EE 513: Stochastic Systems Theory
Fall 2022 Schmid

Homework Assignment 2
Distributed: Friday, September 9, 2022
Due: Wednesday, September 19, 2022 (11 days)

Reference Material:

1. class notes
2. Leon-Garcia, Ch. 1, Ch. 2, Ch. 4
3. J. A. Gubner, Probability and Random Processes for Electrical and Computer Engineers, Cambridge University Press, 2006, Ch. 1, Ch. 2, and Ch. 4.

Three of the following problems will be selected at random and will be graded.

Problem 2.1 (based on Problem 2.127, Leon-Garcia)
In order for a circuit board to work, seven identical chips must be in working order. To improve reliability, an additional chip is included in the board, and the design allows it to replace any of the seven other chips when they fail.

(a) Find the probability p_b that the board is working in terms of the probability p that an individual chip is working.
(b) Suppose that n circuit boards are operatied in parallel, and that we require 99.9% probability that at least one board is working. How many boards are needed?

Problem 2.2

a. The continuous random variable X has a *Laplacian* probability density function $f_X(x) = ae^{-b|x|}$, for $-\infty < x < +\infty$, where a and b are constants. What is the relationship between a and b? Determine and plot the cumulative probability distribution function (cdf) of X for $a = 1/2$.
b. Repeat part (a) if X has the *Cauchy* probaility density

$$f_X(x) = \frac{a}{b^2+x^2} .$$

Problem 2.3 (based on Problem 4.63 on page 221, Leon-Garcia'06)
Let X be Gaussian random variable with $m = 5$ and $\sigma^2 = 16$.

(a) Using tables of the Q-function find the following probabilities $P[X > 4]$, $P[X \geq 7]$, $P[2 < X < 7]$, $P[6 \leq X \leq 8]$.
(b) Find a, if $P[X < a] = 0.8869$.
(c) Find c, if $P[13 < X \leq c] = 0.0123$.
(d) Show that the Q-function satisfies $Q(-x) = 1 - Q(x)$.

Problem 2.4 (based on Problem 3.85 from Leon-Garcia'08)
Use Octave, Matlab or C to generate 200 pairs of numbers (X_i, Y_i), in which the components are independent, and each component is uniform in the set $\{1,2, \dots ,9,10\}$.

(a) Plot the relative frequencies of the X and Y outcomes.
(b) Plot the relative frequencies of the random variable $Z = X + Y$. Can you discern the pmf of Z?
(c) Plot the relative frequencies of $W = XY$. Can you discern the pmf of W?
(d) Plot the relative frequencies of $V = X/Y$. Is the pmf discernable?

Problem 2.5 (based on Problem 3.86 from Leon-Garcia'08)
Use Octave, Matlab or C to generate 200 points of numbers (X_i, Y_i), in which the components are independent, and where X_i are binomial with parameters $n = 8,\ p = 0.5$ and Y_i are binomial with parameters $n = 4,\ p = 0.5$.
(a) Plot the relative frequencies of X and Y outcomes.
(b) Plot the relative frequencies of the random variable $Z = X + Y$. Does this correspond to the pmf you expect? Explain.

Problem 2.6
Generate 200 pairs of points (X_i, Y_i), in which the components are independent Gaussian with mean zero and variance $\sigma^2 = 1$.
(a) Plot the relative frequencies of X and Y outcomes.
(b) Plot the relative frequencies of the random variable $Z = X/Y$. Does this correspond to the pdf you expect?

Problem 2.7
Generate 200 points of numbers X_i in which the components are independent uniform on the interval $(0,1)$. Plot the relative frequencies of the random variable $Z = -\ln(X)$. Can you discern the pdf of Z?

Problem 2.8
Use the 200 points X_i generated in Problem 2.5 to find the sample mean and sample variance and compare them with the analytically derived values of the mean and variance of the Binomial $n = 8,\ p = 0.5$ random variable.
(a) Find the mean and variance of a Binomial random variable with parameters $n = 8,\ p = 0.5$.
(b) Evaluate the sample mean $M_{200} = \frac{1}{200}\sum_{i=1}^{200} X_i$ and sample variance $V_{200} = \frac{1}{199}\sum_{i=1}^{200}(X_i - M_{200})^2$ from the 200 generated points.

Problem 2.9
Use the 200 points X_i generated in Problem 2.6. Apply the following transformation to each generated point $Y_i = 2X_i + 5$ to generate 200 points drawn from a Gaussian distribution with mean 5 and variance 4. Find the sample mean and sample variance of the new Gaussian and compare them with the true values of the mean and variance.

To find the sample mean and the sample variance use the following equations

$$M_{200} = \frac{1}{200}\sum_{i=1}^{200} Y_i \text{ and } V_{200} = \frac{1}{199}\sum_{i=1}^{200}(Y_i - M_{200})^2.$$