Eddp - Seminar 1

6.10.2011

① Să se determine primitivele funcției

a) $f(x)=\frac{1}{\cos^2 x}$, $x\in\left(0,\frac{\pi}{2}\right)$

b) $f(x)=\frac{1}{\sin^2 x\cos^2 x}$ $x\in\left(0,\frac{\pi}{2}\right)$

c) $f(x)=(x^2+2x)\cdot e^{3x}$ $x\in\mathbb{R}$

d) $f(x)=(x^3+1)\log_2 x$ $x\in(0,\infty)$

e) $f(x)=\sqrt{4-x^2}$ $x\in(-2,2)$

f) $f(x)=\frac{e^{\operatorname{arctg} x}}{x^2+1}$ $x\in\mathbb{R}$

② Să se determine mulțimea sol. ec. dif. $\frac{dx}{dt}=f(t)$ unde

) $f(t)=t\sqrt{t^2+1}$, $t\in\mathbb{R}$

) $f(t)=e^{t}\sin t$, $t\in\mathbb{R}$

① a) $F(x)=\int f(x)\,dx=\int\frac{1}{\cos^2 x}\,dx=\operatorname{tg} x+\mathcal{C}$

b) $F(x)=\int f(x)\,dx=\int\frac{1}{\sin^2 x\cos^2 x}\,dx=$

$=\int\frac{\sin^2 x+\cos^2 x}{\sin^2 x\cos^2 x}\,dx=\int\frac{\cancel{\sin^2 x}}{\cancel{\sin^2 x}\cos^2 x}\,dx+\int\frac{\cancel{\cos^2 x}}{\sin^2 x\cancel{\cos^2 x}}\,dx=$

$=\int\frac{1}{\cos^2 x}\,dx+\int\frac{1}{\sin^2 x}\,dx=\operatorname{tg} x-\operatorname{ctg} x+\mathcal{C}$

c) $F(x)=\int f(x)\,dx=\int(x^2+2x)\,e^{3x}\,dx=\frac{1}{3}\int(x^2+2x)(e^{3x})'\,dx$

$=\frac{1}{3}\left[(x^2+2x)\cdot e^{3x}-\int(x^2+2x)'\cdot e^{3x}\,dx\right]=$

$=\frac{1}{3}\left[(x+2x)\cdot e^{3x}-\int(2x+2)\,e^{3x}\,dx\right]=$

$=\frac{1}{3}\left[(x^2+2x)\cdot e^{3x}-\frac{1}{3}\int(2x+2)(e^{3x})'\,dx\right]=\frac{1}{3}\left[(x^2+2x)\,e^{3x}-\frac{1}{3}\left[(2x+2)\cdot\right.\right.$

$\left.\left.\cdot e^{3x}-2\int e^{3x}\,dx\right]\right]=\frac{1}{3}(x^2+2x)\,e^{3x}-\frac{1}{9}(2x+2)\,e^{3x}+\frac{2}{27}e^{3x}+\mathcal{C}$

$$\log_a x = \frac{\ln x}{\ln a}$$

$$(\log_a x)' = \frac{1}{x \ln a}$$

d) $F(x) = \int f(x)\,dx = \int (x^3+1)\log_2 x \, dx =$

$= \int \left(\frac{x^4}{4} + x\right)' \log_2 x \, dx =$

$= \left(\frac{x^4}{4} + x\right) \log_2 x - \int \cancel{x}\left(\frac{x^3}{4} + 1\right) \underbrace{(\log_2 x)'}_{\frac{1}{\cancel{x}\ln 2}} dx =$

$= \left(\frac{x^4}{4} + x\right) \log_2 x - \frac{1}{\ln 2} \int \left(\frac{x^3}{4} + 1\right) dx =$

$= \left(\frac{x^4}{4} + x\right) \log_2 x - \frac{1}{\ln 2} \left(\frac{1}{4} \frac{x^4}{4} + x\right) + C$

$= \left(\frac{x^4}{4} + x\right) \log_2 x - \frac{1}{\ln 2} \left(\frac{x^4}{16} + x\right) + C$

e) $F(x) = \int f(x)\,dx = \int \sqrt{4-x^2}\,dx =$

$= \int x' \sqrt{4-x^2}\,dx = x\sqrt{4-x^2} - \int x \left(\sqrt{4-x^2}\right)' dx =$

$= x\sqrt{4-x^2} - \int x \frac{1}{2\sqrt{4-x^2}} \cdot (4-x^2)' \, dx =$

$= x\sqrt{4-x^2} - \int x \frac{1}{\cancel{2}\sqrt{4-x^2}} \cdot (-\cancel{2}x)\, dx =$

$= x\sqrt{4-x^2} - \int \frac{4-x^2}{\sqrt{4-x^2}}\,dx = x\sqrt{4-x^2} - \int \frac{(4-x^2)-4}{\sqrt{4-x^2}}\,dx =$

$= x\sqrt{4-x^2} - \left(\int \sqrt{4-x^2}\,dx - \int \frac{4}{\sqrt{4-x^2}}\,dx \right) =$

$= x\sqrt{4-x^2} - \int \sqrt{4-x^2}\,dx + \int \frac{4}{\sqrt{4-x^2}}\,dx =$

$= x\sqrt{4-x^2} - \underbrace{\int \sqrt{4-x^2}\,dx}_{F(x)} + 4 \int \frac{1}{\sqrt{4-x^2}}\,dx \Rightarrow$

$$\Rightarrow F(x) = x\sqrt{4-x^2} - F(x) + 4\arcsin\frac{x}{2} \Rightarrow$$

$$\Rightarrow F(x) = \frac{x}{2}\sqrt{4-x^2} + 2\arcsin\frac{x}{2} + \mathcal{C}$$

f) $$F(x) = \int f(x)\,dx = \int \frac{e^{\operatorname{arctg} x}}{x^2+1}\,dx = e^{\operatorname{arctg} x} + \mathcal{C}$$

$$\operatorname{arctg} x = t \Rightarrow \frac{1}{x^2+1}\,dx = dt$$

$$\Rightarrow \int e^t\,dt = e^t$$

② a) $$\frac{dx}{dt} = t\sqrt{t^2+1}$$

$$\Rightarrow x(t) = \frac{1}{2}\int 2t\sqrt{t^2+1}\,dt = \frac{1}{2} = \frac{1}{\cancel{2}}\frac{\cancel{2}}{3}(t^2+1)^{3/2} + \mathcal{C}$$

$$t^2+1 = u \Rightarrow 2t\,dt = du$$

$$\int\sqrt{u}\,du = \int u^{1/2}\,du = \frac{u^{3/2}}{3/2} + \mathcal{C}$$

b) $$\frac{dx}{dt} = e^t \sin t$$

$$x(t) = \underbrace{\int e^t \sin t\,dt} = \int (e^t)' \sin t\,dt = e^t \sin t - \int e^t \underbrace{(\sin t)'}_{\cos t}\,dt =$$

$$= e^t \sin t - \int (e^t)' \cos t\,dt = e^t \sin t - e^t \cos t + \int e^t \underbrace{(\cos t)'}_{-\sin t}\,dt =$$

$$= e^t \sin t - e^t \cos t - \underbrace{\int e^t \sin t\,dt} \Rightarrow$$

$$\Rightarrow x(t) = \frac{e^t \sin - e^t \cos t}{2} = \frac{1}{2}e^t(\sin t - \cos t) + \mathcal{C}$$

$$\frac{dx}{dt} = f(t, x)$$ nu se scrie $x(t) = \ldots$ ⚠

③ Să se det mult. sol ec. dif.

$$\frac{dx}{dt} = \frac{1}{t}(x^2-9) \quad x \in \mathbb{R},\ t>0$$

$f:(0,\infty)\times\mathbb{R} \to \mathbb{R}$

$$f(t,x) = \underbrace{\frac{1}{t}}_{a(t)}\underbrace{(x^2-9)}_{b(x)} \Rightarrow \text{ec. este cu variabile separabile}$$

Sol. staționare

I) $b(x)=0 \Rightarrow x^2-9=0 \Rightarrow x=\pm 3 \Rightarrow \begin{cases} \varphi_1(t) = 3 & \forall t>0 \\ \varphi_2(t) = -3 & \forall t>0 \end{cases}$ (1)

II) $b(x)\neq 0 \Rightarrow x \in \underbrace{(-\infty,-3)}\cup\underbrace{(-3,3)}\cup\underbrace{(3,+\infty)}$

$$\frac{dx}{x^2-9} = \frac{1}{t}dt$$

$$B(x) = \int \frac{dx}{x^2-9} = \frac{1}{6}\ln\left|\frac{x-3}{x+3}\right|$$

$$A(t) = \int \frac{1}{t}dt = \ln|t|$$

Sol în formă implicită: $\frac{1}{6}\ln\left|\frac{x-3}{x+3}\right| = \ln|t| + \ln c,\ c>0$

$$\ln\left|\frac{x-3}{x+3}\right| = 6\ln|t|c \Rightarrow$$

$$\Rightarrow \ln\left|\frac{x-3}{x+3}\right| = \ln(c|t|)^6$$

$$\left|\frac{x-3}{x+3}\right| = \underbrace{c^6}_{>0}|t|^6 \Rightarrow$$

$$\Rightarrow \frac{x-3}{x+3} = \pm c^6 t^6 \Rightarrow \frac{x-3}{x+3} = c_1 t^6 \Rightarrow$$

$c_1 \neq 0$

$$\Rightarrow x-3 = xc_1t^6 + 3c_1t^6 \Rightarrow x(1-c_1t^6) = 3c_1t^6+3 \Rightarrow$$

$$\Rightarrow \boxed{x = \frac{3(c_1t^6+1)}{1-c_1t^6}} \quad c_1 \neq 0 \quad (2)$$

Mulț. sol = (1) ∪ (2)

(4) Să se det mulț. sol ec dif.

a) $\frac{dx}{dt} = \frac{\cos^2 x}{t^2+3}$ $\quad t \in \mathbb{R}$, $x \in \mathbb{R}$

b) $\frac{dx}{dt} = \frac{1}{t}(x^2-4x+3)$ $\quad t>0$, $x \in \mathbb{R}$

c) $\frac{dx}{dt} = (tg^2 t+1)\frac{1}{\cos^2 x}$, $t \in \mathbb{R}$, $x \in (0, \frac{\pi}{2})$

d) $\frac{dx}{dt} = \frac{e^{arctg\, t}}{(t^2+1)(2x+1)e^{x^2+x}}$ $\quad x \in (-\frac{1}{2}, \infty)$, $t \in \mathbb{R}$

a) $f: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$

$$f(t,x) = \underbrace{\frac{1}{t^2+3}}_{a(t)} \cdot \underbrace{\cos^2 x}_{b(x)}$$

I) $b(x) = 0 \Rightarrow \cos^2 x = 0 \Rightarrow x \in \{(2n+1)\frac{\pi}{2} \mid n \in \mathbb{Z}\}$

$\varphi_n(t) = (2n+1)\frac{\pi}{2}$ $\quad \forall n \in \mathbb{Z}, t \in \mathbb{R}$

II $b(x) \neq 0 \Rightarrow \frac{dx}{\cos^2 x} = \frac{1}{t^2+3}dt$

$B(x) = \int \frac{1}{\cos^2 x}dx = tg\, x$

$A(t) = \int \frac{1}{t^2+(\sqrt{3})^2}dt = \frac{1}{\sqrt{3}} arctg \frac{t}{\sqrt{3}}$

$\Rightarrow tg\, x = \frac{1}{\sqrt{3}} arctg \frac{t}{\sqrt{3}} + c$

$x = arctg\left(\frac{1}{\sqrt{3}} arctg \frac{t}{\sqrt{3}}\right) + c$

$$\int \frac{1}{a^2-t^2}\,dt = \cancel{\text{arctg}}\ \text{arcth}\,\frac{t}{a} + \mathcal{C}$$

$$\int \frac{1}{\sqrt{t^2-a^2}}\,dt = \ln\left|t+\sqrt{t^2-a^2}\right| + \mathcal{C}$$

$$\int \frac{1}{t^2-a^2}\,dt = \frac{1}{2a}\ln\left|\frac{t-a}{t+a}\right| + \mathcal{C}$$

$$\int \frac{1}{\sqrt{t^2+a^2}}\,dt = \ln\left(t+\sqrt{t^2+a^2}\right) + \mathcal{C}$$

$$\int \frac{1}{t^2+a^2}\,dt = \frac{1}{a}\,\text{arctg}\,\frac{t}{a} + \mathcal{C}$$

13.10.2011

Ecuații cu variabile separabile (din temă)

1) $\frac{dx}{dt} = \frac{e^{\operatorname{arctg} t}}{(t^2+1)(2x+1)\cdot e^{x^2+x}}$ $x \in \left(-\frac{1}{2}; +\infty\right)$, $t \in \mathbb{R}$

$a(t) = \frac{e^{\operatorname{arctg} t}}{t^2+1}$

$b(x) = \frac{1}{(2x+1)\, e^{x^2+x}}$

I) $b(x) = 0 \Rightarrow \frac{1}{(2x+1)\, e^{x^2+x}} = 0$ nu are soluții reale $\Rightarrow$ ec. nu are sol. staționare

II) $b(x) \neq 0$

$(2x+1)\, e^{x^2+x}\, dx = \frac{e^{\operatorname{arctg} t}}{t^2+1}\, dt$

$B(x) = \int (2x+1)\, e^{x^2+x}\, dx = \int (e^{x^2+x})'\, dx = e^{x^2+x}$

$A(t) = \int \frac{e^{\operatorname{arctg} t}}{t^2+1}\, dt = -\int (e^{\operatorname{arctg} t})'\, dt = -e^{\operatorname{arctg} t}$

$B(x) = A(t) + \mathcal{C} \Rightarrow$

$e^{x^2+x} = -e^{\operatorname{arctg} t} + \mathcal{C}$. (sol. ec. în formă implicită)

Dacă se cere sol care verif condiția inițială $x(1) = 0$

înloc pe x cu 0 și pe t cu 1

$e^0 = -e^{\operatorname{arctg} 1} + \mathcal{C}$

$1 = -e^{\frac{\pi}{4}} + \mathcal{C} \Rightarrow \mathcal{C} = 1 + e^{\frac{\pi}{4}}$

Sol. pb. Cauchy $\begin{cases} \frac{dx}{dt} = \frac{e^{\operatorname{arctg} t}}{(t^2+1)(2x+1)\, e^{x^2+x}} \\ x(1) = 0 \end{cases}$ (la care adăugăm)

este sub forma implicită $e^{x^2+x} = -e^{\operatorname{arctg} t} + 1 + e^{\frac{\pi}{4}}$

Ecuația liniară

$$\frac{dx}{dt} = a(t)\cdot x + b(t) \qquad (1)$$

$a, b : I \subset \mathbb{R} \to \mathbb{R}$ funcții continue

I. Rez. ec. liniară omogenă atașată: $\frac{d\bar{x}}{dt} = a(t)\bar{x}$

$\Rightarrow \bar{x}(t) = C e^{A(t)}$, unde $A(\cdot)$ este o primitivă pt $a(\cdot)$.

II Variația constantelor

- determinăm $C(\cdot)$ a.î. $x(t) = C(t) e^{A(t)}$ să verifice ecuația (1)

$$\Rightarrow \frac{d}{dt}\left(C(t)\cdot e^{A(t)}\right) = a(t)\, C(t)\, e^{A(t)} + b(t) \Rightarrow$$

$\Rightarrow$ pt $C(\cdot)$ o ecuație de tip primitivă $\Rightarrow C(t) = \ldots \Rightarrow$ se scrie sol ec. (1)

Exemple

1. Să se determine mulțimea soluțiilor ec.:

$$x' = \frac{x - \ln t}{t \ln t}, \quad t > 1$$

Aflați soluția care verifică $x(e) = 1$

$$x' = \frac{x}{t \ln t} - \frac{1}{t}$$

$$a(t) = \frac{1}{t \ln t}$$

$$b(t) = -\frac{1}{t}$$

I. $\frac{d\bar{x}}{dt} = \frac{1}{t\ln t}\,\bar{x}$

$\bar{x}(t) = C e^{A(t)}$

$$A(t) = \int \frac{1}{t \ln t}\, dt \Leftrightarrow \int \cancel{\ldots}\, u\, du = \frac{u^2}{2} \Leftrightarrow \frac{1}{2t^2}$$

~~$\ln t = u \Rightarrow \frac{1}{t}\,dt = du$~~ $\ln t = u \Rightarrow \frac{1}{t}\, dt = du$

$$\int \frac{du}{u} = \ln|u|$$

$$A(t) = \ln|\underbrace{\ln t}_{>0}| = \ln(\ln(t))$$

$$\bar{x}(t) = C \cdot e^{\ln(\ln(t))} = C \cdot \ln t$$

II. $x(t) = c(t) \cdot \ln t$

$$\frac{d}{dt}\left(c(t)\cdot \ln(t)\right) = \frac{1}{t\cancel{\ln t}}\cdot c(t)\cancel{\ln t} - \frac{1}{t}$$

$$(c(t))' \cdot \ln t + \cancel{c(t)\cdot \frac{1}{t}} = \cancel{\frac{1}{t} c(t)} - \frac{1}{t}$$

$$c'(t)\cdot \ln t = -\frac{1}{t} \Rightarrow c'(t) = \frac{-1}{t\ln t} \Rightarrow$$

$$\Rightarrow c(t) = -\int \frac{1}{t\ln t}\,dt = -\ln(\ln t) \Rightarrow c(t) = -\ln(\ln t) + c_1$$

$$x(t) = (-\ln(\ln t) + c_1)\cdot \ln t$$

$x=1$, $t=e$ $\Rightarrow (-\ln(\ln e) + c_1)\ln e = 1 \Rightarrow c_1 = 1$

Temă. Det. mult. sol. ec.

a) $x' = \operatorname{tg} t\, x + \cos t$, $t \in (0, \frac{\pi}{4})$

b) $t(x' - t\cos t) - x = 0$ (expr. x') , $t > 0$

c) $x' = \dfrac{2x + \ln t}{t\ln t}$, $t > 1$

Ec. omogenă (2)

$$\boxed{\frac{dx}{dt} = g\left(\frac{x}{t}\right)} \text{ sau } \frac{dx}{dt} = f(t,x) \text{ cu } f(\alpha t, \alpha x) = f(t,x) \quad \forall \alpha \in \mathbb{R}$$

Rezolvarea. presupune schimbarea de variabila

$$\boxed{y = \frac{x}{t}} \quad \left(y(t) = \frac{x(t)}{t}\right) \Leftrightarrow \boxed{x = ty}$$

În ec (2) $\Rightarrow \frac{d}{dt}(yt) = g(y) \Rightarrow$ ec. cu var. separabile $\Rightarrow y(t) = \ldots \Rightarrow$

$\Rightarrow x(t) = ty$

Exemplu

(2) Determinați mulțimea sol. ec. urm.

$$\frac{dx}{dt} = \frac{x^2 - 2t^2}{xt + t^2}, \quad t \in (0, \infty)$$

Aflați soluția ce verifică condiția inițială: $x(1) = 2$

$$\frac{dx}{dt} = \frac{\cancel{t^2}\left(\frac{x^2}{t^2} - 2\right)}{\cancel{t^2}\left(\frac{x}{t} + 1\right)}$$

$$\frac{dx}{dt} = \frac{\left(\frac{x}{t}\right)^2 - 2}{\frac{x}{t} + 1}$$

$$g(y) = \frac{y^2 - 2}{y + 1} \quad \left(\text{schimbare de var } y = \frac{x}{t} \Leftrightarrow x = ty\right)$$

$$\frac{d}{dt}(ty) = g(y) = \frac{y^2 - 2}{y + 1} \Rightarrow$$

$$c'(t) \Leftrightarrow \frac{dc}{dt}$$

$$\frac{dy}{dt} \Leftrightarrow y'$$

$$\Rightarrow \underbrace{t'y + ty'}_{\cancel{y + ty}}$$

$$1 \cdot y + t\frac{dy}{dt} = \frac{y^2 - 2}{y + 1}$$

$$t\frac{dy}{dt} = \frac{\cancel{y^2} - 2 - \cancel{y^2} - y}{y + 1} \Leftrightarrow \frac{dy}{dt} = \underbrace{-\frac{y+2}{y+1}}_{b(y)} \cdot \underbrace{\frac{1}{t}}_{a(t)} \quad \text{ec cu var separabile}$$

$$b(y) = 0 \Rightarrow -\frac{y+2}{y+1} = 0 \Rightarrow y + 2 = 0 \Rightarrow y = -2 \Rightarrow \boxed{x(t) = -2t}$$

$$b(y) \neq 0 \Rightarrow -\frac{y+1}{y+2}dy = \frac{1}{t}dt$$

$$B(y) = -\int \frac{y+1+1-1}{y+2}dy = -\int \frac{y+2-1}{y+2}dy = -\int 1\, dy + \int \frac{1}{y+2}dy =$$

$$= -y + \ln|y+2|$$

$A(t) = \int \frac{1}{t}\,dt = \ln \underset{>0}{|t|} = \ln t$

$-y + \ln|y+2| = \ln t + c$

$y = \frac{x}{t} \Rightarrow -\frac{x}{t} + \ln\left|\frac{x}{t} + 2\right| = \ln t + c$ sol. implicită

$x(1) = 2 \Rightarrow -\frac{2}{1} + \ln 4 = \ln 1 + c \Rightarrow$

$t = 1$

$x = 2$ $\Rightarrow c = \ln 4 - 2$

Temă. Mulț. sol. ec.

1) $\frac{dx}{dt} = \frac{2x+t}{x-t}$ $t \in \mathbb{R}$, $x \neq t$

2) $\frac{dx}{dt} = \frac{xt}{x^2 + t^2}$ $t, x > 0$

$$\frac{dx}{dt} = g\left(\frac{at+bx+c}{\alpha t + \beta x + \gamma}\right)$$

$\Delta = a\beta - b\alpha$ $|a| + |\alpha| > 0$

$|b| + |\beta| > 0$

$\Delta = 0 \Rightarrow$ pt $b \neq 0$ se face s.v. $y = at + bx$ sau

pt $\beta \neq 0$ se face s.v. $y = \alpha t + \beta x$ $\Rightarrow$ ec. cu var. sep.

$\Delta \neq 0 \Rightarrow$ s.v. $\begin{cases} s = t - t_0 \\ y = x - x_0 \end{cases}$ unde (t_0, x_0) sol. sist. $\begin{cases} at + bx + c = 0 \\ \alpha t + \beta x + \gamma = 0 \end{cases}$ $\Rightarrow$ ec. omogenă

Exemple

③ $\frac{dx}{dt} = \frac{x+t-1}{t-x+2}$

④ $\frac{dx}{dt} = \frac{2x-t-1}{x+t-2}$

⑤ $\frac{dx}{dt} = \left(\frac{x+t-3}{t-x+1}\right)^2$

$a = -1$

$b = 2$

$\alpha = 1$

$\beta = 1$

$\Delta = -1\cdot 1 - 2\cdot 1 \neq 0$

$\begin{cases} 2x - t - 1 = 0 \\ x + t - 2 = 0 \end{cases} \Leftrightarrow \begin{cases} 2x - t = 1 \\ x + t = 2 \end{cases}$

$3x = 3 \Rightarrow x = 1 \Rightarrow t = 1$

$$\begin{cases} s = t-1 \\ y = x-1 \end{cases} \quad y(s) = x(t(s)) - 1$$

$\Updownarrow$

$$\begin{cases} t = s+1 \\ x = y+1 \end{cases} \quad \boxed{x(t) = y(s(t)) + 1}$$

în ec. princ.

$$\frac{d}{dt}\left(y(s(t))+1\right) = \frac{2(y+1)-(s+1)-1}{y+1+s+1+2}$$

$$\frac{dy}{ds}\cdot \frac{ds}{dt} = \frac{2y-s}{y+s} \Rightarrow \frac{dy}{ds} = \frac{2\frac{y}{s}-1}{\frac{y}{s}-1} \quad \text{ec. omogenă}$$

$s(t) = t-1$

$\frac{ds}{dt} = 1$

$v = \frac{y}{s} \Leftrightarrow y = sv$

$v(s) = \frac{y(s)}{s} \Rightarrow$ Temă

$$x(t) \xrightarrow[\substack{t=s+1 \\ x=y+1}]{} y(s) \;(\Rightarrow \text{din ec. omog}) \xrightarrow[y(s)=sv(s)]{} v(s)$$

$x(t) = y(s(t)) + 1$

EAAP - Seminar 3

17.10.2011

din temă

1)

$$\frac{dx}{dt} = \frac{xt}{x^2+t^2}$$

$$\frac{dx}{dt} = \frac{x\cancel{t}}{t^{\cancel{2}}\left(\left(\frac{x}{t}\right)^2+1\right)} \Rightarrow \frac{dx}{dt} = \frac{x}{t\left[\left(\frac{x}{t}\right)^2+1\right]} \Rightarrow \frac{dx}{dt} = \frac{\frac{x}{t}}{\left(\frac{x}{t}\right)^2+1}$$

$$\frac{x}{t} = y \Rightarrow x = ty \qquad x(t) = ty(t)$$

$$\frac{d}{dt}(ty) = \frac{y}{y^2+1}$$

$$y + t\frac{dy}{dt} = \frac{y}{y^2+1}$$

$$\frac{dy}{dt} = \frac{\frac{y}{y^2+1} - y}{t} \Rightarrow \frac{dy}{dt} = \frac{y - y^3 - y}{t(y^2+1)} \Rightarrow$$

$$\Rightarrow \frac{dy}{dt} = \underbrace{\frac{1}{t}}_{a(t)} \cdot \underbrace{\left(\frac{-y^3}{y^2+1}\right)}_{b(y)}$$ ec. cu var. separabile

i) $b(y) = 0 \Rightarrow \frac{-y^3}{y^2+1} = 0 \Rightarrow y = 0$ sol. staționară $\Rightarrow$

$\Rightarrow \cancel{x(t)} = 0$

ii) $b(y) \neq 0 \Rightarrow$ separăm variabilele

$$\frac{y^2+1}{-y^3}\,dy = \frac{1}{t}\,dt$$

$$B(y) = -\int \frac{y^2+1}{y^3}\,dy = -\int \frac{y^2}{y^3}\,dy - \int \frac{1}{y^3}\,dy = -\ln|y| - \frac{y^{-2}}{-2} =$$

$$= -\ln|y| + \frac{1}{2y^2} + c$$

$$A(t) = \int \frac{1}{t}\,dt = \ln\underset{>0}{|t|} = \ln t + c$$

$$-\ln|y| + \frac{1}{2y^2} = \ln|t| + c$$ sol în formă implicită pt y

$$-\ln\left|\frac{x}{t}\right| + \frac{1}{2\left(\frac{x}{t}\right)^2} = \ln|t| + c$$ formă implicită pt ec în x

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2) $$\frac{dx}{dt} = \frac{x-t-1}{t-x+1}$$ soluția generală

Ec. de tipul $$\frac{dx}{dt} = g\left(\frac{at+bx+c}{\alpha t+\beta x+\gamma}\right)$$

$$\begin{cases} |a|+|\alpha| > 0 \\ |\beta|+|b| > 0 \end{cases}$$

$$\left.\begin{array}{l} a=-1 \\ b=1 \\ \alpha=1 \\ \beta=-1 \end{array}\right| \Rightarrow \Delta = 1-1 = 0$$

Avem $b \neq 0 \Rightarrow$ se face schimbarea de var $y = x-t$

$x = y+t$

$x(t) = y(t)+t$

$$\frac{d}{dt}(y+t) = \frac{-t+(y+t)-1}{t-(y+t)+2}$$

$$\frac{dy}{dt} + 1 = \frac{y-1}{-y+2}$$

$$\frac{dy}{dt} = \frac{y-1}{-y+2} - 1 \Rightarrow \frac{dy}{dt} = \frac{y-1+y-2}{-y+2} \Rightarrow \frac{dy}{dt} = \underbrace{\frac{2y-3}{-y+2}}_{b(y)} \cdot \underbrace{1}_{a(t)}$$

i) $b(y) = 0 \Rightarrow 2y-3 = 0 \Rightarrow y = \frac{3}{2}$ $\Rightarrow x = \frac{3}{2} + t$

$y=\frac{3}{2}$: sol staționară pt ec în y

$x = \frac{3}{2}+t$: sol. particulară

II) $b(y)\neq 0 \Rightarrow \frac{-y+2}{2y-3}\,dy = 1\,dt$

$$B(y)=\int \frac{-y+2}{2y-3}\,dy = -\frac{1}{2}\int \frac{2y-3+1}{2y-3}\,dy =$$

$$= -\frac{1}{2}\left(y - \int \frac{1}{2y-3}\,dy\right) =$$

$$= -\frac{1}{2}y + \frac{1}{2}\,\frac{\ln|2y-3|}{2}$$

$A(t)=t$

$B(y)=A(t)+c$

$-\frac{1}{2}y + \frac{1}{4}\ln|2y-3| = t+c$ soluţie implicită a ec. în y

$-\frac{1}{2}(x-t)+\frac{1}{4}\ln|2x-2t-3| = t+c$

$$\frac{dx}{dt}=\frac{x+t-3}{t-x+1}$$

$a=1$
$b=1$
$c=-3$
$\alpha=1$
$\beta=-1$
$\gamma=1$

$\Delta=-1-1=-2\neq 0$

$$\begin{cases} t+x-3=0\\ t-x+1=0\end{cases}$$

$2t-2=0 \Rightarrow t=1 \Rightarrow t_0=1$

$x=2 \Rightarrow x_0=2$

Schimbare de var

$$\begin{vmatrix} y = x-x_0 \\ s = t-t_0 \end{vmatrix} \begin{matrix} = x-2 \Rightarrow x=y+2 \\ = t-1 = s(t) \end{matrix}$$

$y(s)=x(t(s))$

$y(s(t)) = x(t)-2$

$x=y+2 \Rightarrow$

$\Rightarrow x(t) = y(s(t))+2$

$$\frac{d}{dt}\left(y(s(t))+2\right) = \frac{y+\cancel{2}+s+\cancel{1}-\cancel{3}}{s+1-y-\cancel{2}+\cancel{1}}$$

$\| $

$y(s(t))'\cdot(s(t))'$

(termenii liberi trebuie să se reducă!)

$$\frac{dy}{ds}\cdot\underbrace{\frac{ds}{dt}}_{1} = \frac{y+s}{s-y}$$

$$\frac{dy}{ds} = \frac{s+y}{s-y}$$

$$\frac{dy}{ds} = \frac{1+\frac{y}{s}}{1-\frac{y}{s}}$$

$$\frac{y}{s} = w \Rightarrow \frac{y(s)}{s} = w(s) \Rightarrow y(s) = s\cdot w(s)$$

Ex.: $t = e^{s}$, $t>0$

$x(t) = y(s(t))$

$s = \ln t \Rightarrow \frac{ds}{dt} = \frac{1}{t}$

$\frac{dx}{dt} = \frac{dy}{ds}\cdot\frac{ds}{dt} = \frac{1}{t}\frac{dy}{ds}$

$x(t) \xrightarrow[s=t-1]{} y(s) \xrightarrow{y(s)=sw(s)} w(s)$

$y = x-2$ $(y(s(t)) = x(t)-2)$

$$\frac{d}{ds}(s\cdot w) = \frac{1+w}{1-w}$$

$$w + s\cdot\frac{dw}{ds} = \frac{1+w}{1-w}$$

$$\frac{dw}{ds} = \frac{1}{s}\cdot\left(\frac{1+w}{1-w} - w\right)$$

$$\frac{dw}{ds} = \underbrace{\frac{1}{s}}_{a(s)}\cdot\underbrace{\frac{1+\cancel{w}-\cancel{w}+w^2}{1-w}}_{b(w)}$$

I) $b(w) = 0 \Rightarrow \frac{1+w^2}{1-w} = 0 \Rightarrow$ nu are soluții.

II $b(w) \neq 0$

$B(w) = A(s) + C$

$$B(w) = \int\frac{1-w}{1+w^2}\,dw = -\frac{1}{2}\int\frac{2w-2}{w^2+1}\,dw = -\frac{1}{2}\left(\int\frac{(w^2+1)'}{w^2+1}\,dw - 2\int\frac{1}{w^2+1}\,dw\right) =$$

$= -\frac{1}{2}\left(\ln(w^2+1) - 2\operatorname{arctg} w\right)$

$A(s) = \ln|s|$

$-\frac{1}{2}\left\{\ln(w^2+1) + 2\operatorname{arctg} w\right\} = \ln|s| + c$

$-\frac{1}{2}\ln\left(\left(\frac{y}{s}\right)^2+1\right) + 2\operatorname{arctg}\frac{y}{s} = \ln|s| + c$

$-\frac{1}{2}\ln\left(\left(\frac{x-2}{t-1}\right)^2+1\right) + \operatorname{arctg}\left(\frac{x-2}{t-1}\right) = \ln|t-1| + c$

Ec. Bernoulli

$\frac{dx}{dt} = a(t)x + b(t)x^{\alpha}$

sau $\alpha \in \mathbb{R}\setminus\{0,1\}$

o ecuație răsturnată de tip Bernoulli

(4) $\frac{dx}{dt} = \frac{1}{t^2 e^x - 2t}$

$\frac{dt}{dx} = a_1(x)t + b_1(x)t^{\alpha}$

(*) $\frac{dt}{dx} = -2t + e^x \cdot t^{2}$ $\alpha = 2$

$a_1(x) = -2$

$b_1(x) = e^x$

$\alpha = 2$

$\frac{d\bar{t}}{dx} = a_1(x)\cdot\bar{t}$

$\bar{t} = c\cdot e^{A_1(x)}$ unde $A_1(x)$ primitivă a lui $a_1(x)$

$A_1(x) = \int a_1(x)\,dx = -\int 2\,dx = -2x$

$\bar{t} = c\cdot e^{-2x}$

Aflăm $c(x)$ a.î. $t(x) = c(x)\cdot e^{-2x}$ să verif. ec inițială (*)

$\frac{d}{dx}\left(c(x)e^{-2x}\right) = -2\left(c(x)e^{-2x}\right) + e^x\left(c(x)\cdot e^{-2x}\right)^2$

$\frac{dc}{dx}\cdot e^{-2x} + \cancel{c(x)\cdot(-2\cdot e^{-2x})} = \cancel{-2\left(c(x)\cdot e^{-2x}\right)} + e^x c^2(x)\cdot e^{-4x}$

$e^{-2x}\frac{dc}{dx} = \underbrace{e^x\cdot c^2(x)\cdot e^{-4x}}_{e^{-3x}}$

$$\frac{dc(x)}{dx} = e^{-x} \cdot c^2(x)$$

$$\frac{dc}{dx} = \underbrace{e^{-x}}_{a_1(x)} \cdot \underbrace{c^2}_{b_1(c)}$$

$b_1(c) = 0 \Rightarrow c^2 = 0 \Rightarrow c = 0$

$t(x) = 0$ sol. stat.

$b_1(c) \neq 0$

$B_1(c) = A_1(x) + C$

$B_1(c) = \int \frac{1}{c^2} dc = \frac{c^{-1}}{-1} + k = \frac{-1}{c}$

$A_1(x) = \int e^{-x} dx = -e^{-x} = -\frac{1}{e^x}$

$-\frac{1}{c} = -\frac{1}{e^x} + k \Rightarrow \cancel{c(x)} =$

$\frac{1}{c} = \frac{1}{e^x} - \frac{e^x}{k} \Rightarrow C = \frac{e^x}{1 - e^x \cdot k} \Rightarrow$

$\Rightarrow t(x) = \frac{e^x}{1 - e^x \cdot k} \cdot \left(-e^{-2x}\right) \quad k \in \mathbb{R}$

Să se det. sol ec. urm.: (ec. Bernoulli)

Temă I.1) $\underset{\frac{dx}{dt}}{x'} = \frac{2tx - x^2}{t^2}$

2) $x' = x\,tg\,t + x^4 \cos t$

3) $x' = \frac{tx^2 + t^3 - t}{2x(t^2 - 1)}$

II. Se dă ec $x' = at^\alpha + bx^\beta$ unde α, β, a, b sunt constante reale. Să se det. α și β a.î. schimbarea de var $x = y^m$ să conducă la ec. omogenă în y.

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3) Să se det. mulț. sol. ec.

$$x' = \frac{tx^2 + t^3 - t}{2x(t^2-1)}, \quad t \in (1, +\infty)$$. Să se det sol care verif. $x(2) = 1$

$$x' = \frac{tx^2}{2x(t^2-1)} + \frac{t\cancel{(t^2-1)}}{2x\cancel{(t^2-1)}} \Rightarrow x' = \underbrace{\frac{t}{2(t^2-1)}}_{a(t)} x + \underbrace{\frac{t}{2}}_{b(t)} \cdot x^{-1} \quad (1)$$

$\alpha = -1$

$$\frac{d\bar{x}}{dt} = a(t)\bar{x} \Rightarrow \bar{x} = c \cdot e^{A(t)} = c \cdot e^{\ln(t^2-1)^{\frac{1}{4}}} = c \cdot (t^2-1)^{\frac{1}{4}}$$

$$\frac{dx}{dt} = a(t)x \Rightarrow$$
$$\Rightarrow x(t) = c \cdot e^{\int a(t)dt}$$

$$A(t) = \int \frac{t}{2(t^2-1)} dt =$$

$$= \frac{1}{4} \ln|t^2-1| = \ln(t^2-1)^{\frac{1}{4}}$$

Metoda var const: det funcția $c(t)$ a.î. $x(t) = c(t)(t^2-1)^{\frac{1}{4}}$ să verif. ec. Bernoulli (1)

$$\frac{d}{dt}\left[c(t)(t^2-1)^{\frac{1}{4}}\right] = \frac{t}{2(t^2-1)}\left[c(t)(t^2-1)^{\frac{1}{4}}\right] + \frac{t}{2} \cdot \frac{1}{c(t)(t^2-1)^{\frac{1}{4}}} \Rightarrow$$

$$\Rightarrow c'(t)\cdot(t^2-1)^{\frac{1}{4}} + \cancel{c(t)\cdot\frac{1}{4}\cdot(t^2-1)^{-\frac{3}{4}}\cdot 2t} = \cancel{\frac{t}{2}\cdot c(t)\cdot(t^2-1)^{-\frac{3}{4}}} + \frac{t}{2}\cdot\frac{1}{c(t)\cdot(t^2-1)^{\frac{1}{4}}}$$

(totdeauna se simplifică termenul cu $c(t)$)

$$\frac{dc}{dt}(t^2-1)^{\frac{1}{4}} = \frac{t}{2(t^2-1)} \cdot \frac{1}{c} \Rightarrow$$

$$\Rightarrow \frac{dc}{dt} = \underbrace{\frac{t}{2(t^2-1)^{\frac{1}{2}}}}_{a_1(t)} \cdot \underbrace{\frac{1}{c}}_{b_1(c)}$$

$b_1(c) = 0 \Rightarrow \frac{1}{c} = 0 \Rightarrow$ nu are sol. staționare

Separarea variabilelor

$$\frac{dc}{\frac{1}{c}} = \frac{t}{2\sqrt{t^2-1}}\,dt$$

$$\int c\,dc = \frac{c^2}{2}$$

$$\int \frac{t}{2\sqrt{t^2-1}}\,dt = \frac{1}{2}\int \underbrace{\left(\sqrt{t^2-1}\right)'}_{=\frac{1}{2\sqrt{t^2-1}}\cdot 2t}dt = \frac{1}{2}\sqrt{t^2-1}$$

$c_1 \in \mathbb{R}$

Sol. implicită $\Rightarrow$ $\frac{c^2}{2} = \frac{1}{2}\sqrt{t^2-1} + \frac{c_1}{2}$, $\Rightarrow$

$$\Rightarrow c^2 = \sqrt{t^2-1} + c_1 \Rightarrow c = \pm\sqrt{\sqrt{t^2-1}+c_1}, \quad c_1 \in \mathbb{R}$$

$$x(t) = \pm\sqrt{\sqrt{t^2-1}+c_1}\cdot\left(t^2-1\right)^{\frac{1}{4}} \quad c_1 \in \mathbb{R}$$

$x(2) = 1$
$t = 2$
$x = 1$

$$\pm\sqrt{\sqrt{3}+c_1}\cdot\sqrt[4]{3} = 1 \quad \text{(doar partea cu +)}$$

$$\sqrt{\sqrt{3}+c_1}\cdot\sqrt[4]{3} = 1 \Big|^2$$

$$\left(\sqrt{3}+c_1\right)\cdot\sqrt{3} = 1$$

$$3 + \sqrt{3}c_1 = 1 \Rightarrow c_1 = \frac{-2}{\sqrt{3}}$$

PC: $\begin{cases} (1) \\ x(2) = 1 \end{cases} \Rightarrow x(t) = \sqrt{\sqrt{t^2-1} - \frac{2}{\sqrt{3}}}\,\sqrt[4]{t^2-1}$

Ec. Riccati

$$\frac{dx}{dt} = a(t)x^2 + b(t) + c(t)$$

$a, b, c : I \subset \mathbb{R} \to \mathbb{R}$ funcții continue

Rezolvarea presupune cunoașterea unei sol. particulare φ_0.

Se efectuează schimbarea de var $x(t) = y(t) + \varphi_0(t)$, care
$(x = y + \varphi_0)$

conduce la ec. Bernoulli

Ex: Fie ecuația $x' = -x^2 + 2tx + 5 - t^2$ (2) $t \in \mathbb{R}$

~~a) Stabiliți tipul ecuației~~

~~b)~~ Determinați $m, n \in \mathbb{R}$ a.î. $\varphi_0(t) = mt + n$ să fie sol. a ec. (2)

c) Det. mulț. sol. ec.(2)

d) Det. soluția care verifică condiția $x(1) = 3$

a) Ec. Riccati

$a(t) = -1$

$b(t) = 2t$

$c(t) = 5 - t^2$

b) Înlocuim φ_0 în ec. (pe x cu φ_0)

$$\underset{(mt+n)'}{m} = -(mt+n)^2 + 2t(mt+n) + 5 - t^2 \quad \cancel{\forall t \in \mathbb{R}}$$

$$m = -(m^2t^2 + 2mnt + n^2) + 2mt^2 + 2nt + 5 - t^2$$

$$m = -m^2t^2 - 2mnt - n^2 + 2mt^2 + 2nt + 5 - t^2$$

$$m = t^2(-m^2 + 2m - 1) + t(-2mn + 2n) + 5 - n^2$$

$$\begin{cases} -m^2 + 2m - 1 = 0 \\ -2mn + 2n = 0 \\ 5 - n^2 = m \end{cases} \iff \begin{cases} -(m-1)^2 = 0 \Rightarrow m = 1 \\ -2mn + 2n = 0 \quad (A) \\ 5 - n^2 = m \Rightarrow n^2 = 4 \Rightarrow n = \pm 2 \end{cases}$$

$\varphi_0(t) = t + 2$ sau $\varphi_0(t) = t - 2$

c) S.n.:

$x = y + \varphi_0 \Rightarrow x = y + t + 2$

$$\frac{d}{dt}(y + t + 2)' = -(y + t + 2)^2 + 2t(y + t + 2) + 5 - t^2$$

$$(y' + 1) = -(y^2 + t^2 + 4 + 2yt + 4t + 4y) + 2ty + 2t^2 + 4t + 5 - t^2$$

$$y' + 1 = -y^2 - \cancel{t^2} - 4 - \cancel{2yt} - \cancel{4t} - 4y + \cancel{2ty} + \cancel{2t^2} + \cancel{4t} + 5 - \cancel{t^2}$$

$$y' \cancel{+1} = -y^2 - 4y + \cancel{1}$$

$\cancel{y^2+5y=0} \Rightarrow \cancel{y(y+5)=0} \Rightarrow \cancel{y=0}$

$\cancel{y=-5}$

$y' = -y^2 - 4y$ ec. bernoulli

$\frac{dy}{dt} = 1(-4y - y^2)$

$a_1(t) = 1$

$b_1(y) = -4y - y^2$

I) $b_1(y) = 0 \Rightarrow -4y - y^2 = 0 \Rightarrow 4y + y^2 = 0 \Rightarrow$

$\Rightarrow y(y+4) = 0 \Rightarrow y = 0$

$y = -4$

Sol în x: $\boxed{x_1 = t+2}$ $x(1) = 3$

$x_2 = t - 2$

II $b_1(y) \neq 0 \Rightarrow y^2 + 4y \neq 0 \Rightarrow y \in \mathbb{R} \setminus \{-4, 0\}$

$dy \frac{1}{-4y - y^2} = dt \Rightarrow \frac{dy}{y^2+4y} = -dt$

$\int \frac{dy}{y^2+4y} = \cancel{\int \frac{1}{y(y+4)} dy}$

$= \int \frac{(y+2)'}{(y+2)^2 - 2^2} = \frac{1}{2 \cdot 2} \ln \left| \frac{y+2-2}{y+2+2} \right| = \frac{1}{4} \ln \left| \frac{y}{y+4} \right|$

Sol implicită: $\frac{1}{4} \ln \left| \frac{y}{y+4} \right| = -t + \frac{C_1}{4}$ $| \cdot 4$

$\ln \left| \frac{y}{y+4} \right| = -4t + C_1$

$\left| \frac{y}{y+4} \right| = e^{-4t} \cdot \underbrace{e^{C_1}}_{C_2 \in \mathbb{R}, \, C_2 > 0}$

$\frac{y}{y+4} = C_2 \, e^{-4t}$, $C_2 \in \mathbb{R}^*$

$y = (y+4) \cdot C_2 e^{-4t}$

$y(1 - C_2 e^{-4t}) = 4 C_2 e^{-4t}$

$$y = \frac{4 C_2 e^{-4t}}{1 - \cancel{4} C_2 e^{-4t}}$$

$$x(t) = \frac{4 C_2 e^{-4t}}{1 - \cancel{4} C_2 e^{-4t}} + t + 2 \qquad C_2 \in \mathbb{R}^*$$

$x(1) = 3 \Rightarrow \quad \dfrac{4 C_2 e^{-4}}{1 - C_2 e^{-4}} + \cancel{3} = \cancel{3}$

$t = 1$

$$\frac{4 C_2 e^{-4}}{1 - C_2 e^{-4}} = 0 \Rightarrow C_2 = 0$$

Temă

Det. mult. sol. ec. urm.:

1) $\dfrac{dx}{dt} = x^2 - \dfrac{x}{t} - \dfrac{4}{t^2}, \quad t \in (-\infty, 0)$

dacă are sol. particulară $\varphi_0(t) = \dfrac{\alpha}{t}$ cu $\alpha \in \mathbb{R}$ convenabil determin.
(ca ex. ant.) $\rightarrow \alpha$ trb. det

Găsiți soluția care verif. $x(-1) = 0$

2) $\dfrac{dx}{dt} = x^2 - \dfrac{3}{t} x + \dfrac{1}{t^2}, \quad t \in (0, \infty)$

(ac. cerințe)

3) $\dfrac{dx}{dt} = x^2 + x - 4e^{2t}; \quad \varphi_0 = \alpha e^t, \ \alpha \in \mathbb{R}$