Extra Problems

MATH 543 — Linear Algebra

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Let $\dim V = n$. If $T \in \mathcal{L}(V)$ is nilpotent, then $T^n = 0$.

$T^j = 0$ for some integer $j \geq 1$ thus $\operatorname{null} T^j = V$.
Growth of $\operatorname{null} T^k$ with increase of k (non-negative integer) stops at $k = n$.

- $\mathbf{j} < \mathbf{n}$: since $\operatorname{null} T^j \subseteq \operatorname{null} T^n$ then $\operatorname{null} T^n = V$.
- $\mathbf{j} \geq \mathbf{n}$: having $\operatorname{null} T^n \neq V$ is a contradiction.

therefore $\operatorname{null} T^n = V$ so $T^n = 0$.

If λ is an eigenvalue for $T \in \mathcal{L}(V)$ then λ^n is an eigenvalue for T^n.

For $v \neq 0$ we have $Tv = \lambda v$:

$$\begin{aligned}
T^2 v &= T(Tv) = T(\lambda v) = \lambda T v = \lambda^2 v \\
T^3 v &= T(T^2 v) = T(\lambda^2 v) = \lambda^2 T v = \lambda^3 v \\
&\vdots \\
T^n v &= T(T^{n-1} v) = T(\lambda^{n-1} v) = \lambda^{n-1} T v = \lambda^n v
\end{aligned}$$

thus λ^n is an eigenvalue for T^n.

If λ^n is an eigenvalue for $T^n \in \mathcal{L}(V)$ then λ is an eigenvalue for T?

For $v \neq 0$ we have $T^n v = \lambda^n v$:

$$\begin{aligned}
(T^n - \lambda^n I)v &= 0 \\
p(T)v &= 0
\end{aligned}$$

where $p(z) = z^n - \lambda^n \in \mathcal{P}(\mathbb{C})$.
We use factorization $p(z) = (z - \lambda)(z^{n-1} + z^{n-2}\lambda + \cdots + z\lambda^{n-2} + \lambda^{n-1})$:

$$p(T)v = (T - \lambda)(T^{n-1} + \lambda T^{n-2} + \cdots + \lambda^{n-2} T + \lambda^{n-1})v = 0$$

thus if $(T^{n-1} + \lambda T^{n-2} + \cdots + \lambda^{n-2} T + \lambda^{n-1})v \neq 0$, then λ must be an eigenvalue for T.
