EE 568: Information Theory

Fall 2022 Schmid

Homework Assignment 3
Distributed: Tuesday, October 4, 2022
Due: Tuesday, October 18, 2022

Reference Material:

1. class notes
2. T. M. Cover and J.A. Thomas, *Elements of Information Theory*, Ch. 5.

General Comments: Solutions for this homework assignment are due in 2 weeks.

Problems: Two of the following problems will be selected at random for grading and should be turned in on a due date.

Problem 3.1 (Based on Problem 5.4 on p. 143, CT'06)
Huffman Coding. Consider the random variable

$$X = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 0.49 & 0.26 & 0.12 & 0.04 & 0.04 & 0.03 & 0.02 \end{pmatrix}$$

(a) Find a binary Huffman code for X.
(b) Find the expected codelength for this coding.
(c) Find a ternary Huffman Code for X.

Problem 3.2 (Based on Problem 5.12 on p. 145, CT'06)
Shannon codes and Huffman codes. Consider a random variable X which takes on four values with probabilities $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{12}\right)$.

(a) Construct a Huffman code for this random variable.
(b) Show that there exist two sets of optimal lengths for the codewords, namely, show that codeword length assignments $(1,2,3,3)$ and $(2,2,2,2)$ are both optimal.
(c) Conclude that there are optimal codes with codeword lengths for some symbols that exceed Shannon code length $\left\lceil \log \frac{1}{p(x)} \right\rceil$.

Problem 3.3 (Based on Problem 5.30, p. 151, CT'06) *Relative entropy is cost of miscoding.*
Let the random variable X have five positive outcomes $\{1,2,3,4,5\}$. Consider two distributions $p(x)$ and $q(x)$ on this random variable.

Symbol	$p(x)$	$q(x)$	$C_1(x)$	$C_2(x)$
1	1/2	1/2	0	0
2	1/4	1/8	10	100
3	1/8	1/8	110	101
4	1/16	1/8	1110	110
5	1/16	1/8	1111	111

(a) Calculate $H(p)$, $H(q)$, $D(p \| q)$, and $D(q \| p)$.
(b) The last two columns represent codes for the random variable. Verify that the average length of C_1 under p is equal to the entropy $H(p)$. Thus, C_1 is optimal for p. Verify that C_2 is optimal for q.
(c) Now assume that we use code C_2 when the distribution is p. What is the average length of the codewords. By how much does it exceed the entropy p?
(d) What is the loss if we use code C_1 when the distribution is q?

Problem 3.4 Find the *Shannon-Fano-Elias* binary code for the random variable X with probabilities

$$p = \left(\frac{1}{25}, \frac{2}{25}, \frac{4}{25}, \frac{5}{25}, \frac{6}{25}, \frac{7}{25} \right).$$

Find the average codeword length $L(X)$ and compare it to the entropy of the random variable X.

Problem 3.5 *(Kuhn-Tucker)*
Determine conditions for minimizing the function

$$J(d_1, d_2) = \frac{1}{2} \ln \frac{\sigma_1^2}{d_1} + \frac{1}{2} \ln \frac{\sigma_2^2}{d_2},$$

subject to the constraints that $d_1 + d_2 = D$ (where D is known and fixed) and also to the constraints that $d_i \le \sigma_i^2, \quad i = 1,2.$
Comment: We will see this result again when we will study the rate-distortion theory. Here D is the total distortion due to lossy coding, d_i is a distortion in the i-th channel due to lossy coding, and σ_i^2 is the variance of Gaussian noise in the i-th channel.

Problem 3.6
Maximize $14x - x^2 + 6y - y^2 + 7$ subject to $x + y \le 2, \quad 2x + 2y \le 3.$