EE 568: Information Theory
Fall 2022 Schmid

Homework Assignment 5

Distributed: Saturday, November 19, 2022

Due: Thursday, December 8, 2022

Reference Material:

1. class notes
2. T. M. Cover and J. A. Thomas, 2006, *Elements of Information Theory,* Chs. 5-11.

General Comments: This is the last homework assignment. Solutions for this homework assignment are due on December 8.

Problems: Three of the following problems will be selected at random for grading and should be turned in on the due date.

Problem 5.1 (Based on Problem 8.1, pp. 256-257, CT'06) *Differential Entropy.*
Evaluate the differential entropy $h(X) = -\int f \ln f$ of the following:

(a) The exponential density, $f(x) = \lambda \exp(-\lambda x), \quad x \geq 0.$

(b) The Laplace density, $f(x) = \frac{1}{2} \lambda \exp(-\lambda |x|).$

(c) The sum of X_1 and X_2, where X_1 and X_2 are independent normal random variables with mean μ_i and variances $\sigma_i^2, i = 1,2.$

Problem 5.2 *Mutual Information for correlated normal.* Find the mutual information $I(X;Y)$ where

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}\left(\mathbf{0}, \begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{bmatrix}\right).$$

Evaluate $I(X;Y)$ for $\rho = 1$, $\rho = 0$, $\rho = -1$, and comment.

Problem 5.3 (Based on Problem 10.1, CT'06) *One-bit quantization of a single Gaussian RV.*
Let X be $N(0, \sigma^2)$ and let the distortion measure be squared error. Here we do not allow block descriptions. Show that the optimum reproduction points for one-bit quantization are $\pm\sqrt{\frac{2}{\pi}}\sigma$, and that the expected distortion for one-bit quantization is $\frac{\pi-2}{\pi}\sigma^2$. Compare this with the distortion rate bound $D = \sigma^2 2^{-2R}$ for $R = 1$.

Problem 5.4 (based on Problem 6.3, Blahut'88)
Suppose a binary equiprobable source is to be compressed. The distortion matrix is

$$d = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

That is, it is twice as serious to reproduce a "0" by a "1" as it is reproduce a "1" by a "0." Find $R(D)$.

Problem 5.5 (based on Problem 11.1, C&T'06) *Chernoff-Stein lemma*
Consider the two-hypothesis test

$$H_1 : f = f_1 \;\; vs. \; H_2 : f = f_2$$

Find $D(f_1||f_2)$ if

(a) $f_i(x) = N(0, \sigma_i^2), \; i = 1, 2.$
(b) $f_i(x) = \lambda_i \exp(-\lambda_i x), \; x \geq 0, \quad i = 1, 2.$
(c) f_1 is uniform on $(0,1)$ and f_2 is uniform on $(a, a+1)$, where $0 < a < 1$.
(d) f_1 corresponds to a fair coin and f_2 corresponds to a two-headed coin.

Problem 5.6 (based on Problem 11.16, C&T'06) *Hypothesis testing*
Let $\{X_i\}$ be i.i.d. ~ $p(x)$, $x \in \{1,2,\ldots\}$. Consider two hypotheses, $H_0 : p(x) = p_0(x)$ vs. $H_1 : p(x) = p_1(x)$, where $p_0(x) = \left(\frac{1}{2}\right)^x$ and $p_1(x) = qp^{x-1}$, $x = 1,2,3,\ldots$.

(a) Find $D(p_0||p_1)$.
(b) Let $\Pr\{H_0\} = 1/2$. Find the minimum probability of error test for H_0 vs. H_1 given data $X_1, X_2, \ldots, X_n \sim p(x)$.

Problem 5.7 (based on Problem 11.14, C&T'06) *Sanov's theorem*
Let X_i be i.i.d. ~ $N(0, \sigma^2)$.

(a) Find the exponent in the behavior of $Pr\left\{\frac{1}{n}\sum_{i=1}^{n} X_i^2 \geq \alpha^2\right\}$. This can be done by using Sanov's Theorem.
(b) What do the data look like if $\frac{1}{n}\sum_{i=1}^{n} X_i^2 \geq \alpha$? That is, what is the P^* that minimizes $D(P||Q)$?