Homework 1

MATH 564 — Intermediate Differential Equations

Name: **Ali Zafari** Fall 2023

1.1 A Simple Mass-Spring System

Exercise 1.1.2

At equilibrium, velocity is zero, there is no air resistance and net force is zero.

$$\begin{aligned} F_1 + F_2 &= 0 \\ mg - k(y+a) &= 0 \\ a &= \frac{mg}{k} \end{aligned}$$

Then

$$\begin{aligned} F_1 + F_2 + F_3 &= m\frac{d^2y}{dt^2} \\ mg - k(y+a) - b\frac{dy}{dt} &= m\frac{d^2y}{dt^2} \\ -ky - b\frac{dy}{dt} &= m\frac{d^2y}{dt^2} \\ \frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{k}{m}y = 0 \end{aligned}$$

1.3 Systems of First-Order Equations

Exercise 1.3.8

(a) i. $\phi(t)$ is differentiable on $t \in (-1, 1)$

ii. $(t, \phi(t)) \in D$ for each $t \in (-1, 1)$, where $D = \{(t, y) | -1 < t < 1, y > 0\}$

iii. $\phi'(t) = t\phi^3(t)$

(b) i. $\phi(t)$ is differentiable on $t \in (-1, 1)$

ii. $(t, \phi(t)) \in D$ for each $t \in (-1, 1)$, where $D = \{(t, y) | -1 < t < 1, y < 0\}$

iii. $\phi'(t) = t\phi^3(t)$

(c) i. $\phi_1(t), \phi_2(t)$ are differentiable on $t \in \mathbb{R}$

ii. $(t, \phi_1(t), \phi_2(t)) \in D$ for each $t \in \mathbb{R}$, where $D = \{(t, y, z) | t \in \mathbb{R}, y > 0, z > 0\}$

iii. $\phi_1'(t) = \phi_2(t)$ and $\phi_2'(t) = \phi_1(t)$

(d) i. $\phi_1(t), \phi_2(t)$ are differentiable on $t \in \mathbb{R}$

ii. $(t, \phi_1(t), \phi_2(t)) \in D$ for each $t \in \mathbb{R}$, where $D = \{(t, y, z) | t \in \mathbb{R}, y > 0, z < 0\}$

iii. $\phi_1'(t) = \phi_2(t)$ and $\phi_2'(t) = \phi_1(t)$

Exercise 1.3.9

(a) $\phi(t)$ is not differentiable at $t = 0$, therefore is not a solution on $I = (-\infty, \infty)$

(b) Yes. $\phi(t)$ is continuous on $I = (-\infty, \infty)$.

(c) No. $\phi'(t)$ is not continuous at $t = 0$.

Exercise 1.3.10

According to plot of $\phi(t)$ in Fig. 1 we have:

- $\boldsymbol{D_1 = \{(t, y) | -\infty < t < -1, |y| < \infty\}}$
  $\phi(t)$ is not differentiable at $t = -1$ which is not included in D_1.

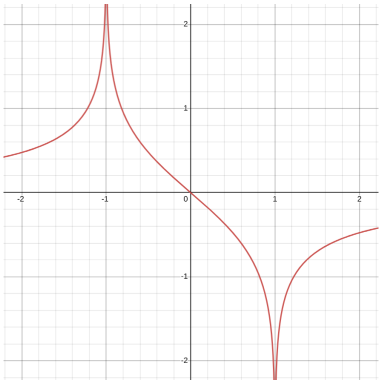

Figure 1: $\phi(t)$

- $\boldsymbol{D_2 = \{(t, y)| -1 < t < -1, |y| < \infty\}}$
  $\phi(t)$ is not differentiable at $t = -1$ and $t =$ which is not included in D_2.

- $\boldsymbol{D_3 = \{(t, y)|1 < t < \infty, |y| < \infty\}}$
  $\phi(t)$ is not differentiable at $t = 1$ and $t =$ which is not included in D_3.

Exercise 1.3.11

According to plot of $\phi(t)$ in Fig. 2 we have:

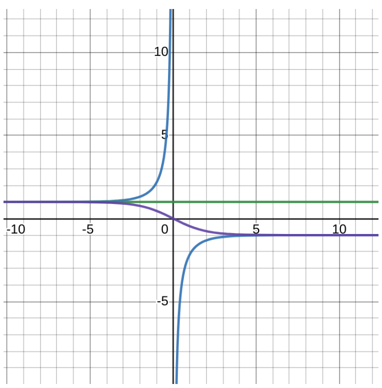

Figure 2: $\phi(t) = \frac{1+ce^t}{1-ce^t}$ for $c = -1, 0, 1$

i. $\phi(t)$ is not differentiable at $t = -\ln c$ when $c > 0$ and is differentiable everywhere when $c \leq 0$

ii. 3 Possible regions of solution will be:

- $c > 0$: $D = \{(t, y) | -\infty < t < -\ln c, 1 < y < \infty\}$
- $c > 0$: $D = \{(t, y) | -\ln c < t < \infty, -\infty < y < 1\}$
- $c \leq 0$: $D = \{(t, y) | -\infty < t < \infty, -1 < y < 1\}$

iii. $\phi'(t) = \frac{\phi^2 - 1}{2}$

Exercise 1.3.17

$$y_1' = +3y_1^2 + 3y_2^2 - 2y_1 y_2$$
$$y_2' = -2y_1^2 - 2y_2^2 + 2y_1 y_2$$

1.4 Vector-Matrix Notation for Systems

Exercise 1.4.2

To show $\forall \boldsymbol{y} \in E_n \quad ||\boldsymbol{y}|| \leq |\boldsymbol{y}| \leq \sqrt{n}||\boldsymbol{y}||$

- LHS Inequality

$$\begin{aligned}
\iff \quad & \sum_i |y_i|^2 \leq (\sum_i |y_i|)^2 \\
\iff \quad & \sum_i |y_i|^2 \leq \sum_i |y_i|^2 + \sum_{i \neq j} |y_i||y_j| \\
\iff \quad & 0 \leq \sum_{i \neq j} |y_i||y_j|
\end{aligned}$$

- RHS Inequality

$$\begin{aligned}
\iff \quad & (\sum_i |y_i|)^2 \leq n \sum_i |y_i|^2 \\
\iff \quad & \sum_i |y_i|^2 + \sum_{i \neq j} |y_i||y_j| \leq n \sum_i |y_i|^2 \\
\iff \quad & \sum_{i \neq j} |y_i||y_j| \leq (n-1) \sum_i |y_i|^2 \qquad (*)
\end{aligned}$$

To prove the last inequality $(*)$, we use modified AM-GM inequality, i.e., for each y_i and y_j we have $|y_i||y_j| \leq \frac{1}{2}(|y_i|^2 + |y_j|^2)$:

$$\sum_{i \neq j} |y_i||y_j| \leq \frac{1}{2} \sum_{i \neq j} |y_i|^2 + |y_j|^2 = \frac{1}{2} 2(n-1) \sum_i |y_i|^2$$

therefore inequality $(*)$ holds true.

Exercise 1.4.3

(i)

$$\begin{aligned}
\iff \quad & ||\boldsymbol{y}|| \geq 0 \\
\iff \quad & \sum_i |y_i|^2 \geq 0
\end{aligned}$$

sum of non-negative elements is equal to zero iff all the elements equal to zero. Therefore $||\boldsymbol{y}|| = 0$ iff $\boldsymbol{y} = 0$.

(ii)

$$\begin{aligned}
&\iff && ||c\boldsymbol{y}|| = |c|||\boldsymbol{y}|| \\
&\iff && \sum_i |cy_i|^2 = |c|^2||\boldsymbol{y}||^2 \\
&\iff && \sum_i |c|^2|y_i|^2 = |c|^2||\boldsymbol{y}||^2 \\
&\iff && |c|^2 \sum_i |y_i|^2 = |c|^2||\boldsymbol{y}||^2
\end{aligned}$$

(iii)

$$\begin{aligned}
||\boldsymbol{y} + \boldsymbol{z}||^2 &= \sum_i |y_i + z_i|^2 \\
&= \sum_i (y_i + z_i)\overline{(y_i + z_i)} \\
&= \sum_i |y_i|^2 + \sum_i |z_i|^2 + 2\sum_i \Re\{y_i\overline{z_i}\} \\
&\leq \sum_i |y_i|^2 + \sum_i |z_i|^2 + 2\sum_i |y_i||z_i| \\
&\leq \sum_i |y_i|^2 + \sum_i |z_i|^2 + 2\sqrt{\sum_i |y_i|^2}\sqrt{\sum_i |z_i|^2} \quad (*) \\
&= (||\boldsymbol{y}|| + ||\boldsymbol{z}||)^2
\end{aligned}$$

where the inequality $(*)$ holds by Schwarz inequality, $|\sum_i |y_i||z_i||^2 \leq \sum_i |y_i|^2 \sum_i |z_i|^2$.

Exercise 1.4.8

Solution of first equation:

$$y_1 = Ce^{-t} \xrightarrow{\phi(0)=(2,1)} y_1 = 2e^{-t}$$

Then for the second equation:

$$y_2'' = y_1' + y_2' = -y1 + y_1 + y_2 = y_2 \rightarrow y_2 = C_1 e^{-t} + C_2 e^t$$

Using $\boldsymbol{\phi}(0) = (2, 1)$:

$$\begin{aligned} C_1 + C_2 &= 1 \\ -C_1 + C_2 &= 3 \end{aligned}$$

therefore $y_2 = -1e^{-t} + 2e^t$.
And the valid interval of t is $I = (-\infty, +\infty)$.

Exercise 1.4.11

Let $v := y_1 + y_2$:

$$v' = 2v + f(t)$$

then the homogeneous solution will be:

$$v = Ce^{2t} \xrightarrow{\boldsymbol{\phi}(0)=(0,0)} v = 0 \rightarrow y_1 = -y_2$$

1.6 Existence, Uniqueness and Continuity

Exercise 1.6.5

For a collections of points to construct a region (open subset), we must have for each point in the subset a ball of radius $\epsilon > 0$ centered on that point, in the region.

(a) Is a region.

(b) Is NOT a region. Points on the boundary cannot have a neighborhood.

(c) IS a region.

(d) IS a region.

Exercise 1.6.8

(a) (i) $\phi(t)$ is differentiable over $t \in (-\infty, 0)$

(ii) $(t, \phi(t)) \in D$ for each $t \in (-\infty, 0)$, where $D = \{(t, y) | -\infty < t < 0, y > 0\}$

(iii) $\phi'(t) = \phi^2(t)$

(b) For the defined D in part (a), both $f = y^2$ and $f' = 2y$ are continuous. Given the initial value $(-1, 1)$ the solution of $\phi(t) = -1/t$ is unique.

(c) As mentioned in part (a), the largest interval for t is $(-\infty, 0)$.

Exercise 1.6.10

Since $\boldsymbol{f}$ and $\partial \boldsymbol{f} / \partial y_k$ are continuous, then for a given initial value in the defined region of D, then a unique solution $\boldsymbol{\phi}$ exists.

Exercise 1.6.16

(a) $\phi(t)$ is not a solution since at $t = 0$ it is not differentiable.

(b) Yes, it is continuous.

(c) No, it is not continuous at $t = 0$.

(d) ?

1.7 The Gronwall Inequality

Exercise 1.7.2

Let $K = 0$ and $g(t) = 1 \quad \forall t \in [0,1]$ in Gronwall inequality, then:

$$f(t) \leq 0 \exp(\int_0^t 1 ds) = 0$$

The only continuous function on $t \in [0,1]$ satisfying $f(t) \leq \int_0^t f(s)ds$ is $f(t) = 0 \quad \forall t \in [0,1]$.

Exercise 1.7.3

Let $U(t) = K_1 + \epsilon(t-\alpha) + K_2 \int_\alpha^t f(s)ds = K_1 + K_2 \int_\alpha^t (\frac{\epsilon}{K_2} + f(s))ds$, then we have $f(t) \leq U(t)$ and $U(\alpha) = K_1$:

$$\begin{aligned}
U'(t) = \epsilon + K_2 f(t) = K_2(f(t) + \frac{\epsilon}{K_2}) &\leq K_2(U(t) + \frac{\epsilon}{K_2}) \\
U'(t) - K_2(U(t) + \frac{\epsilon}{K_2}) &\leq 0 \\
U'(t)\exp(-K_2(t-\alpha)) - K_2(U(t) + \frac{\epsilon}{K_2})\exp(-K_2(t-\alpha)) &\leq 0 \\
[(U(t) + \frac{\epsilon}{K_2})\exp(-K_2(t-\alpha))]' &\leq 0 \\
(U(t) + \frac{\epsilon}{K_2})\exp(-K_2(t-\alpha)) - U(\alpha) - \frac{\epsilon}{K_2} &\leq 0 \\
(U(t) + \frac{\epsilon}{K_2})\exp(-K_2(t-\alpha)) - K_1 - \frac{\epsilon}{K_2} &\leq 0 \\
U(t) \leq -\frac{\epsilon}{K_2} + K_1 \exp(K_2(t-\alpha)) &+ \frac{\epsilon}{K_2}\exp(K_2(t-\alpha)) \\
U(t) \leq K_1 \exp(K_2(t-\alpha)) &+ \frac{\epsilon}{K_2}(\exp(K_2(t-\alpha)) - 1)
\end{aligned}$$

since $f(t) \leq U(t) \quad \forall t \in [\alpha, \beta]$:

$$f(t) \leq K_1 \exp(K_2(t-\alpha)) + \frac{\epsilon}{K_2}(\exp(K_2(t-\alpha)) - 1)$$

2.3 Linear Homogeneous Systems

Exercise 2.3.6

Suppose they are linearly dependent, then there exists $a_1, a_2, a_3 \in \mathbb{E}$ not all zero such that:

$$a_1v_1 + a_2v_2 + a_3v_3 = 0$$
$$\begin{pmatrix} a_1 \\ a_2 + a_3 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

then $a_1 = a_3 = a_2 = 0$ which is a contradiction and v_1, v_2, v_3 are linearly independent.

Exercise 2.3.8

Suppose they are linearly dependent, then there exists $a_1, a_2 \in \mathbb{E}$ not all zero such that:

$$a_1e^{r_1t} + a_2e^{r_2t} = 0 \quad (\forall t \in \mathbb{R})$$

since $r_1 \neq r_2$, then both $a_1 = a_2 = 0$ which is contradiction and v_1, v_2 are linearly independent.

Exercise 2.3.23

First we show that $\mathbf{\Phi}$ is a solution matrix:

- $$\begin{pmatrix} -\sin t \\ -\cos t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$$

- $$\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

now that $\mathbf{\Phi}$ is a solution matrix, if $\det \mathbf{\Phi} \neq 0 \quad \forall t \in I$, then $\mathbf{\Phi}$ is a fundamental matrix. Since $\det \mathbf{\Phi} = 1 \quad \forall t \in I$ then $\mathbf{\Phi}$ is a fundamental matrix.

Exercise 2.3.24

First we show that $\mathbf{\Phi}$ is a solution matrix:

-
$$\begin{pmatrix} r_1\exp(r_1t) \\ r_1^2\exp(r_1t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix} \begin{pmatrix} \exp(r_1t) \\ r_1\exp(r_1t) \end{pmatrix}$$

-
$$\begin{pmatrix} r_2\exp(r_2t) \\ r_2^2\exp(r_2t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix} \begin{pmatrix} \exp(r_2t) \\ r_2\exp(r_2t) \end{pmatrix}$$

the equation above holds since r_1, r_2 are solutions of $z^2 + a_1z + a_2 = 0$. Now that $\mathbf{\Phi}$ is a solution matrix, if $\det\mathbf{\Phi} \neq 0 \quad \forall t \in I$, then $\mathbf{\Phi}$ is a fundamental matrix.
Using Abel's Formula, since $\det\mathbf{\Phi}(0) = r_2 - r_1 \neq 0$ then $\det\mathbf{\Phi} \neq 0 \quad \forall t \in I$ and therefore $\mathbf{\Phi}$ is a fundamental matrix.

Exercise 2.3.27

(a) First we show that $\mathbf{\Phi}$ is a solution matrix:

-
$$\begin{pmatrix} 2t \\ 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2/t^2 & 2/t \end{pmatrix} \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$$

-
$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2/t^2 & 2/t \end{pmatrix} \begin{pmatrix} t \\ 1 \end{pmatrix}$$

Now that $\mathbf{\Phi}$ is a solution matrix, if $\det\mathbf{\Phi} \neq 0 \quad \forall t \in I$, then $\mathbf{\Phi}$ is a fundamental matrix. Using Abel's Formula, since $\det\mathbf{\Phi}(1) = 2 \neq 0$ then $\det\mathbf{\Phi} \neq 0 \quad \forall t \in I$ and therefore $\mathbf{\Phi}$ is a fundamental matrix.

(b) No, since the interval I on which solution Φ is defined, does not include zero.

Exercise 2.3.28

Since we can write

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = \begin{pmatrix} e^{it} & e^{-it} \\ ie^{it} & -ie^{-it} \end{pmatrix} \begin{pmatrix} 0.5 & -0.5i \\ 0.5 & 0.5i \end{pmatrix}$$

then it is also a fundamental matrix.
Another real fundamental matrix:

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t + \cos t & \cos t \end{pmatrix}$$

2.4 Linear Nonhomogeneous Systems

Exercise 2.4.5

From **Exercise 2.3.27** we know a fundamental matrix of the homogeneous equation on $I = \mathbb{R} \setminus \{0\}$:

$$\mathbf{\Phi}(t) = \begin{pmatrix} t^2 & t \\ 2t & 1 \end{pmatrix}$$

where its inverse will be:

$$\mathbf{\Phi}^{-1}(t) = \frac{-1}{t^2} \begin{pmatrix} 1 & -t \\ -2t & t^2 \end{pmatrix}$$

The particular solution $\psi(t)$ with initial conditions of $\psi(2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ will be calculated using Variation of Constants formula:

$$\begin{aligned}
\psi(t) &= \mathbf{\Phi}(t) \int_2^t \mathbf{\Phi}^{-1}(s) g(s) ds \\
&= \mathbf{\Phi}(t) \int_2^t \begin{pmatrix} -1 & s \\ 2s & -s^2 \end{pmatrix} \begin{pmatrix} s^2 \\ s \end{pmatrix} ds \\
&= \mathbf{\Phi}(t) \int_2^t \begin{pmatrix} 0 \\ s^3 \end{pmatrix} ds \\
&= \mathbf{\Phi}(t) \begin{pmatrix} 0 \\ t^4/4 - 4 \end{pmatrix} \\
&= \begin{pmatrix} t^5/4 - 4t \\ t^4/4 - 4 \end{pmatrix}
\end{aligned}$$

Now we find the homogeneous solution $\phi_h(t)$ using the intial condition of $\phi_h(2) = \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \mathbf{\Phi}(2)\boldsymbol{c}$:

$$\boldsymbol{c} = \mathbf{\Phi}^{-1}(2) \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \frac{-1}{4} \begin{pmatrix} -7 \\ 12 \end{pmatrix}$$

therefore

$$\phi_h(t) = \frac{-1}{4}\begin{pmatrix} 7t^2 + 12t \\ -14t + 12 \end{pmatrix}$$

hence the general solution will be $\phi(t) = \phi_h(t) + \psi(t)$.

Exercise 2.4.6

(a) Let $y_1 = y$ and $y_2 = y'$ then:

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -q(t)y_1 - p(t)y_2 + f(t) \end{aligned}$$

then $\begin{pmatrix} \phi_1 \\ \phi_1' \end{pmatrix}$ and $\begin{pmatrix} \phi_2 \\ \phi_2' \end{pmatrix}$ are the solutions of this system of differential equations. Since ϕ_1 and ϕ_2 are linearly independent (? not sure about vectors!), then the matrix $\mathbf{\Phi}(t) = \begin{pmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{pmatrix}$ is a fundamental matrix.

(b) By having $g(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}$, using Variation of Constants formula:

$$\begin{aligned} \psi(t) &= \mathbf{\Phi}(t) \int_2^t \mathbf{\Phi}^{-1}(s) g(s) ds \\ &= \mathbf{\Phi}(t) \int_2^t \frac{1}{\det \mathbf{\Phi}} \begin{pmatrix} \phi_2' & -\phi_2 \\ -\phi_1' & \phi_1 \end{pmatrix} \begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds \end{aligned}$$

(c)

$$\begin{aligned} \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix} &= \mathbf{\Phi}(t) \int_2^t \mathbf{\Phi}^{-1}(s) g(s) ds \\ &= \mathbf{\Phi}(t) \int_2^t \frac{f(s)}{\phi_1\phi_2' - \phi_2\phi_1'} \begin{pmatrix} -\phi_2 \\ \phi_1 \end{pmatrix} ds \end{aligned}$$

and we know that $\psi_1' = \psi_2$ by definition of the ODE system.

2.5 Linear Systems with Constant Coefficients

Exercise 2.5.3

$$
\begin{aligned}
\exp M.\exp P =& (I + \frac{M}{1!} + \frac{M^2}{2!} + \dots)(I + \frac{P}{1!} + \frac{P^2}{2!} + \dots) \\
&= I + (P+M) + \frac{M^2+P^2+2MP}{2!} + \frac{M^3+P^3+3MP^2+3PM^2}{3!} + \dots \\
&= \exp{(M+P)}
\end{aligned}
$$

last equality holds since M and P are commutative.

Exercise 2.5.5

The power series of $\exp tA$ is convergent for $t \in (-\infty, +\infty)$, so its integration/differentiation is allowed. We use the same differentiation rule of power series for the matrix power series, therefore:

$$
\begin{aligned}
(\exp tA)' &= A + \frac{tA^2}{1!} + \frac{t^2A^3}{2!} + \dots \\
&= A(I + \frac{tA}{1!} + \frac{t^2A^2}{2!} + \dots) \\
&= A\exp tA
\end{aligned}
$$

Exercise 2.5.6

The exponents are commutable: $(tA)(-tA) = (-tA)(tA)$.
Since $(\exp tA)(\exp -tA) = (\exp -tA)(\exp tA) = \exp(tA - tA) = \exp(0A) = I$, $\exp -tA$ is the inverse of $\exp tA$.
