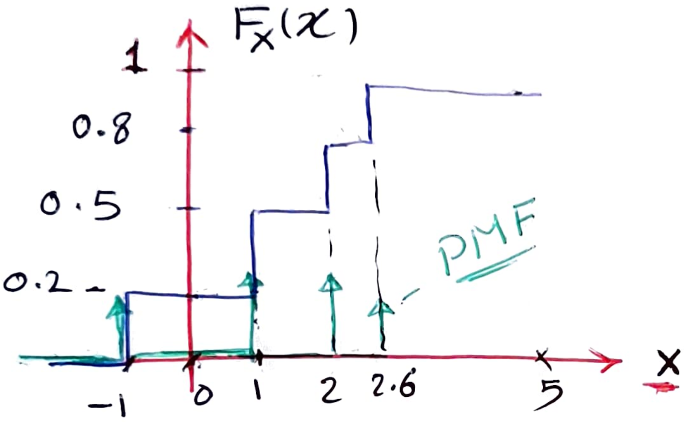
Probability session 4, Mansoura 25/12/2024 (1)

- Continuous R.V.'s
- " Probability Distributions

Review; Discrete Probability distributions



$P_X(-1) =$ nonzero

$P_X(1) =$ nonzero

$P_X(2) =$ "

$P_X(2.6) =$ "

$P_X(0) = F_X(0^+) - F_X(0^-) = 0$

$P_X(1) = 0.5 - 0.2 = 0.3$

$P_X(5) = 0$

$P(2 < x \leq 5) = 1 - 0.8 = 0.2$

$P(0 < X) = 0.8$

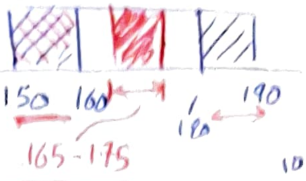
$P(x \leq 0) = F_X(0) = 0.2$

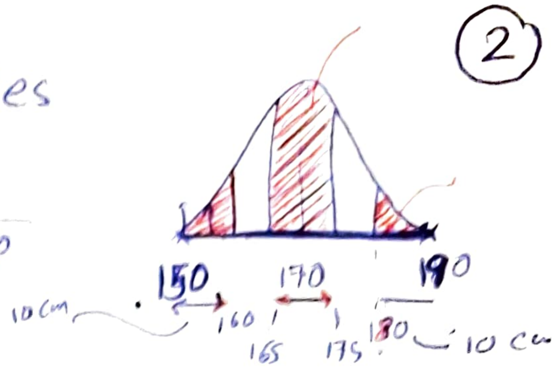
$P(X < 0) = 0.2$

$P(x > 2) \neq P(X \geq 2)$

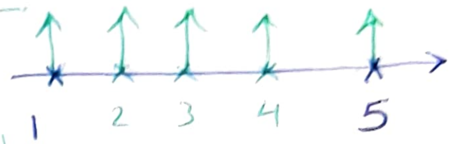
$P(x \geq 2) = P(x > 2) + P_X(2)$

Continuous Random Variables



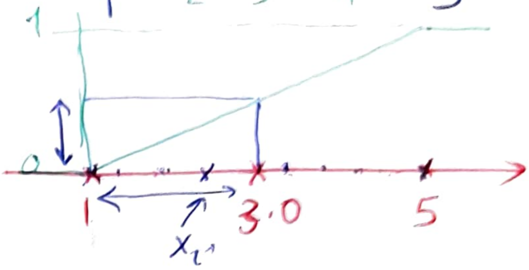


$X \in \{1, \dots, 5\}$ 1/5



real number

$Y \in [1, 5]$



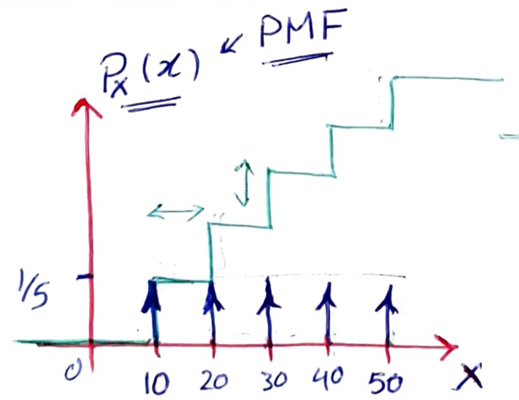
x_i is impossible $\neq$ $P_X(x_i) = 0$

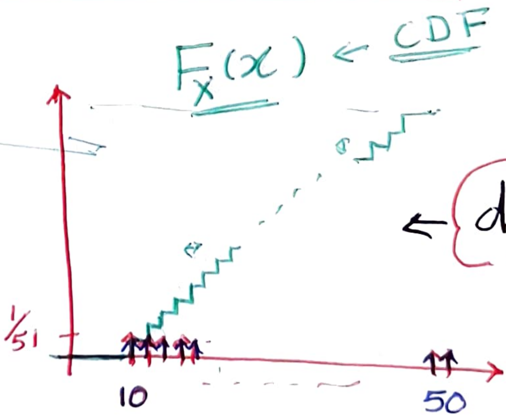
infinite number of possible outcomes.

$P_X(x_1 < X \leq x_2) = \checkmark$

$$\boxed{P_X(175\,\text{cm}) \stackrel{?}{\equiv} P(174.5 < X \leq 175.5) = \checkmark}$$

ex.



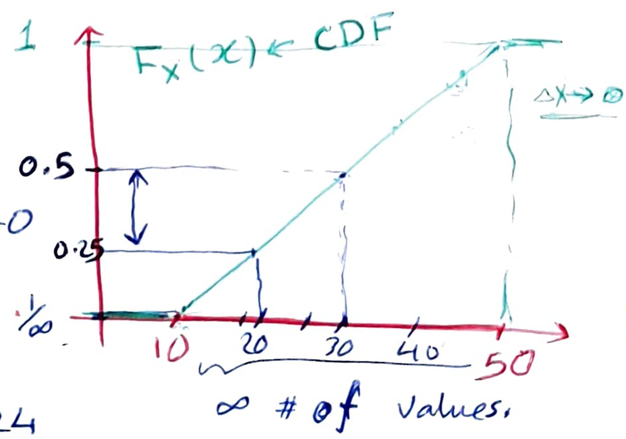


← discrete

$P_X(30) = 0$

$P_X(10 < X \leq 30) = 0.5 - 0$
$= 0.5$

$P_X(20 < X \leq 30)$
$= 0.5 - 0.25 = 0.24$

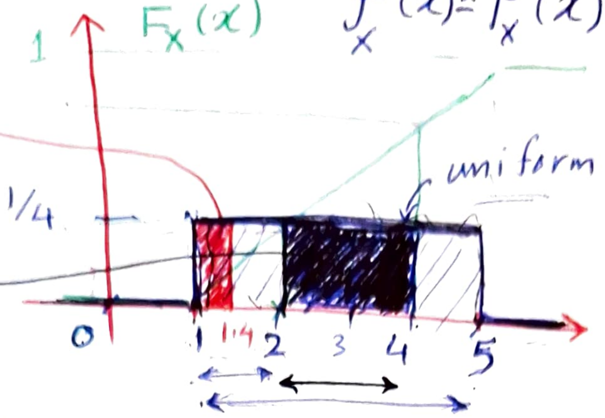


uniform distribution

continuous

this area $= P(1 < X < 1.4) = 0.1$

$$f_X(x) = P_X(x) = \frac{d}{dx} F_X(x)$$



$$\frac{\Delta y}{\Delta x} = \frac{1}{5-1} = 1/4$$

this area
$= P(2 < X < 4)$
$= 2 \times 1/4 = 1/2$

$f_X(x) = P_X(x)$;

Probability
Density
Function

PDF

$$P(2 < X < 4) = 3/4 - 1/4 = 1/2$$

$$= F_X(4) - F_X(2)$$

$$= \int_2^4 f_X(x)\,dx$$

$$= F_X(x)\Big|_2^4 = F_X(4) - F_X(2)$$

$$= \int_2^4 f_X(x)\,dx = \int_2^4 1/4\,dx = 1/4\,x\Big|_2^4$$

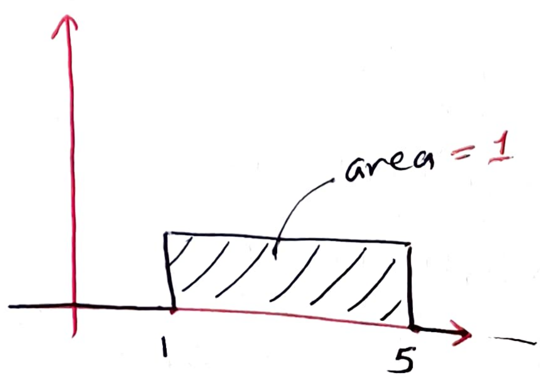
$$= 1/4 \times 4 - 1/4 \times 2$$

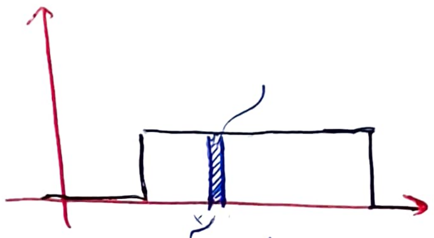
$$= 1 - 1/2 = 1/2$$

$$f_X(x) = \frac{d}{dx} F_X(x)$$

$$F_X(x) = \int_{-\infty}^{x} f_X(\alpha)\,d\alpha$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

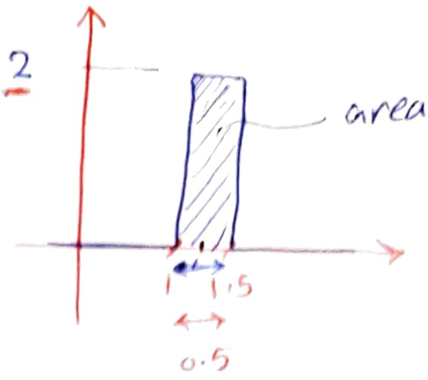




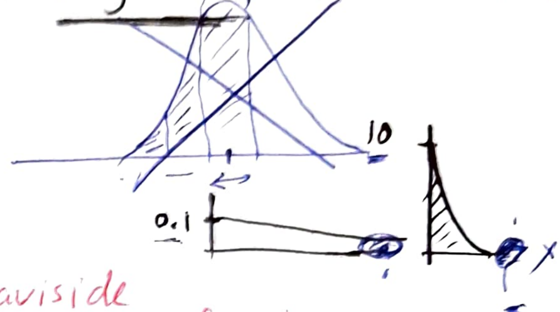
density x range = Prob.

↓

Uniform distribution



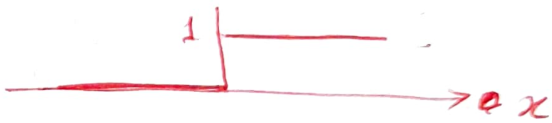
→ exponential distribution



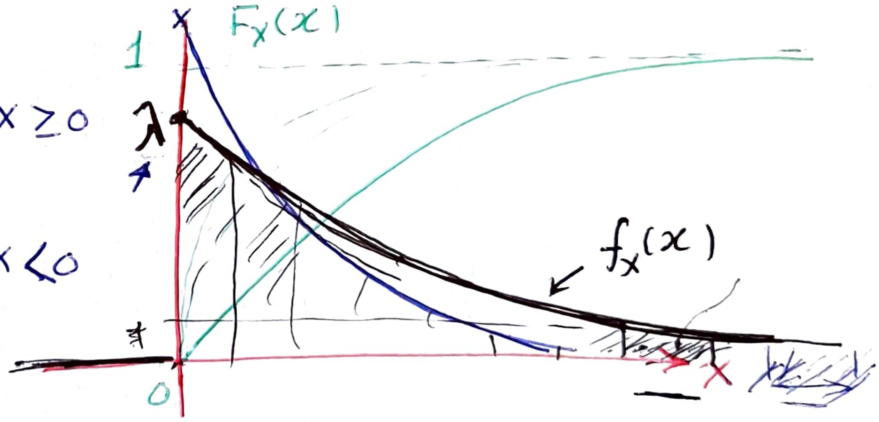
$$F_X(x) = (1 - e^{-\lambda x})\, u(x)$$

Heaviside unit step function

$$= \begin{cases} 1 - e^{-\lambda x} & ; x \geq 0 \\ 0 & ; x < 0 \end{cases}$$



$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & ; x \geq 0 \\ 0 & ; x < 0 \end{cases}$$



$$\frac{d}{dx}(1 - e^{-\lambda x}) = 0 - (-\lambda e^{-\lambda x}) = \lambda e^{-\lambda x}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(\alpha)\, d\alpha$$

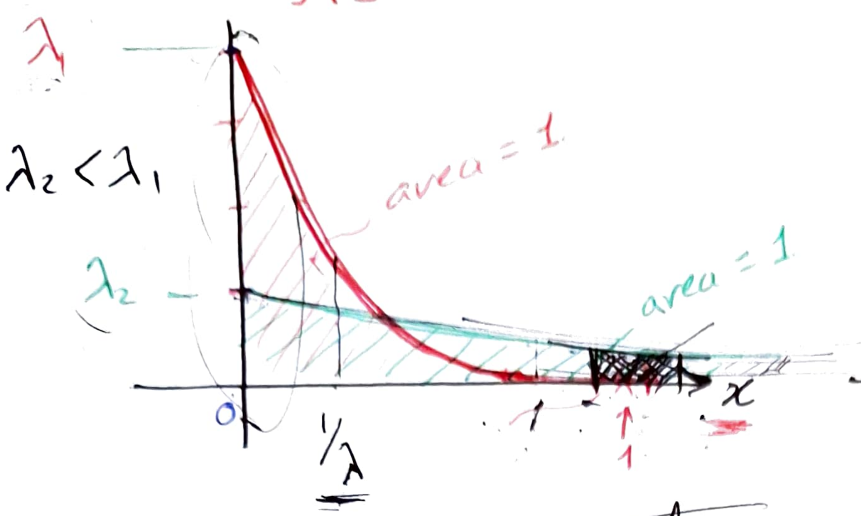
for $x < 0 \Rightarrow F_X(x) = 0$

for $x \geq 0$

$$F_X(x) = \int_0^x \lambda e^{-\lambda \alpha}\, d\alpha$$

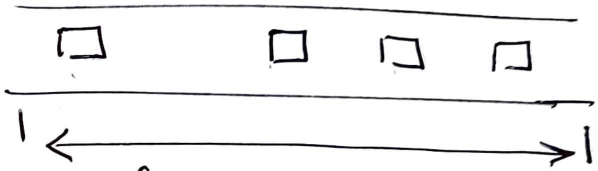
$$= \lambda \frac{e^{-\lambda \alpha}}{-\lambda} \Big|_0^x = -e^{-\lambda x} - (-e^{-\lambda 0}) = 1 - e^{-\lambda x}$$

$\lambda = 20$ $\lambda = 0.5$



$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0 \\ 0 & x < 0 \end{cases}$$

arrival of cars



$X \sim 1\,Km$ avg.

$P(0.5 < X < 1.5)$

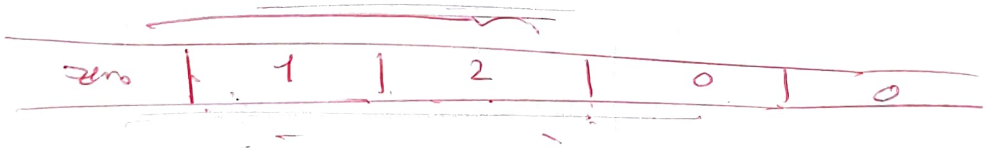
\# of cars ~ Poisson dist.

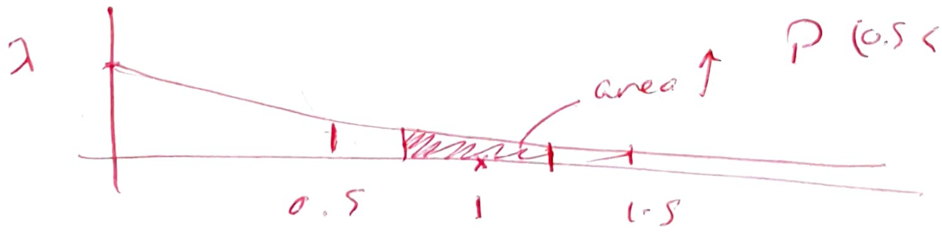
$\lambda \equiv$ density cars/Km

road #1 : 20 cars/Km

road #2 : 0.5 cars/Km

"1 cars per 2 Km)





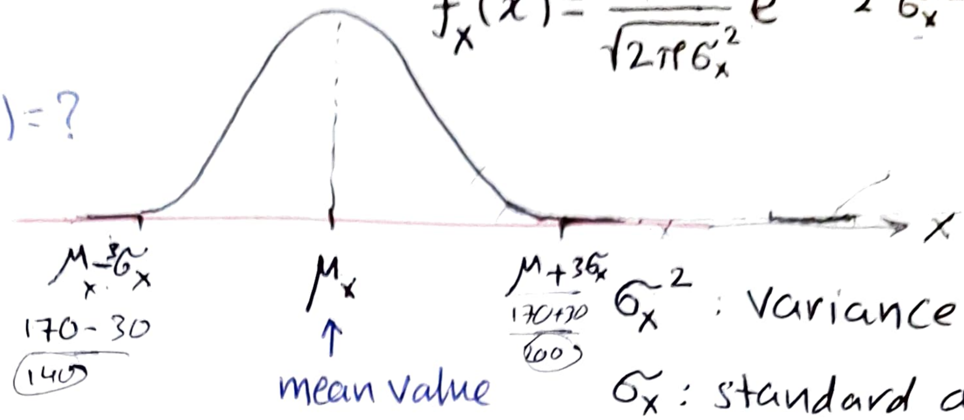
$P(0.5 < X < 1.5) \uparrow$

Gaussian "Normal" Distribution

$F_X(x) = ?$

$P(x_1 < X \leq x_2) = ?$

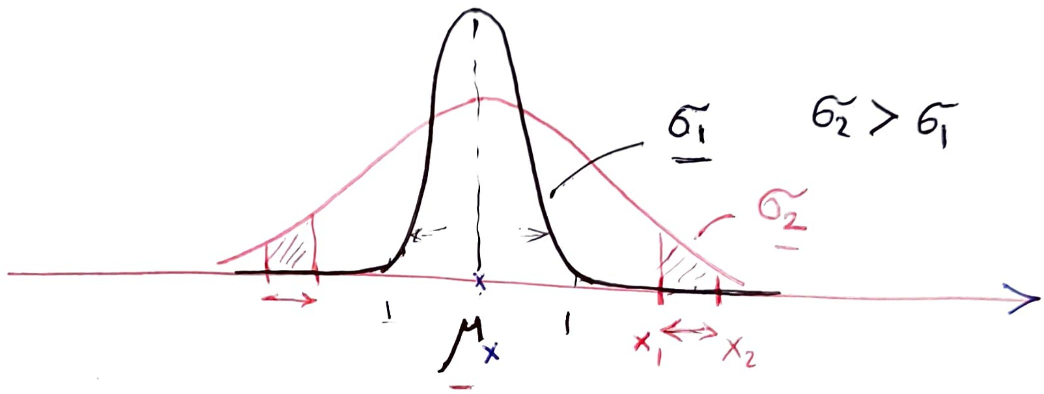
$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}}$$



$\frac{0.3}{100}$

σ_x^2 : variance

σ_x : standard deviation



σ_1 $\quad \sigma_2 > \sigma_1$

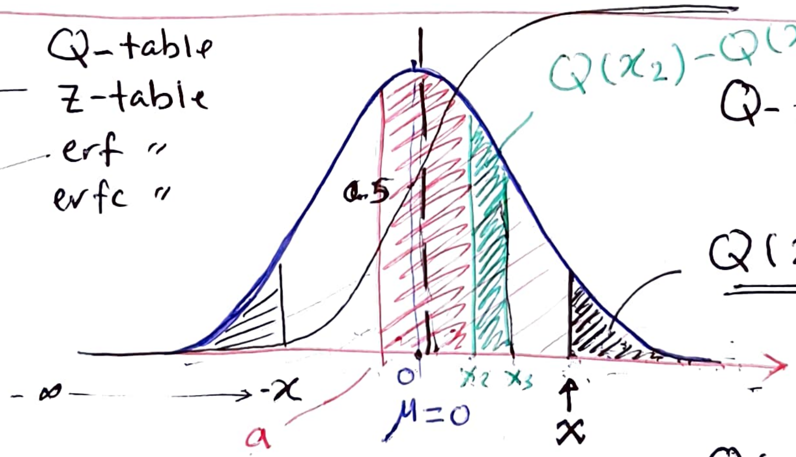
σ_2

x	0	0.1
0	0.5	0.499
1		

Q-table
Z-table
erf "
erfc "

$Q(x_2) - Q(x_3)$

Q- function



$Q(x) = P(X > x)$

$Q(0) = 0.5$

$\sigma = 1$

$$\boxed{Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\alpha^2/2} \, d\alpha}$$

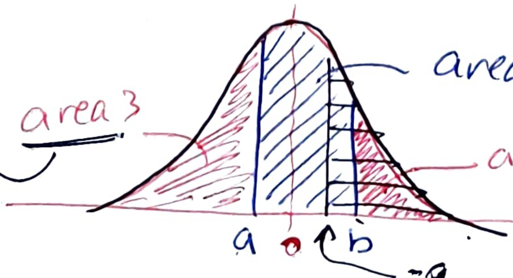
$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

0

$\mu = 0$

$\sigma_x = 1$

$Q(-a)$

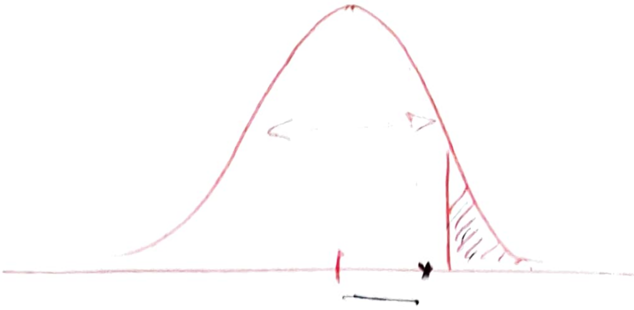


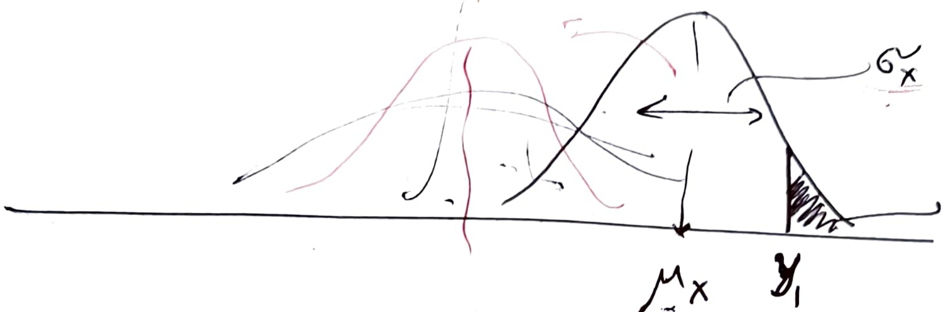
area = 1 − area$_2$ − area$_3$

$Q(b) - Q(-a)$

area 2 +

$Q(b)$





$$Y \sim N(\mu_Y, \sigma_Y^2)$$

$$P(Y > y_1) = Q\left(\frac{y_1 - \mu_Y}{\sigma_Y}\right)$$

68 - 95 - 99.7 Rule

six-sigma

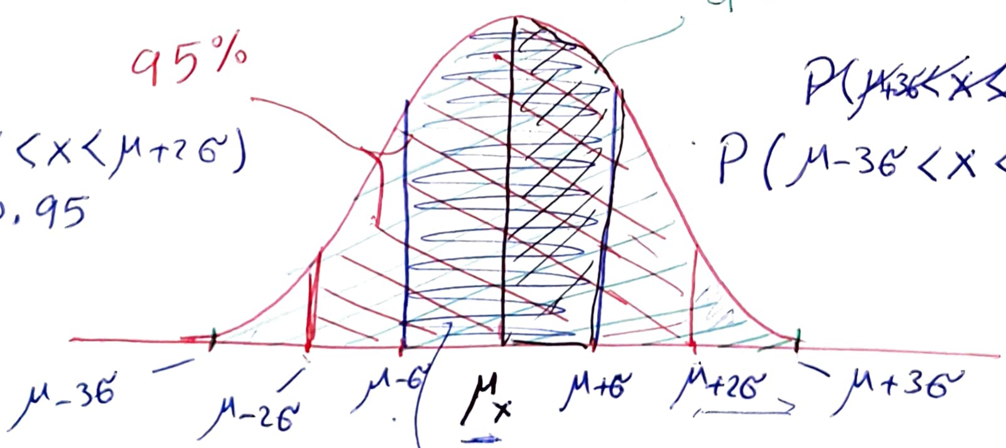
99.7 %

95%

$P(\mu - 2\sigma < X < \mu + 2\sigma)$
$\simeq 0.95$

~~$P(\mu + 3\sigma < X < \mu$~~

$P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.997$



68%

$P(\mu - \sigma < X < \mu + \sigma) \approx .68$

$P(\mu < X < \mu + \sigma) \simeq 0.34$

μ_X, σ_X^2

statistical tools / moments

for continuous Prob. Distributions.

Continuous 

$$\mu_X \equiv \bar{X} \equiv \int_{-\infty}^{\infty} x \, \underline{f_X(x)} \, dx$$

P.D.F.

$$\overline{x^2} = \int_{-\infty}^{\infty} x^2 \, \underline{f_X(x)} \, dx$$

$$\sigma_X^2 = \int_{-\infty}^{\infty} (x - \bar{x})^2 f_X(x) \, dx$$

Disrete distributions

$$\mu_X \equiv \bar{X} = \sum_{i=1}^{n} x_i \, \underline{P_X(x_i)}$$

$$\sigma_X^2 = \overline{x^2} - \bar{x}^2$$

$$\overline{x^2} = \sum_{i=1}^{n} x_i^2 \, P_X(x_i)$$

$$\sigma_X^2 = \overline{(x_i - \bar{x})^2}$$

$$= \sum_{i} \underline{(x_i - \bar{x})^2} \, \underline{P_X(x)}$$

CLT

Central Limit theorem

indep.

X is an R.V.

Y is an R.V.

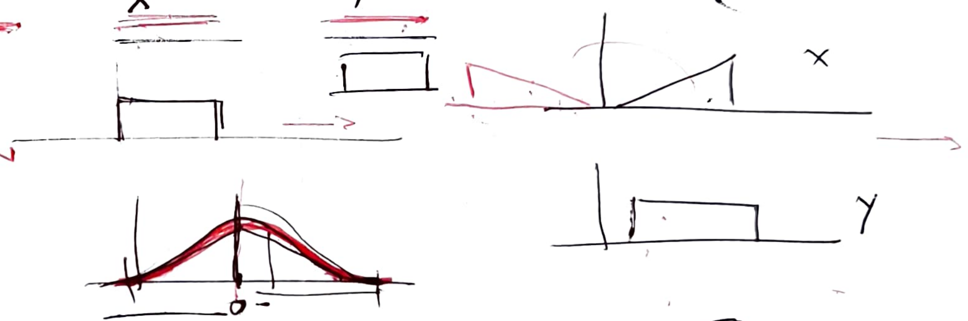
CNN

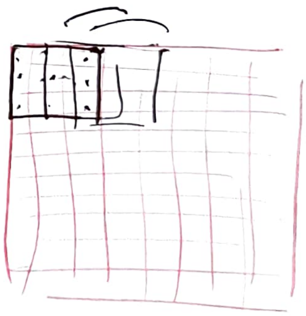
convolutional

$Z = X + Y$

convolution!

$$f_Z(z) = f_X(x) * f_Y(y)$$





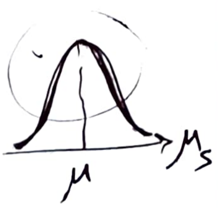
CLT

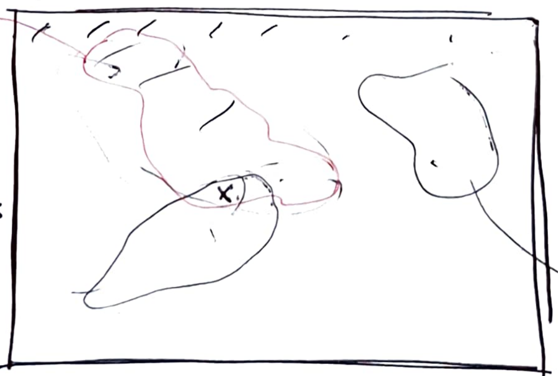
sample mean

sample #1

$\mu_{s_1} = \sim 168$

$\mu_{s_2} = - 171$





Population

(45)

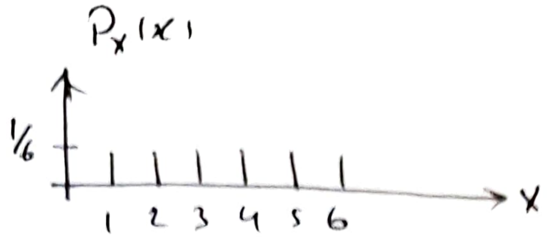
μ_{s_3}

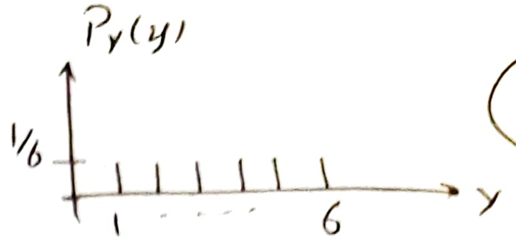
173

170

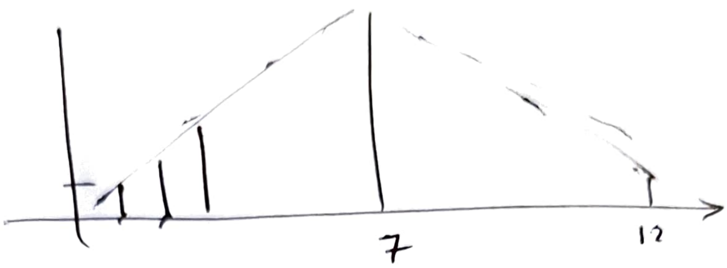
CLT

ex.

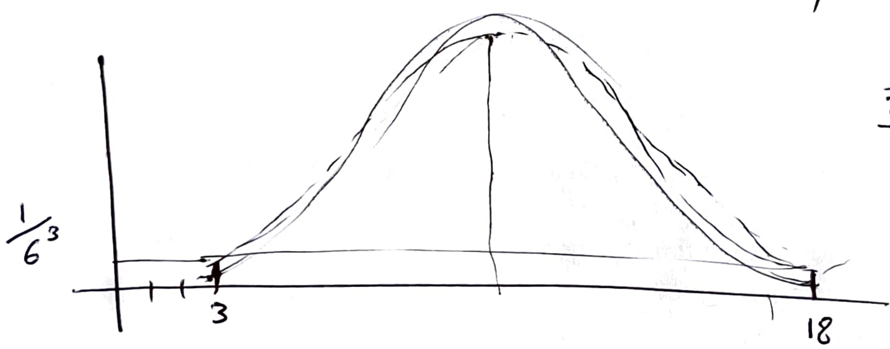




$Z = X + Y$



$Z = X + Y + W + \ldots$



poisson Bernoulli