ME46002 Chapter 4: Tutorial questions

Question 1:

For a set of four data points

i	0	1	2	3
x_i	1	2.2	2.8	3.5
$f(x_i)$	1	2.4	3.2	4.1

(a) Employ Newton's divided difference interpolating polynomial $f_3(x)$ to fit the data

$$f_3(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2).$$

Find the coefficients of $f_3(x)$.

(b) Least-square regression is employed to fit the data

$$f(x)=a_0+ a_1x.$$

Find the fitting coefficients a_0 and a_1.

Solution:

(a)

The interpolating polynomial is written as

$$f_3(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$$

The divided differences can be computed as:

i	x_i	y_i	$g[x_{i+1},x_i]$	$g[x_{i+2},x_{i+1},x_i]$	$g[x_{i+3},x_{i+2},x_{i+1},x_i]$
0	1	1	1.1667	0.0926	-0.0517
1	2.2	2.4			
2	2.8	3.2			
3	3.5	4.1			

The coefficients of the interpolating polynomial are

$a_0=1$

$a_1=1.1667$

$a_2=0.0926$

$a_3=-0.0517$

(b)
$f(x) = a_0 + a_1 x$ is the fitting function

$$e = \sum_{i=1}^{n} [f(x_i) - y_i]^2, \qquad \frac{\partial e}{\partial a_0} = 0, \qquad \frac{\partial e}{\partial a_1} = 0$$

Resulting in

$$\sum_{i=1}^{n} y_i - na_0 - a_1(\sum_{i=1}^{n} x_i) = 0$$

$$\sum_{i=1}^{n} (x_i y_i) - a_0 \sum_{i=1}^{n} x_i - a_1 \sum_{i=1}^{n} x_i^{\ 2} = 0$$

Solving the equation we have

$$a_0 = (\sum^{n} y_i \sum^{n} x_i^2 - \sum^{n} x_i \sum^{n} x_i y_i) / (n \sum^{n} x_i^2 - (\sum^{n} x_i)^2)$$

=-0.2713

$$a_1 = (n \sum^{n} x_i y_i - \sum^{n} y_i \sum^{n} x_i) / (n \sum^{n} x_i^2 - (\sum^{n} x_i)^2)$$

=1.2405

Question 2:
For a set of 4 data points, employ the Newton's divided difference interpolating polynomial to fit the data. Determine the coefficients of the interpolating formulas.

i	0	1	2	3
x_i	1.5	2.5	3.5	4.5
y_i	3	5	3	7

Answers:

$$f_3(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$$

a_0=3
a_1=2
a_2=-2
a_3=1.667

Question 3:
For a set of 4 data points, employ the linear least-square regression to fit the data points

i	0	1	2	3
x_i	1.5	2.5	3.5	4.5
y_i	3	5	7.2	8.6

Answers:

$f(x) = a_0 + a_1 x$ is the fitting function

$$a_0 = (\sum_{=}^{n} y_i \sum_{=}^{n} x_i^2 - \sum_{=}^{n} x_i \sum_{=}^{n} x_i y_i) / (n\sum_{=}^{n} x_i^2 - (\sum_{=}^{n} x_i)^2)$$

=-0.25

$$a_1 = (n\sum_{=}^{n} x_i y_i - \sum_{=}^{n} y_i \sum_{=}^{n} x_i) / (n\sum_{=}^{n} x_i^2 - (\sum_{=}^{n} x_i)^2)$$

=1.9

Question 4:
Answer the following questions

(a) For a set of three data points (x_i, y_i), $i = 0, 1, 2$, with $x_i = -2, 0, 2$ and $y_i = -6, 0, 6$, if the data are fitted with cubic splines $f(x) = a + bx + cx^2 + dx^3$ with the constrains that f ' (-2) = 11; f ' (2) = 11, determine the cubic functions.

(b) Are the following functions cubic splines? Why?

$1 \le x \le 2$: $f_1(x) = 11 - 24x + 18x^2 - 4x^3$

$2 \le x \le 3$: $f_2(x) = -54 + 72x - 30x^2 + 4x^3$

Solution:

(a)

$$f_1(x) = a_1 + b_1 x + c_1 x^2 + d_1 x^3$$
$$f_2(x) = a_2 + b_2 x + c_2 x^2 + d_2 x^3$$

Equations can be generated to determine 8 unknowns in the cubic splines.

At interior points

$$f_1(x_1) = 0$$
$$f_2(x_1) = 0$$

At two endpoints

$$f_1(x_0) = -6$$

$$f_2(x_2) = 6$$

First derivatives are continuous at the interior point $x_1=0$

$$f_1'(x_1) = f_2'(x_1)$$

Second derivatives are continuous at the interior point $x_1=0$

$f_1''(x_1) = f_2''(x_1)$,

At the endpoints the constrains are given by

$f_1'(-2) = 11$; $f_2'(2) = 11$,

which leads to two equations

Solving the 8 equations we have

$a_1=a_2=0$

$b_1=b_2=-1$

$c_1=c_2=0$

$d_1=d_2=1$

therefore

$$f_1(x) = -x + x^3$$

$$f_2(x) = -x + x^3$$

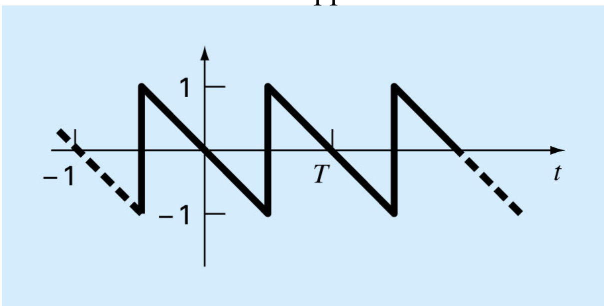
(b)

No.

Because $f_1(x_1) \neq f_2(x_1)$

Question 5:

Use the Continuous Fourier Series to approximate the saw-tooth form functions



Solution:

$a_0 = 0$

$$a_k = \frac{2}{T^2}\int_{-T/2}^{T/2} -2t\cos(k\omega_0 t)\,dt$$

$$= -\frac{4}{T^2}\left[\frac{1}{(k\omega_0)^2}\cos(k\omega_0 t) + \frac{t}{k\omega_0}\sin(k\omega_0 t)\right]_{-T/2}^{T/2}$$

$$b_k = \frac{2}{T^2}\int_{-T/2}^{T/2} -2t\sin(k\omega_0 t)\,dt$$

$$= -\frac{4}{T^2}\left[\frac{1}{(k\omega_0)^2}\sin(k\omega_0 t) - \frac{t}{k\omega_0}\cos(k\omega_0 t)\right]_{-T/2}^{T/2}$$

On the basis of these, all a's = 0. For k = odd,

$$b_k = \frac{-2}{k\pi}$$

For k = even,

$$b_k = \frac{2}{k\pi}$$

Therefore, the series is

$$f(t) = -\frac{2}{\pi}\sin(\omega_0 t) + \frac{1}{\pi}\sin(2\omega_0 t) - \frac{2}{3\pi}\sin(3\omega_0 t) + \frac{1}{2\pi}\sin(4\omega_0 t) + \cdots$$

The first 4 terms are plotted below along with the summation: