Homework 3

Due: Feb, 20, midnight

Problem 1. Consider the following first-order system:

$$\dot{y} = -0.5y + 2u, \quad y(0) = 0 \tag{1}$$

(a) (5 points) First, consider a proportional control law $u(t) = K_p\big(r(t) - y(t)\big)$ where $r(t)$ is the reference command. As mentioned in class, it is typically important, for practical reasons, that $u(t)$ does not get too large. Consider a unit step command:

$$r(t) = \begin{cases} 0 & t < 0\,\text{sec} \\ 1 & t \geq 0\,\text{sec} \end{cases} \tag{2}$$

For what gains K_p is $|u(t)| \leq 1$ for all time? (Hint: The largest value of $|u(t)|$ will occur at $t = 0$.)

(b) (5 points) Choose the gain K_p that satisfies the constraint in part (a) and minimizes the steady-state error due to the unit step command. What is the time constant of the closed-loop system for this gain?

(c) (5 points) Next, consider a proportional-integral (PI) control law:

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau)\, d\tau \tag{3}$$

where $e(t) = r(t) - y(t)$ is the tracking error. Combine the system model (Equation 1) and the PI controller (Equation 3) to obtain a model of the closed-loop system in the form:

$$\ddot{y} + a_1\dot{y} + a_0 y = b_1\dot{r} + b_0 r \tag{4}$$

How do the damping ratio and natural frequency depend on K_p and K_i? What is the steady-state error if r is a unit step?

(d) (10 points) Keep the value of K_p designed in part (b) and choose K_i to obtain a damping ratio of $\zeta = 0.7$. For these PI gains, what are the estimated maximum overshoot and 5% settling time (neglecting the effect of the zero)?

(e) (5 points) Plot the output response $y(t)$ due to a unit step r for both the P and PI controllers. The closed-loop with the PI controller has a zero due to the term $b_1\dot{r}$. Briefly explain how this zero affects the response.

Problem 2. (20 points) Consider the following first-order system:

$$\ddot{y} - 2\dot{y} + y = u, \quad y(0) = 0$$

with a PD controller in the form

$$u_t = K_p\big(r(t) - y(t)\big) - K_d\dot{y}(t).$$

(a) What is the ODE model for the closed-loop system from r to y?

(b) Choose (K_p, K_d) so that the closed-loop system is stable and has $(\omega_n, \zeta) = (2, 0.5)$.

(c) What is the steady-state error if r is a unit step reference?

(d) Would you increase or decrease K_p to reduce the steady-state error?

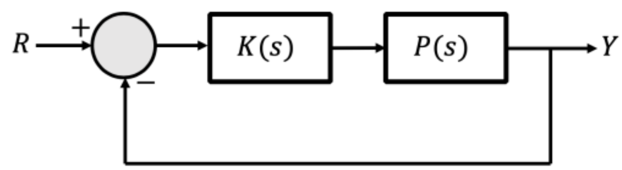

Figure 1: A diagram of a unity feedback system.

Problem 3. (20 points) Consider the unity feedback system in Figure 1. Let the plant's transfer function be given by:

$$P(s) = \frac{6.32}{s^2 - 0.12}$$

Suppose our controller is given by $K(s) = 4$. Can we choose $K(s)$ as a PI controller to stabilize the closed-loop system from r to y? Apply the Routh-Hurwitz criterion to determine this.

Problem 4. Figure 2 below shows the key forces on a car. By Newton's second law, the longitudinal motion of the car is modeled by the following first-order ODE:

$$m\dot{v}(t) = F_{\text{net}}(t) - F_{\text{aero}}(t) - F_{\text{roll}} - F_{\text{grav}}(t) \tag{5}$$

where v is the velocity $(\frac{m}{\text{sec}})$, $m = 2085\,\text{kg}$ is the mass, and the forces are given by:

- F_{net} is the net engine force. For simplicity, assume this force is proportional to the throttle angle: $F_{\text{net}} = ku$ where $u :=$ engine throttle input (deg) and $k = 40\,\frac{N}{\text{deg}}$ is the force constant. The engine throttle is physically limited to remain within $0° \le u \le 90°$.
- F_{aero} is the aerodynamic drag force. For this problem we will model this as $F_{\text{aero}} = b_0 + b_1 v$ where $b_0 = -336.4\,\text{N}$ and $b_1 = 23.2\,\frac{\text{N}\cdot\text{sec}}{\text{m}}$. This approximation is accurate for velocities near $v = 29\,\frac{\text{m}}{\text{sec}}$.

- $F_{\text{roll}} = 228\,\text{N}$ is the rolling resistance force due to friction at the interface of the tire and road.
- F_{grav} is the force due to gravity. This is given by $F_{\text{grav}} = mg\sin(\theta)$ where θ is the slope of the road (rads) and $g = 9.81\,\frac{\text{m}}{\text{sec}^2}$ is the gravitational constant.

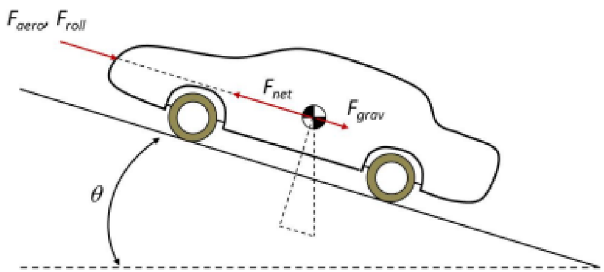

Figure 2: Free body diagram for a car.

[1] Additional details on the model are given in Exame 2.1 of the book [FPE]. Putting these pieces together yields the following first-order ODE:

$$2085\dot{v}(t) + 23.2v(t) = 40u(t) + 108.4 - F_{\text{grav}}(t) \tag{6}$$

The input is the throttle u and the output is the velocity v. The gravitational force F_{grav} is a disturbance. The homework contains a Simulink diagram `CruiseControlSim.mdl` that implements the vehicle dynamics. You can either implement the dynamics by yourself or use the provided Simulink model. For your convenience, there is also an m-file `CruiseControlPlots.m` that can be used as a template for your answers (you can also just use your own template).

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[1] A better approximation for the aerodynamic drag is $F_{\text{aero}} = c_D v^2$, where $c_D = 0.4\,\frac{\text{N}\cdot\text{sec}^2}{\text{m}^2}$. This is a nonlinear function of the velocity. For simplicity, we can approximate this by the linear function:

$$c_D v^2 \approx b_0 + b_1 v$$

This linear approximation is obtained by performing a Taylor series around the velocity $\bar{v} = 29\,\frac{\text{m}}{\text{sec}}$.

(a) (5 points) Assume the car is on flat road so that $\theta(t) = 0\,\text{rads}$ and $F_{\text{grav}}(t) = 0\,\text{N}$. What is the open-loop (constant) input $\bar{u}$ required to maintain a desired velocity of $v_{\text{des}} = 29\,\frac{\text{m}}{\text{sec}}$?

(b) (5 points) Simulate the system with the input $\bar{u}$, initial condition $v(0) = 29\,\frac{\text{m}}{\text{sec}}$, and the following gravitational force:

$$F_{\text{grav}}(t) = \begin{cases} 0\,\text{N} & t < 10\,\text{sec} \\ 350\,\text{N} & t \geq 10\,\text{sec} \end{cases}$$

Submit a plot of velocity v versus time t. Note that the gravitational force of 350 N corresponds to a very small road slope of $\approx 1°$. Observe that this small slope causes a large deviation in the vehicle velocity.

(c) (10 points) Let $e(t) = v_{\text{des}} - v(t)$ denote the tracking error between the desired velocity $v_{\text{des}} = 29\,\frac{\text{m}}{\text{sec}}$ and actual velocity $v(t)$. Consider a PI controller of the following form:

$$u(t) = \bar{u} + K_p e(t) + K_i \int_0^t e(\tau)\,d\tau \tag{7}$$

where $\bar{u}$ is the open-loop input computed in part (a). Choose the PI gains so that the cruise control system is stable and rejects disturbances due to changing road slopes within $\approx 10\,\text{sec}$. The closed-loop should also be over or critically damped as oscillations are uncomfortable for the driver.

Hint: Note that $\bar{u}$ is chosen to maintain a desired velocity $v_{\text{des}} = 29\,\frac{\text{m}}{\text{sec}}$ when on flat road $\theta = 0°$. In other words, $\bar{u}$ is chosen to satisfy:

$$23.2 v_{\text{des}} = 40\bar{u} + 108.4$$

Thus substituting the expression for $u(t)$ (Equation 8) into the longitudinal dynamics (Equation 7) yields:

$$2085\dot{v}(t) + 23.2 v(t) = 23.2 v_{\text{des}} + 40\left(K_p e(t) + K_i \int_0^t e(\tau)\,d\tau\right) - F_{\text{grav}}(t)$$

This closed-loop ODE can be used to select your gains.

(d) (10 points) Modify the Simulink diagram to include your PI controller. Simulate the closed-loop system with your PI controller, initial condition $v(0) = 29\,\frac{\text{m}}{\text{sec}}$, and the following gravitational force:

$$F_{\text{grav}}(t) = \begin{cases} 0\,\text{N} & t < 10\,\text{sec} \\ 1400\,\text{N} & t \geq 10\,\text{sec} \end{cases}$$

Note that the gravitational force of 1400 N corresponds to a road slope of $\approx 4^\circ$. You will need to update the Simulink block that generates this gravitational force.

Submit plots of velocity v versus time t and throttle input u versus t. Verify that the throttle input remains within the physical limits. You should also submit the Simulink diagram modified to include your PI controller.