2E Exercise 1

Suppose $f_1, f_2, \ldots$ is a sequence of functions from X to $\mathbb{R}$, point-wise converging to f for each $x \in X$.
Therefore, for each $x_i \in X$ and $\epsilon > 0$ there exists $n_i \in \mathbb{Z}^+$ such that $|f_k(x_i) - f(x_i)| < \epsilon$ for all $k \geq n_i$. Since X is a finite set, there exists

$$N = \max\{n_i \mid n_i \text{ is the smallest positive integer satisfying } |f_k(x_i) - f(x_i)| < \epsilon,\ \forall k \geq n_i, \epsilon > 0, \forall x_i \in X\}$$

and by having N defined as above, the convergence of f_k to f will be independent of the chosen $x \in X$. Therefore f_k is uniformly convergent to f.

2E Exercise 4

Uniform continuity of f_k: $\forall \epsilon > 0$ there exists $\delta > 0$ such that $|f_k(a) - f_k(b)| < \epsilon$, $\forall k \in \mathbb{Z}^+$, for all $a, b \in A$ with $|a - b| < \delta$.

Uniform convergence of f_k to f: $\forall \epsilon > 0$ there exists $n \in \mathbb{Z}^+$ we have $\forall x \in A$ that $|f_k(x) - f(x)| < \epsilon$ for all integers $k \geq n$.

Suppose there exists $\delta > 0$ and $a, b \in A$ with $|a - b| < \delta$ such that $|f_k(a) - f_k(b)| < \epsilon'$ for $\epsilon' > 0$. Also suppose that for the same $\epsilon' > 0$ there exists $n \in \mathbb{Z}^+$ we have $|f_k(x) - f(x)| < \epsilon'$ for all integers $k \geq n$ and $\forall x \in A$.

Therefore we can write:

$$\begin{aligned} |f(a) - f(b)| &= |f(a) - f_n(a) + f_n(a) - f_n(b) + f_n(b) - f(b)| \\ &\leq |f(a) - f_n(a)| + |f_n(a) - f_n(b)| + |f_n(b) - f(b)| \\ &\leq \epsilon' + \epsilon' + \epsilon' = \epsilon \end{aligned}$$

where the last equality holds by choosing $\epsilon' = \frac{\epsilon}{3}$. Therefore the function f is also uniformly continuous.

2E Exercise 5

Suppose sequence of functions $f_1, f_2, \dots$ from $[0, \infty]$ to $\{0, 1\}$ be defined as:

$$f_k(x) = \begin{cases} 1 & \text{if } x \in [k-1, k] \\ 0 & \text{otherwise} \end{cases}$$

hence this sequence if pointwise convergent to zero function.

Suppose Egorov's theorem is true for this case and let $E \subseteq \mathbb{R}$ such that this sequence converges uniformly to zero function on E, i.e.,

$$\forall \epsilon > 0 \;\; \exists n \in \mathbb{Z}^+ \;\; \forall k \geq n \;\; \forall x \in E \text{ such that } |f_k(x) - 0| < \epsilon$$

Now assume $\epsilon = 1$, then by definition of f_k we must have $x \notin [n, \infty)$. As a result we must have $E \subseteq [0, n)$. We can see that $\mu(\mathbb{R} \setminus E) > \mu(\mathbb{R} \setminus [0, n)) = \mu([n, \infty)) = \infty$ and violates the Egorov's theorem. Therefore the hypothesis $\mu(X) < \infty$ is necessary for the Egorov's theorem.

2E Exercise 15

By 2.89 of Measure, Integration & Real Analysis there exists a sequence $f_1, f_2, \ldots$ of Lebesgue measurable functions from B to $\mathbb{R}$ converging pointwise on B to f. Suppose $k \in \mathbb{Z}^+$. Then there exists $c_1, \ldots, c_n \in B$ and disjoint Lebesgue measurable sets $A_1, \ldots, A_n \subseteq B$ such that

$$f_k = c_1\chi_{B_1} + \cdots + c_n\chi_{B_n}.$$

For each $j \in \{1, \ldots, n\}$, there exists a Borel set $B_j \subseteq A_j$ such that $|A_j \setminus B_j| = 0$.
Let

$$g_k = c_1\chi_{B_1} + \cdots + c_n\chi_{B_n}.$$

Then g_k is a Borel measurable function and $|\{x \in B : g_k(x) \neq f_k(x)\}| = 0$.
If $x \notin \bigcup_{k=1}^{\infty}\{x \in B : g_k(x) \neq f_k(x)\}$, then $g_k(x) = f_k(x)$ for all $k \in \mathbb{Z}^+$ and hence $\lim_{k\to\infty} g_k(x) = f(x)$. Let

$$E = \{x \in B : \lim_{k\to\infty} g_k(x) \text{ exists in } \mathbb{R}\}.$$

Then E is a Borel subset of B by 2B Exercise 14 of Measure, Integration & Real Analysis. Also,

$$B \setminus E \subseteq \bigcup_{k=1}^{\infty}\{x \in B : g_k(x) \neq f_k(x)\}$$

and thus $|B \setminus E| = 0$. For $x \in B$, let

$$g(x) = \lim_{k\to\infty}(\chi_E g_k)(x) \tag{1}$$

If $x \in E$, then the limit above exists by the definition of E; if $x \in B \setminus E$, then the limit above exists because $(\chi_E g_k)(x) = 0$ for all $k \in \mathbb{Z}^+$. For each $k \in \mathbb{Z}^+$, then function $\chi_E g_k$ is Borel measurable. Thus (1) implies that g is a Borel measurable function (by

2.48 of Measure, Integration & Real Analysis). Because

$$\{x \in B : g_k(x) \neq f_k(x)\} \subseteq \bigcup_{k=1}^{\infty} \{x \in B : g_k(x) \neq f_k(x)\},$$

we have $|\{x \in B : g_k(x) \neq f_k(x)\}| = 0$, completing the proof.