Homework 1
MATH 402 Applied Stochastic Processes
Habib University – Fall 2023

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1. A number U is selected at random from the unit interval. Let the events A and B be: $A =$ "U differs from $\frac{1}{2}$ by more than $\frac{1}{4}$" and $B =$ "$1 - U$ is less than $\frac{1}{2}$." Find the events:
   (a) $A \cap B$

   Solution: $A \cap B = \left\{U \in [0, 1] \mid \left|U - \frac{1}{2}\right| > \frac{1}{4} \text{ and } 1 - U < \frac{1}{2}\right\}$

   (b) $A \cup B$

   Solution: $A \cup B = \left\{U \in [0, 1] \mid \left|U - \frac{1}{2}\right| > \frac{1}{4} \text{ or } 1 - U < \frac{1}{2}\right\}$

   (c) $A^c \cap B$

   Solution: $A^c \cap B = \left\{U \in [0, 1] \mid \left|U - \frac{1}{2}\right| \leq \frac{1}{4} \text{ and } 1 - U < \frac{1}{2}\right\}$

2. Let A, B, and C be events. Find expressions for the events:
   (a) Exactly one of the three events occurs.

   Solution: $(A \cap B' \cap C') \cup (A' \cap B \cap C') \cup (A' \cap B' \cap C)$

   (b) Exactly two of the events occur.

   Solution: $(A \cap B \cap C') \cup (A \cap B' \cap C) \cup (A' \cap B \cap C)$

   (c) One or more of the events occur.

   Solution: $A \cup B \cup C$

(d) Two or more of the events occur.

Solution: $(A \cap B) \cup (A \cap C) \cup (B \cap C) \cup (A \cap B \cap C)$

(e) None of the events occur.

Solution: $A' \cap B' \cap C'$

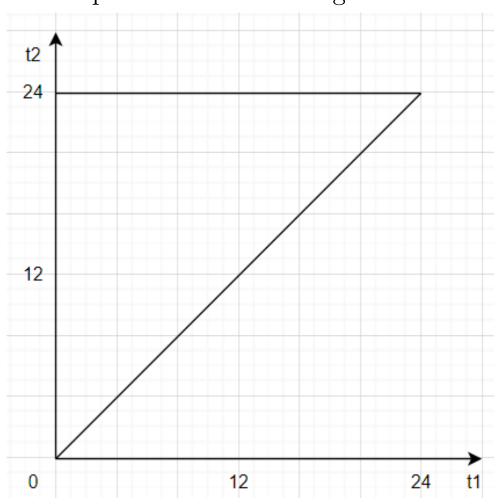
3. In a specified 8-AM-to-8-AM 24-hour period, a student wakes up at time t_1 and goes to sleep at some later time t_2.

(a) Find the sample space and sketch it on the x-y plane if the outcome of this experiment consists of the pair (t_1, t_2)

Solution: The sample space for this experiment consists of all possible pairs of times (t_1, t_2) when the student wakes up at time t_1 and goes to sleep at time t_2. Since it's a 24-hour period, both t_1 and t_2 can range from 0 (midnight) to 24 (midnight of the next day). However, t_2 should be greater than or equal to t_1 since the student goes to sleep after waking up.

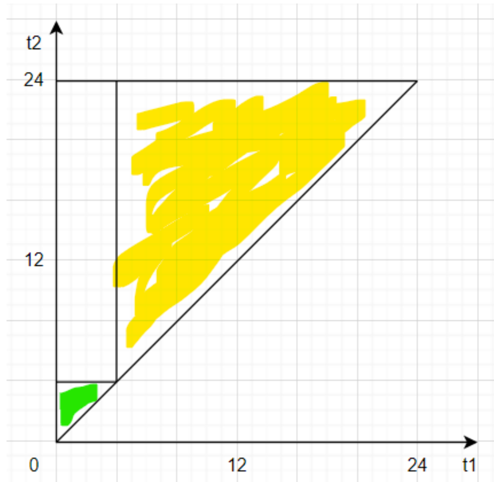
So, the sample space is the set of all pairs (t_1, t_2) where $0 \leq t_1 \leq 24$ and $t_1 \leq t_2 \leq 24$.

The sample space for this experiment is the triangle shown in the figure below.



(b) Specify the set A and sketch the region on the plane corresponding to the event "student is asleep at noon."

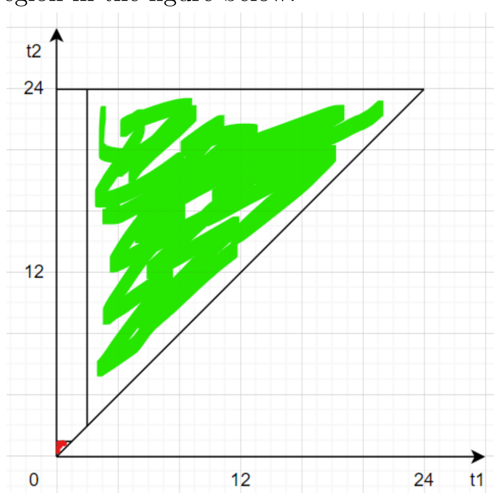
Solution: The event "student is asleep at noon" corresponds to when the student wakes up after noon or wakes up before noon and goes to sleep before noon as well. The set of all pairs (t_1, t_2) where $4 < t_1 < t_2$ or $t_1 < t_2 < 4$. The set A is the shaded region in the figure below.



The yellow shaded region corresponds to the set $4 < t_1 < t_2$ and the green shaded region corresponds to the set $t_1 < t_2 < 4$, and the union of these two regions is the set A.

(c) Specify the set B and sketch the region on the plane corresponding to the event "student sleeps through breakfast (9–10 AM)."

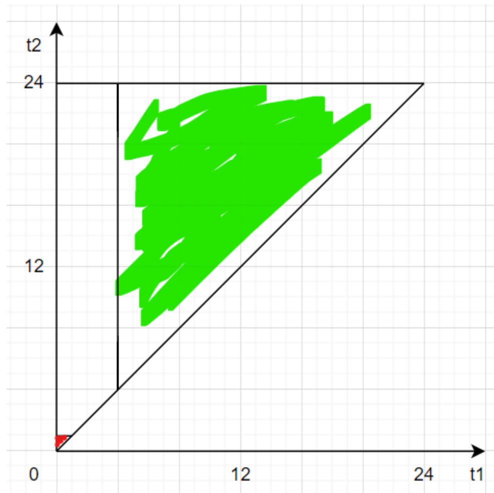
Solution: The event "student sleeps through breakfast (9-10 AM)" corresponds to when the student wakes up after 10 AM or wakes up before 9 AM and goes to sleep before 9 AM as well. The set of all pairs (t_1, t_2) where $2 < t_1 < t_2$ or $t_1 < t_2 < 1$. The set B is the shaded region in the figure below.



The green shaded region corresponds to the set $2 < t_1 < t_2$ and the red shaded region corresponds to the set $t_1 < t_2 < 1$, and the union of these two regions is the set B.

(d) Sketch the region corresponding to $A \cap B$ and describe the corresponding event in words.

Solution: The set $A \cap B$ is the intersection of the sets A and B where the student wakes up after noon or wakes up before 9 AM and goes to sleep before 9 AM as well. The set $A \cap B$ is the set of all pairs (t_1, t_2) where $4 < t_1 < t_2$ or $t_1 < t_2 < 1$. The set $A \cap B$ is the shaded region in the figure below.



The green shaded region corresponds to the set $4 < t_1 < t_2$ and the red shaded region corresponds to the set $t_1 < t_2 < 1$, and the union of these two regions is the set $A \cap B$.

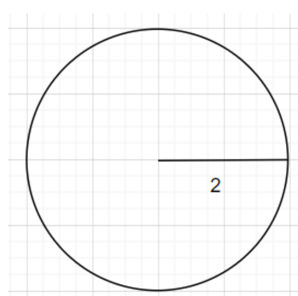
4. A dart is equally likely to land at any point inside a circular target of radius 2. Let R be the distance of the landing point from the origin.

(a) Find the sample space S and the range if R, S_R;

Solution: The sample space S is the set of all points inside the circular target of radius 2. The range of S_R is the set of all real numbers between 0 and 2, inclusive.

$S_R = [0, 2]$

$S = \{(x, y) | x^2 + y^2 \leq 4\}$

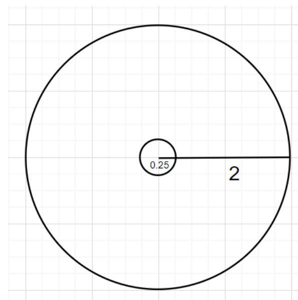


(b) Show the mapping from S to S_R;

Solution: The mapping from S to S_R is the function $f(x, y) = \sqrt{x^2 + y^2}$.

(c) The "bull's eye" is the central disk in the target of radius 0.25. Find the event A in S_R corresponding to "dart hits the bull's eye." Find the equivalent event in S and $P(A)$.

Solution: The event A in S_R corresponding to "dart hits the bull's eye" is the set of all real numbers between 0 and 0.25, inclusive.
$A = [0, 0.25]$

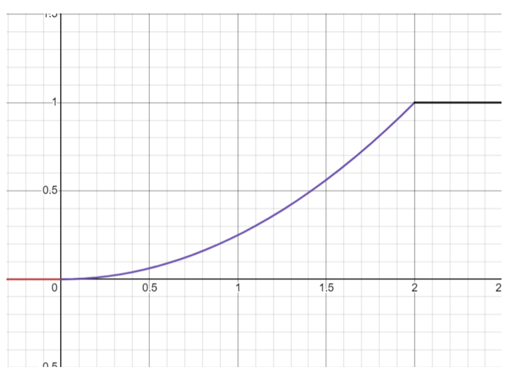


The equivalent event in S is the set of all points inside the circular target of radius 0.25.
$A = \{(x, y)|x^2 + y^2 \leq 0.25\}$
$P(A) = \dfrac{0.25^2\pi}{2^2\pi} = \dfrac{1}{16}$

(d) Find and plot the cdf of R.

Solution: The cdf of R is given by

$$F_R(r) = \begin{cases} 0, & r < 0, \\ \dfrac{r^2}{4}, & 0 \leq r \leq 2, \\ 1, & r > 2 \end{cases}$$



5. A voltage X is uniformly distributed in the set $\{-3, -2, \ldots, 3, 4\}$.

(a) Find the mean and variance of X.

Solution: The mean of X is given by

$$\begin{aligned}\mu_X &= \sum_{x \in S_X} x P(X = x) \\ &= \frac{1}{8} \sum_{x=-3}^{4} x \\ &= \frac{1}{8} (-3 + -2 + \cdots + 3 + 4) \\ &= \frac{1}{8} (-3 + -2 + -1 + 0 + 1 + 2 + 3 + 4) \\ &= \frac{1}{8} (4) \\ &= \frac{1}{2}\end{aligned}$$

The variance of X is given by

$$\begin{aligned}\sigma_X^2 &= \sum_{x \in S_X} (x - \mu_X)^2 P(X = x) \\ &= \frac{1}{8} \sum_{x=-3}^{4} (x - \mu_X)^2 \\ &= \frac{1}{8} \left((-3 - \mu_X)^2 + (-2 - \mu_X)^2 + \cdots + (3 - \mu_X)^2 + (4 - \mu_X)^2 \right) \\ &= \frac{1}{8}((-3 - \frac{1}{2})^2 + (-2 - \frac{1}{2})^2 + (-1 - \frac{1}{2})^2 + (0 - \frac{1}{2})^2 + \\ &\quad (1 - \frac{1}{2})^2 + (2 - \frac{1}{2})^2 + (3 - \frac{1}{2})^2 + (4 - \frac{1}{2})^2) \\ &= \frac{1}{8} \left((-\frac{7}{2})^2 + (-\frac{5}{2})^2 + (-\frac{3}{2})^2 + (-\frac{1}{2})^2 + (\frac{1}{2})^2 + (\frac{3}{2})^2 + (\frac{5}{2})^2 + (\frac{7}{2})^2 \right) \\ &= \frac{1}{8} \left(\frac{49}{4} + \frac{25}{4} + \frac{9}{4} + \frac{1}{4} + \frac{1}{4} + \frac{9}{4} + \frac{25}{4} + \frac{49}{4} \right) \\ &= \frac{1}{8} \left(\frac{168}{4} \right) = \frac{42}{8} = \frac{21}{4}\end{aligned}$$

(b) Find the mean and variance of $Y = -2X^2 + 3$.

Solution: The mean of Y is given by

$$\begin{aligned}
\mu_Y &= \sum_{y\in S_Y} yP(Y=y) \\
&= \frac{1}{8}\sum_{x=-3}^{4}(-2x^2+3) \\
&= \frac{1}{8}\left(-2(-3)^2+3+-2(-2)^2+3+\cdots+-2(3)^2+3+-2(4)^2+3\right) \\
&= \frac{1}{8}\left(-2(9)+3+-2(4)+3+\cdots+-2(9)+3+-2(16)+3\right) \\
&= \frac{1}{8}\left(-18+3+-8+3+\cdots+-18+3+-32+3\right) \\
&= \frac{1}{8}\left(-15+-5+-5+-5+-5+-5+-5+-5\right) \\
&= \frac{1}{8}\left(-50\right) \\
&= -\frac{25}{4}
\end{aligned}$$

The variance of Y is given by

$$\begin{aligned}
\sigma_Y^2 &= \sum_{y\in S_Y}(y-\mu_Y)^2P(Y=y) \\
&= \frac{1}{8}\sum_{x=-3}^{4}(-2x^2+3-\mu_Y)^2 \\
&= \frac{1}{8}((-2(-3)^2+3-\mu_Y)^2+(-2(-2)^2+3-\mu_Y)^2+\cdots+ \\
&\quad (-2(3)^2+3-\mu_Y)^2+(-2(4)^2+3-\mu_Y)^2) \\
&= \frac{1}{8}((-2(9)+3+\frac{25}{4})^2+(-2(4)+3+\frac{25}{4})^2+\cdots+(-2(9)+3+ \\
&\quad \frac{25}{4})^2+(-2(16)+3+\frac{25}{4})^2) \\
&= \frac{1}{8}((-18+3+\frac{25}{4})^2+(-8+3+\frac{25}{4})^2+\cdots+(-18+3+\frac{25}{4})^2+ \\
&\quad (-32+3+\frac{25}{4})^2)
\end{aligned}$$

(c) Find the mean and variance of $Z=\cos(\pi X/8)$.

Solution: The mean of Z is given by

$$\begin{aligned}
\mu_Z &= \sum_{z \in S_Z} z P(Z = z) \\
&= \frac{1}{8} \sum_{x=-3}^{4} \cos\left(\frac{\pi x}{8}\right) \\
&= \frac{1}{8}\left(\cos\left(\frac{\pi(-3)}{8}\right) + \cos\left(\frac{\pi(-2)}{8}\right) + \cdots + \cos\left(\frac{\pi(3)}{8}\right) + \cos\left(\frac{\pi(4)}{8}\right)\right) \\
&= 0.628
\end{aligned}$$

The variance of Z is given by

$$\begin{aligned}
\sigma_Z^2 &= \sum_{z \in S_Z} (z - \mu_Z)^2 P(Z = z) \\
&= \frac{1}{8} \sum_{x=-3}^{4} (\cos\left(\frac{\pi x}{8}\right) - \mu_Z)^2 \\
&= \frac{1}{8}((\cos\left(\frac{\pi(-3)}{8}\right) - \mu_Z)^2 + (\cos\left(\frac{\pi(-2)}{8}\right) - \mu_Z)^2 + \cdots + \\
&\quad (\cos\left(\frac{\pi(3)}{8}\right) - \mu_Z)^2 + (\cos\left(\frac{\pi(4)}{8}\right) - \mu_Z)^2) \\
&= \frac{1}{8}((\cos\left(\frac{\pi(-3)}{8}\right) - 0.628)^2 + \cdots + \\
&\quad (\cos\left(\frac{\pi(4)}{8}\right) - 0.628)^2) \\
&= 0.105
\end{aligned}$$

(d) Find the mean and variance of $W = \cos^2(\pi X/8)$.

Solution: The mean of W is given by

$$\begin{aligned}
\mu_W &= \sum_{w \in S_W} w P(W = w) \\
&= \frac{1}{8} \sum_{x=-3}^{4} \cos^2\left(\frac{\pi x}{8}\right) \\
&= \frac{1}{8}\left(\cos^2\left(\frac{\pi(-3)}{8}\right) + \cdots + \cos^2\left(\frac{\pi(4)}{8}\right)\right) \\
&= 0.5
\end{aligned}$$

The variance of W is given by

$$\begin{aligned}
\sigma_W^2 &= \sum_{w \in S_W} (w - \mu_W)^2 P(W = w) \\
&= \frac{1}{8} \sum_{x=-3}^{4} (\cos^2\left(\frac{\pi x}{8}\right) - \mu_W)^2 \\
&= \frac{1}{8} \left((\cos^2\left(\frac{\pi(-3)}{8}\right) - \mu_W)^2 + \cdots + (\cos^2\left(\frac{\pi(4)}{8}\right) - \mu_W)^2 \right) \\
&= 0.125
\end{aligned}$$

6. A random variable X has pdf:

$$f_X(x) = \begin{cases} cx(1-x^2), & 0 \le x \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find c and plot the pdf and the cdf of X.

Solution: Since $f_X(x)$ is a pdf, it must satisfy the following condition:

$$\begin{aligned}
\int_{-\infty}^{\infty} f_X(x)dx &= 1 \\
\int_{-\infty}^{\infty} cx(1-x^2)dx &= 1 \\
c\int_{0}^{1} x(1-x^2)dx &= 1 \\
c\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 &= 1 \\
c\left[\frac{1}{2} - \frac{1}{4}\right] &= 1 \\
c\left[\frac{1}{4}\right] &= 1 \\
c &= 4
\end{aligned}$$

The pdf of X is given by

$$f_X(x) = \begin{cases} 4x(1-x^2), & 0 \le x \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

The cdf of X is given by

$$\begin{aligned}
F_X(x) &= \int_{-\infty}^{x} f_X(x)dx \\
&= \int_{-\infty}^{x} 4x(1-x^2)dx \\
&= \left[2x^2 - \frac{4x^4}{4}\right]_0^x \\
&= \left[2x^2 - x^4\right]_0^x \\
&= 2x^2 - x^4
\end{aligned}$$

The blue curve is the pdf of X and the red curve is the cdf of X.

(b) Find $P(0 \leq X \leq 0.5)$, $P(X = 1)$, and $P(0.25 \leq X \leq 0.5)$.

Solution:

$$\begin{aligned}
P(0 \leq X \leq 0.5) &= F_X(0.5) - F_X(0) \\
&= 2(0.5)^2 - (0.5)^4 - 2(0)^2 - (0)^4 \\
&= 0.5
\end{aligned}$$

$$\begin{aligned}
P(X = 1) &= F_X(1) - F_X(1) \\
&= 2(1)^2 - (1)^4 - 2(1)^2 - (1)^4 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
P(0.25 \leq X \leq 0.5) &= F_X(0.5) - F_X(0.25) \\
&= 2(0.5)^2 - (0.5)^4 - 2(0.25)^2 - (0.25)^4 \\
&= 0.5 - 0.140625 \\
&= 0.359375
\end{aligned}$$

7. Consider two RVs, X and Y, and an RV, Z, such that $P[Z = X] = p$ and $P[Z = Y] = 1 - p$.
   (a) Show that the pdf of Z is given by

$$f_Z(z) = pf_X(z) + (1-p)f_Y(z).$$

Solution:

$$\begin{aligned} f_Z(z) &= P[Z = z] \\ &= P[Z = z \mid X = z]P[X = z] + P[Z = z \mid Y = z]P[Y = z] \\ &= pP[X = z] + (1-p)P[Y = z] \\ &= pf_X(z) + (1-p)f_Y(z) \end{aligned}$$

   (b) Calculate the cdf of two-sided exponential RV that has PDF given by

$$f_Z(z) = \begin{cases} p\lambda e^{\lambda z}, & z < 0, \\ (1-p)\lambda e^{-\lambda z}, & z \geq 0. \end{cases}$$

where $\lambda > 0$ and $0 < p < 1$.

Solution:

$$\begin{aligned} F_Z(z) &= \int_{-\infty}^{z} f_Z(z)dz \\ &= \begin{cases} \int_{-\infty}^{z} p\lambda e^{\lambda z}dz, & z < 0, \\ \int_{-\infty}^{0} p\lambda e^{\lambda z}dz + \int_0^z (1-p)\lambda e^{-\lambda z}dz, & z \geq 0. \end{cases} \end{aligned}$$

$$F_Z(z) = \begin{cases} p\lambda \int_{-\infty}^{z} e^{\lambda z}dz, & z < 0, \\ p\lambda \int_{-\infty}^{0} e^{\lambda z}dz + (1-p)\lambda \int_0^z e^{-\lambda z}dz, & z \geq 0. \end{cases}$$

$$F_Z(z) = \begin{cases} p\lambda \left[\frac{e^{\lambda z}}{\lambda}\right]_{-\infty}^{z}, & z < 0, \\ p\lambda \left[\frac{e^{\lambda z}}{\lambda}\right]_{-\infty}^{0} + (1-p)\lambda \left[-\frac{e^{-\lambda z}}{\lambda}\right]_0^z, & z \geq 0. \end{cases}$$

$$F_Z(z) = \begin{cases} p \left[\frac{e^{\lambda z}}{\lambda}\right]_{-\infty}^{z}, & z < 0, \\ p \left[\frac{e^{\lambda z}}{\lambda}\right]_{-\infty}^{0} + (1-p) \left[-\frac{e^{-\lambda z}}{\lambda}\right]_0^z, & z \geq 0 \end{cases}$$