BILKENT UNIVERSITY
DEPARTMENT OF ELECTRICAL AND ELECTRONICS ENGINEERING

# EEE 321
# Signals and Systems
# Lab Assignment 5

*due: 26.04.2024, Friday at 23:55 on Moodle*

Fourier Transform is one of the most widely used methods in signal processing in various domains. In this lab assignment, you will apply your knowledge of the Fourier transform to find the equivalent impulse response of a system. Furthermore, you will use the Fourier transform to estimate the location of an object using a transmitter-receiver (T/R) setup and use it with the Doppler effect to estimate the velocity of a moving object in this T/R setup.

Please work on this assignment before coming to the laboratory. At the end of the laboratory session, you must show your completed work for **ALL** parts. After the lab session, you will have more time to format your completed work as a report and submit it on Moodle. Some parts will be performed by hand, and others will be done using MATLAB. Please address all the questions asked in the assignment and include all your codes as text and derivations for all parts. Before the submission deadline, make sure to upload your work as a readable, well-formatted single PDF file. Note that the MATLAB codes will be tested on MATLAB R2023a. Ensure your code does not raise an error or a warning in earlier versions. Do NOT forget to add proper captions, axis labels, and titles for any plot you provide.

## Part 1

Please derive the expressions or values that are asked in the following questions, showing all relevant steps.

a) A block diagram of a continuous-time linear time-invariant (LTI) system is provided in Figure 1. This system consists of cascaded subsystems where,

$$h_1(t) = \frac{d}{dt}\left[\frac{\sin(\omega_c t)}{2\pi t}\right]$$

$$h_2(t) = -\frac{1}{2}e^{j\omega_1 t} + \cos(\omega_1 t)$$

$$H_3(j\omega) = e^{-j\frac{2\pi}{\omega_2}\omega}$$

$$h_4(t) = u(t)$$

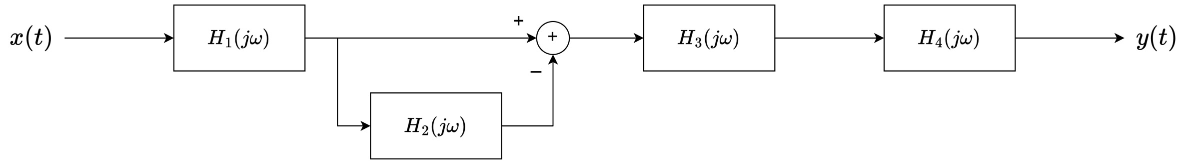


Figure 1: Block diagram of the continuous-time LTI system.

First, sketch the magnitude response of the system, $|H_1(j\omega)|$, and show all the important values on your sketch. Then, find the equivalent frequency response function of the overall system, $H_{\text{eq}}(j\omega)$. Also, derive the impulse response $h_{\text{eq}}(t)$ of the equivalent system. (**Note:** Please do not leave the resulting expressions as convolution of two terms; evaluate the convolutions and simplify as much as possible). For the input $x(t) = \sin(50t) + \cos(200t)$, sketch the magnitude of the Fourier transform of $x(t)$, $|X(j\omega)|$. For the parameter values $\omega_c = 100$ rad/s, $\omega_1 = 75$ rad/s, $\omega_2 = 2\pi 10$ rad/s, and the given $x(t)$, obtain $y(t)$. What does the equivalent system do on the frequency content of the input signal (i.e., what kind of filter is $H_{\text{eq}}(j\omega)$)?

b) The following are known about a signal $x(t)$ and its Fourier transform $X(j\omega)$:

- $x(t)$ is real and even.
- $X(j\omega) = c$ for $\omega \in [0, 2\pi]$, where $c$ is an unknown constant.
- $X(j\omega) = ce^{2\omega}$ for $\omega < -2\pi$.
- $\int_{-\infty}^{\infty} x(t)dt = 1$.

Using the provided information and without any calculations, find the constant $c$. Then, find and plot $X(j\omega)$. Finally, compute the time-domain signal $x(t)$ from its Fourier-domain counterpart $X(j\omega)$. (**Note:** Express exponential expressions that are a function of $t$ in terms of cosine and sine functions and simplify your expressions as much as possible).

# Part 2

## 2.1 Implementing the Fourier Transform

As you have learned in the lectures, the Fourier transform can be expressed as follows,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{1}$$

In this part, you will write a function on MATLAB that implements the Fourier transform as in Eqn. (1), without using built-in Fourier transform functions found in MATLAB (such as **fft** or **fftshift** and similar).

Similar to the work that you have done in the previous lab assignments, we need to discretize the provided continuous-time equation to implement it on MATLAB. For this, first, substitute the integral with a summation operation. Also, as you cannot compute an infinite summation on a PC with finite memory, you need to compute the summation from

$-N$ to $N$ where $2N$ is the number of samples in your signal. Then, you will be summing up the multiplication of the exponential function with the samples of your signal that are sampled with $T_s$. To ensure that the frequency-domain and time-domain signals have the same duration (length), compute the frequency-domain signal using the frequency values from $-F_s/2$ to $F_s/2$ by dividing this frequency interval into $2N$ parts, where $F_s = 1/T_s$ is your sampling frequency. In the end, your function should look like:

**function [frequency_array] = FourierTransform(x, t, $\mathbf{T_s}$)**

- **frequency_array:** An array that contains the Fourier transform of the time-domain signal from the frequencies $-F_s/2$ to $F_s/2$ and has the length $2N$, the same length as the time-domain signal **x**.
- **x:** Time-domain signal that is sampled with the sampling period $\mathbf{T_s}$.
- **t:** The sampled time variable that is used to define your signal **x**. It is sampled with step size $\mathbf{T_s}$.
- $\mathbf{T_s}$**:** The sampling period of the signal **x**.

## 2.2 Testing the Function

To ensure that your function works as intended, you should try it with a signal whose Fourier transform you know of. For this purpose, first, derive the Fourier transform of $\cos(2\pi 30t)$ and sketch it roughly, showing the important values on your plots. Then, using the function that you have implemented in Part 2.1, obtain the Fourier transform of $\cos(2\pi 30t)$. You may use $T_s = 0.01$ s and define the variable $t$ from –10 s to 10 s. Plot the magnitude of the frequency-domain signal that you obtained as the output of your function, **FourierTransform**. Make sure that the $x$-axis values of your plot are appropriate. Compare your sketch and the plot you obtained. Comment.

# Part 3

In this part, you will apply your Fourier transform knowledge to a real-world application. Figure 2 shows two transmitter-receiver (T/R) pairs (shown with squares) and an object (shown as a sphere) in a two-dimensional (2D) scene. In this setup, your aim is to estimate the location of the object using these two T/R pairs.

## 3.1 Derivation of Relations

In this setup, to estimate the location of the object, each transmitter emits a signal for a short duration while the receivers listen to capture the reflected signals from the object. Here, both transmitters emit pulses concurrently. As a result, the received signals are the superposition of the delayed versions of the emitted signals from both transmitters. When

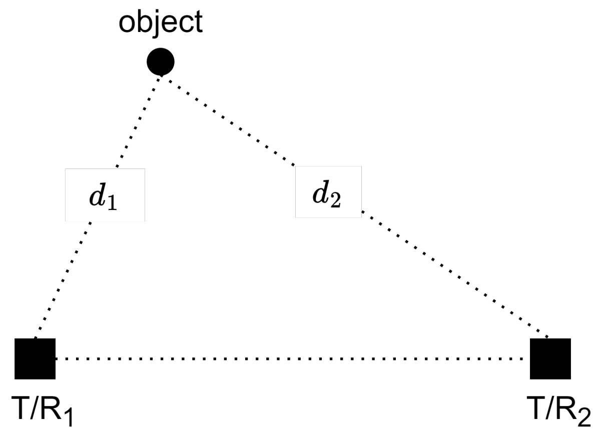


Figure 2: The setup with two T/R pairs, shown with squares, and an object in the scene, shown with a circle.

these signals are reflected from the object, we assume that the reflected signals' amplitude is unchanged. Note that the signals are delayed in proportion to their propagation path. To discriminate between the signals, the two transmitters emit signals with two distinct frequencies.

For the implementation, first, you will derive some relations analytically. Then, you will implement them on MATLAB and estimate the distances of the T/R pairs to the object, $d_1$ and $d_2$, numerically. For this purpose, let us name the signal transmitted from transmitter 1 as $x_1(t)$ and from transmitter 2 as $x_2(t)$. As mentioned before, these two signals have two distinct frequencies called, $f_1$ and $f_2$, respectively. Also, let us call the received signal from the receiver 1 as $r_1(t)$ and from the receiver 2 as $r_2(t)$. Let the propagation speed of the transmitted signal in this environment be a constant $c$. Using this nomenclature, express $r_1(t)$ in terms of the transmitted signals, the distances of the T/R pairs to the object, and the propagation speed of the signal. Repeat this for $r_2(t)$. Then, using your knowledge of the Fourier transform properties, express the Fourier transforms of $r_1(t)$ and $r_2(t)$, called $R_1(j\omega)$ and $R_2(j\omega)$, in terms of the Fourier transforms of $x_1(t)$ and $x_2(t)$, called $X_1(j\omega)$ and $X_2(j\omega)$, and other parameters that are defined above.

To estimate the distance of the object, each receiver uses a band-pass filter (BPF) around the frequency that its transmitter emits. In this way, it removes the frequency components in the received signal that are not transmitted by its own transmitter and obtains the delayed version of the transmitted signal its transmitter has emitted. Let us call these filtered signals $y_1(t)$ and $y_2(t)$ for each receiver. Assume that the frequencies $f_1$ and $f_2$ are separated clearly and ideal BPFs centered at the respective frequencies $f_1$ and $f_2$ with a bandwidth of $\omega_{\text{pass}}$ are used on the received signals $R_1(j\omega)$ and $R_2(j\omega)$. Derive the relationship between the frequency-domain signals $Y_1(j\omega)$ and $X_1(j\omega)$ in terms of the distance $d_1$, the propagation speed of the signal $c$, the center frequency $f_1$, and the pass-band width $\omega_{\text{pass}}$. Repeat for the relationship between $Y_2(j\omega)$ and $X_2(j\omega)$ in terms of the distance $d_2$, the propagation speed of the signal $c$, the center frequency $f_1$, and the pass-band width $\omega_{\text{pass}}$. From these relations, obtain expressions for $d_1$ and $d_2$ by leaving them alone on one side of the equation.

## 3.2 Estimating Distances

Now that you have an analytical expression that can provide you with the distances $d_1$ and $d_2$, you can implement this setup and find an estimate for the values of $d_1$ and $d_2$. For this, let us define the transmitted signals $x_1(t)$ and $x_2(t)$ as follows,

$$\begin{aligned} x_1(t) &= \cos(2\pi f_1 t) * \Pi_T(t - T/2) \\ x_2(t) &= \cos(2\pi f_2 t) * \Pi_T(t - T/2) \end{aligned}$$

where $\Pi_T(t)$ is a rectangular pulse defined as,

$$\Pi_T(t) = \begin{cases} 1 & \text{for } -T/2 < t < T/2 \\ 0 & \text{otherwise} \end{cases}$$

In other words, each transmitted signal is a sinusoidal waveform confined in time by a rectangular pulse. Each transmitted sinusoidal signal starts from $t = 0$ and lasts until $t = T$. Implement $x_1(t)$, $x_2(t)$, $r_1(t)$, and $r_2(t)$ on MATLAB by using the expressions provided above for $x_1(t)$ and $x_2(t)$, and the ones that you have derived for the received signals $r_1(t)$ and $r_2(t)$ in Part 3.1. For this, assume the duration of each transmitted pulse, $T$, is equal to 1 second, and the total duration of received signals, $r_1(t)$ and $r_2(t)$, are 2 seconds. Set the frequencies of the transmitted signals to $f_1 = 100$ Hz and $f_2 = 800$ Hz, the distances to $d_1 = 0.05$ m and $d_2 = 0.1$ m, the propagation speed of the signal as $c = 343$ m/s, and the sampling period to $T_s = 0.0001$ s. Plot $x_1(t)$, $x_2(t)$, $r_1(t)$, and $r_2(t)$ on the same figure using a $2 \times 4$ **subplot**.

Then, using the **FourierTransform** function that you have coded in Part 2.1, obtain the frequency-domain signals for $x_1(t)$, $x_2(t)$, $r_1(t)$, and $r_2(t)$. Plot the magnitudes of $X_1(j\omega)$ and $R_1(j\omega)$ on top of each other, using the **hold on** command on MATLAB. You can see better if you plot $|R_1(j\omega)|$ first, and then $|X_1(j\omega)|$ on top of it. On a second figure, plot the magnitudes of $X_2(j\omega)$ and $R_2(j\omega)$ in the same way. Use the **legend** command to indicate the names of each plotted signal. Make sure that the values on the $x$-axis (the frequency values) of your plot are appropriate.

On the frequency-domain signals $R_1(j\omega)$ and $R_2(j\omega)$, apply an ideal BPF that is centered at their transmitters' frequency, $f_1$ and $f_2$, respectively. Both BPFs should have a bandwidth of 50 Hz, an amplitude of 1 in the pass-band, and should be two-sided (for instance, it should filter both around $f_1$ and $-f_1$). As a result of these operations, you will obtain the filtered signals $Y_1(j\omega)$ and $Y_2(j\omega)$ from the received signals $R_1(j\omega)$ and $R_2(j\omega)$, respectively. Plot the magnitudes of $Y_1(j\omega)$ and $Y_2(j\omega)$ on the same figure using the **subplot** command. Finally, using the result of the derivation you made for $d_1$ and $d_2$ in Part 3.1 and the frequency-domain signals you obtained on MATLAB, compute the values of $d_1$ and $d_2$. (**Note:** Compute the phase value at $\omega = 2\pi f_1$ for $d_1$ and $\omega = 2\pi f_2$ for $d_2$). Compare the estimated $d_1$ and $d_2$ values with the true values. Comment on your findings. Propose a way to improve this system's location estimation performance.

## Part 4

In this part, using a similar setup to that of Part 3, you will estimate the velocity of a moving object at the time of the measurement.

When an object moves with a constant velocity $v$, the reflected signal's frequency, $f_r$, is found to be $f_r = f_t\left(\frac{c+v}{c-v}\right)$ due to the Doppler effect. Here, $f_t$ is the transmitted signal's frequency, and $c$ is the propagation speed of the signal. In the setup shown in Figure 2, since the two T/R pairs will observe relative components of the object's velocity vector, their observed frequency scaling will be different, $f_{r1} = f_t\left(\frac{c+v_1}{c-v_1}\right)$, and $f_{r2} = f_t\left(\frac{c+v_2}{c-v_2}\right)$, respectively. Here, $v_1$ is the relative velocity of the object with respect to receiver 1, and $v_2$ is the relative velocity of the object with respect to receiver 2. This is illustrated in Figure 3. Using this information, your knowledge of Fourier transform properties and the relations that you have derived in Part 3.1, express the received signals, $r_1(t)$ and $r_2(t)$, first in the time domain, in terms of the transmitted signals $x_1(t)$ and $x_2(t)$ and other parameters such as object's relative velocity $v_1$ or $v_2$, the propagation speed of the signal $c$, and the distances $d_1$ and $d_2$. Then, in the frequency domain, derive the expressions for $R_1(j\omega)$ and $R_2(j\omega)$, in terms of the transmitted signals' Fourier transforms $X_1(j\omega)$, $X_2(j\omega)$ and other parameters.

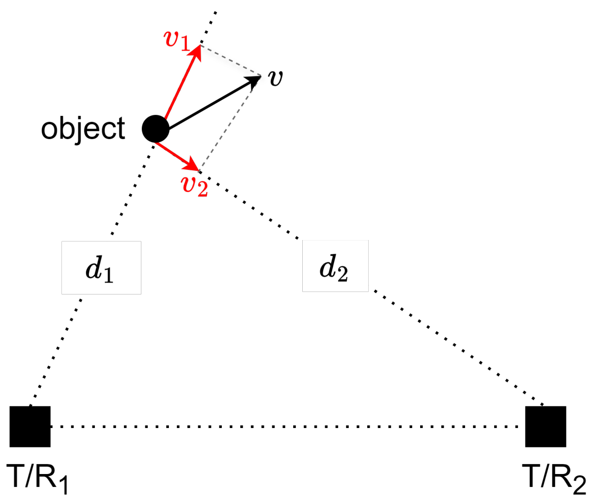


Figure 3: The setup with two T/R pairs, shown with squares, and a moving object in the scene, shown with a circle.

To estimate the velocity of the signal, first, implement the transmitted and received signals $x_1(t)$, $x_2(t)$, $r_1(t)$, and $r_2(t)$ on MATLAB as you have done in Part 3.2, but also add the changes due to the velocity of the object. Set the relative velocities as $v_1 = -30$ m/s and $v_2 = 40$ m/s. You may use the values of the other parameters as set in Part 3.2. Then, by computing the Fourier transform of these received signals with the **FourierTransform** function you have written in Part 2.1, obtain $X_1(j\omega)$, $X_2(j\omega)$, $R_1(j\omega)$, and $R_2(j\omega)$. Plot the magnitude of $X_1(j\omega)$ on top of the magnitude of $R_1(j\omega)$ on the same plot, using the **hold on** command. On another figure, repeat this for $X_2(j\omega)$ and $R_2(j\omega)$. On the plot, detect the peak of $X_1(j\omega)$ and the peak of $R_1(j\omega)$ that is closest to the peak of $X_1(j\omega)$. You should see that the peaks do not coincide as the frequencies of $R_1(j\omega)$ are scaled. Note down and show frequencies that correspond to the peaks on the plots. Using the proportion of the frequencies corresponding to the peaks, obtain the scaling factor $\frac{c+v_1}{c-v_1}$. Using this value, estimate the relative velocity $v_1$. Repeat these steps for $X_2(j\omega)$ and $R_2(j\omega)$, to estimate the relative velocity $v_2$. Compare your calculations with the true values of the velocities. What can be another method to obtain the velocity $v_1$ using the signals you obtained $X_1(j\omega)$ and $R_1(j\omega)$?

If you thought the topics in this laboratory assignment were interesting, and would like to learn more about real-life applications of location and velocity estimation using T/R antennas, "Understanding Radar Principles" video series of MATLAB on YouTube can be a good start.

# 5 Final Remarks

Throughout this assignment, you are **NOT** allowed to use symbolic operations in MATLAB. Submit the results of your own work in the form of a well-documented lab report on Moodle. Borrowing full or partial code from your peers or elsewhere is **NOT** allowed and will be punished. The axes of all plots should be scaled and labeled. To modify the styles of the plots, add labels, and scale the plots, use only MATLAB commands; do **NOT** use the GUI of the figure windows. When your program is executed, the figures must appear exactly the same as you provide in your solution. You need to write your MATLAB codes not only correctly but efficiently as well. Please include all evidence (plots, screen dumps, MATLAB codes, MATLAB command window print-outs, etc.) as needed in your report. Append your MATLAB code at the end of your assignment as text, not as an image, and do **NOT** upload it separately. You can use the "Publish" menu of MATLAB to generate a PDF file from your codes and their outputs and append it to the end of your report. If you do this, please also indicate the part that the code corresponds to with a label. Typing your report instead of handwriting some parts will be better. If you decide to write some parts by hand, please use plain white paper. Please do not upload any photos/images of your report. Your complete report should be uploaded on Moodle as a single good-quality PDF file by the given deadline. Please try to upload several hours before the deadline to avoid last-minute problems that may cause you to miss the deadline. Please DO **NOT** submit files by e-mail or as hard copies.

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