The language of the first-order logic (FOL)

Three things define a declarative language:

- syntax – what groups of symbols are valid and in what order
  "the car that I drive"
  "drive the car I that"
- semantics – what the well-formed sentences mean – some expressions may not mean anything
  "blue holiday runs"
- pragmatics – how meaningful expressions are used
  "There is someone behind you"

The syntax in FOL

There are two types of symbols: logical and nonlogical.

1. The logical symbols (like the reserved words in a programming language)
   a) punctuation: "(", ")", "."
   b) connectives: $\neg$, $\wedge$, $\vee$ (in descending order of priority)
      quantifiers $\exists$, $\forall$
      $=$ logical equality (special symbol, not a predicate)
   c) variables: an infinit set of symbols (denoted by x, y, z with/without subscripts and superscripts)

2. The nonlogical symbols (like the identifiers in a programming language)
   - they have an application-dependent meaning or use
   - they have arity
   a) function symbols – start with a lower-case letter; written in mixed case

      examples: bestFriend
      a, b, c, f, g, h (with/without sub/superscripts)
      by convention – if arity is zero, we use symbols a, b, c – they are called constants.

b) predicate symbols – start with an upper-case letter; written in mixed case

examples: OlderThan

P, Q, R (with/without sub/superscripts)

- predicate symbols of arity 0 are called propositional symbols.

In FOL there are two types of valid syntactic expressions

1) Terms

a) every variable is a term;

b) if $t_1, \dots, t_n$ are terms and f is a function symbol with n arguments, then $f(t_1, \dots, t_n)$ is a term

2) Formulas

called atomic formulas (or atoms)

a) if $t_1, \dots, t_n$ are terms and P is a predicate symbol with n arguments, then $P(t_1, \dots, t_n)$ is a formula

b) if t_1 and t_2 are terms, then $t_1 = t_2$ is a formula

c) if α and β are formulas and x is a variable, then $\neg\alpha$, $\alpha \wedge \beta$, $\alpha \vee \beta$, $\forall x. \alpha$ and $\exists x. \alpha$ are formulas

The propositional subset of FOL is the language with no terms or quantifiers, but only propositional symbols are used:

$$(P \wedge Q) \vee \neg R$$

Abbreviations:

$\alpha \supset \beta$ for $(\neg \alpha \vee \beta)$ (also denoted by $\rightarrow$ or $\Rightarrow$)

$\alpha \equiv \beta$ for $((\alpha \supset \beta) \wedge (\beta \supset \alpha))$ (also denoted by $\Leftrightarrow$)

A variable occurrence is bound in a formula if it lies within the scope of a quantifier. Otherwise it is free.

$$\forall y.\ P(x) \ \wedge \ \exists x\ [P(y) \vee Q(x)]$$

free

bound

bound

Obs. In some books, the occurrence of the variable just after the quantifiers is neither free nor bound.

Definition. A sentence in FOL is any formula without free variables

$$\forall y . \exists x [P(y) \vee Q(x)]$$

The semantics in FOL

The meaning of a sentence derives from the interpretation of the nonlogical symbols involved.
Nonlogical symbols are application-dependent, so no definitive answers can be offered.
We are looking for a clear specification of the meaning of a sentence as a function of the interpretation of the predicate and function symbols.
For example, the meaning of "DemocraticCountry" can be specified by "objects" that represent those countries that we consider to be democratic. We may agree or not on which those countries are, but in this case we just talk about different interpretations. In terms of FOL, we are not interested to say what "DemocraticCountry" means according to a dictionary (free elections, representative government etc.)

Interpretations in FOL

An interpretation $\mathcal{I}$ is a pair $<D, I>$, where

- D is a non-empty set of objects, called the domain of the interpretation (it can be anything)
- I is the interpretation mapping, that assigns a meaning to the predicate and function symbols.

If P is a predicate symbol of arity n, then

$$I[P] \subseteq \underbrace{D \times \dots \times D}_{n \text{ times}} \qquad I[P] \text{ is a } n\text{-ary relation over } D.$$

if f is a function symbol of arity n, then

$I[f] \in [\underbrace{\Delta \times \ldots \times \Delta}_{n \text{ times}} \to \Delta]$ $I[f]$ is a n-ary function over Δ

Examples

$\Delta = \{d_1, d_2, d_3$, ana, maria, costel, george, ion, $\ldots\}$

Dog - unary predicate symbol

$I[\text{Dog}] = \{d_1, d_2, d_3\}$ - the set of dogs in this interpretation

OlderThan - binary predicate symbol

$I[\text{OlderThan}] = \{(\text{ana}, \text{maria}), (\text{costel}, \text{maria})\}$

bestFriend - unary function symbol

$I[\text{bestFriend}] : \Delta \to \Delta$

$I[\text{bestFriend}](\text{ana}) = \text{maria}$

firstChildOf - binary function symbol

$I[\text{firstChildOf}] : \Delta \times \Delta \to \Delta$

$I[\text{firstChildOf}](\text{maria}, \text{vasile}) = \text{george}$

$I[\text{motherOfThreeChildren}] : \Delta \times \Delta \times \Delta \to \Delta$

$I[\text{motherOfThreeChildren}](\text{george}, \text{ion}, \text{ana}) = \text{maria}.$

Obs. johnSmith is a constant $I[\text{johnSmith}] = \text{ion} \in \Delta$

A useful alternative to interpret predicates symbols is in terms of their characteristic function. Thus, for P a predicate of arity n, we view $I[P]$ as an n-ary function to $\{0,1\}$

$$I[P] \in [\underbrace{\Delta \times \ldots \times \Delta}_{n \text{ times}} \to \{0,1\}]$$

The two specifications are related as following: a tuple of objects is considered to be in the relation over Δ iff the characteristic function over those objects has value 1.

if P is a predicate of arity 0, $I[P]$ is either 0 or 1 (false/true)

For the propositional subset of FOL, the domain Δ can be ignored and see the interpretation as a mapping I from the propositional symbols to either 0 or 1.

Denotation

It means to indicate which element of Δ is denoted by a term in an interpretation $\mathcal{J} = <\Delta, I>$

1. if a term does not contain any variable

$$\|bestFriend(johnSmith)\|_{\mathcal{J}} = I[bestFriend](I(johnSmith))$$

2. if a term contains variables, we first define μ - a variable assignment over Δ.

for a variable x, $\mu[x] \in \Delta$.

The denotation of a term t, given $\mathcal{J}$ and μ, is written $\|t\|_{\mathcal{J},\mu}$ and it is defined by the rules:

a) if x is a variable, then $\|x\|_{\mathcal{J},\mu} = \mu[x]$;

b) if $t_1, \dots, t_n$ are terms and f is a function symbol of arity n, then

$$\|f(t_1, \dots, t_n)\|_{\mathcal{J},\mu} = I[f](\|t_1\|_{\mathcal{J},\mu}, \dots, \|t_n\|_{\mathcal{J},\mu})$$

$$\|bestFriend(x)\|_{\mathcal{J},\mu} = I[bestFriend](\mu[x])$$

if $\mu[x] = ion$ then $I[bestFriend](ion)$

The denotation of a term is an element of Δ.

Satisfaction

We can say which sentences in FOL are true and which are false, according to an interpretation $\mathcal{J}$ and a variable assignment μ

For example, Dog(bestFriend(johnSmith)) is true in $\mathcal{J}$ iff

1. we use I to obtain the subset of Δ denoted by "Dog"
2. find the object in Δ denoted by "bestFriend(johnSmith)"
3. the object found at step 2 belongs to the subset found at 1.

Given $\mathcal{J}$ and μ, we say that the formula α is satisfied in $\mathcal{J}, \mu$ and we write $\mathcal{J}, \mu \models \alpha$ according to the following rules:

1. $\mathcal{I},\mu \models P(t_1,\dots t_n)$ iff $< \|t_1\|_{\mathcal{I},\mu},\dots,\|t_n\|_{\mathcal{I},\mu} > \in I[P]$;
2. $\mathcal{I},\mu \models t_1 = t_2$ iff $\|t_1\|_{\mathcal{I},\mu}$ & $\|t_2\|_{\mathcal{I},\mu}$ are the same element of Δ;
3. $\mathcal{I},\mu \models \neg\alpha$ iff it is not the case that $\mathcal{I},\mu \models \alpha$;
4. $\mathcal{I},\mu \models (\alpha \wedge \beta)$ iff $\mathcal{I},\mu \models \alpha$ and $\mathcal{I},\mu \models \beta$;
5. $\mathcal{I},\mu \models (\alpha \vee \beta)$ iff $\mathcal{I},\mu \models \alpha$ or $\mathcal{I},\mu \models \beta$;
6. $\mathcal{I},\mu \models \exists x.\alpha$ iff $\mathcal{I},\mu' \models \alpha$ for some variable assignment μ' that differs from μ on at most x
7. $\mathcal{I},\mu \models \forall x.\alpha$ iff $\mathcal{I},\mu' \models \alpha$ for every variable assignment μ' that differs from μ on at most x

- If α is a sentence, satisfaction does not depend on any μ. We write $\mathcal{I} \models \alpha$ and read "α is true in the interpretation $\mathcal{I}$"
- For the propositional subset of FOL, we write $I[\alpha]=1$ or 0 according to whether $\mathcal{I} \models \alpha$ or not
- if S is a set of sentences, we write $\mathcal{I} \models S$ if all the sentences in S are true in $\mathcal{I}$. We say that $\mathcal{I}$ is a logical model of S.