THE HONG KONG POLYTECHNIC UNIVERSITY

Department of Mechanical Engineering

Subject Title	:	Numerical Methods for Engineers	**Subject Code**	:	ME 46002
Session	:	Semester 1, 2017/2018	**Programme**	:	BEng(Hons) in Mechanical Engineering (43460)
Date	:	**14 Dec 2017**	**Time**	:	**19:00 ~ 22:00**
Time Allowed	:	3 hours	**Subject Examiner**	:	T.Y. Ng

This question paper has 5 pages (this page included)

Instructions to Candidates: This paper has 5 questions. Answer all questions.

Constants :

Others :

Available from Invigilator :

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Question 1 (12 marks)
Given the integral

$$I = \int_0^3 x^2 e^x dx$$

(a) Use composite trapezoidal rule to approxmate I using six subintervals. [4 marks]

(b) Use the Gaussian quadrature formula with $n = 2$ and $n = 3$ to find an approximation to the integral.

n	w_i	x_i
2	$w_1 = 1.000000000$	$x_1 = -0.577350269$
	$w_2 = 1.000000000$	$x_2 = 0.577350269$
3	$w_1 = 0.555555556$	$x_1 = -0.774596669$
	$w_2 = 0.888888889$	$x_2 = 0$
	$w_3 = 0.555555556$	$x_3 = 0.774596669$

[4 marks]

(c) Given $\epsilon = 0.001$, estimate the number of subintervals n required to approximate I when using the composite trapezoidal rule. The error term is given by

$$E(f) = -\frac{(b-a)h^2}{12} f''(\xi)$$

where h is the length of the subintervals and a and b represent the lower and upper limits of the integral, respectively. [4 marks]

Question 2 (8 marks)
For the given function $f(x) = \sin \pi x$, let $x_0 = 1$, $x_1 = 1.25$ and $x_2 = 1.6$. Construct a Lagrange interpolation polynomial of degree at most two to approximate $f(1.4)$. Also given the error term of degree n formula as follows:

$$E = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

Find the error bound for this approximation and the absolute error. [8 marks]

Question 3 (45 marks)

Symbol	Name	Unit	–
k	thermal conductivity	W/(m·°C)	400
ρ	density	kg/m^3	9000
c	specific heat	J/(kg·°C)	380
h_∞	convection coefficient	W/(m^2·°C)	150

(a) Given a solid rod of total length $L = 1$ m and cross-sectional area $A = 0.002$ m^2. Suppose that the heat generation function is

$$G(x,t) = \sin t$$

and the boundary conditions are given as follows:

$$u(0,t) = 100°\text{C} \quad \text{and} \quad -kA\frac{\partial u}{\partial x}\bigg|_{x=L} = 0\,\text{W}$$

With the initial conditions $u(L,0) = 50°$C, $\beta = 2/3$, $\Delta t = 3$ s, determine the temperature u and the rate of change of temperature $\partial u/\partial t$ at $x = L$, and the heat flux Q at $x = 0$ for the instant $t = 0$ s and $t = 3$ s by using a 1-element finite element model with $x_1 = 0$ and $x_2 = L$.

i	t_i	$u(1,t) = u_{2,i}$	$\frac{\partial}{\partial t}u(1,t) = \dot{u}_{2,i}$	$Q(0,t_i)$
0				
1				

[25 marks]

(b) Given the data for a 2-element FE model of a solid rod as follows:

e	1	2	Unit
A	$0.003 - 0.001x$	0.002	m^2
L	1	1	m

Assume the domain of the solid rod is $\Omega = [0,2]$. The prescribed temperature at $x = 0$ is $u(0) = 100°$C and there is a heat loss due to heat convection at $x = 2$ where the far field temperature u_∞ is 10°C. Assume that we are interested in the steady-state solution of the problem. Determine the temperatures at node 2 and node 3, and the heat flux at $x = 0$ using the 2-element FE model.

[20 marks]

Question 4 (20 marks)
Given a linear system

$$\begin{aligned} 5x_1 + 2x_2 - x_3 &= 0 \\ -x_1 + 8x_2 + 2x_3 &= -15 \\ 2x_1 - 3x_2 + 10x_3 &= 18 \end{aligned}$$

(a) Use the Jacobi's iterative method (Don't swap rows) to obtain an approximate solution to the linear system by carrying out 5 iterations (use 4 decimal places for answers) using $\mathbf{x}^{(0)} = (0,1,0)^T$ and then compute $||\mathbf{x}^{(5)} - \mathbf{x}^{(4)}||_2$. Show the details for the first iteration.

[10 marks]

(b) Use the Gaussian Elimination Algorithm to solve the system.

[10 marks]

Question 5 (15 marks)
Given an equation

$$2x + 3\sin x - e^x = 0 \quad \text{with} \quad 0 \le x \le 1$$

(a) Use bisection method to obtain an approximate solution for the equation on the interval with the stopping criterion given by

$$\frac{|b_n - a_n|}{2} < 0.05 \quad n \in \mathbb{N}$$

Provide detailed calculations for the first 3 iterations.

[10 marks]

(b) Use Newton-Raphson method to find an approximate solution with the given initial guess $x_0 = 1.0$. Use $|x_i - x_{i-1}| < 0.002$ as the stopping criterion for this part.

[5 marks]

Note: Use 6 decimal places for all numerical values.

Formulas

Cardinal Polynomial

$$L_i(x) = \prod_{j \neq i, j=0}^{n} \left(\frac{x - x_j}{x_i - x_j} \right) \qquad 0 \leq i \leq n$$

Shape Functions

$$\Psi_1(x) = \frac{x_{e2} - x}{L_e} \qquad \Psi_2(x) = \frac{x - x_{e1}}{L_e}$$

Elemental Equations

$$m_{ij}^e = \int_{x_{e1}}^{x_{e2}} \rho c A \Psi_i \Psi_j dx$$

$$k_{ij}^e = \int_{x_{e1}}^{x_{e2}} k A \Psi_i' \Psi_j' dx$$

$$g_i^e = \int_{x_{e1}}^{x_{e2}} G A \Psi_i dx$$

$$h_i^e = \begin{cases} Q(x_{e1}, t) & \text{if } i = 1 \\ -Q(x_{e2}, t) & \text{if } i = 2 \end{cases}$$

Let

$$a_1 \equiv (1 - \beta)\Delta t \qquad a_2 \equiv \beta \Delta t \qquad a_3 \equiv \frac{\beta - 1}{\beta} \qquad a_4 \equiv \frac{1}{\beta \Delta t}$$

then

$$\mathbf{A}\mathbf{u}_{i+1} = \mathbf{B}\mathbf{u}_i + \mathbf{c}_i$$

$$\dot{\mathbf{u}}_{i+1} = a_3 \dot{\mathbf{u}}_i + a_4(\mathbf{u}_{i+1} - \mathbf{u}_i)$$

where

$$\mathbf{A} = \mathbf{M} + a_2\mathbf{K} \qquad \mathbf{B} = \mathbf{M} - a_1\mathbf{K} \qquad \mathbf{c}_i = a_1\mathbf{f}_i + a_2\mathbf{f}_{i+1}$$

Inverse of 2×2 matrix:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

- END -