矩阵论　第八次作业

第 1 章　线性空间和线性变换

1.3　两个特殊的线性空间

ppt 例题

P8　设矩阵空间 $R^{2\times 2}$ 的子空间为

$$V = \{X = (x_{ij})_{2\times 2} \mid x_{11} + x_{12} + x_{21} = 0,\ x_{ij} \in R\}$$

V 中的线性变换为 $T(X) = X + X^T$.
求 $(T^3)(X),\ X = \begin{bmatrix} 4 & -4 \\ 0 & -3 \end{bmatrix} \in V$.
求 $(T^k)(X),\ \forall X \in V$.

解　任意找一组基，利用 Schmidt 正交化方法得到 V 的一组标准正交基 $e_1, \cdots, e_n$，$x = k_1e_1 + \cdots + k_ne_n$，其中 $k_i = (x,\ e_i)$.
令 $x_{11} = -x_{12} - x_{21}$

$$X = \begin{bmatrix} -x_{12} - x_{21} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}, \quad Y = \begin{bmatrix} -y_{12} - y_{21} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$$

定义 V 的内积为 $(X,\ Y) = tr(XY^T) = (x_{12} + x_{21})(y_{12} + y_{21}) + x_{12}y_{12} + x_{21}y_{21} + x_{22}y_{22}$.
任意找一组基

$$X = \begin{bmatrix} -x_{12} - x_{21} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = x_{12}\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + x_{21}\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} + x_{22}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = x_{12}X_1 + x_{21}X_2 + x_{22}X_3$$

则，

$$\begin{aligned}
Y_1' &= X_1 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \\
Y_2' &= X_2 - \frac{(X_2, Y_1')}{(Y_1', Y_1')}Y_1' \\
&= \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix} \\
Y_3' &= X_3 - \frac{(X_3, Y_2')}{(Y_2', Y_2')}Y_2' - \frac{(X_3, Y_1')}{(Y_1', Y_1')}Y_1' \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 \end{bmatrix} - 0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

单位化，

$$
\begin{aligned}
e_1 &= \frac{1}{|Y_1'|}Y_1' = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \\
e_2 &= \frac{1}{|Y_2'|}Y_2' = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix} \\
e_3 &= \frac{1}{|Y_3'|}Y_3' = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

则，

$$
x = \begin{bmatrix} 4 & -4 \\ 0 & -3 \end{bmatrix} = (e_1, e_2, e_3)\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \begin{cases} k_1 = (x, e_1) = -4\sqrt{2} \\ k_2 = (x, e_2) = 0 \\ k_3 = (x, e_3) = -3 \end{cases}
$$

$$
Te_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}, \quad Te_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix}, \quad Te_3 = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}
$$

解得，

$$
\begin{aligned}
Te_1 &= (e_1, e_2, e_3)\begin{bmatrix} 2 \\ \sqrt{3} \\ 0 \end{bmatrix} \\
Te_2 &= (e_1, e_2, e_3)\begin{bmatrix} \sqrt{3} \\ 0 \\ 0 \end{bmatrix} Te_3 &= (e_1, e_2, e_3)\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}
\end{aligned}
$$

得到

$$
T(e_1 \cdots e_n) = (e_1 \cdots e_n)\begin{bmatrix} 2 & \sqrt{3} & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} = (e_1 \cdots e_n)A_0
$$

其中 $A_0 = PJP^{-1}$，J 是 Jordan 标准型 $\Rightarrow T(e_1 \cdots e_n) = (e_1 \cdots e_n)PJP^{-1}$.

$$\begin{aligned}
\lambda I - A_0 &= \begin{bmatrix} \lambda-2 & -\sqrt{3} & 0 \\ -\sqrt{3} & \lambda & 0 \\ 0 & 0 & \lambda-3 \end{bmatrix} \\
&\to \begin{bmatrix} -\sqrt{3} & \lambda-2 & 0 \\ \lambda & -\sqrt{3} & 0 \\ 0 & 0 & \lambda-3 \end{bmatrix} \to \begin{bmatrix} -\sqrt{3} & 0 & 0 \\ \lambda & \frac{\lambda-2}{\sqrt{3}}\lambda-\sqrt{3} & 0 \\ 0 & 0 & \lambda-3 \end{bmatrix} \\
&\to \begin{bmatrix} -\sqrt{3} & 0 & 0 \\ \lambda & \frac{1}{\sqrt{3}}(\lambda+1)(\lambda-3) & 0 \\ 0 & 0 & \lambda-3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \lambda-3 \\ 0 & (\lambda+1)(\lambda-3) & 0 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda-3 & 0 \\ 0 & 0 & (\lambda+1)(\lambda-3) \end{bmatrix}
\end{aligned}$$

不变因子：$d_1(\lambda) = 1,\ d_2(\lambda) = \lambda - 3,\ d_3(\lambda) = (\lambda+1)(\lambda-3)$
初等因子：$(\lambda-3);\ (\lambda+1),\ (\lambda-3)$
初等因子组：$(\lambda-3),\ (\lambda+1),\ (\lambda-3)$

Jordan 块：$J_1(\lambda_1) = (3),\ J_2(\lambda_2) = (-1),\ J_3(\lambda_3) = (3)$ Jordan 标准型：$J = \begin{bmatrix} 3 & & \\ & -1 & \\ & & 3 \end{bmatrix}$ 则

$P = (x_1, x_2, x_3),\ PJ = A_0P \Rightarrow (3x_1, -x_2, 3x_3) = (A_0x_1, A_1x_2, A_0x_3)$.

$$(3I - A_0)x_1 = \begin{bmatrix} 1 & -\sqrt{3} & \\ -\sqrt{3} & 3 & \\ & & 0 \end{bmatrix} x_1 = 0$$

$$(-I - A_0)x_2 = \begin{bmatrix} -3 & -\sqrt{3} & \\ -\sqrt{3} & -1 & \\ & & -4 \end{bmatrix} x_2 = 0$$

$$(3I - A_0)x_3 = \begin{bmatrix} 1 & -\sqrt{3} & \\ -\sqrt{3} & 3 & \\ & & 0 \end{bmatrix} x_3 = 0$$

解得

$$x_1 = \begin{bmatrix} \sqrt{3} \\ 1 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} -1 \\ \sqrt{3} \\ 0 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

因此

$$P = (x_1, x_2, x_3) = \begin{bmatrix} \sqrt{3} & -1 & 0 \\ 1 & \sqrt{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} \frac{\sqrt{3}}{4} & \frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

有

$$E_1=(e_1,e_2,e_3)\begin{bmatrix}\sqrt{3}\\1\\0\end{bmatrix}=\frac{2}{\sqrt{6}}\begin{bmatrix}-2&1\\1&0\end{bmatrix}$$

$$E_2=(e_1,e_2,e_3)\begin{bmatrix}-1\\\sqrt{3}\\0\end{bmatrix}=\sqrt{2}\begin{bmatrix}0&-1\\1&0\end{bmatrix}$$

$$E_3=(e_1,e_2,e_3)\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

且 $T(E_1,E_2,E_3)=(E_1,E_2,E_3)J$，通过坐标变换得到

$$x=(E_1,\cdots,E_n)P^{-1}\begin{bmatrix}k_1\\\vdots\\k_n\end{bmatrix}=(E_1,\cdots,E_n)\begin{bmatrix}l_1\\\vdots\\l_n\end{bmatrix}$$

$$\begin{aligned}x=\begin{bmatrix}4&-4\\0&-3\end{bmatrix}&=(e_1,e_2,e_3)\begin{bmatrix}-4\sqrt{2}\\0\\-3\end{bmatrix}\\&=(E_1,E_2,E_3)P^{-1}\begin{bmatrix}-4\sqrt{2}\\0\\-3\end{bmatrix}=(E_1,E_2,E_3)\begin{bmatrix}-\sqrt{6}\\\sqrt{2}\\-3\end{bmatrix}\end{aligned}$$

于是

$$(T^3)(x)=(E_1,E_2,E_3)\begin{bmatrix}27&&\\&-1&\\&&27\end{bmatrix}\begin{bmatrix}-\sqrt{6}\\\sqrt{2}\\-3\end{bmatrix}=\begin{bmatrix}108&-52\\-56&-81\end{bmatrix}$$

$$(T^k)(x)=(E_1,E_2,E_3)\begin{bmatrix}3^k&&\\&(-1)^k&\\&&3^k\end{bmatrix}\begin{bmatrix}\frac{\sqrt{3}}{4}&\frac{1}{4}&0\\-\frac{1}{4}&\frac{\sqrt{3}}{4}&0\\0&0&1\end{bmatrix}\begin{bmatrix}(x,e_1)\\(x,e_2)\\(x,e_3)\end{bmatrix}$$

例题

例 1.36 在欧式空间 $R^{2\times2}$ 中，矩阵 $\boldsymbol{A}$ 和 $\boldsymbol{B}$ 的内积定义为 $(\boldsymbol{A},\boldsymbol{B})=tr(\boldsymbol{A}^T\boldsymbol{B})$，子空间

$$V=\left\{\boldsymbol{X}=\begin{bmatrix}x_1&x_2\\x_3&x_4\end{bmatrix}\ \middle|\ x_3-x_4=0\right\}$$

V 中的线性变换为

$$T(\boldsymbol{X})=\boldsymbol{X}\boldsymbol{B}+\boldsymbol{X}^T\quad(\forall\boldsymbol{X}\in V),\quad \boldsymbol{P}=\begin{bmatrix}1&2\\1&1\end{bmatrix}$$

(1) 求 V 的一个标准正交基;

(2) 验证 T 是 V 中的对称变换;

(3) 求 V 的一个标准正交基，使 T 在该基下的矩阵为对角矩阵.

解

(1) 设 $\boldsymbol{X} \in V$，则

$$\boldsymbol{X} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

故 V 的一个标准正交基为

$$\boldsymbol{X}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{X}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{X}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

(2) 计算基象组：

$$T(\boldsymbol{X}_1) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = 1\boldsymbol{X}_1 + 2\boldsymbol{X}_2 + 0\boldsymbol{X}_3$$

$$T(\boldsymbol{X}_2) = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} = 2\boldsymbol{X}_1 + 1\boldsymbol{X}_2 + 0\boldsymbol{X}_3$$

$$T(\boldsymbol{X}_3) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 \\ 3 & 3 \end{bmatrix} = 0\boldsymbol{X}_1 + 0\boldsymbol{X}_2 + 3\boldsymbol{X}_3$$

设 $T(\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3) = (\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3)\boldsymbol{A}$，则

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$\boldsymbol{A}$ 为对称矩阵，因此 T 是对称变换.

(3) 求正交矩阵 $\boldsymbol{Q}$ 使得 $\boldsymbol{Q}^{-1}\boldsymbol{A}\boldsymbol{Q} = \boldsymbol{A}$，解得

$$\boldsymbol{A} = \begin{bmatrix} 3 & & \\ & 3 & \\ & & -1 \end{bmatrix} \quad \boldsymbol{Q} = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{bmatrix}$$

令 $(\boldsymbol{Y}_1, \boldsymbol{Y}_2, \boldsymbol{Y}_3) = (\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3)Q$，求得标准正交基

$$\boldsymbol{Y}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad \boldsymbol{Y}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{Y}_3 = -\frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

且有 $T(\boldsymbol{Y}_1, \boldsymbol{Y_2}, \boldsymbol{Y_3}) = (\boldsymbol{Y}_1, \boldsymbol{Y_2}, \boldsymbol{Y_3})\boldsymbol{A}$.

习题

习题 1.3.15 在欧式空间 $R^{2\times 2}$ 中，矩阵 $\boldsymbol{A}$ 和 $\boldsymbol{B}$ 的内积定义为 $(\boldsymbol{A},\boldsymbol{B})=tr(\boldsymbol{A}^T\boldsymbol{B})$，子空间

$$V=\left\{\boldsymbol{X}=\begin{bmatrix}x_1 & x_2\\ x_3 & x_4\end{bmatrix}\ \middle|\ \begin{matrix}x_1-x_4=0\\ x_2-x_3=0\end{matrix}\right\}$$

V 中的线性变换为

$$T(\boldsymbol{X})=\boldsymbol{X}\boldsymbol{P}+\boldsymbol{X}^T\quad(\forall\boldsymbol{X}\in V),\quad \boldsymbol{P}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

(1) 求 V 的一个标准正交基;

(2) 验证 T 是 V 中的对称变换;

(3) 求 V 的一个标准正交基，使 T 在该基下的矩阵为对角矩阵.

解

(1) 设 $\boldsymbol{X}\in V$，则

$$\boldsymbol{X}=\begin{bmatrix}x_1 & x_2\\ x_2 & x_1\end{bmatrix}=x_1\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+x_2\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

故 V 的一个标准正交基为

$$\boldsymbol{X}_1=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad \boldsymbol{X}_2=\frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

(2) 计算基象组

$$T(X_1)=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}=1\boldsymbol{X}_1+1\boldsymbol{X}_2$$
$$T(X_2)=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}=1\boldsymbol{X}_1+1\boldsymbol{X}_2$$

设 $T(\boldsymbol{X}_1,\boldsymbol{X}_2)=(\boldsymbol{X}_1,\boldsymbol{X}_2)\boldsymbol{A}$，则

$$\boldsymbol{A}=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$

(3)

$$\begin{aligned}\lambda I-A&=\begin{bmatrix}\lambda-1 & -1\\ -1 & \lambda-1\end{bmatrix}\\ &\to\begin{bmatrix}\lambda & -\lambda\\ -1 & \lambda-1\end{bmatrix}\to\begin{bmatrix}\lambda & 0\\ -1 & \lambda-2\end{bmatrix}\\ &\to\begin{bmatrix}\lambda & 0\\ 0 & \lambda-2\end{bmatrix}\end{aligned}$$

所以

$$
(-\boldsymbol{A})x_1 = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} x_1 = 0
$$

$$
(2I - \boldsymbol{A})x_2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x_2 = 0
$$

解得 $x_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad x_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

所以

$$
\Lambda = \begin{bmatrix} 0 & \\ & 2 \end{bmatrix} \quad P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}
$$

于是有

$$
\boldsymbol{Y}_1 = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}
$$

$$
\boldsymbol{Y}_2 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
$$