3A Exercise 2

First we show that $\int g d\delta_c = g(c)$ for any simple function $g = \sum_{i=1}^{n} e_i \chi_{E_i}$:

$$\int g d\delta_c = \int \left(\sum_{i=1}^{n} e_i \chi_{E_i} \right) d\delta_c = \sum_{i=1}^{n} e_i \delta_c(E_i) = \sum_{i=1}^{n} e_i \chi_c(E_i)(c) = g(c)$$

where we used 3.7 of Measure, Integration & Real Analysis.

Using approximation of an $\mathcal{S}$-measurable function by simple functions and Monotone Convergence Theorem (2.89, 3.11 of Measure, Integration & Real Analysis), suppose $f_1, f_2, \ldots$ is an increasing sequence of simple functions where $f = \lim_{k\to\infty} f_k$. Since each f_k is a simple function, we have $\int f_k d\delta_c = f_k(c)$.

And since $\int f d\delta_c = \lim_{k\to\infty} \int f_k d\delta_c = \lim_{k\to\infty} f_k(c)$, then we have $\int f d\delta_c = f(c)$.

3 A Exercise 3

We have $\{x \in X : f(x) > 0\} = \lim_{k\to\infty}\{x \in X : f(x) > \frac{1}{k}\}$.

$\boxed{\Rightarrow}$

If $\int f d\mu > 0$ then

$$\int_{\{x\in X: f(x)>0} f d\mu > 0$$

therefore we must have $\mu(\{x \in X : f(x) >\}) > 0$ since otherwise will be a contradiction (integration over empty set being greater than zero).

$\boxed{\Leftarrow}$

If $\mu(\{x \in X : f(x) > 0\}) > 0$ then there exists $k_0 \in \mathbb{Z}^+$ such that $\mu(\{x \in X : f(x) > \frac{1}{k_0}\}) > 0$. Therefore:

$$\int_X f d\mu \geq \int_{\{x\in X: f(x)>\frac{1}{k_0}\}} f d\mu \geq \frac{1}{k_0}\mu(\{x \in X : f(x) > \frac{1}{k_0}\}) > 0$$

3A Exercise 9

Three properties of a measure should be verified for the proposed measure of $v(A) = \int \chi_A f d\mu = \int_A f d\mu$:

- Since $v(A) = \int_A f d\mu$ is the supremum over lower Lebesgue sums, and each of the sums are non-negative, $v(A)$ must be non-negative as well. Therefore $v(A) \geq 0 \quad \forall A \in \mathcal{S}$.
- Integral of f over an empty set is zero, since the measure of that set is zero. Therefore $v(\varnothing) = 0$.
- Suppose disjoint sequence of sets $E_1, E_1, \cdots \in \mathcal{S}$:

$$\begin{aligned} v(\bigcup_i E_i) &= \int \chi_{\cup_i E_i} f d\mu \\ &= \sum_i \int \chi_{E_i} f d\mu \\ &= \sum_i v(E_i) \end{aligned}$$

3A Exercise 15

(a) First we prove it for a simple function g. $g(x) = \sum_{i=1}^{n} c_i \chi_{E_i}(x)$ and $g_t(x) = \sum_{i=1}^{n} c_i \chi_{E_i}(x-t) = \sum_{i=1}^{n} c_i \chi_{t+E_i}(x)$.

Then computing the integral:

$$\begin{aligned}
\int g_t d\lambda &= \int \sum_{i=1}^{n} c_i \chi_{t+E_i} d\lambda \\
&= \sum_{i=1}^{n} c_i \lambda(t+E_i) \quad \boxed{\text{3.15 of Measure, Integration \& Real Analysis}} \\
&= \sum_{i=1}^{n} c_i \lambda(E_i) \quad \boxed{\text{2.7 of Measure, Integration \& Real Analysis}} \\
&= \int \sum_{i=1}^{n} c_i \chi_{E_i} d\lambda \\
&= \int g d\lambda
\end{aligned}$$

Using approximation of an $\mathcal{S}$-measurable function by simple functions and Monotone Convergence Theorem (2.89, 3.11 of Measure, Integration & Real Analysis), suppose $f^1, f^2, \ldots$ is an increasing sequence of simple functions where $f = \lim_{k\to\infty} f^k$. Since each f_k is a simple function, we have $\int f_t^k d\lambda = \int f^k d\lambda$.

Taking limit of both sides and using Monotone Convergence Theorem we have $\int f_t d\lambda = \int f d\lambda$.

(b) First we prove it for a simple function g. $g(x) = \sum_{i=1}^{n} c_i \chi_{E_i}(x)$ and $g_t(x) = \sum_{i=1}^{n} c_i \chi_{E_i}(tx) =$

$\sum_{i=1}^{n} c_i \chi_{\frac{1}{t}E_i}(x)$. Then computing the integral:

$$\begin{aligned}
\int g_t d\lambda &= \int \sum_{i=1}^{n} c_i \chi_{\frac{1}{t}E_i} d\lambda \\
&= \sum_{i=1}^{n} c_i \lambda(\frac{1}{t}E_i) \quad \boxed{\text{3.15 of Measure, Integration \& Real Analysis}} \\
&= \frac{1}{|t|} \sum_{i=1}^{n} c_i \lambda(E_i) \quad \boxed{\text{Exercise 2A.2 of Measure, Integration \& Real Analysis}} \\
&= \frac{1}{|t|} \int \sum_{i=1}^{n} c_i \chi_{E_i} d\lambda \\
&= \frac{1}{|t|} \int g d\lambda
\end{aligned}$$

Using approximation of an $\mathcal{S}$-measurable function by simple functions and Monotone Convergence Theorem (2.89, 3.11 of Measure, Integration & Real Analysis), suppose $f^1, f^2, \ldots$ is an increasing sequence of simple functions where $f = \lim_{k \to \infty} f^k$. Since each f_k is a simple function, we have $\int f_t^k d\lambda = \int f^k d\lambda$.

Taking limit of both sides and using Monotone Convergence Theorem we have $\int f_t d\lambda = \int f d\lambda$.