ME46002 Chapter 1: Tutorial questions

QUESTION 1:

(a) Employ the Newton-Raphson method to determine the root for

$$f(x) = \cos(x)\ln(x) + e^x/5 = 0$$

using an initial guess of x_0= 2. Perform the computation until the percentage relative error is less than 1%. x is in unit of radians.

(b) The Newton-Raphson method may diverge or converge on the true root slowly depending on the initial guess. Give *two* examples that the Newton-Raphson method may diverge.

(c) Determine the root of the following equation

$$f(x) = \sin(x) + (1-x)e^x = 0$$

using bisection method with two initial guesses of a=1 and b=2. Perform the computation until the percentage relative error is less than 5%. x is in unit of radians.

Solution

(a)

$$f(x) = \cos(x)\ln(x) + 0.2 * \exp(x)$$
$$f'(x) = -\sin(x)\ln(x) + \cos(x)/x + 0.2 * \exp(x)$$
$$Hence \qquad x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Iteration=1 xi=2.000000 x=0.140058 Error=1327.9%
Iteration=2 xi=0.140058 x=0.366659 Error=61.8%
Iteration=3 xi=0.366659 x=0.569535 Error=35.6%
Iteration=4 xi=0.569535 x=0.625998 Error=9.0%
Iteration=5 xi=0.625998 x=0.628862 Error=0.46%

Hence the root is x=0.625998 with error of 0.46%

(b)
For examples,
When the initial guess is chosen as or close to the inflection point
or the local minima or maxima of the f(x),
the Newton-Raphson method may diverge.

(c)
Based on the bisection method, the estimation of root is given by c=(a+b)/2;

If f(a)*f(c)>0, a=c otherwise b=c in the next iteration.

Iteration
Iteration=1 a=1.000000 b=2.000000 c=1.500000
Iteration=2 a=1.000000 b=1.500000 c=1.250000 Error=20.00%
Iteration=3 a=1.250000 b=1.500000 c=1.375000 Error=9.090%
Iteration=4 a=1.250000 b=1.375000 c=1.312500 Error=4.761%

Hence the root is x= c=1.312500 with an error of 4.761%

QUESTION 2:

Given $f(x) = -2x^6 - 1.5x^4 + 10x + 2$
use bisection to determine the *maximum* of this function. Employ initial guesses of $x_l = 0$ and $x_u = 1$, and perform iterations until the approximate relative error falls below 5%.

Solution:

In order to determine the maximum with a root location technique, we must first differentiate the function to yield

$$f'(x) = -12x^5 - 6x^3 + 10$$

The root of this function represents an extremum. Using bisection and the recommended initial guesses gives:

i	x_l	x_u	x_r	$f(x_l)$	$f(x_r)$	$f(x_l)\times f(x_r)$	ε_a
1	0.00000	1.00000	0.50000	10.00000	8.87500	88.75000	
2	0.50000	1.00000	0.75000	8.87500	4.62109	41.01221	33.33%
3	0.75000	1.00000	0.87500	4.62109	-0.17444	-0.80610	14.29%
4	0.75000	0.87500	0.81250	4.62109	2.53263	11.70351	7.69%
5	0.81250	0.87500	0.84375	2.53263	1.26437	3.20217	3.70%

The maximum can be determined by substituting the root into the original equation to give

$$f(0.84375) = -2(0.84375)^6 - 1.5(0.84375)^4 + 10(0.84375) + 2 = 8.956$$

QUESTION 3:

Use the Newton-Raphson method to find the root of

$$f(x) = e^{-0.5x}(4 - x) - 2=0$$

Employ initial guesses of (a) 2, (b) 6, and (c) 8. Explain your results.

Solution:

$$x_{i+1} = x_i - \frac{e^{-0.5x_i}(4 - x_i) - 2}{-e^{-0.5x_i}(3 - 0.5x_i)}$$

(a)

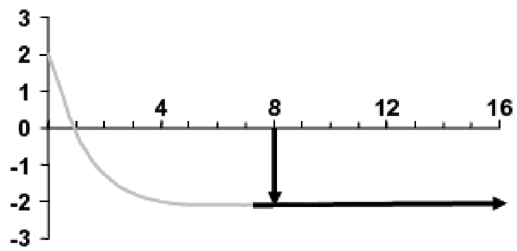
i	x	$f(x)$	$f'(x)$
0	2	-1.26424	-0.73576
1	0.281718	1.229743	-2.48348
2	0.776887	0.18563	-1.77093
3	0.881708	0.006579	-1.64678
4	0.885703	9.13E-06	-1.64221
5	0.885709	1.77E-11	-1.6422
6	0.885709	0	-1.6422

(b) The case does not work because the derivative is zero at $x_0 = 6$.

(c)

i	x	$f(x)$	$f'(x)$
0	8	-2.07326	0.018316
1	121.1963	-2	2.77E-25
2	7.21E+24	-2	0

This guess breaks down because, as depicted in the following plot, the near zero, positive slope sends the method away from the root.



QUESTION 4:

(a) Apply the Newton-Raphson method to the function $f(x) = \tanh(x^2 - 9)$ to evaluate its known real root at $x = 3$. Use an initial guess of $x_0 = 3.2$ and take a minimum of four iterations. **(b)** Did the method exhibit convergence onto its real root? Sketch the plot with the results for each iteration shown.

Solution:

(a)

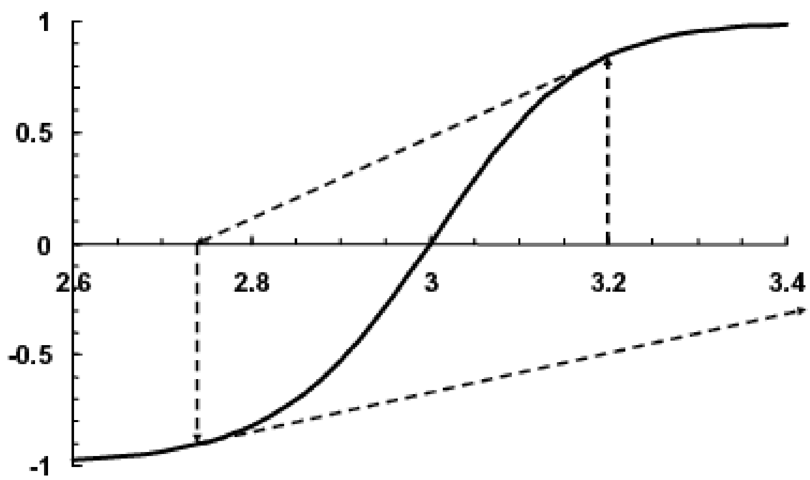
$$x_{i+1} = x_i - \frac{\tanh\left(x_i^2 - 9\right)}{2x_i \operatorname{sech}^2\left(x_i^2 - 9\right)}$$

Using an initial guess of 3.2, the iterations proceed as

iteration	x_i	$f(x_i)$	$f'(x_i)$	ε_a
0	3.2	0.845456	1.825311	
1	2.736816	−0.906910	0.971640	16.924%
2	3.670197	0.999738	0.003844	25.431%
3	−256.413			101.431%

Note that on the fourth iteration, the computation should go unstable.

(b) The solution diverges from its real root of $x = 3$. Due to the concavity of the slope, the next iteration will always diverge. The following graph illustrates how the divergence evolves.



QUESTION 5:

The polynomial $f(x) = 0.0074x^4 - 0.284x^3 + 3.355x^2 - 12.183x + 5$ has a real root between 15 and 20. Apply the Newton-Raphson method to this function using an initial guess of $x_0 = 16.15$. Explain your results.

Solution:

$$x_{i+1} = x_i - \frac{0.0074x_i^4 - 0.284x_i^3 + 3.355x_i^2 - 12.183x_i + 5}{0.0296x_i^3 - 0.852x_i^2 + 6.71x_i - 12.183}$$

Using an initial guess of 16.15, the iterations proceed as

iteration	x_i	$f(x_i)$	$f'(x_i)$	ε_a
0	16.15	−9.57445	−1.35368	
1	9.077102	8.678763	0.662596	77.920%
2	−4.02101	128.6318	−54.864	325.742%
3	−1.67645	36.24995	−25.966	139.852%
4	−0.2804	8.686147	−14.1321	497.887%
5	0.334244	1.292213	−10.0343	183.890%
6	0.463023	0.050416	−9.25584	27.813%
7	0.46847	8.81E−05	−9.22351	1.163%
8	0.46848	2.7E−10	−9.22345	0.002%

As depicted below, the iterations involve regions of the curve that have flat slopes. Hence, the solution is cast far from the roots in the vicinity of the original guess.