Iran university of science and technology

Discrete mathematics
Problem Set #1

Ali Heydari

March 11, 2018

1 Set Theory Warmup

This question is designed to help you get used to the notation and mathematical conventions surrounding sets. Consider the following sets:

$$A = \{1, 2, 3, 4\}$$
$$B = \{2, 2, 2, 1, 4, 3\}$$
$$C = \{1, \{2\}, \{\{3, 4\}\}\}$$
$$D = \{1, 3\}$$
$$E = \mathbb{N}$$
$$F = \{\mathbb{N}\}$$

Answer each of the following questions and briefly justify your answers. No proofs are necessary.

i. Which pairs of the above sets, if any, are equal to one another?

Answer.

$$A = B$$

ii. Is $D \in A$? Is $D \subseteq A$?

Answer.

$$D \notin A \quad , \quad D \subseteq A$$

iii. What is $A \cap C$? How about $A \cup C$? How about $A \Delta C$?

Answer.

$$A \cap C = \{a\}$$

$$A \cup C = \{1, 2, 3, 4, \{2\}, \{\{3, 4\}\}\}$$

$$A \bigtriangleup C = \{2, 3, 4, \{2\}, \{\{3, 4\}\}\}$$

iv. What is $A - C$? How about $\{A - C\}$? Are those sets equal?

Answer.

$$A - C = \{2, 3, 4\} \quad , \quad \{A - C\} = \{\{2, 3, 4\}\} \quad \Rightarrow \quad A - C \neq \{A - C\}$$

v. What is $|B|$? What is $|E|$? What is $|F|$?

Answer.

$$|B| = 4 \quad , \quad |E| = \aleph \quad , \quad |F| = 1$$

vi. What is $E - A$? Express your answer in set-builder notation.

Answer.

$$E - A = \{x \in \mathbb{N} : x \geq 5\} \cup \{0\}$$

vii. Is $0 \in E$? Is $0 \in F$?

Answer.

$$0 \in E \quad , \quad 0 \notin F$$

2 Proof or disproof

Below are a number of claims about sets. For each claim, decide whether the statement is true or false. If it's true, prove it. If it's false, disprove it.

i. For all sets A and B, the following is true: $(A - B) \cup B = A$.

Answer. *Disproof.*

$$A = \{1\} \quad , \quad B = \{2\} \quad \Rightarrow \quad (A - B) \cup B = \{1\} \cup \{2\} = \{1, 2\} \neq \{1\} = A$$

$\square$

ii. For all sets A, B, and C, if $A \subseteq B \cap C$, then $A \subseteq B$ and $A \subseteq C$.

Answer. *Proof.*

$$\forall A, B, C \quad \text{if} \quad A \subseteq B \cap C \quad \forall x \in A \qquad \Rightarrow \qquad x \in B \quad \text{and} \quad x \in C \qquad \Rightarrow \qquad A \subseteq B \quad \text{and} \quad A \subseteq C$$

$\square$

iii. For any set A, if $A \not\subseteq \emptyset$, then $137 \in A$.

Answer. *Disproof.*

$$A = \{1\} \quad \text{But} \quad 137 \notin A$$

□

3 Prove

Prove the statement bellow:

$$\text{if} \quad 0 \leq x \leq 2, \quad \text{then} \quad -x^3 + 4x + 1 > 0.$$

Answer. *Proof.*

$$x(4 - x^2) + 1 > 0 \quad \Rightarrow \quad \overbrace{x(2-x)(2+x)}^{f(x)} + 1 > 0$$

$$f(x) + 1 > 0 \quad \Rightarrow \quad f(x) > -1 \quad \Rightarrow$$

x		-2		0		2	
f(x)	+		-		+		-

$$\Rightarrow \quad \text{if} \quad 0 \leq x \leq 2 \quad \rightarrow \quad f(x) > 0 > -1$$

□

4 Case proof

Answer. *Proof.*

$$C - (A \cap B) = C \cap (A \cap B)' = C \cap (A' \cup B') = (C \cap A') \cup (C \cap B') = (C - A) \cup (C - B)$$

□

5 $n \times n$ chessboard

Consider an $n \times n$ chessboard with the four corners removed. For which values of n can you cover the board with L- tetrominoes as in Figure. 1?

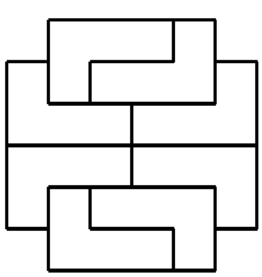

Figure 1:

Answer. *Proof.* There are n^24squares on the board. To cover it with tetrominoes n^24 must be a multiple of 4, i.c., n must be even. But this is not sufficient. To sec this, we color the board as in Figure. 2 An L-tetromino covers three white and one black squares or three black and one white squares. Since there is an equal number of black and white squares on the board, any complete covering uses an equal number of tetrominoes of each kind. Hence, it uses an even number of tetrominoes, that is, n^24 must be a multiple of 8. So, n must have the form $4k + 2$. By actual construction, it is easy to sec that the condition $4k + 2$ is also sufficient. ☐

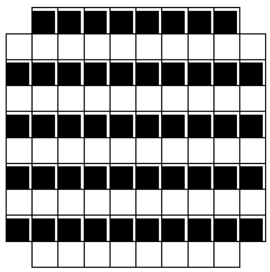

Figure 2:

6 Properties of Sets

Here are some claims about properties of sets. Some of them are true and some of them are false. For each true statement, write a proof that the statement is true. For each false statement, write a ***disproof*** of the statement (take a look at the Proofwriting Checklist for information about how to write a disproof.) You can use any proof techniques you'd like.

i. Prove or disprove: for all sets A, B, and C, if $A \in B$ and $B \in C$, then $A \in C$.

This is your first example of a "prove or disprove" problem. Part of the challenge of approaching a problem like this one is that you'll need to figure out whether or not the statement is even true in the first place, since if it's true you'll want to prove it and if it's false you'll want to disprove it.

Here are two strategies for approaching problems like these. First, try out a lot of examples! You'll want to get a feel for what the symbolic expression above "feels" like in practice. Second, get a sheet of scratch paper and write out both the statement and its negation. One of those statements is true, and your task is to figure out which one it is. Once you have those two statements, think about what you would need to do to prove each of them. In each case, what would you be assuming? What would you need to prove? If you can answer those questions, you can explore both options and seeing which one ends up panning out.

Disproof.

$$A = \{1\} \quad , \quad B = \{\{1\}, 2\} \quad , \quad C = \{\{\{1\}, 2\}, 3\}$$

$$A \in B \quad \text{and} \quad B \in C \quad \text{But} \quad A \notin C \qquad ※$$

☐

ii. Prove or disprove: for all sets A, B, and C, if $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

Proof.

$$\text{if} \quad A \subseteq B \quad \Rightarrow \quad \forall x \in A \quad \Rightarrow \quad x \in B \tag{1}$$
$$\text{if} \quad A \subseteq C \quad \Rightarrow \quad \forall x \in A \quad \Rightarrow \quad x \in C \tag{2}$$

$$1, 2 \quad \Rightarrow \forall x \in A \quad \Rightarrow (x \in B) \wedge (x \in C) \quad \Rightarrow \quad A \subseteq (B \cap C)$$

□

iii. Prove or disprove: for all sets A, B, and C, if $A \subsetneq B$ and $A \subsetneq C$, then $A \subsetneq B \cap C$. (The notation $A \subsetneq B$ says that A is a ***strict subset*** of B, meaning that $A \subseteq B$ and $A \neq B$.)

Proof.

$$\text{if} \quad A \subseteq (B \cap C) \quad \Rightarrow \quad (A \subseteq B) \wedge (A \subseteq C) \qquad ※$$

□

iv. Prove or disprove: there exists a set A where $\wp(A) = \{A\}$.

-

v. Prove or disprove: for all sets A and B, if $\wp(A) = \wp(B)$, then $A = B$.

Look back at Wednesday's lecture. What's a good general way to prove that two sets are equal?

Proof. $A = \{1\} \quad B = \{2\}$
$\Rightarrow \wp(A) = \{1, \{1\}\} \quad , \quad \wp(B) = \{2, \{2\}\} \quad \text{But} \quad A \neq B \qquad ※$ □

Before you turn in these proofs, be sure to read over the Proofwriting Checklist and to go one item at a time through each of your proofs. Here are a few specific things to look for:

- *Make sure that the structures of your proofs match the definitions of the relevant terms. For example, to prove that a set S is a subset of a set T, follow the pattern from lecture: pick an arbitrary $x \in S$, then prove that $x \in T$ by making specific claims about x.*
- *However, avoid restating definitions in the abstract. For example, rather than writing*

  "We know that $S \subseteq T$ if every element of S is an element of T.
  Therefore, since we know that $A \subseteq B$ and $x \in A$, we see that $x \in B$."

  instead remove that first sentence and just write something like this:

  "Since $x \in A$ and $A \subseteq B$, we see that $x \in B$."

  Whoever is reading your proof knows all the relevant definitions. They're more interested in seeing ***how those definitions interact with one another*** *than* ***what those definitions are****.*

- *Make sure you clearly indicate what each variable means and whether it's chosen arbitrarily or chosen to have a specific value. For example, in your answers, if you refer to variables like A, B, or C, you should clearly indicate whether they?re chosen arbitrarily or refer to specific values.*
- *If you're talking about an arbitrary set A, it's often tempting to try to list of the elements of A by writing something like $A = \{x_1, x_2, \ldots, x_n\}$. The problem with this approach is that by writing $A = \{x_1, x_2, \ldots, x_n\}$, you're implicitly saying that the set A is finite, since you're claiming it only has n elements in it. This is a problem if A is an infinite set. In fact, if A is infinite, because of Cantor's theorem you can't necessarily even write $A = \{x_1, x_2, x_3, \ldots\}$, since you might run out of natural numbers with which to name the elements of A without having listed all of them!*