ME46002 Chapter 3: Tutorial questions

Questions

(1) Find the eigenvalues and eigenvectors of

$$[A] = \begin{bmatrix} 1.5 & 0 & 1 \\ -0.5 & 0.5 & -0.5 \\ -0.5 & 0 & 0 \end{bmatrix}$$

Solution

The characteristic equation is given by

$\det\,([A] - \lambda[I]) = 0$

$$\det \begin{bmatrix} 1.5-\lambda & 0 & 1 \\ -0.5 & 0.5-\lambda & -0.5 \\ -0.5 & 0 & -\lambda \end{bmatrix} = 0$$

$$(1.5-\lambda)[(0.5-\lambda)(-\lambda)-(-0.5)(0)] + (1)[(-0.5)(0)-(-0.5)(0.5-\lambda)] = 0$$

$$-\lambda^3 + 2\lambda^2 - 1.25\lambda + 0.25 = 0$$

The roots of the above equation are

$\lambda = 0.5, 0.5, 1.0$

Note that there are eigenvalues that are repeated. Since there are only two distinct eigenvalues, there are only two eigenspaces. But corresponding to $\lambda = 0.5$ there should be two eigenvectors that form a basis for the eigenspace.

To find the eigenspaces, let

$$[X] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Given $[(A - \lambda I)][X] = 0$

$$\begin{bmatrix} 1.5-\lambda & 0 & 1 \\ -0.5 & 0.5-\lambda & -0.5 \\ -0.5 & 0 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda = 0.5$,

$$\begin{bmatrix} 1 & 0 & 1 \\ -0.5 & 0 & -0.5 \\ -0.5 & 0 & -0.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system gives

$x_1 = a$, $x_2 = b$, $x_3 = -a$

So

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ -a \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ -a \end{bmatrix} + \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix}$$

$$= a\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

So the vectors $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ form a basis for the eigenspace for the eigenvalue λ=0.5.

For $\lambda = 1$,

$$\begin{bmatrix} 0.5 & 0 & 1 \\ -0.5 & -0.5 & -0.5 \\ -0.5 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system gives

$x_1 = a$, $x_2 = -0.5a$, $x_3 = -0.5a$

The eigenvector corresponding to $\lambda = 1$ is

$$\begin{bmatrix} a \\ -0.5a \\ -0.5a \end{bmatrix} = a\begin{bmatrix} 1 \\ -0.5 \\ -0.5 \end{bmatrix}$$

Hence the vector $\begin{bmatrix} 1 \\ -0.5 \\ -0.5 \end{bmatrix}$ is a basis for the eigenspace for the eigenvalue of $\lambda = 1$.

(2) For a standard eigenvalue problem

$$\begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

find (a) all eigenvalues λ, and (b) the corresponding eigen-vectors.

Solution

(a)

$$AX = \lambda X, \quad or \quad (A - \lambda I)X = 0$$

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$-(\lambda - 1)(\lambda - 3)(\lambda - 4) = 0$$

The solutions for λ is 1, 3, 4. Therefore three eigenvalues are $\lambda_1=1$, $\lambda_2=3$, $\lambda_3=4$

(b)

<u>For $\lambda_1=1$</u>

$$AX = \lambda X, \quad or \quad (A - \lambda I)X = 0$$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

The solution is the eigenvector

$x_1=a$, $x_2=2a$, $x_3=a$, where a is a real number.

<u>For $\lambda_2=3$</u>

$$AX = \lambda X, \quad or \quad (A - \lambda I)X = 0$$

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

The solution is the eigenvector
x_1=b, x_2=0, x_3=-b,   where b is a real number.

For λ_3=4
$AX = \lambda X, \quad or \quad (A - \lambda I)X = 0$

$$\begin{bmatrix} -1 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

The solution is the eigenvector
x_1=c, x_2=-c, x_3=c,   where c is a real number.

(3) Assume **V** is an eigenvector of matrix **A** that corresponds to the eigenvalue λ. **I** is an identity matrix. If λ=6, find one eigenvalue of the matrix $(\mathbf{A}\text{-}3\mathbf{I})^{-1}$ and its corresponding eigenvector.

Solution
Because AX=λX=6X
Therefore (A-3I)X=AX-3IX=AX-3X=6X-3X=3X

And
$(A-3I)^{-1}$ (A-3I)X=$(A-3I)^{-1}$3X

Therefore
IX=3$(A-3I)^{-1}$X

That is
$(A-3I)^{-1}$X=(1/3)X

Therefore the eigenvalue of $(A-3I)^{-1}$ is 1/3, and X is its eigenvector