2D Exercise 2

I. Set $A \subseteq \mathbb{R}$ is Lebesgue measurable if and only if for every $\epsilon > 0$ there exists closed set $F_\epsilon \subseteq A$ such that $|A \setminus F_\epsilon| < \epsilon$. We want to show that for every closed set F in A we have $|F| \leq |A| - 1$.

To see if A can be a Lebesgue measurable set or not:

Assume A is a Lebesgue measurable set. Then $|F| \leq |A| - 1$ for every closed set F in A which is the same as having $|A \setminus F| \geq 1$ for every closed set F in A, which is a **contradiction** with the definition of Legesgue measurable set.

Therefore the set A is this exercise cannot be a Lebesgue measurable set.

II. Now we construct set A:

Let V be the set defined in proof of 2.18 of Measure, Integration & Real Analysis, then we define $A := \{\frac{k}{|V|}v | v \in V\} = \frac{k}{|V|}V$ for fixed $k \in \mathbb{R}$, $k \geq 1$. Since $V \subseteq [-1, 1]$, then $A \subseteq [\frac{-k}{|V|}, \frac{k}{|V|}]$. Using 2A Exercise 2 of Measure, Integration & Real Analysis, outer measure of set A can be written as $|A| = |\frac{k}{|V|}||V| = k$.

III. Let $B \subseteq A$ a Borel set with $|B| > 0$. Also let $\{r_1, r_2, \dots\} := [-1, 1] \cap \mathbb{Q}$, then again by using the proof of 2.18 of Measure, Integration & Real Analysis we know that the sets $r_1 + B, r_2 + B, \dots$ are disjoint Borel sets.

On the other hand, we have $\bigcup_{i=1}^{\infty}(r_i + B) \subseteq [\frac{-k}{|V|} - 1, \frac{k}{|V|} + 1]$. By measurability of the LHS and subadditivity property of outer measure we have:

$$|\bigcup_{i=1}^{\infty}(r_i + B)| \leq 2(\frac{k}{|V|} + 1)$$

$$\sum_{i=1}^{\infty} |r_i + B| \leq 2(\frac{k}{|V|} + 1)$$

$$\sum_{i=1}^{\infty} |B| \leq 2(\frac{k}{|V|} + 1)$$

$$\infty \leq 2(\frac{k}{|V|} + 1)$$

which is a contradiction. Therefore $|B| = 0$ for any Borel set contained in A.

Therefore for any closed set $F \subseteq A$ (also F is a Borel set), $|A \setminus F| = |A| - |F| = k - 0 \geq 1$.

Hence $|F| \leq |A| - 1$.

2D Exercise 5

Since A is a Lebesgue measurable set, for every $\epsilon > 0$ there exists closed set $F_\epsilon \subseteq A$ such that $|A \setminus F_\epsilon| < \epsilon$.

Now let $k \in \mathbb{Z}^+$, similarly there exists closed set $H_k \subseteq A$ such that $|A \setminus H_k| < \frac{1}{k}$.

Now consider the increasing sequence of sets $\underbrace{H_1}_{F_1}, \underbrace{H_1 \cup H_2}_{F_2}, \underbrace{H_1 \cup H_2 \cup H_3}_{F_3}, \dots, \underbrace{H_1 \cup \cdots \cup H_n}_{F_n}$ where $F_n := \bigcup_{k=1}^{n} H_k$, where we can also define the sequence as:

$$
\begin{aligned}
F_1 &= H_1 \\
F_{n+1} &= F_n \cup H_{n+1} \supseteq F_n \quad \forall n \geq 1
\end{aligned}
$$

F_n is a closed set since union of every finite collection of closed sets is a closed set. (by 0.64 (b) of Supplement for Measure, Integration & Real Analysis)

We have

$$A \setminus \bigcup_{n=1}^{\infty} F_n \subseteq A \setminus F_n \subseteq A \setminus H_k$$

Therefore by order-preserving property of outer measure:

$$\left| A \setminus \bigcup_{n=1}^{\infty} F_n \right| \leq |A \setminus H_k| < \frac{1}{k}$$

since k was chosen arbitrarily, then $|A \setminus \bigcup_{n=1}^{\infty} F_n| = 0$.

2D Exercise 7

Since A is a Lebesgue measurable set, for every $\epsilon > 0$ there exists open set $G_\epsilon \supseteq A$ such that $|G_\epsilon \setminus A| < \epsilon$.

Now let $k \in \mathbb{Z}^+$, similarly there exists open set $H_k \supseteq A$ such that $|H_k \setminus A| < \frac{1}{k}$.

Now consider the decreasing sequence $\underbrace{H_1}_{G_1}, \underbrace{H_1 \cap H_2}_{G_2}, \underbrace{H_1 \cap H_2 \cap H_3}_{G_3}, \dots, \underbrace{H_1 \cap \cdots \cap H_n}_{G_n}$ where $G_n := \bigcap_{k=1}^{n} H_k$, where we can also define the sequence as:

$$G_1 = H_1$$
$$G_{n+1} = G_n \cup H_{n+1} \subseteq G_n \quad \forall n \geq 1$$

G_n is an open set since intersection of every finite collection of open sets is also an open set. (by 0.55 (b) of Supplement for Measure, Integration & Real Analysis)

We have

$$\bigcap_{n=1}^{\infty} G_n \setminus A \subseteq G_n \setminus A \subseteq H_k \setminus A$$

Therefore by order-preserving property of outer measure:

$$\left| \bigcap_{n=1}^{\infty} G_n \setminus A \right| \leq |H_k \setminus A| < \frac{1}{k}$$

since k was chosen arbitrarily, then $|\bigcap_{n=1}^{\infty} G_n \setminus A| = 0$.

2D Exercise 8

Since A is a Lebesgue measurable set, then there exists a Borel set $B \subseteq A$ such that $|A \setminus B| = 0$, by definition.

Since collection of Borel sets is translation invariant, then the set $t + B \subseteq t + A$ is also a Borel set for $t \in \mathbb{R}$. (by 2B Exercise 7 of Measure, Integration & Real Analysis)

By translation invariance property of outer measure we know that $|t + A \setminus B| = |A \setminus B| = 0$.

Using **Remark (♠)**, $t + A \setminus t + B \subseteq t + A \setminus B$, then by order preserving property of outer measure we have $|t + A \setminus t + B| \leq |t + A \setminus B| = 0$.

Therefore, $|t + A \setminus \underbrace{t + B}_{\text{Borel set}}| = 0$ since the outer measure must be greater than or equal to zero.

As a result, the set $t + A$ is Lebesgue measurable by definition.

Remark (♠)

$t + A \setminus t + B \subseteq t + A \setminus B$ for $t \in \mathbb{R}$ and arbitrary sets A and B.

proof. Let $x \in t + A \setminus t + B$ then $x \in t + A$ and $x \notin t + B$, therefore we have $x - t \in A$ and $x - t \notin B$. Hence $x - t \in A \setminus B$ and then $x \in t + A \setminus B$.