Homework 4
EE 668 — Information Theory

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Aside
Throughout the solutions below, we have made extensive use of entropy decomposition equation as below. Although shown for a three dimensional probability vector (α, β, γ), it is generally applicable to any dimensionality as well.

$$\alpha + \beta + \gamma = 1 \longrightarrow H(\alpha, \beta, \gamma) = H(\alpha, 1-\alpha) + (1-\alpha)H(\frac{\beta}{1-\alpha}, \frac{\gamma}{1-\alpha})$$

Problem 4.1

((a)) The distribution on the input to the channel can be assumed a general Bernoulli:

$$p_X(x) = p^x(1-p)^{1-x} \qquad x \in \{0, 1\}$$

then we should find the capacity as defined:

$$C = \max_{p_X(x)} I(X;Y)$$

the mutual information can be decomposed as:

$$I(X;Y) = H(Y) - H(Y|X)$$

now both of the terms above can be calculated separately:

$$\begin{aligned} H(Y|X) &= \sum_x H(Y|X=x)p_X(x) \\ &= pH(1-\epsilon-\alpha, \alpha, \epsilon) + (1-p)H(1-\epsilon-\alpha, \alpha, \epsilon) \\ &= H(1-\epsilon-\alpha, \alpha, \epsilon) \end{aligned}$$

and

$$\begin{aligned} H(Y) &= H(p_Y(1), p_Y(e), p_Y(0)) \\ &= H(p(1-\epsilon-\alpha) + (1-p)\epsilon, \epsilon, (1-p)(1-\epsilon-\alpha) + p\epsilon) \\ &= H(\alpha, 1-\alpha) + (1-\alpha)H(\frac{p(1-\epsilon-\alpha)+(1-p)\epsilon}{1-\alpha}, \frac{(1-p)(1-\epsilon-\alpha)+p\epsilon}{1-\alpha}) \end{aligned}$$

maximum entropy for a Bernoulli random variable equals to one, so:

$$H(Y) \leq H(\alpha, 1-\alpha) + (1-\alpha)$$

to have the equality for the above inequality, the second term entropy should be a uniform Bernoulli, then:

$$\frac{p(1-\epsilon-\alpha)+(1-p)\epsilon}{1-\alpha} = \frac{(1-p)(1-\epsilon-\alpha)+p\epsilon}{1-\alpha} \longrightarrow p = \frac{1}{2}$$

Finally the capacity will be equal to:

$$\begin{aligned}
C &= [H(\alpha, 1-\alpha) + (1-\alpha)] - H(1-\epsilon-\alpha, \alpha, \epsilon) \\
&= [H(\alpha, 1-\alpha) + (1-\alpha)] - [H(\alpha, 1-\alpha) + (1-\alpha)H(\frac{1-\alpha-\epsilon}{1-\alpha}, \frac{\epsilon}{1-\alpha})] \\
&= (1-\alpha) - (1-\alpha)H(\frac{1-\alpha-\epsilon}{1-\alpha}, \frac{\epsilon}{1-\alpha})
\end{aligned}$$

((b)) For $\alpha = 0$; binary symmetric channel:

$$C = 1 - H(\epsilon, 1-\epsilon)$$

((c)) For $\epsilon = 0$; binary erasure channel:

$$C = 1 - \alpha$$

Problem 4.2

((a)) Output of the channel is described as:

$$Y = X + Z \qquad , X \in \{0, 1\}, Z \in \{0, a\}$$

then we should find the capacity as defined:

$$C = \max_{p_X(x)} I(X;Y)$$

the mutual information can be decomposed as:

$$I(X;Y) = H(X) - H(X|Y)$$

depending on the value of a, 4 scenarios can be imagined:

- $a = -1$

  $Y \in \{-1, 0, 1\}$, now the channel is like a binary erasure channel with erasure probability of 0.5. Therefore the capacity will be $C = 0.5$.

- $a = 0$

  $Y = X$, knowing Y will determine X exactly ($H(X|Y) = 0$). Therefore capacity will be $C = \max H(X) = 1$.

- $a = 1$

  $Y \in \{0, 1, 2\}$, now the channel is like a binary erasure channel with erasure probability of 0.5. Therefore the capacity will be $C = 0.5$.

- $a =\notin \{-1, 0, 1\}$

  $Y \in \{0, 1, a, a+1\}$, knowing Y will determine X exactly ($H(X|Y) = 0$). Therefore capacity will be $C = \max H(X) = 1$.

Problem 4.3

Both X and Z are Bernoulli random variables:

$$\begin{aligned} p_X(x) &= p^x(1-p)^{1-x} \qquad & x \in \{0, 1\} \\ p_Z(z) &= \alpha^z(1-\alpha)^{1-z} \qquad & z \in \{0, 1\} \end{aligned}$$

Then the distribution of the output of the channel will be also a Bernoulli:

$$p_Y(y) = (p\alpha)^y(1-p\alpha)^{1-y} \qquad y \in \{0, 1\}$$

$$\begin{aligned} C = \max_{p_X(x)} I(X;Y) &= \max_{p_X(x)}[H(Y) - H(Y|X)] \\ &= \max_{p_X(x)}[H(p\alpha, 1-p\alpha) - (pH(Y|X=1) + (1-p)H(Y|X=0))] \\ &= \max_{p_X(x)}[H(p\alpha, 1-p\alpha) - (pH(\alpha, 1-\alpha) + (1-p)\times 0))] \\ &= \max_{p_X(x)}[H(p\alpha, 1-p\alpha) - pH(\alpha, 1-\alpha))] \\ &= \max_{p_X(x)}[p\alpha \log p\alpha + (1-p\alpha)\log(1-p\alpha) - p\alpha \log \alpha - p(1-\alpha)\log(1-\alpha)] \end{aligned}$$

by setting the derivative equal to zero w.r.t. to variable p, we fine that $p = \frac{1}{\alpha 2^{\frac{H(\alpha,1-\alpha)}{\alpha}} + \alpha}$. finally, the capacity will be:

$$C = \log(2^{\frac{H(\alpha,1-\alpha)}{\alpha}} + 1) - \frac{H(\alpha, 1-\alpha)}{\alpha}$$
