4/12/2024

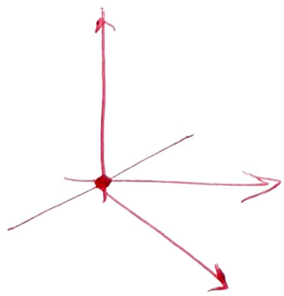
Linear Algebra for AI, Mansoura S3

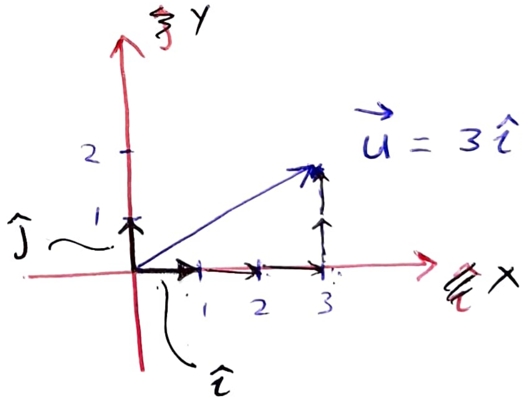
- summary, fundamental spaces
  - column space
  - Row space
  - Null space
- Linear transformation
- Rank
- Determinant
- Matrix inverse

→ Review

Space   Vectors & their linear combinations

$\mathbb{R}^1$ $\mathbb{R}^2$ $\mathbb{R}^3$



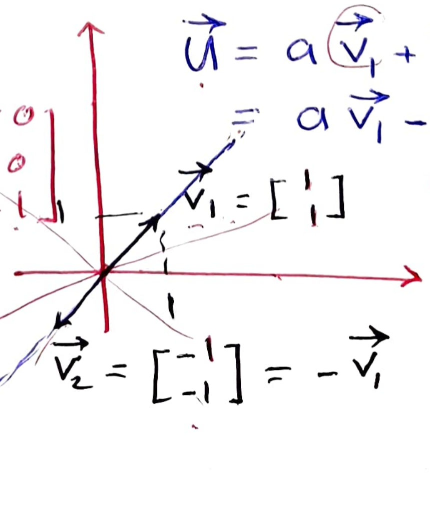


$\vec{u} = 3\hat{i} + 2\hat{j} = \begin{bmatrix}3\\2\end{bmatrix}$

$\begin{array}{c} R\\ G\\ B\end{array}\begin{bmatrix}15\\16\\17\end{bmatrix} = 15\underset{R}{\underline{\begin{bmatrix}1\\0\\0\end{bmatrix}}} + 16\begin{bmatrix}0\\1\\0\end{bmatrix} + 17\begin{bmatrix}0\\0\\1\end{bmatrix}$

$\begin{bmatrix}a\\a\\b\end{bmatrix}$ a $\boxed{x_1 = \begin{bmatrix}1\\1\\0\end{bmatrix} \quad b\,x_2 = \begin{bmatrix}0\\0\\1\end{bmatrix}}$

→ subspace



$\vec{u} = a\vec{v}_1 + b\vec{v}_2$
$= a\vec{v}_1 - b\vec{v}_1 =$
$(a-b)(\vec{v}_1)$

$\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$

$\vec{v}_2 = \begin{bmatrix}-1\\-1\end{bmatrix} = -\vec{v}_1$

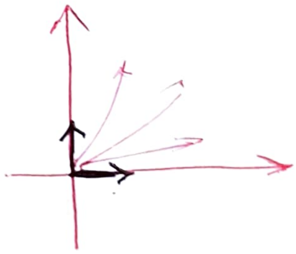
Matrix A

Column space $C(A)$: span(column vectors)

Row space $R(A)$: span(row vectors)

Null space $\mathcal{N}(A)$: span$(\vec{x})$; $A\vec{x}=\vec{0}$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



$C(A) = \mathbb{R}^2$

$R(A) = \mathbb{R}^2$

$\mathcal{N}(A) : \vec{x} \; ; \{ A\vec{x} = \vec{0}$

$\mathcal{N}(A) = \vec{0}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x} = \vec{0}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$1 \cdot x_1 + \cancel{0 x_2} = 0$$

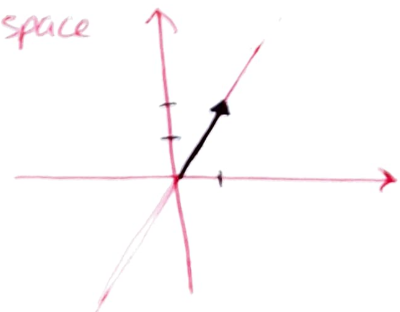
$$\cancel{0 x_1} + 1 \cdot x_2 = 0$$

$$x_1 = 0$$

$$x_2 = 0$$

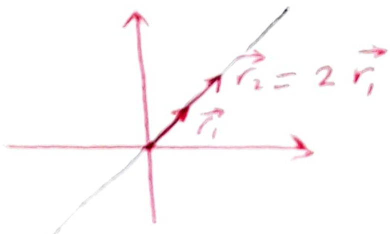
$$\underset{2\times 2}{B} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Column space



$C(B) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

$R(B) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ Row space



$\mathcal{N}(B)$

$\mathcal{N}(B)$

$$B\vec{x} = \vec{0}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 = 0$$

$$2x_1 + 2x_2 = 0$$

$\mathcal{N}(B) = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$

$x_1 = -x_2 \rightarrow$ free variable

$$2x_1 = -2x_2$$

$$\begin{bmatrix} -x_2 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1+1 \\ -2+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Column picture

$$\vec{B}\,\vec{u} = \vec{v}$$

$$\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = u_1\begin{bmatrix} b_1 \\ b_3 \end{bmatrix} + u_2\begin{bmatrix} b_2 \\ b_4 \end{bmatrix}$$

C.V.$_1$ C.V.$_2$

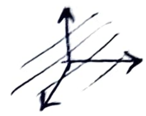
$$A_{3\times 3} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$\vec{c}_3 = \vec{c}_1 + \vec{c}_2$

rank = # of indep. rows
= # of " columns
for square matrices

$$RREF(A) \Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$3\times$



$RREF \neq I_3$

$$RREF(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

row

$a\vec{r}_1 + b\vec{r}_2 + c\vec{r}_3 = [0,0,0] = \vec{0}$

$\vec{r}_3 = -\frac{a}{c}\vec{r}_1 + -\frac{b}{c}\vec{r}_2$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & 3 & 3 \end{bmatrix} \Rightarrow (\vec{r}_1 \times -1) + r_2 \Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \end{bmatrix} \times -3$$

$\vec{r}_3 - 3(r_{2\,new}) = \vec{0}$

$\vec{r}_3 = 3((\vec{r}_1 \times -1) + \vec{r}_2) = 0$

$R(A) = \text{span}\{\vec{r}_1, \vec{r}_2\} = \{(1,0,1), (0,1,1)\}$

$C(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}$

$N(A) = \text{span}\left\{\begin{bmatrix} -1 \\ -1 \\ +1 \end{bmatrix}\right\}$ $\quad A\vec{x} = \vec{0} \Leftrightarrow RREF(A)\,\vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 0 + x_3 = 0$ $\quad \alpha$

$0 + x_2 + x_3 = 0$

~~$0 + 0 + 0 = 0$~~

$x_3 = \alpha$

$x_1 = -\alpha$

$x_2 = -\alpha$

let $\alpha = 1$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} x_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A_{3\times 3} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

$\vec{C_2} = 2\vec{C_1}$

$\vec{C_3} = 3\vec{C_1}$

RREF(A) ⇒

$$\begin{array}{l} \\ r_2 - r_1 \rightarrow \\ r_3 - 2r_1 \rightarrow \end{array} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

rank = 1

$R(A) = \text{span}\{(1, 2, 3)\}$ $\quad \dim(R(A)) = 1$

$C(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}\right\}$ $\quad \dim(C(A)) = 1$

$N(A)$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

both x_2 & x_3 are free variables.

$$1x_1 + \underline{2x_2} + \underline{3x_3} = 0$$

$x_2 = \alpha \qquad x_3 = \beta$

$x_1 + 2\alpha + 3\beta = 0$

let $\alpha = 0$, $\beta = 1$ ⇒ $x_1 = -3$

$$\Rightarrow \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \vec{n_1}$$

let $\alpha = 1$, $\beta = 0$

$x_1 = -2$

$$\Rightarrow \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \vec{n_2}$$

$$N(A) = \left\{\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right\}$$

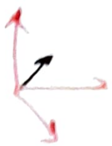
$\dim(N(A)) = 2$

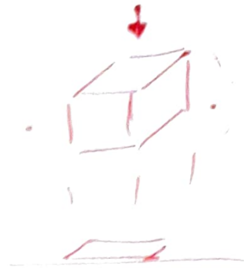
$k_1 A \vec{n_1} = \vec{0} \;/\; k_2 A \vec{n_2} = 0$

$$A\,(\underline{k_1\vec{n_1} + k_2\vec{n_2}}) = 0$$

$$A_{3\times 4} = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 2 & 6 & 7 & 6 \\ 3 & 9 & 9 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$\underbrace{\quad}_{n\text{-dim}}$ $\qquad$ $\underbrace{\quad}_{m\text{-dim}}$





$$\text{RREF}(A) = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\dim(R(A)) = \dim(C(A))$$

$$R(A) = \text{span}\{(1,3,0,0), (0,0,1,0), (0,0,0,1)\}$$

$$C(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} \right\} = \mathbb{R}^3$$

free variable

$$N(A) = ?$$

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \alpha_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 3\alpha_2 = 0$$
$$x_3 = 0$$
$$x_4 = 0$$

$$= \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{bmatrix} -3\alpha \\ \alpha \\ 0 \\ 0 \end{bmatrix}$$

$$N(A) = \text{span}\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\dim(C(A)) = 3 \qquad \dim(R(A)) = 3 \qquad \dim(N(A)) = 1$$

3 rows × 4 colums.

$m \times n$

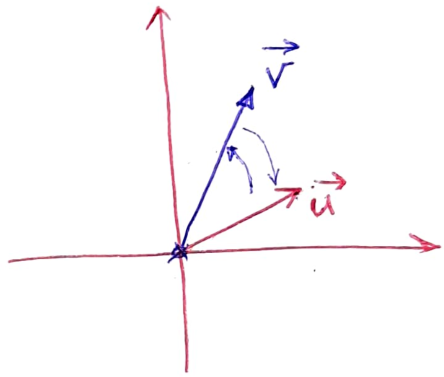
$$\boxed{\begin{aligned} \text{rank} &= \#\text{ of indep. rows} \\ &= \#\text{ of indep. columns} \\ &= \dim(R(A)) \\ &= \dim(C(A)) \end{aligned}}$$

A is invertible if there exists another matrix A^{-1} ; $\boxed{\begin{array}{l} AA^{-1} = I \\ A^{-1}A = I \end{array}}$ for square matrices

$A_{m\times m}$

Transformation matrices

$$\underset{m\times m}{A}\,\underset{m}{\vec{u}} = \underset{m}{\vec{v}}$$

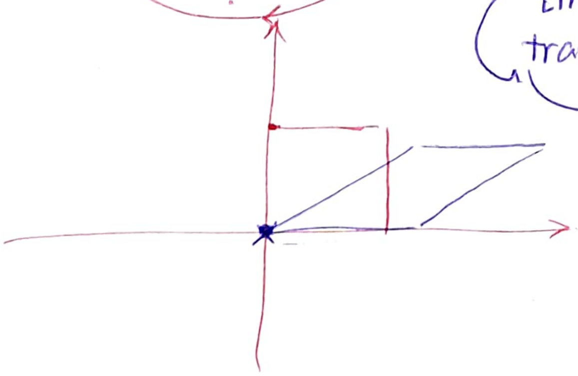


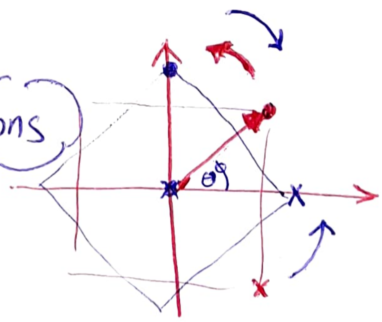
$$x \longrightarrow \boxed{f(x)} \longrightarrow y$$

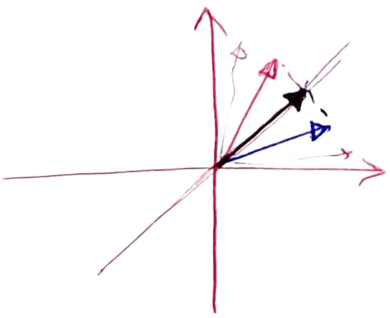
$$\vec{x} \longrightarrow \boxed{A} \longrightarrow \vec{y} = A\,\vec{x}$$

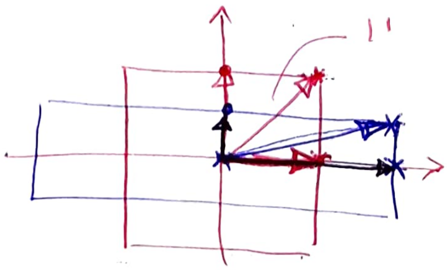
?

Linear transformations

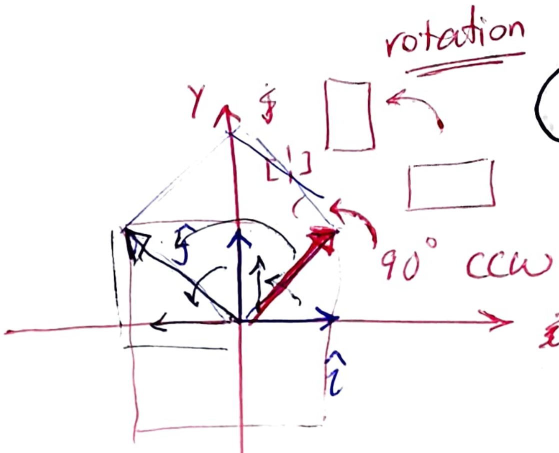








$$A = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0.5 \end{bmatrix}$$

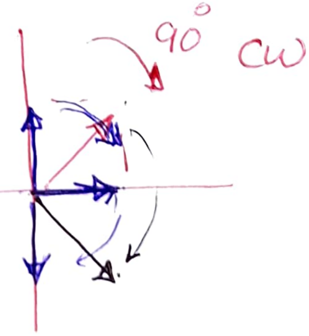


$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

where $\hat{\imath}$ vector lands

where vector $\hat{\jmath}$ goes

$$\hat{\imath} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \hat{\jmath} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} I_2$$

$$\boxed{B = A^{-1}}$$

$x \times x^{-1} = 1 \;;\; x^{-1} = \frac{1}{x}$

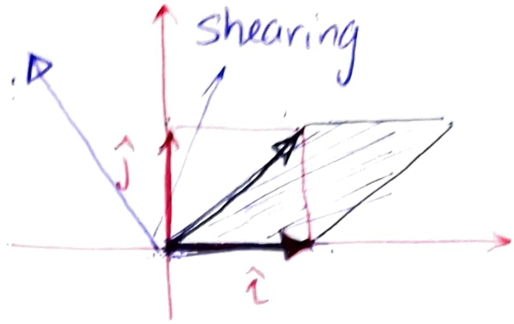
$AA^{-1} = I$

$A^{-1}A = I$

Shearing transformations

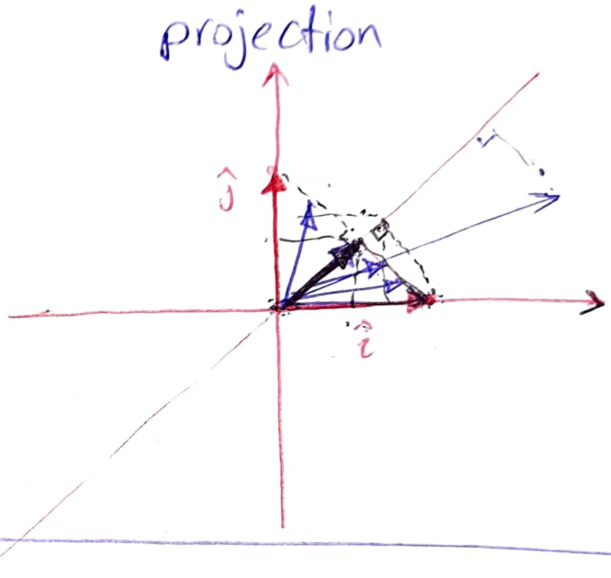
$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

det (A) = nonzero



$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

det (A) = 0



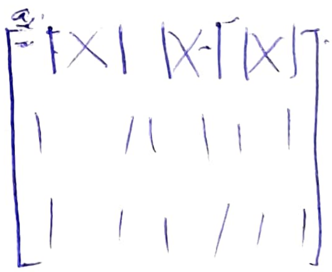
$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

$$\operatorname{adj}(A) = C^{T}$$ transpose

Cofactor

$$C = (-1)^{i+j} M_{i,j}$$

minors

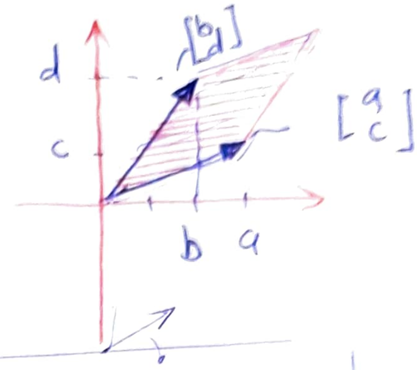


determinant of $A_{2\times2}$ matrix (for square matrices)

$$\det(A_{2\times2}) = \left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right|$$

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

2x2 matrix



det(A) = 0 if column vectors are not independent!

3x3 matrix

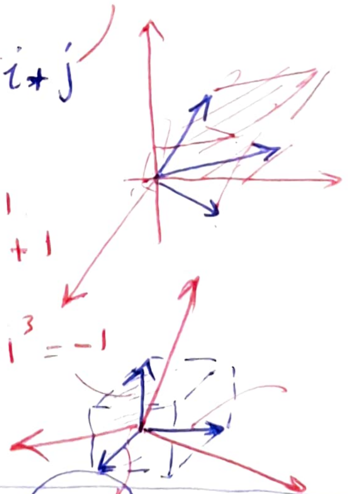
$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

row # column #

$$(-1)^{i+j}$$

$$(-1)^{1+1} = +1$$

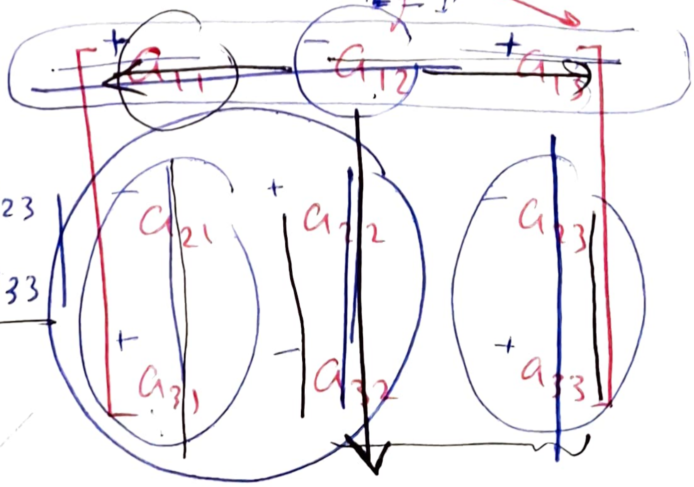
$$(-1)^{1+2} = -1^{3} = -1$$



$$+a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$+a_{11} \underline{M_{11}} + (-)a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+ \underline{a_{13}} \underbrace{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}_{\text{Minor } (a_{13})}$$



$$A_{3\times 3} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$$

$A^{-1} = ?$ det(A) = ? $\begin{cases} 0 & \text{A is not invertible!} \\ \text{non zero} & \checkmark \end{cases}$

adj(A)

$$\begin{bmatrix} \overset{+}{a_{11}} & \overset{-}{b} & \overset{+}{c} \\ \overset{-}{d} & \overset{+}{e} & \overset{-}{f} \\ \overset{+}{g} & \overset{-}{h} & \overset{+}{k} \end{bmatrix} \qquad \begin{bmatrix} e & f \\ h & k \end{bmatrix}$$

$i=2,\ j=3$

$(-1)^{2+3} = (-1)^5 = -1$

$$C_{11} = (-1)^{i+j} \underline{M_{11}} = (ek - fh)(-1)^{i+j}$$

→ inverse of a matrix using Gaussian Elimination.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \Rightarrow \left[\ I \ \middle|\ A^{-1}\ \right]$$

$$\underbrace{}_{A} \qquad \underbrace{}_{I}$$

$$\Rightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \div -2 \Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 1.5 & -0.5 \end{array}\right]$$

$r_1 \times -3 + r_2$

Check

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$$

$$= \begin{bmatrix} \underline{\underline{-2+3}} & \cancel{2}\ 1-1 \\ -6+6 & 3-2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A

I $\qquad$ A^{-1}
