NAME: ALI KHALID

ROLL # MSDS21001

DEEP LEARNING / SPRING 2020

Home Work 1

Q1: Find critical points of following functions

(a) $f(x,y) = x^4 + y^4 + 16xy$

$$\frac{\partial f}{\partial x} = 4x^3 + 16y \qquad \frac{\partial f}{\partial y} = 4y^3 + 16x$$

To find Critical point put gradient equal to 0

$$4x^3 + 16y = 0 \quad \dots \quad (1)$$
$$4y^3 + 16x = 0 \quad \dots \quad (2)$$

From (1)

$$4x^3 + 16y = 0$$
$$16y = -4x^3$$
$$y = \frac{-\cancel{4}x^3}{\cancel{16}4}$$
$$y = \frac{-x^3}{4} \quad \text{put in eq (2)}$$
$$4\left(-\frac{x^3}{4}\right)^3 + 16x = 0$$
$$-\cancel{4}\frac{x^9}{\cancel{64}16} + 16x = 0$$
$$\frac{-x^9}{16} = -16x$$
$$x^9 - 256x = 0$$
$$x(x^8 - 256) = 0$$

$x = 0$ | $x^8 - 256 = 0$

$x^8 = 256$

$x^8 = (2)^8$ | $x^8 = (-2)^8$

So we have 3 values of x. Find corresponding y

- $x = 0$ put in (1)

$$y = \frac{(0)^3}{4} = 0$$
$$(x,y) = (0,0)$$

- $x = 2 \Rightarrow y = \frac{-(2)^3}{4} = \frac{-8}{4} = -2$

$$(x,y) = (2,-2)$$

- $x = -2 \Rightarrow y = -\frac{(-2)^3}{4} = \frac{8}{4} = 2$

$$(x,y) = (-2,2)$$

So we have 3 critical points

$$\boxed{[(0,0), (2,-2), (-2,2)]}$$

B) $f(x,y) = \sqrt{x^2+y^2} + 1$

$$\frac{\partial f}{\partial x} = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2x \qquad \frac{\partial f}{\partial y} = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2y$$

$$= \frac{x}{\sqrt{x^2+y^2}} \qquad = \frac{y}{\sqrt{x^2+y^2}}$$

To find critical points put gradient equal to 0

$$\frac{x}{\sqrt{x^2+y^2}} = 0 \quad \cdots (1) \qquad \frac{y}{\sqrt{x^2+y^2}} = 0 \quad \cdots (2)$$

from (1) and (2)

$$\frac{x}{\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}} \Rightarrow x = y$$

also $\frac{x}{\sqrt{x^2+y^2}} = 0$ which is ~~only~~ not true for any value of x or y. But the above equation (gradient) is undefined at $(x,y) = (0,0)$.

So the point where gradient is not defined is also included in critical points.

Hence, this function has only one critical point

$[(0,0)]$.

c) $f(x, y) = e^{-(x^2+y^2+2x)}$

$$\frac{\partial f}{\partial x} = e^{-(x^2+y^2+2x)} \cdot -(2x+2) \qquad \frac{\partial f}{\partial y} = e^{(x^2+y^2+2x)} \cdot -2y$$

$$= (-2x-2)\, e^{-(x^2+y^2+2x)} \qquad \frac{\partial f}{\partial y} = -2y \cdot e^{-(x^2+y^2+2x)}$$

To find critical point put gradient equal to 0

$$(-2x-2)\, e^{-(x^2+y^2+2x)} = 0 \quad \cdots (1)$$

$$-2y\, e^{-(x^2+y^2+2x)} = 0 \quad \cdots (2)$$

from (1) & (2)

$$(-2x-2)\, \cancel{e^{-(x^2+y^2+2x)}} = -2y\, \cancel{e^{-(x^2+y^2+2x)}}$$

$$-2x-2 = -2y$$

$$2x+2 = 2y$$

$$\cancel{2}(x+1) = \cancel{2}y$$

$$x = y-1 \quad \rightarrow \text{put in} \quad (1)$$

$$-2(y-1)-2 \; e^{-((y-1)^2+y^2+2(y-1))} = 0$$

$$-2y+\cancel{2}-\cancel{2} \cdot e^{-(y^2+1-\cancel{2y}+y^2+\cancel{2y}-2)} = 0$$

$$-2y\, e^{-(2y^2-1)} = 0$$

$-2y = 0$ because $e^{-(2y^2-1)}$ is never 0

$y = 0 \quad \rightarrow$ put in $x = y-1 \Rightarrow x = -1$

so

critical point $= [(-1, 0)]$

Question 2: Find the critical points, extreme values and saddle point using second derivative test.

A) $f(x,y) = xe^y - e^x$

$\frac{\partial f}{\partial x} = e^y - e^x$ $\qquad$ $\frac{\partial f}{\partial y} = xe^y$

$\frac{\partial f^2}{\partial x^2} = -e^x$ $\qquad$ $\frac{\partial f^2}{\partial y \partial x} = e^y$ $\qquad$ $\frac{\partial f^2}{\partial y^2} = xe^y$ $\qquad$ $\frac{\partial f^2}{\partial x \partial y} = e^y$

Hessian Matrix $= \begin{bmatrix} -e^x & e^y \\ e^y & xe^y \end{bmatrix}$

Put gradient equal to 0 to find critical points

$e^y - e^x = 0$ — ① $\qquad$ $xe^y = 0$ — ②

from ① $\qquad$ from ②

$x = y$ $\qquad$ $x = 0$

so critical point $= [(0,0)]$

Now put $(0,0)$ in Hessian Matrix

$$H = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

$\det(H) = (-1)(0) - (1)(1) = -1$

As $\det(H) < 0$, $(0,0)$ is a saddle point

Critical points $= [\ (0,0) \to$ saddle point$]$

B) $f(x,y) = x\sin(y)$

$$\frac{\partial f}{\partial x} = \sin(y) \qquad \frac{\partial f}{\partial y} = x\cos(y)$$

$$\frac{\partial f^2}{\partial x^2} = 0, \quad \frac{\partial f^2}{\partial y \partial x} = \cos(y), \quad \frac{\partial f^2}{\partial y^2} = -x\sin(y), \quad \frac{\partial f}{\partial x \partial y} = \cos(y)$$

<u>Hessian Matrix</u> $= \begin{bmatrix} 0 & \cos(y) \\ \cos(y) & -x\sin(y) \end{bmatrix}$

To find critical points put gradient equal to 0

$\sin(y) = 0$ — ① $\qquad x\cos(y) = 0$ — ②

from ① $\qquad$ from ②

$y = n\pi \qquad x = 0$

what ever multiple of π we consider, $x = 0$ always.

So lets take one odd and one even muliple of π

$y = 0, \pi$

Critical points $= [(0,0), (0,\pi)]$

put $(x,y) = (0,0)$ in hessian Matri

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \det(H) = -1$$, As $\det(H) < 0$ it is a saddle point

put $(x,y) = (0,\pi)$ in hessian Matrix

$$H = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \Rightarrow \det(H) = -1$$, As $\det(H) < 0$ it is a saddle point.

So,

critical point $= [\,(0,0) \rightarrow$ sadlle point,

$(0,\pi) \rightarrow$ saddle point $]$

c) $f(x,y) = 4xy - x^4 - y^4$

$\frac{\partial f}{\partial x} = 4y - 4x^3$, $\frac{\partial f}{\partial y} = 4x - 4y^3$

$\frac{\partial f^2}{\partial x^2} = -12x^2$, $\frac{\partial f^2}{\partial y \partial x} = 4$, $\frac{\partial f^2}{\partial y^2} = -12y^2$, $\frac{\partial f^2}{\partial x \partial y} = 4$

Hessian Matrix = $\begin{bmatrix} -12x^2 & 4 \\ 4 & -12y^2 \end{bmatrix}$

To find critical points, put gradient equal to 0

$4y - 4x^3 = 0$ — ①

$y = x^3$

Put in ②

$4x - 4y^3 = 0$ ②

$4x - 4(x^3)^3 = 0$

$4x - 4x^9 = 0$

$x(1 - x^8) = 0$

$x = 0$ $1 - x^8 = 0$

$x^8 = 1^8$

$x = 1$, $x = -1$

Put in $y = x^3$

$x = 0$, $y = 0$ $(0,0)$

$x = 1$, $y = 1$ $(1,1)$

$x = -1$, $y = -1$ $(-1,-1)$

So we have 3 critical points = $[(0,0), (1,1), (1,-1)]$

$(x,y) = (0,0)$

Hessian = $\begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix} \Rightarrow \det(H) = -16$

As $\det(H) < 0$, it is saddle point

$(x,y) = (1,1)$

Hessian = $\begin{bmatrix} -12 & 4 \\ 4 & -12 \end{bmatrix} \xrightarrow{\det(H)} = 144 - 16 = 128$, as $\det(H) > 0$ and $\frac{\partial f^2}{\partial x^2}$ or $\frac{\partial f^2}{\partial y^2} < 0$, it is maxima

$(x,y) = (-1,-1)$

Hessian = $\begin{bmatrix} -12 & 4 \\ 4 & -12 \end{bmatrix} \Rightarrow \det(H) = 144 - 16 = 128$, As $\det(H) > 0$ and $\frac{\partial f^2}{\partial x^2}$ or $\frac{\partial f^2}{\partial y^2} < 0$, it is maxima

critical points = [$(0,0) \rightarrow$ saddle point,

$(1,1) \rightarrow$ point of Maxima,

$(-1,-1) \rightarrow$ point of Maxima]