Homework 4

MATH 543 — Linear Algebra

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4 Polynomials

Exercise 5

Let $T : \mathcal{P}_m(\mathbb{F}) \to \mathbb{F}^{m+1}$ be defined as $T(p) = (p(z_1), \dots, p(z_{m+1}))$. T is trivially linear. Since $\dim \mathcal{P}_m(\mathbb{F}) = \dim \mathbb{F}^{m+1}$, there is an isomorphism between the two spaces.

So, there is an invertible T that:

- by *surjectivity* of T, $\quad \forall (w_1, \dots, w_{m+1}) \in \mathbb{F}^{m+1}$ there exists a $p \in \mathcal{P}_m(\mathbb{F})$.
- and p is unique due to *injectivity* of T.

Exercise 6

$\Longrightarrow$

$p \in \mathcal{P}(\mathbb{C})$ has m distinct zeros.
p can be factorized: $p = c(z - \lambda_1)\dots(z - \lambda_m)$ s.t. $c, \lambda_1, \dots, \lambda_m \in \mathbb{C}$ (λ_i's are distinct).
p can be written as $p(z) = (z - \lambda_i)q(z)$ s.t. $q(\lambda_i) \neq 0$. Differentiating results in $p'(z) = q(z) + (z - \lambda_i)q'(z)$. Therefore for <u>no i</u>, $p'(\lambda_i) = 0$.

$\Longrightarrow$

p and p' have no zeros in common.
By definition p has m roots. Let p have a root at $z = \lambda_i$ so $p(z) = (z - \lambda_i)q_i(z)$, where $q_i(z)$ could be zero at $z = \lambda_i$. Then $p'(z) = q_i(z) + (z - \lambda_i)q_i'(z)$. If we have $q_i(\lambda_i) = 0$, then p and p' share the root λ_i, a *contradiction*. Therefore p has m <u>distinct</u> roots.

5.A Invariant Subpspaces

Exercise 1

(a) Let $u \in U$, so $u \in \operatorname{null} T$ and $Tu = 0 \in U$ (U is a subspace). U in invariant under T.

(b) Let $u \in U$, then $Tu \in \operatorname{range} T$ so $Tu \in U$. U in invariant under T.

Exercise 3

Let $v \in \operatorname{range} S$, so exists $u \in V$ s.t. $Su = v$. Then $Tv = T(Su) = TSu = STu = S(Tu) \in \operatorname{range} S$. Therefore $\operatorname{range} S$ is invariant under T.

Exercise 7

$$\begin{aligned} T(x, y) &= \lambda(x, y) \\ (-3y, x) &= (\lambda x, \lambda y) \end{aligned}$$

Then $-3y = \lambda x$ and $x = \lambda y$. So $\lambda^2 = -3$, and there is no $\lambda \in \mathbb{R}$ satisfying the equation. T has no eigenvalues and thus no eigenvectors.

Exercise 9

$$\begin{aligned} T(z_1, z_2, z_3) &= \lambda(z_1, z_2, z_3) \\ (2z_2, 0, 5z_3) &= (\lambda z_1, \lambda z_2, \lambda z_3) \end{aligned}$$

then we have $2z_2 = \lambda z_1$, $0 = \lambda z_2$ and $5z_3 = \lambda z_3$. $\lambda = 5$ and $\lambda = 0$ are eigenvalues with eigenvectors $(0, 0, y) \quad \forall y \in \mathbb{F}, y \neq 0$ and $(x, 0, 0) \quad \forall x \in \mathbb{F}, x \neq 0$, respectively.

Exercise 10

(a)

$$\begin{aligned} T(x_1, x_2, x_3, \ldots, x_n) &= \lambda(x_1, x_2, x_3, \ldots, x_n) \\ (x_1, 2x_2, 3x_3, \ldots, nx_n) &= (\lambda x_1, \lambda x_2, \lambda x_3, \ldots, \lambda x_n) \end{aligned}$$

Eigenvalue	Eigenvector
1	$(x, 0, 0, \ldots, 0) \quad \forall x \in \mathbb{F}, x \neq 0$
2	$(0, x, 0, \ldots, 0) \quad \forall x \in \mathbb{F}, x \neq 0$
3	$(0, 0, x, \ldots, 0) \quad \forall x \in \mathbb{F}, x \neq 0$
$\ldots$	$\ldots$
n	$(0, 0, 0, \ldots, x) \quad \forall x \in \mathbb{F}, x \neq 0$

(b) All invariant subspaces:

Invariant Subspaces Under T
$\{(x, 0, 0, \ldots, 0) \mid x \in \mathbb{F}, x \neq 0\}$
$\{(0, x, 0, \ldots, 0) \mid x \in \mathbb{F}, x \neq 0\}$
$\{(0, 0, x, \ldots, 0) \mid x \in \mathbb{F}, x \neq 0\}$
$\ldots$
$\{(0, 0, 0, \ldots, x) \mid x \in \mathbb{F}, x \neq 0\}$

Exercise 11

Let $p \in \mathcal{P}(\mathbb{R})$ of degree m. Then Tp will be of degree at most $m-1$. So

$$a_1 + 2a_2 z \cdots + m a_m z^{m-1} = \lambda(a_0 + a_1 z + \cdots + a_m z^m)$$

is satisfied only when eigenvalue $\lambda = 0$ and eigenvector $p = a \quad \forall a \in \mathbb{F}, a \neq 0$ (constant polynomial).

Exercise 21

(a) If $Tv = \lambda v$, since T is invertible, $\frac{1}{\lambda} v = T^{-1} v$. Thus $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

Proof of the other way is the same.

(b) As showed in part (a), every eigenvector of T is also an eigenvector for T^{-1}, and vice-versa.

Exercise 22

Aside:

Remark 1. If λ is an eigenvalue for T, λ^2 is an eigenvalue for T^2.
Remark 2. If λ is an eigenvalue for T^2, either $+\sqrt{\lambda}$ or $-\sqrt{\lambda}$ is an eigenvalue for T.
(*proof.* $T^2 v = \lambda v \to (T - \sqrt{\lambda} I)(T + \sqrt{\lambda} I)v = 0$)

$$\begin{aligned} Tv &= 3w \\ T^2 v &= 3Tw \\ T^2 v &= 9v \end{aligned}$$

9 is an eigenvalue for T^2, therefore either -3 or 3 is an eigenvalue for T.

Exercise 25

Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{F}$ be eigenvalues corresponding to u, v and $u+v$, respectively.

$$\begin{aligned} T(u+v) &= \lambda_3(u+v) \\ Tu + Tv &= \lambda_3 u + \lambda_3 v \\ \lambda_1 u + \lambda_2 v &= \lambda_3 u + \lambda_3 v \\ (\lambda_1 - \lambda_3)u + (\lambda_2 - \lambda_3)v = 0 \end{aligned}$$

u and v are linearly independent, thus $\lambda_1 = \lambda_2 = \lambda_3$.

Exercise 29

Let T has n distinct eigenvalues, with corresponding eigenvectors $v_1, \ldots, v_n$. Trivially $v_1, \ldots, v_n \in \operatorname{range} T$ for nonzero eigenvalues. Thus if only one of the eigenvalues be zero, there will be $n-1$ linearly independent eigenvectors.

Thus $n - 1 \leq \dim \operatorname{range} T = k$, meaning that $n \leq k+1$.

Exercise 30

Let $u, v, W \in V$ be eigenvectors corresponding to $-4, 5, \sqrt{7}$, respectively. Thus exists $a_1, a_2, a_3 \in R$ s.t. $x = a_1u + a_2v + a_3w$.

$$\begin{aligned} T(a_1u + a_2v + a_3w) - 9(a_1u + a_2v + a_3w) &= (-4, 5, \sqrt{7}) \\ -13a_1u - 4a_2v + (\sqrt{7} - 9)a_3w &= (-4, 5, \sqrt{7}) \end{aligned}$$

since u, v, w are linearly independent in $\mathbb{R}^3$, a_1, a_2, a_3 are uniquely determined, thus x exists.

Exercise 33

Let $v + \operatorname{range} T \in V/\operatorname{range} T$, we have

$$T/\operatorname{range} T(v + \operatorname{range} T) = \underbrace{Tv + \operatorname{range} T}_{\in \operatorname{range} T}$$

thus $T/\operatorname{range} T(v + \operatorname{range} T) = 0$ and since $v + \operatorname{range} T \in V/\operatorname{range} T$ is arbitrary chosen, $T/\operatorname{range} T = 0$.

Exercise 34

$\Longrightarrow$

null $T \cap \operatorname{range} T = \{0\}$. Let $v + \operatorname{null} T$ be an arbitrary element in $\operatorname{null}(T/\operatorname{null} T)$.

$$T/\operatorname{null} T(v + \operatorname{null} T) = Tv + \operatorname{null} T = 0 + \operatorname{null} T$$

thus $Tv \in \operatorname{null} T$, and $Tv \in \operatorname{null} T \cap \operatorname{range} T = \{0\}$, thus $Tv = 0$. Therefore $v \in \operatorname{null} T$. So, $\operatorname{null}(T/\operatorname{null} T) = \operatorname{null} T$ meaning that $T/\operatorname{null} T$ is injective.

$\Longleftarrow$

$T/\operatorname{null} T$ is injective. Let $u \in \operatorname{null} T \cap \operatorname{range} T$ be an arbitrary element s.t. $u = Tv$ for a $v \in V$. Then

$$(T/\operatorname{null} T)(v + \operatorname{null} T) = Tv + \operatorname{null} T = \underbrace{u}_{\in \operatorname{null} T} + \operatorname{null} T = \operatorname{null} T$$

injectivity results in $v + \operatorname{null} T = \operatorname{null} T$, so $v \in \operatorname{null} T$. Thus $y = Tv = 0$.
