4B Exercise 1

(Notation: $f_I = \frac{1}{|I|}\int_I f$)

Since $f \in \mathcal{L}^1(\mathbb{R})$, by 4.10 of Measure, Integration & Real Analysis there exits $b \in \mathbb{R}$ such that

$$\lim_{t \downarrow 0} \frac{1}{2t}\int_{b-t}^{b+t} |f - f(b)| = 0$$

on the other hand we have:

$$\begin{aligned}
\frac{1}{2t}\int_{b-t}^{b+t} |f - f_{[b-t,b+t]}| &= \frac{1}{2t}\int_{b-t}^{b+t} |f - \frac{1}{2t}\int_{b-t}^{b+t} f| \\
&\leq \frac{1}{2t}\int_{b-t}^{b+t} |f - \frac{1}{2t}\int_{b-t}^{b+t} f(b)| \\
&= \frac{1}{2t}\int_{b-t}^{b+t} |f - f(b)|
\end{aligned}$$

where taking $\lim_{t\downarrow 0}$ of both sides completes the proof.

4B Exercise 2

(Notation: $f_I = \frac{1}{|I|}\int_I f$)

Since $f \in \mathcal{L}^1(\mathbb{R})$, by $\boxed{\text{4.10 of Measure, Integration \& Real Analysis}}$ there exits $b \in \mathbb{R}$ such that

$$\lim_{t\downarrow 0} \frac{1}{2t}\int_{b-t}^{b+t} |f - f(b)| = 0$$

Now let I be an interval such that $b \in I$. Then:

$$\begin{aligned}
\left|\left(\frac{1}{|I|}\int_I f\right) - f(b)\right| &= \left|\left(\frac{1}{|I|}\int_I f\right) - \frac{1}{|I|}\int_I f(b)\right| \\
&= \left|\frac{1}{|I|}\int_I (f - f(b))\right| \\
&\leq \frac{1}{|I|}\int_I |f - f(b)| \\
&\leq \frac{1}{|I|}\int_{b-|I|}^{b+|I|} |f - f(b)|
\end{aligned}$$

therefore:

$$\lim_{t\downarrow 0} \sup\{f_I - f(b) : I \text{ is an interval of length } t \text{ containing } b\} = 0$$

which implies the desired result:

$$\lim_{t\downarrow 0} \sup\{\frac{1}{|I|}\int_I |f - f_I| : I \text{ is an interval of length } t \text{ containing } b\} = 0$$

4B Exercise 3

(Notation: $f_I = \frac{1}{|I|}\int_I f$)

Since $f^2 \in \mathcal{L}^1(\mathbb{R})$, we can assume that in a local neighborhood of $b \in \mathbb{R}$ we also have $f \in \mathcal{L}^1(\mathbb{R})$. We have chosen b by 4.10 of Measure, Integration & Real Analysis such that:

$$\lim_{t\downarrow 0} \frac{1}{2t}\int_{b-t}^{b+t} |f - f(b)| = 0$$

and also:

$$\lim_{t\downarrow 0} \frac{1}{2t}\int_{b-t}^{b+t} |f^2 - f^2(b)| = 0$$

which implies that $\lim_{t\downarrow 0} \frac{1}{2t}\int_{b-t}^{b+t} f = f(b)$ and $\lim_{t\downarrow 0} \frac{1}{2t}\int_{b-t}^{b+t} f^2 = f^2(b)$.

We have

$$\frac{1}{2t}\int_{b-t}^{b+t} |f - f(b)|^2 = \frac{1}{2t}\int_{b-t}^{b+t} f^2 - 2f(b)\frac{1}{2t}\int_{b-t}^{b+t} f + f^2(b)$$

where taking $\lim_{t\downarrow 0}$ of both sides completes the proof.

4B Exercise 6

Since $h \in \mathcal{L}^1(\mathbb{R})$, by $\boxed{\text{4.19 of Measure, Integration \& Real Analysis}}$ we can define $g : \mathbb{R} \to \mathbb{R}$ such that $g(s) = \int_{-\infty}^{s} h$ and $g'(b) = f(b)$ for almost every $b \in \mathbb{R}$.

By the hypothesis, since $\int_{-\infty}^{s} h = 0$ for all $s \in \mathbb{R}$, then $g(s) = 0$ for all $s \in \mathbb{R}$.

As a result, $g'(b) = 0 = h(b)$ for almost every $b \in \mathbb{R}$.