Chapter 2. Simultaneous Linear Equations

Consider n simultaneous equations (constraints) with n unknowns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots\cdots\cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots\cdots\cdots + a_{2n}x_n = b_2$$

....

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots\cdots\cdots + a_{nn}x_n = b_n$$

In Matrix notation

$$\begin{bmatrix} a_{11} & a_{12} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n2} & a_{n2} & a_{n3} & \vdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$$

$$[a_{ij}][x_i] = [b_i]$$

$$\mathbf{AX} = \mathbf{B}$$

or

$$A_{n\times n}\, X_{n\times 1} = B_{n\times 1}$$

where i is row number; j is column number

X & **B** are called column vectors

1. Examples

Two equations

$$3x_1 + 2x_2 = 18$$
$$-x_1 + 2x_2 = 2$$

then

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 18 \\ 2 \end{bmatrix}$$

Three equations

$$\begin{aligned} 2x_2 + 5x_3 &= 1 \\ 2x_1 + x_2 + x_3 &= 0 \\ 3x_1 + x_2 &= 5 \end{aligned}$$

then

$$A = \begin{bmatrix} 0 & 2 & 5 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$$

2. Revision on Matrix

Product rule:

$$A = [a_{ik}]; \qquad B = [b_{kj}]; \qquad C = [c_{nl}]$$

if

$$A_{n\times m}\ B_{m\times l} = C_{n\times l}$$

then

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

or

$$\mathbf{C} = \mathbf{AB}$$

eg. $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21}) & (a_{11}b_{12} + a_{12}b_{22}) & (a_{11}b_{13} + a_{12}b_{23}) \\ (a_{21}b_{11} + a_{22}b_{21}) & (a_{21}b_{12} + a_{22}b_{22}) & (a_{21}b_{13} + a_{22}b_{23}) \end{bmatrix}$

Determinants

For matrix

$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

The determinant of A is a value

$$|A| = \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1 A_1 + b_1 B_1 + c_1 C_1$$
$$= a_2 A_2 + b_2 B_2 + c_2 C_2$$
$$= a_3 A_3 + b_3 B_3 + c_3 C_3$$

,

where A_1 is called the cofactor of a_1. The cofactor A_1 equals to $(-1)^{i+j}$ times the **minor of a_1**. Here *i* and *j* are the row and column number of a_1 respectively, and the **minor of a_1** is simply the determinant of a submatrix obtained by striking the *i*-th row and the *j*-th column of A, defining

$$\text{Minor of } a_1 \text{ is } \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = (b_2 c_3 - b_3 c_2)$$

$$\text{Minor of } b_1 \text{ is } \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$$

$$\text{Minor of } c_2 \text{ is } \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

For example: $\Delta = \begin{vmatrix} 0 & 2 & 5 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \end{vmatrix}$

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = a_1 A_1 + a_2 A_2 + a_3 A_3 = 1$$

3. Cramer's rule

Cramer's rule states that the *n* unknowns are given by

$$x_1 = \frac{|C_1|}{|A|} \quad ; \quad x_2 = \frac{|C_2|}{|A|} \quad ; \quad x_3 = \frac{|C_3|}{|A|}$$

For example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 2 & 5 \end{bmatrix} \; ; \; B = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix} \; ; \; |A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 2 & 5 \end{vmatrix} \; ; \; |C_1| = \begin{vmatrix} 4 & 2 & 3 \\ 5 & 3 & 4 \\ 1 & 2 & 5 \end{vmatrix}$$

$$|C_2| = \begin{vmatrix} 1 & 4 & 3 \\ 2 & 5 & 4 \\ 4 & 1 & 5 \end{vmatrix} \; ; \; |C_3| = \begin{vmatrix} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 4 & 2 & 1 \end{vmatrix}$$

|A| is the determinant of **A** and |A| is not 0; C_1 , C_2 and C_3 are obtained by replacing the first, second and third column of matrix **A**, respectively.

Example 1:

$$3x_1 + 2x_2 = 18$$

$$-x_1 + 2x_2 = 2$$

Solution

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix} \; ; \; B = \begin{bmatrix} 18 \\ 2 \end{bmatrix} \; ; \; |A| = \begin{vmatrix} 3 & 2 \\ -1 & 2 \end{vmatrix} = 3(2) - 2(-1) = 8$$

$$|C_1| = \begin{vmatrix} 18 & 2 \\ 2 & 2 \end{vmatrix} \; ; \; |C_2| = \begin{vmatrix} 3 & 18 \\ -1 & 2 \end{vmatrix}$$

$$x_1 = \frac{|C_1|}{|A|} = \frac{32}{8} = 4 \quad ; \quad x_2 = \frac{|C_2|}{|A|} = \frac{24}{8} = 3$$

4. Gaussian elimination method

For equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots\dots\dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots\dots\dots + a_{2n}x_n = b_2$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots\dots\dots + a_{nn}x_n = b_n$$

the first equation times a_{21}/a_{11} is subtracted from the second equation to eliminate the first term of the second equation and likewise for other equations, resulting in:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots\dots\dots + a_{1n}x_n = b_1$$

$$0 \quad + a'_{22}x_2 + a'_{23}x_3 + \dots\dots\dots + a'_{2n}x_n = b'_2$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$0 \quad + a'_{n2}x_2 + a'_{n3}x_3 + \dots\dots\dots + a'_{nn}x_n = b'_n$$

where

$$a'_{ij} = a_{ij} - (a_{i1}/a_{11})a_{1j}$$

with

the first equation UNCHANGED

Next, the second term of every equation with *i*>2 is eliminated by subtracting the second equation times

$$a'_{i2}/a'_{22}$$

And so on for the rest of the equations using similar method. Finally we have

$$a_{11}^{1}x_1 + a_{11}^{1}x_2 + a_{11}^{1}x_3 + \dots\dots\dots + a_{1n}^{1}x_n = b_1^{1}$$

$$0 \quad + a_{22}^{2}x_2 + a_{23}^{2}x_3 + \dots\dots\dots + a_{2n}^{2}x_n = b_2^{2}$$

$$\dots\dots\dots\dots \; \dots\dots\dots\dots \; \dots\dots\dots\dots \; \dots\dots\dots\dots \; \dots\dots\dots\dots \; \dots\dots\dots$$

$$0 \quad + 0 \quad + 0 \quad + \dots\dots\dots + a_{nn}^{n}x_n = b_n^{n}$$

In matrix notation, an upper-triangular matrix is formed

$$\begin{bmatrix} a_{11}^{1} & a_{11}^{1} & a_{11}^{1} & \dots & a_{11}^{1} \\ 0 & a_{22}^{2} & a_{23}^{2} & \dots & a_{2n}^{2} \\ : & : & : & ::: & : \\ : & : & : & ::: & : \\ 0 & 0 & 0 & ::: & a_{nn}^{n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ : \\ : \\ x_n \end{bmatrix} = \begin{bmatrix} b_1^{1} \\ b_2^{2} \\ : \\ : \\ b_n^{n} \end{bmatrix}$$

The unknowns are then obtained by reverse the order

$$x_n = b_n^{n} \Big/ a_{nn}^{n}$$

$$x_i = \left(b_i^{i} - \sum_{j=i+1}^{n} a_{ij}^{i} x_j \right) \Big/ a_{ii}^{i}$$

Example 2:

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$

$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

Eliminate x_1 of second and third equation

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

$$0 \quad + 7.003x_2 - 0.293x_3 = -19.56$$

$$0 \quad - 0.190x_2 + 10.02x_3 = 70.615$$

Eliminate x_2 of third equation

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

$$0 \quad + 7.003x_2 - 0.293x_3 = -19.56$$

$$0 \quad + 0 \quad + 10.012x_3 = 70.0843$$

The unknowns are obtained

$$x_3 = 7$$
$$x_2 = -2.5$$
$$x_1 = 3$$

Solution:

Using Matrix notation

$$10x_1 + 2x_2 - x_3 = 27$$

$$-3x_1 - 6x_2 + 2x_3 = -61.5$$

$$x_1 + x_2 + 5x_3 = -21.5$$

$$A = \begin{bmatrix} 10 & 2 & -1 \\ -3 & -6 & 2 \\ 1 & 1 & 5 \end{bmatrix} \quad ; \quad B = \begin{bmatrix} 27 \\ -61.5 \\ -21.5 \end{bmatrix}$$

The system is first expressed as a matrix, called **augment matrix**:

$$Aug = \begin{bmatrix} 10 & 2 & -1 & 27 \\ -3 & -6 & 2 & -61.5 \\ 1 & 1 & 5 & -21.5 \end{bmatrix}$$

Using **forward elimination**:

a_{21} is eliminated by multiplying row 1 by –3/10 and subtracting the result from row 2.

a_{31} is eliminated by multiplying row 1 by 1/10 and subtracting the result from row 3.

$$Aug = \begin{bmatrix} 10 & 2 & -1 & 27 \\ 0 & -5.4 & 1.7 & -53.4 \\ 0 & 0.8 & 5.1 & -24.2 \end{bmatrix}$$

a_{32} is eliminated by multiplying row 2 by 0.8/(–5.4) and subtracting the result from row 3.

$$Aug = \begin{bmatrix} 10 & 2 & -1 & 27 \\ 0 & -5.4 & 1.7 & -53.4 \\ 0 & 0 & 5.351852 & -32.1111 \end{bmatrix}$$

Back substitution:

$$x_3 = \frac{-32.1111}{5.351852} = -6$$

$$x_2 = \frac{-53.4 - 1.7(-6)}{-5.4} = 8$$

$$x_1 = \frac{27 - (-1)(-6) - 2(8)}{10} = 0.5$$

5. Other Elimination Methods

(a) After performing the Gaussian elimination (forward elimination) the same method is applied backward (called backward elimination or **Gaussian - Jordan elimination**). The Gaussian-Jordon reduces all the off-diagonal to zero. The results can be obtained without the need for backward substitution.

$$\begin{bmatrix} a_{11}^{1} & 0 & 0 & \dots & 0 \\ 0 & a_{22}^{2} & 0 & \dots & 0 \\ 0 & 0 & a_{kk}^{k} & ::: & 0 \\ : & : & : & ::: & 0 \\ 0 & 0 & 0 & ::: & a_{nn}^{n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_k \\ : \\ x_n \end{bmatrix} = \begin{bmatrix} b_1^1 \\ b_2^2 \\ : \\ : \\ b_n^n \end{bmatrix}$$

Examples 3:

Gaussian-Jordon method in solving equations

$$2x_1 + x_2 - x_3 = 1$$

$$5x_1 + 2x_2 + 2x_3 = -4$$

$$3x_1 + x_2 + x_3 = 5$$

Solution:

$$Aug = \begin{bmatrix} 2 & 1 & -1 & 1 \\ 5 & 2 & 2 & -4 \\ 3 & 1 & 1 & 5 \end{bmatrix}$$

Normalize the first row as before

$$Aug = \begin{bmatrix} 1 & 0.5 & -0.5 & 0.5 \\ 0 & -0.5 & 4.5 & -6.5 \\ 0 & -0.5 & 2.5 & 3.5 \end{bmatrix}$$

finally the augment matrix becomes:

$$Aug = \begin{bmatrix} 1 & 0 & 0 & 14 \\ 0 & 1 & 0 & -32 \\ 0 & 0 & 1 & -5 \end{bmatrix}$$

(b) Gaussian elimination does not work if the first coefficient of the first equation is zero. *The pivot element* a_{kk}^k *should not be zero.* **Pivoting** is used to change the sequential order of the equations to prevent diagonal coefficient from becoming zero and to make the diagonal coefficient larger than any other coefficient below it. **The row containing the largest pivot element can be chosen as pivot row.** Usually all rows are normalised by dividing with their pivoting element. The method is called **Gaussian elimination with pivoting**.

Example 4:

Gaussian elimination with pivoting in solving

$$8x_1 + 2x_2 - 2x_3 = -2$$

$$10x_1 + 2x_2 + 4x_3 = 4$$

$$12x_1 + 2x_2 + 2x_3 = 6$$

Solution:

change $Aug = \begin{bmatrix} 8 & 2 & -2 & -2 \\ 10 & 2 & 4 & 4 \\ 12 & 2 & 2 & 6 \end{bmatrix}$ to

$$\begin{bmatrix} 12 & 2 & 2 & 6 \\ 10 & 2 & 4 & 4 \\ 8 & 2 & -2 & -2 \end{bmatrix}$$

and perform the **naive Gaussian elimination**.

6. LU Decomposition method

In the elimination methods, we have seen that it is necessary to solve an upper triangular matrix system. Suppose that Gaussian elimination, without row interchanges (pivoting), can be performed successfully to solve the linear equations **AX=B**. Then the matrix **A** can be factored as the product of a lower-triangular matrix **L** and an upper-triangular matrix **U:**

(a) Decomposition **A=LU**

Furthermore, **L** can be constructed to have 1's on its diagonal and **U** will have nonzero diagonal elements.

After finding **L** and **U**, the solution **X** is computed in two steps

(b) Solve **LD=B** for **D** using forward substitution;

(c) Solve **UX=D** for **X** using back substitution.

LU decomposition based on the Gaussian elimination:

Assume

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The first step in Gaussian elimination is to multiply row 1 of **A** by the factor

$$f_{21} = \frac{a_{21}}{a_{11}}$$

and subtract the result from the second row to eliminate a_{21}. To eliminate a_{31}, row 1 is multiplied by

$$f_{31} = \frac{a_{31}}{a_{11}}$$

and the results are subtracted to the third row. Then **A** becomes

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & a'_{32} & a'_{33} \end{bmatrix},$$

The final step is to multiply the modified second row by

$$f_{32} = \frac{a'_{32}}{a'_{22}}$$

and subtract the result from the third row to eliminate a'_{32}. Finally **A** becomes

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix}$$

The L and U can be constructed based on these Gaussian elimination steps:

Let $U = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix}$ and $L = \begin{bmatrix} 1 & 0 & 0 \\ f_{21} & 1 & 0 \\ f_{31} & f_{32} & 1 \end{bmatrix}$

Then **A=LU**

Example 5: LU decomposition of matrix $A = \begin{bmatrix} 10 & 2 & -1 \\ -3 & -6 & 2 \\ 1 & 1 & 5 \end{bmatrix}$

Using Gaussian elimination, A can be changed into an upper-triangular matrix

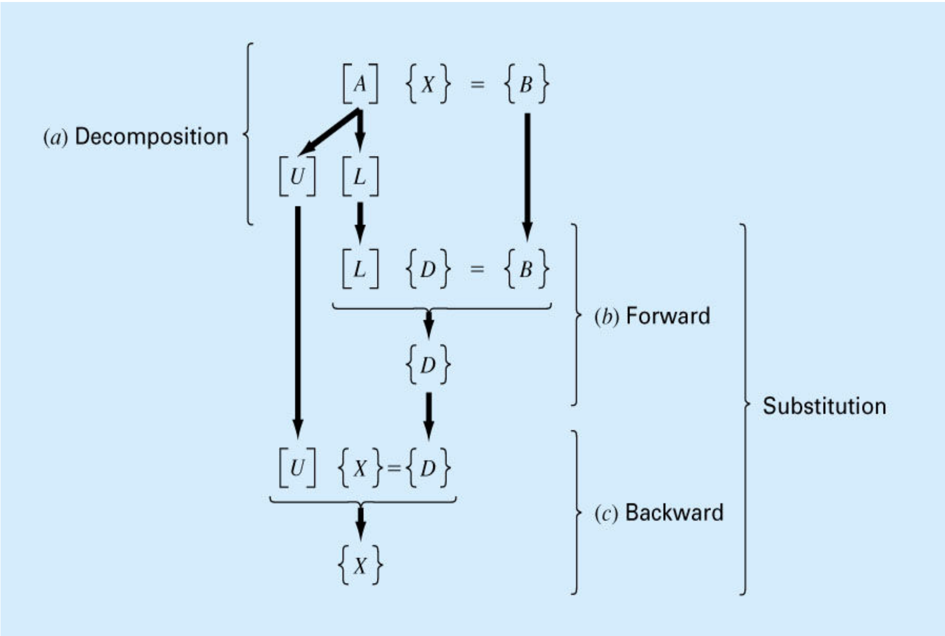
$$U = \begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.351852 \end{bmatrix}$$

Using f_{21} =–3/10; f_{31} =1/10; and f_{32} =0.8/(–5.4).

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.148148 & 1 \end{bmatrix}$$

We can confirm that $LU = \begin{bmatrix} 10 & 2 & -1 \\ -3 & -6 & 2 \\ 1 & 1 & 5 \end{bmatrix} = A$

Using LU decomposition, the linear equation can be solved following the procedure (a)-(c) as follows



Example 6: Solve equations

$$10x_1 + 2x_2 - x_3 = 27$$

$$-3x_1 - 6x_2 + 2x_3 = -61.5 \qquad \text{Or AX=B with } A = \begin{bmatrix} 10 & 2 & -1 \\ -3 & -6 & 2 \\ 1 & 1 & 5 \end{bmatrix} \; ; \; B = \begin{bmatrix} 27 \\ -61.5 \\ -21.5 \end{bmatrix}$$

$$x_1 + x_2 + 5x_3 = -21.5$$

using LU decomposition

Solution:

<u>*Step (a)*</u>*:* Decomposition **A=LU**

$$U = \begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.351852 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.148148 & 1 \end{bmatrix}$$

<u>*Step (b)*</u>*:* Find matrix **D** in the linear equation **LD=B**

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.148148 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 27 \\ -61.5 \\ -21.5 \end{bmatrix}$$

by forward substitution that **D**=[27 -53.4 -32.111]$^{\mathbf{T}}$.

<u>*Step (c)*</u>*:* Find matrix **X** in the linear equation **UX=D**

$$\begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.351852 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 27 \\ -53.4 \\ -32.111 \end{bmatrix}$$

by backward substitution that **X**=[0.5 8 -6]$^{\mathbf{T}}$.

7. Gauss-Seidel method

Iterative or approximate methods provide an alternative to the elimination methods. The Gauss-Seidel method is the most commonly used iterative method in solving linear equations.

The system ***AX***=***B*** is reshaped by solving the first equation for x_1, the second equation for x_2, and the third for x_3, …and nth equation for x_n. For conciseness, we will limit ourselves to a *3-by-3* set of equations.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 = b_3$$

, and

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$

$$x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}}$$

$$x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$$

Now we can start the solution process by choosing guesses for the x's. A simple way to obtain initial guesses is to assume that they are zero. These zeros can be substituted into x_1 equation to calculate a new $x_1=b_1/a_{11}$.

New x_1 is substituted to calculate x_2 and x_3. The procedure is repeated until the convergence criterion is satisfied:

$$\left|\varepsilon_{a,i}\right| = \left|\frac{x_i^j - x_i^{j-1}}{x_i^j}\right| \times 100\% < \varepsilon_s$$

for all unknowns x_i (i=1, 2, 3),where *j-th* and *(j-1)-th* are the present and previous iterations; ε_s is the prescribe error.

As each new x value is computed for the Gauss-Seidel method in the *j-th* iteration, it is immediately used in the next equation to determine another *x* value, until all the unknowns **X** are determined in the *j-th* iteration. Alternatively, in the *j-th* iteration all unknowns **X** can be computed on the basis of a set of old unknowns obtained in the *(j-1)-th* iteration. As shown in Figure 1(b), this alternative approach in solving the linear equations is called Jacobi method.

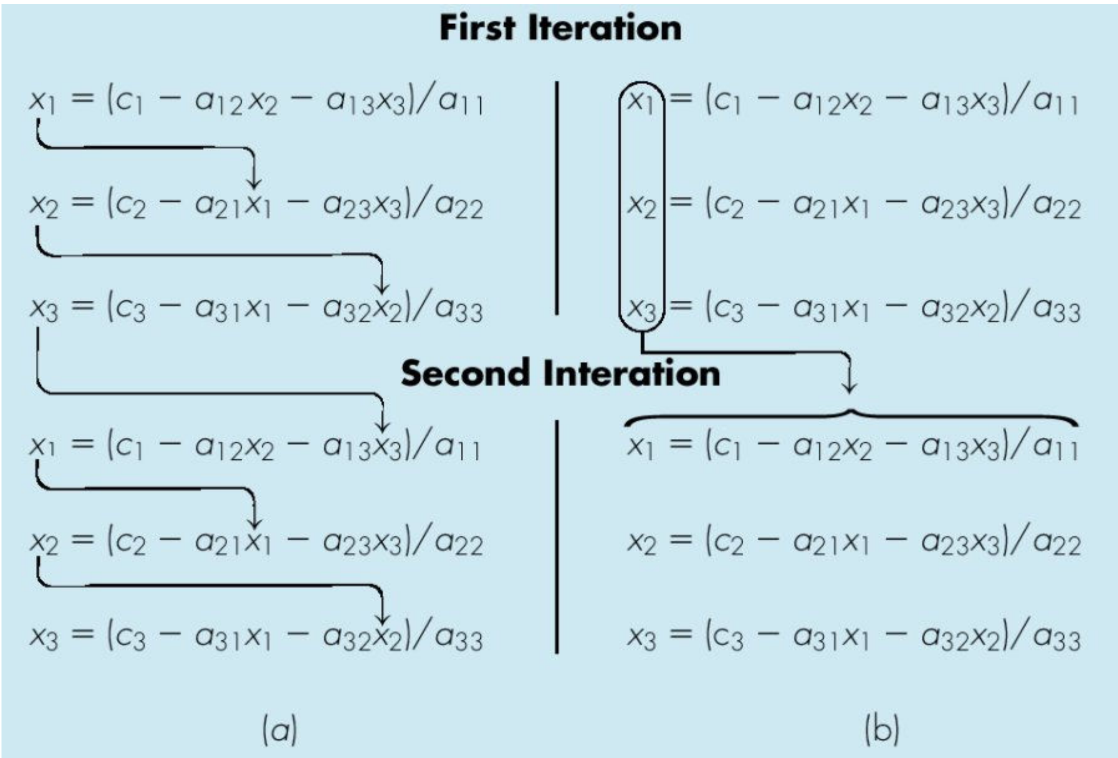


Figure 1. Differences in Gauss-Seidel method (a) and Jacobi method (b)

Convergence Criterion for Gauss-Seidel Method

The Gauss-Seidel method has two fundamental problems as any iterative method:

- It is sometimes nonconvergent, and
- If it converges, converges very slowly.

Using a 2-by-2 linear equation as an example:

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2 \quad ,$$

we define two linear functions shown as straight lines in Figure 2.

$$u(x_1, x_2) = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2$$

$$v(x_1, x_2) = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1$$

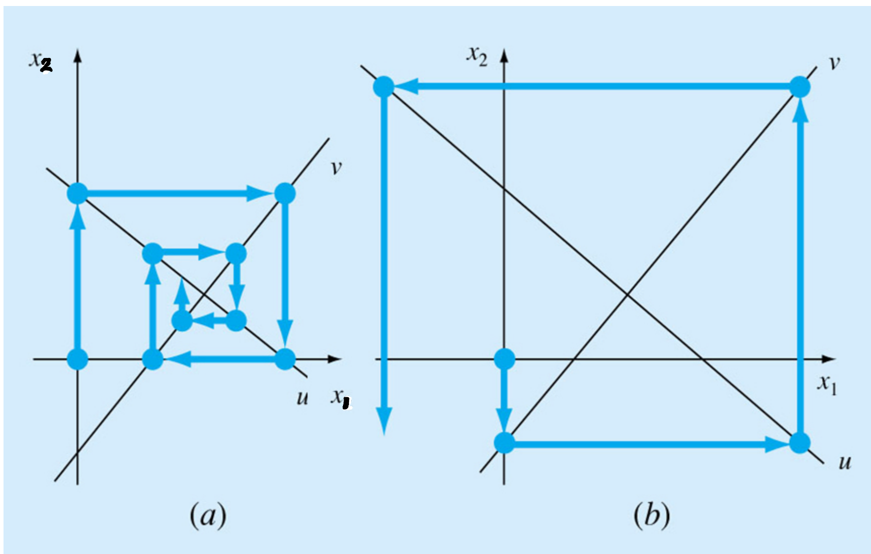


Figure 2. Convergence (a) and divergence (b) of Gauss-Seidel method

For the Gauss-Seidel method, the sufficient conditions for convergence of two linear equations is the relations related with *u(x,y)* and *v(x,y)*, as follows:

$$\left|\frac{\partial u}{\partial x}\right| + \left|\frac{\partial u}{\partial y}\right| < 1$$

$$\left|\frac{\partial v}{\partial x}\right| + \left|\frac{\partial v}{\partial y}\right| < 1$$

$$\frac{\partial u}{\partial x_1} = 0 \qquad \frac{\partial u}{\partial x_2} = -\frac{a_{12}}{a_{11}}$$

$$\frac{\partial v}{\partial x_1} = -\frac{a_{21}}{a_{22}} \qquad \frac{\partial v}{\partial x_2} = 0$$

Therefore, the convergence criteria for the Gauss-Seidel method is

$$\left|\frac{a_{12}}{a_{11}}\right| < 1 \quad and \qquad \left|\frac{a_{21}}{a_{22}}\right| < 1$$

In other words, the absolute values of the slopes of the lines shown in Figure 2 must be less than unity for convergence:

$$|a_{11}| > |a_{12}|$$
$$|a_{22}| > |a_{21}|$$

For n-by-n linear equations with **A**=[a_{ij}], the sufficient conditions for convergence of Gauss-Seidel method is expressed for all rows of **A**

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{i,j}|$$

where a_{ii} are the diagonal elements of **A,** j is the index of column of **A**. These convergence conditions can be simply stated that **A** should be diagonally dominant.

Example 7:

Use the Gauss-Seidel method to solve the following system until the percent relative error falls below $\varepsilon_s = 5\%$,

$$10x_1 + 2x_2 - x_3 = 27$$
$$-3x_1 - 6x_2 + 2x_3 = -61.5$$
$$x_1 + x_2 + 5x_3 = -21.5$$

<u>Solution:</u>

The first iteration can be implemented as

$$x_1 = \frac{27 - 2x_2 + x_3}{10} = \frac{27 - 2(0) + 0}{10} = 2.7$$

$$x_2 = \frac{-61.5 + 3x_1 - 2x_3}{-6} = \frac{-61.5 + 3(2.7) - 2(0)}{-6} = 8.9$$

$$x_3 = \frac{-21.5 - x_1 - x_2}{5} = \frac{-21.5 - (2.7) - 8.9}{5} = -6.62$$

Second iteration:

$$x_1 = \frac{27 - 2(8.9) - 6.62}{10} = 0.258$$

$$x_2 = \frac{-61.5 + 3(0.258) - 2(-6.62)}{-6} = 7.914333$$

$$x_3 = \frac{-21.5 - (0.258) - 7.914333}{5} = -5.934467$$

The error estimates can be computed as

$$\varepsilon_{a,1} = \left|\frac{0.258 - 2.7}{0.258}\right| \times 100\% = 947\%$$

$$\varepsilon_{a,2} = \left|\frac{7.914333 - 8.9}{7.914333}\right| \times 100\% = 12.45\%$$

$$\varepsilon_{a,3} = \left|\frac{-5.934467 - (-6.62)}{-5.934467}\right| \times 100\% = 11.55\%$$

The remainder of the calculation proceeds until all the errors fall below the stopping criterion of 5%. The entire computation can be summarized as

iteration	unknown	value	ε_a	maximum ε_a
1	x_1	2.7	100.00%	
	x_2	8.9	100.00%	
	x_3	-6.62	100.00%	100%
2	x_1	0.258	946.51%	
	x_2	7.914333	12.45%	
	x_3	-5.93447	11.55%	946%
3	x_1	0.523687	50.73%	
	x_2	8.010001	1.19%	
	x_3	-6.00674	1.20%	50.73%
4	x_1	0.497326	5.30%	
	x_2	7.999091	0.14%	
	x_3	-5.99928	0.12%	5.30%
5	x_1	0.500253	0.59%	
	x_2	8.000112	0.01%	
	x_3	-6.00007	0.01%	0.59%

Thus, after 5 iterations, the maximum error is 0.59% and we arrive at the result: x_1 = 0.500253, x_2 =8.000112 and x_3 = 6.00007. Note that the exact result is [0.5 8 6].

Example 8:

Use the Gauss-Seidel method to solve the following system until the percent relative error falls below $\varepsilon_s = 5\%$. Rearrange the equations to achieve convergence

$$-3x_1 + x_2 + 12x_3 = 50$$

$$6x_1 - x_2 - x_3 = 3$$

$$6x_1 + 9x_2 + x_3 = 40$$

Solution:

The equations should first be rearranged so that they are diagonally dominant,

$$6x_1 - x_2 - x_3 = 3$$

$$6x_1 + 9x_2 + x_3 = 40$$

$$-3x_1 + x_2 + 12x_3 = 50$$

Each can be solved for the unknown on the diagonal as

$$x_1 = \frac{3 + x_2 + x_3}{6}$$

$$x_2 = \frac{40 - 6x_1 - x_3}{9}$$

$$x_3 = \frac{50 + 3x_1 - x_2}{12}$$

(a) The first iteration can be implemented as

$$x_1 = \frac{3 + 0 + 0}{6} = 0.5$$

$$x_2 = \frac{40 - 6(0.5) - 0}{9} = 4.11111$$

$$x_3 = \frac{50 + 3(0.5) - 4.11111}{12} = 3.949074$$

(b) Second iteration:

$$x_1 = \frac{3 + 4.11111 + 3.949074}{6} = 1.843364$$

$$x_2 = \frac{40 - 6(1.843364) - 3.949074}{9} = 2.776749$$

$$x_3 = \frac{50 + 3(1.843364) - 2.776749}{12} = 4.396112$$

The error estimates can be computed as

$$\varepsilon_{a,1} = \left| \frac{1.843364 - 0.5}{1.843364} \right| \times 100\% = 72.88\%$$

$$\varepsilon_{a,2} = \left| \frac{2.776749 - 4.11111}{2.776749} \right| \times 100\% = 48.05\%$$

$$\varepsilon_{a,3} = \left| \frac{4.396112 - 3.949074}{4.396112} \right| \times 100\% = 10.17\%$$

The remainder of the calculation proceeds until all the errors fall below the stopping criterion of 5%. The entire computation can be summarized as

iteration	unknown	value	ε_a	maximum ε_a
1	x_1	0.5	100.00%	
	x_2	4.111111	100.00%	
	x_3	3.949074	100.00%	100.00%
2	x_1	1.843364	72.88%	
	x_2	2.776749	48.05%	
	x_3	4.396112	10.17%	72.88%
3	x_1	1.695477	8.72%	
	x_2	2.82567	1.73%	
	x_3	4.355063	0.94%	8.72%
4	x_1	1.696789	0.08%	
	x_2	2.829356	0.13%	
	x_3	4.355084	0.00%	0.13%

Thus, after 4 iterations, the maximum error is 0.13% and we arrive at the result: $x_1 = 1.696789$, $x_2 = 2.829356$ and $x_3 = 4.355084$.