3 A Exercise 2

Suppose $f_1 = e^{-|x|}, f_2 = e^{\frac{-|x|}{2}}, f_3 = e^{\frac{-|x|}{3}}, \ldots$.

Then we have

$$\lim_{k\to\infty} f_k(x) = \lim_{k\to\infty} e^{\frac{-|x|}{k}} = 0$$

Now we calculate the limit of the integral:

$$\lim_{k\to\infty} \int f_k \, d\lambda = \lim_{k\to\infty} \int_{-\infty}^{\infty} e^{\frac{-|x|}{k}} \, dx = \lim_{k\to\infty} 2 \int_0^{\infty} e^{\frac{-x}{k}} \, dx = \infty$$

3A Exercise 3

WLOG suppose $b \geq a$:

$$\begin{aligned}
|g(a) - g(b)| &= \left| \int_{(-\infty,a)} f \, d\lambda - \int_{(-\infty,b)} f \, d\lambda \right| \\
&= \left| \int_{(-\infty,a)} f \, d\lambda - \int_{(-\infty,a)} f \, d\lambda - \int_{(a,b)} f \, d\lambda \right| \\
&= \left| \int_{(a,b)} f \, d\lambda \right| \\
&\leq \lambda((a,b)) \sup_{(a,b)} |f|
\end{aligned}$$

where the last inequality holds by 3.25 of Measure, Integration & Real Analysis.
Since $f \in \mathcal{L}^1(\lambda)$, then we can assume $\sup_{(a,b)} |f| = M < \infty$.

Then to have $|g(a) - g(b)| < \epsilon$, we must have $(b-a)M < \epsilon$.
Therefore, for fixed $\epsilon > 0$ it suffices that we choose $\delta > \frac{\epsilon}{M}$ to have $|g(a) - g(b)| < \epsilon$ for all $a, b \in \mathbb{R}$ such that $|b - a| < \delta$. And by doing so, g is uniformly continuous on $\mathbb{R}$.

3 A Exercise 5

Let $f_1 = f\chi_{[-1,1]}, f_2 = f\chi_{[-2,2]}, f_3 = f\chi_{[-3,3]}, \ldots$ be sequence of Borel measurable functions. We can see that

$$\lim_{k\to\infty} f_k(x) = f(x)$$

for any $x \in X$.

Also $|f|$ is Borel measurable function from X to $[0, \infty]$ satisfying two conditions:

- $\int |f| \, d\lambda < \infty$
- $|f_k(x)| \leq |f(x)|$

for arbitrary $k \in \mathbb{Z}^+$ and any $x \in X$.

By using Dominated Convergence Theorem we have:

$$\lim_{k\to\infty} \int f_k \, d\lambda = \int f \, d\lambda$$

3A Exercise 12

Let $f_1 = \frac{(1-x)\cos(x^{-1})}{\sqrt{x}}, f_2 = \frac{(1-x)^2\cos(x^{-2})}{\sqrt{x}}, f_3 = \frac{(1-x)^3\cos(x^{-3})}{\sqrt{x}}, \ldots$ be sequence of measurable functions from $(0,1)$ to $[-\infty,\infty]$. We can see that

$$\lim_{k\to\infty} f_k(x) = 0$$

for any $x \in (0,1)$.

Now let $g_j(x) = \frac{1}{\sqrt{x}}\chi_{(\frac{1}{j},1)}$, where g_j is a non-negative increasing sequence of functions from $(0,1)$ to $[0,\infty]$. We also have

$$g(x) = \lim_{j\to\infty} g_j(x)$$

by using Monotone Convergence Theorem we have

$$\lim_{j\to\infty}\int_{(0,1)} g_j \; d\mu = \int_{(0,1)} g \; d\mu$$

Suppose $g(x) = \frac{1}{\sqrt{x}}$. Then g is a Borel measurable function from $(0,1)$ to $[0,\infty]$ satisfying two conditions:

- $\int_0^1 |g| \; d\mu = \int_{(0,1)} g d\mu = \lim_{j\to\infty}\int_{(0,1)} g_j \; d\mu = \lim_{j\to\infty}\int_0^1 g_j = \lim_{j\to\infty}\int_0^{\frac{1}{j}} x^{-1/2}dx = \lim_{j\to\infty} 2 - 2\sqrt{\frac{1}{j}} = 2 < \infty$
- $|f_k(x)| \le |g(x)|$

for arbitrary $k \in \mathbb{Z}^+$ and any $x \in X$.

By using Dominated Convergence Theorem we have:

$$\lim_{k\to\infty}\int_{(0,1)} f_k \; d\lambda = \int_{(0,1)} 0 \; d\lambda = 0$$