Chapter 1. Computer Solution of Nonlinear Equations

Numerical methods

Finding the roots of the equation *f(x)=0* by numerical method

Two basic methods for finding the roots (zeros of a constraint),

1. The **Bisection method** (**Closed or Bracketing method**) and
2. The **Newton-Raphson method** (**Open method**).

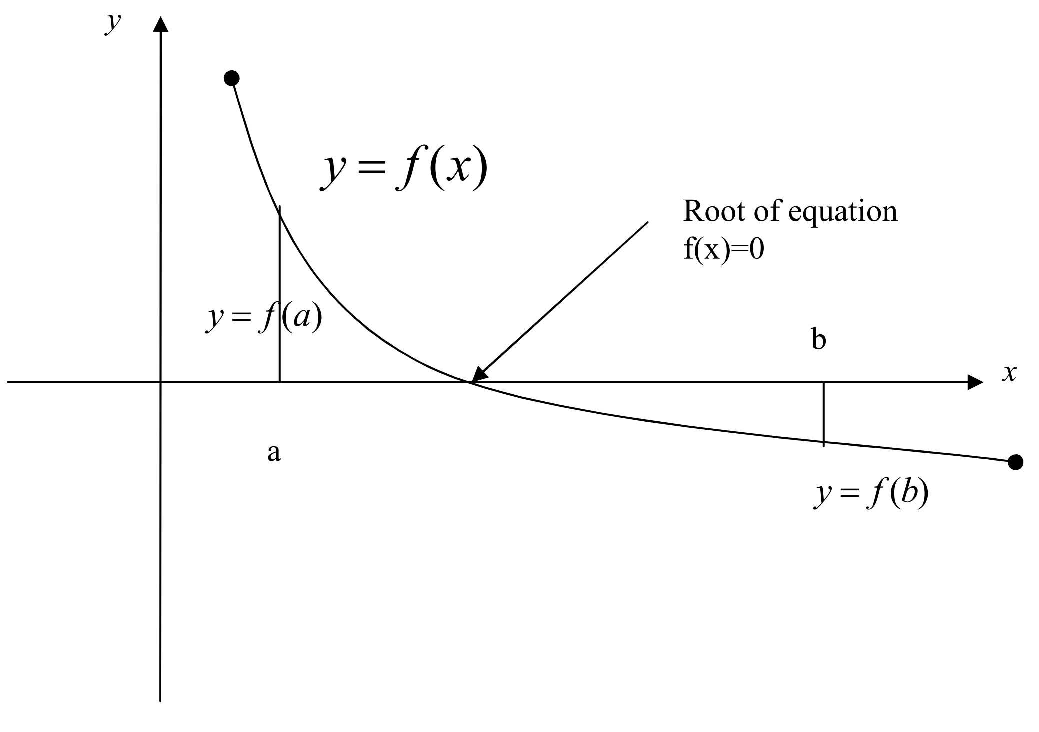


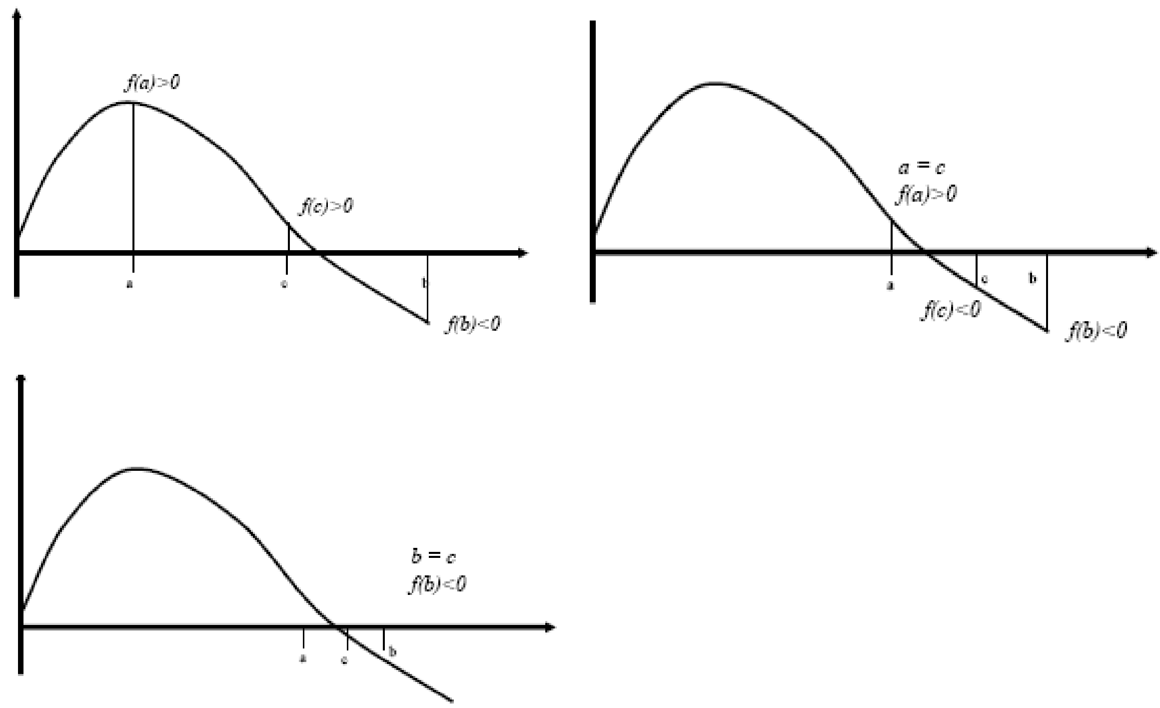
Fig. 1 Numerical method (Bisection method) in finding a root

1. Bracketing Method

HALF-INTERVAL SEARCH (binary search method or *bisection method*)

If ***f(a)*f(b)*<0**, then there is at least one real root between a and b. Using bisection method, the solution is **c=(a+b)/2**, and slowly converges to the root.

- c=(a+b)/2



Let us solve the transcendental equation:

$$e^{-x} - \sin(\pi x/2) = 0$$

$$or$$

$$f(x) = e^{-x} - \sin(\pi x/2) = 0$$

This equation has a single solution between 0 and 1.
Using the bisection method, we should find f(x) at 0.5 and determine whether the root is between 0 - 0.5 or 0.5 -1.

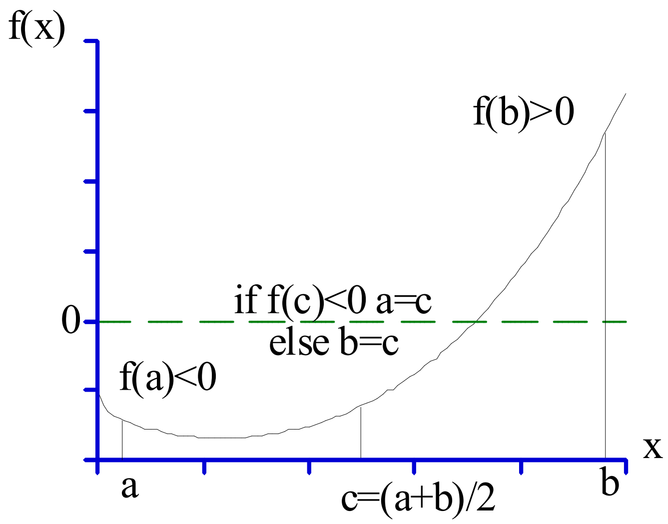
By computing f(x) at the midpoint of the known interval x=c, the root can be found to within desired accuracy ε, said 0.01.

Iteration I	c	f(c)
1	0.5	-0.100576
2	0.25	0.396117
3	0.375	0.131719
4	0.4375	0.011255
5	0.46875	-0.04577495
6	0.45125	-0.0175341
7	0.4453125	-0.0032075

This method starts with a search interval [*a,b*] over which the function to be solved changes sign, or *f(a)*f(b)*<0, and continues to reduce the interval by ½ until |b-a| is so small that all values of *c* in [*a*<*c*<*b*] satisfy |*f(c)*|<ε.

Procedures:

Check if the mid point *c* = *(a+b)/2* is close enough to satisfy |*f(c)*|<ε,
if yes *c* is the solution,
if not
set *a*=*c* if *f(c)*f(a)*>0
else *b*=*c*.



Now the search interval is reduced by ½, and the process can start again until the interval is so small that *f(c)* must be less than the tolerance ε.
Given an ε=0.01, we can be sure that no more than 7 iterations will be needed for a root between [a,*b*].

Summary of Bisection Method Algorithms

Step 1: Choose lower **a** and upper **b** guesses for the root such that the function changes sign over the interval. f(a)f(b)<0.
Step 2: An estimate of the root c where c=(a+b)/2.
Step 3: Determine which interval the root lies by:
(i) if f(a)f(c)<0, the root lies in the lower interval, set b=c.
(ii) if f(a)f(c)>0, the root lies in the upper interval, set a=c.
(iii) if f(a)f(c)=0, the root equal to c, terminate the computation.

Termination criteria

At the beginning of computation, the initial guess for the root is *a*.
At the first iteration:

$$x_1^{new} - x_1^{old} = \frac{b-a}{2}$$

$$x_1^{new} = \frac{a+b}{2}$$

At the r-th iteration, *a* and *b* are changed. The above relations still apply.

Define the approximate percentage relative error at the r - th iteration

$$\varepsilon_a = \left| \frac{x_r^{new} - x_r^{old}}{x_r^{new}} \right| \times 100\%$$

ε_a tends to zero when iteration ***r*** increases and can be used as termination criteria.
For example, if $\varepsilon_a < \varepsilon_s$, the tolerance of error (said 0.01), stop the computation. Otherwise repeat the process.

Example 1:

$f(x) = 5x^3 - 5x^2 + 6x - 2 = 0$. Find the root between (0, 1)

<u>Solution:</u>

1st iteration:

Present new root is $c = \dfrac{a+b}{2} = \dfrac{0+1}{2} = 0.5$

$$\varepsilon_a = \left|\frac{b-a}{b+a}\right| \times 100\% = \left|\frac{1-0}{1+0}\right| \times 100\% = 100\%$$

$$f(a)f(c) = f(0)f(0.5) = -2(0.375) = -0.75$$

Therefore, the new bracket is $a = 0$ and $b = 0.5$.

The process can be repeated until the approximate relative error falls below 10% and after 5 iterations yielding a root estimate of 0.40625.

iteration	*a*	*b*	*c (root)*	*f*(*a*)	*f*(c)	*f*(a)×*f*(c)	ε_a
1	0	1	0.5	-2	0.375	-0.75	
2	0	0.5	0.25	-2	-0.73438	1.46875	100.00%
3	0.25	0.5	0.375	-0.73438	-0.18945	0.13913	33.33%
4	0.375	0.5	0.4375	-0.18945	0.08667	-0.01642	14.29%
5	0.375	0.4375	0.40625	-0.18945	-0.05246	0.009939	7.69%

2. Newton-Raphson Method *(Open Method)*

Revision on Taylor's Expansion

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''}{3!}(x-a)^3 \cdots + \frac{f^m}{m!}(x-a)^m$$

Taylor expansion on f(x) = sin(x) about x=0 gives:

$$\sin(x) = \sin(0) + (x-0)\cos(0) - (x-0)^2\frac{\sin(0)}{2!} + (x-0)^3\frac{\cos(0)}{2!} + \cdots + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \cdots$$

Similarly

$$e^x = 1 + x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \frac{x^4}{4\,!} + \cdots + \cdots$$

and

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \cdots$$

Solving an equation f(x)=0

Based on Taylor expansion

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \cdots + R$$

$$0 = f(x_i) + f'(x_i)(x_{i+1} - x_i)$$

Hence

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Usually instead of exact differentiation $f'(x)$, numerical approximation is used.

Therefore $f'(x) = \dfrac{f(x_{i+h}) - f(x_i)}{h}$

is used to evaluate f ’(x).

Graphically the Newton-Raphson method can be described as follows

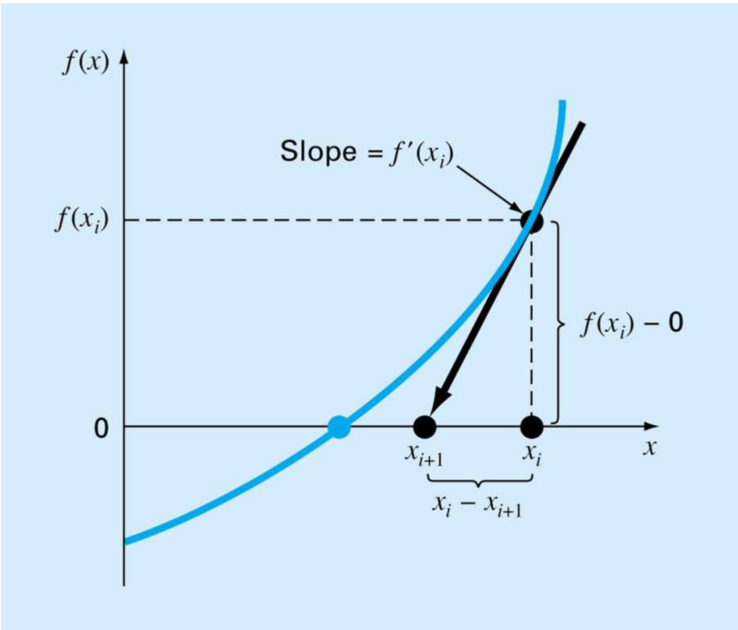


Fig. 2 Numerical method (Newton-Raphson method) in finding a root

If the initial guess x_0 is in the proximity of the root, the search could converge much faster than the bisection method.

Example 2:

Use Newton-Raphson Method to estimate the root of equation

$$f(x) = e^{-x} - x$$

By exact differentiating, we have $f'(x) = -e^{-x} - 1$

$$Hence \qquad x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{e^{-x_i} - x_i}{-e^{-x_i} - 1}$$

Starting an initial guess at x_0=0, we have x_i=0.56714329 after 4 iteration.

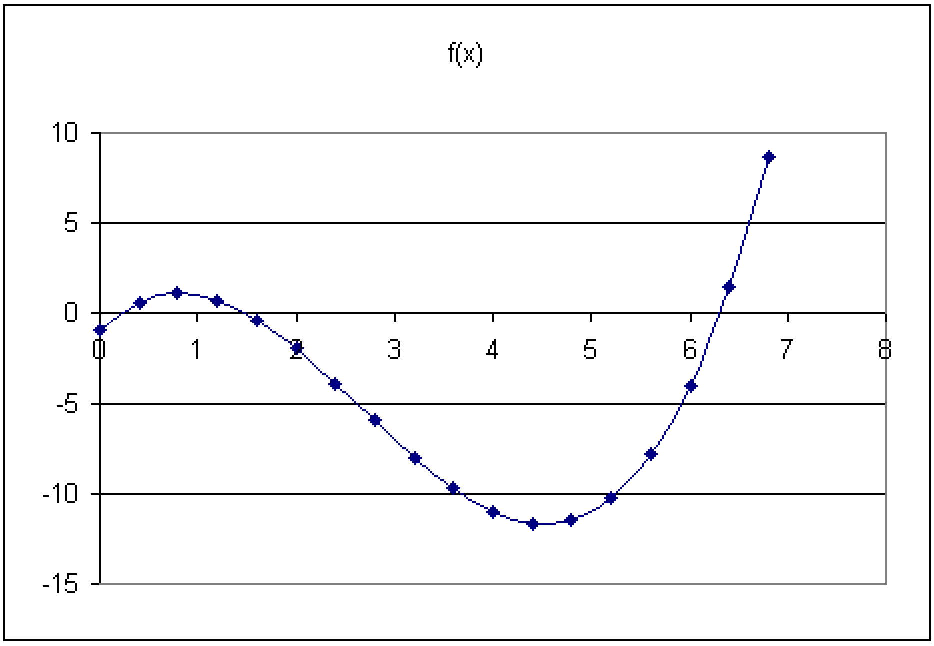
Example 3:

Use Newton-Raphson method to estimate the roots of equation

$$f(x) = -1 + 5.5x - 4x^2 + 0.5x^3$$

a) By graphical solution, 3 real roots at approximately 0.2, 1.5 and 6.3.

x	f(x)
0	-1
0.4	0.592
0.8	1.096
1.2	0.704
1.6	-0.392
2	-2
2.4	-3.928
2.8	-5.984
3.2	-7.976
3.6	-9.712
4	-11
4.4	-11.648
4.8	-11.464
5.2	-10.256
5.6	-7.832
6	-4
6.4	1.432
6.8	8.656



b) The Newton-Raphson method can be set up as

$$x_{i+1} = x_i - \frac{-1 + 5.5x_i - 4x_i^2 + 0.5x_i^3}{5.5 - 8x_i + 1.5x_i^2}$$

The three roots as summarized in the following tables:

i	x_i	*f*(*x*)	*f'*(*x*)	ε_a
0	0	-1	5.5	
1	0.181818	-0.12923	4.095041	100.00%
2	0.213375	-0.0037	3.861294	14.789%
3	0.214332	-3.4E-06	3.85425	0.4466%
4	0.214333	-2.8E-12	3.854244	0.000408%

i	x_i	*f*(*x*)	*f'*(*x*)	ε_a
0	2	-2	-4.5	
1	1.555556	-0.24143	-3.31481	28.57%
2	1.482723	-0.00903	-3.06408	4.912%
3	1.479775	-1.5E-05	-3.0536	0.199%
4	1.479769	-4.6E-11	-3.05358	0.00034%

i	x_i	*f*(*x*)	*f'*(*x*)	ε_a
0	6	-4	11.5	
1	6.347826	0.625955	15.15974	5.479%
2	6.306535	0.009379	14.7063	0.6547%
3	6.305898	2.22E-06	14.69934	0.010114%
4	6.305898	1.42E-13	14.69934	0.000002%

Therefore, the roots are 0.214333, 1.479769, and 6.305898.