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Good work.

Homework 1

EE 513 — Stochastic Systems Theory

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Problem 1.1

a. Each link in the network can have 2 states, exist or not. The sample space of the network with 4 links can be written as:

#	1	2	3	4	5	6	7
S	None	a	b	c	d	ab	ac
P	$(1-p)^4$	$p(1-p)^3$	$p(1-p)^3$	$p(1-p)^3$	$p(1-p)^3$	$p^2(1-p)^2$	$p^2(1-p)^2$

8	9	10	11	12	13	14
ad	bc	bd	cd	abc	abd	acd
$p^2(1-p)^2$	$p^2(1-p)^2$	$p^2(1-p)^2$	$p^2(1-p)^2$	$p^3(1-p)^1$	$p^3(1-p)^1$	$p^3(1-p)^1$

15	16
bcd	abcd
$p^3(1-p)^1$	p^4

The sigma-algebra $\mathcal{F}$ includes all the possible subsets of the sample space with their corresponding probabilities. The total number of elements covered in $\mathcal{F}$ (including the empty set, i.e., $\varnothing$) will be 2^{16}.

b. We would consider all the outcomes which result in the event that nodes 1 and 4 have a connection. To have established the connection, link a must exist, then one links d and cb must at least exist.

$$A = \{ad, abc, abd, acd, abcd\}$$

outcomes in A are mutually exclusive:

$$\begin{aligned} P(A) &= P(ad) + P(abc) + P(abd) + P(acd) + P(abcd) \\ &= p^2(1-p)^2 + p^3(1-p)^1 + p^3(1-p)^1 + p^3(1-p)^1 + p^4 \\ &= p^2 - 2p^3 + p^4 + 3p^3 - 3p^4 + p^4 \\ &= p^2 + p^3 - p^4 \end{aligned}$$

c. To have connection between nodes 2 and 3, either links c or bd must exist.

$$B = \{c, ac, bc, cd, bd, abc, abd, acd, bcd, abcd\}$$

outcomes in B are mutually exclusive:

$$\begin{aligned}P(B) &= P(c) + P(ac) + P(bc) + P(cd) + P(bd) \\ &\quad + P(abc) + P(abd) + P(acd) + P(bcd) + P(abcd) \\ &= p(1-p)^3 + 4 \times p^2(1-p)^2 + 4 \times p^3(1-p)^1 + p^4 \\ &= p + p^2 - p^3\end{aligned}$$

d.

$$A \cap B = \{abc, acd, abd, abcd\}$$

outcomes in $A \cap B$ are mutually exclusive:

$$\begin{aligned}P(A \cap B) &= P(abc) + P(acd) + P(abd) + P(abcd) \\ &= 3 \times p^3(1-p)^1 + p^4 \\ &= 3p^3 - 2p^4\end{aligned}$$

To check if two events A and B are independent or not, we calculate $P(A) \times P(B)$:

$$\begin{aligned}P(A)P(B) &= (p^2 + p^3 - p^4)(p + p^2 - p^3) \\ &= p^3 + p^4 - p^5 + p^4 + p^5 - p^6 - p^5 - p^6 + p^7 \\ &= p^3 + 2p^4 - p^5 - 2p^6 + p^7 \\ &\neq P(A \cap B)\end{aligned}$$

Therefore A and B are not independent.

e. Availability of c partitions the sample space, i.e., $S = C \cup \bar{C}$ therefore:

$$\begin{aligned}P(A) &= P(A \cap S) \\ &= P(A \cap (C \cup \bar{C})) \\ &= P((A \cap C) \cup (A \cap \bar{C})) && \text{[mutually exclusive]} \\ &= P(A \cap C) + P(A \cap \bar{C}) && \text{[Bayes rule]} \\ &= P(C)P(A|C) + P(\bar{C})P(A|\bar{C}) \\ &= pP(A|C) + (1-p)P(A|\bar{C}) \\ &= p(2p^2(1-p) + p^3) + (1-p)(p^2(1-p) + p^3) \\ &= 2p^3 - p^4 + p^2 - p^3 \\ &= p^2 + p^3 - p^4\end{aligned}$$

f. By moving the link c in-between nodes 1 and 2, the event A_{new} (nodes 1 and 4 can communicate) have outcomes as follows:

$$A_{new} = \{ad, cd, abd, acd, cbd, abcd\}$$

outcomes in A_{new} are mutually exclusive:

$$\begin{aligned}
P(A_{new}) &= P(ad) + P(cd) + P(abd) + P(acd) + P(cbd) + P(abcd) \\
&= 2 \times p^2(1-p)^2 + 3 \times p^3(1-p)^1 + p^4 \\
&= 2p^2 - 4p^3 + 2p^4 + 3p^3 - 3p^4 + p^4 \\
&= 2p^2 - p^3
\end{aligned}$$

To verify $P(A_{new}) >^{?} P(A)$, we calculate their difference:

$$\begin{aligned}
P(A_{new}) - P(A) &= 2p^2 - p^3 - (p^2 + p^3 - p^4) \\
&= p^2 - 2p^3 + p^4 \\
&= p^2(1 - 2p + p^2) \\
&= p^2(1-p)^2
\end{aligned}$$

$p^2(1-p)^2$ is always greater than zero, therefore $P(A_{new}) > P(A)$.

Problem 1.2

We assume that x divides the unit length stick at random ($x \in [0,1]$), then y is chosen to be less than x ($0 < y < x$) as depicted in Figure 1. This assumption makes the sample space as shown in Figure 2 (solid lines).

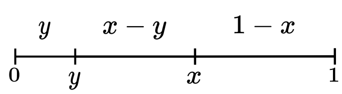

Figure 1: stick of unit length

For the 3 dividened lengths to create a triangle, all 3 triangle inequalities must be satisfied simultaneously:

i.

$$y + x - y > 1 - x \quad \longrightarrow \quad x > 0.5$$

ii.

$$y + 1 - x > x - y \quad \longrightarrow \quad 2y - 2x + 1 > 0$$

iii.

$$x - y + 1 - x > y \quad \longrightarrow \quad y < 0.5$$

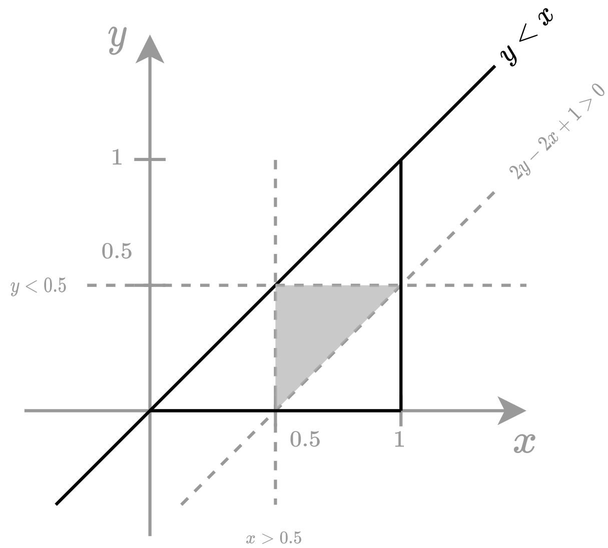

Figure 2: Sample space (solid lines) and target event (shaded area)

So we can calculate the probability based on the ratio of area of the event and area of the sample space:

$$P(triangle) = \frac{\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}}{\frac{1}{2} \times 1 \times 1} = \frac{1}{4} = 0.25$$

Problem 1.3

Probability of intersection of events A_n and B_m can be written in terms of conditional probabilities using the product rule:

$$\begin{aligned} P(A_n \cap B_m) &= P(A_n)P(B_m|A_n) \\ &= P(B_m)P(A_n|B_m) \end{aligned}$$

so by re-arranging, we have:

$$P(B_m|A_n) = \frac{P(A_n|B_m)P(B_m)}{P(A_n)} \tag{$*$}$$

As B_i's partition the sample space, we have $\bigcup_{i=1}^{M} B_i = S$. So we can conclude:

$$\begin{aligned}
P(A_n) &= P(A_n \cap S) \\
&= P(A_n \cap (\bigcup_{i=1}^{M} B_i)) \\
&= P(\bigcup_{i=1}^{M} (A_n \cap B_i)) \qquad \text{[mutually exclusive]} \\
&= \sum_{i=1}^{M} P(A_n \cap B_i) \qquad \text{[conditional probability]} \\
&= \sum_{i=1}^{M} P(A_n|B_i)P(B_i) \qquad (\dagger)
\end{aligned}$$

Now we only need to replace $(\dagger)$ in equation $(*)$ to have:

$$P(B_m|A_n) = \frac{P(A_n|B_m)P(B_m)}{\sum_{i=1}^{M} P(A_n|B_i)P(B_i)}$$

Problem 1.4

a. With the law of total probability:

$$\begin{aligned}
P(out = 0) &= P(out = 0 \cap in = 0) + P(out = 0 \cap in = 1) \\
&= P(out = 0|in = 0)P(in = 0) + P(out = 0|in = 1)P(in = 1) \\
&= (1 - \epsilon) \times 0.4 + 0 \times 0.6 \\
&= 0.4(1 - \epsilon)
\end{aligned}$$

$$\begin{aligned}
P(out = 1) &= P(out = 1 \cap in = 0) + P(out = 1 \cap in = 1) \\
&= P(out = 1|in = 0)P(in = 0) + P(out = 1|in = 1)P(in = 1) \\
&= 0 \times 0.4 + (1 - \epsilon) \times 0.6 \\
&= 0.6(1 - \epsilon)
\end{aligned}$$

$$\begin{aligned}
P(out = erasure) &= P(out = erasure \cap in = 0) + P(out = erasure \cap in = 1) \\
&= P(out = erasure|in = 0)P(in = 0) + P(out = erasure|in = 1)P(in = 1) \\
&= \epsilon \times 0.4 + \epsilon \times 0.6 \\
&= \epsilon
\end{aligned}$$

b. Probability of input being 0 given output is 1, utilizing Bayes rule:

$$
\begin{aligned}
P(in = 0|out = 1) &= \frac{P(out = 1|in = 0)P(in = 0)}{P(out = 1)} \\
&= \frac{0 \times 0.4}{0.6(1-\epsilon)} = 0
\end{aligned}
$$

Probability of input being 1 given output is 1, utilizing Bayes rule:

$$
\begin{aligned}
P(in = 1|out = 1) &= \frac{P(out = 1|in = 1)P(in = 1)}{P(out = 1)} \\
&= \frac{(1-\epsilon) \times 0.6}{0.6(1-\epsilon)} = 1
\end{aligned}
$$

Problem 1.5

a. Sample space of received signal Y from the transmitted signal X written as tuples of (x, y):

$$S_{(x,y)} = \{(-2,-2), (-2,-1), (-2,0), (+2,0), (+2,+1), (+2,+2)\}$$

b.

$$\{\text{X is definitely } +2\}_{(x,y)} = \{(+2,+1), (+2,+2)\}$$

c. To have outcome $Y = 0$, two different situations can be hypothesized to cover the corresponding outcomes in the sample space, i.e., $\{(-2,0), (+2,0)\}$

   i. Transmitted signal was -2 and the numbers of head in 2 tosses of the coin is 2.

   ii. Transmitted signal was $+2$ and the numbers of head in 2 tosses of the coin is 2.

Problem 1.6

a. The probability distribution is a binomial with $N = 100$ and $p = 10^{-2}$:

$$
\begin{aligned}
P(\text{block accepted}) &= P(\text{2 or fewer errors in the received block}) \\
&= P(\text{no error}) + P(\text{1 bit error}) + P(\text{2 bits error}) \\
&= \binom{100}{0}(0.01)^0(0.99)^{100} + \binom{100}{1}(0.01)^1(0.99)^{99} + \binom{100}{2}(0.01)^2(0.99)^{98} \\
&= 0.9206
\end{aligned}
$$

b. The re-transmit probability is:

$$\begin{aligned} P(\text{Re-transmit}) &= P(\text{more than 2 errors in the received block}) \\ &= 1 - P(\text{2 or fewer errors in the received block}) \\ &= 1 - 0.9206 = 0.0794 \end{aligned}$$

To have M re-transmissions, the block should be accepted at the $(M+1)^{th}$ transmission, so the probability will be:

$$\begin{aligned} P(\text{M-Retransmissions}) &= P(\text{Re-transmit})^M \times P(\text{block accepted at } (M+1)^{th}) \\ &= (0.0794)^M (0.9206) \end{aligned}$$
