Strategic Exercise with Market Crisis

Huangyu Chen* Binhan Chi

Abstract

We consider a symmetric duopoly market in a real option setting with exercise interaction. In particular we use jump process to analyze the impact of a market crisis.

1 Model Setting

At outset, local real estate market has two identical buildings owned by two distinct individuals, each collecting R units of rent per unit time. Both owners have rights to develop a new superior building with an immediate cost of construction I.

A new building will yield a rental rate according to a downward-sloping inverse demand function:

$$P(t) = X(t) \cdot D[Q(t)]$$

where $X(t)$ is multiplicative demand shock, $P(t)$ is the rent flow at time t, $Q(t)$ is the supply of improved buildings at time t, and $D(\cdot)$ is a differentiable function with $D' < 0$. The Grenadier 1996 assumes that the shock follows a Geometric Brownian Motion, and we add an additional jump term to consider the case of a crisis. Thus, we suppose:

$$dX = \mu X dt + \sigma X dW_t + \int_R -\phi X \, N(dt, dx)$$

where $\phi \in [0, 1]$ describes the severity of the crisis, the jump has a jump size of $-\phi X$ and an intensity λ, when $\phi = 1$ the market is under risk of utter deterioration.

In order to make this model more realistic, the development of superior building will last for δ years and during which no rent will be collected. On the other hand, whenever somebody has started developing a new building, the rent for old building will decrease to $(1 - \gamma)R$. We assume this change is immediate for the sake of tractability, no results of the model are altered if we assume γ is a function of stage of new building construction.

2 Value of Leader and Follower

In the model, the individual who chooses to develop first will become the Leader, and the other will become the Follower. In this section, the value function of both parties' strategies will be derived.

*correspondent author, email:rencochy@bu.edu — This is an extension to 1996 paper of Professor Steven Grenadier from Stanford: *The Strategic Exercise of Options: Development Cascades and Overbuilding in Real Estate Markets.*

The value function of Follower's strategy will be discussed first. We will follow the standard in solving dynamic games and we will work backward in a dynamic-programming fashion. Therefore, we assume that one player has already begun the construction.

The value function of the Follower strategy contains two parts, including existing building (old building) and an option to exchange the existing building for a new building. The existing building can generate a return of $(1-\gamma)\cdot R$. Since this payment is deterministic, its discounted value is $(1-\gamma)\cdot R/r$, because it is a permanent annuity. In addition, the follower can also choose to exercise the option, which requires to pay a construction cost I, and the new building needs δ years until completion. Denote the value of this option as $W(X)$. Thus, the total value of the Follower strategy, $F(X)$, will be the portfolio value: $F(X) = (1-\gamma)\cdot R/r + W(X)$.

We now derive the value for $W(X)$. An American Option like this should satisfy the following PDP problem:

$$W(X_t) = Max\{g(X_t), E[e^{-\int_t^{t+\varepsilon} r_s ds} W(X_{t+\varepsilon})] | X_{t-} = x)\} \tag{1}$$

where $g(X_t)$ is the intrinsic value for the option:

$$g(X_t) = \left[\frac{D(2)}{r-\mu} e^{-(r-\mu)\delta}\right] \cdot X_t - \frac{(1-\gamma)R}{r} - I \tag{2}$$

Such American Option problem has a trigger value, below which it's optimal to wait and above which it's optimal to exercise. For follower's problem, there exists a trigger value X_F. When $X_t < X_F$, the option is still in it's continuation region:

$$W(X_t) = E[e^{-\int_t^{t+\varepsilon} r_s ds} W(X_{t+\varepsilon}) | X_{t-} = x)]$$

This gives corresponding HJB equation as follows:

$$\frac{1}{2}\sigma^2 X_t^2 W''(X_t) + \mu X_t W'(X_t) + \lambda \cdot (W((1-\phi)X_t) - W(X_t)) - rW(X_t) = 0$$

Or,

$$\frac{1}{2}\sigma^2 X_t^2 W''(X_t) + \mu X_t W'(X_t) - (r+\lambda)\cdot W(X_t) + \lambda \cdot W((1-\phi)X_t) = 0 \tag{3}$$

with boundary conditions:

$$\begin{aligned} W(X_F) &= \left[\frac{D(2)}{r-\mu} \cdot e^{-\delta(r-\mu)}\right] \cdot X_F - \frac{(1-\gamma)R}{r} - I \\ W'(X_F) &= \frac{D(2)}{r-\mu} e^{-\delta(r-\mu)} \end{aligned} \tag{4}$$

The solution of this PDE has the form of $W(X_t) = C \cdot {X_t}^{\beta}$. Taking this into equation (3) gives:

$$\frac{1}{2}\sigma^2\beta^2 + (\mu - \frac{1}{2}\sigma^2)\beta - (r+\lambda) + \lambda(1-\phi)^{\beta} = 0 \tag{5}$$

This equation has no closed form solution unless $\phi = 1$ or $\phi = 0$. In the rest of this section, we'll assume $\phi = 1$ to discuss results analytically, the numerical results of the general solutions will be discussed in section 4.

Given that $\phi = 1$ and the boundary conditions, we can find the analytical solution.

$$W(X_t) = \begin{cases} \left(\dfrac{I + (1-\gamma)R/r}{\beta - 1}\right) \cdot \left(\dfrac{X_t}{X_F}\right)^{\beta}, & \text{if } X_t \leq X_F \\ \left[\dfrac{D(2)}{r-\mu} e^{-\delta(r-\mu)}\right] \cdot X_t - \dfrac{(1-\gamma)R}{r} - I, & \text{if } X_t > X_F \end{cases} \tag{6}$$

where

$$\beta = \frac{-(\mu - \frac{1}{2}\sigma^2) + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2(r+\lambda)\sigma^2}}{\sigma^2} > 1$$

$$X_F = \left(\frac{\beta}{\beta - 1}\right) \cdot \left(\frac{r-\mu}{D(2)}\right) \cdot e^{(r-\mu)\delta} \cdot \left[I + \frac{(1-\gamma)R}{r}\right]$$

Follower will chose to exercise her option the moment that X_t hits X_F, we define such time as:

$$T_F = \inf\left\{t \geq 0 : X_t \geq \left(\frac{\beta}{\beta - 1}\right) \cdot \left(\frac{r-\mu}{D(2)}\right) \cdot e^{(r-\mu)\delta} \cdot \left[I + \frac{(1-\gamma)R}{r}\right]\right\}$$

Now, let's consider the value function of the Leader strategy, conditioning on the Follower pursuing the optimal Follower strategy. Assume the Leader has already exercised his construction option and now has τ years until completion, where $\tau \in [0, \delta]$. For the next τ years, Leader will not receive any rental from this real estate. Once the construction of the new building is complete, Leader will earn monopoly rental rate of $X(t) \cdot D(1)$. However, when the Follower also chooses to construct new building, the Leader can only receive duopoly cash flow $X(t) \cdot D(2)$.

Therefore, the value of the Leader strategy can be replicated by following portfolio of options:

1. Long a call option on the building which pays a perpetual dividend rate of $X(t) \cdot D(1)$. This call option has a 0 exercise price, and a fixed expiration date of τ

2. Long a call option on the building which pays a perpetual dividend rate of $X(t) \cdot [D(2) - D(1)]$. This call option has a zero exercise price, and a stochastic expiration date of $T_F + \delta$

The value of this portfolio can be derived in a similar way as the derivation for the value of the Follower strategy. Denote the value of the Leader strategy, $L(X, \tau)$.

The solution for $L(X, \tau)$ can be expressed as:

$$L(X, \tau) \tag{7}$$

$$= \begin{cases} \dfrac{e^{-(r-\mu)\tau}}{r-\mu} X D(1) + \dfrac{\beta}{\beta - 1} \dfrac{D(2) - D(1)}{D(2)} \left(I + \dfrac{(1-\gamma)R}{r}\right) \left(\dfrac{X}{X_F}\right)^{\beta}, & \text{if} \quad X < X_F \\ \dfrac{e^{-(r-\mu)\tau}}{r-\mu} X D(1) + \dfrac{X e^{-(r-\mu)\delta}}{r-\mu} [D(2) - D(1)], & \text{if} \quad X \geq X_F \end{cases}$$

Finally, we can compare the relative value of the Leader and Follower strategy at the moment the Leader begins the construction. The Leader receives a payoff of $L(X,\delta) - I$ and the Follower receives $F(X)$. Depending on the initial entry time (which is determined by equilibrium considerations and derived in the next section), the Leader's return may be larger or smaller than that of the Follower. The following statement describes the relative valuations:

There exists a unique point, $X_L \in (0, X_F)$ with following properties:

$$\begin{aligned}
L(X,\delta) - I &< F(X) && \text{for} \quad X < X_L \\
L(X,\delta) - I &= F(X) && \text{for} \quad X = X_L \\
L(X,\delta) - I &> F(X) && \text{for} \quad X_L < X < X_F \\
L(X,\delta) - I &= F(X) && \text{for} \quad X \geq X_F
\end{aligned}$$

This demonstrates that there is a unique value of $X \in (0, X_F)$, called X_L, at which the payoffs to both Leader and Follower are equal. At any point below X_L, each developer would prefer being the Follower. At any point above X_L, each developer would prefer being the leader. We will assume that when both developers want to be leader, one of them will be randomly selected to be the leader, and the another one has to begin a moment later. This could be explained by the fact that real estate development must be approved by local government. If both developers want to exercise despite being Leader or Follower, then we will describe such exercise as Simultaneous Exercise. Therefore, when X hits X_L from below, there will only be one developer building.

3 Equilibrium Exercise Strategy

Given Leader's Optimal Strategy and Follower's Optimal Strategy, we summarize the market equilibriums as follow and X_0 denotes the initial value of X:

3.1 Equilibrium With Sequential Exercise: $X_0 < X_F$

A sequential exercise equilibrium means two developers begin developing at different time.

If $X_0 \leq X_L$, no developer exercises. When X hits X_L from below, both developers want to be Leaders, but only one of them will win the race and starts to build first, the other will have to wait until $X \geq X_F$.

If $X_L \leq X_0 \leq X_F$, then one of the developer begins construction at beginning, and the other will be a follower.

3.2 Equilibrium With Simultaneous Exercise: $X_0 \geq X_F$

If $X_0 \geq X_F$ then there exists infinite amount of equilibrium strategies. Above X_F both developers are less prone to exercise. This is because that if one exercises, the other will exercise too, and this leaves no advantage for the leader. We will prove in more detailed paper that there exists a Pareto optimal equilibrium corresponds to a value of X_J. If $X_0 \in (X_F, X_J)$, then nobody exercises, but the moment that

X hits the boundary, both start to exercise immediately. The difference is, when X hits X_F from above, both exercise because of the fear of preemption (this could help to explain why real estate market often boom when the economic environment is deteriorating). When X hits X_J from below, both developers start to exercise because this maximizes their profit.

4 Implications

In the original paper of Steven Grenadier(1996) *"The Strategic Exercise of Options: Development Cascades and Overbuilding in Real Estate Markets"*, Grenadier had discussed the impact of σ on market equilibrium. Our contribution is the consideration of market crisis, which is depicted by two parameters λ and ϕ. Assuming that ϕ is constant and we consider the impact of λ, or the intensity of a random arriving crisis. Using our closed form solutions we can prove that

$$\frac{\partial X_F}{\partial \lambda} < 0$$

We can also see this from the following graph.

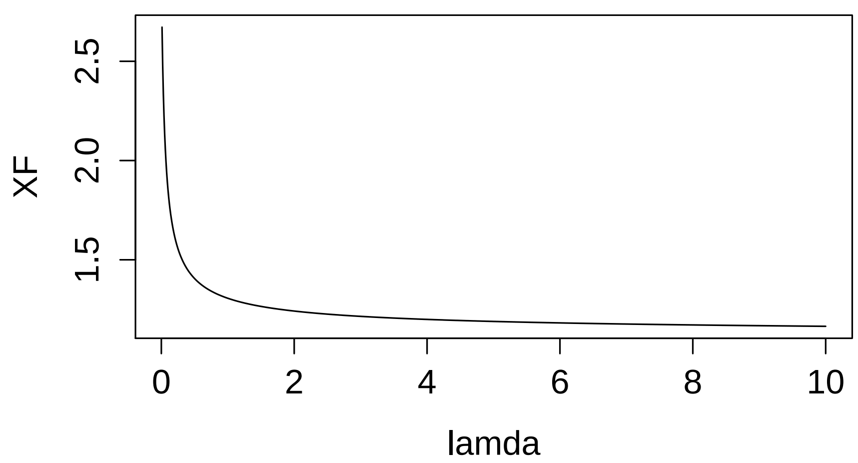

Figure 1: X_F Versus λ

There are tow conflicting forces here:

1. A bigger λ means higher risk of losing everything after exercising, this should makes developers more reluctant of entering and makes X_F higher.

2. A bigger λ also means options become price 0 even before exercising, this makes developers more eager to exercise options earlier.

The net effect of λ is making developers eager to enter the market with a lower boundary.

We use `Mathematica` to produce the surface and contour for the X_F given different values for λ and ϕ in the following.

For the rest of the paper, we work the numerical results for the problem assuming the following parameter values: $\mu = 0.05, \sigma = 0.2, r = 0.1, D(2) = 4, \delta = 5, I = 2, \gamma = 0.2, R = 5$.

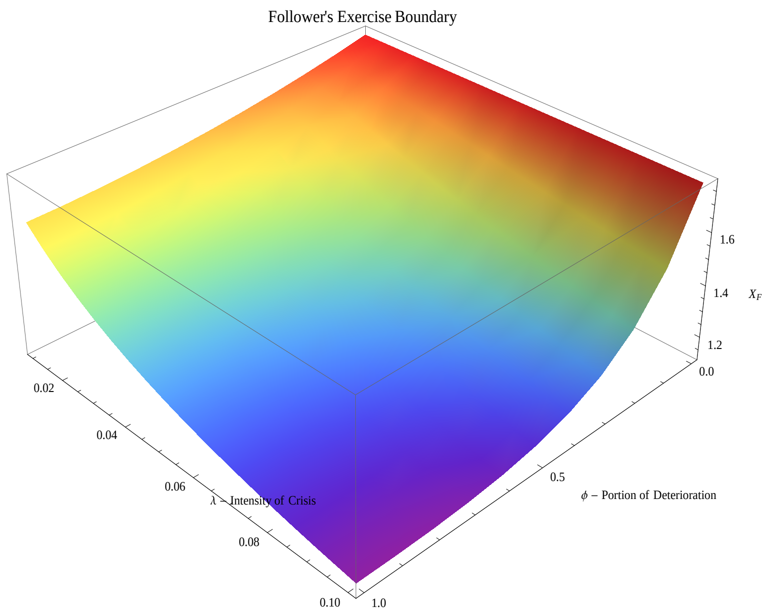

Figure 2: Follower's Exercise Boundary

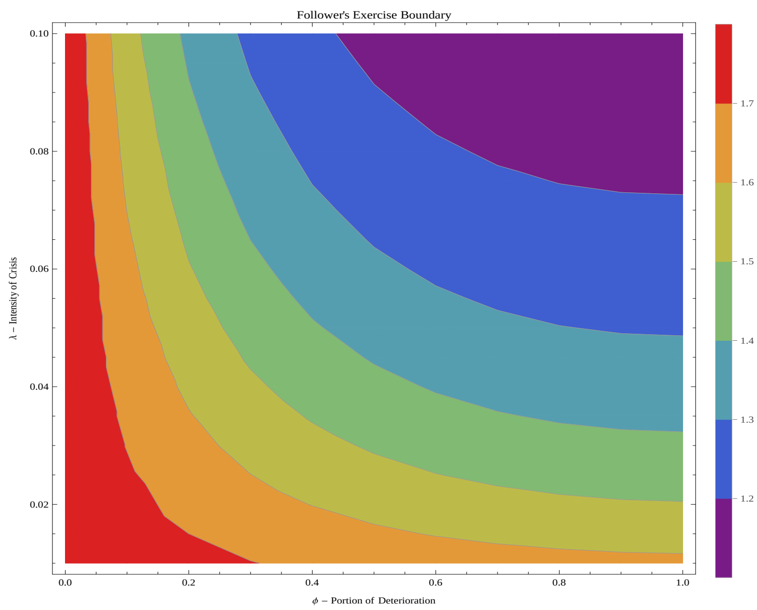

Figure 3: Follower's Exercise Boundary

For a jump process, we're not able to find the analytical expected passage time between Leader's and Follower's construction. Instead, we use Monte Carlo method to produce the median passage time between Leader's and Follower's construction, this reflects how long the leader could remain monopolist in new market. Results are summarized as following. When the passage time is longer than 50 years, we take it as Infinity, meaning it will not happen in limited time.

Median Passage Time between Leader's and Follower's Construction (year)

	$\lambda = 0$	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16
$\phi = 0$	38.20	32.74	29.04	29.78	23.05	19.20	28.78	25.27	28.36
0.1	33.17	36.56	36.40	47.35	47.31	55.90	56.39	65.33	∞
0.2	28.78	28.49	49.92	77.09	61.65	∞	∞	∞	∞
0.3	30.85	49.49	59.24	∞	∞	∞	∞	∞	∞
0.4	33.44	46.48	∞	∞	∞	∞	∞	∞	∞
0.5	28.46	40.59	85.73	∞	∞	∞	∞	∞	∞
0.6	37.22	62.01	∞	∞	∞	∞	∞	∞	∞

5 References

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