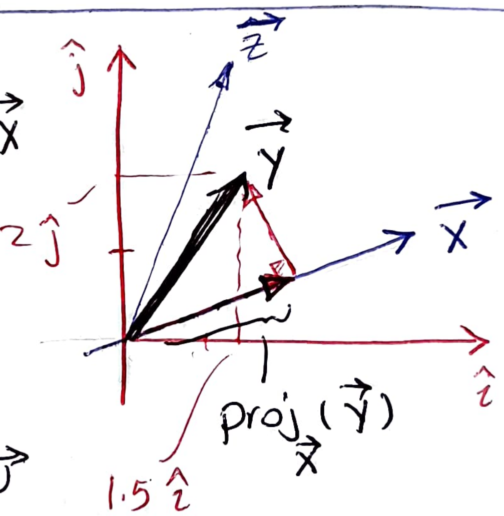
Linear Algebra, AI Mansoura, 11/12/2024

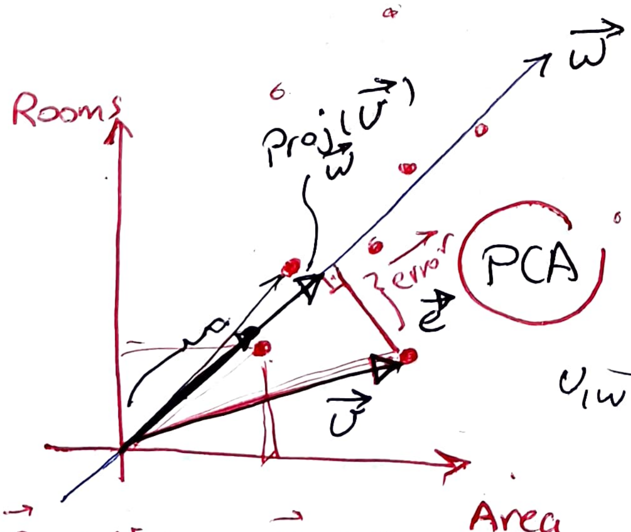
session 4:

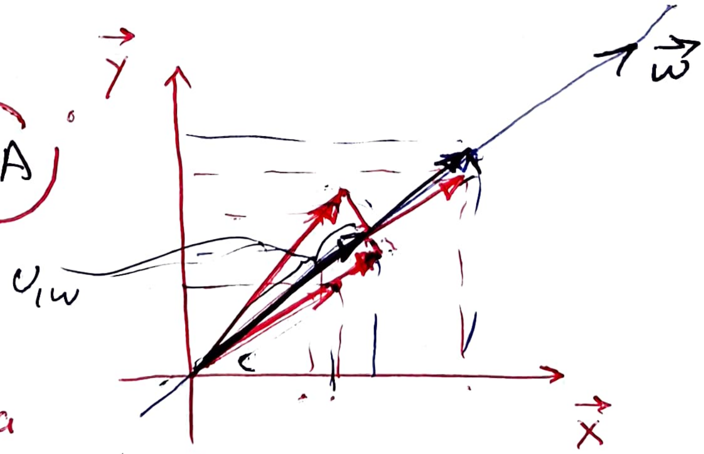
- Projection of vectors
- Gram Schmidt orthonormalization
- Eigenvectors & eigenvalues
- Eigen Decomposition [Diagonalization]
- Introduction to PCA ← session 5

Projection of $\vec{y}$ on $\vec{x}$

$\text{Proj}_{\vec{x}}(\vec{y})$





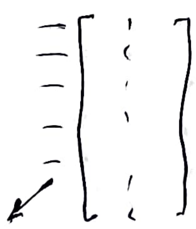


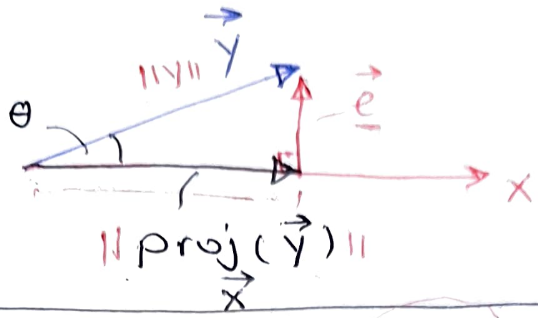
$\vec{v}_1 = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$ $\quad \vec{v}_2 = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix}$

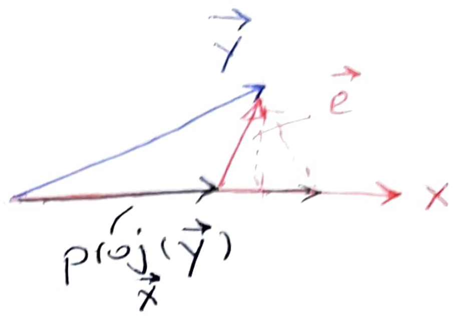
$\vec{v}_1 \approx [w_1]$

$\vec{v}_1 = \begin{bmatrix} v_{1A} \\ v_{1PS} \end{bmatrix}$

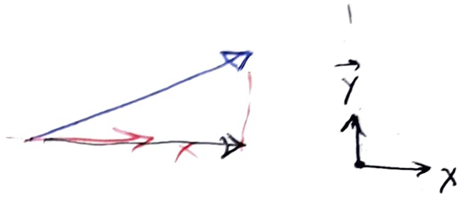
$\vec{v}_1 \approx [v_{1w}]$





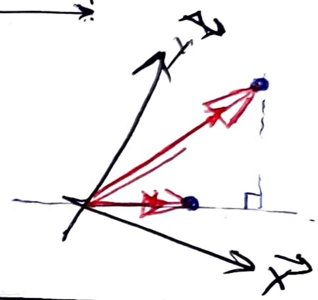


$$\boxed{\mathrm{proj}_{\vec{x}}(\vec{y}) = \frac{\vec{y}\cdot\vec{x}}{\vec{x}\cdot\vec{x}}\,\vec{x}}$$

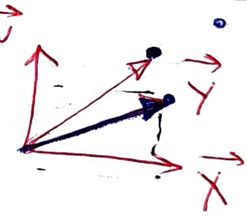


$$= \frac{\|y\|\cancel{\|x\|}\cos\theta}{\|x\|^{\cancel{2}}}\,\vec{x}$$

$$= \|y\|\cos\theta\,\frac{\vec{x}}{\|x\|} \rightsquigarrow \hat{x}$$



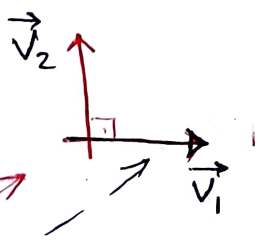
→ Gram-Schmidt Orthonormalization



Any Basis $\Longrightarrow$ Orthonormal basis

Orthogonal + normal

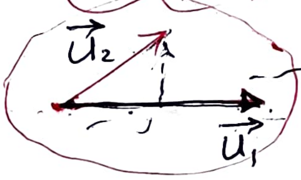
any set of n-linearly indep. vectors are basis of $\mathbb{R}^n$



$\|x_i\| = 1$

$\frac{\vec{X}}{\|X\|} = \hat{X}$

$\|\hat{X}\| = 1$



given a set of vectors

$\vec{x}_1, \vec{x}_2, \vec{x}_3 \cdots \vec{x}_n$ (non-orthogonal basis of $\mathbb{R}^n$)

→ Come up with a new set of vectors

$\vec{y}_1, \vec{y}_2, \vec{y}_3, \cdots \vec{y}_n$; (orthogonal basis of $\mathbb{R}^n$)

1. let, $\vec{y}_1 = \vec{x}_1$

2. $\vec{y}_2 = \vec{x}_2 - \text{Proj}_{\vec{y}_1}(\vec{x}_2)$

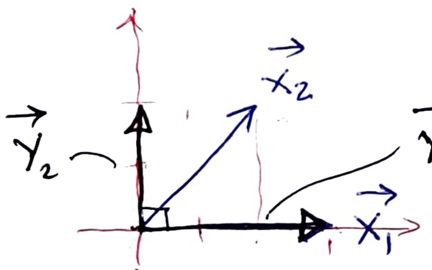
3. $\vec{y}_3 = \vec{x}_3 - \text{Proj}_{\vec{y}_1}(\vec{x}_3) - \text{Proj}_{\vec{y}_2}(\vec{x}_3)$

$\vdots$

$$\vec{y}_j = \vec{x}_j - \sum_{k=1}^{j-1} \text{proj}_{\vec{y}_k}(\vec{x}_j)$$

$$= \vec{x}_j - \sum_{k=1}^{j-1} \left(\frac{\vec{x}_j \cdot \vec{y}_k}{\vec{y}_k \cdot \vec{y}_k} \right) \vec{y}_k$$

Orthogonalization.



$\vec{x}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ $\Rightarrow$ $\vec{y}_1 = \vec{x}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $\vec{y}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$\vec{x}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $\quad$ $\vec{y}_2 = \vec{x}_2 - \text{proj}_{\vec{y}_1}(\vec{x}_2)$

$$\vec{y}_2 = \begin{bmatrix} 2 - \frac{2}{3} \times 3 \\ 2 - 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \frac{\langle 2, 2 \rangle \cdot \langle 3, 0 \rangle \to 6}{\langle 3, 0 \rangle \cdot \langle 3, 0 \rangle \to 9} \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

step-by-step;

$$\vec{Y}_2 = \vec{X}_2 - \text{Proj}_{\vec{Y}_1}(\vec{X}_2)$$

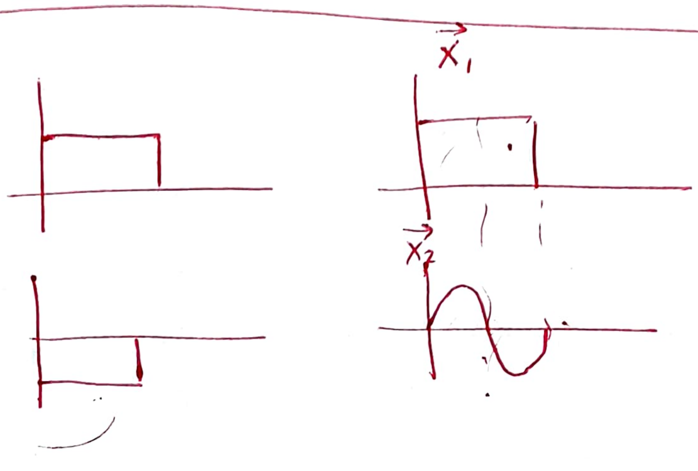
$$= \vec{X}_2 - \frac{\vec{X}_2 \cdot \vec{Y}_1}{\vec{Y}_1 \cdot \vec{Y}_1}\,\vec{Y}_1$$

$$= \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \frac{\begin{bmatrix} 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \end{bmatrix}}{\begin{bmatrix} 3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \end{bmatrix}} \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \frac{\overset{\cancel{6}\quad 2}{2\times 3 + 2\times 0}}{\underset{\cancel{9}\quad 3}{3\times 3 + 0\times 0}} \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\vec{Y}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad \vec{Y}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \qquad \vec{Y}_1 \cdot \vec{Y}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 0 \checkmark$$



$\vec{x}_1 \cdot \vec{x}_2$

next, normalization

$$\hat{e}_1 = \hat{y}_1 = \frac{\vec{y}_1}{\|y_1\|}$$

$$\hat{e}_2 = \hat{y}_2 = \vec{y}_2 / \|y_2\|$$

$$\vdots \qquad \vdots$$

$$\hat{e}_n = \hat{y}_n = \vec{y}_n / \|y_n\|$$

ortho-normal vectors

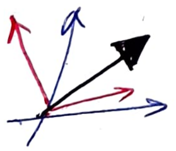
$$\hat{e}_1 = \frac{\begin{bmatrix}3\\0\end{bmatrix}}{\sqrt{3^2+0^2}} = \frac{\begin{bmatrix}3\\0\end{bmatrix}}{3} = \begin{bmatrix}1\\0\end{bmatrix}$$

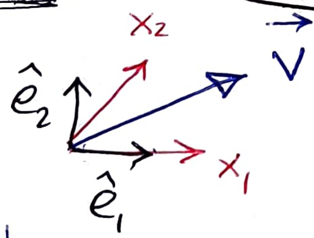
$$\hat{e}_2 = \frac{\begin{bmatrix}0\\2\end{bmatrix}}{2} = \begin{bmatrix}0\\1\end{bmatrix}$$

$\begin{bmatrix}0\\1\end{bmatrix} \begin{bmatrix}1\\0\end{bmatrix}$ or $\begin{bmatrix}1\\1\end{bmatrix} \begin{bmatrix}2\\1\end{bmatrix}$

example

$$\vec{v} = \begin{bmatrix}6\\3\end{bmatrix}$$

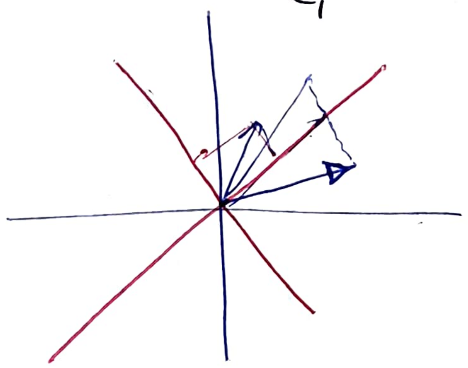




$$\boxed{\vec{v} = a\,\vec{x}_1 + b\,\vec{x}_2}$$

$$\vec{v} = \alpha\,\hat{e}_1 + \beta\,\hat{e}_2$$

$$v = \alpha\,\hat{e}_1$$



$$\begin{bmatrix} \uparrow & \uparrow \\ x_1 & x_2 \\ \downarrow & \downarrow \end{bmatrix} \begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}\uparrow\\v\\\downarrow\end{bmatrix}$$

$$\begin{bmatrix}3 & 2\\0 & 2\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}6\\3\end{bmatrix}$$

$$\begin{bmatrix} \uparrow & \uparrow \\ e_1 & e_2 \\ \downarrow & \downarrow \end{bmatrix} \begin{bmatrix}\alpha\\\beta\end{bmatrix} = \begin{bmatrix}\uparrow\\v\\\downarrow\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix} = \begin{bmatrix}6\\3\end{bmatrix}$$

Standard basis $\hat{i}, \hat{j}$

two basis $\cancel{\hat{u}}, \cancel{\hat{v}}$ $\hat{e}_1, \hat{e}_2$

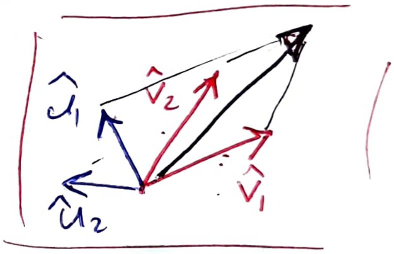
Change of basis matrix

$$\vec{v} = v_x \hat{i} + v_y \hat{j}$$

$$\equiv \underline{v_1} \hat{e}_1 + \underline{v_2} \hat{e}_2$$

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \underbrace{\begin{bmatrix} \hat{i}\cdot\hat{e}_1 & \hat{i}\cdot\hat{e}_2 \\ \hat{j}\cdot\hat{e}_1 & \hat{j}\cdot\hat{e}_2 \end{bmatrix}}_{\left[\text{Change of basis matrix}\right]} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$B_{standard}$ $\qquad$ B_e



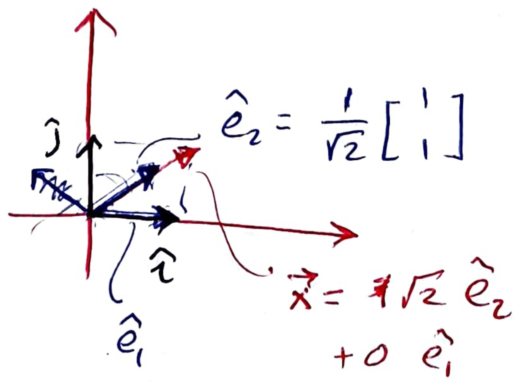
$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \hat{i}\cdot\hat{e}_1 & \hat{i}\cdot\hat{e}_2 \\ \hat{j}\cdot\hat{e}_1 & \hat{j}\cdot\hat{e}_2 \end{bmatrix}^{-1} \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

$\hat{i}\cdot\hat{e}_1 = 1$

$\hat{i}\cdot\hat{e}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = 1/\sqrt{2}$

$\hat{j}\cdot\hat{e}_1 = 0$

$\hat{j}\cdot\hat{e}_2 = 1/\sqrt{2}$



$\vec{x} = 0\,\hat{e}_1 + \sqrt{2}\,\hat{e}_2$

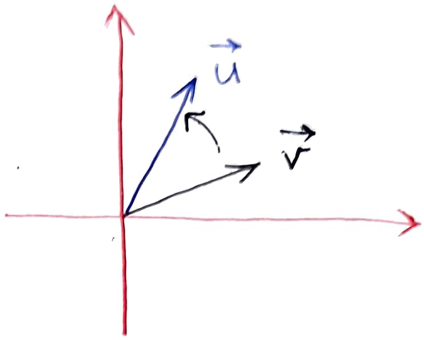
$\vec{x} = 1\,\hat{i} + 1\,\hat{j}$

eigenvectors & eigenvalues.

A as a transformation

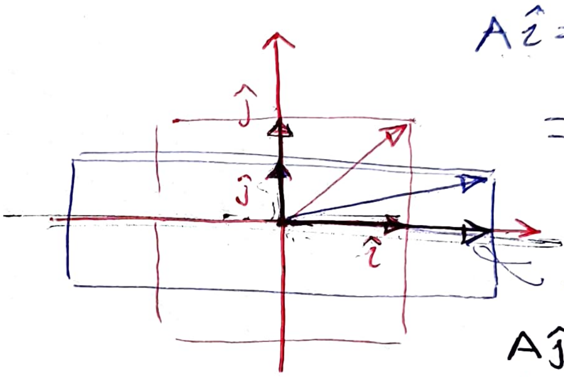
$A\vec{v} = \vec{u}$

$\vec{v} \to \boxed{A} \to \vec{u}$



ex.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}$$



$$A\hat{i} = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A\hat{j} = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

scaling -!

if $A\vec{v} = \lambda\vec{v}$

$\vec{v}$ → eigenvector, λ → eigenvalue

Invertibility:	$A_{m\times m}$ has m- nonzero eigenvalues
non-invertible	⟺ if matrix has some eigenvalues = 0

→ To compute eigenvalues & eigenvectors;

$$A\vec{v} - \lambda\vec{v} = \vec{0} \Rightarrow A\vec{v} - \lambda I\vec{v} = \vec{0}$$

$$\boxed{(A - \lambda I)\vec{v} = \vec{0}}$$

$$|(A - \lambda I)| = 0$$

→ solve for λ

ex.

① finding eigen values.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$\underline{|A - \lambda I|} = \underline{0}$$

$$= \left| \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$$

$$= \begin{vmatrix} 2-\lambda & 0 \\ 0 & 0.5-\lambda \end{vmatrix} = \underline{(2-\lambda)(0.5-\lambda)} = 0$$

$$\lambda_1 = 2 \qquad \underline{\lambda_2 = 0.5}$$

② finding eigenvectors.

for $\lambda_1 = 2$ $\qquad (A - \lambda_1 I)\vec{v_1} = \vec{0}$

$$\left(\begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) \vec{v_1} = \vec{0}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & -1.5 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$0\, v_{11} - 0\, v_{12} = 0$ $\qquad v_{11}$ is a free variable

$\boxed{0\, v_{11} - 1.5\, v_{12} = 0}$ $\qquad v_{12} = 0$

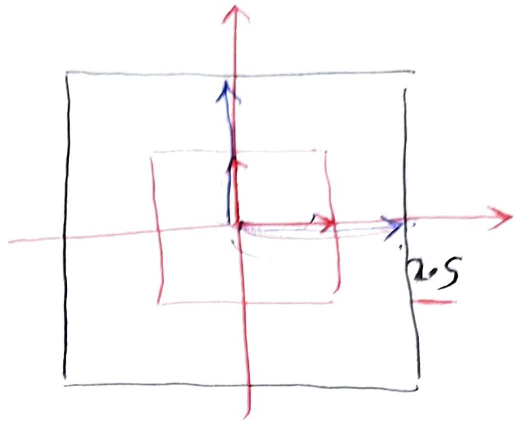
let $\underline{v_{11} = 1}$ $\Rightarrow$ $\vec{v_1} = \begin{bmatrix} \cancel{v_{11}}\ 1 \\ 0 \end{bmatrix}$ $\qquad \lambda_1 = 2 \quad v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$A v_1 = \lambda_1 v_1$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

repeat for λ_2 to find $\vec{v_2}$

$$A = \begin{bmatrix} 2.5 & 0 \\ 0 & 2.5 \end{bmatrix}$$



$\lambda = 2.5$ with algebraic multiplicity of two

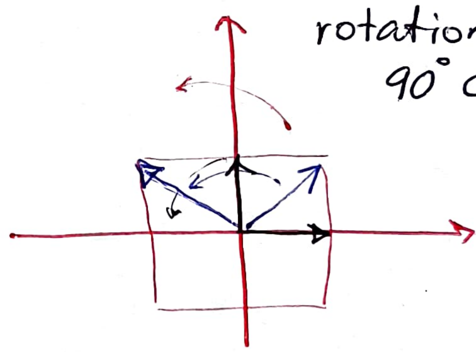
$$|A - \lambda I| = 0 \Leftrightarrow |\lambda I - A| = 0$$

$$\left| \begin{bmatrix} 2.5-\lambda & 0 \\ 0 & 2.5-\lambda \end{bmatrix} \right| = 0 = (2.5-\lambda)^2 = 0$$

$$\lambda_{1,2} = 2.5$$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

rotation 90° ccw.



$$|\lambda I - A| = 0$$

$$\begin{vmatrix} \lambda-0 & +1 \\ -1 & \lambda-0 \end{vmatrix} = \lambda^2 + 1 = 0$$

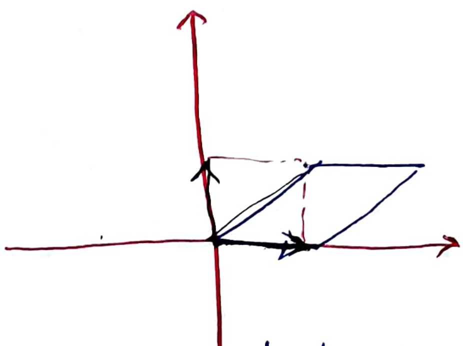
$$\lambda^2 = -1 \Rightarrow \lambda = \pm\sqrt{-1}$$

$$= \pm i$$

$$i = \sqrt{-1}$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

defective



$$|\lambda I - A| = 0$$

$$\begin{vmatrix} \lambda-1 & 1 \\ 0 & \lambda-1 \end{vmatrix} = (\lambda-1)^2 - 0 = 0$$

$\lambda_{1,2} = 1$ $\lambda_1 = 1$ with algebraic multiplicity of two

v_1, geometric multiplicity of two.

Diagonalization

if $A_{m\times m} = \overset{?}{P}_{m\times m} \underbrace{\overset{?}{D}_{m\times m}}_{\text{Diagonal}} P^{-1}_{m\times m}$

$\Rightarrow$ A is diagonalizable

A^n "Powers of matrices"

if A is diagonalizable $\Rightarrow$

$$A = PDP^{-1}$$

$$A^n = AAAA = (\underbrace{PD\cancel{P^{-1}}}_{A})(\underbrace{\cancel{P}D\cancel{P^{-1}}}_{A})(\underbrace{\cancel{P}D\cancel{P^{-1}}}_{A}\cdots\underbrace{\cancel{P}DP^{-1}}_{A})$$

$$= \underline{P D^n P^{-1}}$$

$$\begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix}^n = \begin{bmatrix} a^n & & \\ & b^n & \\ & & c^n \end{bmatrix}$$

$\boxed{\text{Markov chains}}$

eigendecomposotion

A_m : eigenvalues : $\lambda_1, \ldots, \lambda_m$

eigenvectors ; $v_1, \ldots, v_m$

$$A = \overset{Q_1}{\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{bmatrix}} \overset{\Lambda}{\boxed{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{bmatrix}}} \begin{bmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{bmatrix}^{-1}$$

eigenvectors

eigenvalues in the diagonal

$$\boxed{A = Q \Lambda Q^{-1}}$$

Capital Greek letter Lambda