4A Exercise 1

Let $c, p > 0$:

$$\begin{aligned}
\mu(\{x \in X : |h(x)| \geq c\}) &= \int_{\{x \in X : |h(x)| \geq c\}} d\mu \\
&= \frac{1}{c^p} \int_{\{x \in X : |h(x)| \geq c\}} c^p \, d\mu \\
&\leq \frac{1}{c^p} \int_{\{x \in X : |h(x)| \geq c\}} |h|^p \, d\mu \\
&\leq \frac{1}{c^p} \int |h|^p \, d\mu
\end{aligned}$$

4A Exercise 2

Let $c > 0$:

$$\begin{aligned}
\mu\left(\left\{x \in X : \left|h(x) - \int h\, d\mu\right| \geq c\right\}\right) &= \int_{\left\{x \in X : \left|h(x) - \int h\, d\mu\right| \geq c\right\}} d\mu \\
&= \frac{1}{c^2} \int_{\left\{x \in X : \left|h(x) - \int h\, d\mu\right| \geq c\right\}} c^2\, d\mu \\
&\leq \frac{1}{c^2} \int_{\{x \in X : |h(x)| \geq c\}} \left|h(x) - \int h\, d\mu\right|^2\, d\mu \\
&\leq \frac{1}{c^2} \int \left|h(x) - \int h\, d\mu\right|^2\, d\mu \\
&= \frac{1}{c^2} \int \left[|h|^2 - 2h(x) \int h\, d\mu + \left(\int h\, d\mu\right)^2\right] d\mu \\
&= \frac{1}{c^2} \left[\int h^2\, d\mu - 2(\int h\, d\mu) \int h\, d\mu + \left(\int h\, d\mu\right)^2 \int d\mu\right] \\
&= \frac{1}{c^2} \left[\int h^2\, d\mu - 2\left(\int h\, d\mu\right)^2 + \left(\int h\, d\mu\right)^2 (1)\right] \\
&= \frac{1}{c^2} \left[\int h^2\, d\mu - \left(\int h\, d\mu\right)^2\right]
\end{aligned}$$

4A Exercise 4

Assume arbitrary large $n > 0$. Let $I_1 = (0, 1)$ and $I_2 = (1 - \frac{1}{n}, 2 - \frac{1}{n})$, their union will be $I_1 \cup I_2 = (0, 2 - \frac{1}{n})$.
Let $v > 0$ be the Vitalli constant, then:

$$v * I_1 = (-0.5v + 0.5, 0.5v + 0.5)$$
$$v * I_2 = (-0.5v + 1.5 - \frac{1}{n}, 0.5v + +1.5 - \frac{1}{n})$$

to have at least one of $v * I_1$ or $v * I_1$ be able to cover the union of I_1 and I_2, we must have:

$$-0.5v + 1.5 - \frac{1}{n} < 0$$
$$v > 3 - \frac{2}{n}$$

or

$$0.5v + 0.5 > 2 - \frac{1}{n}$$
$$v > 3 - \frac{2}{n}$$

Since n was chosen arbitrarily large, the only way to have Vitali Covering Lemma be correct, is to have its constant at least 3.

4A Exercise 8

To find $h^*(b) = \sup_{t>0} \frac{1}{2t}\int_{b-t}^{b+t} |h|$, we separately find it over partitions on b:

- $\mathbf{b < 0}$

  If $b + t \leq 0$ then $h^*(b) = 0$.

  Else if $b + t > 0$, we can assume two partitions:

  $\mathbf{b + t > 1}$

  $$\frac{1}{2t}\int_0^1 x\,dx = \frac{1}{4t} < \frac{1}{4(1-b)}$$

  $\mathbf{1 > b + t > 0}$

  $$\frac{1}{2t}\int_0^{b+t} x\,dx = \frac{(b+t)^2}{4t} < \frac{1}{4(1-b)}$$

  hence in this region $h^*(b) = \frac{1}{4(1-b)}$.

- $\mathbf{b > 1}$

  If $b - t \geq 1$ then $h^*(b) = 0$.

  Else if $b - t < 1$, we can assume two partitions:

  $\mathbf{b - t < 0}$

  $$\frac{1}{2t}\int_0^1 x\,dx = \frac{1}{4t} < \frac{1}{4b}$$

  $\mathbf{0 < b - t < 1}$

  $$\frac{1}{2t}\int_{b-t}^1 x\,dx = \frac{1-(b-t)^2}{4t}$$

  To find the supremum of the above, we look for $t > 0$ which makes the derivative zero:

  $$t = \sqrt{b^2 - 1}$$

then we replace it into the expression above:

$$\frac{1-(b-t)^2}{4t} \leq \frac{b-\sqrt{b^2-1}}{2}$$

- $\mathbf{0 < b < 1}$
  In this case, $\frac{1}{2t}\int_{b-t}^{b+t} x\, dx = b$.

To conclude, the Hardy-Littlewood maximal function is:

$$h^*(b) = \begin{cases} \frac{1}{4(1-b)} & \text{if } b \in (-\infty, 0) \\ b & \text{if } b \in [0,1] \\ \frac{b-\sqrt{b^2-1}}{2} & \text{if } b \in (1,\infty) \end{cases} \tag{1}$$