2B Exercise 18

$\forall x \in \mathbb{R}$ derivative of f exists as

$$f'(x) = \lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta}$$

There exist $k \in \mathbb{N}$ such that $\frac{1}{k} < \delta$ for each $\delta > 0$, thus if $\delta \to 0$ then $k \to \infty$.
Let sequence of $g_1, g_2, \dots$ defined as:

$$g_k(x) = \frac{f(x+\frac{1}{k}) - f(x)}{\frac{1}{k}}$$

Since differentiability implies continuity, $f(x)$ is a continuous function therefore $\frac{f(x+\frac{1}{k})}{\frac{1}{k}}$ and $\frac{f(x)}{\frac{1}{k}}$ are continuous functions and also measurable $\forall k \in \mathbb{N}$ by 2.41 of Measure, Integration & Real Analysis.

Then g_k is also measurable by 2.46 of Measure, Integration & Real Analysis.

As a result, since $f'(x)$ is a point-wise limit of g_k, we have $\forall x \in \mathbb{R}$

$$f'(x) = \lim_{k \to \infty} g_k(x)$$

then by 2.48 of Measure, Integration & Real Analysis $f'(x)$ is also Borel measurable.

2B Exercise 25

Let $f_k \quad \forall k \in \mathbb{N}$ be defined as

$$f_k(x) := f(x) + \frac{x}{k}$$

$f_k(x)$ is trivially an strictly increasing function where its point-wise limit equals $f(x)$, i.e. $f(x) = \lim_{k\to\infty} f_k(x)$, and satisfies the conditions mentioned in the problem.

2C Exercise 2

Let $w_k \quad \forall k \in E$ be defined as

$$w_k := \mu(\{k\})$$

which satisfies the conditions mentioned in the problem, when $E_k := \{k\}$

$$\mu(E) = \mu(\bigcup_{k \in E} E_k) = \sum_{k \in E} \mu(E_k) = \sum_{k \in E} \mu(\{k\}) = \sum_{k \in E} w_k$$

2C Exercise 9

- $\mu + \nu : \mathcal{S} \to [0, \infty]$
- $(\mu + \nu)(\varnothing) = \mu(\varnothing) + \nu(\varnothing) = 0$
- For every disjoint sequence of sets $E_1, E_2, \cdots \in \mathcal{S}$:

$$\begin{aligned}
(\mu + \nu)(\bigcup_k E_k) &= \mu(\bigcup_k E_k) + \nu(\bigcup_k E_k) \\
&= \sum_k \mu(E_k) + \sum_k \nu(E_k) \\
&= \sum_k \mu(E_k) + \sum_k \nu(E_k) \\
&= \sum_k (\mu + \nu)(E_k)
\end{aligned}$$