EE 568: Information Theory

Fall 2022 Schmid

Homework Assignment 2

Distributed: Thursday, September 8, 2022

Due (TENTATIVE): Thursday, September 22, 2022

Reference Material:

1. class notes
2. T. M. Cover and J.A. Thomas, *Elements of Information Theory*, Ch. 3. (Ed. 2006)
3. Leon-Garcia has a few chapters on infinite sequences, convergence, and laws of large numbers.

Suggested Problems:

The following problems are exercises for self improvement. They will not be graded and do not need to be turned in. The solutions are attached.

Exercise 2.1 (Based on Problem 3.6, CT'06) *An AEP-like limit.*
Let $X_1, X_2, \ldots$ be i.i.d. drawn according to probability mass function $p(x)$. Find

$$\lim_{n\to\infty} [p(X_1, X_2, \ldots, X_n)]^{1/n}.$$

Exercise 2.2 (Based on Problem 3.7, CT'06) *The AEP and source coding.*
A discrete memoryless source emits a sequence of statistically independent binary digits with probabilities $p(1) = 0.005$ and $p(0) = 0.995$. The digits are taken 100 at a time and a binary codeword is provided for every sequence of 100 digits containing three or fewer ones.

(a) Assuming that all codewords are the same length, find the minimum length required to provide codewords for all sequences with three or fewer ones.
(b) Calculate the probability of observing a source sequence for which no codeword has been assigned.
(c) Use Chebychev inequality to bound the probability of observing a source sequence for which no codeword has been assigned. Compare this bound with the actual probability computed in part (b).

Problems:

Two of the following problems will be selected at random for grading and should be turned in on a due date.

Problem 2.1
Consider a sequence $X_1, X_2, \ldots, X_n, \ldots$ of i.i.d. exponentially distributed random variables with p.d.f.

$$f_X(x:\lambda) = \lambda \exp(-\lambda x), \quad x > 0.$$

A sequence of sample means (estimates of the true mean of the random variable) is formed as

$$M_n = \frac{1}{n}\sum_{k=1}^{n} X_k, \quad n = 2,3,\ldots.$$

(a) Find the following probability $P[M_n > 2]$. For simplicity, choose $\lambda = 1$. Plot the probability as a function of n.

(b) Develop Chernoff bound on $P[M_n > 2]$. Plot it as a function of n. Compare the results in (a) and (b).

In case if you are not familiar with Chernoff bound, develop Chebychev inequality for $P[M_n > 2]$. Plot it as a function of n.

Problem 2.2 (Based on Problem 3.8 from CT'06, page 66) *Products.*
Let

$$X = \begin{cases} 1, & 1/2 \\ 2, & 1/4 \\ 3, & 1/4 \end{cases}$$

Let $X_1, X_2, \ldots$ be drawn i.i.d. according to this distribution.
(a) Find the limiting behavior of the product

$$(X_1 X_2 \ldots X_n)^{1/n}.$$

(b) Calculate the entropy of the source?

Problem 2.3 (Based on Problem 3.9 from CT'06, page 67) *AEP.*
Let $X_1, X_2, \ldots$ be i.i.d. random variables drawn according to the probability mass function $p(x), x \in \{1,2,\ldots,m\}$. Thus $p(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} p(x_i)$. We know that $-1/n \log p(X_1, X_2, \ldots, X_n) \to H(X)$ in probability. Let $q(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} q(x_i)$, where q is another probability mass function on $\{1,2,\ldots,m\}$.

(a) Evaluate the limit

$$\lim_{n\to\infty} \left[-\frac{1}{n} \log q(X_1, X_2, \ldots, X_n) \right], \text{ where } X_1, X_2, \ldots \text{ are i.i.d. } p(x).$$

(b) Now evaluate the limit of the loglikelihood ratio $\frac{1}{n} \log \frac{q(X_1, \ldots, X_n)}{p(X_1, \ldots, X_n)}$ when $X_1, X_2, \ldots$ are drawn i.i.d. $p(x)$. Thus the odds favoring q are exponentially small when p is true.

Problem 2.4 (Based on Problem 3.10 from CT'06, page 67) *Random box size.*
An n-dimensional rectangular box with sides $X_1, X_2, X_3, \ldots, X_n$ is to be constructed. The volume is $V_n = \prod_{i=1}^{n} X_i$. The edge length l of a n-cube with the same volume as the random box is $l = V_n^{1/n}$. Let $X_1, X_2, \ldots$ be i.i.d. uniform random variables over the unit interval [0,1]. Find $\lim_{n\to\infty} V_n^{1/n}$ and compare to $(EV_n)^{1/n}$. Clearly the expected edge length does not capture the idea of the volume of the box.