Homework 2

MATH 564 — Intermediate Differential Equations

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2.5 Linear Systems with Constant Coefficients

Exercise 2.5.15

Calculating eigenvalues by $\det(A - \lambda I) = 0$

$$\lambda_1, \lambda_2, \lambda_3 = -3, 2 + \sqrt{7}, 2 - \sqrt{7}$$

Their corresponding eigenvectors are:

$$v_1 \in \left\{ \alpha \begin{pmatrix} -3/4 \\ 7/4 \\ 1 \end{pmatrix} | \alpha \in \mathbb{F} \right\}$$
$$v_2 \in \left\{ \beta \begin{pmatrix} -(1+\sqrt{7})/2 \\ (3\sqrt{7}+11)/2 \\ 1 \end{pmatrix} | \alpha \in \mathbb{F} \right\}$$
$$v_3 \in \left\{ \gamma \begin{pmatrix} (-1+\sqrt{7})/2 \\ -(3\sqrt{7}-11)/2 \\ 1 \end{pmatrix} | \alpha \in \mathbb{F} \right\}$$

If we find the initial condition as a linear combination of those eigenvectors:

$$\eta = \begin{pmatrix} 0 \\ -2 \\ -7 \end{pmatrix} = \alpha \begin{pmatrix} -3/4 \\ 7/4 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} -(1+\sqrt{7})/2 \\ (3\sqrt{7}+11)/2 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} (-1+\sqrt{7})/2 \\ -(3\sqrt{7}-11)/2 \\ 1 \end{pmatrix}$$

then α, β, γ are known.

Since there are 3 distinct eigenvalues, fundamental matrix will be collection of these 3 columns:

$$\phi_1(t) = e^{-3t}\alpha \begin{pmatrix} -3/4 \\ 7/4 \\ 1 \end{pmatrix}$$

$$\phi_2(t) = e^{(2+\sqrt{7})t}\beta \begin{pmatrix} -(1+\sqrt{7})/2 \\ (3\sqrt{7}+11)/2 \\ 1 \end{pmatrix}$$

$$\phi_3(t) = e^{(2-\sqrt{7})t}\gamma \begin{pmatrix} (-1+\sqrt{7})/2 \\ -(3\sqrt{7}-11)/2 \\ 1 \end{pmatrix}$$

Exercise 2.5.16

$$(A-I)^2 v = \begin{pmatrix} 2 & -1 & -4 & 2 \\ 2 & 2 & -2 & -4 \\ 2 & -1 & -4 & 2 \\ 1 & 2 & -1 & -4 \end{pmatrix}^2 \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \zeta \end{pmatrix} = \begin{pmatrix} -4 & 4 & 8 & -8 \\ 0 & -4 & 0 & 8 \\ -4 & 4 & 8 & -8 \\ 0 & -4 & 0 & 8 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \zeta \end{pmatrix} = 0$$

solving above shows that v has the form of $\begin{pmatrix} 2\gamma \\ 2\zeta \\ \gamma \\ \zeta \end{pmatrix}$.

$$(A-I)^2 w = \begin{pmatrix} 4 & -1 & -4 & 2 \\ 2 & 4 & -2 & -4 \\ 2 & -1 & -2 & 2 \\ 1 & 2 & -1 & -2 \end{pmatrix}^2 \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \zeta \end{pmatrix} = \begin{pmatrix} 8 & 0 & -8 & 0 \\ 8 & 8 & -8 & -8 \\ 4 & 0 & -4 & 0 \\ 4 & 4 & -4 & -4 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \zeta \end{pmatrix} = 0$$

solving above shows that w has the form of $\begin{pmatrix} \alpha \\ \beta \\ \alpha \\ \beta \end{pmatrix}$.

To find fundamental matrix, we assume a general initial condition and write it as sum of two vectors in the corresponding eigenspaces:

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{pmatrix} = \begin{pmatrix} 2(\eta_1 - \eta_3) \\ 2(\eta_2 - \eta_4) \\ \eta_1 - \eta_3 \\ \eta_2 - \eta_4 \end{pmatrix} + \begin{pmatrix} 2\eta_3 - \eta_1 \\ 2\eta_4 - \eta_2 \\ 2\eta_3 - \eta_1 \\ 2\eta_4 - \eta_2 \end{pmatrix} = v_1 + w_1$$

the solution can be written as

$$\begin{aligned} \phi(t) = & e^t[I + t(A - I)]v_1 + e^{-t}[I + t(A + I)]w_1 \\ =& e^t \begin{pmatrix} 1+2t & -t & -4t & 2t \\ 2t & 1+2t & -2t & -4t \\ 2t & -t & 1-4t & 2t \\ 1t & 2t & -t & 1-4t \end{pmatrix} \begin{pmatrix} 2(\eta_1 - \eta_3) \\ 2(\eta_2 - \eta_4) \\ \eta_1 - \eta_3 \\ \eta_2 - \eta_4 \end{pmatrix} \\ & + e^{-t} \begin{pmatrix} 1+4t & -t & -4t & 2t \\ 2t & 1+4t & -2t & -4t \\ 2t & -t & 1-2t & 2t \\ t & 2t & -t & 1-2t \end{pmatrix} \begin{pmatrix} 2\eta_3 - \eta_1 \\ 2\eta_4 - \eta_2 \\ 2\eta_3 - \eta_1 \\ 2\eta_4 - \eta_2 \end{pmatrix} \end{aligned} \tag{$*$}$$

By letting only one of $\eta_i = 1$ at each time, we can find 4 set of solutions $\phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t)$. Putting them together as columns will give us the fundamental matrix (e^{tA}) such that $\Phi(0) = I$, as

$$\Phi(t) = \begin{pmatrix} 2e^t - e^{-t} & -te^{-t} & -2e^t + 2e^{-t} & 2te^{-t} \\ 2te^t & 2e^t - e^{-t} & -2te^t & -2e^t + 2e^{-t} \\ e^t - e^{-t} & -te^{-t} & -e^t + 2e^{-t} & 2te^{-t} \\ te^t & e^t - e^{-t} & -te^t & -e^t + 2e^{-t} \end{pmatrix}$$

A particular solution which satisfies $\eta = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$ will be calculated from Eq. $(*)$:

$$\phi(t) = \begin{pmatrix} 4e^t - 3e^{-t} \\ 4te^t \\ 2e^t - 3e^{-t} \\ 2te^t \end{pmatrix}$$

Exercise 2.5.17

First considering the homogeneous system:

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$$

Eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 2$, with corresponding eigenvectors $v_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then a fundamental matrix will be:

$$\Phi(t) = \begin{pmatrix} -e^{-t} & e^{2t} \\ 2e^{-t} & e^{2t} \end{pmatrix}$$

A solution satisfying the initial condition as in $\eta = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0v_1 + v_2$, will be:

$$\phi_h(t) = \Phi(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Then to find the solution of the non-homogeneous part:

$$\begin{aligned}
\psi(t) &= \Phi(t)\int_0^t \frac{-e^{-s}}{3}\begin{pmatrix} e^{2s} & -e^{2s} \\ -2e^{-s} & -e^{-s} \end{pmatrix}\begin{pmatrix} \sin s \\ \cos s \end{pmatrix} \mathrm{d}s \\
&= \frac{1}{3}\Phi(t)\int_0^t \begin{pmatrix} -e^{s}\sin s + e^{s}\cos s \\ 2e^{-2s}\sin s + e^{-2s}\cos s \end{pmatrix} \mathrm{d}s \\
&= \frac{1}{3}\Phi(t)\begin{pmatrix} e^{t}\cos t - 1 \\ -e^{-2t}\cos t + 1 \end{pmatrix}
\end{aligned}$$

Then the solution is:

$$\phi(t) = \phi_h(t) + \psi(t)$$

Exercise 2.5.18

Let $y_1 = y$ and $y_2 = y'$, then $y_2 = y_1'$ and $y_2' = y''$. Then we have $y_2' = -py_2 - qy_1 = 0$ as in:

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Eigenvalues are $\lambda_{1,2} = \frac{-p\pm\sqrt{p^2-4q}}{2}$.

Exercise 2.5.19

Existence of 2 distinct eigenvalues gives us directly the fundamental matrix:

$$\Phi(t) = \left(e^{\lambda_1 t}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad , e^{\lambda_2 t}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right)$$

Exercise 2.5.20

$\lambda = -p/2$ with multiplicity 2. Then the solution and fundamental matrix are as follows:

$$\begin{aligned}\phi(t) &= e^{-pt/2}[I + t(A + \frac{p}{2}I)]\eta \\ &= e^{-pt/2}\begin{pmatrix} 1+\frac{p}{2}t & t \\ -\frac{p^2}{4}t & 1-\frac{p}{2}t \end{pmatrix}\eta \\ &= \Phi(t)\eta\end{aligned}$$

where the last line is true since η is any vector in $\mathbb{F}^2$.
The general solution for $y'' + py' + q = 0$ is linear combination of the first row of fundamental matrix:

$$\rho(t) = \alpha e^{-pt/2}(1 + \frac{p}{2}t) + \beta e^{-pt/2}t$$

Exercise 2.5.24

$$\begin{aligned}A^2 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I \\ A^3 &= A^2A = (-I)A = -A \\ A^4 &= A^2A^2 = I\end{aligned}$$

Therefore:

$$A^m = \begin{cases} (-1)^{m/2}I & \text{if m is even} \\ (-1)^{\lfloor m/2 \rfloor}A & \text{if m is odd} \end{cases}$$

Exercise 2.5.25

$$
\begin{aligned}
e^{tA} &= I + \sum_{m=1}^{\infty} \frac{t^m}{m!} A^m \\
&= I + \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} A^{2n} + \sum_{n=1}^{\infty} \frac{t^{2n+1}}{(2n+1)!} A^{2n+1} \\
&= I + \sum_{n=1}^{\infty} \frac{t^{2n}(-1)^n}{(2n)!} I + \sum_{n=0}^{\infty} \frac{t^{2n+1}(-1)^n}{(2n+1)!} A \\
&= \left(\sum_{n=0}^{\infty} \frac{t^{2n}(-1)^n}{(2n)!} \right) I + \left(\sum_{n=0}^{\infty} \frac{t^{2n+1}(-1)^n}{(2n+1)!} \right) A \\
&= \cos t I + \sin t A \\
&= \begin{pmatrix} \cos t & 0 \\ 0 & \cos t \end{pmatrix} + \begin{pmatrix} 0 & \sin t \\ -\sin t & 0 \end{pmatrix} \\
&= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}
\end{aligned}
$$

Exercise 2.5.27

Let $y_1 = y$ and $y_2 = y'$, then $y_2 = y_1'$ and $y_2' = y''$.

$$
\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}
$$

A fundamental matrix for the homogeneous system is e^{tA}, which from **Exercise 2.5.24-25** we have:

$$
\Phi(t) = e^{tA} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}
$$

therefore

$$
\phi_h(t) = \Phi(t - t_0)\eta
$$

Then to find the solution of the non-homogeneous part:

$$\begin{aligned}
\psi(t) &= \Phi(t) \int_{t_0}^{t} \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \begin{pmatrix} 0 \\ f(s) \end{pmatrix} \mathrm{d}s \\
&= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} \int_{t_0}^{t} -f(s) \sin s \, \mathrm{d}s \\ \int_{t_0}^{t} f(s) \cos s \, \mathrm{d}s \end{pmatrix} \\
&= \begin{pmatrix} \int_{t_0}^{t} f(s) \sin(t-s) \, \mathrm{d}s \\ \int_{t_0}^{t} f(s) \cos(t-s) \, \mathrm{d}s \end{pmatrix}
\end{aligned}$$

Then the solution is:

$$\phi(t) = \phi_h(t) + \psi(t)$$

2.6 Similarity of Matrices and the Jordan Canonical Form

Exercise 2.6.5

Eigenvalues:

$$(1-\lambda)(-\lambda) - 2 = 0$$
$$\lambda_1, \lambda_2 = 2, -1$$

To find transformation matrix:

$$\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\begin{pmatrix} t_{11} + t_{21} & t_{12} + t_{22} \\ 2t_{11} & 2t_{12} \end{pmatrix} = \begin{pmatrix} 2t_{11} & -t_{12} \\ 2t_{21} & -t_{22} \end{pmatrix}$$

Then $T = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$.

Exercise 2.6.6

$$\begin{aligned} T^{-1}AT &= \frac{-1}{3} \begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \\ &= \frac{-1}{3} \begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 2 & 2 \end{pmatrix} \\ &= \frac{-1}{3} \begin{pmatrix} -6 & 0 \\ 0 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

Exercise 2.6.7

Since left hand side of $e^{tA}T = Te^{tJ}$ is a fundamental matrix, then Te^{tJ} also forms a fundamental matrix. Therefore

$$\Phi(t) = Te^{tJ} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{pmatrix} = \begin{pmatrix} e^{2t} & e^{-t} \\ e^{2t} & -2e^{-t} \end{pmatrix}$$
