Homework 3

MATH 564 — Intermediate Differential Equations

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2.8 Autonomous Systems-Phase Space

Exercise 2.8.10

- eigenvalues:

$$\det \begin{pmatrix} -\lambda & v \\ -v & -\lambda \end{pmatrix} = 0 \Longrightarrow \lambda_1, \lambda_2 = +iv, -iv$$

- eigenvectors:
  - $\lambda_1 = +iv$

$$\begin{pmatrix} -iv & v \\ -v & -iv \end{pmatrix} v_1 = 0 \Longrightarrow v_1 = \begin{pmatrix} c_1 \\ ic_1 \end{pmatrix}$$

  - $\lambda_1 = -iv$

$$\begin{pmatrix} iv & v \\ -v & iv \end{pmatrix} v_2 = 0 \Longrightarrow v_2 = \begin{pmatrix} c_2 \\ -ic_2 \end{pmatrix}$$

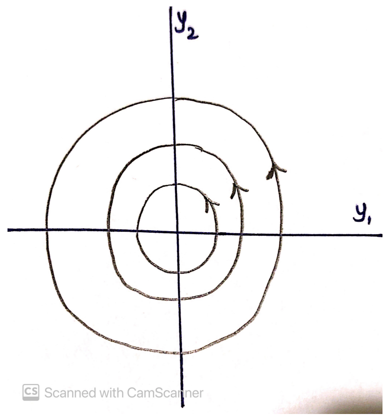
- general solution ($\boldsymbol{\eta} = \phi(0)$):

$$\phi(t) = e^{ivt} \begin{pmatrix} c_1 \\ ic_1 \end{pmatrix} + e^{-ivt} \begin{pmatrix} c_2 \\ -ic_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \cos vt + \eta_2 \sin vt \\ -\eta_1 \sin vt + \eta_2 \cos vt \end{pmatrix}$$

  Using polar coordinates in which $y_1 = r\cos\theta$ and $y_2 = r\sin\theta$, by letting $\rho = \frac{1}{\sqrt{\eta_1^2+\eta_2^2}}$, $\cos\alpha = \frac{\eta_1}{\rho}$, $\sin\alpha = \frac{\eta_2}{\rho}$:

$$\phi(t) = \begin{pmatrix} \rho\cos(-(vt-\alpha)) \\ \rho\sin(-(vt-\alpha)) \end{pmatrix} \Longrightarrow r = \rho, \quad \theta = -(vt-\alpha)$$

- phase space ($v < 0$):



Exercise 2.8.11

- change of variables, $y_1 = x$ and $y_2 = x'$:

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

- eigenvalues:

$$\det\begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = 0 \Longrightarrow \lambda_1, \lambda_2 = +i, -i$$

- eigenvectors:
  - $\lambda_1 = +i$

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} v_1 = 0 \Longrightarrow v_1 = \begin{pmatrix} c_1 \\ ic_1 \end{pmatrix}$$

– $\lambda_1 = -i$

$$\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} v_2 = 0 \Longrightarrow v_2 = \begin{pmatrix} c_2 \\ -ic_2 \end{pmatrix}$$

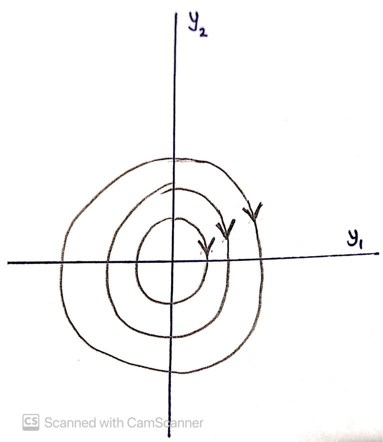
- general solution ($\boldsymbol{\eta} = \phi(0)$):

$$\phi(t) = e^{it} \begin{pmatrix} c_1 \\ ic_1 \end{pmatrix} + e^{-it} \begin{pmatrix} c_2 \\ -ic_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \cos t + \eta_2 \sin t \\ -\eta_1 \sin t + \eta_2 \cos t \end{pmatrix}$$

Using polar coordinates in which $y_1 = r\cos\theta$ and $y_2 = r\sin\theta$, by letting $\rho = \frac{1}{\sqrt{\eta_1^2+\eta_2^2}}$, $\cos\alpha = \frac{\eta_1}{\rho}$, $\sin\alpha = \frac{\eta_2}{\rho}$:

$$\phi(t) = \begin{pmatrix} \rho\cos(-(t-\alpha)) \\ \rho\sin(-(t-\alpha)) \end{pmatrix} \Longrightarrow r = \rho, \quad \theta = -(t-\alpha)$$

- phase space (origin is not an attractor point):



Exercise 2.8.12

- change of variables, $y_1 = x$ and $y_2 = x'$:

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

- eigenvalues:

$$\det\begin{pmatrix} -\lambda & 1 \\ -1 & 3-\lambda \end{pmatrix} = 0 \Longrightarrow \lambda_1, \lambda_2 = \frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}$$

- eigenvectors:
  - $\lambda_1 = \frac{3+\sqrt{5}}{2}$

$$\begin{pmatrix} -\frac{3+\sqrt{5}}{2} & 1 \\ -1 & \frac{3-\sqrt{5}}{2} \end{pmatrix} v_1 = 0 \Longrightarrow v_1 = \begin{pmatrix} \frac{3-\sqrt{5}}{2}c_1 \\ c_1 \end{pmatrix}$$

  - $\lambda_1 = \frac{3-\sqrt{5}}{2}$

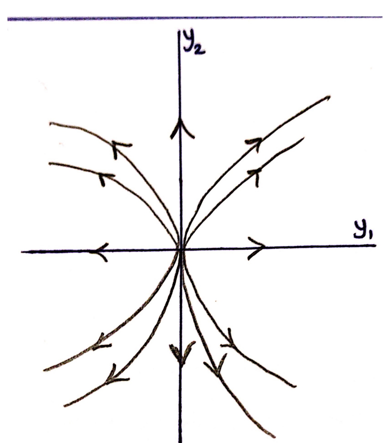
$$\begin{pmatrix} -\frac{3-\sqrt{5}}{2} & 1 \\ -1 & \frac{3+\sqrt{5}}{2} \end{pmatrix} v_2 = 0 \Longrightarrow v_2 = \begin{pmatrix} \frac{3+\sqrt{5}}{2}c_2 \\ c_2 \end{pmatrix}$$

- general solution ($\boldsymbol{\eta} = \phi(0)$):

$$\phi(t) = e^{\frac{3+\sqrt{5}}{2}t}\begin{pmatrix} \frac{3-\sqrt{5}}{2}c_1 \\ c_1 \end{pmatrix} + e^{\frac{3-\sqrt{5}}{2}t}\begin{pmatrix} \frac{3+\sqrt{5}}{2}c_2 \\ c_2 \end{pmatrix}$$

  where $\frac{3-\sqrt{5}}{2}c_1 + \frac{3+\sqrt{5}}{2}c_2 = \eta_1$ and $c_1 + c_2 = \eta_2$.

- phase space (origin is not an attractor point):



Exercise 2.8.17

- change of variables, $y_1 = x$ and $y_2 = x'$:

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

- eigenvalues:

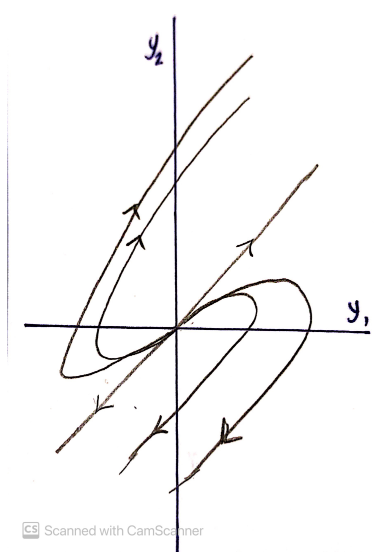
$$\det \begin{pmatrix} -\lambda & 1 \\ -1 & 2-\lambda \end{pmatrix} = 0 \Longrightarrow \lambda_1, \lambda_2 = 1, 1$$

- eigenvectors: Any vector in $\mathbb{F}^2$ can be assumed as generalized eigenvector.

- general solution ($\boldsymbol{\eta} = \phi(0)$):

$$\phi(t) = e^t \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -t & t \\ -t & t \end{pmatrix} \right] \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = e^t \begin{pmatrix} (\eta_2 - \eta_1)t + \eta_1 \\ (\eta_2 - \eta_1)t + \eta_2 \end{pmatrix}$$

- phase space (origin is an attractor point):



Exercise 2.8.19

(a)
- system of differential equations:

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

- eigenvalues:

$$\det \begin{pmatrix} 1-\lambda & -1 \\ 2 & -2-\lambda \end{pmatrix} = 0 \Longrightarrow \lambda_1, \lambda_2 = 0, -1$$

- eigenvectors:
  - $\lambda_1 = 0$

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} v_1 = 0 \Longrightarrow v_1 = \begin{pmatrix} c_1 \\ c_1 \end{pmatrix}$$

  - $\lambda_1 = -1$

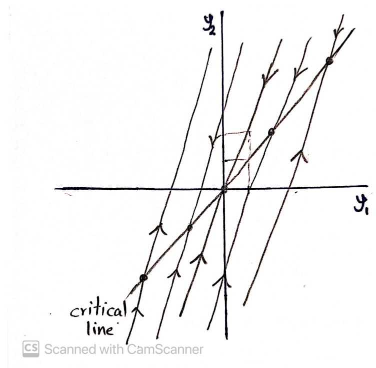
$$\begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} v_2 = 0 \Longrightarrow v_2 = \begin{pmatrix} c_2 \\ 2c_2 \end{pmatrix}$$

- general solution ($\boldsymbol{\eta} = \phi(0)$):

$$\phi(t) = \begin{pmatrix} c_1 \\ c_1 \end{pmatrix} + e^{-t} \begin{pmatrix} c_2 \\ 2c_2 \end{pmatrix} = \begin{pmatrix} 2\eta_1 - \eta_2 + e^{-t}(\eta_2 - \eta_1) \\ 2\eta_1 - \eta_2 + 2e^{-t}(\eta_2 - \eta_1) \end{pmatrix}$$

All points on the line $y_1 = y_2$ are critical points.

(b) phase space:



y_2
y_1
critical line