Solution:

Q1:

1. Using naive Gaussian elimination

3×3 **Matrix**

$$\begin{aligned} 2x_1 + x_2 - x_3 &= -1 \\ x_1 + 3x_2 + 2x_3 &= 13 \\ x_1 - x_2 + 4x_3 &= 11 \end{aligned}$$

Eliminate x_1 **of second and third equation**

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & -1 \\ 1 & 3 & 2 & 13 \\ 1 & -1 & 4 & 11 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 2 & 1 & -1 & -1 \\ 0 & 5/2 & 5/2 & 27/2 \\ 0 & -3/2 & 9/2 & 23/2 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 2 & 1 & -1 & -1 \\ 0 & 5 & 5 & 27 \\ 0 & -3 & 9 & 23 \end{array}\right)$$

Eliminate x_2 **of third equation**

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & -1 \\ 0 & 5 & 5 & 27 \\ 0 & -3 & 9 & 23 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 2 & 1 & -1 & -1 \\ 0 & 5 & 5 & 27 \\ 0 & 0 & 12 & 196/5 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 2 & 1 & -1 & -1 \\ 0 & 5 & 5 & 27 \\ 0 & 0 & 60 & 196 \end{array}\right)$$

After elimination, we have:

$$\begin{aligned} 2x_1 + x_2 - x_3 &= -1 \\ 5x_2 + 5x_3 &= 27 \\ 60x_3 &= 196 \end{aligned}$$

So,

$$x_3 = \frac{196}{60} = \frac{49}{15} \approx 3.2667$$

$$x_2 = (27 - 5\times\frac{49}{15})/5 = \frac{32}{15} \approx 2.1333$$

$$x_1 = (-1 + \frac{49}{15} - \frac{32}{15})/2 = \frac{1}{15} \approx 0.0667$$

Q2:

Use Gaussian elimination

$$A = \begin{bmatrix} 1 & 2 & 4 & 1 \\ 2 & 8 & 6 & 4 \\ 3 & 10 & 8 & 8 \\ 4 & 2 & 10 & 6 \end{bmatrix}$$

is changed into upper triangular matrix

$$U = \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & 4 & -2 & 2 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & -6 \end{bmatrix}$$ with factors 2, 3, 1, 4, 1, 2 for eliminations in each row.

Therefore the lower triangular matrix is

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 4 & 1 & 2 & 1 \end{bmatrix}.$$

For equation LD=B, the solution D is $D = [21 \quad 10 \quad 6 \quad -24]^T$.

For equation UX=D, the solution X is $X = [1 \quad 2 \quad 3 \quad 4]^T$.

Q3:

Using Gaussian-Jordan elimination

The system is first expressed as an augmented matrix:

$$\begin{bmatrix} 1 & 1 & -1 & -3 \\ 6 & 2 & 2 & 2 \\ -3 & 4 & 1 & 1 \end{bmatrix}$$

Normalize the first row, and then eliminate a_{21} and a_{31},

$$\begin{bmatrix} 1 & 1 & -1 & -3 \\ 0 & -4 & 8 & 20 \\ 0 & 7 & -2 & -8 \end{bmatrix}$$

Normalize the second row and eliminate a_{12} and a_{32},

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 12 & 27 \end{bmatrix}$$

Normalize the third row and eliminate a_{13} and a_{23},

$$\begin{bmatrix} 1 & 0 & 0 & -0.25 \\ 0 & 1 & 0 & -0.5 \\ 0 & 0 & 1 & 2.25 \end{bmatrix}$$

The last column is the solution.

Q4:

The equations must first be rearranged so that they are diagonally dominant

$$-8x_1 + x_2 - 2x_3 = -20$$

$$2x_1 - 6x_2 - x_3 = -38$$

$$-3x_1 - x_2 + 7x_3 = -34$$

The first iteration can be implemented as

$$x_1 = \frac{-20 - x_2 + 2x_3}{-8} = \frac{-20 - 0 + 2(0)}{-8} = 2.5$$

$$x_2 = \frac{-38 - 2x_1 + x_3}{-6} = \frac{-38 - 2(2.5) + 0}{-6} = 7.166667$$

$$x_3 = \frac{-34 + 3x_1 + x_2}{7} = \frac{-34 + 3(2.5) + 7.166667}{7} = -2.761905$$

Second iteration:

$$x_1 = \frac{-20 - 7.166667 + 2(-2.761905)}{-8} = 4.08631$$

$$x_2 = \frac{-38 - 2x_1 + x_3}{-6} = \frac{-38 - 2(4.08631) + (-2.761905)}{-6} = 8.155754$$

$$x_3 = \frac{-34 + 3x_1 + x_2}{7} = \frac{-34 + 3(4.08631) + 8.155754}{7} = -1.94076$$

The error estimates can be computed as

$$\varepsilon_{a,1} = \left|\frac{4.08631 - 2.5}{4.08631}\right| \times 100\% = 38.82\%$$

$$\varepsilon_{a,2} = \left|\frac{8.155754 - 7.166667}{8.155754}\right| \times 100\% = 12.13\%$$

$$\varepsilon_{a,3} = \left|\frac{-1.94076 - (-2.761905)}{-1.94076}\right| \times 100\% = 42.31\%$$

The remainder of the calculation proceeds until all the errors fall below the stopping criterion of 5%. The entire computation can be summarized as

iteration	unknown	value	ε_a	maximum ε_a
0	x_1	0		
	x_2	0		
	x_3	0		
1	x_1	2.5	100.00%	
	x_2	7.166667	100.00%	
	x_3	-2.7619	100.00%	100.00%
2	x_1	4.08631	38.82%	
	x_2	8.155754	12.13%	
	x_3	-1.94076	42.31%	42.31%
3	x_1	4.004659	2.04%	
	x_2	7.99168	2.05%	
	x_3	-1.99919	2.92%	2.92%

Thus, after 3 iterations, the maximum error is 2.92% and we arrive at the result: x_1 = 4.004659, x_2 = 7.99168 and x_3 = −1.99919.