2B Exercise 1

We need to show that $\mathcal{S}$ satisfies 3 conditions of σ-algebra:

- $\varnothing \in \mathcal{S}$ when we choose $K = \varnothing \subseteq \mathbb{Z}$.
- Let $E \in \mathcal{S}$. Then, by definition, there exists $K \subseteq \mathbb{Z}$ such that $E = \bigcup_{n \in K}(n, n+1]$. WLOG let $K = \{k_1, k_2, \dots\}$ such that $k_1 > k_2 > \dots$ and $k_1, k_2, \dots$ are pairwise distinct, then $E = (k_1, k_1+1] \cup (k_2, k_2+1] \cup \dots$, therefore $E^c = \dots \cup (k_1-1, k_1] \cup (k_1+1, k_1+2] \cup \dots \cup (k_2-1, k_2] \cup (k_2+1, k_2+2] \cup \dots$.
  We can write E^c as $E^c = \bigcup_{n \in \mathbb{Z} \setminus K}(n, n+1]$. Since $\mathbb{Z} \setminus K \subseteq \mathbb{Z}$ then $E^c \in \mathcal{S}$.
- Let $\bigcup_{n \in K_1}(n, n+1], \bigcup_{n \in K_2}(n, n+1], \dots \in \mathcal{S}$. Then the union of this sequence of elements in $\mathcal{S}$ is $\bigcup_{i=1}^{\infty} \cup_{n \in K_i}(n, n+1]$, which is equal to $\bigcup_{n \in K}(n, n+1]$ where $K = \bigcup_{i=1}^{\infty} K_i$. And $\bigcup_{n \in K}(n, n+1] \in \mathcal{S}$ by definition of $\mathcal{S}$.

2B Exercise 4

Let $\mathcal{S}' = \{(r,s] : r,s \in \mathbb{Q}\}$ be the collection of Borel subsets of $\mathbb{R}$ (Theorem (♠)).
To show that S equals the collection of Borel subsets of $\mathbb{R}$:

"$\Rightarrow$"
Let $r,s \in \mathbb{Q}$. Define $n \in \mathbb{Z}$ such that $n \geq s$.
Then we have $\underbrace{(r,s]}_{\in \mathcal{S}'} = (r,n] \cap (-\infty, s] = \underbrace{\underbrace{(r,n]}_{\in \mathcal{S}} \cap \underbrace{(\mathbb{R} \setminus \underbrace{(s,n+1]}_{\in \mathcal{S}})}_{\in \mathcal{S}}}_{\in \mathcal{S}}$. Therefore $(r,s] \in S$, i.e., each element of $\mathcal{S}'$ is included in S. Using theorem (♠), the collection of Borel subsets of $\mathbb{R}$ is a subset of $\mathcal{S}$.

"$\Leftarrow$"
Let $r \in \mathbb{Q}$ and $n \in \mathbb{Z}$. Then $(r,n] \in \{(r,s] : r,s \in \mathbb{Q}\}$. It means that every element of $\mathcal{S}$ is also an element of $\mathcal{S}'$. Using theorem (♠), $\mathcal{S}$ is a subset of the collection of Borel subsets of $\mathbb{R}$.

Theorem (♠) [2B Exercise 3]

Suppose $\mathcal{S}'$ is the smallest σ-algebra on $\mathbb{R}$ containing $\{(r,s] : r,s \in \mathbb{Q}\}$. Prove that $\mathcal{S}'$ is the collection of Borel subsets of $\mathbb{R}$.
proof.

"$\Rightarrow$"
Let $r,s \in \mathbb{Q}$. Then $(r,s] = \bigcap_{n=1}^{\infty}(r, s+\frac{1}{n})$. Since each $(r, s+\frac{1}{n})$ is a Borel set, then $(r,s]$ is also a Borel set by 2.25 of Measure, Integration & Real Analysis. So $\mathcal{S}'$ is a subset of the collection of Borel subsets of $\mathbb{R}$.

"$\Leftarrow$"
Let $a,b \in \mathbb{Q}$. Then $(a,b) = \bigcup_{n=1}^{\infty}(a, b-\frac{1}{n}] \in \mathcal{S}'$. Let $c,d \in \mathbb{R}$. For any $n \in \mathbb{N}$, there exists $a_n, b_n \in \mathbb{Q}$ such that $c \leq a_n < c + \frac{1}{n}$ and $d - \frac{1}{n} < b_n \leq d$. Then $(c,d) = \bigcup_{n=1}^{\infty}(a_n, b_n)$. Since $(a_n, b_n) \in \mathcal{S}'$ for each $n \in \mathbb{N}$, $(c,d) \in \mathcal{S}'$. A subset of $\mathbb{R}$ is open if and only if it is the union of a disjoint sequence of open intervals by

0.59 of Supplement for Measure, Integration & Real Analysis. So any open subset of $\mathbb{R}$ is included in $\mathcal{S}'$. Therefore, the collection of Borel subsets of $\mathbb{R}$ is a subset of $\mathcal{S}'$.

2B Exercise 8

Conditioning on the value of $\mathbf{t} \in \mathbb{R}$:

- If $\mathbf{t} = \mathbf{0}$, then if $B \subseteq \mathbb{R}$ is a Borel set, $tB = \{0\}$ is also a Borel set since every countable subset of $\mathbb{R}$ is a Borel set (by 2.30 of Measure, Integration & Real Analysis).
- If $\boldsymbol{t} \in \mathbb{R} \setminus \{\mathbf{0}\}$ and by defining function $f : \mathbb{R} \to \mathbb{R}$ as $f(x) = \frac{x}{t}$, we argue that f is a Borel-measurable function since it is continuous. Therefore by definition of Borel-measurable functions, $f^{-1}(B) = tB$ is a Borel set for every Borel set $B \subseteq \mathbb{R}$.

2B Exercise 12

(a) Let $k \in \mathbb{Z}^+$ and $x \in G_k$. By definition, there exists $\delta_x > 0$ such that $|f(b) - f(c)| < \frac{1}{k} \quad \forall b, c \in (x - \delta, x + \delta)$. Given the value of δ, let $\delta_x < \delta$ be such that it builds a ball around x as $\underbrace{(x - \delta_x, x + \delta_x)}_{B(x,\delta_x)} \subset (x - \delta, x + \delta)$ and pick $y \in B(x, \delta_x)$.

Given the value of y and δ_x, now let $\delta_y < \delta_x$ such that $\underbrace{(y - \delta_y, y + \delta_y)}_{B(y,\delta_y)} \subset B(x, \delta_x)$.

Then for any $b', c' \in B(y, \delta_y)$ we have $|f(b') - f(c')| < \frac{1}{k}$, thus we can conclude that $y \in G_k$. Since y was chosen arbitrarily from $B(x, \delta_x)$ then $B(x, \delta_x) \subset G_k$ which means that an open ball around x exists for each element x of G_k. Therefore G_k is an open subset of $\mathbb{R}$.

(b) To show $\bigcap_{k=1}^{\infty} G_k = \underbrace{\{\text{set of points at which } f \text{ is continuous}\}}_{:=C}$:

"$\Rightarrow$"
Let $x \in \bigcap_{k=1}^{\infty} G_k$ and let $\epsilon > 0$. There exists an $n \in \mathbb{Z}^+$ such that $\frac{1}{n} \leq \epsilon$ and $x \in G_n$, so $|f(x') - f(x)| < \frac{1}{n} \leq \epsilon$ for all $x' \in B(x, \delta_x)$, i.e., for all x' such that $|x' - x| < \delta_x$. Therefore $x \in C$.

"$\Leftarrow$"
Let $x \in C$ and $k \in \mathbb{Z}^+$. Then there is δ_x such that $|f(x') - f(x)| < \frac{1}{2k} \quad \forall x' \in B(x, \delta_x)$. Using triangle inequality:

$$|f(x'') - f(x')| < |f(x'') - f(x)| + |f(x) - f(x')| < \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k}$$

for all $x', x'' \in B(x, \delta_x)$, so $x \in G_k$. Since k was arbitrary, then $x \in \bigcap_{k=1}^{\infty} G_k$.

(c) Since G_k's are open subsets of $\mathbb{R}$ then they are Borel sets. Borel sets are closed under intersection, therefore $\bigcap_{k=1}^{\infty} G_k$ is also a Borel set. And as we showed in part (b), this set is equal to the set of points at which f is continuous. Hence the set of points at which f is continuous, is a Borel set.