IRAN UNIVERSITY OF SCIENCE AND TECHNOLOGY

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Discrete Mathematics
Problem Set #5

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1 Connected graph

Prove the theorem blow.

Theorem 1.1. A graph is connected if and only if for every partition of it's vertices into two nonempty sets, there is an edge with endpoints in both sets.

Answer. *Proof.* ($\Rightarrow$) Let G is connected. Take an arbitrary partition $V(G) = X \cup Y of V(G)$ into two non-empty sets. We should show that G has an edge with one endpoint in X and the other in Y. Select vertices x in X, and y in Y. (Possible because X and Y are non-empty.) Because G is connected, there has to be a path in G that joins x to y. Denote this path as follows.

$$x = x_0, x_1, x_2, x_3, \ldots, x_n = y$$

The first vertex x_0 of this path is in X, and the last vertex x_n is in Y. Any one of the others is either in X or Y. Suppose i be the smallest index for which $x_i \in Y$. (Such an i exists, because $x_n \in Y$, so i is at most n.) Now we have $x_{i-1} \in X$ and $x_i \in Y$, so $x_{i-1}x_i$ is an edge of G with one endpoint in X and the other in Y.
($\Leftarrow$) Let that for any partition $V(G) = X \cup Y of V(G)$ into two non-empty sets, G has an edge with one endpoint in X and the other in Y. We need to show G is connected. For the sake of contradiction, suppose G is not connected. Let C be one of its components. Now we have a partition

$$V(G) = V(C) \in (V(G)V(C))$$

of $V(G)$ into two non-empty sets. By assumption, G has an edge with endpoints in each set in this partition. That is to say G has an edge with one endpoint in one of its components and the other endpoint in another component. Contradiction! □

2 n-vertex graph

Prove the theorem blow.

Theorem 2.1. Every n-vertex graph with at least n edges contains a cycle.

Answer. *Proof.* **Assume:** G contains no cycles. Then every connected component of G is a tree.
Claim: The number of edges in a tree on n vertices is $n-1$.
Proof is by induction. The claim is obvious for $n = 1$. Assume that it holds for trees on n vertices. Take a tree on $n+1$ vertices. It's an easy exercise (look at a longest path in G) to show that a tree has at least one terminal vertex (i.e. with degree 1). Removing this terminal vertex along with its edge, we get a tree on n vertices, and induction takes us home. Hence the number of edges in a graph without cycles is $n - k$, where k is the number of connected components. □

3 Connected simple n-vertex graph

Prove the theorem blow.

Theorem 3.1. If l, m, n are nonnegative integers with $l + m = n \geq 1$, then there exists a connected simple n-vertex graph with l vertices of even degree and m vertices of odd degree if and only if m is even, except for $(l, m, n) = (2, 0, 2)$.

Answer. ANSWER HERE.

4 k_cube graph

what is $k\_cube(Qk)$ graph?
Count number of it's Vertices and Edges and prove that it is bipartite.

Answer. In graph theory, the hypercube graph Q_n is the graph formed from the vertices and edges of an n-dimensional hypercube. For instance, the cubical graph Q_3 is the graph formed by the 8 vertices and 12 edges of a three-dimensional cube. Q_n has $2n$ vertices, $2^{n-1}n$ edges, and is a regular graph with n edges touching each vertex.*

Proof. PROOF HERE. □

5 Diameter of graph

diameter of G is length of the longest path between two vertices in it. Show that if G's diameter is greater than 3 its complement's diameter would be less than 3.

Answer. *Proof.* PROOF HERE. □

6 About Graph

Necessary and sufficient conditions for a list d to be graphic when d consists of k copies of a and nk copies of b, with $a \geq b \geq 0$. Since the degree sum must be even, the quantity $k\ a + (nk)b$ must be even. In addition, the inequality $k\ a \leq k(k1) + (nk)\ min\{k, b\}$ must hold, since each vertex with degree b has at most $min\{k, b\}$ incident edges whose other endpoint has degree a. We construct graphs with the desired degree sequence when these conditions hold. Note that the inequality implies $a \leq n1$.

Answer. *Proof.* PROOF HERE. □

7 Graph orientation

Prove the theorem blow.

Theorem 7.1. Every graph G has an orientation such that $|d + (v) - d - (v)| \leq 1$ for all v.

Answer. ANSWER HERE.

8 Strong orientation of a graph

A strong orientation of a graph that has an odd cycle also has an odd (*directed*) cycle. Suppose that D is a strong orientation of a graph G that has an odd cycle $v1, \ldots, v2k + 1$. Since D is strongly connected, for each i there is a $vi, vi + 1 - path$ in D. If for some i every such path has even length, then the edge between vi and $vi + 1$ points from $vi + 1$ to vi, since the other orientation would be a $vi, vi + 1 - path$ of length 1 (*odd*). In this case, we have an odd cycle through vi and $vi + 1$. Otherwise, we have a path of odd length from each vi to $vi + 1$. Combining these gives a closed trail of odd length In a digraph as well as in a graph (by the same proof), a closed odd trail contains the edges of an odd cycle

Answer. ANSWER HERE.

9 The Tournaments

Tournaments with all players kings.
If n is odd, then there is an tournament with n vertices such that every player is a king.

Answer. ANSWER HERE.

10 About Tree

For a tree T with vertex degrees in $\{1, k\}$, the possible values of $n(T)$ are the positive integers that are 2 more than a multiple of $k1$.

Answer. ANSWER HERE.

11 More about Tree

Prove the theorems blow.

I

Theorem 11.1. A tree has exactly one center or has two adjacent centers.

Answer. ANSWER HERE.

II

Theorem 11.2. A tree has exactly one center if and only if its diameter is twice its radius.

Answer. ANSWER HERE.

12 Cutting edge

Prove the theorem blow.

Theorem 12.1. $e = xy$ is a cutting edge if and only if $\begin{cases} G - e \quad \text{does not have a} \quad x \to y \quad \text{path.} \\ \text{it does not belong to any cycle.} \end{cases}$

Answer. ANSWER HERE.

13 $5 - cycles$ in peterson graph

how many $5 - cycles$ does **peterson** graph have?

Answer. ANSWER HERE.

14 Certain bridge club

A certain bridge club has a special rule to the effect that four members may play together only if no two of them have previously partnered one another. At one meeting fourteen members, each of whom has previously partnered five others, turn up. Three games are played, and then proceedings come to a halt because of the club rule. Just as the members are preparing to leave, a new member, unknown to any of them, arrives. Show that at least one more game can now be played.

Answer. ANSWER HERE.