ME46002 Chapter 2: Tutorial questions

QUESTIONS:

(1) Solve the linear equations by Cramer's rule

$$-3x_2 + 7x_3 = 2$$
$$x_1 + 2x_2 - x_3 = 3$$
$$5x_1 - 2x_2 = 2$$

<u>Solution</u>

The determinant of matrix

$$D = \begin{vmatrix} 0 & -3 & 7 \\ 1 & 2 & -1 \\ 5 & -2 & 0 \end{vmatrix} \text{ is -69}$$

Cramer's rule

$$x_1 = \frac{\begin{vmatrix} 2 & -3 & 7 \\ 3 & 2 & -1 \\ 2 & -2 & 0 \end{vmatrix}}{D} = \frac{-68}{-69} = 0.985507$$

$$x_2 = \frac{\begin{vmatrix} 0 & 2 & 7 \\ 1 & 3 & -1 \\ 5 & 2 & 0 \end{vmatrix}}{D} = \frac{-101}{-69} = 1.463768$$

$$x_3 = \frac{\begin{vmatrix} 0 & -3 & 2 \\ 1 & 2 & 3 \\ 5 & -2 & 2 \end{vmatrix}}{D} = \frac{-63}{-69} = 0.913043$$

(2) Solve the linear equations using Gaussian elimination

$$10x_1 + 2x_2 - x_3 = 27$$
$$-3x_1 - 6x_2 + 2x_3 = -61.5$$
$$x_1 + x_2 + 5x_3 = -21.5$$

Solution:

The system is first expressed as an augmented matrix:

$$\begin{bmatrix} 10 & 2 & -1 & 27 \\ -3 & -6 & 2 & -61.5 \\ 1 & 1 & 5 & -21.5 \end{bmatrix}$$

Forward elimination:

a_{21} is eliminated by multiplying row 1 by –3/10 and subtracting the result from row 2. a_{31} is eliminated by multiplying row 1 by 1/10 and subtracting the result from row 3.

$$\begin{bmatrix} 10 & 2 & -1 & 27 \\ 0 & -5.4 & 1.7 & -53.4 \\ 0 & 0.8 & 5.1 & -24.2 \end{bmatrix}$$

a_{32} is eliminated by multiplying row 2 by 0.8/(–5.4) and subtracting the result from row 3.

$$\begin{bmatrix} 10 & 2 & -1 & 27 \\ 0 & -5.4 & 1.7 & -53.4 \\ 0 & 0 & 5.351852 & -32.1111 \end{bmatrix}$$

Back substitution:

$$x_3 = \frac{-32.1111}{5.351852} = -6$$

$$x_2 = \frac{-53.4 - 1.7(-6)}{-5.4} = 8$$

$$x_1 = \frac{27 - (-1)(-6) - 2(8)}{10} = 0.5$$

(3) Solve the linear equations using Gaussian elimination *with pivoting*.

$$8x_1 + 2x_2 - 2x_3 = -2$$
$$10x_1 + 2x_2 + 4x_3 = 4$$
$$12x_1 + 2x_2 + 2x_3 = 6$$

<u>Solution:</u>

The system is first expressed as an augmented matrix:

$$\begin{bmatrix} 8 & 2 & -2 & -2 \\ 10 & 2 & 4 & 4 \\ 12 & 2 & 2 & 6 \end{bmatrix}$$

Forward elimination: First, we pivot by switching rows 1 and 3:

$$\begin{bmatrix} 12 & 2 & 2 & 6 \\ 10 & 2 & 4 & 4 \\ 8 & 2 & -2 & -2 \end{bmatrix}$$

Multiply row 1 by 10/12 = 0.83333 and subtract from row 2 to eliminate a_{21}. Multiply row 1 by 8/12 = 0.66667 and subtract from row 3 to eliminate a_{31}.

$$\begin{bmatrix} 12 & 2 & 2 & 6 \\ 0 & 0.33333 & 2.33333 & -1 \\ 0 & 0.66667 & -3.33333 & -6 \end{bmatrix}$$

Pivot:

$$\begin{bmatrix} 12 & 2 & 2 & 6 \\ 0 & 0.66667 & -3.33333 & -6 \\ 0 & 0.33333 & 2.33333 & -1 \end{bmatrix}$$

Multiply row 2 by 0.33333/0.66667 = 0.5 and subtract from row 3 to eliminate a_{32}.

$$\begin{bmatrix} 12 & 2 & 2 & 6 \\ 0 & 0.66667 & -3.33333 & -6 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$

Back substitution:

$$x_3 = \frac{2}{4} = 0.5$$
$$x_2 = \frac{-6 - (-3.3333)0.5}{0.66667} = -6.5$$
$$x_1 = \frac{6 - 2(0.5) - 2(-6.5)}{12} = 1.5$$

(4) Solve the linear equations

$$2x_1 + x_2 - x_3 = 1$$
$$5x_1 + 2x_2 + 2x_3 = -4$$
$$3x_1 + x_2 + x_3 = 5$$

using Gauss-Jordan method

<u>Solution</u>

The system is first expressed as an augmented matrix:

$$\begin{bmatrix} 2 & 1 & -1 & 1 \\ 5 & 2 & 2 & -4 \\ 3 & 1 & 1 & 5 \end{bmatrix}$$

Normalize the first row and then eliminate a_{21} and a_{31},

$$\begin{bmatrix} 1 & 0.5 & -0.5 & 0.5 \\ 0 & -0.5 & 4.5 & -6.5 \\ 0 & -0.5 & 2.5 & 3.5 \end{bmatrix}$$

Normalize the second row and eliminate a_{12} and a_{32},

$$\begin{bmatrix} 1 & 0 & 4 & -6 \\ 0 & 1 & -9 & 13 \\ 0 & 0 & -2 & 10 \end{bmatrix}$$

Normalize the third row and eliminate a_{13} and a_{23},

$$\begin{bmatrix} 1 & 0 & 0 & 14 \\ 0 & 1 & 0 & -32 \\ 0 & 0 & 1 & -5 \end{bmatrix}$$

Thus, the answers are $x_1 = 14$, $x_2 = -32$, and $x_3 = -5$.

(5) Solve the linear equation $[A]\{X\} = \{B\}$ with

$$[A] = \begin{bmatrix} 2 & -6 & -1 \\ -3 & -1 & 7 \\ -8 & 1 & -2 \end{bmatrix} \qquad \{B\} = \begin{Bmatrix} -38 \\ -34 \\ -20 \end{Bmatrix}$$

using LU decomposition *with pivoting*.

Solution:

As the system is set up, we must first pivot by switching the first and third rows of $[A]$. Note that we must make the same switch for the right-hand-side vector $\{B\}$

$$[A] = \begin{bmatrix} -8 & 1 & -2 \\ -3 & -1 & 7 \\ 2 & -6 & -1 \end{bmatrix} \qquad \{B\} = \begin{Bmatrix} -20 \\ -34 \\ -38 \end{Bmatrix}$$

The coefficient a_{21} is eliminated by multiplying row 1 by $f_{21} = -3/-8 = 0.375$ and subtracting the result from row 2. a_{31} is eliminated by multiplying row 1 by $f_{31} = 2/(-8) = -0.25$ and subtracting the result from row 3. The factors f_{21} and f_{31} can be stored in a_{21} and a_{31}.

$$[A] = \begin{bmatrix} -8 & 1 & -2 \\ 0.375 & -1.375 & 7.75 \\ -0.25 & -5.75 & -1.5 \end{bmatrix}$$

Next, we pivot by switching rows 2 and 3. Again, we must also make the same switch for the right-hand-side vector $\{B\}$

$$[A] = \begin{bmatrix} -8 & 1 & -2 \\ -0.25 & -5.75 & -1.5 \\ 0.375 & -1.375 & 7.75 \end{bmatrix} \qquad \{B\} = \begin{Bmatrix} -20 \\ -38 \\ -34 \end{Bmatrix}$$

a_{32} is eliminated by multiplying row 2 by $f_{32} = -1.375/(-5.75) = 0.23913$ and subtracting the result from row 3. The factor f_{32} can be stored in a_{32}.

$$[A] = \begin{bmatrix} -8 & 1 & -2 \\ -0.25 & -5.75 & -1.5 \\ 0.375 & 0.23913 & 8.108696 \end{bmatrix}$$

Therefore, the *LU* decomposition is

$$[L]=\begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0.375 & 0.23913 & 1 \end{bmatrix} \qquad [U]=\begin{bmatrix} -8 & 1 & -2 \\ 0 & -5.75 & -1.5 \\ 0 & 0 & 8.108696 \end{bmatrix}$$

Forward substitution for $[L]\{D\} = \{B\}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0.375 & 0.23913 & 1 \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix}=\begin{Bmatrix} -20 \\ -38 \\ -34 \end{Bmatrix}$$

The solving yields $d_1 = -20$, $d_2 = -43$, and $d_3 = -16.2174$.

Back substitution to solve $[U]\{X\} = \{D\}$

$$\begin{bmatrix} -8 & 1 & -2 \\ 0 & -5.75 & -1.5 \\ 0 & 0 & 8.108696 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}=\begin{Bmatrix} -20 \\ -43 \\ -16.2174 \end{Bmatrix}$$

$$x_3 = \frac{-16.2174}{8.108696} = -2$$

$$x_2 = \frac{-43+1.5(-2)}{-5.75} = 8$$

$$x_1 = \frac{-20+2(-2)-8}{-8} = 4$$

(6) Solve the linear equations

$$0.8x_1 - 0.4x_2 = 41$$
$$0.4x_1 - 0.8x_2 + 0.4x_3 = -25$$
$$-0.4x_2 + 0.8x_3 = 105$$

using Gauss-Seidel iterative method with a prescribed error of 5%

<u>Solution</u>

The first iteration can be implemented as

$$x_1 = \frac{41+0.4x_2}{0.8} = \frac{41+0.4(0)}{0.8} = 51.25$$

$$x_2 = \frac{25 + 0.4x_1 + 0.4x_3}{0.8} = \frac{25 + 0.4(51.25) + 0.4(0)}{0.8} = 56.875$$

$$x_3 = \frac{105 + 0.4x_2}{0.8} = \frac{105 + 0.4(56.875)}{0.8} = 159.6875$$

Second iteration:

$$x_1 = \frac{41 + 0.4(56.875)}{0.8} = 79.6875$$

$$x_2 = \frac{25 + 0.4(79.6875) + 0.4(159.6875)}{0.8} = 150.9375$$

$$x_3 = \frac{105 + 0.4(150.9375)}{0.8} = 206.7188$$

The error estimates can be computed as

$$\varepsilon_{a,1} = \left|\frac{79.6875 - 51.25}{79.6875}\right| \times 100\% = 35.69\%$$

$$\varepsilon_{a,2} = \left|\frac{150.9375 - 56.875}{150.9375}\right| \times 100\% = 62.32\%$$

$$\varepsilon_{a,3} = \left|\frac{206.7188 - 159.6875}{206.7188}\right| \times 100\% = 22.75\%$$

The remainder of the calculation proceeds until all the errors fall below the stopping criterion of 5%. The entire computation can be summarized as

iteration	unknown	value	ε_a	maximum ε_a
1	x_1	51.25	100.00%	
	x_2	56.875	100.00%	
	x_3	159.6875	100.00%	100.00%
2	x_1	79.6875	35.69%	
	x_2	150.9375	62.32%	
	x_3	206.7188	22.75%	62.32%
3	x_1	126.7188	37.11%	
	x_2	197.9688	23.76%	
	x_3	230.2344	10.21%	37.11%
4	x_1	150.2344	15.65%	
	x_2	221.4844	10.62%	
	x_3	241.9922	4.86%	15.65%
5	x_1	161.9922	7.26%	
	x_2	233.2422	5.04%	
	x_3	247.8711	2.37%	7.26%
6	x_1	167.8711	3.50%	
	x_2	239.1211	2.46%	
	x_3	250.8105	1.17%	3.50%

Thus, after 6 iterations, the maximum error is 3.5% and we arrive at the result: x_1 = 167.8711, x_2 = 239.1211 and x_3 = 250.8105.