Chapter 2

Simultaneous Linear Equations

Finding the roots of N linear equations

$$
\begin{cases}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots\dots\dots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots\dots\dots + a_{2n}x_n = b_2 \\
\dots\dots\dots\dots \quad \dots\dots\dots\dots \quad \dots\dots\dots\dots \quad \dots\dots\dots\dots \quad \dots\dots\dots\dots \quad \dots\dots\dots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots\dots\dots + a_{nn}x_n = b_n
\end{cases}
$$

$$
\begin{bmatrix}
a_{11} & a_{12} & a_{12} & \dots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\
\vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\
a_{n2} & a_{n2} & a_{n3} & \vdots\vdots\vdots & a_{nn}
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}
$$

$$[a_{ij}] \qquad [x_i] = [b_i]$$

$$\mathbf{AX} = \mathbf{B}$$

Noncomputer Methods for Solving Systems of Equations

- For small number of equations ($n \leq 3$) linear equations can be solved readily by simple techniques such as “method of elimination.”
- Linear algebra provides the tools to solve such systems of linear equations.
- Nowadays, easy access to computers makes the solution of large sets of linear algebraic equations possible and practical.

Solving linear Equations

- There are many ways to solve a system of linear equations:
  - Matrix inversion
  - Cramer's rule
  - Methods of elimination
  - LU decomposition

Chapters of textbook to read:
Textbook #1 (Chapra & Canale): Chapters 9.2; 9.3.1; 9.3.2; 9.4.2; 9.7; 10.1.1

2.1 Revision on Matrix

$$
\begin{array}{c}
\text{Column 3} \\
[A] = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\
\cdot & \cdot & & & \cdot \\
\cdot & \cdot & & & \cdot \\
\cdot & \cdot & & & \cdot \\
a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm}
\end{bmatrix} \quad \text{Row 2}
\end{array}
$$

Elements in red region are all zero: A is called **U**pper triangular matrix;

Elements in blue region are all zero: A is called **L**ower triangular matrix;

Elements in red and blue region are all zero: A is called diagonal matrix;

In a diagonal matrix, if all elements are 1, A is called identity matrix **I**.

Determinants and Cramer's Rule

- Determinant can be illustrated for a set of three equations:

$$[A]\{x\} = \{B\}$$

- Where [A] is the coefficient matrix:

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- Assuming all matrices are square matrices, there is a number associated with each square matrix [A] called the determinant, D, of [A]. If [A] is order 1, then [A] has one element:

  $[A]=[a_{11}]$

  $D=a_{11}$

- For a square matrix of order 3, the ***minor*** of an element a_{ij} is the determinant of the matrix of order 2 by deleting row *i* and column *j* of [A].

$$D=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$D_{11}=\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}=a_{22}\,a_{33}-a_{32}\,a_{23}$$

$$D_{12}=\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}=a_{21}\,a_{33}-a_{31}\,a_{23}$$

$$D_{13}=\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}=a_{21}\,a_{32}-a_{31}\,a_{22}$$

$$D = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

- *Cramer's rule* expresses the solution of a systems of linear equations in terms of ratios of determinants of the array of coefficients of the equations. For example, x_1 would be computed as:

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{D}$$

2.2 Methods of Elimination

- The basic strategy is to successively solve one of the equations of the set for one of the unknowns and to eliminate that variable from the remaining equations by substitution.
- The elimination of unknowns can be extended to systems with more than two or three equations; however, the method becomes extremely tedious to solve by hand.

Gaussian elimination method

Example:

$$\begin{cases} 3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \\ 0.1x_1 + 7x_2 - 0.3x_3 = -19.3 \\ 0.3x_1 - 0.2x_2 + 10x_3 = 71.4 \end{cases}$$

Eliminate the 1st terms of Eq. (2) (3)

$$\begin{cases} 3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \\ 0 \quad + 7.003x_2 - 0.293x_3 = -19.56 \\ 0 \quad - 0.190x_2 + 10.02x_3 = 70.615 \end{cases}$$

Eliminate the 2nd term of Eq. (3)

$$\begin{cases} 3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \\ 0 \quad + 7.003x_2 - 0.293x_3 = -19.56 \\ 0 \quad + 0 \quad + 10.012x_3 = 70.0843 \end{cases}$$

$x_3 = 7;\qquad x_2 = -2.5 \qquad x_1 = 3$

The augmented matrix

Example:

$$Aug = \begin{bmatrix} 10 & 2 & -1 & 27 \\ -3 & -6 & 2 & -61.5 \\ 1 & 1 & 5 & -21.5 \end{bmatrix}$$

A **B**

Reduce all the off diagonal elements below the diagonal to zero

$$\rightarrow \begin{bmatrix} 10 & 2 & -1 & 27 \\ 0 & -5.4 & 1.7 & -53.4 \\ 0 & 0.8 & 5.1 & -24.2 \end{bmatrix} \rightarrow \begin{bmatrix} 10 & 2 & -1 & 27 \\ 0 & -5.4 & 1.7 & -53.4 \\ 0 & 0 & 5.351852 & -32.1111 \end{bmatrix}$$

$$x_3 = 6; \qquad x_2 = 8 \qquad x_1 = 0.5$$

Naive Gauss Elimination

- Extension of *method of elimination* to large sets of equations by developing a systematic scheme or algorithm to eliminate unknowns and to back substitute.
- As in the case of the solution of two equations, the technique for *n* equations consists of two phases:
  - Forward elimination of unknowns
  - Back substitution

$$
\left[\begin{array}{ccc|c}
a_{11} & a_{12} & a_{13} & c_1 \\
a_{21} & a_{22} & a_{23} & c_2 \\
a_{31} & a_{32} & a_{33} & c_3
\end{array}\right]
$$

$$\Downarrow$$

$$
\left[\begin{array}{ccc|c}
a_{11} & a_{12} & a_{13} & c_1 \\
 & a'_{22} & a'_{23} & c'_2 \\
 & & a''_{33} & c''_3
\end{array}\right]
$$

Forward elimination

$$\Downarrow$$

$$x_3 = c''_3 / a''_{33}$$

$$x_2 = (c'_2 - a'_{23} x_3) / a'_{22}$$

$$x_1 = (c_1 - a_{12} x_2 - a_{13} x_3) / a_{11}$$

Back substitution

General algorithm:

- The first equation times a_{21}/a_{11} is subtracted from the second equation to eliminate the first term of the second equation and likewise for other equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots\dots\dots + a_{1n}x_n = b_1 \\ 0 \quad + a'_{22}x_2 + a'_{23}x_3 + \dots\dots\dots + a'_{2n}x_n = b'_2 \\ \dots\dots\dots\dots \quad \dots\dots\dots\dots \quad \dots\dots\dots\dots \quad \dots\dots\dots\dots \quad \dots\dots\dots\dots \quad \dots\dots\dots \\ 0 \quad + a'_{n2}x_2 + a'_{n3}x_3 + \dots\dots\dots + a'_{nn}x_n = b'_n \end{cases}$$

$$a'_{ij} = a_{ij} - (a_{i1}/a_{11})a_{1j}$$

with

the first equation UNCHANGED

- Next, the second term of every equation in the third i>2 is eliminated by subtracting the second equation times

$$a'_{i2}/a'_{22}$$

- And so on for the rest of the equations using similar method.

After n-1 steps of elimination:

$$\begin{bmatrix} a_{11}^1 & a_{11}^1 & a_{11}^1 & \cdots & a_{11}^1 \\ 0 & a_{22}^2 & a_{23}^2 & \cdots & a_{2n}^2 \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \vdots & \vdots & a_{kk}^k & \vdots\vdots\vdots & \vdots \\ 0 & 0 & 0 & \vdots\vdots\vdots & a_{nn}^n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1^1 \\ b_2^2 \\ \vdots \\ \vdots \\ b_n^n \end{bmatrix}$$

The solutions are obtained by backward substitution:

$$x_n = b_n^n \Big/ a_{nn}^n$$

$$x_i = \left(b_i^i - \sum_{j=i+1}^{n} a_{ij}^i x_j \right) \Big/ a_{ii}^i \quad , \qquad i=\text{n-1, n-2, …, 1}$$

Gaussian elimination with pivoting

The pivoting step with Gaussian elimination is as follows:

At each elimination step starting from the *k-th* row, a row switching is done based on the following criterion:

Find the maximum of $|a_{k,k}|, |a_{k+1,k}|, \ldots |a_{n,k}|$

which is $|a_{p,k}|$ in the *p-th* row;
Then switch rows p and k.

Example: $\begin{cases} 2x_2 + 5x_3 = 1 \\ 2x_1 + x_2 + x_3 = 0 \\ 3x_1 + x_2 = 5 \end{cases}$

$$Aug = \begin{bmatrix} a_{11} & a_{12} & a_{1,k} & \ldots & a_{1,n+1} \\ 0 & a_{22} & a_{2,k} & \ldots & a_{2,n+1} \\ 0 & 0 & a_{k,k} & \vdots\vdots\vdots & \vdots \\ \vdots & 0 & a_{k+1,k} & \vdots\vdots\vdots & \vdots \\ 0 & 0 & a_{n,k} & \vdots\vdots\vdots & a_{n,n+1} \end{bmatrix}$$

The numerical solution procedure is as follows

$$(1) \textit{Pivoting and elimination}$$

$$Aug = \begin{bmatrix} 0 & 2 & 5 & 1 \\ 2 & 1 & 1 & 0 \\ 3 & 1 & 0 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 1 & 0 & 5 \\ 2 & 1 & 1 & 0 \\ 0 & 2 & 5 & 1 \end{bmatrix}$$

After switching

$$\rightarrow \begin{bmatrix} 3 & 1 & 0 & 5 \\ 0 & 1/3 & 1 & -10/3 \\ 0 & 2 & 5 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 1 & 0 & 5 \\ 0 & 2 & 5 & 1 \\ 0 & 1/3 & 1 & -10/3 \end{bmatrix}$$

After switching

$$\rightarrow \begin{bmatrix} 3 & 1 & 0 & 5 \\ 0 & 2 & 5 & 1 \\ 0 & 0 & 1/6 & -21/6 \end{bmatrix}$$

$$(2) \textit{Backward substitution}$$

$$x_3 = -21$$

$$x_2 = (1 - 5x_3)/2 = 53$$

$$x_1 = (5 - x_2)/3 = -16$$

Example

Assume *any arithmetic computation* must follow the five-digit chopping arithmetic which is defined as follows: given a real number *q* expressed in normalized decimal form as

$$q = \pm(0.d_1 d_2 d_3 d_4 d_5 \cdots d_k d_{k+1} \cdots) \times 10^n$$

the value of this number is ,

$$q = \pm(0.d_1 d_2 d_3 d_4 d_5) \times 10^n$$, where $1 \le d_1 \le 9$, and $0 \le d_k \le 9$ for $k>1$.

Answer the following questions.

(i) Based on the five-digit chopping arithmetic, solve the following simultaneous equation using naïve Gaussian elimination.

$$\begin{bmatrix} 12 & -6 & 0 \\ -3 & 1.501 & 5.626 \\ 5 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6.9 \\ 3.9 \\ 5.375 \end{bmatrix}$$

(ii) Based on the five-digit chopping arithmetic, solve the simultaneous equation using the Gaussian elimination with pivoting.

(iii) Comment on the results obtained using different numerical methods in (i) and (ii) for the same simultaneous equation.

Ans:

(i) The augmented matrix of the simultaneous linear equations is

$$Aug = \begin{bmatrix} 12 & -6 & 0 & 6.9 \\ -3 & 1.501 & 5.626 & 3.9 \\ 5 & 0 & 5 & 5.375 \end{bmatrix}$$

The 1st step of elimination changes the augmented matrix into

$$Aug = \begin{bmatrix} 12 & -6 & 0 & 6.9 \\ 0 & 0.001 & 5.626 & 5.625 \\ 0 & 2.5 & 5 & 2.5 \end{bmatrix}$$

The 2nd step of elimination changes the augmented matrix into

$$Aug = \begin{bmatrix} 12 & -6 & 0 & 6.9 \\ 0 & 0.001 & 5.626 & 5.625 \\ 0 & 0 & -14060 & -14059 \end{bmatrix}$$

Backward substitution gives the solutions

$x_3 = 0.99992$

$x_2 = -0.5$

$x_1 = 0.325$

(ii) The augmented matrix of the simultaneous linear equations is

$$Aug = \begin{bmatrix} 12 & -6 & 0 & 6.9 \\ -3 & 1.501 & 5.626 & 3.9 \\ 5 & 0 & 5 & 5.375 \end{bmatrix}$$

The 1st step of elimination changes the augmented matrix into

$$Aug = \begin{bmatrix} 12 & -6 & 0 & 6.9 \\ 0 & 0.001 & 5.626 & 5.625 \\ 0 & 2.5 & 5 & 2.5 \end{bmatrix}$$

Performing pivoting changes the augmented matrix into

$$Aug = \begin{bmatrix} 12 & -6 & 0 & 6.9 \\ 0 & 2.5 & 5 & 2.5 \\ 0 & 0.001 & 5.626 & 5.625 \end{bmatrix}$$

The 2nd step of elimination changes the augmented matrix into

$$Aug = \begin{bmatrix} 12 & -6 & 0 & 6.9 \\ 0 & 2.5 & 5 & 2.5 \\ 0 & 0 & 5.624 & 5.624 \end{bmatrix}$$

Backward substitution gives the solutions

$x_3 = 1.0000$
$x_2 = -1.0000$
$x_1 = 0.075000$

(iii)

The exact solutions of the equations are

$x_3 = 1.0$
$x_2 = -1.0$
$x_1 = 0.075$

Naive Gaussian elimination in **(i)** solves the equation with very large errors, which is caused by the accumulation of error in the 2nd step of forward elimination.

The Gaussian elimination with pivoting could solve the equation with very small errors.

It is concluded that the Gaussian elimination with pivoting could avoid the error accumulation during the elimination processes.

Gaussian - Jordan elimination method

Example:

$$\begin{cases} 2x_1 + x_2 - x_3 = 1 \\ 5x_1 + 2x_2 + 2x_3 = -4 \\ 3x_1 + x_2 + x_3 = 5 \end{cases}$$

$$\dashrightarrow \begin{bmatrix} 2 & 1 & -1 & 1 \\ 5 & 2 & 2 & -4 \\ 3 & 1 & 1 & 5 \end{bmatrix} \dashrightarrow \begin{bmatrix} 1 & 0.5 & -0.5 & 0.5 \\ 0 & -0.5 & 4.5 & -6.5 \\ 0 & -0.5 & 2.5 & 3.5 \end{bmatrix} \dashrightarrow \begin{bmatrix} 1 & 0 & 0 & 14 \\ 0 & 1 & 0 & -32 \\ 0 & 0 & 1 & -5 \end{bmatrix}$$

The solutions are obtained straightforward:

backward substitution is not needed.

2.3 LU Decomposition

- Provides an efficient way to compute matrix inverse by separating the time consuming elimination of the Matrix [A] from manipulations of the right-hand side {B}.
- *Gauss elimination*, in which the forward elimination comprises the bulk of the computational effort, can be implemented as an LU decomposition.

If

L- lower triangular matrix

U- upper triangular matrix

Then,

$[A]\{X\}=\{B\}$ can be decomposed into two matrices $[L]$ and $[U]$ such that

$[L][U]=[A]$

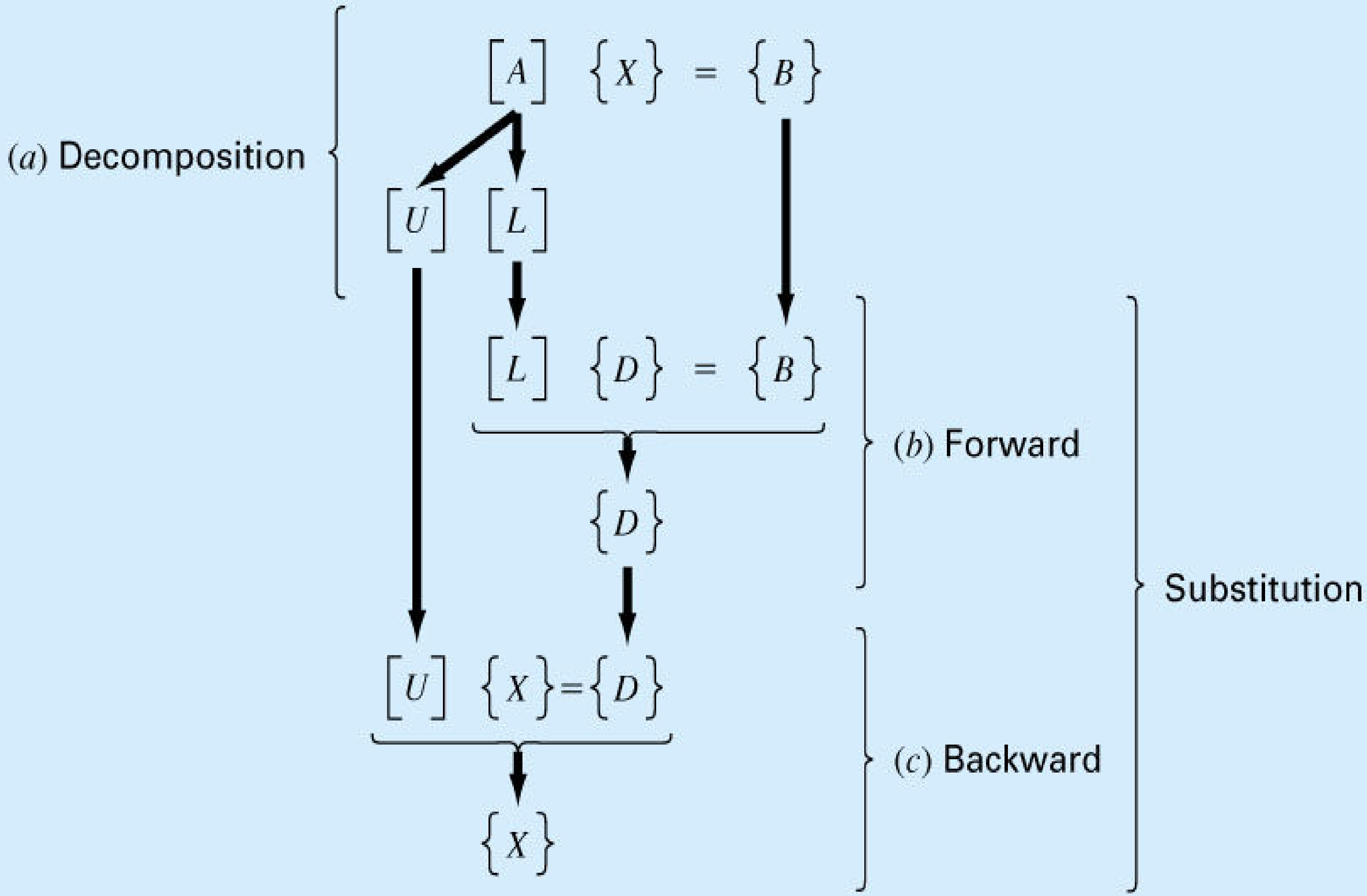
$[L][U]\{X\}=\{B\}$

Similar to first phase of *Gauss elimination*, consider

$[U]\{X\}=\{D\}$

$[L]\{D\}=\{B\}$

- $[L]\{D\}=\{B\}$ is used to generate an intermediate vector $\{D\}$ by forward substitution
- Then, $[U]\{X\}=\{D\}$ is used to get $\{X\}$ by back substitution.


[A] {X} = {B}
(a) Decomposition
[U] [L]
[L] {D} = {B}
(b) Forward
{D}
Substitution
[U] {X}={D}
(c) Backward
{X}

LU decomposition

- requires the same total multiply/divide steps (or floating point operation per second) as for *Gauss elimination.*
- Saves computing time by separating time-consuming elimination step from the manipulations of the right hand side.
- Provides efficient means to compute the matrix inverse