Bootstrapping Time Series Data

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CSP 2015
New Orleans, LA

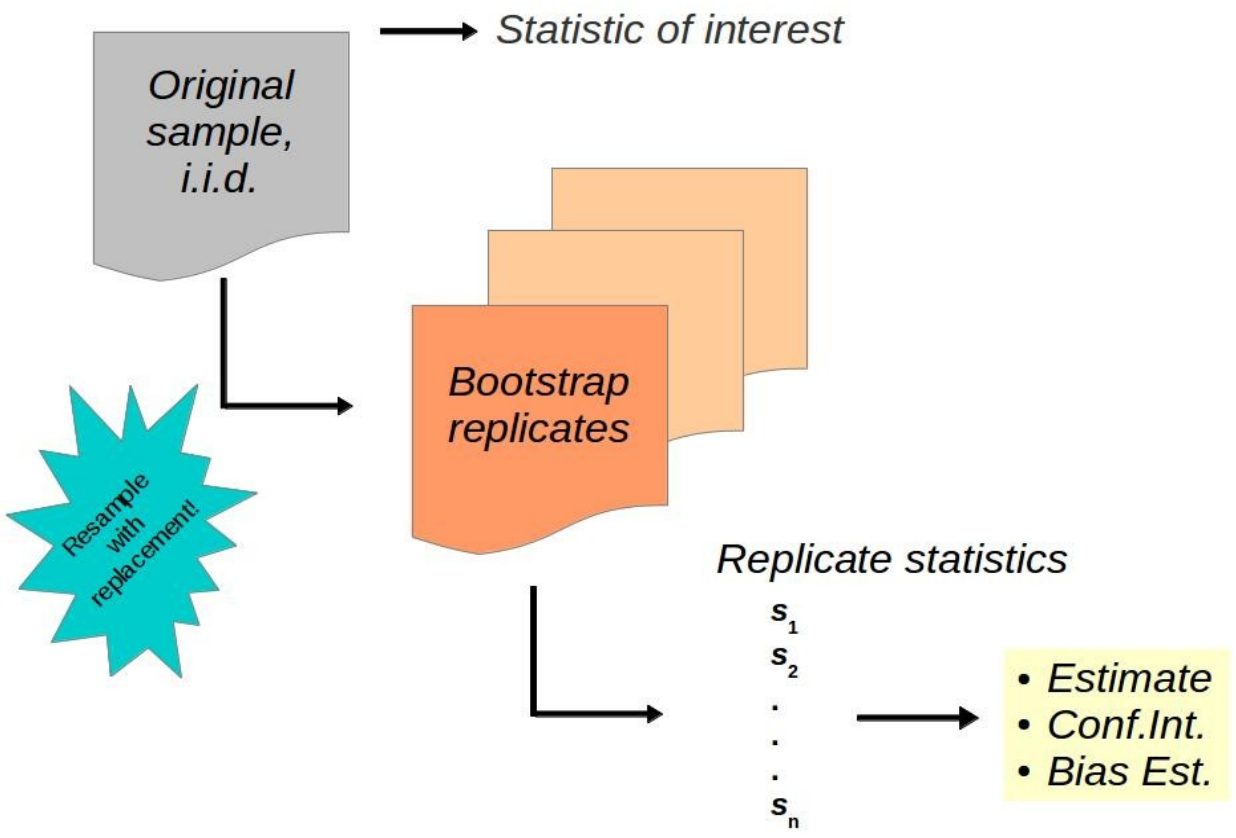
We'll cover a range of bootstrapping procedures today.

- Background on the bootstrap
- Non-parametric: The naïve bootstrap
- Handling dependency: The Moving Block bootstrap
- Honoring a model: Parametric bootstrap
- Balanced approach: The Maximum Entropy bootstrap

When would you bootstrap time series data?

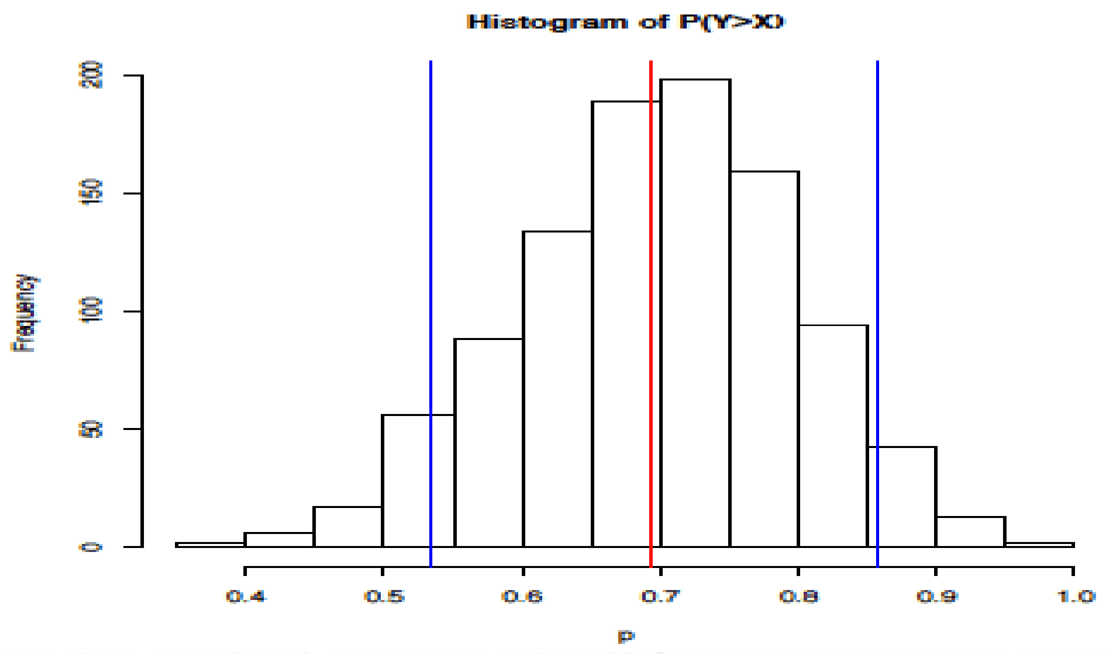
- You have some time series data
- But not much data – *whatever "much" means*
- Want to estimate a statistic – *especially a tricky statistic*
- . . . and its confidence interval
- No closed-form solution

Bootstrapping generates bootstrap replicates and replicate statistics.



Q: How do we get statistic's conf. interval from replicate statistics?

A: The percentiles of the empirical distribution (histogram) give the confidence interval for the statistic. Cool!



Bootstrapping time series data has special challenges.

- Interesting time series are not i.i.d.

  We difference the data.

- How do we generate plausible bootstrap replicates?

  Several ways. That's what this talk is really about.

- How do we deal with dependency structure?

  By choosing the right replication method. Stay tuned.

The bootstrap procedure requires i.i.d. data.

- i.i.d. necessary for resampling with replacement.
- Differencing time series can create i.i.d. data.
- Random walk model, where ε_t are i.i.d., typically $N(\mu, \sigma^2)$:

$$y_t = y_{t-1} + \varepsilon_t$$

- Becomes:

$$\varepsilon_t = y_t - y_{t-1}$$

If differences are i.i.d., we can use the naïve bootstrap.

Procedure:

1) Calculate successive *differences*.
2) Repeatedly,
    1) Resample the differences <u>with</u> replacement.
    2) Sum those differences to construct one replicate time series.
    3) Using that time series, calculate one replicate statistic.
3) From all the replicate statistics, form the estimate and confidence interval:

    Mean of replicate statistics → estimate

    Percentiles of replicate statistics → confidence interval

Toy Example

Given time series:

[1] 10.00 9.67 9.50 8.66 8.33 7.26 7.48 8.03 8.60 8.44

Statistic of interest for given data:

[1] 2.74

Compute differences:

[1] -0.33 -0.17 -0.84 -0.33 -1.07 0.22 0.55 0.57 -0.16

Resample the differences <u>with replacement</u>:

[1] 0.55 -0.16 -0.84 -0.33 0.22 -0.84 0.22 0.22 0.57

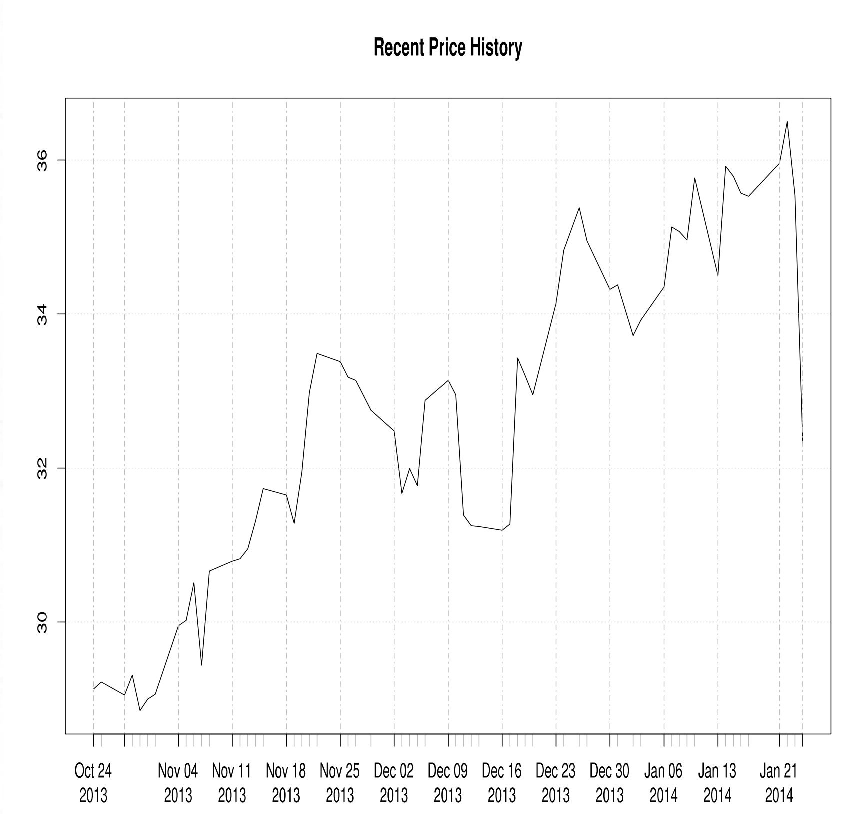
Construct one bootstrap replicate (by summing):

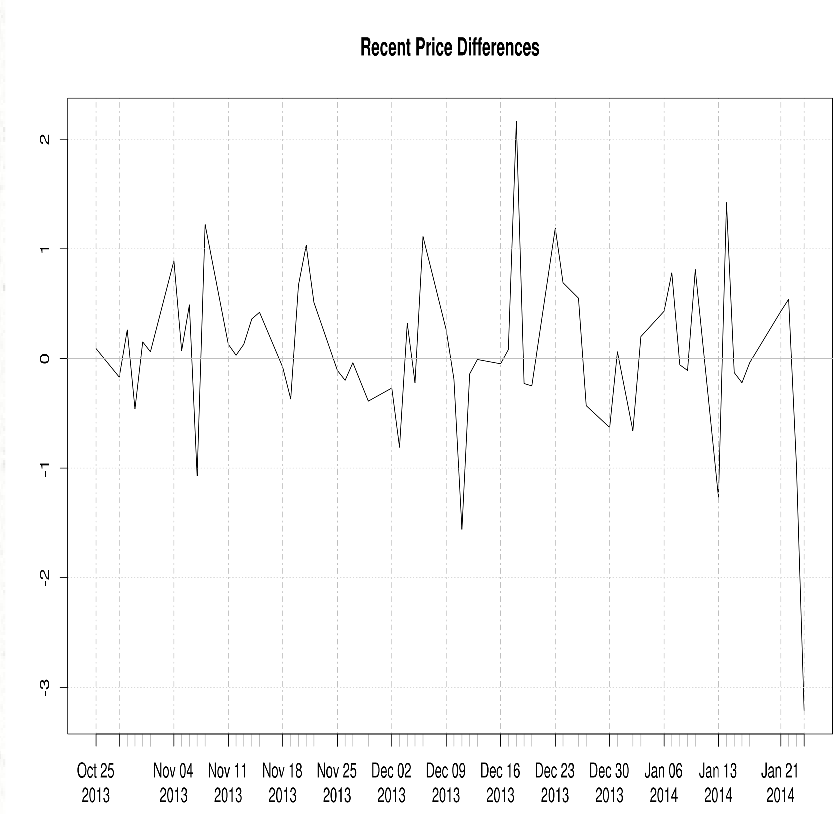
[1] 10.00 10.55 10.39 9.55 9.22 9.44 8.60 8.82 9.04 9.61

Compute one replicate statistic:

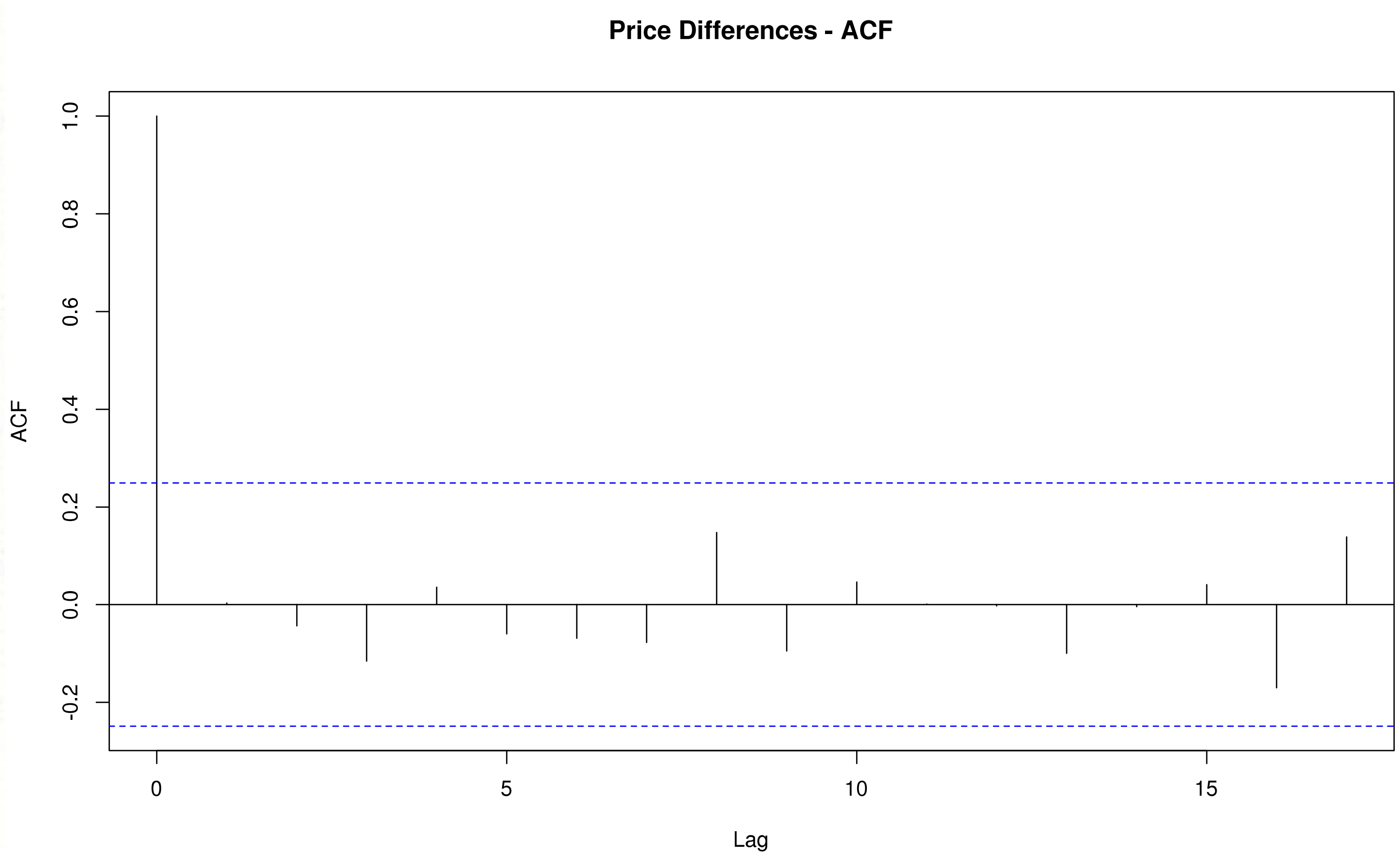
[1] 1.95

Naïve bootstrap example: Stock price, differences, net change

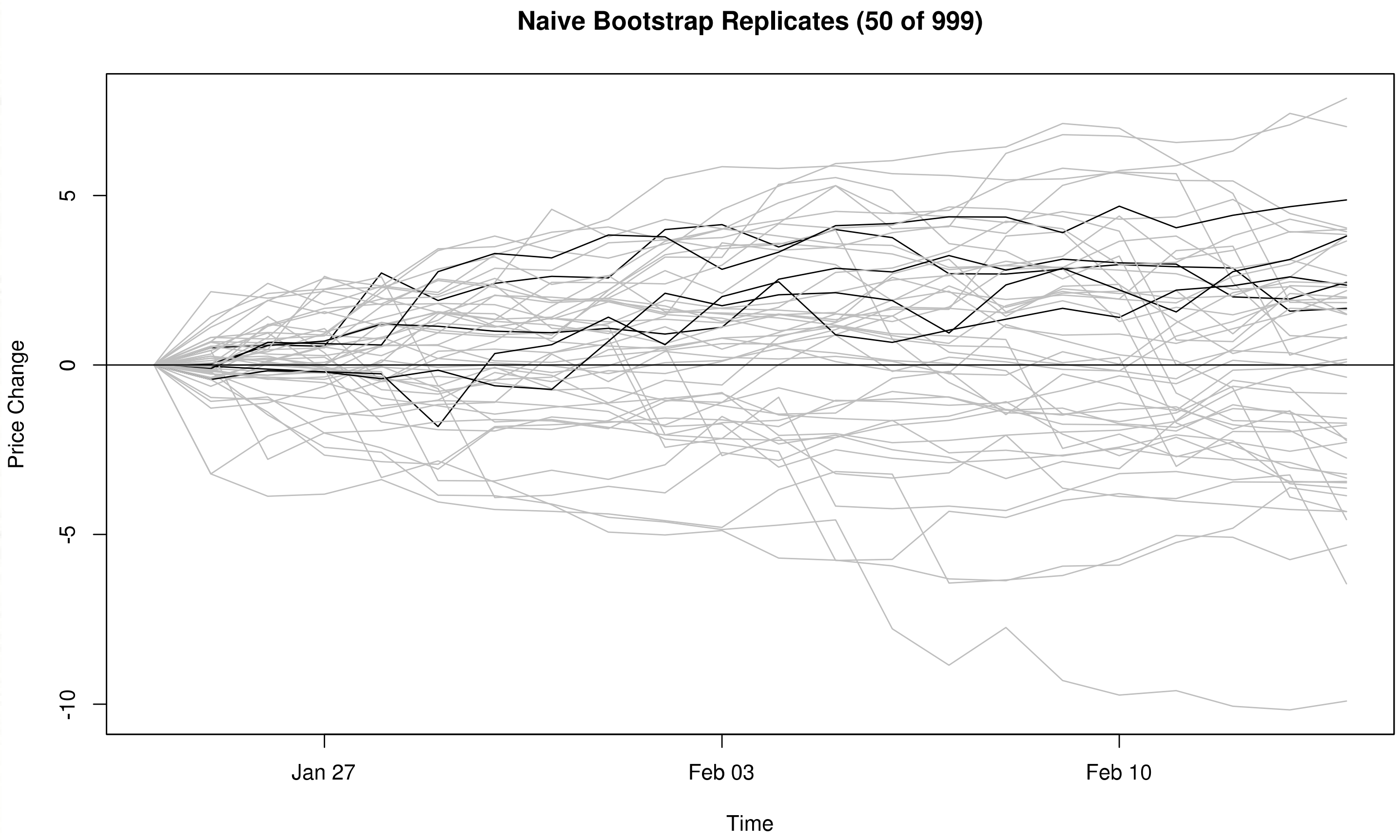




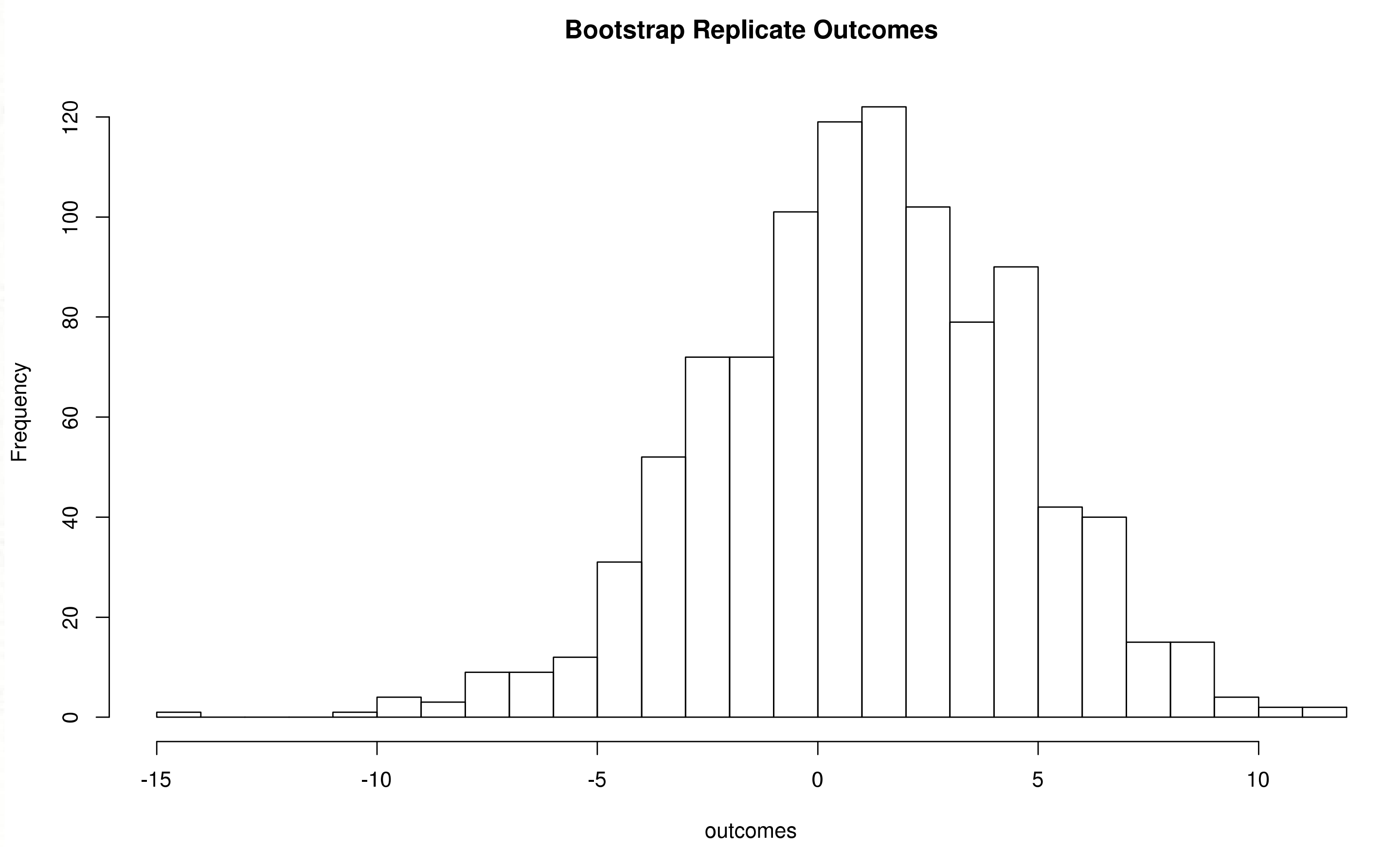
Check: Is it reasonable to assume differences are i.i.d.?



Create replicates by summing resampled differences



Naïve bootstrap example: Statistic of interest is net change



Simple implementation in R

```
diffs = diff(price)
HOR = 21
reps = replicate(999,
          sample(diffs, HOR, replace=TRUE),
          simplify=TRUE)
reps = apply(reps, 2, cumsum)
outcomes = reps[HOR,]
print(
  quantile(outcomes, prob=c(0.025, 0.975)) )
```

Mean and quantiles of replicate statistics give estimate and conf. int.

```
> summary(outcomes)
   Min. 1st Qu.  Median    Mean 3rd Qu. 
Max.
-14.430  -1.225   1.120   1.057   3.445 
11.540
> quantile(outcomes, prob=c(0.025, 0.975))
   2.5%   97.5%
-6.4120  7.6425
```

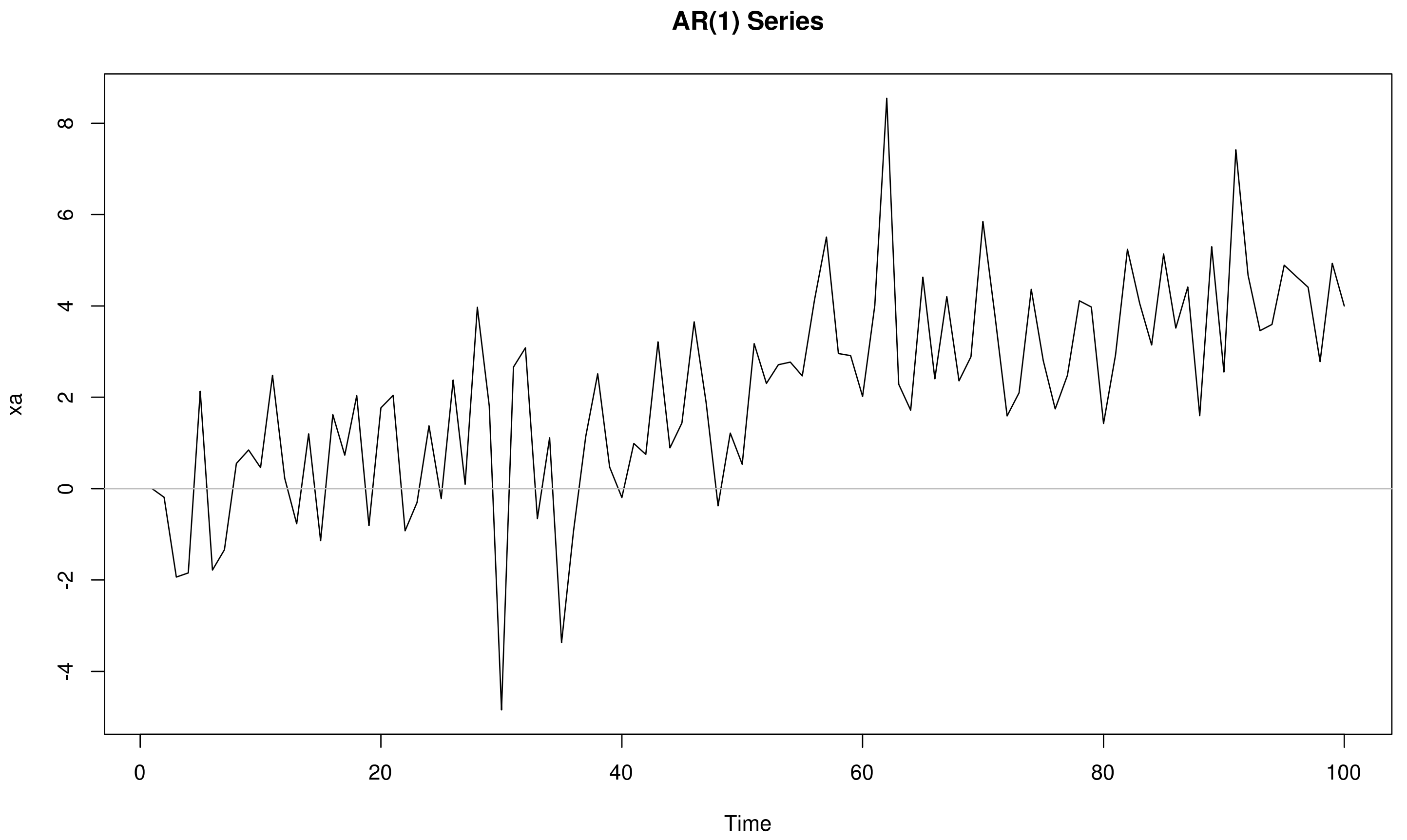
Next problem:
What if the differences are <u>not</u> i.i.d.?

If not, purely random resampling will not capture the structure of the differences.

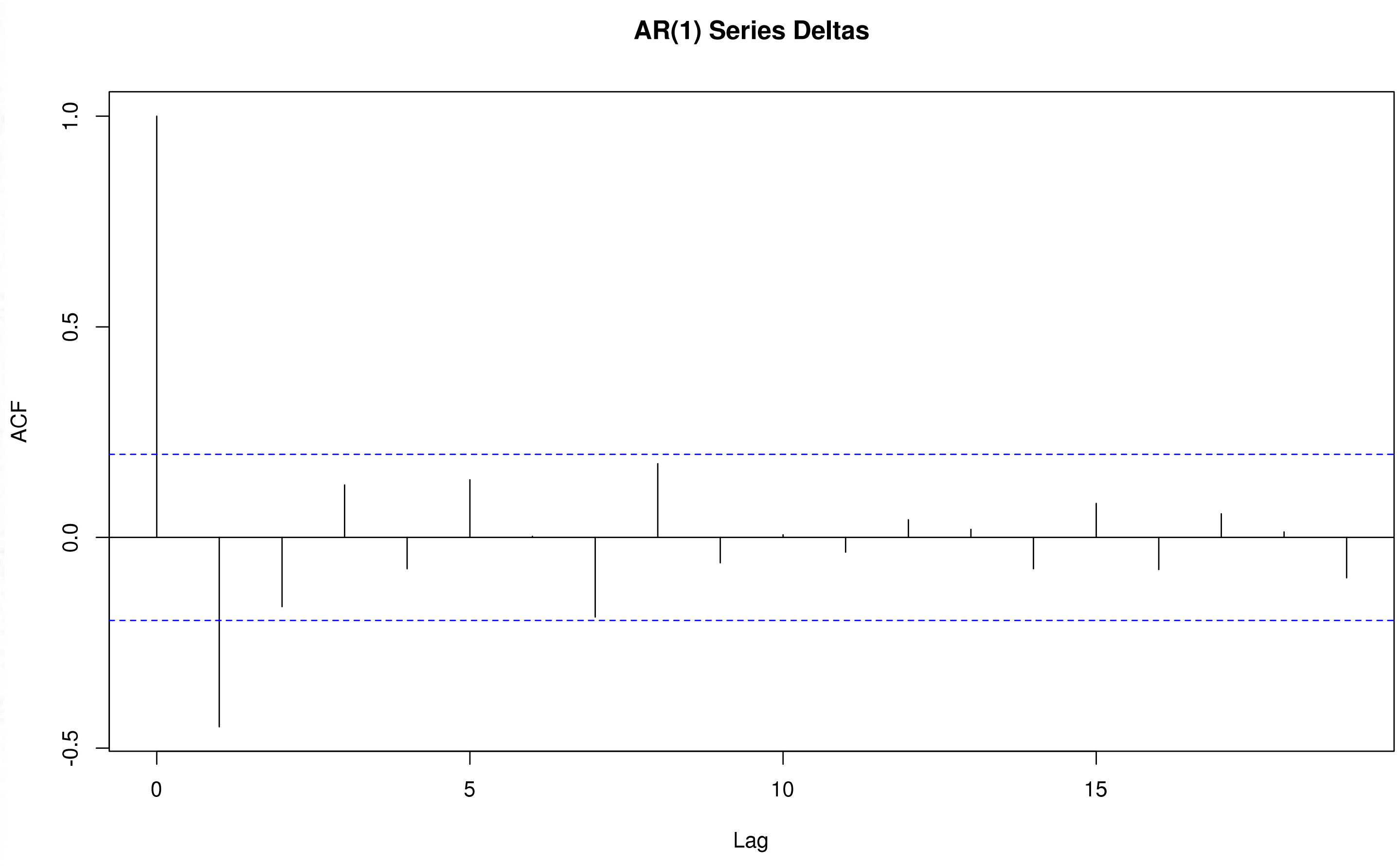
Bootstrap replicates will not resemble our data.

Uh oh.

Example: AR(1) time series



The ACF of this time series reveals a (simple) dependency.



Moving Block Bootstrap preserves the local dependency structure.

- Break time series into little blocks.
- Resample the blocks, not individual points – *kind of "random shuffling", with replacement.*
- <u>Within</u> blocks, structure is preserved.
- Works if structure <u>between</u> blocks is (quasi) i.i.d.

The Moving Block procedure resamples blocks of points, not single points.

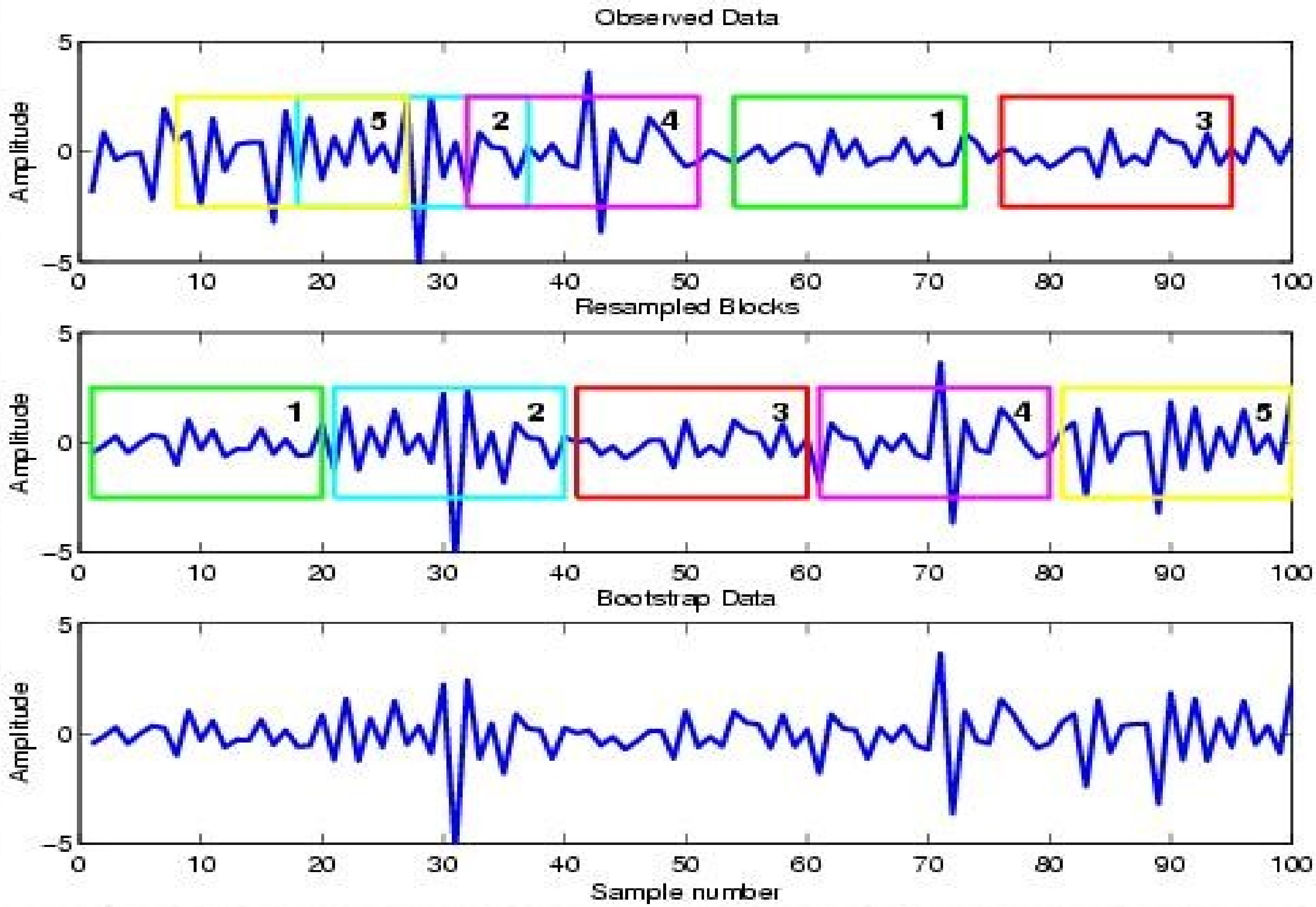


Illustration courtesy of http://www.csp.curtin.edu.au/photos/resample.jpg

In R, tsboot and boot.ci functions together implement a moving block bootstrap.

```
library(boot)
theStatistic = function(x) { . . . }
BLOCK_SIZE = 5        # guess at block size
mbb = tsboot(ts(xa), theStatistic, R=999,
             l=BLOCK_SIZE, sim="fixed")
replStats = as.vector(mbb$t)
print(summary(replStats))   # for estimate
print(
   boot.ci(mbb, type=c("norm","basic","perc"))
)
```

Output from boot.ci (statistic: maximum peak-to-valley distance)

```
*** Confidence Intervals: AR(1) Data, Block Bootstrap
BOOTSTRAP CONFIDENCE INTERVAL CALCULATIONS
Based on 999 bootstrap replicates


Intervals :
Level      Normal              Basic              Percentile
95%   ( 3.174,  9.245 )   ( 4.240,  9.084 )   ( 8.541, 13.385 )
Calculations and Intervals on Original Scale
```

What if you have a time series model of your data?

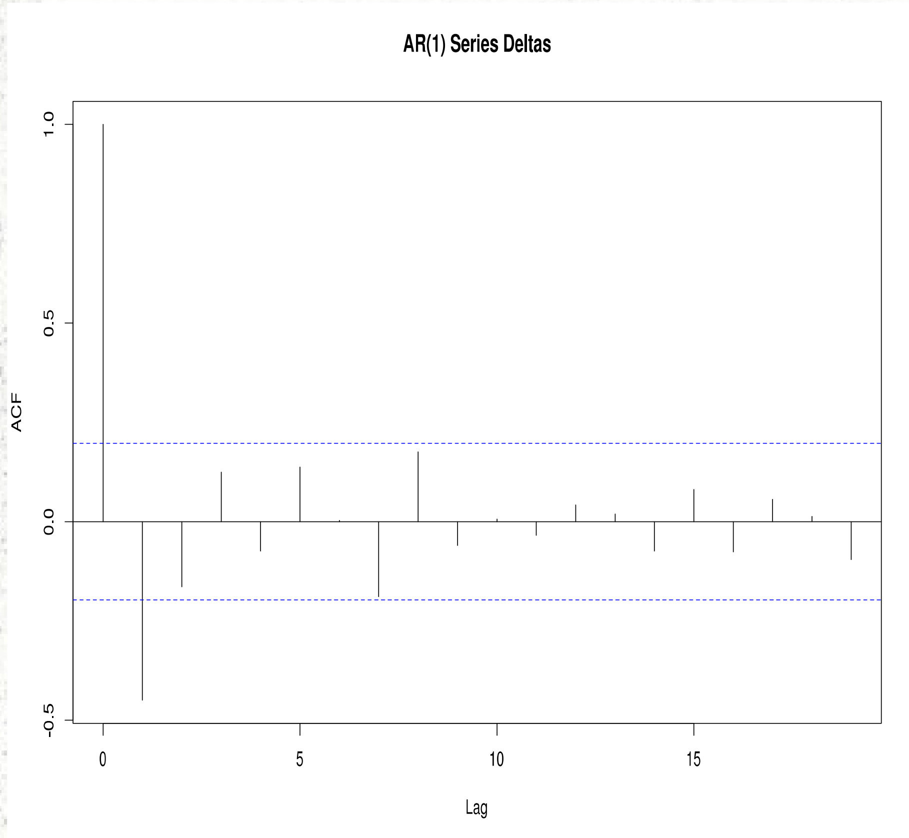
- Example: ARMA, state-space model, or seasonality.
- Model can remove known dependency structure.
- Residuals embody the remaining uncertainty.
- If residuals are i.i.d. time series, we can bootstrap them:

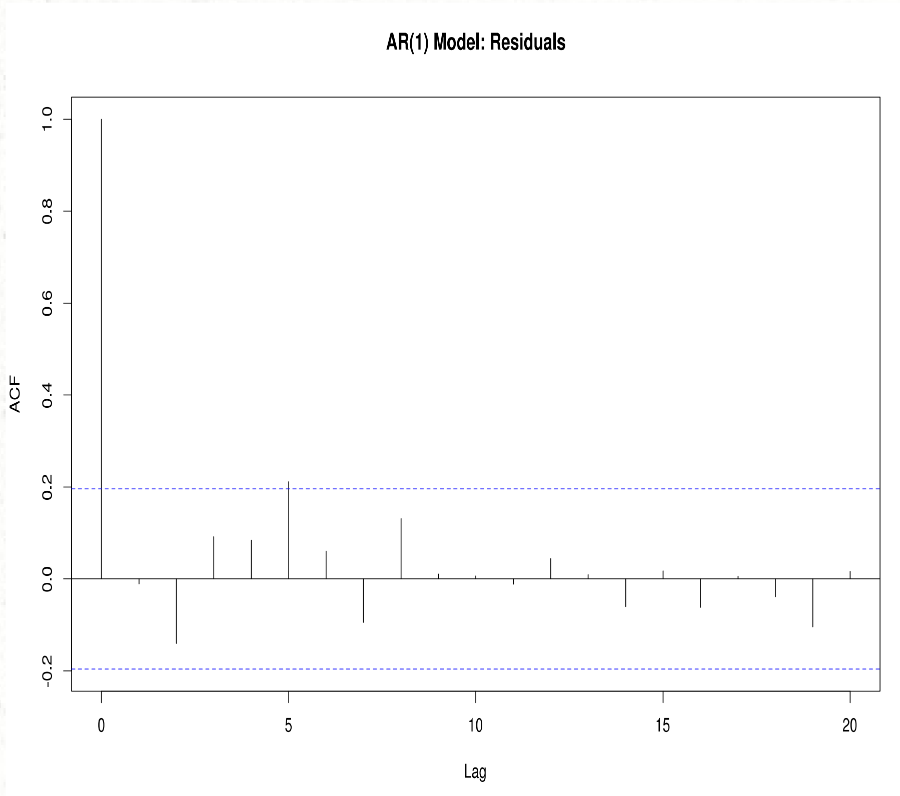
Simulate the model repeatedly, each time substituting resampled residuals for original residuals.

For example, let's fit the AR(1) data to a model (with trend term).

```
*** Fitted AR(1) model:


Call:
arima(x = as.ts(xa), order = c(1, 0, 0), xreg = time,
      include.mean = FALSE)


Coefficients:
          ar1    time
      -0.0329  0.0449
s.e.   0.0995  0.0027


sigma^2 estimated as 2.622:  log likelihood = -190.09,  aic =
386.18
```

Unlike the original AR(1) data, the residuals show no autocorrelation.





Bootstrap the residuals by resampling them and substituting them into AR(1) process.

If residuals are

$$\varepsilon_1 \ldots \varepsilon_T$$

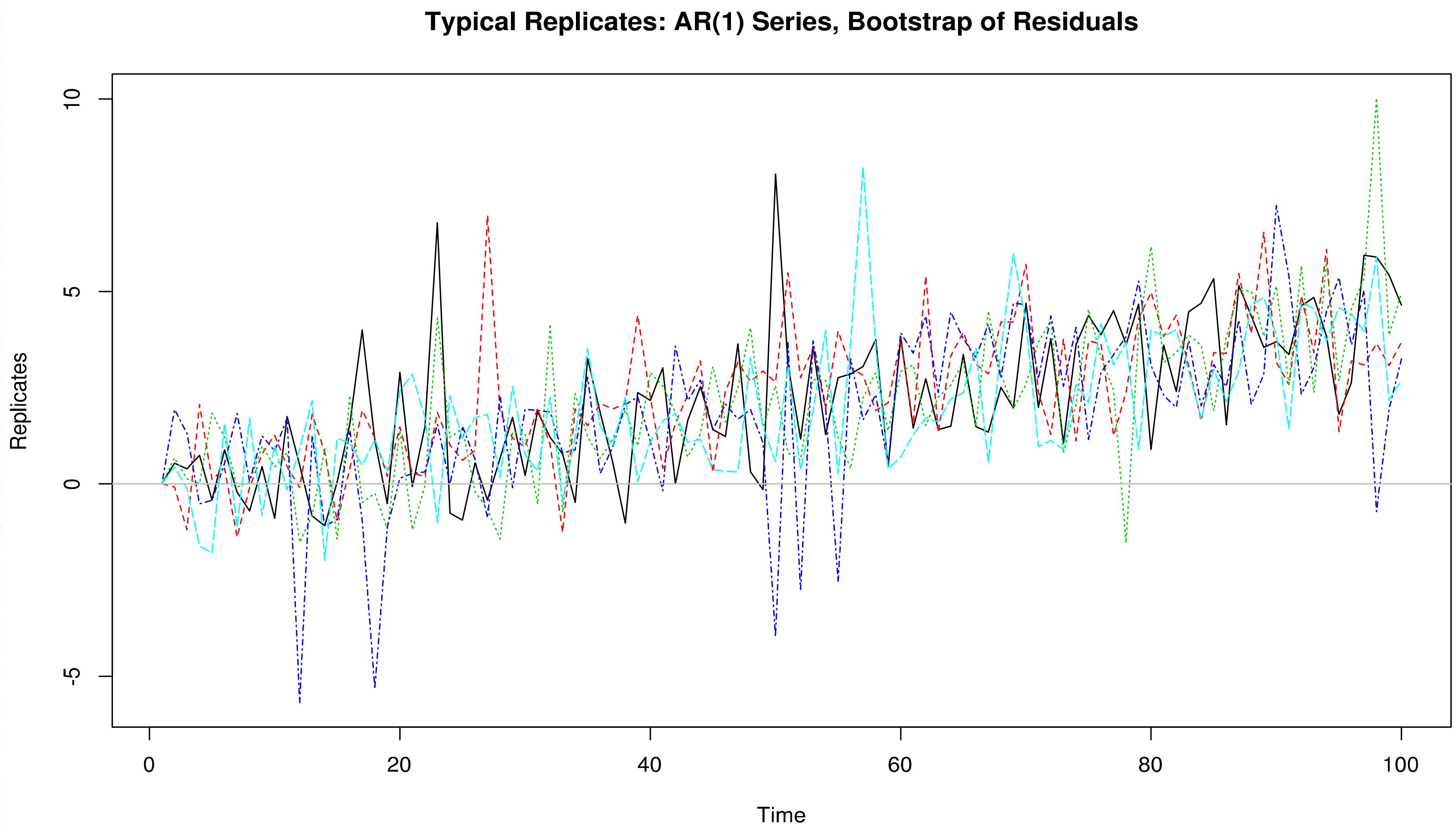
Resample with replacement, giving

$$\varepsilon_1' \ldots \varepsilon_T'$$

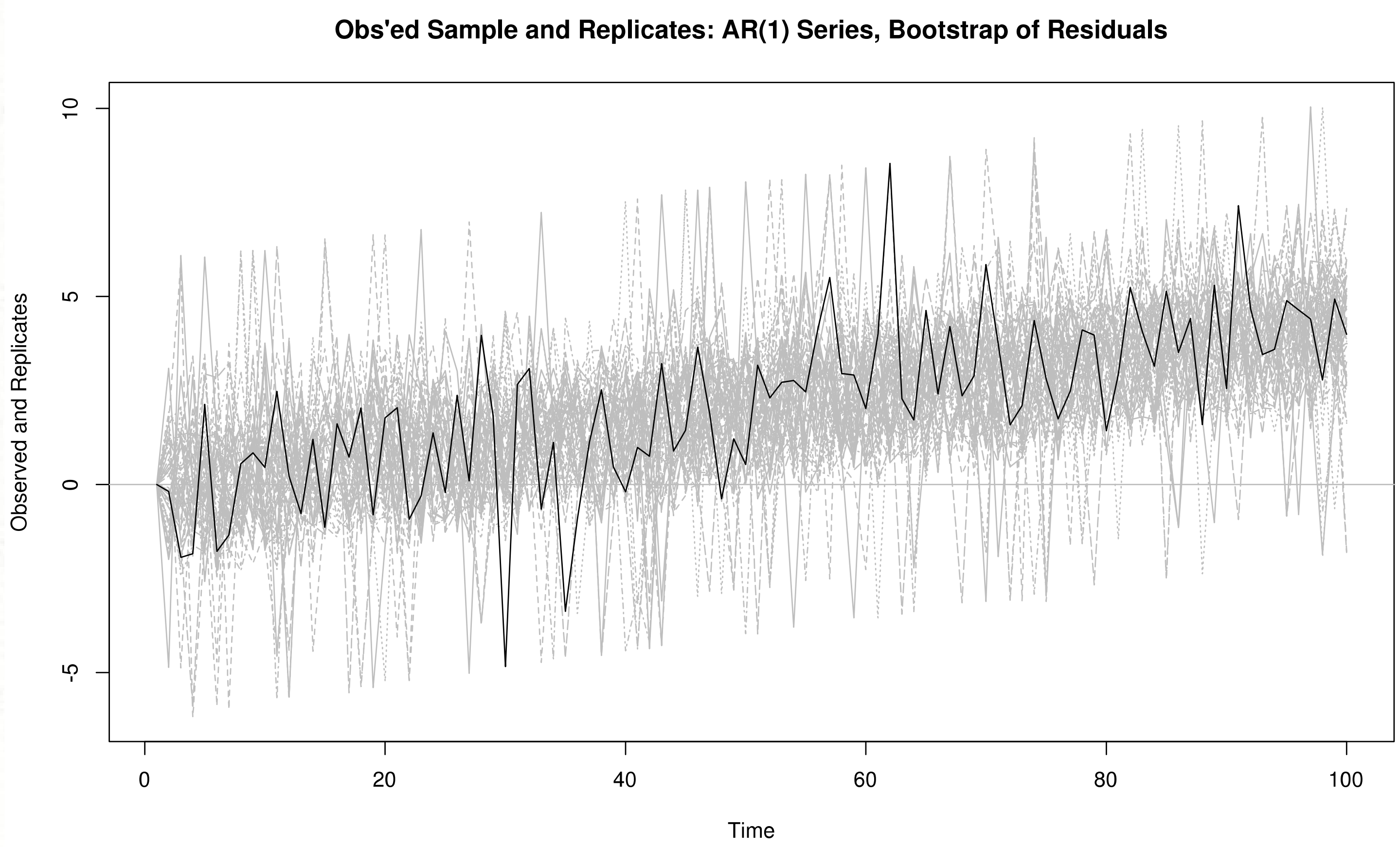
And substitute into the AR(1) process:

$$y_t = \delta + \varphi y_{t-1} + \varepsilon_t'$$

Bootstrap replicates will be plausible variations that conform to the model.



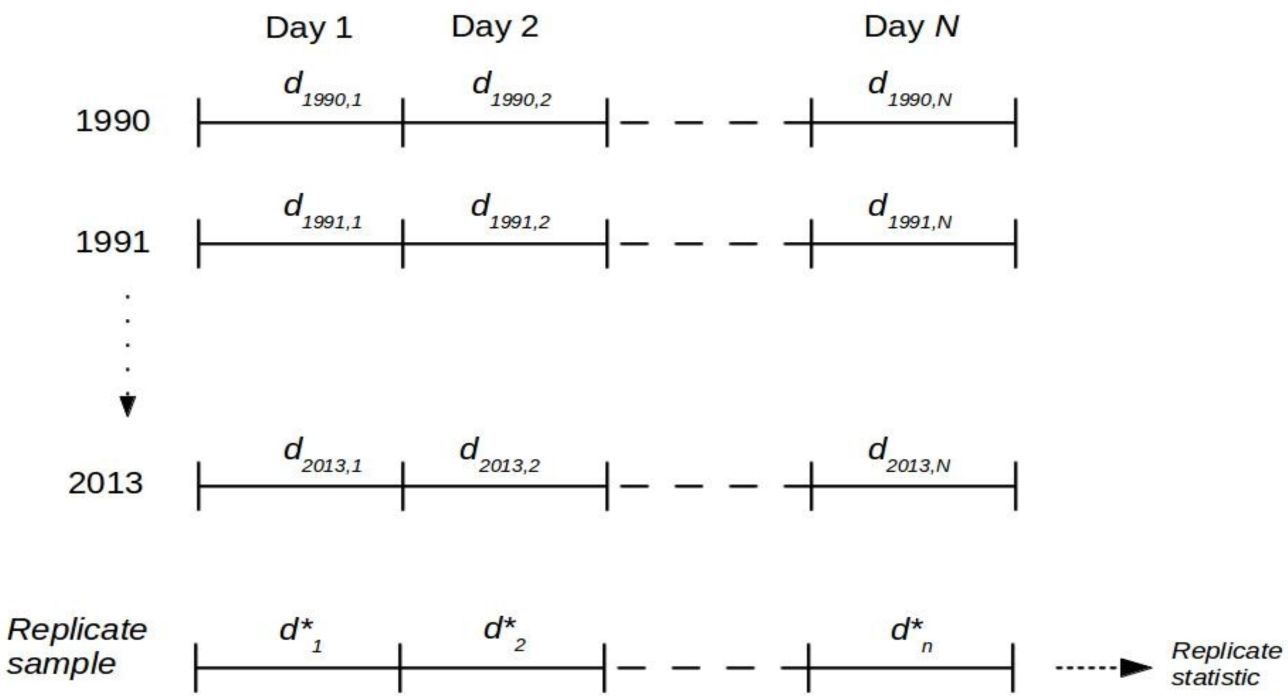
Results of bootstrapping AR(1) residuals



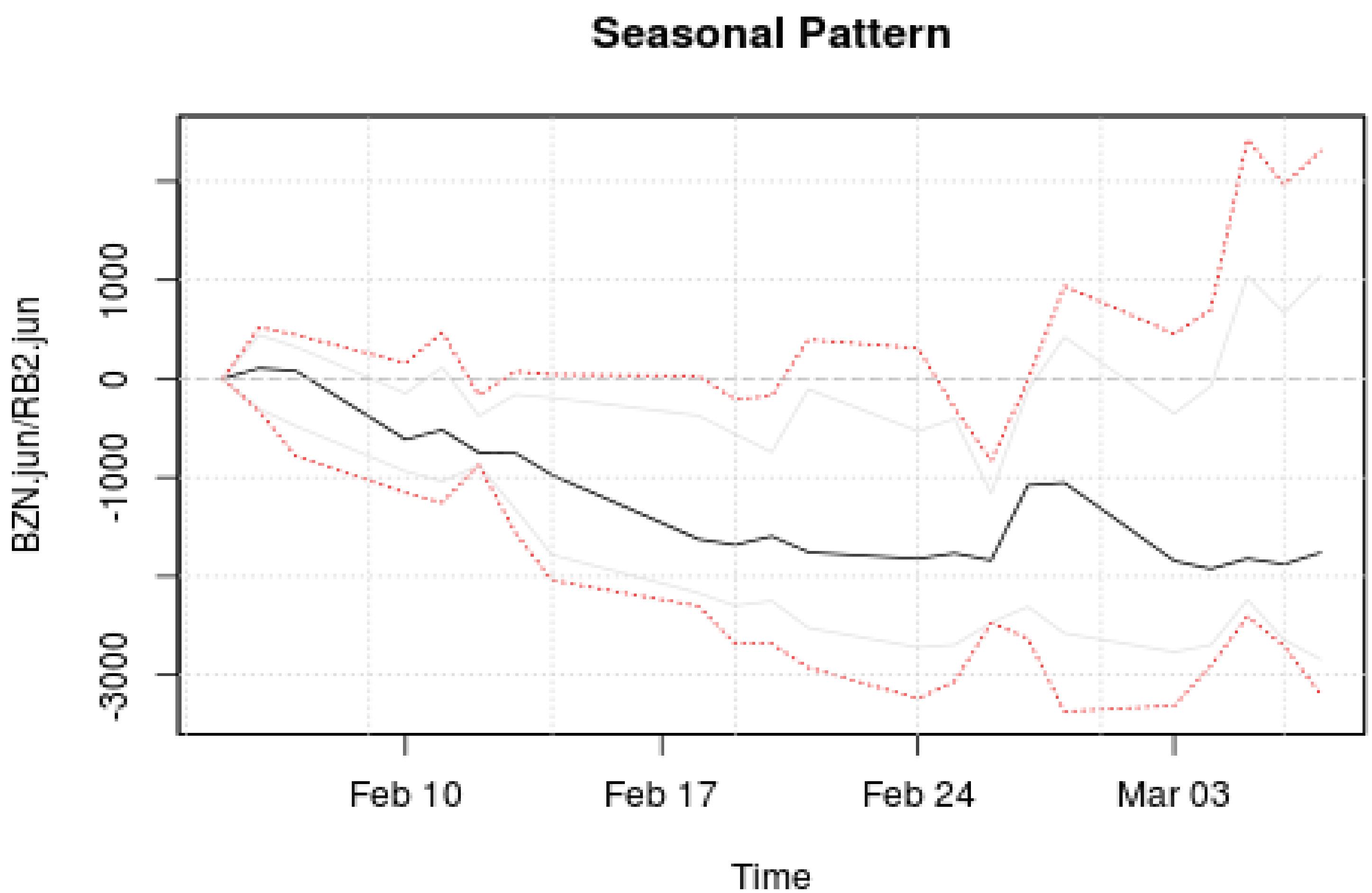
If the model's good, it can tighten the final confidence interval.

```
*** Confidence Intervals: AR(1) Series , Bootstrap of Residuals
BOOTSTRAP CONFIDENCE INTERVAL CALCULATIONS
Based on 999 bootstrap replicates


CALL :
boot.ci(boot.out = bout, type = c("norm", "basic", "perc"))


Intervals :
Level      Normal              Basic              Percentile
95%   ( 5.302, 12.067 )   ( 5.816, 12.507 )   ( 5.118, 11.809 )
Calculations and Intervals on Original Scale
```

Seasonal dependence suggests a different scheme for resampling.



Seasonal replicates example: median and 95% conf. bands



Advanced procedures can handle other dependency structures.

Procedure	**Structure**
Moving block	Stationary; discrete or categorical data
Local bootstrap – *Similar to Monte Carlo*	Short-range dependence, mild distributional assumption.
Markov bootstrap	Stationary, short-range dependence; discrete or categorical data
Sieve bootstrap	AR(n) models

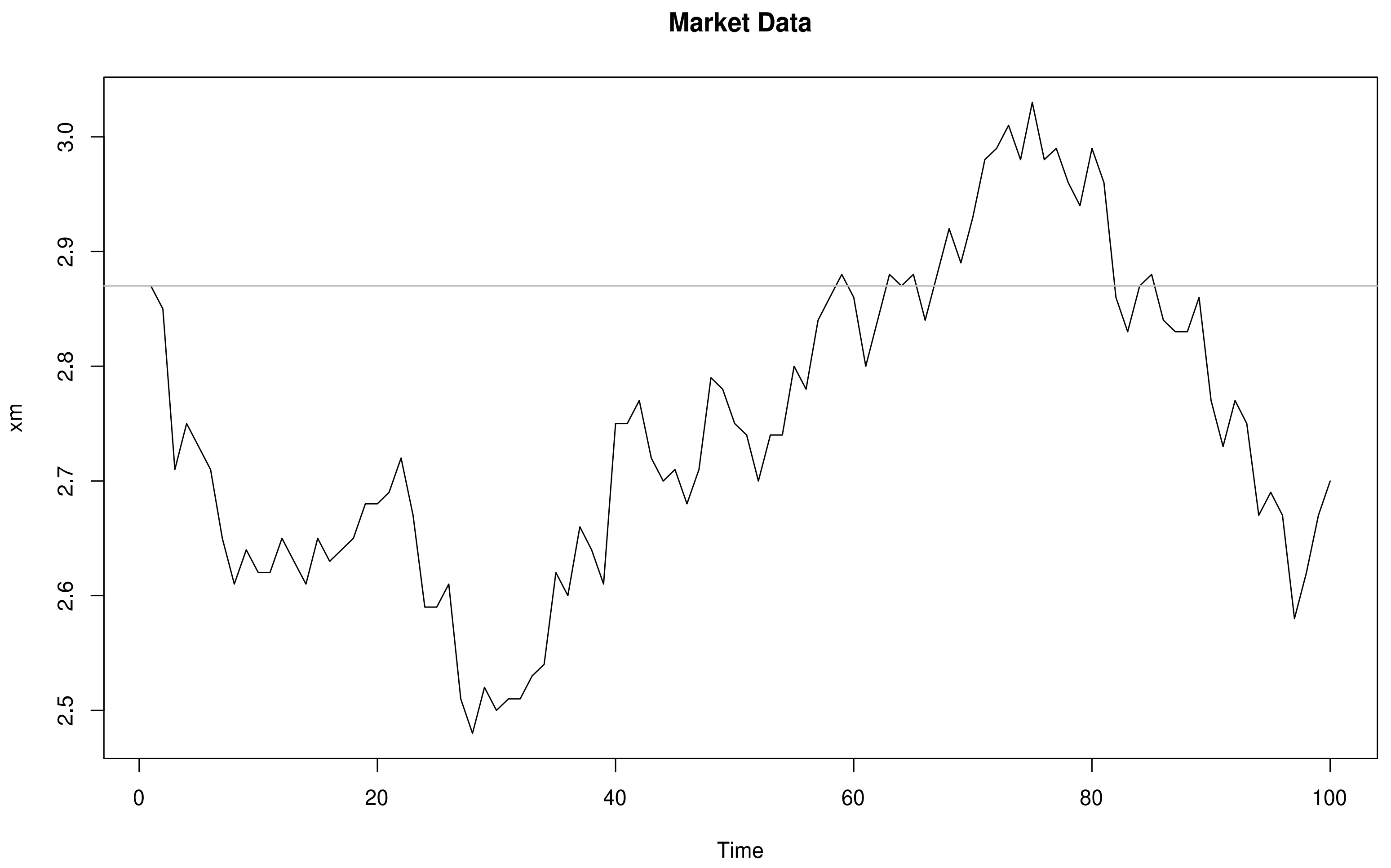
Is there a middle-ground between naïve bootstrap and full model?

- Naïve can be <u>too</u> naïve.
- Model is often unknown.
- Maximum Entropy bootstrap is alternative.
- Parametric bootstrap of differences.
- Maximum entropy distribution of differences – very mild assumption
- Preserves many properties, including shape, seasonality, even some non-stationarity

Vastly oversimplifed outline of maximum entropy bootstrap

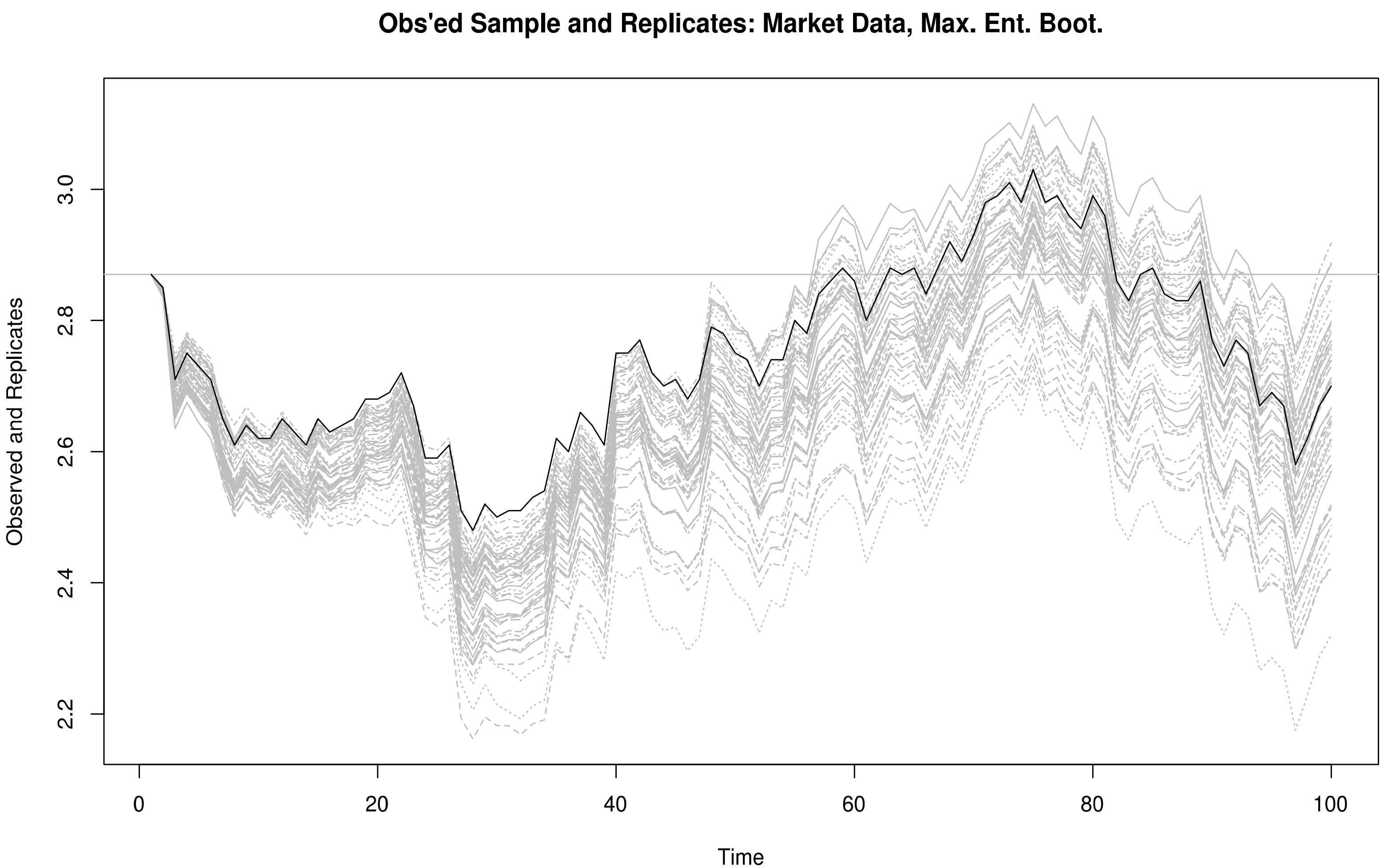
1) Sort the original data.
2) Using sorted data, compute its intermediate points and lower limits for left and right tails.
3) Compute the mean of the maximum entropy density within each interval.
4) Generate uniform random values on [0,1], and compute sample quantiles at those points.
5) Apply to the sample quantiles the correct order to honor the dependence relationships of the observed data.
6) Repeat steps 4 and 5 many times (e.g. 999).

Example: This interest rate data seems to have structure. In fact, the naive bootstrap works poorly.



The maximum entropy bootstrap preserves the gross structure.

Maximum Entropy Replicates



In R, meboot package implements the max. entropy bootstrap.

```
library(meboot)
mebOut = meboot(ts(diff(prices)),
                 reps=999)
mebens = mebOut$ensemble
mebens = rbind(prices[1], mebens)
repls = apply(mebens, 2, cumsum)
# 'repls' is matrix of bootstrap replicates
```

Bootstrapping Time Series Data: Some Limitations

- Problems with sample: non-representative, too small
- Problems from dependency structure: wrong dependency assumption; regime changes; long-term dependency; overlooked completely
- Parametric bootstrap: wrong model; non-stationary (unstable) process, hence unstable parameters
- Problems with certain statistics: “Edge” statistics may require many, many replicates
- Tools: Easy in R, awkward in SAS/Macro
- Finally, Monte Carlo may be better alternative

Some References

- *An Introduction to the Bootstrap* by Efron and Tibshirani
- *Bootstrap Methods and Their Applications* by Davison and Hinkley
- “The Moving Blocks Bootstrap Versus Parametric Time Series Models”, Vogel and Shallcross, *Water Resources Research* (June 1996)
- “Bootstraps for Time Series”, Bühlmann, *Statistical Science* (2002, No. 1)
- “Maximum Entropy Bootstrap for Time Series”, Vinod and López-de-Lacalle, *J. of Stat. Soft.* (Jan 2009)

Thank you!

Talk materials available at

`http://bit.ly/csp2015-teetor`