EE 513: Stochastic Systems Theory
Fall 2022 Schmid

Final (take home)

Distributed on December 15, 2022 (at 9 am)
Due is on December 16, 2022 (at 9 am)

Do your own work. The test is take home and thus open book and notes.

Name:

Pledge: “I have neither given nor received unauthorized aid on this examination.”

Signed:

Problem	Points	Points received
1	13	
2	15	
3	12	
4	15	
5	15	
6	15	
7	15	
Bonus	10	
Total		

Problem 1 (13 points): Assume n i.i.d. Poisson distributed random variables $X_1, \ldots, X_n$ with parameter λ.

(a) Find the maximum likelihood (ML) estimate of λ.
(b) Compute the bias and variance of the estimate.

Problem 2 (15 points): Suppose R is a random variable that, under hypothesis H_0, has pdf

$$p_R(r \mid H_0) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{r^2}{2}\right)$$

and, under hypothesis H_1, has pdf

$$p_R(r \mid H_1) = \begin{cases} 1/5, & \text{if } r \in [0,5] \\ 0, & \text{otherwise} \end{cases}$$

Find the Bayesian rule and the total probability of error for testing H_0 against H_1 when the prior probabilities are set to $\pi_0 = 3/4$ and $\pi_1 = 1/4$.

Problem 3 (12 points): Let $X_1, \ldots, X_n, \ldots$ be a sequence of independent identically distributed random variables. Each X_i is a ternary random variable:

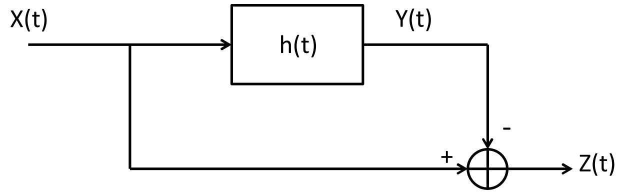
$$X_i = \begin{cases} 1 & \textit{with prob.} \ 1/3 \\ 0 & \textit{with prob.} \ 1/3 \\ -1 & \textit{with prob.} \ 1/3 \end{cases}$$

State and prove the weak law of large numbers for a sequence of sample means:

$$M_n = \frac{1}{n} \sum_{i=1}^{n} X_i \, .$$

Problem 4 (15 points): The number of messages arriving at a multiplexer is a Poisson random variable with mean 15 messages/second. Use the central limit theorem to estimate the probability that more than 950 messages arrive in <u>one minute</u>.

Problem 5 (15 points): A linear time invariant system (LTI) system is shown below.
Let $Y(t) = X(t) * h(t) = \int_{-\infty}^{+\infty} X(\tau) h(t - \tau) d\tau$ and $Z(t) = X(t) - Y(t)$.



$H(f)$ is the frequency response of the filter with the impulse response $h(t)$ is given as

$$H(f) = \frac{1}{1+f^2}$$

The random process $X(t)$ is given as $X(t) = A\cos(2\pi f_c t + \Theta)$, where A and Θ are independent random variables. A is a binary random variable with the probability mass function

$$A = \begin{cases} 4 & with\ probability\ 1/2 \\ 0 & with\ probability\ 1/2 \end{cases}$$ and Θ is uniform over $(0, 2\pi)$.

(a) Find the mean and autocovariance function of $X(t)$.
(b) Prove that $X(t)$ is a wide sense stationary process (WSS).
(c) Find the power spectral density on the output of the system $S_Z(f)$. Assume that $f_c = 1$.
(d) Find $E[Z^2(t)]$.

Hint: Find the frequency response (transfer function) of the overall LTI system, then use it to answer questions (c) and (d).

Problem 6 (15 points): Suppose X and Y have joint pdf $f_{X,Y}(x,y)$ and $W = 2X - Y$ and $Z = X + 2Y$.

(a) Find the pdf of Z. Express it in terms of $f_{X,Y}(x,y)$.
(b) Express the joint pdf of W and Z in terms of $f_{X,Y}(x,y)$.

Problem 7 (15 points): Suppose X and Y are jointly Gaussian random variables such that X is $N(0,16)$, Y is $N(0,9)$, and the correlation coefficient is denoted by ρ. For two jointly Gaussian random variables, they are independent if and only if they are uncorrelated. The solution to the questions below may depend on ρ and may fail to exist for some values of ρ.

(a) For what value(s) of a is X independent of $X + aY$?
(b) For what value(s) of d is $2X + dY$ independent of $(X - dY)^2$? If d exists, providing one value of d is sufficient.

Bonus Problem (10 points)

(a) Develop the tightest right Chernoff bound on the probability $P(X > E[X] + \varepsilon)$, where $\varepsilon > 0$. Assume that the probability density function $f_X(x)$ of X is known.
(b) Demonstrate your result on a Gaussian random variable with mean 5 and variance 4 and compare it with the exact value of the probability.
(c) Plot the bound and the exact probability as a function of ε.