Statistical and Mathematical Methods for Data Analysis

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Textbooks

- **Probability & Statistics for Engineers & Scientists**, Ninth Edition, Ronald E. Walpole, Raymond H. Myer

- **Elementary Statistics: Picturing the World,** 6th Edition, Ron Larson and Betsy Farber

- **Elementary Statistics**, 13th Edition, Mario F. Triola

Reference books

- **Probability Demystified**, Allan G. Bluman
- **Schaum's Outline of Probability and Statistics**
- **MATLAB Primer**, Seventh Edition
- **MATLAB Demystified** by McMahon, David

Reference books

- **Probability and Statistical Inference, Ninth Edition,** Robert V. Hogg, Elliot A. Tanis, Dale L. Zimmerman

- **Probability Demystified**, Allan G. Bluman

- **Practical Statistics for Data Scientists: 50 Essential Concepts,** Peter Bruce and Andrew Bruce

- **Schaum's Outline of Probability,** Second Edition, Seymour Lipschutz, Marc Lipson

- **Python for Probability, Statistics, and Machine Learning,** José Unpingco

References

Readings for these lecture notes:

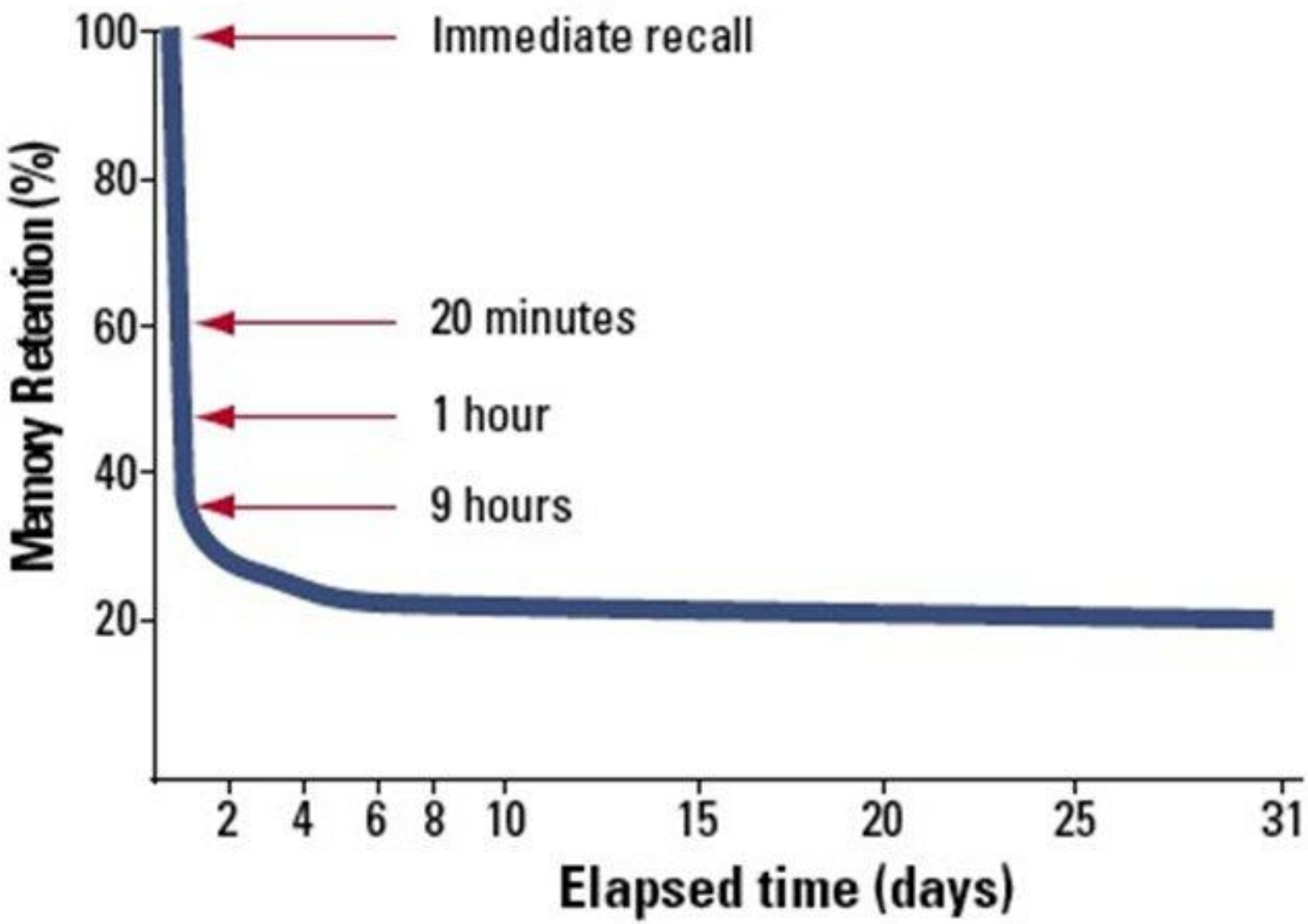
- Probability & Statistics for Engineers & Scientists, Ninth edition, Ronald E. Walpole, Raymond H. Myer
- http://www.statisticshowto.com/geometric-distribution/
- https://peakmemory.me/category/forgetting-curve/

These notes contain material from the above resources.

“If you want to know what a man's like, take a good look at how he treats his inferiors, not his equals.”

— J.K. Rowling, Harry Potter and the Goblet of Fire

Forgetting curve



Poisson Distribution [1]

Example: An automobile manufacturer is concerned about a fault in the braking mechanism of a particular model. The fault can, on rare occasions, cause a catastrophe at high speed. The distribution of the number of **cars per year** that will experience the fault is a Poisson random variable with $\boldsymbol{\lambda = 5}$.

(a) What is the probability that **at most 3** cars per year will experience a catastrophe?

(b) What is the probability that **more than 1** car per year will experience a catastrophe?

Solution: Here $\lambda t = (5)(1) = 5$

$$P(x;\ \lambda t) = \frac{(\lambda t)^{x} e^{-\lambda t}}{x!},\ x = 0,\ 1,\ 2,\ .\ .\ .$$

(a) $P(X \leq 3) = \sum_{x=0}^{x=3} p(x;5)$

$= 0.2650$

(b) $P(X > 1) = 1 - P(x \leq 1)$

$= 1 - \sum_{x=0}^{x=1} p(x;5)$

$= 1 - 0.0404$

$= 0.9596$

Poisson Distribution [2]

Example: Changes in airport procedures require considerable planning. Arrival rates of aircraft are important factors that must be taken into account. Suppose small aircraft arrive at a certain airport, according to a Poisson process, at the rate of **6 per hour**. Thus the Poisson parameter for arrivals for a period of hours is $\boldsymbol{\lambda = 6}$.

(a) What is the probability that **exactly 4** small aircraft arrive during a **1-hour period**?

Poisson Distribution [3]

(b) What is the probability that **at least 4** arrive during a **1-hour period**?

(c) If we define a **working day** as **12 hours**, what is the probability that at least **75 small aircraft** arrive **during a day**?

Poisson Distribution [3]

Here $\lambda t=(6)(1)=6$

$$P(x;\lambda t) = \frac{(\lambda t)^x e^{-\lambda t}}{x!},\ x = 0, 1, 2, \ldots$$

(a) $P(X = 4) = \dfrac{(6)^4 e^{-6}}{4!} =$ **0.1339**

(b) $P(X \geq 4) = 1 - P(x < 4) = 1 - \sum_{x=0}^{x=3} p(x;6) = 1 - 0.1512 =$ **0.8488**

Here **$\lambda t = (6)(12) = 72$**

(c) $P(X \geq 75) = 1 - P(x < 75) = 1 - \sum_{x=0}^{x=74} p(x;72) =$ **0.3773.**

Poisson Distribution using Python

a) What is the probability that **exactly 4** small aircraft arrive during a **1-hour period**?

```
from scipy.stats import  poisson
mu = 6
x = 4
prob = round(poisson.pmf(x, mu), 4)
print('Probability that at least 4
arrive during a1-hour period:', prob)
#0.1339
```

(b) What is the probability that **at least 4** arrive during a **1-hour period**?

```
x = [0, 1, 2, 3]
prob = 1 - round(sum(poisson.pmf(x, mu)), 4)
print('Probability that at least 4 arrive during a 1-hour period:', prob)
#0.8488
```

(c) If we define a **working day** as **12 hours**, what is the probability that at least **75 small aircraft** arrive **during a day**?

```
mu =  12 * 6
x = range(0, 75)
#x = list(range(0, 75))
#print(x)
prob =1 - round(sum(poisson.pmf(x, mu)), 4)
print('Probability that at least 75 small aircraft arrive during a day:', prob)
#0.3773
```

Poisson approximation

The **Binomial distribution** converges towards the **Poisson distribution** as the number of trials goes to **infinity** while the product **np** remains fixed. Therefore the Poisson distribution with parameter **$\lambda = np$** can be used as an approximation to b(n, p) of the binomial distribution if n is sufficiently large and p is sufficiently small.

According to two rules of thumb, this approximation is good if

$n \geq 20$ and $p \leq 0.05$, or if $n \geq 100$ and $np \leq 10$.

Poisson Distribution [4]

Formula:

$f(x) = (e^{-\lambda} \lambda^{x})/x!$, $x = 0, 1, 2, \ldots$

where, λ is an average rate of value, x is a Poisson random variable and e is the base of logarithm(e = 2.718).

Example:

Consider, in an office on **average 2 customers** arrived per day. Calculate the possibilities for exactly 3 customers to be arrived on today.

Step1: Find $e^{-\lambda}$.

where, $\lambda = 2$ and $e = 2.718$, $e^{-\lambda} = (2.718)^{-2} = 0.135$.

Step2: Find $\lambda^{x.}$

where, $\lambda = 2$ and $x = 3$, $\lambda^x = 2^3 = 8$.

Step3: Find f(x).

$f(x) = e^{-\lambda} \lambda^x / x!$

$f(3) = (0.135)(8) / 3! = 0.18$.

Hence there are 18% possibilities for 3 customers to be arrived today

Geometric Distribution [1]

- Suppose we have a sequence of Bernoulli trials, each with a probability **p** of success and a probability **q = 1-p** of failure. How many trials occur **before we obtain a success?**

Example

- A **search engine** goes through a list of sites looking for a **given key phrase**. Suppose the **search terminates** as soon as the **key phrase is found**. The number of sites visited is **Geometric**.

Let the random variable X be the number of trials needed to obtain a success. Then X has values in the range {1,2,…}, and for k ≥1,

$g(x; p) = p\, q^{x-1}, x = 1, 2, 3, \cdots$

Alternative form

$g(x; p) = p\, q^{x}, x = 0, 1, 2, 3, \cdots$

Geometric Distribution [2]

Mean = 1/p and Variance = q/p^2

In the theory of **probability and statistics**, a **Bernoulli trial** is an experiment whose outcome is random and can be either of **two possible** outcomes, **“success” and “failure”.**

Geometric Distribution [3]

Conditions:

An experiment consists of repeating trials **until first success.**

Each trial has **two possible outcomes**.

A success with probability **p**.

A failure with probability **q** = 1 − p.

Repeated trials are **independent.**

x = number of trials to first success

x is a **GEOMETRIC RANDOM VARIABLE.**

$\mathbf{g(x; p) = q^{x-1}p, \; x = 1, 2, 3, \cdots}$

Assumptions for the Geometric Distribution

The three assumptions are:

- There are **two possible outcomes** for each trial (success or failure).
- The trials are **independent**.
- The **probability of success** is the same for each trial.

Example From past experience it is known that **3%** of accounts in a large accounting population are in **error**.

What is the probability that **5 accounts** are audited **before** an account in **error** is found?

Solution:

$g(x; p) = q^{x-1}p, \ x = 1, 2, 3, \cdots$

$P(X = 5) = P(\text{1st 4 correctly stated})\ P(\text{5th in error})$

$= g(x; p) = q^{x-1}p, \ x = 1, 2, 3, \cdots$

$= (0.97)^{5-1}\ (0.03)$

$= 0.0266$

Example: In a certain manufacturing process it is known that, on the average, **1** in every **100**, items is defective. What is the probability that the **fifth item** inspected is the **first defective** item found?

Solution: Using the geometric distribution with x = 5 and

p = 1/100 = 0.01, q = 0.99, we have

$$g(x; p) = p\, q^{x-1}, \quad x = 1, 2, 3, \cdots$$

$$g(5;0.01) = (0.01)(0.99)^{5-1}$$

$$= 0.0096$$

Python code

```
from scipy.stats import geom
p = 1/100
x = 5
prob = round(geom.pmf(x, p), 4)
print('The probability that the fifth
item inspected is the first defective
item found :', prob)
# 0.0096
```

Example: At **"busy time"** a telephone exchange is very near capacity, so callers have difficulty placing their calls. It may be of interest to know the number of attempts necessary in order to gain a connection. Suppose that we let **$p = 0.05$** be the probability of a connection during busy time. We are interested in knowing the probability that **5 attempts** are necessary for a successful call.

Solution:

Using the geometric distribution with **$x = 5$** and **$p = 0.05$** yields

$g(x; p) = p\, q^{x-1}$, $x = 1, 2, 3, \cdots$

$P(X = x) = g(5; 0.05)$

$= (0.05)\,(0.95)^{5-1}$

$= 0.041.$

Python code

```
p = 0.05
x = 5
prob = round(geom.pmf(x, p), 4)

print('The probability that 5
attempts are necessary for a
successful call :', prob)
```

Discrete Uniform Distribution [1]

If a random variable has any of n possible values that are **equally probable**, then it has a discrete uniform distribution. The probability of any outcome k_i **is 1/n**.

A simple example of the discrete uniform distribution is throwing a fair die. The possible values of k are **1, 2, 3, 4, 5, 6**; and each time the die is thrown, the probability of a given score is **1/6**.

Discrete Uniform Distribution [2]

Generating random numbers are the prime application of uniform distribution. The basic random numbers are **0, 1, 2, 3, 4, 5, 6, 7, 8, 9.** Each with probability equal to **1/10**.

For **two digit random numbers** the probability of selecting a particular random variable will be **1/100**.

Discrete Uniform Distribution [3]

If the random variable *X* assumes the values x_1, x_2, x_3, …, x_k with equal probabilities, then the discrete uniform distribution is given by

$$\mathbf{P(x;\, k)} = \frac{\mathbf{1}}{\mathbf{k}}, \qquad \mathbf{x_1, x_2, x_3, \ldots, x_k}$$

Discrete Uniform Distribution [4]

When a light bulb is selected at random from a box that contains a 40-watt bulb, a 60-watt bulb, a 75-watt bulb, and a 100-watt bulb, each element of the sample1 space ***S* = {40, 60, 75, 100}** occurs with probability 1/4. Therefore, we have a uniform distribution, with probability

$$\mathbf{P(x;\,k) = \frac{1}{4}\,,\qquad x = 40, 60, 75, 100}$$