Homework 6

MATH 543 — Linear Algebra

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8.A Generalized Eigenvectors and Nilpotent Operators

Exercise 2

- **Eigenvalues.**

$$\begin{aligned} Tv &= \lambda v \\ T(v_1, v_2) - \lambda(v_1, v_2) &= 0 \\ (-v_2 - \lambda v_1, v_1 - \lambda v_2) &= 0 \\ \lambda &= i, -i \end{aligned}$$

- **Generalized Eigenspaces.** There is enough (2) distinct eigenvalues, so generalized eigenspaces are equal to eigenspaces.

$$E(i, T) = \operatorname{span}(\begin{pmatrix} i \\ 1 \end{pmatrix})$$

$$E(-i, T) = \operatorname{span}(\begin{pmatrix} 1 \\ i \end{pmatrix})$$

Exercise 3

Let $n = \dim V$, by induction on n we prove that $\operatorname{null}(T - \lambda I)^n = \operatorname{null}(T^{-1} - \frac{1}{\lambda} I)^n$.

- $\mathbf{n = 1}$. Let $v \in \operatorname{null}(T - \lambda I)$ then $Tv = \lambda v$ or equivalently $T^{-1}v = \frac{1}{\lambda}v$ thus $v \in \operatorname{null}(T^{-1} - \frac{1}{\lambda} I)$. For $w \in \operatorname{null}(T^{-1} - \frac{1}{\lambda} I)$ same steps leads to $w \in \operatorname{null}(T - \lambda I)$.
- **Let the hypothesis be correct** $\forall i \quad 1 \leq i \leq n - 1$.
  Let $v \in \operatorname{null}(T - \lambda I)^n$:

$$(T - \lambda I)^n v = 0 \Rightarrow (T - \lambda I)^{n-1}((T - \lambda I)v) = 0 \Rightarrow (T - \lambda I)v \in \operatorname{null}(T - \lambda I)^{n-1}$$

from induction hypothesis:

$$(T-\lambda I)v \in \text{null}(T^{-1}-\frac{1}{\lambda}I)^{n-1}$$

so

$$(T^{-1}-\frac{1}{\lambda}I)^{n-1}(T-\lambda I)v = 0 \Rightarrow (T-\lambda I)(T^{-1}-\frac{1}{\lambda}I)^{n-1}v = 0$$

thus

$$(T^{-1}-\frac{1}{\lambda}I)^{n-1}v \in \text{null}(T-\lambda I) = \text{null}(T^{-1}-\frac{1}{\lambda}I)$$

and equivalently

$$\begin{aligned}(T^{-1}-\frac{1}{\lambda}I)(T^{-1}-\frac{1}{\lambda}I)^{n-1}v = 0\\ (T^{-1}-\frac{1}{\lambda}I)^{n}v = 0\\ v \in (T^{-1}-\frac{1}{\lambda}I)^{n}\end{aligned}$$

thus

$$\text{null}(T-\lambda I)^n \subseteq \text{null}(T^{-1}-\frac{1}{\lambda}I)^n$$

the same steps can be followed to show the inclusion in the other direction and we have:

$$\text{null}(T-\lambda I)^n = \text{null}(T^{-1}-\frac{1}{\lambda}I)^n$$

Exercise 4

Suppose $v \neq 0$ and $v \in G(\alpha, T) \cap G(\beta, T)$. Since v is an eigenvector for two generalized eigenspaces of distinct eigenvalues, it is a contradiction with 8.13. Therefore $v = 0$.

(8.13: Generalized eigenvectors corresponding to distinct eigenvalues are linearly independent.)

8.B Decomposition of an Operator

Exercise 1

$G(0, N) = \text{null}(N)^{\dim V}$ and since zero is the only eigenvalue $V = \text{null}(N)^{\dim V}$ or equivalently $N^{\dim V} = 0 \quad \forall v \in V$. Thus N is nilpotent.

Exercise 2

We consider a zero eigenvalue and a pair of complex conjugate eigenvalues:

$$T(x, y, z) = (0, z, -y)$$

Exercise 3

Let $v \in V$ be eigenvector of T corresponding to eigenvalue λ. There exists $u \in V$ such that $v = Su$. (S is surjective)

$$S^{-1}TSu = S^{-1}Tv = S^{-1}(\lambda v) = \lambda u$$

thus every eigenvalue for T is also an eigenvalue for $S^{-1}TS$.

Exercise 10

Look at LADW pp. 264. $N^j = 0$ for some $j \leq n$.
Since N is normal, there exists a basis of V consisting of orthonormal eigenvectors $e_1, \ldots, e_n$?

8.C Characteristic and Minimal Polynomials

Exercise 2

For T

$$1 \leq \dim G(5, T) \leq n - 1$$
$$1 \leq \dim G(6, T) \leq n - 1$$

therefore $(T - 6I)^{n-1}(T - 6I)^{n-1}$ is a multiple of characteristic polynomial $q(T)$ and thus equal to zero.

Exercise 4

$$\chi_T(z) = (z-1)(z-5)^3 \text{ and } p(z) = (z-1)(z-5)^2$$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

Exercise 5

$$\chi_T(z) = z(z-1)^2(z-3) = p(z)$$

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Exercise 6

$$\chi_T(z) = z(z-1)^2(z-3) \text{ and } p(z) = z(z-1)(z-3)$$

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Exercise 8

Let χ_T be the characteristic polynomial of T.

$$\begin{aligned} & T \text{ is invertible} \\ \iff & 0 \text{ is not an eigenvlue of } T \\ \iff & \chi_T(0) \neq 0 \\ \iff & \text{constant term of } \chi_T(z) \text{ is not zero} \end{aligned}$$

Exercise 9

We have $p(T) = 4 + 5T - 6T^2 - 7T^3 + 2T^4 + T^5 = 0$.
Then $T^{-5}p(T) = 4T^{-5} + 5T^{-4} - 6T^{-3} - 7T^{-2} + 2T^{-1} + I = 0$. Thus the minimal polynomial of T^{-1} is $p(T^{-1}) = T^{-5} + 1.25T^{-4} - 1.5T^{-3} - 1.75T^{-2} + 0.5T^{-1} + 0.25I$

8.D Jordan Form

Exercise 1

- **Characteristic Polynomial.** Since N has only zero eigenvalues and multiplicity of it is 4 then:

$$\chi_N(z) = z^4$$

- **Minimal Polynomial.** Since $N^3 \neq 0$, then the minimal polynomial is $p(N) = N^4$.

Exercise 2

- **Characteristic Polynomial.** Since N has only zero eigenvalues and multiplicity of it is 6 then:

$$\chi_N(z) = z^6$$

- **Minimal Polynomial.** Since $N^3 = 0$, then the minimal polynomial is $p(N) = N^3$.

Not from Axler's Book.

Find a Jordan basis and the Jordan normal form for A.

$$A = \begin{pmatrix} 4 & -4 & -11 & 11 \\ 3 & -12 & -42 & 42 \\ -2 & 12 & 37 & -34 \\ -1 & 7 & 20 & -17 \end{pmatrix}$$

- **Eigenvalues.**

$$\begin{aligned} det(A - \lambda I) &= 0 \\ (\lambda - 3)^4 &= 0 \\ \lambda &= 3 \text{ with multiplicity of } 4 \end{aligned}$$

- **Number of blocks.** dimension of eigenspace $E(3, T) = \text{null}(T - 3I)$ determines number of blocks in Jordan form.

$$\begin{aligned} \text{null}(A - 3I) &= \text{null}\begin{pmatrix} 1 & -4 & -11 & 11 \\ 3 & -15 & -42 & 42 \\ -2 & 12 & 34 & -34 \\ -1 & 7 & 20 & -20 \end{pmatrix} \\ &= \text{null}\begin{pmatrix} 1 & -4 & -11 & 11 \\ 0 & -3 & -9 & 9 \\ 0 & 4 & 12 & -12 \\ 0 & 3 & 9 & -9 \end{pmatrix} \quad \text{(row reduction)} \\ &= \text{null}\begin{pmatrix} 1 & -4 & -11 & 11 \\ 0 & -3 & -9 & 9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{(row reduction)} \end{aligned}$$

thus the number of blocks is 2.

$$E(3,T) = \text{span}(\begin{pmatrix} 1 \\ 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \end{pmatrix})$$

- **Jordan Form.** thus the Jordan form is one of the below:

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix} \text{ or } \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

- **Generalized Eigenspace.**

$$\begin{aligned} \text{null}(A-3I)^2 &= \text{null}\begin{pmatrix} 0 & 1 & 3 & -3 \\ 0 & 3 & 9 & -9 \\ 0 & -2 & -6 & 6 \\ 0 & -1 & -3 & 3 \end{pmatrix} \\ &= \text{null}\begin{pmatrix} 0 & 1 & 3 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{(row reduction)} \end{aligned}$$

Since $\dim \text{null}(A-3I)^2 = 3 < 4$, we continue for $\text{null}(A-3I)^3$:

$$\text{null}(A-3I)^3 = \text{null}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus $\dim \text{null}(A-3I)^3 = 4$.

Therefore Jordan Normal form consists of two blocks of 1x1 and 3x3 as below:

$$J = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix} \tag{♣}$$

- **Jordan Basis.** To choose the first eigenvector, we look for $v_1 \neq 0$ such that $(A -$

$3I)^3 v_1 = 0$ and $(A-3I)^2 v_1 \neq 0$. Then we choose $v_4 \neq 0$ such that $(A-3I)v_4 = 0$.

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$v_2 = (A-3I)v_1 = \begin{pmatrix} -4 \\ -15 \\ 12 \\ 7 \end{pmatrix}$$

$$v_3 = (A-3I)^2 v_1 = \begin{pmatrix} 1 \\ 3 \\ -2 \\ -1 \end{pmatrix}$$

$$v_4 = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

To have exactly the Jordan form as in (♣), the order of the elements in matrix P below matters and should be exactly like $[\quad v_4 \quad | \quad (A-3I)^2 v_1 \quad | \quad (A-3I)v_1 \quad | \quad v_1 \quad]$, i.e.,

$$P = \begin{pmatrix} 1 & 1 & -4 & 0 \\ 3 & 3 & -15 & 1 \\ 0 & -2 & 12 & 0 \\ 1 & -1 & 7 & 0 \end{pmatrix}$$

Then we have for sure $J = P^{-1}AP$ with J as showed in (♣).