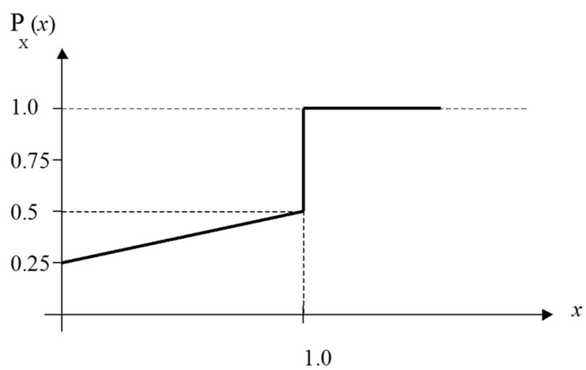
EE 513: Stochastic Systems Theory
Fall 2022 Schmid

Homework Assignment 3
Distributed: Friday, September 23, 2022
Due: Wednesday, October 5, 2022

Reference Material:
1. class notes
2. Leon-Garcia, Chapters 1 – 4
3. J. A. Gubner, Probability and Random Processes for Electrical and Computer Engineers, Cambridge University Press, 2006, Ch. 1 – 4, Sec. 5.1 – 5.5.

Three of the following problems will be selected at random and graded.

Problem 3.1 The cumulative distribution function $P_X(x) = P(X \leq x)$ for a random variable X is shown below. Note that the function $P_X(x)$ is zero for $x < 0$.



(a) What type of variable (discrete, continuous, or mixed) is X?
(b) Evaluate the following probabilities:

$P\left(X < -\frac{1}{2}\right)$ $\qquad$ $P(X < 0)$ $\qquad$ $P(X \leq 0)$

$P\left(\frac{1}{4} \leq X < 1\right)$ $\qquad$ $P\left(\frac{1}{4} \leq X \leq 1\right)$ $\qquad$ $P\left(X > \frac{1}{2}\right)$

$P(X \geq 5)$ $\qquad$ $P(X < 5)$

Problem 3.2 Let $Y = e^X$, where X is a random variable.
(a) Determine the cumulative distribution and probability density functions of Y.
(b) Determine the probability density function of Y if X is a Gaussian random variable $N(m, \sigma^2)$ with mean m and variance σ^2:

$$p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-m)^2/(2\sigma^2)}.$$

Comment: The random variable Y in 3.2 (b) is said to have a lognormal distribution. The amplitude of light that propagates through clear-air turbulence is often modeled as a lognormal random variable. Such light is seen when looking over a heated roadway or through the atmosphere at a star.

Problem 3.3 Let the radius X have the following probability density function (pdf)

$$f_X(x) = \begin{cases} cx(1-x), & 0 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

(a) Find c
(b) Find the pdf of the area covered by a disk with radius X
(c) Find the pdf of the volume of a sphere with radius X
(d) Find the pdf of $Y = X^n$.

Problem 3.4 This problem examines the relationship between the expected value of a random variable, $E[X]$, and the expected value of the reciprocal of that random variable, $E\left[\frac{1}{X}\right]$.

(a) Suppose that

$$X = \begin{cases} -1, & \text{with probability } \dfrac{1}{9} \\ \dfrac{1}{2}, & \text{with probability } \dfrac{4}{9} \\ 2, & \text{with probability } \dfrac{4}{9} \end{cases}$$

Show that the expected value of the reciprocal of X equals the reciprocal of the expected value of X; that is, show that $E\left[\frac{1}{X}\right] = \frac{1}{E[X]}$.

(b) Suppose that X is uniformly distributed on the interval (1,2). Show that $E(1/X) \neq 1/E(X)$.

This problem illustrates that, in general, $E[g(X)]$ *does not equal* $g(E[X])$.

Problem 3.5 Let X be a Laplacian-distributed random variable; that is,

$$f_X(x) = \frac{\alpha}{2} exp(-\alpha|x|), \quad -\infty < x < +\infty$$

(a) Determine the characteristic function $M_X(jv) = E\left[e^{jvX}\right]$ of X.
(b) Determine the mean and the variance of X by applying <u>the moment property of the characteristic function</u>.

Problem 3.6 The sample X of a speech signal is a Laplacian random variable with parameter $\alpha = 1$. Suppose that X is quantized by a nonuniform quantizer consisting of four intervals: $(-\infty, -a]$, $(-a, 0]$, $(0, a]$, and $(a, +\infty)$.

(a) Find the value of a so that X is equally likely to fall in each of four intervals.
(b) Find the representation point $x_1 = q(X)$ for X in $(0, a]$ that minimizes the mean-square error, that is,

$$\int_0^a (x - x_1)^2 f_X(x)dx \quad \text{is minimized.}$$

Hint: Differentiate the above expression with respect to x_1. Find representation points for the other intervals.

(c) Evaluate the mean-square error of the quantizer: $E[(X - q(X))^2]$.