EE 513: Stochastic Systems Theory

Fall 2022 Schmid

Homework Assignment 7

Distributed: Wednesday, November 30, 2022

Due: Wednesday, December 7, 2022

This is the last homework

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Reference Material:

1. class notes
2. Leon-Garcia'08, Ch. 9: Random Processes (Sec. 9.1-9.6), and Ch. 10: Analysis and Processing of Random Signals (Sec.10.1-10.4)

The following exercises are for self-study and will not be graded. The solutions will be provided.

Exercise 7.1 (Leon-Garcia'94, problem 57, page 396)
Let $\{X(t): -\infty < t < \infty\}$ and $\{Y(t): -\infty < t < \infty\}$ be independent, wide-sense stationary random processes with zero means and the same covariance function $K(\tau)$. Let $\{Z(t): -\infty < t < \infty\}$ be defined by:

$$Z(t) = aX(t) + bY(t),$$

where a and b are real numbers.

(a) Determine whether or not $Z(\cdot)$ is also wide-sense stationary.

(b) Determine the probability density function of $Z(t)$ if, in addition, $X(\cdot)$ and $Y(\cdot)$ are jointly Gaussian random processes.

Exercise 7.2 (Leon-Garcia'94, problem 19, page 452) Let $Y(t)$ be a T-second moving time-average of a WSS random process $X(t)$, defined by:

$$Y(t) = \frac{1}{T}\int_{t-T}^{t} X(u)du .$$

Determine the power-density spectrum $S_Y(f)$ of $Y(\cdot)$ in terms of the power-density spectrum $S_X(f)$ of $X(\cdot)$.

Exercise 7.3 (Leon-Garcia'94, problem 19, page 391) Let $X(t)$ be a zero-mean Gaussian random process with covariance function $K(t,u)$. If $X(t)$ is the input to a "square-law detector," then the output is given by

$$Y(t) = X^2(t) .$$

Determine the mean and the covariance functions of the random process $Y(t)$.

Two of the following problems will be selected at random and graded.

Problem 7.1: Given a function $X(t) = 2\sin(2\pi(1000)t + \Theta)$, where Θ is uniformly distributed between $-\pi$ and π

$$f_\Theta(\theta) = \begin{cases} 1/(2\pi) & \text{for } -\pi < \theta < \pi \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the mean of $X(t)$
(b) Find the autocorrelation of $X(t)$
(c) Is $X(t)$ wide sense stationary (WSS)? If so, find the power spectral density (PSD) of $X(t)$.

Problem 7.2: Additive white Gaussian noise $W(t)$ with two-sided spectral density $N_0/2 = 0.1$ W/Hz is input into a linear time invariant filter with frequency response

$$H(f) = \begin{cases} 2, & \text{for } \; -10 < f < 10 \\ 0, & \text{otherwise} \end{cases}.$$

The output is $\widetilde{W}(t) = W(t) * h(t)$, where $h(t)$ is the filter response and "*" stands for convolution.

(a) Determine the autocorrelation function $R_W(\tau)$ of $W(t)$.
(b) Determine $m_{\widetilde{W}}(t) = E[\widetilde{W}(t)]$.
(c) Determine the power spectral density at the filter output $S_{\widetilde{W}}(f)$.
(d) Determine $E[(\widetilde{W}(t))^2]$.
(e) Give the pdf of the output sampled at time t_1.
(f) Give an expression for $P[\widetilde{W}(t_1) > 3]$.

Problem 7.3: The input to a linear time-invariant filter is $X(t) = S(t) + W(t)$, where $W(t)$ is additive white Gaussian noise with two-sided spectral density $N_0/2 = 0.001$ W/Hz, and $S(t)$ is a WSS process with autocorrelation:

$$R_S(\tau) = 6000\,\mathrm{sinc}(3000\tau).$$

The filter has frequency response:

$$H(f) = \begin{cases} 5 & |f| < 1000 \\ 2 & 1000 < |f| < 2000 \\ 0 & \text{otherwise} \end{cases}$$

Determine the SNR at the output of the filter. Express your answer in dB.

Note: The function $\text{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$.

Problem 7.4: Shown below are two sets of functions. The left column contains seven possible correlation functions, and the right column contains seven possible power-spectral densities.

a. For each function in the left column:
   1) Determine if it is a legitimate correlation function for some real-valued, stationary random process.
   2) If it is, determine the power-density spectrum that corresponds to it in the right-hand column. If necessary, apply appropriate scale factors to the ordinate and abscissa or add a suitable function. Determine all scale factors and constants.

b. For each function in the right column:
   1) Determine if it is a legitimate power-spectral density function for some real-valued, stationary random process.
   2) If it is, determine the correlation function that corresponds to it in the left-hand column. If necessary, apply appropriate scale factors to the ordinate and abscissa or add a suitable function. Determine all scale factors and constants.