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Effect of deterioration on two-warehouse inventory model with imperfect quality

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ABSTRACT

The formulation of inventory models is often done with the presumption that all items produced are of perfect quality in nature. However, in real-life scenarios, the lot that arrives in inventory includes a fraction of imperfect quality items due to faulty production or mishandling of goods. It is also a misconception that the goods preserve their characteristics during their presence in the inventory. Thus, these assumptions need to be rectified in the planning/preparation of inventory models. In order to reduce the losses due to deterioration, at times, the retailer is forced to rent other warehouses (RW) with better preserving facilities, owing to the lack of facilities in his own warehouse (OW). The retailer may also rent warehouses voluntarily to store excessive goods that have been obtained at a discounted price or in order to avoid inflation rates. The present research paper is, thus, an attempt to incorporate the aforementioned concepts to develop a new inventory model with two warehouses: for goods that have a fraction of imperfect quality items and those that are deteriorating in nature. The study also states the developed mathematical model along with the solution procedure. Further, the sensitivity analysis of the optimal solution with respect to key parameters of the inventory system has also been performed to develop the managerial insights.

1. Introduction

The traditional Economic Order Quantity (EOQ) model is often based on assumptions that are idealistic and unrealistic in nature. So, the development of the inventory models requires a certain amount of relaxation of these assumptions to represent the actual realistic scenario.

The foremost unrealistic assumption of the EOQ model is that all items produced are of good quality. The truth of the matter is that the goods produced are directly affected by the process of production. The faulty production process introduces a fraction of items that are imperfect in quality and need to be removed by screening so as to meet the demand. A significant amount of work has been done to examine the effect of imperfect quality goods on the inventory. Porteus (1986) followed by Rosenblatt and Lee, had presented the substantial relationship between imperfect quality and lot size. In addition to this, Zhang and Gerchak (1990) had studied a joint lot sizing and inspection policy under an EOQ model, where a random proportion of units were considered defective. Further, Salameh and Jaber (2000) had developed an economic production quantity model for imperfect quality items with a known probability distribution. Therefore, they had suggested that the imperfect quality items are sold as a single batch by the end of the screening process. In the same year, Cárdenas-Barrón (2000) had not deviated from the main idea but had pointed out and rectified an error in the model devised by Salameh and Jaber.

Goyal and Cárdenas-Barrón (2002) had proposed a simple approach to determine the economic production quantity for the model given by Salameh and Jaber (2000). Subsequently, Papachristos and Konstantaras (2006) had examined the issue of non-shortages in the model with proportional imperfect quality, given that the proportion of the imperfects is a random variable. Recently, Sana (2010) had developed a production-inventory model in an imperfect production process. The following year, Khan, Jaber, Guiffrida, and Zolfaghari (2011) extended the work of Salameh and Jaber by assuming that the screening process is not error-free. Jaber, Zanoni, and Zavanella (2013) further proposed an EOQ model with imperfect quality based on entropy. The same year Moussawi-Haidar, Salameh, and Nasr (2013) suggested a model which integrates inventory lot sizing with imperfect quality items and quality control. The ensuing year, Karimi-Nasab and Sabri-Laghaie (2014) formulated a new imperfect production problem that generates defectives randomly. Recent research work about EPQ models with imperfect quality

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items can be studied in the articles published by Sana (2011), Sarkar, Sana, and Chaudhuri (2010), Sarkar and Moon (2011), Sarkar (2012), Yoo, Kim, and Park (2012) and the references enlisted by them, the details of which are provided in the bibliography. Currently, Moussawi-Haidar, Salameh, and Nasr (2014) had considered the effect of deterioration on the instantaneous replenishment of the lot with imperfect quality items.

The second assumption prevalent in many models is that the goods produced preserve their physical characteristics during their stay in the inventory. The reality remains that the items in inventory are subject to several risks such as pilferage, breakage, evaporation and obsolescence. Inventory of certain products such as food items, pharmaceuticals, chemicals, blood, gasoline, and radioactive chemicals deteriorate rapidly over time. Thus, the loss from deterioration cannot be ignored. The phenomenon of deterioration was first introduced by Ghare and Schrader (1963) by assuming the exponential decaying of the items. Covert and Philip (1973) further researched and extended Ghare and Schrader's (1963) model with the assumption of Weibull distribution deterioration. Subsequently, Raafat, Wolfe, and Eldin (1991), Goyal and Giri (2001) and Bakker, Riezebos, and Teunter (2012) surveyed the trends in modeling of continuously deteriorating inventory.

The next problematic presupposition in the development of most of the traditional inventory models is that a single warehouse (OW) has unlimited capacity, which is factually incorrect. The inventory costs (holding cost and deterioration cost) in RW are usually higher than those of OW because of additional cost of maintenance, material handling and better preserving facilities. Despite these facts, the question of survival in an ever-increasing competitive business environment cannot be negated. Discounted price of goods from suppliers, high inflation rates and other such factors often allure the retailer to order an amount that surpasses his own warehouse capacity, thereby compelling him to rent other warehouses.

It is important to substantiate the argument with the help of the research done in this area. The primary notable work can be seen in the Chapter 12 of Hartley (1976) who first proposed a two-warehouse inventory system. This was followed by Das, Maity, and Maiti (2007) who had discussed a joint performance of a supply chain with two warehouse facilities. Hsieh, Dye, and Ouyang (2008) successively built a deterministic inventory model for deteriorating items with two warehouses by minimizing the net present value of the total cost. In continuation, a number of interesting research papers in this area has been published over the last few decades. One may refer to the recent works of Bhunia and Maiti (1998), Zhou and Yang (2005), Lee (2006), Chung, Her, and Lin (2009), Yang (2004, 2006, 2012), Liang and Zhou (2011), Hsieh et al. (2008), Niu and Xie (2008), Lee and Hsu (2009), Jaggi and Verma (2010), Zhong and Zhou (2013) and Bhunia, Jaggi, Sharma, and Sharma (2014) for an extensive study. The complete references have been given in the bibliography.

This paper takes into account a two-warehouse inventory model with items that have a certain percentage of defective items and are of deteriorating in nature. As pointed out earlier, to meet the demand and ensure the supply of perfect quality items, the defective items that result out of a faulty production process, need to be screened out. Thus, if these factors are not taken into consideration, it may lead to shortages and losses for the retailer. In the present analysis, 100% screening is conducted immediately after an order is received, as discussed in Salameh and Jaber (2000). At the end of the screening process, imperfect items are removed from the inventory and then sold as a single batch at a discounted price. The deteriorated items are not replaced as described by Lo, Wee, and Huang (2007). To avoid shortages, it is further assumed that the number of good items that remain after removing defective and deteriorated items meet the demand during screening. Since the rented warehouse has better preserving facilities, the rate of deterioration of goods is less than that of the own warehouse, resulting in higher holding cost of goods in the former. Hence, the model developed takes the LIFO model into consideration, i.e. the items are stocked into own warehouse first and then in rented warehouse. However, while fulfilling the demand the rented warehouse is consumed first. Finally, a numerical example is presented to illustrate the applicability of the proposed model. Sensitivity analysis on key parameters is provided to reveal managerial insights.

The rest of the paper is designed as follows:

Section 2 provides notations and assumptions used in the paper. The models' formulations are studied in Section 3 whereas Section 4 discusses the solution procedure. Section 5 discusses some of the special cases that can be obtained from the proposed inventory model. The proposed model is illustrated numerically along-with Sensitivity analysis in Sections 6 and 7 respectively. Finally, conclusions are made and future research directions are outlined in Section 8.

2. Assumption and notations

The mathematical models of the two-warehouse inventory problems are based on the following assumptions:

1. The own warehouse (OW) has a fixed capacity of w units while the rented warehouse (RW) has unlimited capacity.
2. Lead time is zero and the initial inventory level is also zero.
3. The rate of deterioration of RW (β) is less than the rate of deterioration of OW (α).
4. The screening process and demand proceeds simultaneously, but the screening rate (x) is greater than the demand rate (D), $x > D$.

2.1. Notations

For simplicity, we define the symbols for parameters, decision variables, functions and optimal values accordingly.

Parameters

w	storage capacity of OW (units)
D	annual demand rate known and constant (unit/time unit)
p	percentage of defective items (per unit)
x	screening rate (unit/year)
α	deterioration rate of OW
β	deterioration rate of RW
c	unit purchasing cost per item ($/unit)
k	fixed cost of placing an order ($/cycle)
s	unit selling price per item of good items ($/unit)
v	unit selling price per item of defective items ($/unit)
d	screening cost per unit item ($/unit)
h_r	unit holding cost per unit item per unit time in RW ($/unit/year)
h_o	unit holding cost per unit item per unit time in OW ($/unit/year)
TR_r	total revenue generated per cycle from RW ($)
TR_o	total revenue generated per cycle from OW ($)
HC_r	inventory holding cost of RW ($)
HC_o	inventory holding cost of OW ($)

Decision variables

y	order size per cycle (units)

Functions

$f(p)$ probability density function of p
$I_o(t)$ inventory level of OW at time t
$I_r(t)$ inventory level of RW at time t
$TP(y)$ the total profit
$TPU(y)$ the total profit per unit time

Optimal values

t_w^* optimal screening time of OW (years)
t_s^* optimal screening time of RW (years)
t_r^* optimal time to use up RW (years)
T^* optimal replenishment cycle time (years)
y^* optimal order size per cycle (units)
$E^*[TPU(y)]$ optimal expected total profit per unit time ($/year)

3. Model formulation

We begin with a proposition that a lot size y enters the inventory system out of which w units are kept in OW and $(y - w)$ units are kept in RW. Since the holding cost per unit item per unit time in RW is greater than OW, the demand is initially fulfilled from RW. Once the RW is exhausted, the demand is fulfilled from OW. It is assumed that each lot received contains a random proportion of defective items, p, with a known probability density function $f(p)$, and mean $E(p) = p$. Upon receiving each lot, a 100% screening process is conducted at a screening rate x per unit time. The defective items are stocked and sold at the end of the screening period at a discounted price of v per unit, $v < c$, contrary to the assumption of Rosenblatt and Lee (1986) who assumed that defective items can be reworked instantaneously at a cost. Following Salameh (1999), items in stock deteriorate at a rate which is proportional to the on-hand inventory.

The screening process takes place in OW and RW simultaneously and gets completed at $t_w = w/x$ and $t_s = (y - w)/x$ respectively. The time period when RW and OW get exhausted are denoted by t_r and T. Depending on the values of t_w, t_s and t_r, the following cases are discussed:

3.1. Case I: When $t_w < t_r$

Figs. 1 and 2 show the behavior of the model over the time interval $[0, T]$ when the screening of OW is completed before the exhaustion of RW. The screening of OW and RW gets completed at t_w and t_s respectively. During the time $[0, t_r]$, the on-hand inventory level in RW drops to zero due to demand and deterioration, whereas in OW the depletion is due to deterioration only. From $[t_r, T]$ the OW depletes due to demand as well as deterioration and reduces to zero at time T.

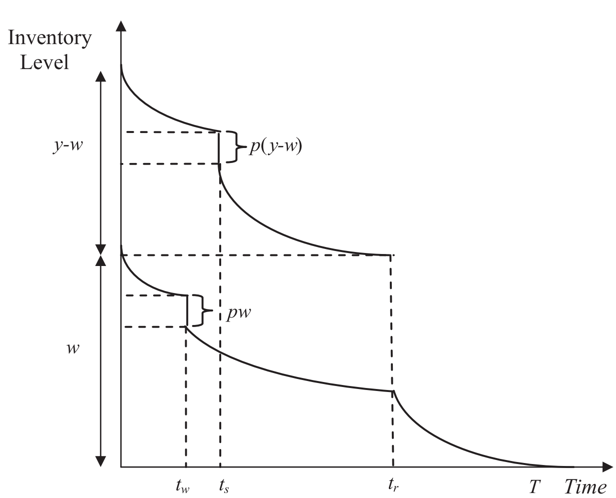

Fig. 1. Inventory level for the two-warehouse system when $t_w < t_s < t_r$.

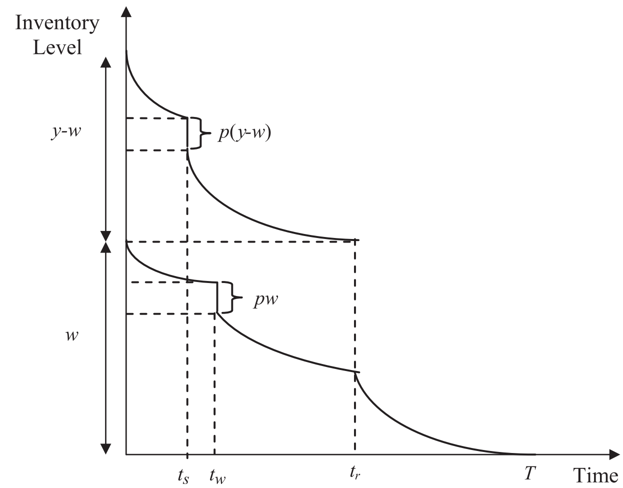

Fig. 2. Inventory level for the two-warehouse system when $t_s < t_w < t_r$.

Therefore, the differential equations that describe the inventory level in both RW and OW at any time t over the period $(0, T)$ are given by:

$$\frac{dI_r(t)}{d(t)} = -D - \beta I_r(t) \quad 0 \leqslant t \leqslant t_r \tag{1}$$

$$\frac{dI_o(t)}{dt} = -\alpha I_o(t) \quad 0 \leqslant t \leqslant t_r \tag{2}$$

$$\frac{dI_o(t)}{dt} = -D - \alpha I_o(t) \quad t_r \leqslant t \leqslant T \tag{3}$$

Solving the above differential equations with the boundary conditions $I_r(0) = y - w$, $I_r(t_s^+) = I_r(t_s) - p(y - w)$, $I_o(0) = w$, $I_o(t_w^+) = I_o(t_w) - pw$ and $I_o(T) = 0$, we get

$$I_r(t) = -\frac{D}{\beta} + \left(y - w + \frac{D}{\beta}\right)e^{-\beta t} \quad 0 \leqslant t \leqslant t_s \tag{4}$$

$$I_r(t) = -\frac{D}{\beta} + ((y - w) + D/\beta - p(y - w)e^{\beta t_s})e^{-\beta t} \quad t_s < t \leqslant t_r \tag{5}$$

$$I_o(t) = we^{-\alpha t} \quad 0 \leqslant t \leqslant t_w \tag{6}$$

$$I_o(t) = \{w - pwe^{\alpha t_w}\}e^{-\alpha t} \quad t_w < t \leqslant t_r \tag{7}$$

$$I_o(t) = \frac{D}{\alpha}(e^{\alpha(T-t)} - 1) \quad t_r \leqslant t \leqslant T \tag{8}$$

Now the value of t_r can be found out by applying the boundary condition $I_r(t_r) = 0$. Thus,

$$t_r = \frac{1}{\beta}\left\{\ln\left((y - w)(1 - pe^{\beta t_s}) + D/\beta\right) - \ln\left(\frac{D}{\beta}\right)\right\} \tag{9}$$

Also considering the continuity of $I_o(t)$ at $t = t_r$, we have,

$$(w - wpe^{\alpha t_w})e^{-\alpha t_r} = \frac{D}{\alpha}\{e^{\alpha(T - t_r)} - 1\}$$

$$T = t_r + \frac{1}{\alpha}\ln\left[\frac{\alpha}{D}(w - pwe^{\alpha t_w})e^{-\alpha t_r} + 1\right] \tag{10}$$

Define $N_r(y,p)$ and $N_o(y,p)$ as the number of good items in each order lot size less the summation of defective and deteriorated items with respect to RW and OWs respectively. Let w_{1r} and w_{1o} be the items deteriorated at time t_r and T in RW and OW respectively.

$$N_r(y,p) = (y-w)(1-p) - w_{1r} \tag{11}$$

$$N_o(y,p) = w(1-p) - w_{1o} \tag{12}$$

Let $I_{01r}(t)$ be the inventory level of RW at time t when we ignore both effects of lot quality and deterioration, so $I_{01r}(t) = -Dt + y - w$. Let $I_{02r}(t)$ be the inventory level of RW when we ignore the effect of deterioration only, so $I_{02r}(t) = -Dt + (y-w)(1-p)$. From Fig. 3 w_{1r} can be inferred as

$$w_{1r} = I_{02r}(t_r) \tag{13}$$

Let $I_{01o}(t)$ be the inventory level of OW at time t when we ignore both effects of lot quality and deterioration, so $I_{01o}(t) = w$. Let $I_{02o}(t)$ and $I_{03o}(t)$ be the inventory level of OW when we ignore the effect of deterioration only, so $I_{02o}(t) = w - pw$ and $I_{03o}(t) = -D(t-t_r) + w - pw$. From Fig. 4 w_{1o} can be inferred as,

$$w_{1o} = I_{03o}(T) \tag{14}$$

To avoid shortages, it is assumed that during screening the number of good items, $N_r(y,p)$ and $N_o(y,p)$ are at least equal to the demand, i.e. t_s and t_w

$$N_r(y,p) \geqslant Dt_s \tag{15}$$

$$N_o(y,p) \geqslant Dt_w \tag{16}$$

Thus from Eqs. (16) and (17) we get

$$p \leqslant \min\left(\frac{-D}{\beta(y-w)} + \left(1 + \frac{D}{\beta(y-w)}\right)e^{-\beta t_s}, \left(1 - \frac{D}{w\alpha}(e^{\alpha t_w} - 1)a^{\alpha t_r}\right)e^{-\alpha t_w}\right) \tag{17}$$

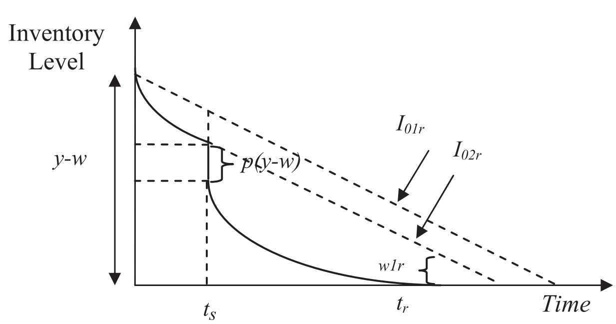

Fig. 3. Inventory level for rented warehouse showing the deteriorated items at time t_r.

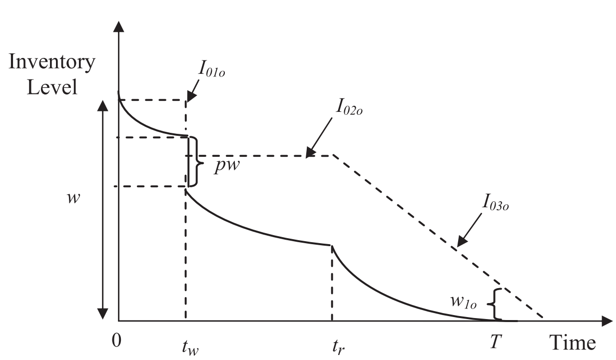

Fig. 4. Inventory level for own warehouse showing the deteriorated items at time T.

TR_r is the sum of the total sales volume of good quality and imperfect quality items from RW and is given as,

$$TR_r = s((y-w)(1-p) - w_{1r}) + v(y-w)p$$

Replacing w_{1r} in Eq. (14) by its expression, TR_r becomes

$$TR_r = sDt_r + v(y-w)p \tag{18}$$

TR_o is the sum of the total sales volume of good quality and imperfect quality items from OW and is given as,

$$TR_o = s(w - wp - w_{1o}) + vwp$$

Replacing w_{1o} in Eq. (14) by its expression, TR_o becomes

$$TR_o = sD(T - t_r) + vwp \tag{19}$$

Thus the total revenue generated from Eqs. (18) and (19) = $TR_r + TR_o = sDT + vyp$

The inventory holding cost per cycle in RW is

$$\begin{aligned} HC_r &= h_r\left\{\int_0^{t_s} I_r(t)d(t) + \int_{t_s}^{t_r} I_r(t)d(t)\right\} \\ &= h_r\left\{\frac{y-w}{\beta}(1-p) - \frac{D}{\beta^2}\left\{\ln\left(\frac{\beta}{D}(y-w)(1-pe^{\beta t_s}) + 1\right)\right\}\right\} \end{aligned} \tag{20}$$

The inventory holding cost per cycle in OW is

$$\begin{aligned} HC_o &= h_o\left\{\int_0^{t_w} I_o(t)dt + \int_{t_w}^{t_r} I_o(t)dt + \int_{t_r}^{T} I_o(t)dt\right\} \\ &= h_o\left\{\frac{w}{\alpha}(1-p) - \frac{D}{\alpha^2}\ln\left\{\frac{\alpha}{D}(w - wpe^{\alpha t_w})e^{-\alpha t_r} + 1\right\}\right\} \end{aligned} \tag{21}$$

Therefore, the total profit $TP(y)$ during the cycle $(0,T)$ is given by

$TP(y)$ = Total Revenue – Purchase cost – screening cost – Holding Costs

$$TP(y) = sDT + vyp - \left\{\begin{array}{l} k + cy + dy + h_r\left\{\frac{y-w}{\beta}(1-p) - \frac{D}{\beta^2}\left\{\ln\left(\frac{\beta}{D}(y-w)(1-pe^{\beta t_s}) + 1\right)\right\}\right\} \\ + h_o\left\{\frac{w}{\alpha}(1-p) - \frac{D}{\alpha^2}\ln\left\{\frac{\alpha}{D}(w - wpe^{\alpha t_w})e^{-\alpha t_r} + 1\right\}\right\} \end{array}\right\} \tag{22}$$

3.2. Case II: When $t_w > t_r$

Fig. 5 shows the behavior of the model over the time interval $[0,T]$ when the screening of OW completes after the exhaustion of RW. The screening of OW and RW gets completed at t_w and t_s respectively. During the time $[0,t_r]$, the on-hand inventory level in RW drops to zero due to demand and deterioration whereas in OW the depletion is due to deterioration only. From $[t_r,T]$ the

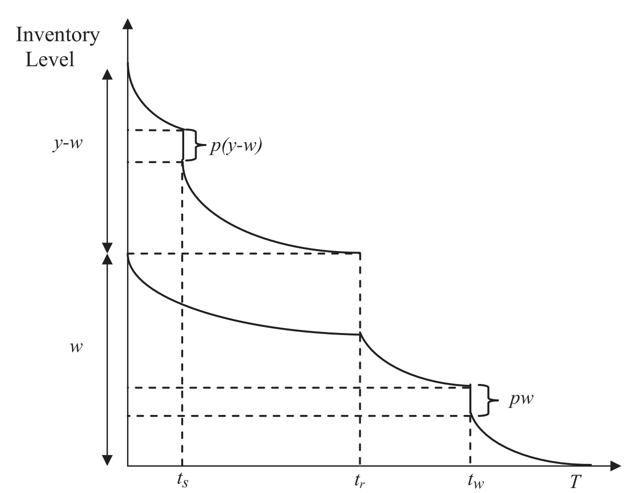

Fig. 5. Inventory level for the two-warehouse system when $t_s < t_r < t_w$.

OW depletes due to demand as well as deterioration and reduces to zero at time T.

The equations for the inventory level of RW are same as in case I. The differential equations and the corresponding solutions for OW are as below,

$$\frac{dI_o(t)}{dt} = -\alpha I_o(t) \quad 0 \leqslant t \leqslant t_r \tag{23}$$

$$\frac{dI_o(t)}{dt} = -D - \alpha I_o(t) \quad t_r \leqslant t \leqslant T \tag{24}$$

Solving the above differential equations with the boundary conditions, $I_o(0) = w$, $I_o(t_w^+) = I_o(t_w) - pw$, and continuity of $I_o(t)$ at $t = t_r$, we get

$$I_o(t) = we^{-\alpha t} \quad 0 \leqslant t \leqslant t_r \tag{25}$$

$$I_o = \frac{-D}{\alpha} + \left(we^{-\alpha t_r} + \frac{D}{\alpha}\right)e^{-\alpha(t-t_r)} \quad t_r \leqslant t \leqslant t_w \tag{26}$$

$$I_o = \frac{-D}{\alpha} + \left(we^{-\alpha t_r} + \frac{D}{\alpha} - pwe^{\alpha(t_w - t_r)}\right)e^{-\alpha(t-t_r)} \quad t_w < t \leqslant T \tag{27}$$

$$\text{Also, } I_o(T) = 0 \text{ implies, } T = t_r + \frac{1}{\alpha}\ln\left[\frac{\alpha w}{D}(1 - pe^{\alpha t_w})e^{-\alpha t_r} + 1\right] \tag{28}$$

Thus, we see the total time cycle T is same in *Case I* (Eq. (10)) and *Case II* (Eq. (26))

Now, the inventory holding cost per cycle in OW is

$$\begin{aligned} HC_o &= h_o\left\{\int_0^{t_w} I_o(t)dt + \int_{t_w}^{t_r} I_o(t)dt + \int_{t_r}^{T} I_o(t)dt\right\} \\ &= h_o\left\{\frac{w}{\alpha}(1-p) - \frac{D}{\alpha^2}\ln\left\{\frac{\alpha}{D}(w - wpe^{\alpha t_w})e^{-\alpha t_r} + 1\right\}\right\} \end{aligned} \tag{29}$$

Therefore, the total profit $TP(y)$ during the cycle $(0, T)$ is given by

$TP(y)$ = Total Revenue – Purchase cost – screening cost – Holding Costs

$$TP(y) = sDT + vyp - \left\{\begin{array}{l} k + cy + dy + h_r\left\{\frac{y-w}{\beta}(1-p) - \frac{D}{\beta^2}\left\{\ln\left(\frac{\beta}{D}(y-w)(1-pe^{\beta t_s}) + 1\right)\right\}\right\} \\ + h_o\left\{\frac{w}{\alpha}(1-p) - \frac{D}{\alpha^2}\ln\left\{\frac{\alpha}{D}(w - wpe^{\alpha t_w})e^{-\alpha t_r} + 1\right\}\right\} \end{array}\right\} \tag{30}$$

Hence, from Eqs. (22) and (30), we see that the profit functions in the case I and case II are the same. Thus, the profit function per unit time would also be same in both the cases and equal to

$$TPU(y) = \frac{1}{T}\left[sDT + vyp - \left\{\begin{array}{l} k + cy + dy + h_r\left\{\frac{y-w}{\beta}(1-p) - \frac{D}{\beta^2}\left\{\ln\left(\frac{\beta}{D}(y-w)(1-pe^{\beta t_s}) + 1\right)\right\}\right\} \\ + h_o\left\{\frac{w}{\alpha}(1-p) - \frac{D}{\alpha^2}\ln\left\{\frac{\alpha}{D}(w - wpe^{\alpha t_w})e^{-\alpha t_r} + 1\right\}\right\} \end{array}\right\}\right] \tag{31}$$

Since p is a random variable with known probability density function, $f(p)$ then the expected total profit per unit time after applying renewal-reward theorem Ross (1996) is

$$E[TPU(y)] = \frac{E[TP(y)]}{E[T]}$$

$$E[TPU(y)] = \frac{1}{E(T)}\left[sDT + vyp - \left\{\begin{array}{l} k + cy + dy + h_r\left\{\frac{y-w}{\beta}(1-E[p]) - \frac{D}{\beta^2}\left\{\ln\left(\frac{\beta}{D}(y-w)(1-E[p]e^{\beta t_s}) + 1\right)\right\}\right\} \\ + h_o\left\{\frac{w}{\alpha}(1-E[p]) - \frac{D}{\alpha^2}\ln\left\{\frac{\alpha}{D}w(1 - E[p]e^{\alpha t_w})e^{-\alpha t_r} + 1\right\}\right\} \end{array}\right\}\right] \tag{32}$$

4. Solution procedure

In this model, the main objective is to obtain the optimal value of y which maximizes the expected total profit function $E[TPU(y)]$. So, in order to determine the optimal values of y which maximizes the expected total profit per unit time, the necessary and sufficient conditions for optimality are:

$$\frac{d(E[TPU(y)])}{dy} = \frac{1}{T^2}\left[\begin{array}{l} \left\{vp - \left(c + d + \frac{h_r}{\beta}(1-p) + D(h_0 - h_r)\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)t_r'\right)\right\}T \\ -\left\{vyp - \left(k + cy + dy + \left(\frac{h_r(y-w)}{\beta} + \frac{h_o w}{\alpha}\right)(1-p) + D\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)(h_o - h_r)t_r\right)\right\}T' \end{array}\right] = 0 \tag{33}$$

and $\frac{d^2(E[TPU(y)])}{dy^2} \leqslant 0$.

Lemma. *For a continuous and differentiable function $f(t)$ on (a, b) is concave on (a, b) if and only if its derivative $\frac{df(t)}{dt}$ is monotonically decreasing on (a, b).*

The proof of this lemma, i.e., the convexity of all profit functions are given in Appendix.

5. Special cases

a. When $\alpha = 0$ and $\beta = 0$ i.e. there is no deterioration.
   Thus, the equations become $t_s = \frac{y-w}{x}$, $t_w = \frac{w}{x}$, $t_r = \frac{(1-p)(y-w)}{D}$ and $T = \frac{(1-p)y}{D}$.
   The total profit per unit time then becomes:

$$\begin{aligned} TPU(y) &= D\left(s - v + \frac{h_r(y-w)^2}{xy} + \frac{h_o w^2}{xy}\right) \\ &\quad + \frac{D}{(1-p)}\left(v - \frac{k}{y} - c - d - \frac{h_r(y-w)^2}{xy} - \frac{h_o w^2}{xy}\right) \\ &\quad - \frac{h_r(y-w)^2(1-p)}{2y} - h_o\left(w(1-p) - \frac{w^2(1-p)}{2y}\right) \end{aligned}$$

   This is the same profit function as obtained by Chung et al. (2009).

b. When w = 0, $h_r = h_w$.
   The total profit per unit time then becomes:

$$TPU(y) = sD + \frac{hD}{\alpha} + \frac{\alpha y\left(vp - c - d - \frac{h}{\alpha}(1-p)\right) - k\alpha}{\ln\left(y + \frac{D}{\alpha} - pye^{\alpha t_s}\right) - \ln\left(\frac{D}{\alpha}\right)}$$

   This is the same profit function as obtained by Moussawi-Haidar et al. (2014).

c. When w = 0, $h_r = h_w$, $\alpha = 0$ and $\beta = 0$.
   The total profit per unit time then becomes:

$$TPU(y) = D\left(s - v + \frac{hy}{x}\right) + \frac{D}{(1-p)}\left(v - \frac{k}{y} - c - d - \frac{hy}{x}\right) - \frac{hy(1-p)}{2}$$

   This is the same profit function as obtained by Salameh and Jaber (2000).

d. When w = 0, $h_r = h_w$, $p = 0$, $\alpha = 0$ and $\beta = 0$ i.e. there is no deterioration.
   The total profit per unit time then becomes:

$$TPU(y) = D(s-c) - \frac{Dk}{y} - \frac{hy}{2}$$

   As a result, the analysis reduces to traditional EOQ model and optimal quantity will be

$$y^* = \sqrt{\frac{2kD}{h}}.$$

6. Numerical example

In this section, a numerical analysis has been done to validate the model developed. The optimal order quantity (y^*) and

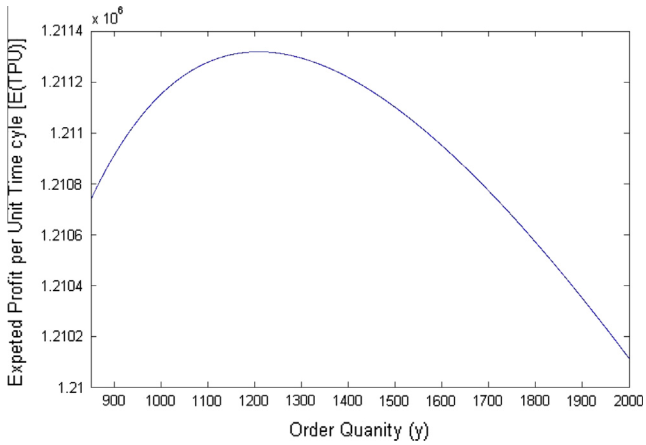

Fig. 6. Graph showing the concave nature of profit per unit time cycle function.

maximum profit per unit time $E^*[TPU(y)]$ are found out for a given set of parameters. Let us consider a situation with the following parameters: $w = 800$ units, $D = 50,000$ units/year, $k = \$100$, $h_o = \$5$/unit/year, $h_r = \$7$/unit/year, $x = 1$ unit/min, $d = \$0.5$/unit, $c = \$25$/unit, $v = \$20$/unit, $s = \$50$/unit, $\alpha = 5\%$, $\beta = 3\%$.

Percentage defective random variable p with its p.d.f is:

$$f(p) = \begin{cases} 25 & 0 \leqslant p \leqslant 0.04 \\ 0 & otherwise \end{cases}, \quad E(p) = 0.02$$

If the inventory operation operates for 8 h per day, for 365 days a year, then the annual screening rate is $x = 175,200$ units. We note that the above chosen parameters satisfy the condition in (17). Then, the optimal lot size y^* is found by maximizing the total profit per unit time function by Eq. (33). We obtain the following optimal solution: $y^* = 1209$ units and corresponding expected annual profit, obtained from (32) as $E^*[TPU(y)] = \$12,11,319$ (see Fig. 6).

7. Sensitivity analysis

In this section, Sensitivity analysis has been performed to study the effects of deterioration (α and β), percentage of defective items (p) and change in the capacity of OW (w) on the expected lot size (y^*) and the expected total profit per unit time $E^*[TPU(y)]$. We have

Table 3
Effect of various parameters on the model.

Parameter (initial value)	Parameter value	y^*	$E^*[TPU(y)]$
$k(100)$	50	893	1,213,748
	150	1457	1,209,403
	200	1671	1,207,770
$D(50,000)$	40,000	1094	968,333
	60,000	1312	1,454,381
	70,000	1410	1,697,504
$s(50)$	25	1209	1,211,319
	75	1209	2,461,319
	100	1209	3,711,319
$v(20)$	15	1208	1,206,214
	20	1208	1,211,318
	25	1208	1,216,424
$c(25)$	20	1224	1,466,567
	30	1194	956,072
	40	1165	445,584

Table 1
Effect of changing of capacity of own-warehouse (w) and percentage of defective items on the optimal solution.

$E[p]$	Capacity of own-warehouse (w)	t_w^*	t_s^*	t_r^*	T^*	y^*	$E^*[TPU(y)]$	Case
0.02	400	0.002	0.0044	0.015	0.023	1165.615	1211027.413	$t_w < t_s < t_r$
	800	0.005	0.0023	0.008	0.024	1208.182	1211318.73	$t_s < t_w < t_r$
	1200	0.007	0.0004	0.001	0.025	1276.047	1211415.897	$t_s < t_r < t_w$
0.05	400	0.002	0.0045	0.015	0.023	1191.920	1202103.01	$t_w < t_s < t_r$
	800	0.005	0.0025	0.008	0.024	1239.831	1202398.451	$t_s < t_w < t_r$
	1200	0.007	0.0007	0.002	0.025	1314.775	1202481.865	$t_s < t_r < t_w$
0.1	400	0.002	0.0048	0.015	0.022	1236.747	1185899.302	$t_w < t_s < t_r$
	800	0.005	0.0028	0.009	0.023	1294.474	1186206.657	$t_s < t_w < t_r$
	1200	0.007	0.001	0.003	0.025	1381.789	1186271.014	$t_s < t_r < t_w$

Table 2
Effect of deterioration rate on the optimal solution.

Deterioration (%) $\alpha > \beta$		$E[p]$	t_s^*	t_r^*	T^*	y^*	$E^*[TPU(y)]$
α	β						
0	0	0.02	0.0028	0.010	0.025	1289.806	1212067.747
		0.06	0.0030	0.010	0.025	1331.694	1200052.655
		0.1	0.0033	0.010	0.025	1374.995	1186966.711
5	3	0.02	0.0023	0.008	0.024	1208.201	1211318.73
		0.06	0.0026	0.008	0.023	1249.936	1199298.23
		0.1	0.0028	0.009	0.023	1293.057	1186206.663
15	10	0.02	0.0015	0.005	0.021	1067.409	1209920.002
		0.06	0.0018	0.006	0.021	1108.738	1197880.885
		0.1	0.0020	0.006	0.021	1151.350	1184770.592
30	20	0.02	0.0007	0.002	0.018	914.624	1208066.252
		0.06	0.0009	0.003	0.018	957.225	1195987.08
		0.1	0.0011	0.004	0.018	1001.047	1182838.268

also observed the change of various parameters including the Ordering cost (k), Demand-rate (D), selling price of good items (s), selling price of defective items (v) and Unit variable cost (c). The observations have been recorded from Tables 1–3.

- From Table 1 it can be deduced that, as the percentage of defective items increases for a fixed capacity of OW, the optimum order quantity increases to meet the demand of quality products with a little or no change in time cycles. But one can observe a decrease in total expected annual profit per unit time. Thus, even though the optimal order quantity increases with increase in percentage of defective items, the stock of good quality products decreases, due to increased screening cost, thereby resulting in lesser profit.
- While keeping the proportion of defective items fixed, as the capacity of OW increases the optimal order quantity, replenishment cycle (T) and the maximum total expected annual profit per unit time increase. This is due to the fact that the holding cost of OW is considered to be lower than that of the RW.
- Table 2 shows the effect of deterioration on the optimal lot size and expected total profit per unit time. When the deterioration is neglected i.e. $\alpha = \beta = 0$, the model reduces to the one described by Chung et al. (2009).
- Now, for a fixed proportion of defective items, we understand, that as the deterioration increases, the time cycles (t_r) and (T), optimal lot size (y^*) and the expected profit per unit time $E^*[TPU(y)]$ decrease significantly. Therefore, it is advisable to place small orders more frequently in deteriorating conditions to avoid the loss due to deterioration.

The following observations have been observed from Table 3:

- By changing the ordering cost (k), the optimum order size (y^*) increases significantly, although the maximum expected profit per unit time $E^*[TPU(y)]$ decreases. So, the retailer should order in greater quantity to avoid the recurring ordering cost.
- It is also a noteworthy observation that changing the selling prices of good items (s) or defective items (v) does not change the optimum order quantity (y^*). But if the cost price per unit item (c) increases, the retailer is forced to order less and accordingly the optimum order quantity (y^*) decreases.
- Furthermore, with increase in demand rate, the optimal order quantity increases, to meet the demand generated.

8. Conclusion

The present study incorporates the concept of two warehouses and the effect of deterioration on the retailer's economic order quantity, invalidating the presumption that there are no imperfect quality items. Hence, it becomes obligatory to screen the lot, to check out the defective items so that the demand can be fulfilled from good quality products. The screening rate is assumed to be more than the demand rate, so that the demand can be fulfilled, out of the products, which are found to be of perfect quality, while the screening is in process. The analytical results illustrate that there exists a unique optimal lot size that maximizes the expected total profit per unit time. Finally, a numerical example is presented to elucidate the applicability of the proposed model followed by the sensitivity analysis.

Further research in this area can be extended in several ways. For instance, we may generalize the demand to stock dependent and stochastic demand. Also, we can extend the model to allow for shortages. Finally, we can consider the effect of trade credit or inflation on the economic order quantity.

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Appendix A

$$\frac{d(E[TPU(y)])}{dy} = -\frac{f_1(y)}{T^2}$$

where

$$f_1(y) = \begin{bmatrix} \left\{ vp - \left(c + d + \frac{h_r}{\beta}(1-p) + D(h_0 - h_r)\left(\frac{1}{\alpha} - \frac{1}{\beta}\right) t'_r \right) \right\} T \\ -\left\{ vyp - \left(k + cy + dy + \left(\frac{h_r(y-w)}{\beta} + \frac{h_o w}{\alpha}\right)(1-p) + D\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)(h_o - h_r) t_r \right) \right\} T' \end{bmatrix}$$

$$\begin{aligned}
\frac{d(f_1(y))}{dy} = \Bigg[\Bigg\{ & 2\left(vp - c - d - \frac{h_r(1-p)}{\beta} - \frac{X}{A}\left(\frac{h_o}{\alpha} - \frac{h_r}{\beta}\right)\right)\left(\frac{X}{DAB}\right) \\
& + \left(y\left(vp - c - d - \frac{h_r}{\beta}(1-p)\right) - \left(\frac{h_o}{\alpha} - \frac{h_r}{\beta}\right)(1-p) \right. \\
& \left. - \frac{D \ln A}{\beta}\left(\frac{h_o}{\alpha} - \frac{h_r}{\beta}\right) - k \right) \left(\frac{-1}{\beta AB}\left(\frac{2\beta^2 p e^{\frac{\beta t_s}{x}}}{Dx} + \frac{\beta^3 (y-w) p e^{\frac{\beta t_s}{x}}}{Dx^2} \right) \right. \\
& \left. - \frac{\beta X^2}{D^2 A^2} + \frac{(\alpha + \beta)(B-1)}{\beta D^2 A^2 B^2} \right) \Bigg\} \\
& - \left(\frac{1}{\beta A}\left(\frac{h_o}{\alpha} - \frac{h_r}{\beta}\right)\left(\frac{2\beta^2 p e^{\frac{\beta t_s}{x}}}{Dx} + \frac{\beta^3 (y-w) p e^{\frac{\beta t_s}{x}}}{Dx^2} \right) \right. \\
& \left. + \frac{\beta X^2}{DA^2}\left(\frac{h_o}{\alpha} - \frac{h_r}{\beta}\right) \right) - \left(y\left(vp - c - d - \frac{h_r}{\beta}(1-p) \right) \right. \\
& \left. - \left(\frac{h_o}{\alpha} - \frac{h_r}{\beta}\right)(1-p) - \frac{D \ln A}{\beta}\left(\frac{h_o}{\alpha} - \frac{h_r}{\beta}\right) - k \right)\left(\frac{X}{DAB}\right)^2 \Bigg] > 0 \\
& if \quad T > 0
\end{aligned}$$

To determine the optimal value of y say y^*, we solve the equation $f_1(y) = 0$.

We obtain $\frac{df_1(y)}{dy} > 0$ if $T > 0$. As $f_1(y)$ is an increasing function on $[0, \infty)$, then $\frac{d(E[TPU(y)])}{dy}$ is a decreasing function on $[0, \infty)$. Using Lemma, $E[TPU(y)]$ is a concave function on $[0, \infty)$.

Also, as $\lim T \to \infty$, then $f_1(y) \to \infty$.

Further,

$$\begin{aligned}
\frac{d(E[TPU(y)])}{dy} &> 0; \quad if \quad T \in [0, T^*), \\
&= 0; \quad if \quad T = T^*, \\
&< 0; \quad if \quad T \in [T^*, \infty).
\end{aligned}$$

By applying the intermediate value theorem, we can state that the optimal solution y^* exists and is unique.

Where $t'_s = \dfrac{1}{x}$, $t'_r = \dfrac{1 - (1 + \beta t_s) p e^{\beta t_s}}{\beta (y - w)(1 - p e^{\beta t_s}) + D}$ *and*

$$T' = \frac{D t'_r}{\alpha w (1 - p e^{\alpha t_w}) e^{-\alpha t_r} + D}$$

$$A = 1 + \frac{\beta (y-w)}{D}(1 - p e^{t_s}), \quad B = 1 + \frac{\alpha w}{D}(1 - p e^{\alpha t_w}) e^{-\alpha t_r} \quad and$$

$$X = 1 - p e^{\frac{\beta (y-w)}{x}} - \frac{\beta}{x}(y - w) p e^{\frac{\beta (y-w)}{x}}.$$

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