EE 568: Information Theory

Fall 2022 Schmid

Homework Assignment 4

Distributed: Thursday, October 27, 2022

Deadline: Thursday, November 10, 2022

Reference Material:

1. Class notes
2. T. M. Cover and J. A. Thomas, 2006, *Elements of Information Theory,* Ch. 7.

General Comments:
Solutions for this homework assignment are due in two weeks.

Suggested Problems:

The following problems are exercises for self-improvement. They will not be graded and do not need to be turned in. The solutions are attached.

Exercise 4.1 (Based on Problem 7.1, page 223 in CT'06) *Preprocessing the output.*
One is given a communication channel with transition probabilities $p(y\,|\,x)$ and channel capacity $C = \max_{p(x)} I(X;Y)$. A helpful statistician preprocesses the output by forming $\widetilde{Y} = g(Y)$. He claims that this will strictly improve the capacity.

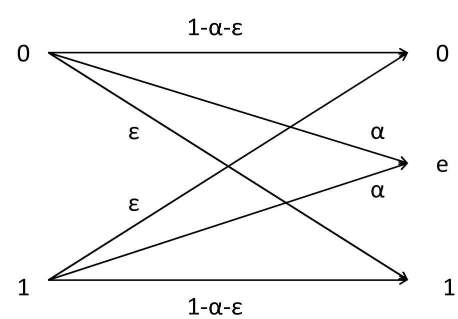
(a) Show that he is wrong.
(b) Under what conditions does he not strictly decrease the capacity?

Problems:

The following problems will be graded.

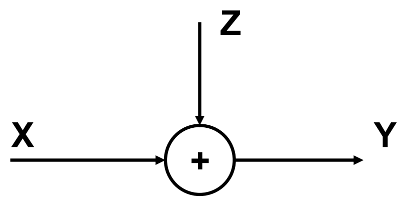
Problem 4.1 (Based on Problem 7.13, CT'06, pp. 226-227) *Erasures and errors in a binary channel.*

Consider a channel with binary inputs that has both erasures and errors. Let the probability of error be ε and the probability of erasure be α, so the channel is follows:



(a) Find the capacity of this channel.
(b) Specialize to the case of the binary symmetric channel ($\alpha = 0$).
(c) Specialize to the case of the binary erasure channel ($\varepsilon = 0$).

Problem 4.2 (Based on Problem 8.2, CT'06, p.224) Find the channel capacity of the following discrete memoryless channel:



where $\Pr\{Z=0\} = \Pr\{Z=a\} = \frac{1}{2}$. The alphabet for X is $\mathcal{X} = \{0,1\}$. Assume that Z is independent of X.

Comment: Observe that the channel capacity depends on the value of a.

Problem 4.3 (Based on Problem 7.29, page 236, CT'06) *Binary Multiplier Channel*

Consider the discrete memoryless channel $Y = XZ$, where X and Z are independent binary random variables that take on values 0 and 1. Let $P(Z=1) = \alpha$. Find the capacity of this channel and the maximizing distribution on X.

Solutions to Exercise Problems

Exercise 4.1 (Problem 7.1, CT'06, p. 223):

(a) The statistic $\tilde{Y} = g(Y)$ is a processed output of a channel Y. Since data transmission and processing form a Markov chain: $X \rightarrow Y \rightarrow \tilde{Y}$, we can use data processing inequality: $I(X;Y) \geq I(X,\tilde{Y})$ for any probability assignment $p(x)$. Suppose $p*(x)$ maximizes $I(X;\tilde{Y})$, then
$\tilde{C} = \max_{p(x)} I(X;\tilde{Y}) = I_{p*(x)}(X;\tilde{Y}) \leq I_{p*(x)}(X;Y) \leq \max_{p(x)} I(X;Y) = C$.

(b) The equality will hold if $X \rightarrow \tilde{Y} \rightarrow Y$ form a Markov chain.