

Assignment 3
Applied Stochastic Processes
Habib University – Fall 2023

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December 10, 2023

1 Bertsekas and Tsitsiklis, Section 7.1

1. Problem 2

Dave fails quizzes with probability $\frac{1}{4}$, independent of other quizzes.

- (a) What is the probability that Dave fails exactly two of the next six quizzes?

Solution:

$$\begin{aligned} P(\text{Dave fails exactly two of the next six quizzes}) &= \binom{6}{2} \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^4 \\ &= 15 \times \frac{1}{16} \times \frac{81}{256} \\ &= \frac{1215}{4096} \\ &= 0.296875 \end{aligned}$$

- (b) What is the expected number of quizzes that Dave will pass before he has failed three times?

Solution:

No. of times he failed = 3

Total no. of quizzes taken to fail 3 times = n

$$\begin{aligned} n * \frac{1}{4} &= 3 \\ n &= 12 \end{aligned}$$

Dave takes 12 quizzes to fail 3 times. Therefore, he passes 9 quizzes.

- (c) What is the probability that the second and third time Dave fails a quiz will occur when he takes his eighth and ninth quizzes, respectively?

Solution:

1st Fail \rightarrow 1 – 7 quizzes

2nd Fail \rightarrow 8th quiz

3rd Fail \rightarrow 9th quiz

$$\begin{aligned}
 P(X) &= P(1 \text{ fail in 7 tests}) \cdot P(2\text{nd fail in 8th test}) \cdot P(3\text{rd fail in 9th test}) \\
 &= \binom{7}{1} \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^6 \cdot \frac{1}{4} \cdot \frac{1}{4} \\
 &= \frac{7 \cdot 3^6}{4^9} = \frac{5103}{262144} \\
 &= 0.0194568634
 \end{aligned}$$

- (d) What is the probability that Dave fails two quizzes in a row before he passes two quizzes in a row?

Solution:

F = Fail, P = Pass

$$\begin{aligned}
 P(X) &= P(\text{Dave fails two quizzes in a row before he passes two quizzes in a row}) \\
 &= P(FF \cup PFF \cup FPFF \cup PFPFF \cup FFPFF \cup \dots) \\
 &= \frac{[P(F)]^2}{1 - P(F) \cdot P(P)} + \frac{P(P) \cdot [P(F)]^2}{1 - P(F) \cdot P(P)} \\
 &= \frac{\left(\frac{1}{4}\right)^2}{1 - \frac{1}{4} \cdot \frac{3}{4}} + \frac{\frac{3}{4} \cdot \left(\frac{1}{4}\right)^2}{1 - \frac{1}{4} \cdot \frac{3}{4}} \\
 &= \frac{7}{52}
 \end{aligned}$$

2. Problem 3

A computer system carries out tasks submitted by two users. Time is divided into slots. A slot can be idle, with probability $P_I = \frac{1}{6}$, and busy with probability $P_B = \frac{5}{6}$. During a busy slot, there is probability $P_{1|B} = \frac{2}{5}$ (respectively, $P_{2|B} = \frac{3}{5}$) that a task from user 1 (respectively, 2)

is executed. We assume that events related to different slots are independent.

T_1 = Task from user 1.

- (a) Find the probability that a task from user 1 is executed for the first time during the 4th slot.

Solution: If a task from user 1 is executed for the first time during the 4th slot, then the task from user 1 is not executed in the first 3 slots (they are busy and not accepting from user 1) and is executed in the 4th slot (4th slot maybe idle and execute or busy and execute).

$$\begin{aligned}
 &P(T_1 \text{ is executed for the first time during the 4th slot}) \\
 &= P(T_1 \text{ is not executed in the first 3 slots}) \cdot P(T_1 \text{ is executed in the 4th slot}) \\
 &= \left(\frac{5}{6} \times \frac{3}{5}\right)^3 \cdot \left[\left(\frac{1}{6} \times 1\right) + \left(\frac{5}{6} \times \frac{2}{5}\right)\right] \\
 &= \left(\frac{1}{2}\right)^3 \cdot \left[\frac{1}{6} + \frac{1}{3}\right] \\
 &= \frac{1}{8} \cdot \frac{1}{2} \\
 &= \frac{1}{16}
 \end{aligned}$$

- (b) Given that exactly 5 out of the first 10 slots were idle, find the probability that the 6th idle slot is slot 12.

Solution: Since exactly 5 out of the first 10 slots were idle, therefore, the 11th slot is busy.

And since the slots are independent,

$$\begin{aligned}
 &P(6\text{th idle slot is slot 12}) \\
 &= P(11\text{th slot is busy}) \cdot P(12\text{th slot is idle}) \\
 &= \frac{5}{6} \times \frac{1}{6} \\
 &= \frac{5}{36} \\
 &= 0.138889
 \end{aligned}$$

- (c) Find the expected number of slots up to and including the 5th task from user 1.

Solution: Probability of a task from user 1.

$$\begin{aligned}
 P(T_1) &= P_I \cdot P_{1|I} + P_B \cdot P_{1|B} \\
 &= \frac{1}{6} \cdot 1 + \frac{5}{6} \cdot \frac{2}{5} \\
 &= \frac{1}{6} + \frac{1}{3} \\
 &= \frac{1}{2}
 \end{aligned}$$

Probability of 5th task from user 1 (first 4 slots are busy and not accepting from user 1).

$$\begin{aligned}
 P(\text{5th task from user 1}) &= \left(\frac{1}{2}\right)^5 \\
 &= \frac{1}{32}
 \end{aligned}$$

Expected number of slots up to and including the 5th task from user 1 is the reciprocal of the probability of 5th task from user 1 = 32

- (d) Find the expected number of busy slots up to and including the 5th task from user 1.

Solution: In busy slots, there is probability $P_{1|B} = \frac{2}{5}$ that a task from user 1 is executed.

Probability of 5th task from user 1 = $\left(\frac{5}{6}\right)^5 \cdot \frac{2}{5} = 0.16075$

Expected number of busy slots up to and including the 5th task from user 1 is the reciprocal of the probability of 5th task from user 1 = $\frac{1}{0.16075} = 6.219$

- (e) Find the PMF, mean, and variance of the number of tasks from user 2 until the time of the 5th task from user 1.

Solution:

$$\binom{k+r-1}{k} p^k (1-p)^r$$

$$p = \left(\frac{1}{6} + \frac{5}{6} \times \frac{3}{5} \right) = \frac{2}{3}$$

$$\binom{k+r-1}{k} \left(\frac{2}{3} \right)^k \left(\frac{1}{3} \right)^r$$

$$\text{Mean} = \frac{pr}{1-p} = 2r$$

$$\text{Variance} = \frac{pr}{(1-p)^2} = 6r$$

The expression $\binom{k+r-1}{k} p^k (1-p)^r$ represents the probability of having k successes and r failures in a sequence of trials, where each trial has a success probability of p and a failure probability of $1-p$ (Binomial distribution).

In this case, p is calculated as $\left(\frac{1}{6} + \frac{5}{6} \times \frac{3}{5} \right) = \frac{2}{3}$.

The mean and variance of this distribution are given by $\text{Mean} = \frac{pr}{1-p} = 2r$ and $\text{Variance} = \frac{pr}{(1-p)^2} = 6r$, respectively.

2 Leon-Garcia, Section 11

1. 11.9

Let X_n be an iid integer-valued random process. Show that X_n is a Markov process and give its one-step transition probability matrix.

Solution: To show that random process X_n is a markov process, I will show that the conditional probability distribution of the future states given present states depends only on the present state and not on sequence of previous states.

Let's denote the one-step transition probability matrix as P , where $P_{ij} = P(X_{n+1} = j | X_n = i)$, i.e, the the probability of transitioning from state i to state j in one step.

Since X_n is an iid. We have:

$$P(X_{n+1} = j | X_n = i, X_{n-1}, X_{n-2}, \dots, X_0) = P(X_{n+1} = j | X_n = i)$$

This is because X_n being iid implies that the future values do not depend on the past values given the current state.

Now let's compute P_{ij} , the probability of transitioning from state i to state j in one step.

$$P_{ij} = P(X_{n+1} = j | X_n = i)$$

Since X_n is iid, this probability is same for all n . Therefore we can simply denote it as $P(X_1 = j|X_0 = i)$ which is one step transition probability.

So the one step transition probability matrix P is given by

$$P = \begin{bmatrix} P(X_1 = 1|X_0 = 1) & P(X_1 = 2|X_0 = 1) & \dots \\ P(X_1 = 1|X_0 = 2) & P(X_1 = 2|X_0 = 2) & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

This matrix will contain the probabilities of transitioning from one state to another in one step, and the independence of the random variables ensures that x_n is a Markov process.

2. 11.20

A certain part of a machine can be in two states: working or undergoing repair. A working part fails during the course of a day with probability a . A part undergoing repair is put into working order during the course of a day with probability b . Let X_n be the state of the part.

- (a) Show that X_n is a two-state Markov chain and give its one-step transition probability matrix P .

Solution: Let X_n be the state of the part at time n . To show that X_n is a two state markov chain, we need to demonstrate that the probability of transitioning to the next state depends only on the current state, and not on the sequence of events that preceded it.

For a one step transition probability matrix P , its entries are given by:

$$P_{ij} = P(X_{n+1} = j|X_n = i)$$

The probability that a working part keeps working is:

$$P_{11} = P(X_{n+1} = 1|X_n = 1) = 1 - a$$

The probability that a part being repaired, remains under repair:

$$P_{22} = P(X_{n+1} = 2|X_n = 2) = 1 - b$$

The probability that a working part goes for repair is

$$P_{21} = P(X_{n+1} = 2|X_n = 1) = a$$

The probability that a part being repaired is restored:

$$P_{12} = P(X_{n+1} = 1|X_n = 2) = b$$

Therefore the one step transition probability matrix P is

$$P = \begin{bmatrix} 1 - a & a \\ b & 1 - b \end{bmatrix}$$

- (b) Find the n -step transition probability matrix P^n .

Solution:

The n -step transition probability matrix P^n is given by:

$$P^n = \underbrace{P \cdot P \cdot \dots \cdot P}_{n \text{ times}}$$

The general formula for each element P^n_{ij} in P^n is obtained by considering all possible paths from state i to state j in n steps. The k -th element of the resulting matrix is given by the sum of products of elements from corresponding positions in matrices P^k , where k varies from 1 to n .

The n -step transition probability matrix P^n is given by:

$$P^n = \begin{bmatrix} (1-a)^n + (ab)^n & a(1-b)^n + (1-a)b^n \\ b(1-a)^n + (1-b)a^n & (1-b)^n + (ab)^n \end{bmatrix}$$

- (c) Find the steady state probability for each of two states.

Solution: The steady state probability vector π satisfies the equation $\pi P = \pi$, where π is a row vector. The steady state probabilities can be found by solving this system of equations.

For the two-state Markov chain, the steady state probability vector π is:

$$\pi = \left(\frac{b}{a+b}, \frac{a}{a+b} \right)$$

This vector represents the long-term proportion of time the system spends in each state.

3. 11.22

A stochastic matrix is defined as a nonnegative matrix for which the elements of each row add to one.

- (a) Show that the transition probability matrix P for a Markov chain is a stochastic matrix.

Solution: To show that the transition probability matrix P for a Markov chain is a stochastic matrix, we need to demonstrate two things:

1. All elements of P are nonnegative.
2. The elements of each row of P add up to one.

Let $P = [p_{ij}]$ be the transition probability matrix for a Markov chain, where p_{ij} represents the probability of transitioning from state i to state j .

1. Nonnegativity: For a stochastic matrix, all elements must be nonnegative. In the context of a transition probability matrix, probabilities must be between 0 and 1. Therefore, $0 \leq p_{ij} \leq 1$ for all i, j .

2. Row Sum: The second condition is that the sum of the elements in each row of P must be equal to 1. Mathematically, for each row i :

$$\sum_j p_{ij} = 1$$

This condition ensures that the system moves to a new state with certainty (probability 1) from the current state, reflecting the Markov property.

Combining both conditions, we can say that P is a stochastic matrix if and only if:

1. $p_{ij} \geq 0$ for all i, j .
2. $\sum_j p_{ij} = 1$ for all i .

Given that P is a valid transition probability matrix, it satisfies both of these conditions and is therefore a stochastic matrix.

(b) Show that if P and Q are stochastic matrices, then PQ is also a stochastic matrix.

Solution: To show that if P and Q are stochastic matrices, then PQ is also a stochastic matrix, we need to verify the two conditions for a matrix to be stochastic:

1. All elements of PQ are nonnegative.
2. The elements of each row of PQ add up to one.

Let P be an $m \times n$ matrix and Q be an $n \times p$ matrix. The product PQ is an $m \times p$ matrix.

1. Nonnegativity: For each element $(PQ)_{ij}$ of PQ , it is obtained by taking the dot product of the i -th row of P and the j -th column of Q . Since both P and Q are stochastic matrices, all their elements are nonnegative. The dot product of nonnegative vectors (rows of P and columns of Q) will also be nonnegative. Therefore, all elements of PQ are nonnegative.

2. Row Sum: For each row i of PQ , the sum of its elements $(PQ)_{ij}$ is given by the dot product of the i -th row of P and all columns of Q . Since the row sums of P and Q are both equal to 1 (due to them being stochastic matrices), the dot product of the rows of P with the columns of Q will also have a sum equal to 1. Therefore, the elements of each row of PQ add up to one.

Combining both conditions, we can conclude that if P and Q are stochastic matrices, then PQ is also a stochastic matrix.

(c) Show that if P is a stochastic matrix, then P^n is also a stochastic matrix.

Solution: To show that if P is a stochastic matrix, then P^n is also a stochastic matrix for any positive integer n , we need to verify the two conditions for a matrix to

be stochastic:

1. All elements of P^n are nonnegative.
2. The elements of each row of P^n add up to one.

Let P be an $m \times m$ stochastic matrix.

1. Nonnegativity: Each element $(P^n)_{ij}$ of P^n is obtained by taking the dot product of the i -th row of P^n and the j -th column of P^n . Since P is a stochastic matrix, all its elements are nonnegative. The dot product of nonnegative vectors (rows of P^n) will also be nonnegative. Therefore, all elements of P^n are nonnegative.

2. Row Sum: For each row i of P^n , the sum of its elements $(P^n)_{ij}$ is given by the dot product of the i -th row of P^n and all columns of P^n . We can express P^n as the product $P \cdot P \cdot \dots \cdot P$, where P is multiplied by itself n times.

Since each row of P adds up to one (as P is stochastic), the dot product of the rows of P^n with the columns of P^n will also have a sum equal to 1. Therefore, the elements of each row of P^n add up to one.

Combining both conditions, we can conclude that if P is a stochastic matrix, then P^n is also a stochastic matrix for any positive integer n .

4. 11.23

Show that if P^k has identical rows, then P^j has identical rows for all $j \geq k$.

Solution: Let P be a transition matrix with identical rows, and let P^k be the matrix obtained by multiplying P by itself k times.

$$P^k = P \cdot \dots \cdot P$$

Since P has identical rows, the i th row of P is equal to the j th row of P for all i, j .

Therefore, the i th row of P^k is equal to the j th row of P^k for all i, j .

This implies that P^k has identical rows.

Since P^k has identical rows, P^{k+1} must also have identical rows.

This implies that P^j has identical rows for all $j \geq k$.

5. 11.24

Prove Eq. (11.14) by induction.

$$P(n) = P^n$$

Solution:

$$P(1) = P^1 = P$$

$$P(2) = P^2 = P \cdot P = P^2$$

$$P(3) = P^3 = P \cdot P^2 = P^3$$

$$P(4) = P^4 = P \cdot P^3 = P^4$$

$$\vdots$$

The base case is $P(1) = P^1 = P$.

Assume that $P(k) = P^k$ for some $k \geq 1$.

Then $P(k+1) = P \cdot P^k = P^{k+1}$.

Therefore, $P(n) = P^n$ for all $n \geq 1$ by induction.

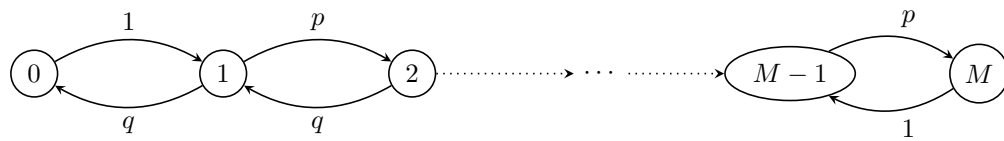
6. 11.30

Consider a random walk in the set $\{0, 1, \dots, M\}$ with transition probabilities

$p_{01} = 1, p_{M,M-1} = 1$, and $p_{i,i-1} = q, p_{i,i+1} = p$ for $i = 1, \dots, M-1$

(a) Sketch the state transition diagram.

Solution:



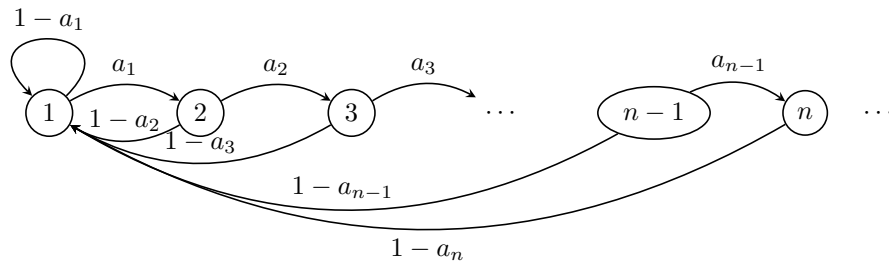
7. 11.37

Consider a Markov chain with state space and the following transition probabilities:

$P_{jj+1} = a_j$ and $p_{j1} = 1 - a_j$ where $0 < a_j < 1$

(a) Sketch the state transition diagram.

Solution:



- (b) Determine whether the Markov chain is irreducible.

Solution: The Markov chain is irreducible if there is a path from every state to every other state. In this case, there is a path from every state to every other state, so the Markov chain is irreducible.