# Assignment 3
# Applied Stochastic Processes
# Habib University – Fall 2023

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## 1 Bertsekas and Tsitsiklis, Section 7.1

1. **Problem 2**

   Dave fails quizzes with probability $\frac{1}{4}$, independent of other quizzes.

   (a) What is the probability that Dave fails exactly two of the next six quizzes?

   **Solution:**

   $$\begin{aligned} P(\text{Dave fails exactly two of the next six quizzes}) &= \binom{6}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^4 \\ &= 15 \times \frac{1}{16} \times \frac{81}{256} \\ &= \frac{1215}{4096} \\ &= 0.296875 \end{aligned}$$

   (b) What is the expected number of quizzes that Dave will pass before he has failed three times?

   **Solution:**
   No. of times he failed = 3
   Total no. of quizzes taken to fail 3 times = $n$

   $$\begin{aligned} n * \frac{1}{4} &= 3 \\ n &= 12 \end{aligned}$$

Dave takes 12 quizzes to fail 3 times. Therefore, he passes 9 quizzes.

(c) What is the probability that the second and third time Dave fails a quiz will occur when he takes his eighth and ninth quizzes, respectively?

**Solution:**
1st Fail $\to 1-7$ quizzes
2nd Fail $\to$ 8th quiz
3rd Fail $\to$ 9th quiz

$$\begin{aligned}P(X) &= P(\text{1 fail in 7 tests}) \cdot P(\text{2nd fail in 8th test}) \cdot P(\text{3rd fail in 9th test})\\ &= \binom{7}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^6 \cdot \frac{1}{4} \cdot \frac{1}{4}\\ &= \frac{7 \cdot 3^6}{4^9} = \frac{5103}{262144}\\ &= 0.0194568634\end{aligned}$$

(d) What is the probability that Dave fails two quizzes in a row before he passes two quizzes in a row?

**Solution:**
$F = \text{Fail}$, $P = \text{Pass}$

$$\begin{aligned}P(X) &= P(\text{Dave fails two quizzes in a row before he passes two quizzes in a row})\\ &= P(FF \cup PFF \cup FPFF \cup PFPFF \cup FPFPFF \cup \dots)\\ &= \frac{[P(F)]^2}{1-P(F)\cdot P(P)} + \frac{P(P)\cdot[P(F)]^2}{1-P(F)\cdot P(P)}\\ &= \frac{\left(\frac{1}{4}\right)^2}{1-\frac{1}{4}\cdot\frac{3}{4}} + \frac{\frac{3}{4}\cdot\left(\frac{1}{4}\right)^2}{1-\frac{1}{4}\cdot\frac{3}{4}}\\ &= \frac{7}{52}\end{aligned}$$

2. **Problem 3**

A computer system carries out tasks submitted by two users. Time is divided into slots. A slot can be idle, with probability $P_I = \frac{1}{6}$, and busy with probability $P_B = \frac{5}{6}$. During a busy slot, there is probability $P_{1|B} = \frac{2}{5}$ (respectively, $P_{2|B} = \frac{3}{5}$) that a task from user 1 (respectively, 2)

is executed. We assume that events related to different slots are independent.
$T_1$ = Task from user 1.

(a) Find the probability that a task from user 1 is executed for the first time during the 4th slot.

**Solution:** If a task from user 1 is executed for the first time during the 4th slot, then the task from user 1 is not executed in the first 3 slots (they are busy and not accepting from user 1) and is executed in the 4th slot (4th slot maybe idle and execute or busy and execute).

$$
\begin{aligned}
&P(T_1 \text{ is executed for the first time during the 4th slot})\\
&= P(T_1 \text{ is not executed in the first 3 slots}) \cdot P(T_1 \text{ is executed in the 4th slot})\\
&= \left(\frac{5}{6} \times \frac{3}{5}\right)^3 \cdot \left[\left(\frac{1}{6} \times 1\right) + \left(\frac{5}{6} \times \frac{2}{5}\right)\right]\\
&= \left(\frac{1}{2}\right)^3 \cdot \left[\frac{1}{6} + \frac{1}{3}\right]\\
&= \frac{1}{8} \cdot \frac{1}{2}\\
&= \frac{1}{16}
\end{aligned}
$$

(b) Given that exactly 5 out of the first 10 slots were idle, find the probability that the 6th idle slot is slot 12.

**Solution:** Since exactly 5 out of the first 10 slots were idle, therefore, the 11th slot is busy.
And since the slots are independent,

$$
\begin{aligned}
&P(\text{6th idle slot is slot 12})\\
&= P(\text{11th slot is busy}) \cdot P(\text{12th slot is idle})\\
&= \frac{5}{6} \times \frac{1}{6}\\
&= \frac{5}{36}\\
&= 0.138889
\end{aligned}
$$

(c) Find the expected number of slots up to and including the 5th task from user 1.

**Solution:** Probability of a task from user 1.

$$P(T_1) = P_I \cdot P_{1|I} + P_B \cdot P_{1|B}$$
$$= \frac{1}{6} \cdot 1 + \frac{5}{6} \cdot \frac{2}{5}$$
$$= \frac{1}{6} + \frac{1}{3}$$
$$= \frac{1}{2}$$

Probability of 5th task from user 1 (first 4 slots are busy and not accepting from user 1).

$$P(\text{5th task from user 1}) = \left(\frac{1}{2}\right)^5$$
$$= \frac{1}{32}$$

Expected number of slots up to and including the 5th task from user 1 is the reciprocal of the probability of 5th task from user 1 = 32

(d) Find the expected number of busy slots up to and including the 5th task from user 1.

**Solution:** In busy slots, there is probability $P_{1|B} = \frac{2}{5}$ that a task from user 1 is executed.
Probability of 5th task from user 1 $= \left(\frac{5}{6}\right)^5 \cdot \frac{2}{5} = 0.16075$
Expected number of busy slots up to and including the 5th task from user 1 is the reciprocal of the probability of 5th task from user 1 $= \frac{1}{0.16075} = 6.219$

(e) Find the PMF, mean, and variance of the number of tasks from user 2 until the time of the 5th task from user 1.

**Solution:**

$$\binom{k+r-1}{k} p^k (1-p)^r$$

$$p = \left(\frac{1}{6} + \frac{5}{6} \times \frac{3}{5}\right) = \frac{2}{3}$$

$$\binom{k+r-1}{k} \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^r$$

$$\text{Mean} = \frac{pr}{1-p} = 2r$$

$$\text{Variance} = \frac{pr}{(1-p)^2} = 6r$$

The expression $\binom{k+r-1}{k} p^k (1-p)^r$ represents the probability of having $k$ successes and $r$ failures in a sequence of trials, where each trial has a success probability of $p$ and a failure probability of $1-p$ (Binomial distribution).
In this case, $p$ is calculated as $\left(\frac{1}{6} + \frac{5}{6} \times \frac{3}{5}\right) = \frac{2}{3}$.
The mean and variance of this distribution are given by Mean $= \frac{pr}{1-p} = 2r$ and Variance $= \frac{pr}{(1-p)^2} = 6r$, respectively.

# 2 Leon-Garcia, Section 11

1. **11.9**

   Let $X_n$ be an iid integer-valued random process. Show that $X_n$ is a Markov process and give its one-step transition probability matrix.

**Solution:** To show that random process $X_n$ is a markov process, I will show that the conditional probability distribution of the future states given present states depends only on the present state and not on sequence of previous states.

Let's denote the one-step transition probability matrix as $P$, where $P_{ij} = P(X_{n+1} = j | X_n = i)$, i.e, the the probability of transitioning from state $i$ to state $j$ in one step.

Since $X_n$ is an iid. We have:
$P(X_{n+1} = j | X_n = i, X_{n-1}, X_{n-2}, \ldots, X_0) = P(X_{n+1} = j | X_n = i)$
This is because $X_n$ being iid implies that the future values do not depend on the past values given the current state.
Now let's compute $P_{ij}$, the probability of transitioning from state $i$ to state $j$ in one step.
$P_{ij} = P(X_{n+1} = j | X_n = i)$

Since $X_n$ is iid, this probability is same for all $n$. Therefore we can simply denote it as $P(X_1 = j|X_0 = i)$ which is one step transition probability.
So the one step transition probability matrix $P$ is given by

$$P = \begin{bmatrix} P(X_1 = 1|X_0 = 1) & P(X_1 = 2|X_0 = 1) & \dots \\ P(X_1 = 1|X_0 = 2) & P(X_1 = 2|X_0 = 2) & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

This matrix will contain the probabilities of transitioning from one state to another in one step, and the independence of the random variables ensures that $x_n$ is a Markov process.

2. **11.23**

   Show that if $P^k$ has identical rows, then $P^j$ has identical rows for all $j \geq k$.

**Solution:** Let $P$ be a transition matrix with identical rows, and let $P^k$ be the matrix obtained by multiplying $P$ by itself $k$ times.

$P^k = P \cdots P$

Since $P$ has identical rows, the $i$th row of $P$ is equal to the $j$th row of $P$ for all $i, j$.

Therefore, the $i$th row of $P^k$ is equal to the $j$th row of $P^k$ for all $i, j$.

This implies that $P^k$ has identical rows.

Since $P^k$ has identical rows, $P^{k+1}$ must also have identical rows.

This implies that $P^j$ has identical rows for all $j \geq k$.

3. **11.24**

   Prove Eq. (11.14) by induction.

   $P(n) = P^n$

**Solution:**

$$\begin{aligned} P(1) &= P^1 = P \\ P(2) &= P^2 = P \cdot P = P^2 \\ P(3) &= P^3 = P \cdot P^2 = P^3 \\ P(4) &= P^4 = P \cdot P^3 = P^4 \\ &\vdots \end{aligned}$$

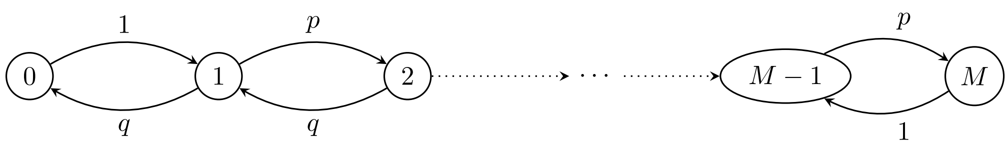
The base case is $P(1) = P^1 = P$.

Assume that $P(k) = P^k$ for some $k \geq 1$.
Then $P(k+1) = P \cdot P^k = P^{k+1}$.
Therefore, $P(n) = P^n$ for all $n \geq 1$ by induction.

4. **11.30**

Consider a random walk in the set $\{0, 1, \ldots, M\}$ with transition probabilities
$p_{01} = 1, p_{M,M-1} = 1, and p_{i,i-1} = q, p_{i,i+1} = p$ for $i = 1, \ldots, M-1$

**Solution:**




This Markov chain has period 2 $\Rightarrow \pi_i(2n) = 2\pi_i$ as $n \to \infty$

$$
\begin{aligned}
\pi_0 &= q\pi_1 \Rightarrow \pi_1 = \frac{1}{q}\pi_0 \\
\pi_1 &= \pi_0 + q\pi_2 \Rightarrow \pi_2 = \left(\frac{1}{q} - 1\right)\pi_0 = \frac{p}{q^2}\pi_0 \\
\pi_2 &= \pi_1 + q\pi_3 \Rightarrow \pi_3 = \left(\frac{1}{q}(\pi_2 - \pi_1)\right) = \frac{p}{q}\left(\frac{p}{q^2} - \frac{1}{q}\right)\pi_0 \\
&\vdots \\
\pi_{M-1} &= p\pi_{M-2} + q\pi_M \\
\pi_M &= p\pi_{M-1} \\
\pi_M &= \pi_{M-1} - p\pi_{M-2} \\
&= \frac{1}{q}\left(\frac{p^2}{q^2}\right)\left(\frac{1}{q} - 1\right)\pi_0 = \frac{p^3}{q^4}\pi_0 \\
&\vdots \\
\pi_M &= \pi_{M-1} - p\pi_{M-2} \\
&= \left(\frac{p^{M-2}}{q^{M-1}} - p\frac{p^{M-3}}{q^{M-2}}\right)\pi_0 \\
&= \frac{p^{M-2}}{q^{M-1}}\left(\frac{1}{q} - 1\right)\pi_0
\end{aligned}
$$