Assignment 3
Applied Stochastic Processes
Habib University – Fall 2023

Ali Asghar Yousuf - ay06993
Muhammad Murtaza - mm06369

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1. Dave fails quizzes with probability $\frac{1}{4}$, independent of other quizzes.

   (a) What is the probability that Dave fails exactly two of the next six quizzes?

   Solution:

   $$\begin{aligned} P(\text{Dave fails exactly two of the next six quizzes}) &= \binom{6}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^4 \\ &= 15 \times \frac{1}{16} \times \frac{81}{256} \\ &= \frac{1215}{4096} \\ &= 0.296875 \end{aligned}$$

   (b) What is the expected number of quizzes that Dave will pass before he has failed three times?

   Solution:
   No. of times he failed = 3
   Total no. of quizzes taken to fail 3 times = n

   $$\begin{aligned} n * \frac{1}{4} &= 3 \\ n &= 12 \end{aligned}$$

   Dave takes 12 quizzes to fail 3 times. Therefore, he passes 9 quizzes.

(c) What is the probability that the second and third time Dave fails a quiz will occur when he takes his eighth and ninth quizzes, respectively?

Solution:
1st Fail $\rightarrow 1-7$ quizzes
2nd Fail $\rightarrow$ 8th quiz
3rd Fail $\rightarrow$ 9th quiz

$$\begin{aligned}
P(X) &= P(\text{1 fail in 7 tests}) \cdot P(\text{2nd fail in 8th test}) \cdot P(\text{3rd fail in 9th test}) \\
&= \binom{7}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^6 \cdot \frac{1}{4} \cdot \frac{1}{4} \\
&= \frac{7 \cdot 3^6}{4^9} = \frac{5103}{262144} \\
&= 0.0194568634
\end{aligned}$$

(d) What is the probability that Dave fails two quizzes in a row before he passes two quizzes in a row?

Solution:
$F = \text{Fail}$, $P = \text{Pass}$

$$\begin{aligned}
P(X) &= P(\text{Dave fails two quizzes in a row before he passes two quizzes in a row}) \\
&= P(FF \cup PFF \cup FPFF \cup PFPFF \cup FPFPFF \cup \dots) \\
&= \frac{[P(F)]^2}{1 - P(F) \cdot P(P)} + \frac{P(P) \cdot [P(F)]^2}{1 - P(F) \cdot P(P)} \\
&= \frac{\left(\frac{1}{4}\right)^2}{1 - \frac{1}{4} \cdot \frac{3}{4}} + \frac{\frac{3}{4} \cdot \left(\frac{1}{4}\right)^2}{1 - \frac{1}{4} \cdot \frac{3}{4}} \\
&= \frac{7}{52}
\end{aligned}$$

2. A computer system carries out tasks submitted by two users. Time is divided into slots. A slot can be idle, with probability $P_I = \frac{1}{6}$, and busy with probability $P_B = \frac{5}{6}$. During a busy slot, there is probability $P_{1|B} = \frac{2}{5}$ (respectively, $P_{2|B} = \frac{3}{5}$) that a task from user 1 (respectively, 2) is executed. We assume that events related to different slots are independent.
$T_1 = \text{Task from user 1}$.

(a) Find the probability that a task from user 1 is executed for the first time during the 4th slot.

Solution: If a task from user 1 is executed for the first time during the 4th slot, then the task from user 1 is not executed in the first 3 slots (they are busy and not accepting from user 1) and is executed in the 4th slot (4th slot maybe idle and execute or busy and execute).

$$
\begin{aligned}
&P(T_1 \text{ is executed for the first time during the 4th slot}) \\
&= P(T_1 \text{ is not executed in the first 3 slots}) \cdot P(T_1 \text{ is executed in the 4th slot}) \\
&= \left(\frac{5}{6} \times \frac{3}{5}\right)^3 \cdot \left[\left(\frac{1}{6} \times 1\right) + \left(\frac{5}{6} \times \frac{2}{5}\right)\right] \\
&= \left(\frac{1}{2}\right)^3 \cdot \left[\frac{1}{6} + \frac{1}{3}\right] \\
&= \frac{1}{8} \cdot \frac{1}{2} \\
&= \frac{1}{16}
\end{aligned}
$$

(b) Given that exactly 5 out of the first 10 slots were idle, find the probability that the 6th idle slot is slot 12.

Solution: Since exactly 5 out of the first 10 slots were idle, therefore, the 11th slot is busy.
And since the slots are independent,

$$
\begin{aligned}
&P(\text{6th idle slot is slot 12}) \\
&= P(\text{11th slot is busy}) \cdot P(\text{12th slot is idle}) \\
&= \frac{5}{6} \times \frac{1}{6} \\
&= \frac{5}{36} \\
&= 0.138889
\end{aligned}
$$

(c) Find the expected number of slots up to and including the 5th task from user 1.

Solution: Probability of a task from user 1.

$$
\begin{aligned}
P(T_1) &= P_I \cdot P_{1|I} + P_B \cdot P_{1|B} \\
&= \frac{1}{6} \cdot 1 + \frac{5}{6} \cdot \frac{2}{5} \\
&= \frac{1}{6} + \frac{1}{3} \\
&= \frac{1}{2}
\end{aligned}
$$

Probability of 5th task from user 1 (first 4 slots are busy and not accepting from user 1).

$$
\begin{aligned}
P(\text{5th task from user 1}) &= \left(\frac{1}{2}\right)^5 \\
&= \frac{1}{32}
\end{aligned}
$$

Expected number of slots up to and including the 5th task from user 1 is the reciprocal of the probability of 5th task from user 1 = 32

(d) Find the expected number of busy slots up to and including the 5th task from user 1.

Solution: In busy slots, there is probability $P_{1|B} = \frac{2}{5}$ that a task from user 1 is executed.
Probability of 5th task from user 1 = $\left(\frac{5}{6}\right)^5 \cdot \frac{2}{5} = 0.16075$
Expected number of busy slots up to and including the 5th task from user 1 is the reciprocal of the probability of 5th task from user 1 = $\frac{1}{0.16075} = 6.219$

(e) Find the PMF, mean, and variance of the number of tasks from user 2 until the time of the 5th task from user 1.

Solution:

$$\binom{k+r-1}{k} p^k (1-p)^r$$

$$p = \left(\frac{1}{6} + \frac{5}{6} \times \frac{3}{5}\right) = \frac{2}{3}$$

$$\binom{k+r-1}{k} \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^r$$

$$\text{Mean} = \frac{pr}{1-p} = 2r$$

$$\text{Variance} = \frac{pr}{(1-p)^2} = 6r$$

The expression $\binom{k+r-1}{k} p^k (1-p)^r$ represents the probability of having k successes and r failures in a sequence of trials, where each trial has a success probability of p and a failure probability of $1-p$ (Binomial distribution).
In this case, p is calculated as $\left(\frac{1}{6} + \frac{5}{6} \times \frac{3}{5}\right) = \frac{2}{3}$.
The mean and variance of this distribution are given by Mean $= \frac{pr}{1-p} = 2r$ and Variance $= \frac{pr}{(1-p)^2} = 6r$, respectively.