# Assignment 3
# Applied Stochastic Processes
# Habib University – Fall 2023

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## 1 Bertsekas and Tsitsiklis, Section 7.1

1. **Problem 2**

   Dave fails quizzes with probability $\frac{1}{4}$, independent of other quizzes.

   (a) What is the probability that Dave fails exactly two of the next six quizzes?

   **Solution:**

$$\begin{aligned} P(\text{Dave fails exactly two of the next six quizzes}) &= \binom{6}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^4 \\ &= 15 \times \frac{1}{16} \times \frac{81}{256} \\ &= \frac{1215}{4096} \\ &= 0.296875 \end{aligned}$$

   (b) What is the expected number of quizzes that Dave will pass before he has failed three times?

   **Solution:**
   No. of times he failed = 3
   Total no. of quizzes taken to fail 3 times = $n$

$$\begin{aligned} n * \frac{1}{4} &= 3 \\ n &= 12 \end{aligned}$$

Dave takes 12 quizzes to fail 3 times. Therefore, he passes 9 quizzes.

(c) What is the probability that the second and third time Dave fails a quiz will occur when he takes his eighth and ninth quizzes, respectively?

**Solution:**
1st Fail $\rightarrow 1 - 7$ quizzes
2nd Fail $\rightarrow$ 8th quiz
3rd Fail $\rightarrow$ 9th quiz

$$\begin{aligned}
P(X) &= P(\text{1 fail in 7 tests}) \cdot P(\text{2nd fail in 8th test}) \cdot P(\text{3rd fail in 9th test}) \\
&= \binom{7}{1} \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^6 \cdot \frac{1}{4} \cdot \frac{1}{4} \\
&= \frac{7 \cdot 3^6}{4^9} = \frac{5103}{262144} \\
&= 0.0194568634
\end{aligned}$$

(d) What is the probability that Dave fails two quizzes in a row before he passes two quizzes in a row?

**Solution:**
$F = \text{Fail}$, $P = \text{Pass}$

$$\begin{aligned}
P(X) &= P(\text{Dave fails two quizzes in a row before he passes two quizzes in a row}) \\
&= P(FF \cup PFF \cup FPFF \cup PFPFF \cup FPFPFF \cup \dots) \\
&= \frac{[P(F)]^2}{1 - P(F) \cdot P(P)} + \frac{P(P) \cdot [P(F)]^2}{1 - P(F) \cdot P(P)} \\
&= \frac{\left(\frac{1}{4}\right)^2}{1 - \frac{1}{4} \cdot \frac{3}{4}} + \frac{\frac{3}{4} \cdot \left(\frac{1}{4}\right)^2}{1 - \frac{1}{4} \cdot \frac{3}{4}} \\
&= \frac{7}{52}
\end{aligned}$$

2. **Problem 3**

A computer system carries out tasks submitted by two users. Time is divided into slots. A slot can be idle, with probability $P_I = \frac{1}{6}$, and busy with probability $P_B = \frac{5}{6}$. During a busy slot, there is probability $P_{1|B} = \frac{2}{5}$ (respectively, $P_{2|B} = \frac{3}{5}$) that a task from user 1 (respectively, 2)

is executed. We assume that events related to different slots are independent.
$T_1$ = Task from user 1.

(a) Find the probability that a task from user 1 is executed for the first time during the 4th slot.

> **Solution:** During each slot, the probability of a task from user 1 is given by,
> $p_1 = p_{1|B} \cdot p_B = (\frac{5}{6})(\frac{2}{5}) = \frac{1}{3}$
> Tasks from user 1 form a Bernoulli process and
>
> $$P(T_1 \text{ is executed in 4th slot}) = p_1(1-p_1)^3 = \frac{1}{3} \cdot \left(\frac{2}{3}\right)^3$$
> $$= \frac{8}{81} = 0.0987654$$

(b) Given that exactly 5 out of the first 10 slots were idle, find the probability that the 6th idle slot is slot 12.

> **Solution:** Since exactly 5 out of the first 10 slots were idle, therefore, the 11th slot is busy.
> And since the slots are independent,
>
> $$\begin{aligned}&P(\text{6th idle slot is slot 12})\\&= P(\text{11th slot is busy}) \cdot P(\text{12th slot is idle})\\&= \frac{5}{6} \times \frac{1}{6}\\&= \frac{5}{36}\\&= 0.138889\end{aligned}$$

(c) Find the expected number of slots up to and including the 5th task from user 1.

> **Solution:** Each slot contains a task from user 1 with probability $p_1 = \frac{1}{3}$, independent of other slots. The time of the 5th task from user 1 is a Pascal random variable of order 5, with parameter $p_1 = \frac{1}{3}$. Its mean is given by
>
> $$\frac{5}{p_1} = \frac{5}{1/3} = 15$$

(d) Find the expected number of busy slots up to and including the 5th task from user 1.

**Solution:** Each busy slot contains a task from user 1 with probability $p_{1|B} = \frac{2}{5}$, independent of other slots. The random variable of interest is a Pascal random variable of order 5, with parameter $p_{1|B} = \frac{2}{5}$. Its mean is

$$\frac{5}{p_{1|B}} = \frac{5}{2/5} = \frac{25}{2} = 12.5$$

(e) Find the PMF, mean, and variance of the number of tasks from user 2 until the time of the 5th task from user 1.

**Solution:** The number $T$ of tasks from user 2 until the 5th task from user 1 is the same as the number $B$ of busy slots until the 5th task from user 1, minus 5. The number of busy slots ("trials") until the 5th task from user 1 ("success") is a Pascal random variable of order 5, with parameter $p_{1|B} = \frac{2}{5}$. Thus,

$$p_B(t) = \binom{t-1}{4}\left(\frac{2}{5}\right)^5\left(1-\frac{2}{5}\right)^{t-5} \qquad t = 5, 6, \ldots$$

Since $T = B - 5$, we have $p_T(t) = p_B(t+5)$, and we obtain

$$p_r(t) = \binom{t+4}{4}\left(\frac{2}{5}\right)^5\left(1-\frac{2}{5}\right)^{t} \qquad t = 0, 1, \ldots$$

Using the formulas for the mean and the variance of the Pascal random variable $B$, we obtain

$$E[T] = E[B] - 5 = \frac{25}{2} - 5 = 7.5$$

and

$$\text{Var}(T) = \text{Var}(B) = \frac{5(1-(2/5))}{(2/5)^2} = 18.75$$

# 2 Leon-Garcia, Section 11

1. **11.9**

   Let $X_n$ be an iid integer-valued random process. Show that $X_n$ is a Markov process and give its one-step transition probability matrix.

**Solution:** To show that random process $X_n$ is a markov process, I will show that the conditional probability distribution of the future states given present states depends only on the present state and not on sequence of previous states.

Let's denote the one-step transition probability matrix as $P$, where $P_{ij} = P(X_{n+1} = j|X_n = i)$, i.e, the the probability of transitioning from state $i$ to state $j$ in one step.

Since $X_n$ is an iid. We have:
$P(X_{n+1} = j|X_n = i, X_{n-1}, X_{n-2}, \dots, X_0) = P(X_{n+1} = j|X_n = i)$
This is because $X_n$ being iid implies that the future values do not depend on the past values given the current state.
Now let's compute $P_{ij}$, the probability of transitioning from state $i$ to state $j$ in one step.
$P_{ij} = P(X_{n+1} = j|X_n = i)$
Since $X_n$ is iid, this probability is same for all $n$. Therefore we can simply denote it as $P(X_1 = j|X_0 = i)$ which is one step transition probability.
So the one step transition probability matrix $P$ is given by

$$P = \begin{bmatrix} P(X_1 = 1|X_0 = 1) & P(X_1 = 2|X_0 = 1) & \dots \\ P(X_1 = 1|X_0 = 2) & P(X_1 = 2|X_0 = 2) & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

This matrix will contain the probabilities of transitioning from one state to another in one step, and the independence of the random variables ensures that $x_n$ is a Markov process.

2. **11.20**

A certain part of a machine can be in two states: working or undergoing repair. A working part fails during the course of a day with probability $a$. A part undergoing repair is put into working order during the course of a day with probability b. Let $X_n$ be the state of the part.

(a) Show that $X_n$ is a two-state Markov chain and give its one-step transition probability matrix $P$.

**Solution:** Let $X_n$ be the state of the part at time $n$. To show that $X_n$ is a two state markov chain, we need to demonstrate that the probability of transitioning to the next state depends only on the current state, and not on the sequence of events that preceded it.
For a one step transition probability matrix $P$, its entries are given by:
$$P_{ij} = P(X_{n+1} = j|X_n = i)$$
The probability that a working part keeps working is:
$$P_{11} = P(X_{n+1} = 1|X_n = 1) = 1 - a$$
The probability that a part being repaired, remains under repair:
$$P_{22} = P(X_{n+1} = 2|X_n = 2) = 1 - b$$
The probability that a working part goes for repair is

$$P_{21} = P(X_{n+1} = 2 | X_n = 1) = a$$

The probability that a part being repaired is restored:

$$P_{12} = P(X_{n+1} = 1 | X_n = 2) = b$$

Therefore the one step transition probability matrix $P$ is

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

(b) Find the n-step transition probability matrix $P^n$.

**Solution:**
The $n$-step transition probability matrix $P^n$ is given by:

$$P^n = \underbrace{P \cdot P \cdot \ldots \cdot P}_{\text{n times}}$$

The general formula for each element $P_{ij}^n$ in $P^n$ is obtained by considering all possible paths from state $i$ to state $j$ in $n$ steps. The $k$-th element of the resulting matrix is given by the sum of products of elements from corresponding positions in matrices $P^k$, where $k$ varies from 1 to $n$.

The $n$-step transition probability matrix $P^n$ is given by:

$$P^n = \begin{bmatrix} (1-a)^n + (ab)^n & a(1-b)^n + (1-a)b^n \\ b(1-a)^n + (1-b)a^n & (1-b)^n + (ab)^n \end{bmatrix}$$

(c) Find the steady state probability for each of two states.

**Solution:** The steady state probability vector $\pi$ satisfies the equation $\pi P = \pi$, where $\pi$ is a row vector. The steady state probabilities can be found by solving this system of equations.
For the two-state Markov chain, the steady state probability vector $\pi$ is:

$$\pi = \left( \frac{b}{a+b}, \frac{a}{a+b} \right)$$

This vector represents the long-term proportion of time the system spends in each state.

3. **11.22**

A stochastic matrix is defined as a nonnegative matrix for which the elements of each row add to one.

(a) Show that the transition probability matrix $P$ for a Markov chain is a stochastic matrix.

**Solution:** To show that the transition probability matrix $P$ for a Markov chain is a stochastic matrix, we need to demonstrate two things:

1. All elements of $P$ are nonnegative.
2. The elements of each row of $P$ add up to one.

Let $P = [p_{ij}]$ be the transition probability matrix for a Markov chain, where $p_{ij}$ represents the probability of transitioning from state $i$ to state $j$.

**1. Nonnegativity:** For a stochastic matrix, all elements must be nonnegative. In the context of a transition probability matrix, probabilities must be between 0 and 1. Therefore, $0 \leq p_{ij} \leq 1$ for all $i, j$.

**2. Row Sum:** The second condition is that the sum of the elements in each row of $P$ must be equal to 1. Mathematically, for each row $i$:

$$\sum_j p_{ij} = 1$$

This condition ensures that the system moves to a new state with certainty (probability 1) from the current state, reflecting the Markov property.

Combining both conditions, we can say that $P$ is a stochastic matrix if and only if:

1. $p_{ij} \geq 0$ for all $i, j$.
2. $\sum_j p_{ij} = 1$ for all $i$.

Given that $P$ is a valid transition probability matrix, it satisfies both of these conditions and is therefore a stochastic matrix.

(b) Show that if $P$ and $Q$ are stochastic matrices, then $PQ$ is also a stochastic matrix.

**Solution:** To show that if $P$ and $Q$ are stochastic matrices, then $PQ$ is also a stochastic matrix, we need to verify the two conditions for a matrix to be stochastic:

1. All elements of $PQ$ are nonnegative.
2. The elements of each row of $PQ$ add up to one.

Let $P$ be an $m \times n$ matrix and $Q$ be an $n \times p$ matrix. The product $PQ$ is an $m \times p$ matrix.

**1. Nonnegativity:** For each element $(PQ)_{ij}$ of $PQ$, it is obtained by taking the dot product of the $i$-th row of $P$ and the $j$-th column of $Q$. Since both $P$ and $Q$ are stochastic matrices, all their elements are nonnegative. The dot product of nonnegative vectors (rows of $P$ and columns of $Q$) will also be nonnegative. Therefore, all elements of $PQ$ are nonnegative.

**2. Row Sum:** For each row $i$ of $PQ$, the sum of its elements $(PQ)_{ij}$ is given by the dot product of the $i$-th row of $P$ and all columns of $Q$. Since the row sums of $P$ and $Q$ are both equal to 1 (due to them being stochastic matrices), the dot product of the rows of $P$ with the columns of $Q$ will also have a sum equal to 1. Therefore, the elements of each row of $PQ$ add up to one.

Combining both conditions, we can conclude that if $P$ and $Q$ are stochastic matrices, then $PQ$ is also a stochastic matrix.

(c) Show that if $P$ is a stochastic matrix, then $P^n$ is also a stochastic matrix.

**Solution:** To show that if $P$ is a stochastic matrix, then $P^n$ is also a stochastic matrix for any positive integer $n$, we need to verify the two conditions for a matrix to be stochastic:

1. All elements of $P^n$ are nonnegative.
2. The elements of each row of $P^n$ add up to one.

Let $P$ be an $m \times m$ stochastic matrix.

**1. Nonnegativity:** Each element $(P^n)_{ij}$ of $P^n$ is obtained by taking the dot product of the $i$-th row of $P^n$ and the $j$-th column of $P^n$. Since $P$ is a stochastic matrix, all its elements are nonnegative. The dot product of nonnegative vectors (rows of $P^n$) will also be nonnegative. Therefore, all elements of $P^n$ are nonnegative.

**2. Row Sum:** For each row $i$ of $P^n$, the sum of its elements $(P^n)_{ij}$ is given by the dot product of the $i$-th row of $P^n$ and all columns of $P^n$. We can express $P^n$ as the product $P \cdot P \cdot \ldots \cdot P$, where $P$ is multiplied by itself $n$ times.

Since each row of $P$ adds up to one (as $P$ is stochastic), the dot product of the rows of $P^n$ with the columns of $P^n$ will also have a sum equal to 1. Therefore, the elements of each row of $P^n$ add up to one.

Combining both conditions, we can conclude that if $P$ is a stochastic matrix, then $P^n$ is also a stochastic matrix for any positive integer $n$.

4. **11.23**

Show that if $P^k$ has identical rows, then $P^j$ has identical rows for all $j \geq k$.

**Solution:** Let $P$ be a transition matrix with identical rows, and let $P^k$ be the matrix obtained by multiplying $P$ by itself $k$ times.

$P^k = P \cdots P$

Since $P$ has identical rows, the $i$th row of $P$ is equal to the $j$th row of $P$ for all $i, j$.

Therefore, the $i$th row of $P^k$ is equal to the $j$th row of $P^k$ for all $i, j$.

This implies that $P^k$ has identical rows.

Since $P^k$ has identical rows, $P^{k+1}$ must also have identical rows.
This implies that $P^j$ has identical rows for all $j \geq k$.

5. **11.24**

   Prove Eq. (11.14) by induction.
   $P(n) = P^n$

   **Solution:**

$$\begin{aligned}
P(1) &= P^1 = P \\
P(2) &= P^2 = P \cdot P = P^2 \\
P(3) &= P^3 = P \cdot P^2 = P^3 \\
P(4) &= P^4 = P \cdot P^3 = P^4 \\
&\vdots
\end{aligned}$$

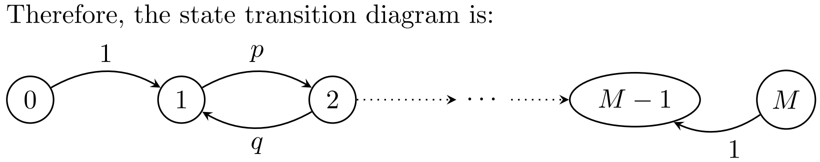
   The base case is $P(1) = P^1 = P$.
   Assume that $P(k) = P^k$ for some $k \geq 1$.
   Then $P(k+1) = P \cdot P^k = P^{k+1}$.
   Therefore, $P(n) = P^n$ for all $n \geq 1$ by induction.

6. **11.30**

   Consider a random walk in the set $\{0, 1, \ldots, M\}$ with transition probabilities
   $p_{01} = 1, p_{M,M-1} = 1,$ and $p_{i,i-1} = q, p_{i,i+1} = p$ for $i = 1, \ldots, M-1$

   (a) Sketch the state transition diagram.

   **Solution:**
   Since the transition probabilities of $p_{01}$ and $p_{M,M-1} = 1 \implies p_{10}$ and $p_{M-1,M} = 0$
   Therefore, the state transition diagram is:

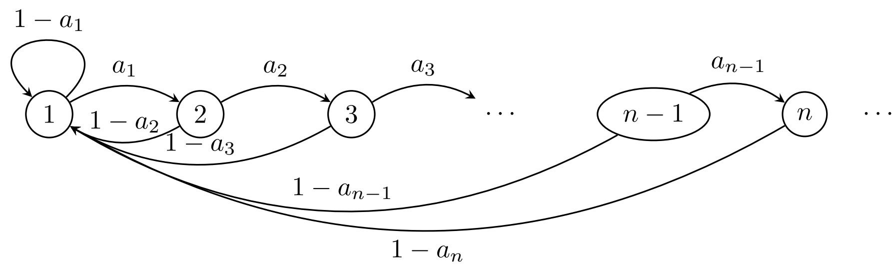



7. **11.37**

   Consider a Markov chain with state space and the following transition probabilities:
   $P_{jj+1} = a_j$ and $p_{j1} = 1 - a_j$ where $0 < a_j < 1$

(a) Sketch the state transition diagram.

**Solution:**




(b) Determine whether the Markov chain is irreducible.

**Solution:** The Markov chain is irreducible if there is a path from every state to every other state. In this case, there is a path from every state to every other state with non zero probability. Therefore, the Markov chain is irreducible.

8. **11.41**

Consider the simple queueing system discussed in Example 11.36.

(a) Use the results in Example 11.36 to find the state transition probability matrix

**Solution:** From Ex 11.36 we have,

$$p_o(t) = \frac{\beta}{\alpha+\beta} + (p_o(0) - \frac{\beta}{\alpha+\beta})e^{-(\alpha-\beta)t}$$
$$p_1(t) = \frac{\beta}{\alpha+\beta} + (p_1(0) - \frac{\beta}{\alpha+\beta})e^{-(\alpha-\beta)t}$$

Now if we suppose the initial state is 0, then $p_0(0) = 1 \implies$

$$p_{00}(t) = \frac{\beta}{\alpha+\beta} + (1 - \frac{\beta}{\alpha+\beta})e^{-(\alpha-\beta)t} = \frac{\beta + \alpha e^{-(\alpha+\beta)t}}{\alpha+\beta}$$
$$p_{01}(t) = 1 - p_{00}(t) = \frac{\alpha(1-e^{-(\alpha+\beta)t})}{\alpha+\beta}$$

If the initial state is 1, then $p_1(0) = 1 \implies$

$$p_{11}(t) = \frac{\alpha}{\alpha+\beta} + (1 + \frac{\alpha}{\alpha+\beta})e^{-(\alpha+\beta)t} = \frac{\alpha + \beta e^{-(\alpha+\beta)t}}{\alpha+\beta}$$
$$p_{10}(t) = 1 - p_{11}(t) = \frac{\beta(1-e^{-(\alpha+\beta)t})}{\alpha+\beta}$$
$$P(t) = \frac{1}{\alpha+\beta}\begin{bmatrix} \beta + \alpha e^{-(\alpha+\beta)t} & \alpha(1-e^{-(\alpha+\beta)t}) \\ \beta(1-e^{-(\alpha+\beta)t}) & \alpha + \beta e^{-(\alpha+\beta)t} \end{bmatrix}$$

(b) Find the following probabilities:

$$P[X(1.5) = 1, X(3) = 1 | X(0) = 0]$$
$$P[X(1.5) = 1, X(3) = 1]$$

**Solution:**

$$\begin{aligned}
&= P[X(3) = 1 | X(1.5) = 1, X(0) = 0] P[X(1.5) = 1 | X(0) = 0] \\
&= P[X(3) = 1 | X(1.5) = 1] P[X(1.5) = 1 | X(0) = 0] \\
&= p_{11}(1.5) p_{01}(1.5) \\
P[X(1.5) = 1, X(3) = 1] &= P[X(3) = 1 | X(1.5) = 1] P[X(1.5) = 1] \\
&= p_{11}(1.5) \left[ \frac{\alpha}{\alpha+\beta} + \left( p_1(0) - \frac{\alpha}{\alpha+\beta} \right) e^{-(\alpha+\beta)1.5} \right]
\end{aligned}$$

9. **11.42**

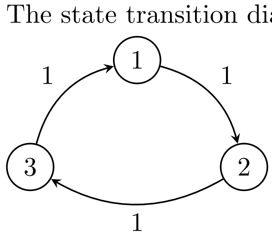
A rechargeable battery in a depot is in one of three states: fully charged, in use, or recharging. Assume the mean time in each of these states is: $\frac{1}{\lambda}$; 1 hour; 3 hours. Batteries are not put into use unless they are fully charged.

(a) Find a Markov model for the battery states and sketch the state transition diagram.

**Solution:** Let $X_n$ be the state of the battery at time $n$. The state space is $\{1, 2, 3\}$, where 1 represents the fully charged state, 2 represents the in use state, and 3 represents the recharging state. The transition probabilities are given by:

$$P_{12} = 1$$
$$P_{23} = 1$$
$$P_{31} = 1$$

The state transition diagram is:




(b) Find the stationary pmf. Explain how the pmf varies with $\lambda$

**Solution:** The transition probability matrix $P$ is given by:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Let $\pi_1, \pi_2, \pi_3$ be the stationary probabilities for states Fully Charged, In Use, and Recharging, respectively.
The stationary distribution $\pi$ satisfies the equation $\pi P = \pi$. Solving for $\pi$, we get:

$$\begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix}$$

Now, let's consider the system of equations derived from the Markov chain:

1. $\pi_1 = \pi_3$
2. $\pi_2 = \pi_1$
3. $\pi_3 = \pi_2$

Substituting $\pi_1 = \pi_3$ from equation (1) into equations (2) and (3):

2. $\pi_2 = \pi_1 = \pi_3$
3. $\pi_3 = \pi_2 = \pi_1$

This implies that $\pi_1 = \pi_2 = \pi_3$, and a possible solution is $\pi_1 = \frac{1}{3}$, $\pi_2 = \frac{1}{3}$, $\pi_3 = \frac{1}{3}$.
As $\lambda$ increases, the mean time in the fully charged state decreases. Therefore, the stationary probabilities $\pi_1, \pi_2, \pi_3$ are influenced by $\lambda$, and an increase in $\lambda$ would lead to a decrease in $\pi_1$ and an increase in $\pi_2$ and $\pi_3$. The system tends to distribute more time across the in-use and recharging states as $\lambda$ increases.