Homework 5

MATH 543 — Linear Algebra

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5.B Eigenvectors and Upper-Triangular Matrices

Exercise 1

(a) For $I - T$ to be invertible, its composition (both right and left) with its inverse must be identity.

$$\begin{aligned}(I-T)(I+T+\cdots+T^{n-1})v &= (I-T)(v+Tv+\cdots+T^{n-1}v)\\ &= v - Tv + Tv - T^2v + T^2v - \cdots - T^{n-1}v + T^n v\\ &= v\end{aligned}$$

and

$$\begin{aligned}(I+T+\cdots+T^{n-1})(I-T)v &= (I+T+\cdots+T^{n-1})(v-Tv)\\ &= v - Tv + Tv - T^2v + T^2v - \cdots + T^{n-1}v - T^n v\\ &= v\end{aligned}$$

so $(I-T)$ is invertible with inverse $(I-T)^{-1} = (I+T+\cdots+T^{n-1})$.

(b) Sum of a geometric sequence $1, r, r^2, \ldots, r^{n-1}$ is:

$$\begin{aligned}1 + r + r^2 + \cdots + r^{n-1} &= \frac{1-r^n}{1-r}\\ 1 + r + r^2 + \cdots + r^{n-1} &= (1-r)^{-1} \qquad (r^n = 0)\end{aligned}$$

replacing r with T and having $T^n = 0$ resembles what we show in part (a).

Exercise 3

$T^2 = I$ implies either $+1$ or -1 is an eigenvalue for T (Exercise 5.A.22). Thus $+1$ is an eigenvalue for T (hypothesis). Thus $\forall v \in V, v \neq 0$:

$$\begin{aligned}Tv &= v\\ T &= I\end{aligned}$$

Exercise 5

$p \in \mathcal{P}(\mathbb{F})$ can be written as

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$

where $a_0, \dots, a_m, z \in \mathbb{F}$, so

$$p(T) = a_0 I + a_1 T + a_2 T^2 + \cdots + a_m T^m$$

Then

$$\begin{aligned} Sp(T)S^{-1} &= a_0 SIS^{-1} + a_1 STS^{-1} + a_2 ST^2 S^{-1} + \cdots + a_m ST^m S^{-1} \\ &= a_0 I + a_1 STS^{-1} + a_2 ST^2 S^{-1} + \cdots + a_m ST^m S^{-1} \\ &= a_0 I + a_1 STS^{-1} + a_2 (STS^{-1})^2 + \cdots + a_m (STS^{-1})^m \\ &= p(STS^{-1}) \end{aligned}$$

where for the third equality we used $(STS^{-1})^m = ST^m S^{-1}$.

Exercise 13

Consider 2 cases separately:
If $U \subset W$ is a zero subspace of W, there exists no eigenvalues for T and the condition holds. If $U \subset W$ is a non-zero finite-dimensional subspace, then $T_{|U} \in \mathcal{L}(U)$ has at least one eigenvalue (Theorem 5.21) which is a contradiction. Thus U is infinite-dimensional.

Exercise 20

By theorem 5.27, there exists basis $v_1, \dots, v_{\dim V}$ such that T has an upper-triangular matrix with respect to it.

By theorem 5.26, $\operatorname{span}(v_1, \dots, v_k)$ of dimension k is invariant under T for each $k \in \{1, \dots, \dim V\}$.

5.C Eigenspaces and Diagonal Matrices

Exercise 1

Since V is diagonalizable, there exists a diagonal matrix $\mathcal{M}(T)$ defined on a basis $v_1, \ldots, v_n$. Thus V is finite-dimensional.

Then for each v_i exists a λ_i such that $Tv_i = \lambda_i v_i$. Now we separate λ_i's indices into two disjoint sets: $\lambda_i = 0 \quad \forall i \in \{1, \ldots, m\}$ and $\lambda_i \neq 0 \quad \forall i \in \{m+1, \ldots, n\}$.

Thus

$$V = \operatorname{span}(v_1, \ldots, v_n) = \operatorname{span}(v_1, \ldots, v_m) \oplus \operatorname{span}(v_{m+1}, \ldots, v_n)$$

since $v_1, \ldots, v_m, v_{m+1}, \ldots, v_n$ is a basis.

- $\operatorname{span}(v_1, \ldots, v_m) = \operatorname{null} T$?
  $\operatorname{span}(v_1, \ldots, v_m) = E(0, T) = \operatorname{null}(T)$

- $\operatorname{span}(v_{m+1}, \ldots, v_n) = \operatorname{range} T$?
  For arbitrary $v \in V$ we have $v = a_1 v_1 + \cdots + a_n v_n$ $(a_i \in \mathbb{F})$ then

  $$\begin{aligned} Tv &= a_1 T v_1 + \cdots + a_n T v_n \\ &= \underbrace{a_1 T v_1 + \cdots + a_m T v_m}_{=0} + \underbrace{a_{m+1} T v_{m+1} + \cdots + a_n T v_n}_{\in \operatorname{span}(v_{m+1}, \ldots, v_n)} \end{aligned}$$

  thus $\operatorname{range} T \subset \operatorname{span}(v_{m+1}, \ldots, v_n)$.

  On the other hand, for every v_i where $i \in \{m+1, \ldots, n\}$ we have $v_i = T(\frac{1}{\lambda_i} v_i) \in \operatorname{range} T$ thus $\operatorname{span}(v_{m+1}, \ldots, v_n) \subset \operatorname{range} T$.

Exercise 2

Aside:

For any invertible $T \in \mathcal{L}(V)$, we have $V = \operatorname{null} T \oplus \operatorname{range} T$.

(*proof.* invertible $T \in \mathcal{L}(V)$ implies $\operatorname{null} T = \{0\}$ and $\operatorname{range} T = V$ and obviously $\operatorname{null} T \cap \operatorname{range} T = \{0\}$.)

Counterexample. Matrix of invertible $S \in \mathcal{L}(\mathbb{R}^2)$

$$\mathcal{M}(S) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

is non-diagonal, although $\mathbb{R}^2 = \operatorname{null} S \oplus \operatorname{range} S$.

Exercise 3

(a) $V = \operatorname{null} T \oplus \operatorname{range} T$

(b) $V = \operatorname{null} T + \operatorname{range} T$

(c) $\operatorname{null} T \cap \operatorname{range} T = \{0\}$

1. **a ⇒ b** Trivial by definition of direct sum.
2. **b ⇒ c** Sum of two subspaces of finite-dimensional V has dimension

$$\begin{aligned}\dim V &= \dim(\operatorname{null} T) + \dim(\operatorname{range} T) - \dim(\operatorname{null} T \cap \operatorname{range} T) \\ &= \dim V - \dim(\operatorname{null} T \cap \operatorname{range} T)\end{aligned}$$

   where Fundamental Theorem of Linear Maps is used for last equality. Thus $\dim(\operatorname{null} T \cap \operatorname{range} T) = 0$ so $\operatorname{null} T \cap \operatorname{range} T = \{0\}$.

3. **c ⇒ a** For $\operatorname{null} T + \operatorname{range} T$ we have

$$\begin{aligned}\dim(\operatorname{null} T + \operatorname{range} T) &= \dim(\operatorname{null} T) + \dim(\operatorname{range} T) - \dim(\operatorname{null} T \cap \operatorname{range} T) \\ &= \dim V - 0\end{aligned}$$

   thus $\operatorname{null} T + \operatorname{range} T = V$ and since $\operatorname{null} T \cap \operatorname{range} T = \{0\}$ the sum is direct.

Exercise 5

$\Longrightarrow$

T is diagonalizable. So there is a basis on which $\mathcal{M}(T)$ is diagonal. For $\lambda \in \mathbb{C}$:

$$\mathcal{M}(T - \lambda I) = \mathcal{M}(T) - \lambda \mathcal{M}(I)$$

since both $\mathcal{M}(T)$ and $\mathcal{M}(I)$ are diagonal, then $\mathcal{M}(T - \lambda I)$ is diagonal.

By using the result of Exercise 5.C.1 we have

$$V = \operatorname{null}(T - \lambda I) \oplus \operatorname{range}(T - \lambda I).$$

$\Longleftarrow$

?

Exercise 8

Suppose 2 and 6 are also eigenvalues of T,

$$\begin{aligned} \dim E(8,T) + \dim E(6,T) + \dim E(2,T) &\leq \dim \mathbb{F}^5 \\ \dim E(6,T) + \dim E(2,T) &\leq 1 \end{aligned}$$

then either $\dim E(6,T) = 0$ or $\dim E(2,T) = 0$. Thus either 6 or 2 is an NOT an eigenvalue of T, so either $T - 6I$ or $T - 2I$ is invertible, respectively, by theorem 5.6.

Exercise 10

Let $0, \lambda_1, \ldots, \lambda_m$ denote the full set of the distinct eigenvalues. Then

$$\begin{aligned} \underbrace{\dim E(0,T)}_{=\dim \operatorname{null} T} + \dim E(\lambda_1,T) + \cdots + \dim E(\lambda_m,T) &\leq \dim V \\ \dim \operatorname{null} T + \dim E(\lambda_1,T) + \cdots + \dim E(\lambda_m,T) &\leq \dim \operatorname{null} T + \dim \operatorname{range} T \\ \dim E(\lambda_1,T) + \cdots + \dim E(\lambda_m,T) &\leq \dim \operatorname{range} T \end{aligned}$$

where the RHS of 2nd inequality comes from Fundamental Theorem of Linear Maps.

Exercise 15

Since T is not diagonalizable, 8 cannot be an eigenvalue of it (Theorem 5.44). Thus $T - 8I$ is invertible/injective/surjective (Theorem 5.6). By surjectivity for $v = (17, \sqrt{5}, 2\pi)$ there exists $(x, y, z) \in \mathbb{C}^3$ such that

$$\begin{aligned} (T - 8I)(x,y,z) &= (17, \sqrt{5}, 2\pi) \\ T(x,y,z) &= (17 + 8x, \sqrt{5} + 8y, 2\pi + 8z) \end{aligned}$$

Exercise 16

(a) *proof by induction.*
$\mathbf{n = 1}$: $T^1(0,1) = (1,1) = (F_1, F_2)$
Assume it holds for $\mathbf{n \leq m - 1}$.
$\mathbf{n = m}$: $T^m(0,1) = TT^{m-1}(0,1) = T(F_{m-1}, F_m) = (F_m, F_{m-1} + F_m) = (F_m, F_{m+1})$.

(b)

$$\begin{aligned}
T(x,y) &= (\lambda x, \lambda y)\\
(y, x+y) &= (\lambda x, \lambda y)\\
x + \lambda x &= \lambda(\lambda x)\\
\lambda^2 - \lambda - 1 &= 0 \quad (x \neq 0, y \neq 0)\\
\lambda &= \frac{1 \pm \sqrt{5}}{2}
\end{aligned}$$

(c)

$$\begin{aligned}
E(\frac{1+\sqrt{5}}{2}, T) &= \text{null}(T - (\frac{1+\sqrt{5}}{2})I)\\
&= \{(x,y)|(y - \frac{1+\sqrt{5}}{2}x, x + \frac{1-\sqrt{5}}{2}y) = 0\}\\
&= \{(x,y)|y = x(\frac{1+\sqrt{5}}{2})\}
\end{aligned}$$

then we choose $v_1 = (1, \frac{1+\sqrt{5}}{2})$.

$$\begin{aligned}
E(\frac{1-\sqrt{5}}{2}, T) &= \text{null}(T - (\frac{1-\sqrt{5}}{2})I)\\
&= \{(x,y)|(y - \frac{1-\sqrt{5}}{2}x, x + \frac{1+\sqrt{5}}{2}y) = 0\}\\
&= \{(x,y)|y = x(\frac{1-\sqrt{5}}{2})\}
\end{aligned}$$

then we choose $v_2 = (1, \frac{1-\sqrt{5}}{2})$.

(d) $(0,1) = \frac{1}{\sqrt{5}}v_1 - \frac{1}{\sqrt{5}}v_2$ in terms of the new basis. Thus

$$\begin{aligned}
T^n(0,1) &= (F_n, F_{n+1})\\
&= T^n(\frac{1}{\sqrt{5}}v_1 - \frac{1}{\sqrt{5}}v_2)\\
&= \frac{1}{\sqrt{5}}(T^n v_1 - T^n v_2)\\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n v_1 - \left(\frac{1-\sqrt{5}}{2}\right)^n v_2\right]\\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n, \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]
\end{aligned}$$

thus $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$.

(e) It suffices to show the magnitude of distance from F_n to $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$ is less than $\frac{1}{2}$. Let

$$\begin{aligned} d := & \left|\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n\right| \\ = & \left|\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n\right| \\ = & \frac{1}{\sqrt{5}}\left|\frac{1-\sqrt{5}}{2}\right|^n \\ = & \frac{1}{\sqrt{5}}\left|\frac{2}{1+\sqrt{5}}\right|^n \end{aligned}$$

be the distance.

Since $\sqrt{5} > 2$ then $\frac{1}{\sqrt{5}} < \frac{1}{2}$ and $\frac{2}{1+\sqrt{5}} < \frac{2}{3}$, thus

$$\frac{1}{\sqrt{5}}\left|\frac{2}{1+\sqrt{5}}\right|^n < \frac{1}{2}(\frac{2}{3})^n < \frac{1}{2}\frac{2}{3} < \frac{1}{2}$$
