Midterm 2

MATH 543 — Linear Algebra

Name: **Ali Zafari** Spring 2023

Problem 1

(a) $\mathcal{M}(T) = A$.

Eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} -\frac{1}{2} - \lambda & 0 & -\frac{3}{2} \\ -\frac{3}{2} & -\lambda & -\frac{3}{2} \\ 0 & 0 & 1 - \lambda \end{vmatrix} = (-\frac{1}{2} - \lambda)(-\lambda)(1 - \lambda) = 0$$

thus eigenvalues are $\lambda_1 = 0$, $\lambda_2 = -\frac{1}{2}$ and $\lambda_3 = 1$.

Eigenspaces and Eigenvectors:

- $E(0, T) = \text{null}(T - 0I)$=?

$$\begin{bmatrix} -\frac{1}{2} & 0 & -\frac{3}{2} \\ -\frac{3}{2} & 0 & -\frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow E(0, T) = span((0, 1, 0)) \Longrightarrow v_1 = (0, 1, 0)$$

- $E(-\frac{1}{2}, T) = \text{null}(T + \frac{1}{2}I)$=?

$$\begin{bmatrix} 0 & 0 & -\frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow E(-\frac{1}{2}, T) = span((1, -3, 0)) \Longrightarrow v_2 = (1, -3, 0)$$

- $E(1, T) = \text{null}(T - 1I)$=?

$$\begin{bmatrix} -\frac{3}{2} & 0 & -\frac{3}{2} \\ -\frac{3}{2} & -1 & -\frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow E(1, T) = span((1, 0, -1)) \Longrightarrow v_3 = (1, 0, -1)$$

Since $\dim V = \Sigma_{i=1}^{3} \dim E(\lambda_i, T) = 3$ then A is diagonalizable.

(b) Since A is diagonalizable, the equality $D = S^{-1}AS$ holds when S consists of the eigenvectors, and D is diagonal matrix with corresponding eigenvalues. Thus:

$$\begin{aligned}
\lim_{n\to\infty} A^n = \lim_{n\to\infty} SD^nS^{-1} &= \lim_{n\to\infty} \begin{bmatrix} 0 & 1 & 1 \\ 1 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}^n \begin{bmatrix} 0 & 1 & 1 \\ 1 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{-1} \\
&= \lim_{n\to\infty} \begin{bmatrix} 0 & 1 & 1 \\ 1 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & (-\frac{1}{2})^n & 0 \\ 0 & 0 & 1^n \end{bmatrix} \begin{bmatrix} 3 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 1 \\ 1 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

Problem 2

The only distinct eigenvalue of $T \in \mathcal{L}(\mathbb{C}^n)$ is $\lambda = 0$.

$\Longrightarrow$

If T is diagonalizable then:

$\dim \mathbb{C}^n = \dim E(0,T) = \dim \operatorname{null}(T - 0I) = \dim \operatorname{null} T$ therefore $\dim \operatorname{null} T = n$.

Since $\operatorname{null} T \subseteq \mathbb{C}^n$, we have $\operatorname{null} T = \mathbb{C}^n$. Therefore $\forall v \in \mathbb{C}^n \quad Tv = 0$ meaning that $T = 0$.

$\Longleftarrow$

If $T = 0$ then:

$E(0,T) = \operatorname{null}(T - 0I) = \operatorname{null} T = \{v | Tv = 0, v \in \mathbb{C}^n\} = \mathbb{C}^n$.

Since $\dim \mathbb{C}^n = \dim E(0,T) = n$ then T is diagonalizable.

Problem 3

(a) $v_1 = (-1, 0, 1)$ and $v_2 = (0, 1, -1)$, using Gram-Schmidt procedure:

$$e_1 = \frac{v_1}{\|v_1\|} = \frac{(-1,0,1)}{\sqrt{2}} = \frac{1}{\sqrt{2}}(-1,0,1)$$

and

$$
\begin{aligned}
e_2 &= \frac{v_2 - \langle v_2, e_1\rangle e_1}{\|v_2 - \langle v_2, e_1\rangle e_1\|} \\
&= \frac{(0,1,-1) - (-\frac{1}{\sqrt{2}})\frac{1}{\sqrt{2}}(-1,0,1)}{\|(0,1,-1) - (-\frac{1}{\sqrt{2}})\frac{1}{\sqrt{2}}(-1,0,1)\|} \\
&= \sqrt{\frac{2}{3}}(-\frac{1}{2}, 1, -\frac{1}{2})
\end{aligned}
$$

(b) Since the origin is in U, the minimum distance to it from U (length of orthogonal projection $P_U\mathbf{0}$) is 0.

$$P_U\mathbf{0} = \langle (0,0,0), e_1\rangle e_1 + \langle (0,0,0), e_2\rangle e_2 = (0,0,0)$$

Problem 4

Let $M, N, P, Q \in \mathbb{R}^2$ denote the vertices of the MNPQ quadrilateral.

$\Longrightarrow$

If $\mathbf{PM} \perp \mathbf{NQ}$, then:

Diagonals can be written as follows:

$$
\begin{aligned}
PM &= M - P \\
NQ &= Q - N
\end{aligned}
$$

Then:

$$
\begin{aligned}
\langle M - P, Q - N\rangle &= 0 \\
\langle M, Q\rangle - \langle M, N\rangle - \langle P, Q\rangle + \langle P, N\rangle &= 0 \\
2\langle M, Q\rangle - 2\langle M, N\rangle - 2\langle P, Q\rangle + 2\langle P, N\rangle &= 0 \\
\left\{\begin{array}{ccccccccc}
 & 2\langle M, Q\rangle & - & 2\langle M, N\rangle & - & 2\langle P, Q\rangle & + & 2\langle P, N\rangle \\
+ & \langle N, N\rangle & - & \langle N, N\rangle & + & \langle M, M\rangle & - & \langle M, M\rangle \\
+ & \langle Q, Q\rangle & - & \langle Q, Q\rangle & + & \langle P, P\rangle & - & \langle P, P\rangle
\end{array}\right\} &= 0 \\
\left\{\begin{array}{ccccccccc}
 & \langle N, N\rangle & - & \langle N, M\rangle & - & \langle M, N\rangle & + & \langle M, M\rangle \\
+ & \langle Q, Q\rangle & - & \langle Q, P\rangle & - & \langle P, Q\rangle & + & \langle P, P\rangle \\
- & \langle P, P\rangle & + & \langle P, N\rangle & + & \langle N, P\rangle & - & \langle N, N\rangle \\
- & \langle Q, Q\rangle & + & \langle Q, M\rangle & + & \langle M, Q\rangle & - & \langle M, M\rangle
\end{array}\right\} &= 0 \quad \text{(re-arranging)} \\
\left\{\begin{array}{cc}
 & \langle N - M, N - M\rangle \\
+ & \langle Q - P, Q - P\rangle \\
- & \langle P - N, P - N\rangle \\
- & \langle Q - M, Q - M\rangle
\end{array}\right\} &= 0
\end{aligned}
$$

Thus:

$$\langle N-M, N-M\rangle + \langle Q-P, Q-P\rangle = \langle P-N, P-N\rangle + \langle Q-M, Q-M\rangle$$
$$|MN|^2 + |PQ|^2 = |NP|^2 + |MQ|^2$$

$\Longleftarrow$

If $|\mathbf{MN}|^{\mathbf{2}} + |\mathbf{PQ}|^{\mathbf{2}} = |\mathbf{NP}|^{\mathbf{2}} + |\mathbf{MQ}|^{\mathbf{2}}$, then:

Let the sides of the quadrilateral be written in terms of its vertices as $MN = N - M$, $PQ = Q - P$, $NP = P - N$ and $MQ = Q - M$ then:

$$\langle N-M, N-M\rangle + \langle Q-P, Q-P\rangle = \langle P-N, P-N\rangle + \langle Q-M, Q-M\rangle$$
$$\langle N-M, N-M\rangle + \langle Q-P, Q-P\rangle - \langle P-N, P-N\rangle - \langle Q-M, Q-M\rangle = 0$$

$$\left\{\begin{array}{ccccccccc} & \langle N,N\rangle & - & \langle N,M\rangle & - & \langle M,N\rangle & + & \langle M,M\rangle \\ & \langle Q,Q\rangle & - & \langle Q,P\rangle & - & \langle P,Q\rangle & + & \langle P,P\rangle \\ - & \langle P,P\rangle & + & \langle P,N\rangle & + & \langle N,P\rangle & - & \langle N,N\rangle \\ - & \langle Q,Q\rangle & + & \langle Q,M\rangle & + & \langle M,Q\rangle & - & \langle M,M\rangle \end{array}\right\} = 0$$

$$-2\langle N,M\rangle - 2\langle Q,P\rangle + 2\langle P,N\rangle + 2\langle M,Q\rangle = 0$$
$$-\langle M,N\rangle - \langle P,Q\rangle + \langle P,N\rangle + \langle M,Q\rangle = 0$$
$$\langle M, Q-N\rangle - \langle P, Q-N\rangle = 0$$
$$\langle M-P, Q-N\rangle = 0$$
$$PM \perp NQ$$