Math 202 A01
Spring 2019
Test 2 Version A
19/03/2019
Instructor: W. Thompson
Time Limit: 50 Minutes

Name (Print): Solutions

♪Hey I just met you,
and this is crazy, ♫
but here's my,

Student Number V00 ____________

Tutorial Section T0 ____________

This exam contains 7 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, calculator, or notes on this exam.
You are required to show your work on each problem on this exam. The following rules apply:

- **Understand what each question is asking.** There is sometimes more than one way to solve a problem. However, part of this course is making sure that you know how to solve a problem using the tools that are taught and not just on obtaining the correct answer. If a question asks you to solve something in a very specific way, you must do so.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- **Fully evaluate when possible.** If it is possible to evaluate an expression further, do so. For example, $\sin(\pi/4) = 1/\sqrt{2}$ is fully evaluated.
- **Don't simplify your answer if the question does not ask you to.** For the purposes of marking, it is easier to determine the accuracy of a method prior to simplification. Only simplify when told to do so.
- **Solutions to Differential Equations may be implicit or explicit.** When you solve a differential equation you may obtain an implicit solution. You do not need to solve explicitely unless told otherwise.
- **State the interval of validity when necessary.** When you obtain an explicit solution to an initial value problem be sure to always state the interval of validity.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	10	
2	8	
3	6	
4	6	
5	6	
Total:	36	

Do not write in the table above.

1. Answer the following short problems. Simplify all algebra where applicable.

(a) (2 points) Solve $\frac{1}{2x}y' = 1 + y^2$.

Split $\Rightarrow$ $\int \frac{1}{1+y^2}\,dy = \int 2x\,dx$

$\Rightarrow \underline{\underline{\arctan(y) = x^2 + C}}$

(b) (2 points) Solve $y'' + 18y' + 81y = 0$.

aux $\Rightarrow m^2 + 18m + 81 = 0$

$\Rightarrow (m+9)^2 = 0$ so $m = -9$ repeated

$\underline{\underline{y(x) = C_1 e^{-9x} + C_2 x e^{-9x}}}$

(c) (3 points) Solve $y'' + 6y' + 25y = 0$.

aux $\Rightarrow m^2 + 6m + 25 = 0$

$\Rightarrow m = \frac{-6 \pm \sqrt{36-100}}{2} = \frac{-6 \pm i\sqrt{64}}{2} = -3 \pm 4i$

Complex

$\alpha = -3,\ \beta = 4.$

$\underline{\underline{y(x) = C_1 e^{-3x}\cos(4x) + C_2 e^{-3x}\sin(4x).}}$

(d) (3 points) Find the equation of the tangent plane to $z = \ln(x^2 + y^2)$ at the point $(1, 0, 0)$.

Rewrite $\Rightarrow$ $\underbrace{e^z - x^2 - y^2}_{F(x,y,z)} = 0$ so $\nabla F = \langle -2x, -2y, e^z \rangle$.

$\nabla F(P) = \langle -2, 0, 1 \rangle$

gives $\nabla F(P) \cdot \langle x-1, y, z \rangle = 0$

$\Rightarrow \underline{\underline{-2(x-1) + z = 0}}$

or $-2x + z = -2$.

2. (8 points) Using Lagrange Multipliers (and no other method) find the maximum and minimum values of the function $f(x, y) = x^2 + x + 2y^2$ on the unit circle $x^2 + y^2 = 1$.

$$(\mathcal{L}) \Rightarrow \begin{cases} 2x+1 = 2\lambda x & (1) \\ 4y = 2\lambda y & (2) \\ x^2+y^2=1 & (3) \end{cases} \quad \left.\begin{matrix} f_x = \lambda g_x \\ f_y = \lambda g_y \end{matrix}\right\} \text{ where } g(x,y) = x^2+y^2=1$$

From (2) $\Rightarrow$ $2y(2-\lambda) = 0$ so $y=0$ or $\lambda = 2$.

If $y=0$:

$\begin{cases} 2x+1=2\lambda x \\ x^2=1 \end{cases}$ gives $x=\pm 1$ and we obtain $(1,0)$ & $(-1,0)$.
(we don't care about λ).

If $\lambda = 2$:

$\begin{cases} 2x+1=4x & (1) \\ x^2+y^2=1 & (2) \end{cases}$ (1) gives $1=2x \Rightarrow x=\frac{1}{2}$ so ...

$(2) \Rightarrow y^2 = 1-x^2 = 1-\frac{1}{4} = \frac{3}{4}$

$\Rightarrow y = \pm\frac{\sqrt{3}}{2}$.

and we obtain $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ & $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

Testing

$f(1,0) = 1+1+0 = 2$

$f(-1,0) = 1-1+0 = 0$

$f\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right) = \frac{1}{4} + \frac{1}{2} + 2\left(\frac{3}{4}\right) = \frac{1}{4} + \frac{1}{2} + \frac{3}{2} = \frac{1}{4} + 2 = \frac{9}{4}$

Thus max at $\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$ min at $(-1,0)$ as above.

3. (6 points) Solve the following initial value problem

$$xy' + 2y = \cos(x), \qquad y(\pi) = 4$$

for an **explicit** solution.

Standard

$$\Rightarrow y' + \frac{2}{x} y = \frac{\cos(x)}{x}$$

so the integrating factor is

$$\mu(x) = e^{\int 2/x \, dx} = e^{2\ln|x|} = e^{\ln x^2} = x^2.$$

Then

$$x^2 + 2xy = x\cos(x)$$

collect $\Rightarrow \frac{d}{dx}[x^2 y] = x\cos(x)$

$$\Rightarrow x^2 y = \int x\cos(x)\,dx = x\sin(x) - \int \sin(x)\,dx.$$

$u = x \quad du = dx$ $\qquad = x\sin(x) + \cos(x) + C.$

$dv = \cos(x)\,dx \quad v = \sin(x)$

Thus $y = \frac{\sin(x)}{x} + \frac{\cos(x) + C}{x^2}$

use $4 = y(\pi) = \frac{\sin(\pi)}{\pi} + \frac{\cos(\pi) + C}{\pi^2} = \frac{-1 + C}{\pi^2}$

$$\Rightarrow 4\pi^2 + 1 = C.$$

Thus

$$\underline{y(x) = \frac{\sin(x)}{x} + \frac{\cos(x) + 4\pi^2 + 1}{x} \quad \text{on } I = (0, \infty)}$$

4. (6 points) Solve the following differential equation

$$\underbrace{(2xy - 9x^2)}_{M}dx + \underbrace{(2y + x^2 + 1)}_{N}dy = 0$$

Test $M_y = 2x + 0$ $\quad N_x = 0 + 2x + 0$ are equal.

so there exists an $f(x,y)$ such that

$$\begin{cases} f_x = M = 2xy - 9x^2 \\ f_y = N = 2y + x^2 + 1. \end{cases}$$

Compute $f(x,y) = \int (2xy - 9x^2)dx + g(y)$

$= x^2y - 3x^3 + g(y)$

Compare to solve for $g(y)$,

$$f_y = x^2 + 0 + g'(y) = N = \underbrace{2y + x^2 + 1}_{\text{From above}}$$

so $\cancel{x^2} + g'(y) = 2y + \cancel{x^2} + 1$

$\Rightarrow g'(y) = 2y + 1$

$\Rightarrow g(y) = y^2 + y$ works.

Thus the general solution satisfies.

$$\underline{\underline{x^2y - 3x^3 + y^2 + y = C}}.$$

5. Consider the following differential equation

$$4x^2y'' + y = 0$$

valid for $x > 0$.

(a) (4 points) Given that $y_1(x) = x^{1/2}\ln(x)$ is a solution find a second solution $y_2(x)$.

Standard form $\Rightarrow$ $y'' + \frac{1}{4x^2}y = 0$ here $P(x) = 0$.

so $y_2(x) = y_1 \int \frac{e^{-\int P dx}}{y_1^2}\,dx$

$= x^{1/2}\ln(x) \int \frac{e^{-\int 0\,dx}}{(x^{1/2}\ln(x))^2}\,dx$

$= x^{1/2}\ln(x) \int \frac{1}{x\ln^2(x)}\,dx$ Let $u = \ln(x)$, $du = \frac{1}{x}dx$

$= x^{1/2}\ln(x) \int \frac{du}{u^2}$

$= x^{1/2}\ln(x)\left(-\frac{1}{u}\right)$

$= x^{1/2}\cancel{\ln(x)}\left(-\frac{1}{\cancel{\ln(x)}}\right)$

$= -x^{1/2}$

(b) (2 points) Assuming these are solutions, show that they form a fundamental set of solutions.

$$W(y_1, y_2) = \begin{vmatrix} x^{1/2}\ln(x) & -x^{1/2} \\ \frac{1}{2}x^{-1/2}\ln(x) + x^{-1/2} & -\frac{1}{2}x^{-1/2} \end{vmatrix}$$

$= -\frac{1}{2}\ln(x) + \frac{1}{2}\ln(x) + 1 = 1 \neq 0$

So they are LI. as there are two solutions then they form a fundamental set.

-This Page is for Scratch Work-