Homework 1

MATH 543 — Linear Algebra

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1.A $\mathbb{R}^n$ and $\mathbb{C}^n$

Exercise 10

$$\begin{aligned}
(4,-3,1,7)+2x &= (5,9,-6,8) \\
2x &= (5,9,-6,8)+(-4,+3,-1,-7) && \text{(Additive Inverse)} \\
x &= \frac{1}{2}(1,12,-7,1) && \text{(Scalar Multiplication)} \\
x &= (0.5,6,-3.5,0.5)
\end{aligned}$$

Exercise 11

Assume $\lambda \in \mathbb{C}$, then we can write λ as $\lambda = a + bi$ where $a, b \in \mathbb{R}$. By using the the scalar multiplication property in $\mathbb{C}^3$ and the fact that two list are equal if their elements are equal in the same order, we have:

$$2a+3b+(-3a+2b)i = 12-5i \tag{1}$$
$$5a-4b+(4a+5b)i = 7+22i \tag{2}$$
$$-6a-7b+(7a-6b)i = -32-9i \tag{3}$$

from equation 1: $a = 3$, $b = 2$
from equation 2: $a = 3$, $b = 2$
from equation 3: $a = 1.5176$, $b = 3.2706$

All the three equations cannot be satisfied simultaneously.

1.B Vector Space

Exercise 1

By additive inverse property of V:

$$\begin{aligned} -v + (-(-v)) &= 0 \\ v - v + (-(-v)) &= 0 + v \\ (-(-v)) &= 0 + v \\ -(-v) &= v \end{aligned}$$

Exercise 2

I. By use of scalar multiplication distributive property:

$$\begin{aligned} av &= 0 \\ av &= av + (-av) \\ av &= (a + (-a))v \\ av &= 0v \\ a &= 0 \end{aligned}$$

II. By use of vector addition distributive property:

$$\begin{aligned} av &= 0 \\ av &= av + (-av) \\ av &= a(v + (-v)) \\ av &= a0 \\ v &= 0 \end{aligned}$$

Exercise 4

Additive Identity. There must exist an element $0 \in V$ such that $v + 0 = v \quad \forall v \in V$, but the empty set has no element at all.

Exercise 6

By using a counterexample we will prove that $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ is not a vector space over $\mathbb{R}$. Assume $\lambda, \gamma \in \mathbb{R}$ and $v = \infty$. By the distributive property of a vector space:

$$(\lambda + \gamma)\infty = \lambda\infty + \gamma\infty$$

if we set $\lambda > 0, \gamma < 0$ and $\lambda + \gamma \neq 0$:

- LHS: ∞ or $-\infty$, depending on the value of $\lambda + \gamma$
- RHS: 0

It says that the additive identity is equal to ∞ or $-\infty$. This violates property (1.25) discussed in book which tells that **in a vector space the additive identity must be unique**.

1.C Subspaces

Exercise 1

(a) $x_1+2x_2+3x_3 = 0$ is the equation of a two dimensional plane in $\mathbb{F}^3$. It includes zero (0) and any linear combination of two points on this plane stays on the plane. Therefore it is a subspace of $\mathbb{F}^3$.

(b) $x_1+2x_2+3x_3 = 4$ is the equation of a two dimensional plane in $\mathbb{F}^3$. It does not contain the zero (0) vector. So it is not a valid subspace of $\mathbb{F}^3$.

(c) $x_1x_2x_3 = 0$ is the equation of points on 3 separate two dimensional planes in $\mathbb{F}^3$. The linear combination of two vectors chosen on 2 different planes may not necessarily stays on the planes. Therefore it is not a valid subspace of $\mathbb{F}^3$.

(d) $x_1 = 5x_3$ is a plane crossing the origin in $\mathbb{F}^3$. This space contains zero and any linear combination of its elements stays in this space. Hence this is a subspace of $\mathbb{F}^3$.

Exercise 4

We call the target space U. For U to be a subspace must have a unique additive identity, $0 + f = f$ such that $0, f \in U$. The condition of vectors in this space must be satisfied even for the additive identity:

$$\int_0^1 0 = b$$

therefore we must have $b = 0$ for this space to be a subspace of $\mathbb{R}^{[0,1]}$. (closure on the linear combination is obvious when $b = 0$.)

Exercise 6

(a) In this case, $a^3 = b^3$ implies $a = b$, hence we will have $\{(a, b, c) \in \mathbb{R}^3 : a = b\}$ and this space includes zero and is closed on any linear combination of its members. This is a subspace of $\mathbb{R}^3$.

(b) In the complex numbers case, solution to the equation $a^3 = b^3$ could also be $a = b\frac{1\pm\sqrt{3}i}{2}$, other than the obvious $a = b$.
If we assume $b = 1$, we will have two vectors $(\frac{1+\sqrt{3}i}{2}, 1, 0)$ and $(\frac{1-\sqrt{3}i}{2}, 1, 0)$ which their addition vector $(1, 2, 0)$ does not stay in the same space. Therefore this is not a subspace of $\mathbb{R}^3$.

Exercise 8

$$U = \{(x, y) \in \mathbb{R}^2 : x + y = 0 \vee x - y = 0\}$$

Exercise 10

If U_1 and U_2 are subspaces of V, they both contain zero, so their intersection at least includes zero. Any linear combination is closed on U_1 or U_2, and since any vector in $U_1 \cap U_2$ is also a member of U_1 then the linear combination is also closed for the intersection.
Finally, by having the zero and linear combination closure, $U_1 \cap U_2$ is a subspace of V.

Exercise 12

U and W are subspaces of V. We prove it in two steps:

$\Longrightarrow$

Here, we assume $U \subseteq W$ and try to show that $U \cup W$ is a subspace of V. By U being a subset of W we have $U \cup W = W$. Since W is a subspace of V, then $U \cup W$ will be a subspace of V as well. (same proof can be derived when $W \subseteq U$)

$\Longleftarrow$

We prove it by contradiction. Assuming $U \cup W$ is a subspace of V then we try to show that neither $U \subseteq W$ nor $W \subseteq U$. By having $U \nsubseteq W$ and $W \nsubseteq U$ it is obvious that the subtract of sets are non-empty. Now let $u \in U \setminus W$ and $w \in W \setminus U$. Now assume $u + w \in U$, then we must have $u + w - u \in U$. It is a contradiction since we had assumed that $w \in W \setminus U$. The same statements can be derived when we assume $u + w \in W$. So this is a contradiction. As a result either $U \subseteq W$ or $W \subseteq U$.

Exercise 23

Counterexample: Let's have $U_1 = \{(x, 0) \in \mathbb{R}^2 : x \in \mathbb{R}\}$ and $U_2 = \{(x, 1) \in \mathbb{R}^2 : x \in \mathbb{R}\}$. Then by letting $W = \{(0, y) \in \mathbb{R}^2 : y \in \mathbb{R}\}$, the vector space $V = \mathbb{R}^2$ can be written as direct sums $U_1 \oplus W$ or $U_2 \oplus W$, however $U_1 \neq U_2$.

Exercise 24

$\mathbb{R}^{\mathbb{R}}$ denotes any function $f : \mathbb{R} \to \mathbb{R}$. The function f can be written as sum of even and odd functions:

$$\begin{aligned} f(x) &= \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} \\ &= f_e(x) + f_o(x) \end{aligned}$$

where $x \in \mathbb{R}$. Since for any $f \in \mathbb{R}^{\mathbb{R}}$ there exists a unique $f_e + f_o \in U_e + U_o$, we can conclude that $\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o$.

2.A Span and Linear Independence

Exercise 3

To have a list of 3 linearly dependent vectors v_1, v_2, v_3 there should exist a nonzero set of coefficients $a_1, a_2, a_3 \in \mathbb{R}$ which make their linear combination equal to zero:

$$a_1(3, 1, 4) + a_2(2, -3, 5) + a_3(5, 9, t) = 0$$

We must choose t such that avoid trivial zero solution for a_1, a_2, a_3. If $a_1 = 3$ and $a_2 = -2$ and $a_3 = 1$ then we can see that by letting $t = 2$ vectors will be linearly dependent.

Exercise 6

We know that:

$$a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = 0$$

holds only when $a_1 = a_2 = a_3 = a_4 = 0$ ($a_1, \dots, a_4 \in \mathbb{F}$). Then linear combination for list $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ will be ($b_1, \dots, b_4 \in \mathbb{F}$):

$$\begin{aligned} b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4v_4 &= b_1v_1 + (b_2 - b_1)v_2 + (b_3 - b_2)v_3 + b_4v_4 \\ &= a_1'v_1 + a_2'v_2 + a_3'v_3 + a_4'v_4 \end{aligned}$$

since the list $v_1, \dots, v_4$ is linearly independent, the linear combination of the above will be equal to zero only if $a_1' = a_2' = a_3' = a_4' = 0$ ($a_1', \dots, a_4' \in \mathbb{F}$). Therefore the list $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ is linearly independent as well.

Exercise 9

Counterexample: assume the list $w_1, \dots, w_m$ be equal to $-v_1, \dots, -v_m$. Then list $v_1 + w_1, \dots, v_m + w_m$ is equal to $v_1 - v_1, \dots, v_m + -v_m$ which only contains the vector 0 and it is linearly dependent.

Exercise 11

We prove it in two steps. The list of vectors $v_1, \ldots, v_m$ is linearly independent.

$\Longrightarrow$

Now we try to prove that if $w \notin span(v_1, \ldots, v_m)$, then the list of vectors $v_1, \ldots, v_m, w$ is linearly dependent. For the list of dependent vectors we have:

$$a_1 v_1 + \cdots + a_m v_m + a_{m+1} w = 0$$

where not all the a_i's are equal to zero.

Conditioning on the value of a_{m+1}:

- if $a_{m+1} = 0$: In this case all other a_i's must be zero which shows that $v_1, \ldots, v_m, w$ are not linearly dependent. (contradiction)
- if $a_{m+1} \neq 0$: In this case we can write $w = -\frac{1}{a_{m+1}}(a_1 v_1 + \cdots + a_m v_m)$, which tells us that $w \in span(v_1, \ldots, v_m)$, violating the hypothesis.

Therefore the list of vectors $v_1, \ldots, v_m, w$ is linearly independent.

$\Longleftarrow$

Proving that if the list of vectors $v_1, \ldots, v_m, w$ is linearly independent then $w \in span(v_1, \ldots, v_m)$. From the definition of span, we know that w can be written as:

$$w = a_1 v_1 + \cdots + a_m v_m$$

where $a_1, \ldots, a_m \in \mathbb{F}$. Then we can see that $a_1 v_1 + \cdots + a_m v_m - w = 0$, which violates the linear independence of $v_1, \ldots, v_m, w$. As a result $w \notin span(v_1, \ldots, v_m)$.

Exercise 17

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