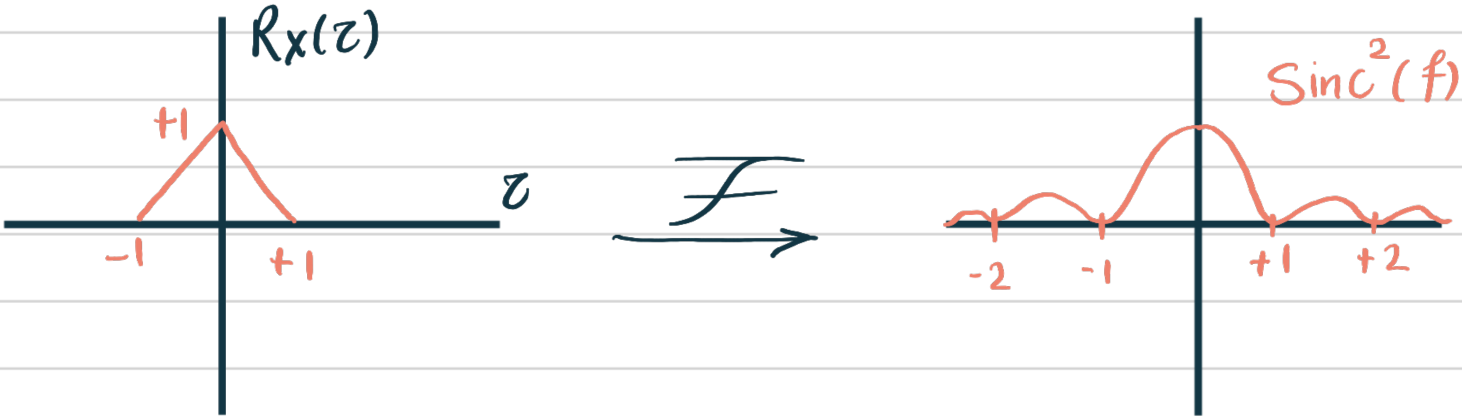
Before answering the problem, some few properties of autocorrelation function and its Fourier transform

For a real-valued stationary process:

- $R_X(\tau)$ is continuous.
- $R_X(\tau)$ is even.
- $R_X(\tau) \leq R_X(0)$.
- $R_X(\tau)$ is real-valued.

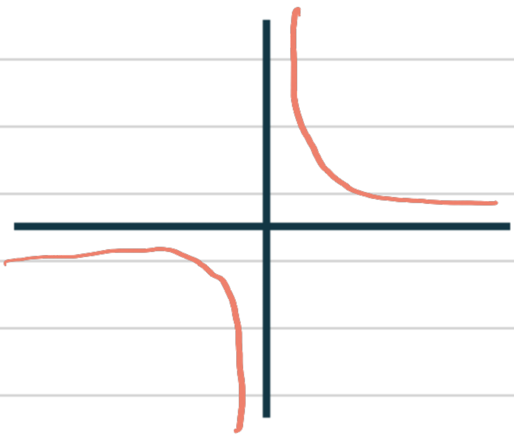
- $S_X(f)$ is real-valued.
- $S_X(f)$ is even.
- $S_X(f) \geq 0$

((a))

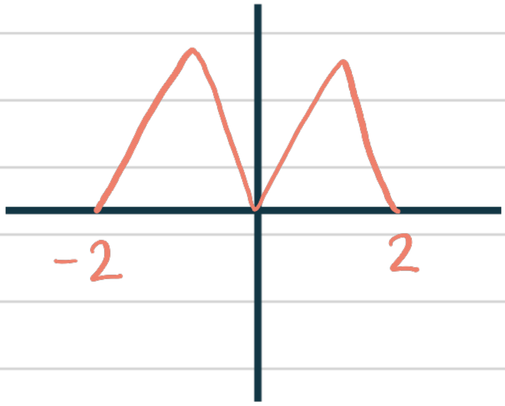


Legitimate Autocorrelation Function. ✓

1. continuous
2. $|R_x(\tau)| \leq R_x(0) = 1$
3. even function
4. real

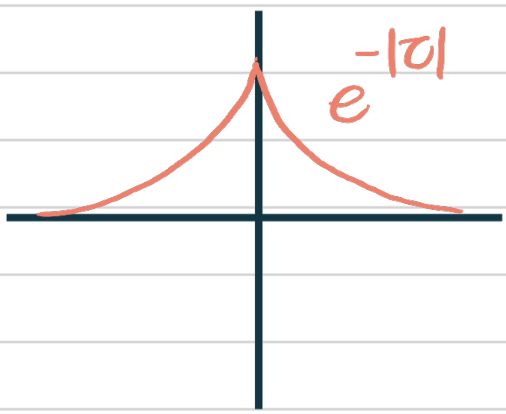


Not a legitimate autocorrelation function, since it's not continuous at $\tau = 0$. ✓

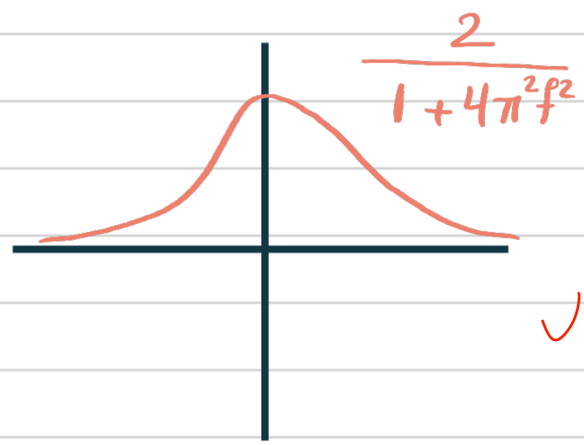


Not a legitimate autocorrelation function, since

$$R_X(\tau) \nleq R_X(0)$$



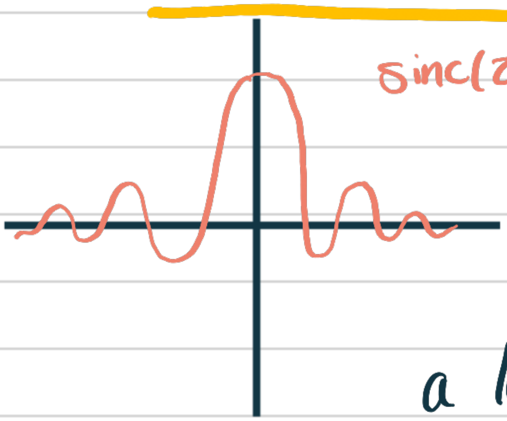
$\xrightarrow{\mathcal{F}}$



A legitimate autocorrelation function.

1. continuous
2. $R_X(\tau) < R_X(0) = 1$
3. even

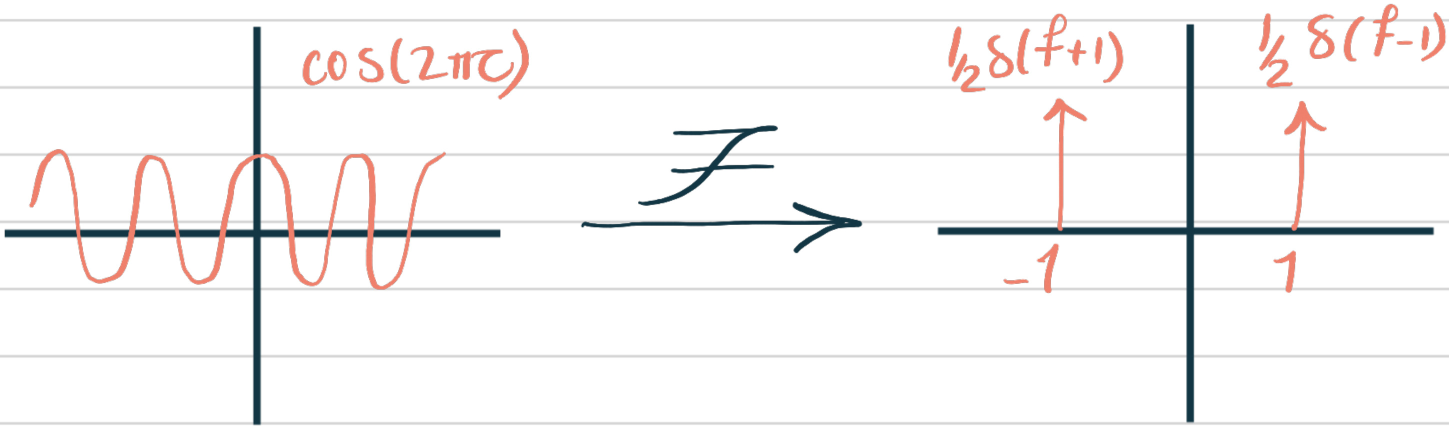
4. real



$\xrightarrow{\mathcal{F}}$

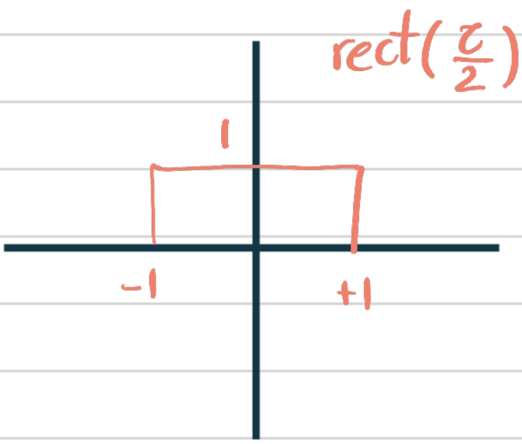
a legitimate autocorrelation function

1. continuous
2. $R_X(\tau) \leq R_\tau(0) = 1$
3. even
4. real



A legitimate autocorrelation function.

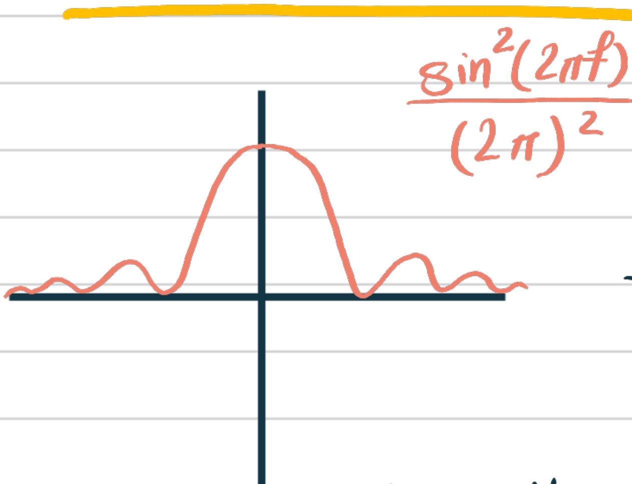
1. continuous
2. $R_X(\tau) \leq R_X(0) = 1$
3. even
4. real



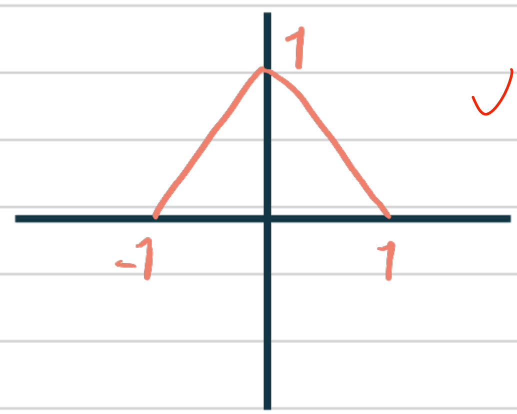
Not a legitimate autocorrelation function, since it is not continuous.

((b))

Not a legitimate PSD function, since it has negative values.

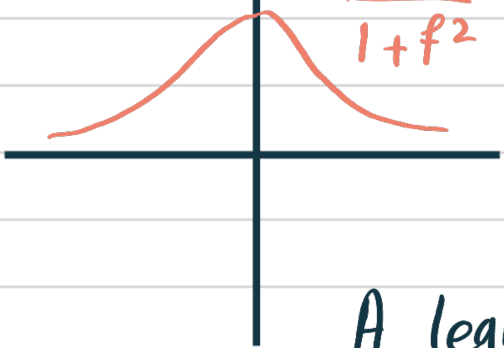


$$\xrightarrow{\mathcal{F}^{-1}}$$

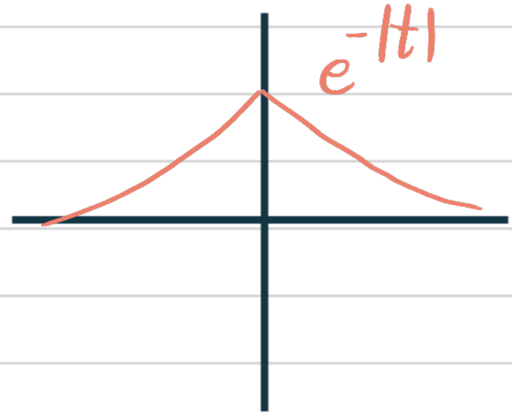


A legitimate PSD.

1. even
2. real-valued
3. positive-valued

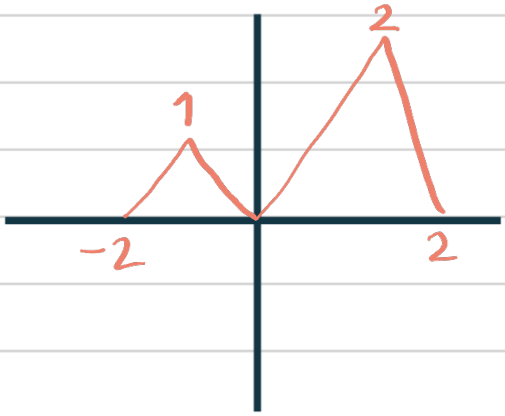


$\xrightarrow{\mathcal{F}^{-1}}$



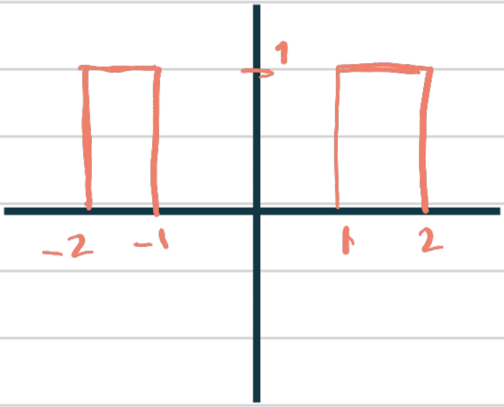
A legitimate PSD.

1. even
2. real-valued
3. positive-valued



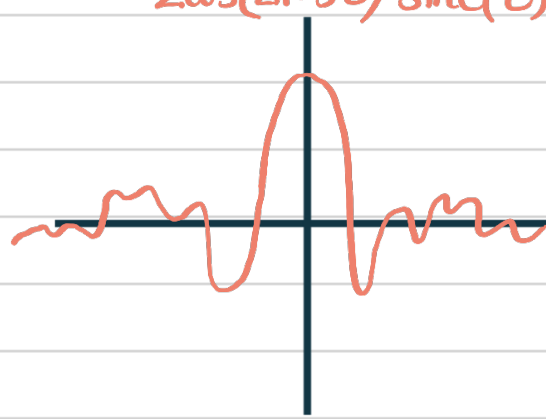
Not a legitimate PSD, since it's not even.

$\text{rect}(f-1.5) + \text{rect}(f+1.5)$



$\xrightarrow{\mathcal{F}^{-1}}$

$2\cos(2\pi 1.5\tau)\,\text{sinc}(\tau)$

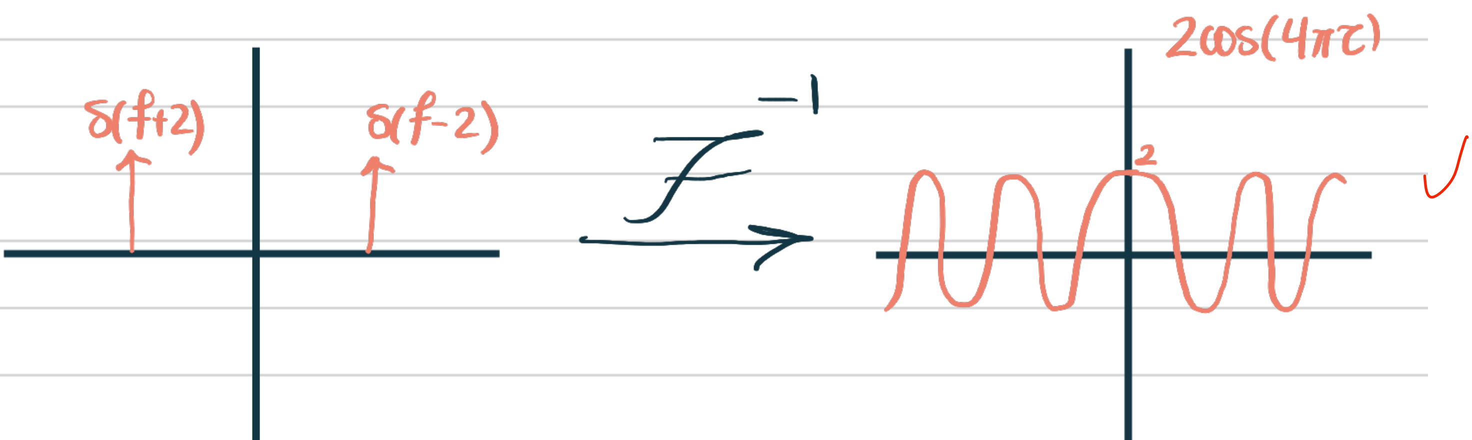


$$\xrightarrow{\mathcal{F}^{-1}} e^{j2\pi 1.5\tau}\,\text{sinc}(\tau) + e^{-j2\pi 1.5\tau}\,\text{sinc}(\tau) = 2\cos(2\pi 1.5\tau)\,\text{sinc}(\tau)$$

A legitimate PSD $\left\{\begin{array}{l} \text{1. even} \\ \text{2. real valued} \\ \text{3. positive valued} \end{array}\right.$

A legitimate PSD.

1. even
2. real-valued
3. positive-valued



A legitimate PSD.

1. even
2. real-valued
3. positive-valued