Chapter 3. Eigenvalue Problems

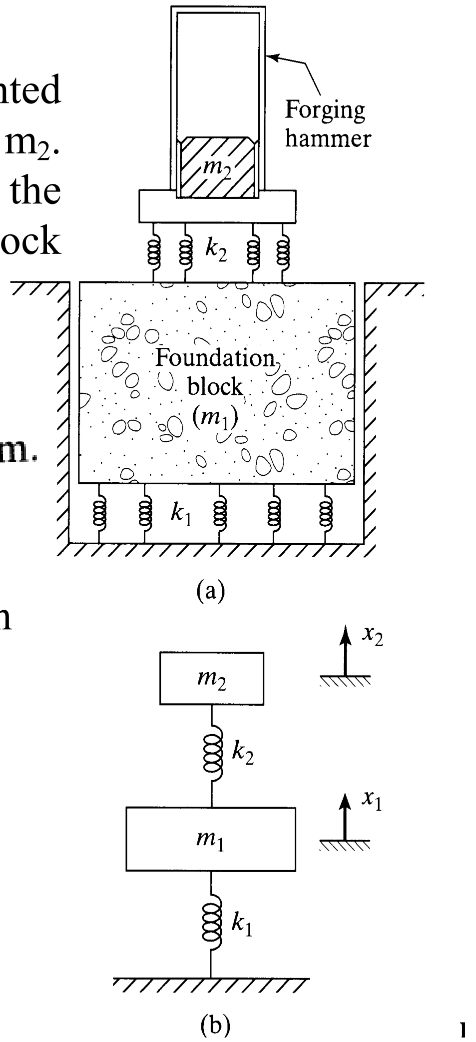
The word eigenvalue comes from the German word "Eigenwert" where Eigen means "characteristic" and Wert means "value". In eigenvalue problem of a system, the eigenvalue is associated with an eigenvector. They describe the characteristics of the system.

1. Practical problems

~~Example 1:~~

A forging hammer of mass m_1 is mounted on a concrete foundation block of mass m_2. The stiffness of the springs underneath the forging hammer and the foundation block are given by k_2 and k_1 respectively.

$m_1 = 20,000$ kg, $m_2 = 5,000$ kg,
$k_1 = 10^7$ N/m, and $k_2 = 5 \times 10^6$ N/m.



The motions of hammer and foundation block are described by

$$m_1 \frac{d^2 x_1}{dt^2} = -k_1 x_1 + k_2 (x_2 - x_1)$$

$$m_2 \frac{d^2 x_2}{dt^2} = -k_2 (x_2 - x_1)$$

If the system undergoes harmonic motion with a natural frequency ω, what is ω and the corresponding x_1 and x_2?

From vibration theory, the solutions can be of the form

$x_i = A_i Sin\ (\omega t - \emptyset)$

where

A_i = amplitude of the vibration of mass 'i'

ω = frequency of vibration

$\emptyset$ = phase shift

then

$$\frac{d^2 x_i}{dt^2} = -A_i w^2 Sin(\omega t - \emptyset)$$

Substituting x_i and $\frac{d^2 x_i}{dt^2}$ in equations, we have the equations of motion written is matrix form:

$$\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix},$$

where ω^2 is the eigenvalue $\vec{X} = \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}$ is the eigenvector (displacement pattern) of the system.

Hence,

$$\begin{bmatrix} 15 \times 10^6 & -5 \times 10^6 \\ -5 \times 10^6 & 5 \times 10^6 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \omega^2 \begin{bmatrix} 20,000 & 0 \\ 0 & 5000 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}$$

The solution of these equations yields

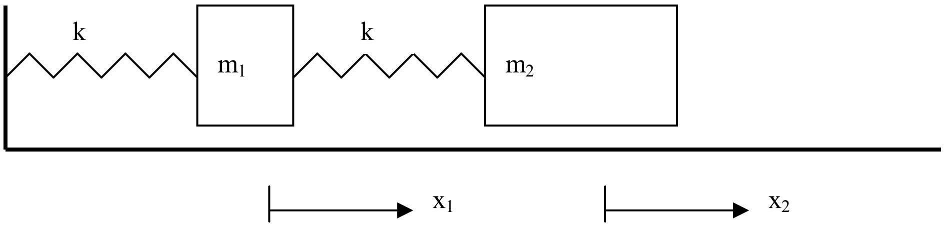
$$\omega_1 = 18.9634 \text{ rad/sec}, \ \vec{X}^{(1)} = \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 1.0 \\ 1.5615 \end{Bmatrix}$$

and

$$\omega_2 = 37.2879 \text{ rad/sec}, \ \vec{X}^{(2)} = \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 1.0 \\ -2.5615 \end{Bmatrix}.$$

Example 2:

A spring-mass system is shown in the figure as follows.



Assume each of the two mass-displacements to be denoted by x_1 and x_2, and let us assume each spring has the same spring constant 'k'. Then by applying Newton's 2nd and 3rd law of motion to develop a force-balance for each mass we have

$$m_1 \frac{d^2 x_1}{dt^2} = -kx_1 + k(x_2 - x_1)$$

$$m_2 \frac{d^2 x_2}{dt^2} = -k(x_2 - x_1)$$

Rewriting the equations, we have

$$m_1 \frac{d^2 x_1}{dt^2} - k(-2x_1 + x_2) = 0$$

$$m_2 \frac{d^2 x_2}{dt^2} - k(x_1 - x_2) = 0$$

$$Let\ m_1 = 10, m_2 = 20, k = 15$$

$$10\frac{d^2 x_1}{dt^2} - 15(-2x_1 + x_2) = 0$$

$$20\frac{d^2 x_2}{dt^2} - 15(x_1 - x_2) = 0$$

From vibration theory, the solutions can be of the form

$$x_i = A_i Sin\ (\omega t - \emptyset)$$

where

A_i = amplitude of the vibration of mass 'i'

ω = frequency of vibration

$\emptyset$ = phase shift

then

$$\frac{d^2x_i}{dt^2} = -A_i w^2 Sin(\omega t - \emptyset)$$

Substituting x_i and $\frac{d^2x_i}{dt^2}$ in equations,

$-10A_1\omega^2 - 15(-2A_1 + A_2) = 0$

$-20A_2\omega^2 - 15(A_1 - A_2) = 0$

gives

$(-10\ \omega^2 + 30)\ A_1 - 15A_2 = 0$

$-15A_1 + (-20\ \omega^2 + 15)\ A_2 = 0$

or

$(-\omega^2 + 3)\ A_1 - 1.5A_2 = 0$

$-0.75A_1 + (-\omega^2 + 0.75)\ A_2 = 0$

In matrix form, these equations can be rewritten as

$$\begin{bmatrix} -\omega^2 + 3 & -1.5 \\ -0.75 & -\omega^2 + 0.75 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1.5 \\ -0.75 & 0.75 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} - \omega^2 \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let $\omega^2 = \lambda$

$$[A] = \begin{bmatrix} 3 & -1.5 \\ -0.75 & 0.75 \end{bmatrix}$$

$$[X] = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

Hence,

$[A]\,[X] - \lambda\,[X] = 0$

$[A]\,[X] = \lambda\,[X]$

In the above equation, 'λ' is the eigenvalue and [X] is the eigenvector corresponding to λ. If we know 'λ' for the above example, we can calculate the natural frequency of the vibration, as

$$\omega = \sqrt{\lambda}$$

This frequency is important since a forcing force on the spring-mass system close to it would make the amplitude A_i very large and make the system unstable.

2. General definition of eigenvalue problems

The eigenvalue problems can be simplified by solving N simultaneous linear equations with (N+1) unknowns, in matrix form as:

$$[A][X] - \lambda[B][X] = 0$$

or

$$\mathbf{([A] - \lambda[B])[X] = 0} \tag{1}$$

where [A] and [B] are N-by-N constant (known) matrices. [X] is a vector consisting of N unknowns. λ is an unknown value.

Eq. (1) is a ***general definition of eigenvalue problems***, i.e., the eigenvalue problem is equivalent to the finding of solution of *nontrivial* vector [X] and the associated value λ in Eq. (1).

These simultaneous equations (1) can be reduced to a simpler expression

$$([A] - \lambda[I])[X] = 0$$

or

$$\mathbf{[A][X] = \lambda[X]} \quad . \tag{2}$$

For example, in the forging hammer problem, [B] is a diagonal matrix.

Eq. (2) is a ***standard form of eigenvalue problems***, i.e., the eigenvalue problem is equivalent to finding the ***eigenvector*** [X] and the ***eigenvalue*** λ of matrix [A].

3. Finding the eigenvalues and eigenvectors of a square matrix

To find the eigenvalues of an N-*by-N* matrix [A], we have

$$[A][X] = \lambda[X]$$

$$[A][X] - \lambda[X] = 0$$

$$[A][X] - \lambda[I][X] = 0$$

Thus

$$([A] - \lambda[I])[X] = 0$$

Now for the above set of equations to have a nonzero solution, we have

$$\det([A] - \lambda[I]) = 0$$

The left hand side of the above equation can be expanded to give a polynomial in λ. Solving the above equation would give us values of the eigenvalues. The above equation is called the **characteristic equation** of [A].

For an N-*by*-*N* matrix [A], the characteristic polynomial of [A] is of degree *N*, as follows,

$$\det([A] - \lambda[I]) = 0 ,$$

gives

$$\lambda^n + c_1 \lambda^{n-1} + \ldots + c_n = 0 \qquad (3)$$

Hence this polynomial can have *N* roots.

After finding the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ from Eq. (3), each eigenvalue λ_i is substituted into Eq. (2) to find the corresponding eigenvector $[X]^i = [x_1, x_2, \ldots, x_n]^i$.

Example 3:

Find the eigenvalues and eigenvectors of the matrix

$$[A] = \begin{bmatrix} 3 & -1.5 \\ -0.75 & 0.75 \end{bmatrix}$$

Solution

$$[A] - \lambda[I] = \begin{bmatrix} 3-\lambda & -1.5 \\ -0.75 & 0.75-\lambda \end{bmatrix}$$

$$\det([A] - \lambda[I]) = (3-\lambda)(0.75-\lambda) - (-0.75)(-1.5) = 0$$

$$2.25-0.75\lambda-3\lambda+\lambda^2-1.125=0$$

$$\lambda^2-3.75\lambda+1.125=0$$

$$\lambda=\frac{-(-3.75)\pm\sqrt{(-3.75)^2-4(1)(1.125)}}{2(1)}$$

$$=\frac{3.75\pm 3.092}{2}$$

=3.421, and 0.3288

So the eigenvalues are 3.421 and 0.3288.

Let

$$[X]=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

be the eigenvector corresponding to

$$\lambda_1=3.421$$

Hence

$([A]-\lambda_1[I])[X]=0$

$$\left\{\begin{bmatrix} 3 & -1.5 \\ -0.75 & 0.75 \end{bmatrix}-3.421\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0$$

$$\begin{bmatrix} -0.421 & -1.5 \\ -0.75 & -2.671 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If $x_1 = s$,

Then

$-0.421s-1.5x_2=0$

$x_2 = -0.2807\ s$

The eigenvector corresponding to $\lambda_1 = 3.421$ then is

$$[X] = \begin{bmatrix} s \\ -0.2807s \end{bmatrix} = s\begin{bmatrix} 1 \\ -0.2807 \end{bmatrix}.$$

The eigenvector corresponding to $\lambda_1 = 3.421$ is $\begin{bmatrix} 1 \\ -0.2807 \end{bmatrix}$

Similarly, the eigenvector corresponding to $\lambda_2 = 0.3288$ is

$$\begin{bmatrix} 1 \\ 1.781 \end{bmatrix}$$

4. Theorems of eigenvalues and eigenvectors

Theorem 1:

If [A] is an N-*by*-N triangular matrix – upper triangular, lower triangular or diagonal, the eigenvalues of [A] are the diagonal entries of [A].

Example

What are the eigenvalues of

$$[A] = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 7 & 3 & 0 & 0 \\ 9 & 5 & 7.5 & 0 \\ 2 & 6 & 0 & -7.2 \end{bmatrix}$$

Solution

Since the matrix [A] is a lower triangular matrix, the eigenvalues of [A] are the diagonal elements of [A]. The eigenvalues are

$$\lambda_1 = 6, \lambda_2 = 3, \lambda_3 = 7.5, \lambda_4 = -7.2$$

Theorem 2:

λ = 0 is an eigenvalue of [A] if [A] is a singular (noninvertible) matrix.

Example

One of the eigenvalues of

$$[A]=\begin{bmatrix} 5 & 6 & 2 \\ 3 & 5 & 9 \\ 2 & 1 & -7 \end{bmatrix}=\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

is zero. Is [A] invertible?

Solution

Because $\lambda = 0$ is an eigenvalue of [A], therefore $|\det(A)|$=0.

That implies [A] is singular and is not invertible, based on the definition of matrix inversion $\mathbf{A}^{-1} = adj(A)/\det(A)$, where adj(A) is the adjoint matrix of A.

$$\text{adj(A)}=\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix};$$

A_1 is the cofactor of a_1; B_1 is the cofactor of b_1, and so on.

<u>Theorem 3:</u>

[A] and $[A]^T$ have the same eigenvalues.

Example

Given the eigenvalues of

$$[A]=\begin{bmatrix} 2 & -3.5 & 6 \\ 3.5 & 5 & 2 \\ 8 & 1 & 8.5 \end{bmatrix}$$

are

$$\lambda_1 = -1.546, \lambda_2 = 12.33, \lambda_3 = 4.711$$

What are the eigenvalues if $[B]=\begin{bmatrix} 2 & 3.5 & 8 \\ -3.5 & 5 & 1 \\ 6 & 2 & 8.5 \end{bmatrix}$

Solution

Since [B] = $[A]^T$, the eigenvalues of [A] and [B] are the same. Hence eigenvalues of [B] also are

$$-\lambda_1 = 1.546, \lambda_2 = 12.33, \lambda_3 = 4.711$$

<u>Theorem 4:</u>

$|\det(A)|$ is the product of the absolute values of the eigenvalues of [A].

Example

Given the eigenvalues of

$$[A] = \begin{bmatrix} 2 & -3.5 & 6 \\ 3.5 & 5 & 2 \\ 8 & 1 & 8.0 \end{bmatrix}$$

are

$$\lambda_1 = -1.546, \lambda_2 = 12.33, \lambda_3 = 4.711$$

Calculate the magnitude of the determinant of the matrix.

Solution

Since

$$|\det[A]| = |\lambda_1||\lambda_2||\lambda_3|$$

$$= |-1.546||12.33||4.711| = 89.80$$

Chapter of textbook to read:
Chapter 4.1 of Textbook #2 (*Singiresu S. Rao, Applied numerical methods for engineers and scientists, Prentice Hall, c2002)*