Homework 3

MATH 543 — Linear Algebra

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3.D Invertibility and Isomorphic Vector Spaces

Exercise 7

(a) Three conditions for E to be a subspace $\mathcal{L}(V, W)$:

1. **additive identity** Obviously linear map $0 \in \mathcal{L}(V, W)$ is a member of E.
2. **closed under addition** Let $T_1, T_2 \in E$:
$$(T_1 + T_2)v = T_1v + T_2v = 0 + 0 = 0$$
therefore $T_1 + T_2 \in E$.
3. **closed under scalar multiplication** Let $T \in E$ and $\lambda \in \mathbb{F}$
$$(\lambda T)v = \lambda Tv = \lambda 0 = 0$$
therefore $\lambda T \in E$.

(b) Let's extend v to a basis $v, v_2, \ldots, v_n$. And assume $w_1, \ldots, w_m$ is a basis for W. Then, $\mathcal{M}$ is an isomorphism between $\mathcal{L}(V, W)$ and $\mathbb{F}^{m,n}$. Having $Tv = 0$ is equivalent to have first column of $\mathcal{M}$ be zero. Therefore $\dim E = m(n-1)$.

Exercise 10

$\Longrightarrow$

$$ST = I \Longrightarrow TS = I$$

Let $v \in V$ then $Tv = u \in V$:

$$\begin{aligned}
Tv &= u \\
STv &= Su \\
Iv &= Su \\
v &= Su \\
Tv &= TSu \\
u &= TSu
\end{aligned}$$

therefore $TS = I$.

$\Longleftarrow$

$$TS = I \Longrightarrow ST = I$$

Let $v \in V$ then $Sv = u \in V$:

$$\begin{aligned} Sv &= u \\ TSv &= Tu \\ Iv &= Tu \\ v &= Tu \\ Sv &= STu \\ u &= STu \end{aligned}$$

therefore $ST = I$.

Exercise 19

(a) Assume $p \in \mathcal{P}_n(\mathbb{R})$ is of degree n. Then arbitrary $T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ can be thought as $T_n : \mathcal{P}_n(\mathbb{R}) \to \mathcal{P}_n(\mathbb{R})$ due to $\deg Tp \leq \deg p$.
$\mathcal{P}_n(\mathbb{R})$ is finite dimensional and T_n is an operator on it, so injectivity of T_n implies its surjectivity.

(b) *proof by contradiction.*
Let $\deg Tp \neq \deg p$ for **invertible** operator $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$. Then $\deg Tp \leq \deg p$ translates to $\deg Tp < \deg p$.
But any linear map to a smaller dimensional space is **not injective** (*contradiction*).
Therefore we must have $\deg Tp = \deg p$.

3.E Products and Quotients of Vector Spaces

Exercise 2

Since $V_1 \times \cdots \times V_m$ is finite-dimensional it has a basis of length

$$\begin{aligned} \dim V_1 + \cdots + \dim V_m &< \infty \\ \dim V_i &< \infty \quad (\dim V_i \geq 0) \quad \forall i \end{aligned}$$

Exercise 8

$\Longrightarrow$

A is an affine subset of V. Then exists subspace U of V such that $A = a + U$ where $a \in V$.

For $v, w \in A$ there exists $u_1, u_2 \in U$ such that $v = a + u_1$ and $w = a + u_2$.

$\forall \lambda \in \mathbb{F}$:

$$\lambda v + (1-\lambda)w = \lambda(a + u_1) + (1-\lambda)(a + u_2) = \underbrace{a + \underbrace{\lambda u_1 + (1-\lambda)u_2}_{\in U}}_{\in A}$$

$\Longleftarrow$

$\lambda v + (1-\lambda)w \in A \quad \forall v, w \in A, \forall \lambda \in \mathbb{F}$ by choosing $a \in A$ we define $U \triangleq -a + A$.

For U to be a subspace of V:

1. **additive identity**.

   Obviously $-a \in A$, so $0 \in U$.

2. **closed scalar multiplication**.

   Let $u \in U$ then exists $b \in A$ such that $u = -a + b$:

$$\begin{aligned} \lambda b + (1-\lambda)a &\in A \\ a + \lambda(-a + b) &\in A \\ \lambda(\underbrace{-a + b}_{=u}) &\in \underbrace{-a + A}_{=U} \end{aligned}$$

3. **closed addition**.

Let $u_1, u_2 \in U$, then exist $a_1, a_2 \in A$ such that $u_1 = -a + a_1$ and $u_2 = -a + a_2$.

$$u_1 + u_2 = -2a + a_1 + a_2 = 2(-a + \frac{1}{2}a_1 + \frac{1}{2}a_2) = \underbrace{2\underbrace{(-a + \underbrace{\frac{1}{2}a_1 + (1 - \frac{1}{2})a_2}_{\in A})}_{\in U}}_{\in U \text{ (closed scalar mult.)}}$$

Exercise 11

(a) Let $v, w \in A$

$$v = a_1 v_1 + \cdots + a_m v_m$$
$$w = b_1 v_1 + \cdots + b_m v_m$$

where $a_1, \ldots, a_m, b_1, \ldots, b_m \in \mathbb{F}$ and $\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} b_i = 1$.

$\forall \lambda \in \mathbb{F}$:

$$\begin{aligned} \lambda v + (1-\lambda)w &= \lambda(a_1 v_1 + \cdots + a_m v_m) + (1-\lambda)(b_1 v_1 + \cdots + b_m v_m) \\ &= \underbrace{(\lambda a_1 + (1-\lambda)b_1)v_1 + \cdots + (\lambda a_m + (1-\lambda)b_m)v_m}_{\in A \text{ (by definition of } A\text{, since } \sum_{i=1}^{m} \lambda a_i + (1-\lambda)b_i = 1)} \end{aligned}$$

By using the result of **Exercise 8**, A will be an affine subset of V.

(b) Let $v \in V$ and U is a subspace of V, such that $v_1, \ldots, v_m \in v + U$. Therefore exists $u_1, \ldots, u_m \in U$ such that $v_i = v + u_i \quad \forall i = 1, \ldots, n$.

Assume $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$ such that $\sum_{i=1}^{m} \lambda_i = 1$, then

$$\begin{aligned} \underbrace{\lambda_1 v_1 + \cdots + \lambda_m v_m}_{\text{arbitrary element in } A} &= \lambda_1(v + u_1) + \cdots + \lambda_m(v + u_m) \\ &= (\lambda_1 + \cdots + \lambda_m)v + \lambda_1 u_1 + \cdots + \lambda_m u_m \\ &= \underbrace{v + \underbrace{\lambda_1 u_1 + \cdots + \lambda_m u_m}_{\in U}}_{\in v+U} \end{aligned}$$

Therefore $A \subset v + U$.

(c) Let $v \in A$:

$$\begin{aligned} v = a_1 v_1 + \cdots + a_m v_m &= (1 - \sum_{i=2}^{m} a_i)v_1 + a_2 v_2 + \cdots + a_m v_m \\ &= v_1 + a_2(v_2 - v_1) + a_3(v_3 - v_1) + \cdots + a_m(v_m - v_1) \end{aligned}$$

Let $U \triangleq span(v_2 - v_1, \dots, v_m - v_1)$. Therefore $v = v_1 + U$. As a result $A \subset v_1 + U$. Now suppose $w \in v_1 + U$ and $b_2, \dots, b_m \in \mathbb{F}$

$$\begin{aligned} w &= v_1 + b_2(v_2 - v_1) + \cdots + b_m(v_m - v_1) \\ &= (1 - \sum_{i=2}^{m} b_i)v_1 + b_2 v_2 + \cdots + b_m v_m \end{aligned}$$

since $1 - \sum_{i=2}^{m} b_i + b_2 + \cdots + b_m = 1$, then $w \in A$ meaning that $v_1 + U \subset A$. As a result $A = v_1 + U$ and obviously $\dim U \leq m - 1$.

Exercise 12

V/U has basis of $v_1 + U, \dots, v_n + U$. $\forall v \in V \quad \exists a_1, \dots, a_n \in \mathbb{F}$ such that

$$v + U = \sum_{i=1}^{n} a_i(v_i + U) \Rightarrow v - \sum_{i=1}^{n} a_i v_i \in U$$

Let's define $T : V \to U \times V/U$ then $Tv = (v - \sum_{i=1}^{n} a_i v_i, \sum_{i=1}^{n} a_i(v_i + U)) \quad \forall v \in V$.

We will show T is an isomorphism.

- **linearity**. $\forall x, y \in V$:

$$\begin{aligned} x + U &= \sum_{i=1}^{n} b_i(v_i + U) \\ y + U &= \sum_{i=1}^{n} c_i(v_i + U) \end{aligned}$$

  where $b_1, \dots, b_n, c_1, \dots, c_n \in \mathbb{F}$. Their linear combination is $d_1(x + U) + d_2(y + U) = \sum_{i=1}^{n}(d_1 b_i + d_2 c_i)(v_i + U)$ where $d_1, d_2 \in \mathbb{F}$.
  So

$$\begin{aligned} T(d_1 x + d_2 y) &= (d_1 x + d_2 y - \sum_{i=1}^{n}(d_1 b_i + d_2 c_i)v_i, \sum_{i=1}^{n}(d_1 b_i + d_2 c_i)(v_i + U)) \\ &= d_1 Tx + d_2 Ty \end{aligned}$$

- **injectivity**.

If $Tv = 0$, then $v = 0$, as shown below

$$Tv = \underbrace{(v - \sum_{i=1}^{n} a_i v_i, \underbrace{\sum_{i=1}^{n} a_i(v_i + U)}_{a_i=0 \quad \forall i \text{ (basis)}}}_{v=0}) = (0,0)$$

Therefore null$T = \{0\}$.

- **surjectivity**. Let for $u \in U$ we had $Tv = (u, \sum_{i=1}^{n} a_i(v_i + U)) \in U \times V/U$ where $v \in V$. It is clear that v is uniquely determined as $v = u + \sum_{i=1}^{n} a_i v_i$.

Exercise 17

Let $v_1 + U, \ldots, v_n + U$ be a basis of V/U. Define the spanning list $W \triangleq span(v_1, \ldots, v_n)$. Let $a_1, \ldots, a_n \in \mathbb{F}$ such that

$$a_1 v_1 + \cdots + a_n v_n = 0$$

since $v_1 + U, \ldots, v_n + U$ are linearly independent, $a_1(v_1 + U) + \cdots + a_n(v_n + U) = a_1 v_1 + \cdots + a_n v_n + U$ is zero only when $a_1 = \cdots = a_n = 0$. Therefore $\dim W = \dim V/U$.

To have $V = U \oplus W$:

1. $\mathbf{V} = \mathbf{U} + \mathbf{W}$.

   For $v \in V$ exists $b_1, \ldots, b_n \in \mathbb{F}$ such that $v + U = b_1(v_1 + U) + \cdots + b_n(v_n + U)$. Therefore $v - \sum_{i=1}^{n} b_i v_i \in U$. So

   $$v = \underbrace{(v - \sum_{i=1}^{n} b_i v_i)}_{\in U} + \underbrace{\sum_{i=1}^{n} b_i v_i}_{\in W}$$

   meaning that $v \in U + W$ hence $V \subset U + W$.

   Since U and W are subspaces of V, $U + W \subset V$.

2. $\dim \mathbf{V} = \dim \mathbf{U} + \dim \mathbf{W}$.

   We showed $V = U + W$. We also know

   $$\begin{aligned} \dim V/U &= \dim V - \dim U \\ \dim W &= \dim V - \dim U \\ \dim V &= \dim U + \dim W \end{aligned}$$

   therefore $V = U \oplus W$.
