Homework 3

EE 513 — Stochastic Systems Theory

Name: Ali Zafari **Student Number:** 800350381 **Fall 2022**

Problem 3.1

((a)) **Mixed**. As the CDF is is not continuous (jump at $x = 0$) the random variable is not continuous. And since the CDF has not a staircase shape, the random variable is not discrete.

((b))

$$P(X < -\frac{1}{2}) = 0$$
$$P(X < 0) = 0$$
$$P(X \leq 0) = 0.25$$
$$P(\frac{1}{4} \leq X < 1) = P(X < 1) - P(X < \frac{1}{4}) = 0.5 - 0.3125 = 0.1875$$
$$P(\frac{1}{4} \leq X \leq 1) = P(X \leq 1) - P(X < \frac{1}{4}) = 1 - 0.3125 = 0.6875$$
$$P(X > \frac{1}{2}) = 1 - P(X \leq \frac{1}{2}) = 1 - 0.375 = 0.625$$
$$P(X \geq 5) = 1 - P(X < 5) = 1 - 1 = 0$$
$$P(X < 5) = P(X \leq 5) - P(X == 5) = 1 - 0 = 1$$

Problem 3.2

((a))

$$Y = g(X) = e^X \qquad \text{(monotonic function)}$$

For $y < 0$:

$$F_Y(y) = 0 \qquad \text{and} \qquad f_Y(y) = 0$$

For $y \geq 0$:

$$F_Y(y) = P(Y \leq y) = P(e^X \leq y) = P(x \leq \ln y) = F_X(\ln y)$$
$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}F_X(\ln y) = \frac{d \ln y}{dy}\frac{d}{dy}F_X(\ln y) = \frac{1}{y}F'_X(\ln y) = \frac{1}{y}f_X(\ln y)$$

((b))

$$X \sim \mathcal{N}(m, \sigma^2)$$

For $y \geq 0$:

$$PDF: \qquad f_Y(y) = \frac{1}{y} f_X(\ln y) = \frac{1}{y} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln y - m)^2}{2\sigma^2}}$$

Problem 3.3

((a))

$$\int_{-\infty}^{+\infty} f_X(x)dx = 1$$
$$\int_0^1 cx(1-x)dx = 1$$
$$c[\frac{x^2}{2} - \frac{x^3}{3}]\Big|_0^1 = 1$$
$$c = 6$$

((b))

$$W = \pi X^2 \qquad \text{(monotonic function on [0,1])}$$

$$x = \sqrt{\frac{w}{\pi}}$$

$$f_W(w) = \frac{d}{dw}(\sqrt{\frac{w}{\pi}}) f_X(\sqrt{\frac{w}{\pi}})$$

$$f_W(w) = \begin{cases} \frac{6}{2\sqrt{\pi w}} \sqrt{\frac{w}{\pi}}(1 - \sqrt{\frac{w}{\pi}}) = \frac{3}{\pi}(1 - \sqrt{\frac{w}{\pi}}) & 0 \leq w \leq \pi \\ 0 & o.w. \end{cases}$$

((c))

$$Z = \frac{4}{3}\pi X^3 \qquad \text{(monotonic function on [0,1])}$$

$$x = \sqrt[3]{\frac{3z}{4\pi}}$$

$$f_Z(z) = \frac{d}{dz}(\sqrt[3]{\frac{3z}{4\pi}})f_X(\sqrt[3]{\frac{3z}{4\pi}})$$

$$f_Z(z) = \begin{cases} 2\sqrt[3]{\frac{9}{16\pi^2 z}}(1 - \sqrt[3]{\frac{3z}{4\pi}}) & 0 \le z \le \frac{4}{3}\pi \\ 0 & o.w. \end{cases}$$

((d))

$$Y = X^n \qquad \text{(monotonic function on [0,1])}$$

$$x = \sqrt[n]{y}$$

$$f_Y(y) = \frac{d}{dy}(\sqrt[n]{y})f_X(\sqrt[n]{y})$$

$$f_Y(y) = \begin{cases} \frac{6}{n}y^{\frac{2-n}{n}}(1 - \sqrt[n]{y}) & 0 \le y \le 1 \\ 0 & o.w. \end{cases}$$

Problem 3.4

((a))

$$E[X] = (-1)\frac{1}{9} + (\frac{1}{2})\frac{4}{9} + (2)\frac{4}{9} = 1 \qquad \checkmark$$
$$E[\frac{1}{X}] = (-1)\frac{1}{9} + (2)\frac{4}{9} + (\frac{1}{2})\frac{4}{9} = 1 \qquad \checkmark$$

thus $E[\frac{1}{X}] = \frac{1}{E[X]}$, in this case.

((b))

$$E[X] = \int_1^2 x \times 1 dx = 1.5$$
$$E[\frac{1}{X}] = \int_1^2 \frac{1}{x} \times 1 dx = \ln 2$$

thus $E[\frac{1}{X}] \neq \frac{1}{E[X]}$, in general.

Problem 3.5

((a))

$$\begin{aligned}
M_X[jv] &= E[e^{jvX}] \\
&= \int_{-\infty}^{+\infty} \frac{\alpha}{2} e^{-\alpha|x|} e^{jvx} dx \\
&= \frac{\alpha}{2}[\int_{-\infty}^{0} e^{(\alpha+jv)x} dx + \int_{0}^{+\infty} e^{(-\alpha+jv)x} dx] \\
&= \frac{\alpha}{2}[\frac{1}{\alpha+jv} e^{(\alpha+jv)x}\Big|_{-\infty}^{0} + \frac{1}{-\alpha+jv} e^{(-\alpha+jv)x}\Big|_{0}^{+\infty}] \\
&= \frac{\alpha}{2}[\frac{1}{\alpha+jv} + \frac{-1}{-\alpha+jv}] \\
&= \frac{\alpha}{2}[\frac{1}{\alpha+jv} + \frac{-1}{-\alpha+jv}] \\
&= \frac{\alpha^2}{\alpha^2-(jv)^2}
\end{aligned}$$

((b))

$$\begin{aligned}
E[X] &= \frac{d}{djv} M_X[jv]\Big|_{v=0} \\
&= \frac{d}{djv}(\frac{\alpha^2}{\alpha^2-(jv)^2})\Big|_{v=0} \\
&= \frac{2jv\alpha^2}{(\alpha^2-(jv)^2)^2}\Big|_{v=0} \\
&= 0
\end{aligned}$$

$$
\begin{aligned}
E[X^2] &= \frac{d^2}{d(jv)^2} M_X[jv]\Big|_{v=0} \\
&= \frac{d}{djv}\left(\frac{2jv\alpha^2}{(\alpha^2-(jv)^2)^2}\right)\Big|_{v=0} \\
&= \frac{2\alpha^2(\alpha^2-(jv)^2)^2 - 2(\alpha^2-(jv)^2)(-2jv(2jv\alpha^2))}{(\alpha^2-(jv)^2)^4}\Bigg|_{v=0} \\
&= \frac{2\alpha^6}{\alpha^8} \\
&= \frac{2}{\alpha^2} \quad \longrightarrow \quad Var[X] = E[X^2] - E[X]^2 = \frac{2}{\alpha^2}
\end{aligned}
$$

Problem 3.6

((a)) To have quantized values of X, i.e. $q(X)$, have same probability we should have the below equality:

$$P(X \leq -a) = P(-a < X \leq 0) = P(0 < X \leq a) = P(X > a) = \frac{1}{4}$$

so we can write:

$$
\begin{aligned}
P(X < -a) &= \frac{1}{4} \\
\int_{-\infty}^{-a} \frac{1}{2} e^{-|x|} dx &= \frac{1}{4} \\
\frac{1}{2} e^{x}\Big|_{-\infty}^{-a} &= \frac{1}{4} \\
e^{-a} &= \frac{1}{2} \\
a &= \ln 2 \\
a &= 0.6931
\end{aligned}
$$

((b))

$$x_1 = \underset{x_1}{\operatorname{argmin}} \int_0^a (x - x_1)^2 \frac{1}{2} e^{-|x|} dx$$

To minimize, we calculate the derivative w.r.t x_1 and find the value which makes it zero:

$$\frac{d}{dx_1}\int_0^a (x-x_1)^2\frac{1}{2}e^{-x}dx = 0$$

$$\int_0^a -2(x-x_1)\frac{1}{2}e^{-x}dx = 0$$

$$\int_0^a xe^{-x}dx - \int_0^a x_1e^{-x}dx = 0$$

$$-e^{-x}(x+1)\Big|_0^a - x_1(-e^{-x})\Big|_0^a = 0$$

$$x_1 = \frac{1-e^{-a}(a+1)}{1-e^{-a}}$$

$$a = \ln 2 \quad\longrightarrow\quad x_1 = \frac{1-(1/2)(\ln 2+1)}{1/2} \quad\longrightarrow\quad x_1 = 0.3069$$

((c)) First, we need to know the values of $q(X)$ for interval of $x \in (a, +\infty)$. We use the result of previous part:

$$-e^{-x}(x+1)\Big|_a^{+\infty} - x_2(-e^{-x})\Big|_a^{+\infty} = 0$$

$$x_2 = \frac{e^{-a}(a+1)}{e^{-a}}$$

$$a = \ln 2 \quad\longrightarrow\quad x_2 = \frac{(1/2)(\ln 2+1)}{1/2} \quad\longrightarrow\quad x_2 = 1.6931$$

The distribution of $q(X)$ is symmetric, so to summarize:

$$q(x) = \begin{cases} -1.6931 & x \le -\ln 2 \\ -0.3069 & -\ln 2 < x \le 0 \\ 0.3069 & 0 < x \le \ln 2 \\ 1.6931 & x > \ln 2 \end{cases}$$

Let's now evaluate the MSE:

$$E[(X-q(X))^2] = E[X^2] - 2E[Xq(X)] + E[q(X)^2]$$

$E[X^2] = Var[X] + E[X]^2 = 2 + 0 = 2$

$E[Xq(X)] = -1.6931(-0.4233) - 0.3069(-0.0767) + 0.3069(0.0767) + 1.6931(0.4233) = 1.4803$

$E[q(X)^2] = \frac{1}{4}(1.6931^2 + 0.3069^2 + 0.3069^2 + 1.6931^2) = 1.4803$

$$\textbf{MSE}: \qquad E[(X-q(X))^2] = 2 - 2(1.4803) + 1.4803 = 0.5197$$