1 A Exercise 12

Since f is Riemann integrable, for $\epsilon > 0$ there exists partition $P = \{x_0, x_1, \dots, x_n\}$ on $[a, b]$ such that $U(f, P, [a, b]) - L(f, P, [a, b]) < \epsilon$ (using Theorem (♠) proven below).

Using reverse triangular inequality, $\forall x, y \in [x_{i-1}, x_i]$ where $i \in \{1, 2, \dots, n\}$:

$$\Big| |f(x)| - |f(y)| \Big| \le \Big| f(x) - f(y) \Big|$$
$$\sup_{x,y\in[x_{i-1},x_i]} \Big| |f(x)| - |f(y)| \Big| \le \sup_{x,y\in[x_{i-1},x_i]} \Big| f(x) - f(y) \Big|$$
$$\sup_{[x_{i-1},x_i]} |f| - \inf_{[x_{i-1},x_i]} |f| \le \sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f \qquad (*)$$

where last line follows from Lemma (♣) proven below.

Now we have

$$\begin{aligned} U(|f|, P, [a, b]) - L(|f|, P, [a, b]) &= \sum_{i=1}^{n} (x_i - x_{i-1}) \Big(\sup_{[x_{i-1},x_i]} |f| - \inf_{[x_{i-1},x_i]} |f| \Big) \\ &\le \sum_{i=1}^{n} (x_i - x_{i-1}) \Big(\sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f \Big) && \text{using } (*) \\ &= U(f, P, [a, b]) - L(f, P, [a, b]) < \epsilon && \text{Theorem (♠)} \end{aligned}$$

therefore

$$U(|f|, P, [a, b]) - L(|f|, P, [a, b]) < \epsilon$$

and again according to Theorem (♠), $|f|$ is Riemann integrable.

In addition, since $-|f| \le f \le |f|$ and all f, $|f|$ and $-|f|$ are Riemann integrable,

$$-\int_a^b |f| \le \int_a^b f \le \int_a^b |f| \quad \Longrightarrow \quad \Big| \int_a^b f \Big| \le \int_a^b |f|$$

Lemma (♣)

Suppose $g : [a, b] \to \mathbb{R}$ is a bounded function. Let $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[a, b]$. Then for each $i \in \{1, 2, \dots, n\}$,

$$\sup_{[x_{i-1},x_i]} g - \inf_{[x_{i-1},x_i]} g = \sup_{x,y\in[x_{i-1},x_i]} \Big| g(x) - g(y) \Big|$$

proof.

- Let $x, y \in [x_{i-1}, x_i]$ and WLOG $g(x) \geq g(y)$. Therefore $\sup_{[x_{i-1},x_i]} g \geq g(x)$ and $\inf_{[x_{i-1},x_i]} g \leq g(y)$, as a result

$$\begin{aligned}
\sup_{[x_{i-1},x_i]} g - \inf_{[x_{i-1},x_i]} g &\geq g(x) - g(y) \\
\sup_{[x_{i-1},x_i]} g - \inf_{[x_{i-1},x_i]} g &\geq |g(x) - g(y)| \qquad (g(x) \geq g(y)) \\
\sup_{[x_{i-1},x_i]} g - \inf_{[x_{i-1},x_i]} g &\geq \sup_{x,y\in[x_{i-1},x_i]} |g(x) - g(y)| \qquad (1)
\end{aligned}$$

- let $\epsilon > 0$. $\exists x, y \in [x_{i-1}, x_i]$ such that $g(x) > \sup_{[x_{i-1},x_i]} g - \frac{\epsilon}{2}$ and $g(y) < \inf_{[x_{i-1},x_i]} g + \frac{\epsilon}{2}$. Therefore $g(x) - g(y) > \sup_{[x_{i-1},x_i]} g - \inf_{[x_{i-1},x_i]} g - \epsilon$. Then equivalently $|g(x) - g(y)| > \sup_{[x_{i-1},x_i]} g - \inf_{[x_{i-1},x_i]} g - \epsilon$:

$$\begin{aligned}
\sup_{x,y\in[x_{i-1},x_i]} |g(x) - g(y)| &> \sup_{[x_{i-1},x_i]} g - \inf_{[x_{i-1},x_i]} g - \epsilon \qquad \forall \epsilon > 0 \\
\sup_{x,y\in[x_{i-1},x_i]} |g(x) - g(y)| &\geq \sup_{[x_{i-1},x_i]} g - \inf_{[x_{i-1},x_i]} g \qquad (2)
\end{aligned}$$

By having (1) and (2), the equality holds true.

Theorem (♠) **[1A Exercise 3]**

Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function. f is Riemann integrable if and only if for each $\epsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(f, P, [a, b]) - L(f, P, [a, b]) < \epsilon$$

proof.

"⇐" Suppose the condition holds. Let $\epsilon > 0$ and choose a partition P which satisfies the condition. Since $U(f, [a, b]) \leq U(f, P, [a, b])$ and $L(f, [a, b]) \geq L(f, P, [a, b])$

$$0 \leq U(f, [a, b]) - L(f, [a, b]) \leq U(f, P, [a, b]) - L(f, P, [a, b]) < \epsilon$$

since ϵ is chosen arbitrarily, $U(f, [a, b]) - L(f, [a, b]) = 0$ and f is Riemann integrable.

"⇒" Suppose f is Riemann integrable. Given $\epsilon > 0$, $\exists$ partitions Q, R such that $U(f, Q, [a, b]) < U(f, [a, b]) + \frac{\epsilon}{2}$ and $L(f, R, [a, b]) > L(f, [a, b]) - \frac{\epsilon}{2}$. Let P be a common refinement of Q and R, then

$$U(f, P, [a, b]) - L(f, P, [a, b]) \leq U(f, Q, [a, b]) - L(f, R, [a, b]) < \underbrace{U(f, [a, b]) - L(f, [a, b])}_{=0} + \epsilon$$

therefore $U(f, P, [a, b]) - L(f, P, [a, b]) < \epsilon$.

1 A Exercise 13

Let $\epsilon > 0$. Assume equidistant partition $P_\epsilon = \{x_0, x_1, \ldots, x_n\}$ defined on $[a, b]$ such that $x_i - x_{i-1} = x_j - x_{j-1} < \frac{\epsilon}{f(b)-f(a)} \quad \forall i, j \in \{1, \ldots, n\}$, then

$$
\begin{aligned}
U(f, P_\epsilon, [a, b]) - L(f, P_\epsilon, [a, b]) &= \sum_{i=1}^{n} (x_i - x_{i-1})(\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f) \\
&= (x_i - x_{i-1}) \sum_{i=1}^{n} \sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f \\
&= (x_i - x_{i-1}) \sum_{i=1}^{n} f(x_i) - f(x_{i-1}) \qquad (f \text{ increasing}) \\
&= (x_i - x_{i-1})(f(b) - f(a)) \\
&< \frac{\epsilon}{f(b) - f(a)} (f(b) - f(a)) = \epsilon
\end{aligned}
$$

therefore according to Theorem (♠) stated below, f is Riemann integrable.

Theorem (♠)

Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function. f is Riemann integrable if and only if for each $\epsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(f, P, [a, b]) - L(f, P, [a, b]) < \epsilon$$

proof. Proven as part of the solution for 1A Exercise 12.

1B Exercise 1

Since set of irrational numbers is dense, no matter of how we choose a partition P on $[0,1]$, an irrational number exists in $[x_{i-1}, x_i]$, therefore $\inf_{[x_{i-1},x_i]} f = 0$ and then $L(f, P, [0,1]) = 0$.

According to Theorem (♠) stated below, if we show $U(f, P, [a,b]) < \epsilon$ for arbitrary choice of ϵ, then f will be Riemann integrable.
Let $A_n = \{x : f(x) \geq \frac{1}{n}\}$ then for any $x = \frac{i}{j} \in A_n$, $i, j \leq n$. Therefore A_n is a finite set ($|A_n| < \infty$).
Let $\epsilon > 0$. Then assume $\frac{1}{n} < \frac{\epsilon}{2}$. We choose P such that each member of A_n falls into $[x_{i-1}, x_i]$ by having

$$x_i - x_{i-1} < \frac{\epsilon}{2|A_n|}$$

Let $B := \{i : A_n \cap [x_{i-1}, x_i] = \emptyset\}$ so $|B| < |A_n|$.

- if $i \in B$ then $\sup_{[x_{i-1},x_i]} f < \frac{1}{n} < \frac{\epsilon}{2}$
- if $i \notin B$ then $\sup_{[x_{i-1},x_i]} f = 1$

then we have

$$\begin{aligned} U(f, P, [a,b]) &= \sum_{i \in B} (x_i - x_{i-1}) \sup_{[x_{i-1},x_i]} f + \sum_{i \notin B} (x_i - x_{i-1}) \sup_{[x_{i-1},x_i]} f \\ &< \sum_{i \in B} (x_i - x_{i-1}) \frac{\epsilon}{2} + \sum_{i \notin B} (x_i - x_{i-1})(1) \\ &< \frac{\epsilon}{2} + |A_n| \frac{\epsilon}{2|A_n|} = \epsilon \end{aligned}$$

therefore f is Riemann integrable and $\int_0^1 f = 0$.

Theorem (♠)

Suppose $f : [a,b] \to \mathbb{R}$ is a bounded function. f is Riemann integrable if and only if for each $\epsilon > 0$, there exists a partition P of $[a,b]$ such that

$$U(f, P, [a,b]) - L(f, P, [a,b]) < \epsilon$$

proof. Proven as part of the solution for 1A Exercise 12.

1B Exercise 5

Let continuous f_n be defined as

$$f_n(x) = \begin{cases} 4n^2x & 0 \le x \le \frac{1}{2n} \\ 4n - 4n^2x & \frac{1}{2n} < x \le \frac{1}{n} \\ 0 & \frac{1}{n} < x \le 1 \end{cases}$$

as shown in Figure 1.

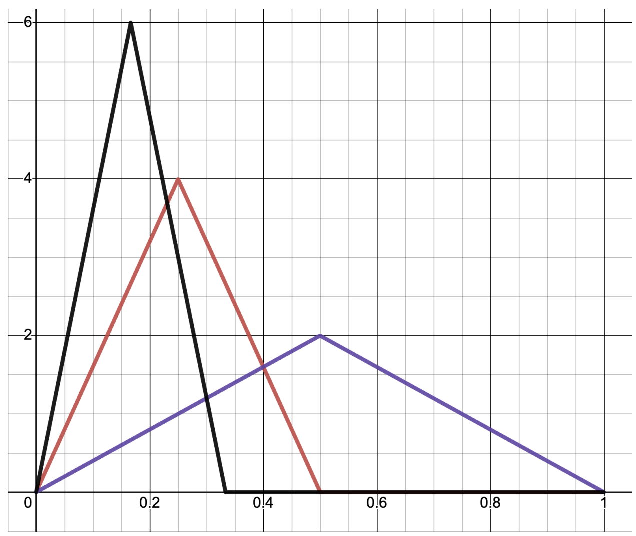

Figure 1: Function f_n defined on $[0, 1]$ plotted for $n = 1, 2, 3$.

Also we have

$$f(x) = \lim_{n \longrightarrow \infty} f_n(x) = 0 \quad 0 \le x \le 1$$

Now for integral of each f_n we have

$$\lim_{n \longrightarrow \infty} \int_0^1 f_n = \lim_{n \longrightarrow \infty} 1 = 1$$

while for the integral of limit of f_n

$$\int_0^1 f = \int_0^1 \lim_{n \longrightarrow \infty} f_n = \int_0^1 0 = 0$$

Therefore

$$\int_0^1 f \ne \lim_{n \longrightarrow \infty} \int_0^1 f_n$$

for this choice of f_n.