2A Exercise 3

$$B \subseteq B = B \cap (A \cup A^c) = (B \cap A) \cup (B \cap A^c) = (B \cap A) \cup (B \setminus A)$$

by using order-preserving property of outer measure:

$$|B| \leq |(B \cap A) \cup (B \setminus A)|$$

and using subadditiviy property of outer measure:

$$|B| \leq |B \cap A| + |B \setminus A| \tag{1}$$

we also have $A \cap B \subseteq A$, so again by subadditiviy property of outer measure we have $|A \cap B| \leq |A|$, hence by replacing in Eq. 1:

$$|B| \leq |A| + |B \setminus A|$$
$$|B \setminus A| \geq |B| - |A|$$

2A Exercise 6

We know from 2.14 of Measure, Integration & Real Analysis that $|[a,b]| = b-a$.

- $\mathbf{|(a,b)| = b-a}$
  - To show $|(a,b)| \le b-a$:
  $$(a,b) \subseteq [a,b] \implies |(a,b)| \le |[a,b]| \implies |(a,b)| \le b-a$$
  - To show $|(a,b)| \ge b-a$:
  We have $[a,b] = \{a\} \cup \{b\} \cup (a,b)$ and by using subaditivity property of outer measure:
  $$\underbrace{|[a,b]|}_{b-a} \le \underbrace{|\{a\}|}_{0} + \underbrace{|\{b\}|}_{0} + |(a,b)| \implies |(a,b)| \ge b-a$$
  Therefore, $|(a,b)| = b-a$.
- $\mathbf{|[a,b)| = b-a}$
  - To show $|[a,b)| \le b-a$:
  $$[a,b) \subseteq [a,b] \implies |[a,b)| \le |[a,b]| \implies |[a,b)| \le b-a$$
  - To show $|[a,b)| \ge b-a$:
  We have $[a,b] = \{b\} \cup [a,b)$ and by using subaditivity property of outer measure:
  $$\underbrace{|[a,b]|}_{b-a} \le \underbrace{|\{b\}|}_{0} + |[a,b)| \implies |[a,b)| \ge b-a$$
  Therefore, $|[a,b)| = b-a$.
- $\mathbf{|(a,b]| = b-a}$
  - To show $|(a,b]| \le b-a$:
  $$(a,b] \subseteq [a,b] \implies |(a,b]| \le |[a,b]| \implies |(a,b]| \le b-a$$
  - To show $|(a,b]| \ge b-a$:
  We have $[a,b] = \{a\} \cup (a,b]$ and by using subaditivity property of outer measure:
  $$\underbrace{|[a,b]|}_{b-a} \le \underbrace{|\{a\}|}_{0} + |(a,b]| \implies |(a,b]| \ge b-a$$
  Therefore, $|(a,b]| = b-a$.

2A Exercise 10

We know from 2.14 of Measure, Integration & Real Analysis that $|[0,1]| = 1$.

- To show $|[0,1] \setminus Q| \leq 1$:

$$[0,1] \setminus Q \subseteq [0,1] \implies |[0,1] \setminus Q| \leq 1$$

- To show $|[0,1] \setminus Q| \geq 1$:
  By using the proved result of 2A Exercise 3 and knowing that $|Q| = 0 < \infty$:

$$|[0,1] \setminus Q| \geq |[0,1]| - \underbrace{|Q|}_{0} \implies |[0,1] \setminus Q| \geq 1$$

Therefore$|[0,1] \setminus Q| = 1$.

2A Exercise 12

$$F = \mathbb{R} \setminus \bigcup_{k=1}^{\infty} (r_k - \frac{1}{2^k}, r_k + \frac{1}{2^k})$$

(a) Let $A := \bigcup_{k=1}^{\infty} \left(r_k - \frac{1}{2^k}, r_k + \frac{1}{2^k}\right)$.
We know from 0.55 of Supplement for Measure, Integration & Real Analysis that union of open subsets in $\mathbb{R}$ is an open subset. Therefore, since F is the complement of A and must be a closet subset of $\mathbb{R}$ by 0.61 of Supplement for Measure, Integration & Real Analysis.

(b) By definition, there exists no rational number in F.
Let $a, b \in I \subseteq F$ where a and b are two arbitrary distinct irrational numbers ($a \neq b$).
WLOG we assume $b > a$. Since I is supposed to be an interval, then $(a, b) \subseteq I$.
By 0.30 of Supplement for Measure, Integration & Real Analysis there must exist a rational number in (a, b), which is a contradiction with the definition of F.
Therefore a and b cannot be distinct and I contains at most one element.

(c) To be able to use the proved result of 2A Exercise 3 we must first show that $|A| < \infty$:

$$|A| = \left|\bigcup_{k=1}^{\infty} (r_k - \frac{1}{2^k}, r_k + \frac{1}{2^k})\right| < \sum_{k=1}^{\infty} \left|(r_k - \frac{1}{2^k}, r_k + \frac{1}{2^k})\right| = 2 < \infty$$

So we have:

$$|F| = \left|\mathbb{R} \setminus \bigcup_{k=1}^{\infty} (r_k - \frac{1}{2^k}, r_k + \frac{1}{2^k})\right| \geq |\mathbb{R}| - 2 > \infty$$