Statistical and Mathematical Methods for Data Analysis

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Textbooks

- **Probability & Statistics for Engineers & Scientists**, Ninth Edition, Ronald E. Walpole, Raymond H. Myer

- **Elementary Statistics: Picturing the World,** 6th Edition, Ron Larson and Betsy Farber

- **Elementary Statistics**, 13th Edition, Mario F. Triola

Reference books

- **Probability and Statistical Inference, Ninth Edition,** Robert V. Hogg, Elliot A. Tanis, Dale L. Zimmerman

- **Probability Demystified**, Allan G. Bluman

- **Practical Statistics for Data Scientists: 50 Essential Concepts,** Peter Bruce and Andrew Bruce

- **Schaum's Outline of Probability,** Second Edition, Seymour Lipschutz, Marc Lipson

- **Python for Probability, Statistics, and Machine Learning,** José Unpingco

References

- Probability & Statistics for Engineers & Scientists, Ninth edition, Ronald E. Walpole, Raymond H. Myer

- Elementary Statistics, Tenth Edition, Mario F. Triola

These notes contain material from the above resources.

Correlation

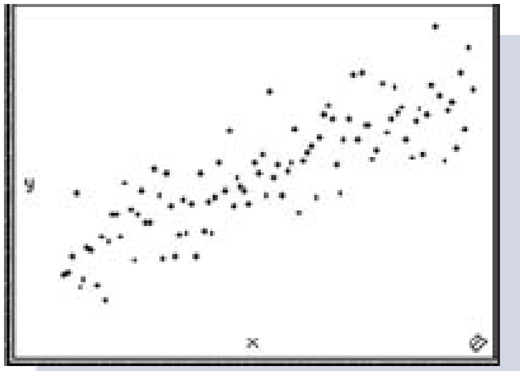
A **correlation** exists between **two variables** when **one** of them is **related** to the other in some way.

Exploring the Data

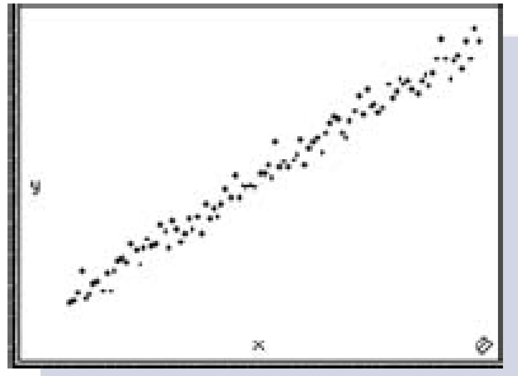
We can often see a **relationship between two variables** by constructing a **scatterplot**. When we examine a **scatterplot**, we should study the **overall pattern** of the plotted points. If there is a pattern, we should note its **direction**.

- An **uphill direction** suggests that as **one variable increases**, the **other also increases**.
- A **downhill direction** suggests that as **one variable increases**, the **other decreases**.
- We should look for **outliers**, which **are points that lie** very **far away** from all of the **other points**.

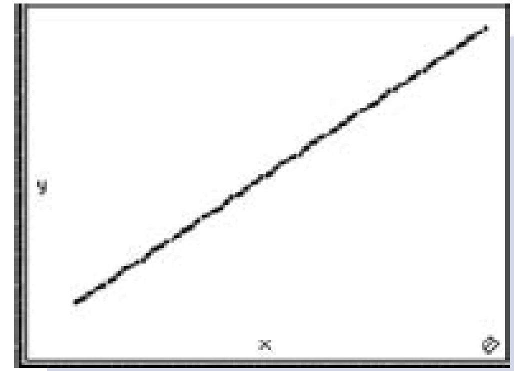
Scatter plots



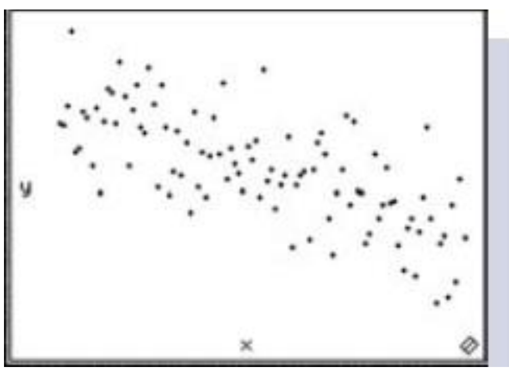
Positive correlation:
***r* = 0.851**



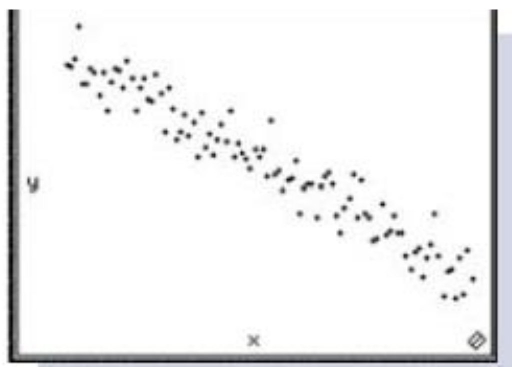
Positive correlation:
***r* = 0.991**



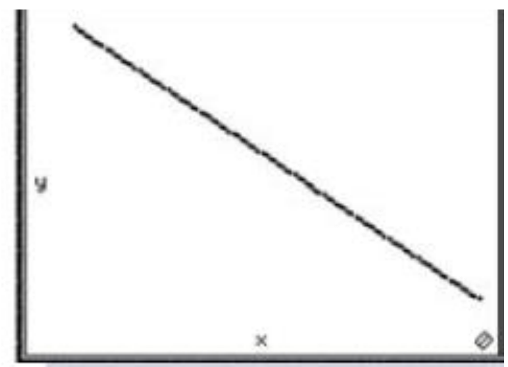
Perfect positive correlation:
***r* = 1**



Negative correlation:
***r* = – 0.702**

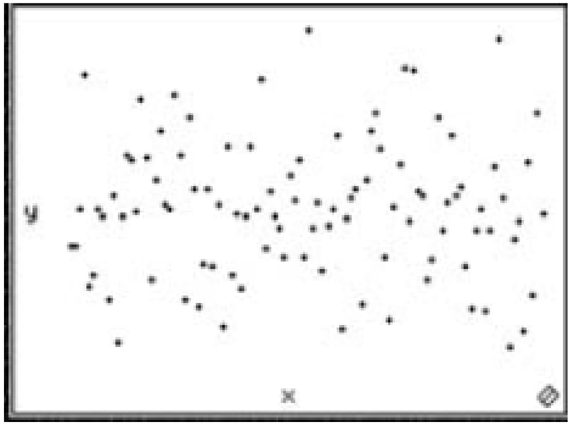


Negative correlation:
***r* = – 0.965**

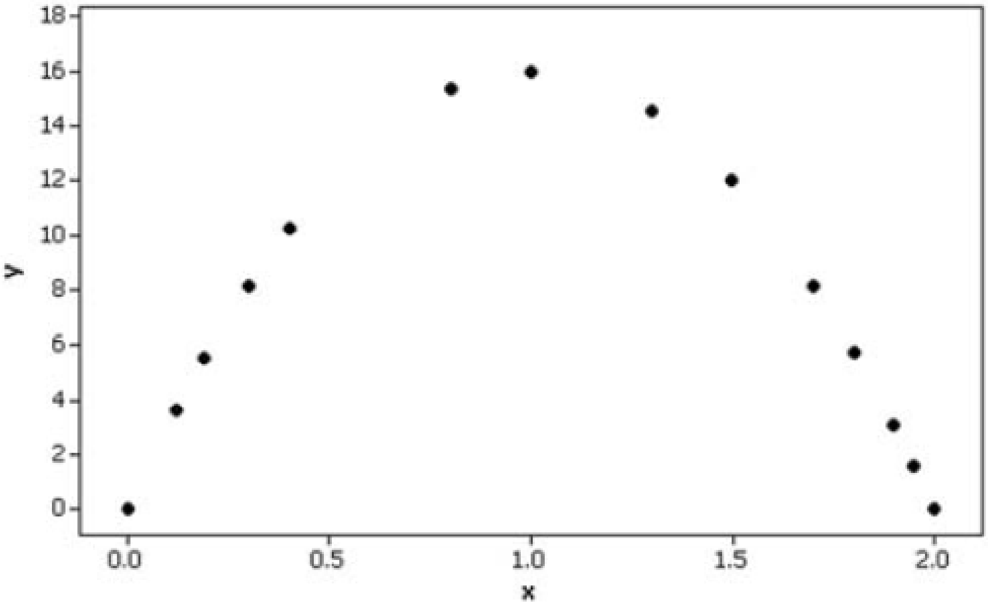


Perfect negative correlation:
***r* = –1**

Scatter plots



No correlation: *r* = 0



Nonlinear relationship: *r* = – 0.087

Palm reading

Palm reading

- Some **people believe** that the **length of their palm's lifeline** can be used to **predict longevity**.

- In a letter published in the **Journal of the American Medical Association**, authors M. E. Wilson and L. E. Mather **refuted that belief** with a **study of cadavers**.

- **Ages at death** were recorded, along with the **lengths** of **palm lifelines**. The authors concluded that **there is no significant correlation between age at death** and **length of lifeline**. Palmistry lost, hands down.

Requirements

Given any **collection of sample paired data**, the linear **correlation coefficient *r*** can always be computed, but the following requirements should be satisfied when **testing hypotheses** or making other **inferences about *r***.

1. The sample of **paired (*x*, *y*)** data is a ***random* sample** of **independent quantitative data**.
2. **Visual examination** of the **scatterplot** must confirm that the points approximate a **straight-line pattern**.
3. **Any outliers** must be removed if they are known to **be errors**. The effects of any other outliers should be considered by **calculating *r* with** and **without the outliers** included.

- ***Note*: Requirements 2 and 3** above are simplified attempts at checking this formal requirement:
- The **pairs of (*x*, *y*)** data must have **a bivariate normal distribution.** (This assumption basically requires that for any **fixed value of *x***, the **corresponding values of *y*** have a distribution that is **bell-shaped**, and for any fixed value of *y*, the values of *x* have a **distribution that is bell-shaped**.)
- This requirement is usually difficult to check, so for now, we will use Requirements 2 and 3 as listed above.

Notation for the Linear Correlation Coefficient

- **n**: represents the **number of pairs** of data present.

- **r**: represents the **linear correlation coefficient** for a **sample**.

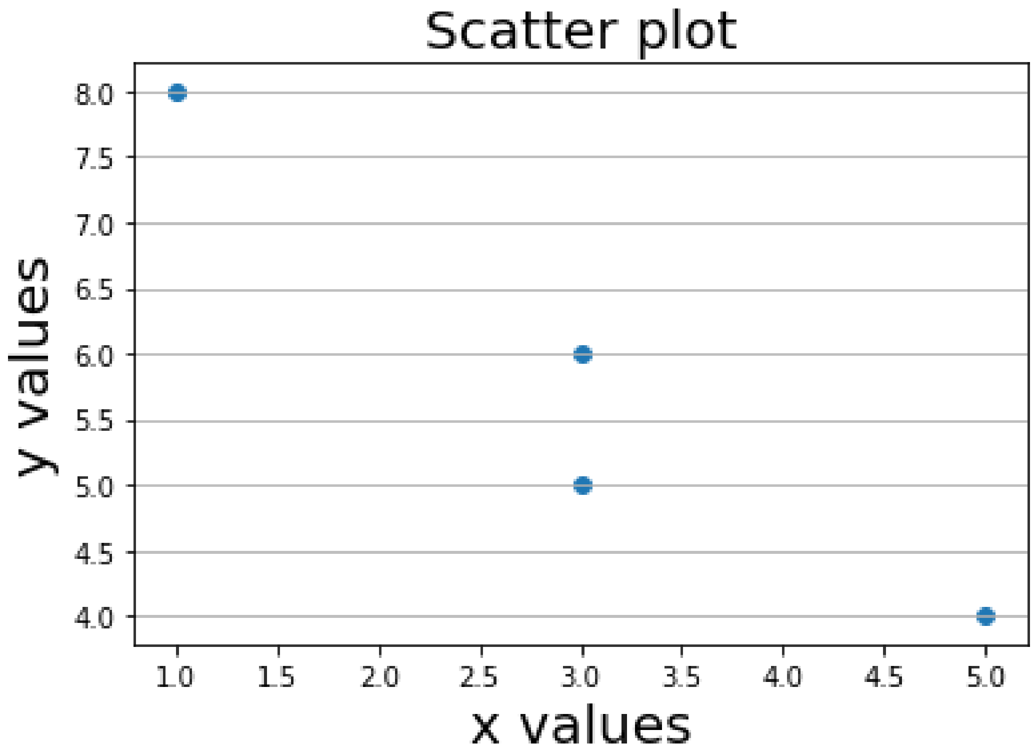
- **ρ**: Greek letter **rho** used to represent the **linear correlation coefficient** for a **population**.

$$\mathbf{r} = \frac{n(\sum xy) - (\sum x)(\sum y)}{\sqrt{n(\sum x^2) - (\sum x)^2}\sqrt{n(\sum y^2) - (\sum y)^2}}$$

Example Calculating *r* Using the simple random sample of data given in the table, find the value of the **linear correlation coefficient *r***.

x	3	1	3	5
y	5	8	6	4

REQUIREMENT The data are a **simple random sample**. The accompanying **Python-generated scatterplot** shows **a pattern of points** that does appear to be a **straight-line pattern.** There are no outliers. We can proceed with the calculation of the linear correlation coefficient ***r*.**



x	y	xy	x^2	y^2
3	5	15	9	25
1	8	8	1	64
3	6	18	9	36
5	4	20	25	16
$\sum x = 12$	$\sum y = 23$	$\sum xy = 61$	$\sum x^2 = 44$	$\sum y^2 = 141$

$$r = \frac{n(\sum xy) - (\sum x)(\sum y)}{\sqrt{n(\sum x^2) - (\sum x)^2}\sqrt{n(\sum y^2) - (\sum y)^2}}$$

$$r = \frac{4(61) - (12)(23)}{\sqrt{4(44) - (12)^2}\sqrt{4(141) - (23)^2}}$$

$$r = \frac{-32}{\sqrt{32}\sqrt{35}} = -0.956$$

- These calculations get quite messy with larger data sets, so it's **fortunate** that the **linear correlation coefficient** can be **found automatically** with many different **calculators** and **computer programs**

Interpreting the Linear Correlation Coefficient

- We need to interpret a calculated **value of *r***, such as the value of **–0.956** found in the preceding example.

- The value of *r* must always fall between **-1 and +1** inclusive.

- If ***r* is close to 0**, we conclude that **there is no linear correlation** between *x* and *y*, but if *r* is close **-1 to or +1** we conclude that there is a **linear correlation between *x* and *y***.

Properties of the Linear Correlation Coefficient *r*

1. The value of r is always between **-1** and **+1** inclusive. That is, $-1 \leq r \leq +1$

2. The value of r does not change if all values of either variable are **converted** to a **different scale**.

3. The value of **r** is **not affected** by the choice of **x** or **y**. Interchange all x- and y-values and the value of r will not change.

4. r measures the strength of a **linear relationship**. It is **not designed** to measure the **strength of a relationship** that is **not linear**.

Hypothesis Test for Correlation

Assume: **r = 0.926**, **n = 8**

1. **We state our hypothesis as**:

H_0: $\rho = 0$ (There is no linear correlation.)

H_1: $\rho \neq 0$ (There is a linear correlation.)

2. **The level of significance is set** $\alpha = 0.05$.

3. **Test statistic to be used is t_{cal}** $= \dfrac{r}{\sqrt{\dfrac{1 - r^2}{n-2}}}$

4. **Calculations:**

$$t_{cal} = \frac{0.926}{\sqrt{\frac{1-(0.926)^2}{8-2}}} = 6.008$$

5. Critical region:

$|t_{cal}| > t_{tab}$, where $t_{tab} = t_{(\alpha/2,\ n-2)}$

$t_{tab} = t_{(0.0250,\ 6)} = 2.447$

$6.008 > 2.447$ (True)

6. Conclusion: Since t_{cal} is greater than the t_{tab}, so we reject H_o.

- There is **sufficient evidence** to support the claim of a linear correlation.

Examples: Applications of correlation

Buying a TV Audience The *New York Post* published the **annual salaries (in millions)** and the **number of viewers (in millions),** with results given below for **Oprah Winfrey**, **David Letterman, Jay Leno, Kelsey Grammer, Barbara Walters, Dan Rather, James Gandolfini, and Susan Lucci**, repsectively. Is there **a correlation between salary and number of viewers**?

Salary	100	14	14	35.2	12	7	5	1
Viewers	7	4.4	5.9	1.6	10.4	9.6	8.9	4.2

Examples: Applications of correlation

Parent Child Heights Listed below are **heights (in inches)** of **mothers** and **heights (in inches)** of their daughters (based on data from the National Health Examination Survey). Does there appear to be a **linear correlation between mother's heights** and the **heights of their daughters**?

Mother's height	63	67	64	60	65	67	59	60
Daughter's height	58.6	64.7	65.3	61.0	65.4	67.4	60.9	63.1

Basic Concepts of Regression

- In some cases, **two variables** are related in a **deterministic way**, meaning that given **a value for one variable**, the value of the **other variable** is **automatically determined** without any **error**.

- For example, the **total cost y** of an item with a list price of **x** and a **sales tax of 5%** can be found by using the deterministic equation $y = 1.05x$. If an item is priced at **$100**, its total cost is **$105**.

Probabilistic Models

- In **probabilistic models**, meaning that one variable is **not determined** completely by the **other variable**.

- For example, a **child's height** is not determined completely by the **height of the father (or mother).**

- **Sir Francis Galton (1822–1911)** studied the phenomenon of heredity and showed that when **tall or short couples have children**, the heights of those children **tend to regress, or revert** to the more typical **mean height** for people of the **same gender**.

Notations

- The **regression equation** expresses a relationship between ***x*** (called the **explanatory variable**, or **predictor variable**, or **independent variable**) and (called the **response variable**, or **dependent variable**).

- The typical equation of a straight line ***y = mx + b*** is expressed in the form $\hat{y} = b_0 + b_1\mathrm{x}$ or $\hat{y} = a + \mathrm{bx}$, where b_0 **or a** is the ***y*-intercept** and b_1 or **b** is the **slope**.

- The given notation shows that b_0 and b_1 are **sample statistics** used to estimate the population parameters β_0 and β_1.

- We will use **paired sample data** to **estimate the regression equation**. Using only sample data, we can't find the **exact values** of the population parameters β_0 and β_1, but we can use the sample data to estimate them with b_0 and b_1

Requirements

1. The sample of **paired (*x*, *y*) data** is a ***random* sample** of **quantitative data**.

2. Visual examination of the **scatterplot shows** that the **points** approximate **a straight-line pattern**.

3. Any **outliers** must be **removed** if they are known to be errors. Consider the effects of any outliers that are not known errors.

Requirements

Note: Requirements 2 and 3 above are simplified attempts at checking these formal requirements for regression analysis:

- For each **fixed value of *x***, the **corresponding values of *y*** have a distribution that is **bell-shaped**.
- For the **different fixed values of *x***, the distributions of the corresponding ***y*-values all have the same variance**.
- For the different fixed values of *x*, the distributions of the corresponding *y*-values have **means that lie along the same straight line**.
- The *y* values are independent.

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Requirements

- Results are **not seriously affected** if departures from **normal distributions** and equal variances are not too extreme.

Definitions

Given a collection of paired sample data, the **regression equation**

$\hat{y} = \boldsymbol{b_0} + \boldsymbol{b_1}\mathbf{x}$

algebraically describes the relationship **between the two variables**. The graph of the **regression equation** is called the **regression line** (or ***line of best fit***, or ***least-squares line***).

Notation for Regression Equation

	Population Parameter	Sample Statistic
y-intercept of regression equation	β_0	$\mathbf{b_0}$
Slope of regression equation	β_1	$\mathbf{b_1}$
Equation of the regression line	$Y = \beta_0 + \beta_1 x$	$\hat{y} = \mathbf{b_0 + b_1 x}$

Finding the slope b_1 and y-intercept b_0 in the regression equation $\hat{y} = \mathbf{b_0 + b_1 x}$

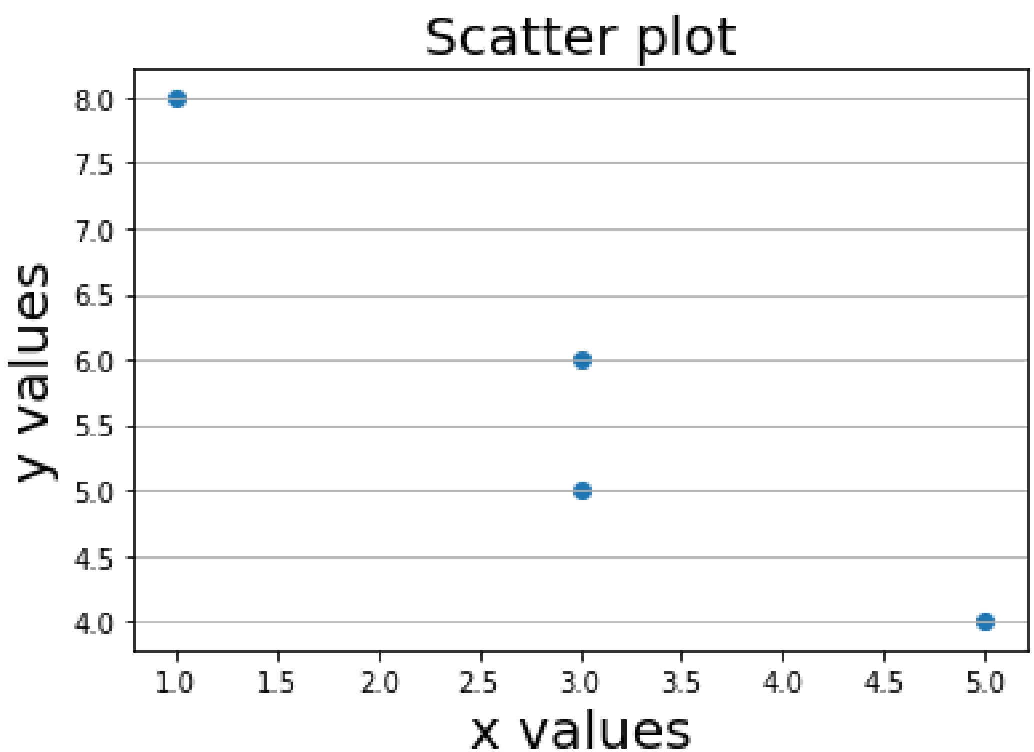
Slope	$\mathbf{b_1} = \dfrac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2}$
***y*-intercept:**	$\mathbf{b_0} = \bar{y} - \mathbf{b_1}\bar{x}$ **or** $\mathbf{b_0} = \dfrac{(\sum y)(\sum x^2) - (\sum x)(\sum xy)}{n(\sum x^2) - (\sum x)^2}$

Example Finding the Regression Equation

Use the given sample data to find the regression equation.

x	3	1	3	5
y	5	8	6	4

REQUIREMENT The data are a simple random sample. The accompanying Python-generated scatterplot shows a pattern of points that does appear to be a straight-line pattern. There are no outliers. We can proceed to find the slope and intercept of the regression line.



x	y	xy	x^2	y^2
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$$\mathbf{b_1} = \frac{\mathbf{n(\sum xy) - (\sum x)(\sum y)}}{\mathbf{n(\sum x^2) - (\sum x)^2}}$$

$$\mathbf{b_1} = \frac{4(61) - (12)(23)}{4(44) - (12)^2} = \frac{-32}{32} = -1$$

$$\bar{x} = \frac{12}{4} = 3$$

$$\bar{y} = \frac{23}{4} = 5.75$$

$$\mathbf{b_0 = \bar{y} - b_1\bar{x}}$$

$$b_0 = 5.75 - (-1)(3)$$

$$\mathbf{b_0} = 8.75$$

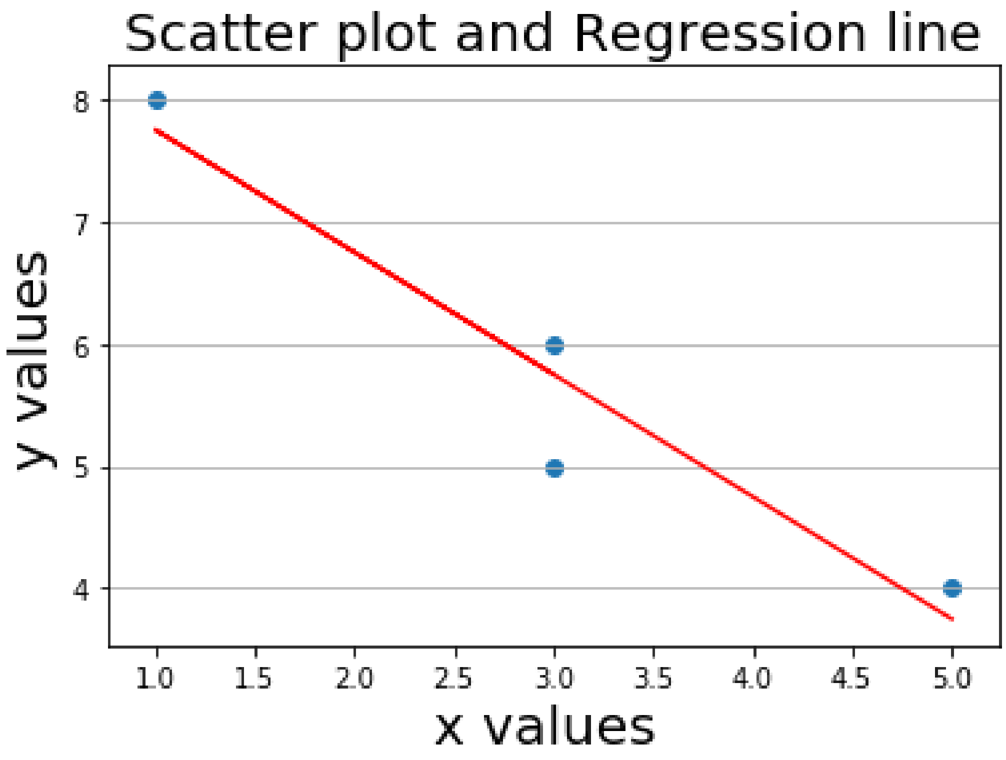
- Knowing the slope ***b_1*** and ***y*-intercept *b_0***, we can now express the **estimated equation** of the **regression line** as

$$\hat{y} = \mathbf{b_0 + b_1 x}$$

$$\hat{y} = \mathbf{8.75 - 1x}$$

- We should realize that this equation is an ***estimate*** of the **true regression equation $Y = \beta_0 + \beta_1 x$.** This estimate is based on one **particular set of sample data**, but another sample drawn from the same population would probably lead to a slightly different equation.

Scatter plot and Regression line



Using the Regression Equation for Predictions

- **Regression equations** are often useful for ***predicting*** the **value of one variable,** given **some particular value of the other variable**.

- If the **regression line fits** the data quite well, then it makes sense to use its **equation for predictions**, provided that we don't go beyond the scope of the available values.

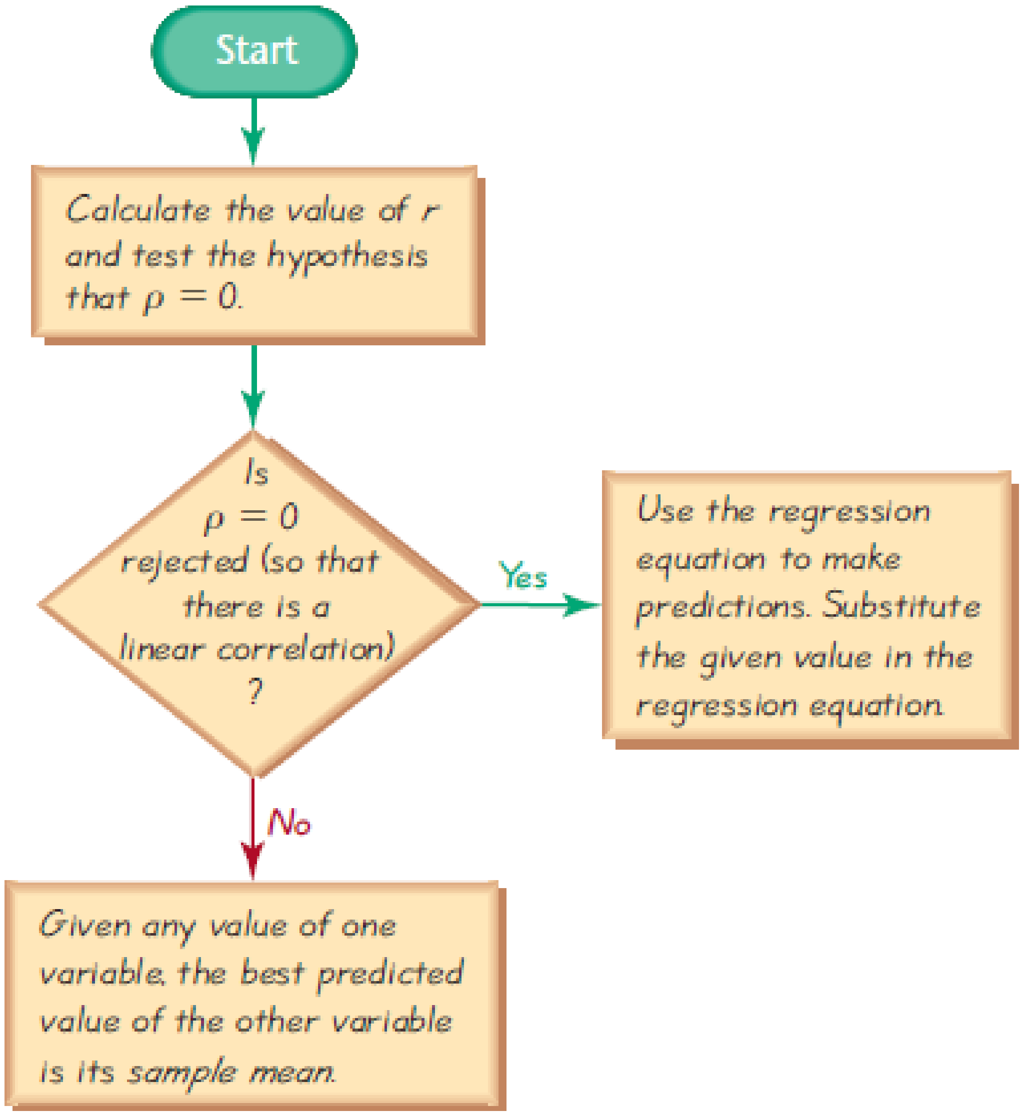
Using the Regression Equation for Predictions

In **predicting a value** of ***y*** **based** on some given value of ***x*** . . .

1. If there is ***not*** **a linear correlation**, the best predicted ***y*-value** is $\bar{y}$.

2. If there is a **linear correlation**, the **best predicted *y*-value** is found by **substituting the *x*-value** into the **regression equation**.

Procedure for Predicting



Guidelines for Using the Regression Equation

1. If there is **no linear correlation**, don't use the **regression equation** to **make predictions**.

2. When using the **regression equation for predictions**, stay within the **scope of the available sample data**. If you find a regression equation that **relates women's heights** and **shoe sizes**, it's absurd to predict the shoe size of a woman who is **10 ft tall**.

Guidelines for Using the Regression Equation

3.A **regression equation** based **on old data** is not **necessarily valid now**. The regression equation relating **used-car prices** and **ages of cars** is no longer usable

if it's based on data from the 1990s.

4.Don't **make predictions** about **a population** that is different **from the population** from which the s**ample data were drawn**. If we collect sample data from men and develop a regression equation **relating age and TV remote-control usag**e, the results don't necessarily apply to women. If we use state averages to develop a regression equation relating SAT math scores and SAT verbal scores, the results don't necessarily apply to individuals.