

IRAN UNIVERSITY OF SCIENCE AND TECHNOLOGY

Discrete mathematics Problem Set #2

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1 Two Is Irrational?

In lecture, we proved that $\sqrt{2}$ is irrational, and in the checkpoint problem you proved that $\sqrt{3}$ is irrational. Below is a purported proof that $\sqrt{4}$ is irrational:

Theorem 1.1. $\sqrt{4}$ is irrational.

Proof. Assume for the sake of contradiction that $\sqrt{4}$ is rational. Then there must exist integers p and q where $q \neq 0$, where $p/q = \sqrt{4}$, and where p and q have no common factors other than 1 and -1 . Since $p/q = \sqrt{4}$, we have $p^2/q^2 = 4$, so $p^2 = 4q^2$. Since q^2 is an integer, we see that p^2 is a multiple of four, and therefore p is a multiple of four. Thus $p = 4n$ for some integer n . Since $4q^2 = p^2$ and $p = 4n$, we have $4q^2 = (4n)^2 = 16n^2$, so $q^2 = 4n^2$. Since n^2 is an integer, we see that q^2 is a multiple of four, so q is a multiple of four as well. But since both p and q are multiples of four, we see that p and q share a common divisor other than 1 and -1 , contradicting our initial assumption. We have reached a contradiction, so our assumption must have been incorrect. Thus $\sqrt{4}$ is irrational. \square

This proof has to be wrong, because $\sqrt{4} = 2 = 2/1$, which is indeed rational! What error does this proof make that lets it conclude $\sqrt{4}$ is irrational? Why doesn't this error occur in the similar proofs that $\sqrt{2}$ and $\sqrt{3}$ are irrational?

Answer. Just because p^2 is divisible by 4, that doesn't mean p is.

2 Modular Arithmetic

Many different numbers yield the same remainder when divided by some number. For example, the numbers 2, 5, 8, 11, 14, and 17, all leave a remainder of two when divided by three, while the numbers 1, 12, 23, 34, and 45 all leave a remainder of one when divided by eleven. To formalize this relationship between numbers, we'll introduce a relation \equiv_k that, intuitively, indicates that two numbers leave the same remainder when divided by k . For example, we'd say that $1 \equiv_{11} 12$ and that $8 \equiv_3 11$.

- i. Prove that for any integer x and any integer x that $x \equiv_k x$.

Answer. *Proof.*

$$\forall x \in \mathbb{Z} : k|x - x \Rightarrow x \equiv_k x$$

\square

- ii. Prove that for any integers x and y and any integer k that if $x \equiv_k y$, then $y \equiv_k x$.

Answer. *Proof.*

$$\forall x, y \in \mathbb{Z}$$

$$x \equiv_k y \Rightarrow k|x - y \quad (1)$$

$$k|x - y = k|y - x \quad (2)$$

$$\text{from 2, 1} \Rightarrow y \equiv_k x$$

□

- iii. Prove that for any integers x, y , and z and any integer k that if $x \equiv_k y$ and $y \equiv_k z$, then $x \equiv_k z$.

Answer. *Proof.*

$$\forall x, y, z \in \mathbb{Z}$$

$$y \equiv_k z \Rightarrow k|y - z \quad (3)$$

$$\text{from 1, 3} \Rightarrow k|x - z \Rightarrow x \equiv_k z$$

□

3 Pigeonhole Principle 1

There are 51 senators in a senate. The senate needs to be divided into n committees such that each senator is on exactly one committee. Each senator hates exactly three other senators. (If senator A hates senator B , then senator B does *not* necessarily hate senator A .)

Answer. *Proof.* In the worst case, consider that senator S hates a set of 3 senators, while he himself is hated by a completely different set of 3 other senators. Thus, given one senator, there may be a maximum of 6 other senators whom he cannot work with. If we have a minimum of 7 committees, there should be at least one committee suitable for the senator S after the assignment of the 6 conflicting senators. □

Find the smallest n such that it is always possible to arrange the committees so that no senator hates another senator on his or her committee.

4 Pigeonhole Principle 2

Let Ω be a set of points in the plane. Each point in Ω is a midpoint of two points in Ω . Show that Ω is an infinite set.

Answer. First proof:

Proof. Suppose Ω is a finite set. Then Ω contains two points A, B with maximal distance $|AB| = m$. B is a midpoint of some segment CD with $C, D \in \Omega$. Figure shows that $|AC| > |AB|$ or $|AD| > |AB|$. □

Proof. We consider all points in Ω farthest to the left, and among those the point M farthest down. M cannot be a midpoint of two points $A, B \in \Omega$ since one element of $\{A, B\}$ would be either left of M or on the vertical below M . \square

5 Pigeonhole Principle 3

A plain is colored blue and red in any way. prove that there exists a rectangle with vertices of same color.

Answer. *Proof.* Choose any 7 collinear points. At least 4 of these points are of the same color, say red. Call them R_1, R_2, R_3, R_4 . We project these points onto two lines parallel to the first line to S_1, \dots, S_4 and T_1, \dots, T_4 . If two S-points or two T-points are red, then we have a red rectangle. Otherwise, there exist 3 blue S-points and 3 T-points, and hence, a blue rectangle. \square

6 Pigeonhole Principle 3

Prove that having 100 whole numbers, one can choose 15 of them so that the difference of any two is divisible by 7

Proof. For the difference to be a multiple of 7, the integers must have equal modulo 7 residues. To avoid having 15 with the same residue, 14 numbers with different modulo 7 residues can be picked ($14 * 7 = 98$). Thus, two numbers are left over and have to share a modulo 7 residue with the other numbers under the pigeonhole principle. \square

7 Contradiction every where

$2n + 1$ persons are placed in the plane so their mutual distances are all distinct. Then everyone shoots his nearest neighbor. Prove that (a) at least one person survives (b) nobody is hit by more than 5 bullets (c) the set of segments formed by the bullet paths does not contain a closed polygon.

Answer.

Proof. (a) All mutual distances are different. Hence there exist two persons A and B with minimum distance. These two persons will shoot each other. If any other person shoots at A or B , someone will survive since A and B have used up three bullets. If not, we can ignore A and B . We are left with the same problem with n replaced by $n-1$. Repeating the argument, we either find a pair at whom three shots are fired, or if not, we arrive, finally, at three persons, and for this case ($n = 1$), the theorem is obvious. \square

Proof. (b) Suppose the persons A, B, C, D, \dots shoot at P (Fig. 3.12). A shoots at P and not at B , so $|AP| < |AB|$. B shoots at P and not at A , so $|BP| < |AB|$. Thus, AB is the largest side in the triangle ABP . The largest angle lies opposite the largest side. Hence, $\gamma > \alpha, \gamma > \beta$ or $2\gamma > \alpha + \beta, 3\gamma > \alpha + \beta + \gamma, \gamma > 60^\circ$. Thus any two bullet paths meeting at P make an angle greater than 60° . Since $360^\circ / 60^\circ = 6$, five bullet paths at most can meet at P . \square

Proof. (c) Suppose the paths of two bullets cross with A shooting at B and C shooting at D (Fig. 3.13). Then $|AB| < |AD|$ and $|CD| < |CB|$ imply $|AB| + |CD| < |AD| + |CB|$. On the other hand, by the triangle inequality, $|AS| + |SD| > |AD|$ and $|BS| + |SC| > |BC| \Rightarrow |AB| + |CD| > |AD| + |BC|$. Contradiction! \square

Proof. (d) Suppose there is a closed polygon $ABCDE \dots MN$ (Fig. 3.14). Let $|AN| < |AB|$, that is, N is the nearest neighbor of A . Then $|AB| < |BC|, |BC| < |CD|, |CD| < |DE|, \dots, |MN| < |NA|$, that is, $|AB| < |NA|$. Contradiction! The assumption $|AN| > |AB|$ also leads to a contradiction. \square