EE 301 Fall 2018-2019

HW 3

Group Number:66

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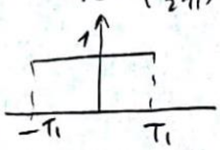
1) a)

If we calculate wrt $rect(\frac{t}{2T_1})$ and period T



And we can calculate X_k for 1 period.

$$X_k = \frac{1}{T}\int_{<T>} rec\left(\frac{t}{2T_1}\right)\cdot e^{-jw_0kt}\, dt$$

$$X_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jw_0kt}\, dt \Rightarrow \boxed{X_0 = \frac{2T_1}{T}}$$

$$X_k = \frac{1}{T}\frac{1}{(-jw_0k)}\, e^{-jw_0kt}\Big|_{-T_1}^{T_1} = \frac{1}{-jTw_0k}\left(e^{-jw_0kT_1} - e^{jw_0kT_1}\right)$$

$$X_k = \frac{2}{Tw_0k}\sin(w_0kT_1) \quad \text{and} \quad w_0 = \frac{2\pi}{T}$$

$$X_k = \frac{1}{\pi k}\sin\left(2\pi k\frac{T_1}{T}\right) \quad \text{and} \quad d = \frac{2T_1}{T}$$

$$\boxed{X_k = \frac{1}{\pi k}\sin(\pi k d)}$$

b)

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j\omega_0 k t} \quad \text{where} \quad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$z(t) = \int_0^1 x(t-\tau)d\tau + \int_3^4 x(t-\tau)\,d\tau$$

$$= \int_0^1 \sum_{k=-\infty}^{\infty} X_k e^{j\omega_0 k(t-\tau)} d\tau + \int_3^4 \sum_{k=-\infty}^{\infty} X_k e^{j\omega_0 k(t-\tau)} d\tau$$

$$= \sum_{k=-\infty}^{\infty} X_k e^{j\omega_0 k t}\left(e^{-j\omega_0 k} - 1\right) + \sum_{k=-\infty}^{\infty} X_k e^{j\omega_0 k t}\left(e^{-j4\omega_0 k} - e^{-j\omega_0 3k}\right)$$

$$= \sum_{k=-\infty}^{\infty} X_k e^{j\omega_0 k t}\left(e^{-j\omega_0 k} - 1\right)\left(1 + e^{-j\omega_0 3k}\right)$$

$$\boxed{Z_k = X_k \left(e^{-j\omega_0 k} - 1\right)\left(1 + e^{-j\omega_0 3k}\right)}$$

c) $y(t) = \frac{d}{dt} z(t) = \frac{d}{dt}\int_0^4 x(\tau)\, x(t-\tau) d\tau = x(4)\, x(t-4) - x(0)\, x(t)$

d) $Y_k = X_4 X_k \cdot e^{-j\omega_0 k 4} - X_0 X_k.$

e) $Z_k = \frac{Y_k}{j\omega_0 k} = \frac{X_4 X_k e^{-j4\omega_0 k} - X_0 X_k}{j\omega_0 k}$

2)		$$f[k] = \sum_{n=0}^{N-1} \left(e^{j\frac{2\pi}{N}k}\right)^n = \frac{1-\left(e^{j\frac{2\pi}{N}k}\right)^N}{1-e^{j\frac{2\pi}{N}k}}$$ $$= \frac{1-e^{j2\pi k}}{1-e^{j\frac{2\pi}{N}k}} \quad \text{, for } k \text{ integer}$$ $$f[k] = 0$$ b) $$f[k] = \sum_{n=0}^{M+N-1} \left(e^{j\frac{2\pi}{N}k}\right)^n - \sum_{n=0}^{M-1} \left(e^{j\frac{2\pi}{N}k}\right)^n$$ $$= \frac{1-e^{j\frac{2\pi}{N}k(M+N)}}{1-e^{j\frac{2\pi}{N}k}} - \frac{1-e^{j\frac{2\pi}{N}kM}}{1-e^{j\frac{2\pi}{N}k}}$$ $$= \frac{e^{j\frac{2\pi}{N}k.M}\left(1-e^{j\frac{2\pi}{N}k.N}\right)}{1-e^{j\frac{2\pi}{N}k}}$$ k is integer $$\boxed{f[k] = 0}$$

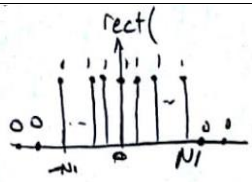
3)	a)	System is <u>causal</u> since it depends on only present value. $h(t)=0,\ t<0$ If system is stable, $\int_{-\infty}^{\infty} h(\tau)d\tau < \infty$. $\int_{-\infty}^{\infty} e^{-\tau} u[\tau] d\tau = \int_{0}^{\infty} e^{-\tau} d\tau = 1$ Then, system is <u>stable.</u> $x(t) = j^t = e^{j\frac{\pi}{2}t}$ $y(t) = H(j\frac{\pi}{2}) e^{j\frac{\pi}{2}t}$ $H(s) = \int_{-\infty}^{\infty} e^{-s\tau} h(\tau) d\tau$ $= \int_{-\infty}^{\infty} e^{-s\tau} \cdot e^{-\tau} u(\tau) d\tau = \int_{0}^{\infty} e^{-s\tau} \cdot e^{-\tau} d\tau$ $H(s) = \frac{1}{s+1}$ $y(t) = \frac{1}{j\frac{\pi}{2}+1} e^{j\frac{\pi}{2}t}$

	b)	* System is <u>not</u> causal since it depends on future value. $h(t) \neq 0,\ t<0$ * $\sum_{k=-\infty}^{\infty} h[k] < \infty$ $\sum_{k=-\infty}^{\infty} 2^{-k} u[k+1] = \sum_{k=-1}^{\infty} 2^{-k}$ $= 2 + \sum_{k=0}^{\infty} 2^{-k} = 2 + \left.\frac{1-\left(\frac{1}{2}\right)^{k+1}}{1-1/2}\right\|_{k=\infty}$ $= 4$ System is <u>stable</u>. $x[n] = z^n$ $y[n] = z^n \sum_{k=-\infty}^{\infty} h[k] z^{-k}$ $y[n] = j^n \sum_{k=-\infty}^{\infty} 2^{-k} u[k+1] j^{-k} = j^n \sum_{k=-1}^{\infty} (2j)^{-k}$ $= j^n \left(2j + \left.\frac{1-\left(\frac{1}{2j}\right)^{k+1}}{1-1/2j}\right\|_{k=\infty} \right)$ $= j^n \left(2j + \frac{2j}{2j-1}\right) = j^n \left(\frac{4+8j}{5}\right)$

4)	a)	i)	If periodic $x(t)=x(t+T)$ $\sin(2t)+\cos(3t)=\sin(2t+2T)+\cos(3t+3T)$ $2T=2\pi k \qquad 3T=2\pi k$ $T=\pi k \qquad T=\frac{2\pi}{3}k$ we should find greatest common divisor. And, that value is our fundamental period. $GCD(\pi, \frac{2\pi}{3})=2\pi$ $\boxed{T_0=2\pi}$ $\sin(2t)+\cos(3t)=\frac{e^{j2t}-e^{-j2t}}{j2}+\frac{e^{j3t}+e^{-j3t}}{2}$ $a_2=\frac{1}{2j}, a_{-2}=-\frac{1}{2j}, a_3=a_{-3}=\frac{1}{2}$ o.w. $a_k=0$
		ii)	ii) Shortly, we can say $\frac{\pi}{2}T=2\pi k \qquad \frac{\pi}{3}T=2\pi k$ $T=4k \qquad T=6k$ Then, $GCD(4,6)=12$ $\boxed{T_0=12}$ $\sin(\frac{\pi}{2}t)+\cos(\frac{\pi}{3}t)=\frac{e^{j\frac{\pi}{2}t}-e^{-j\frac{\pi}{2}t}}{2j}+\frac{e^{j\frac{\pi}{3}t}+e^{-j\frac{\pi}{3}t}}{2}$ $w_0=\frac{2\pi}{T_0}=\frac{\pi}{6}$ $a_2=a_{-2}=\frac{1}{2}, a_3=\frac{1}{2j}, a_{-3}=-\frac{1}{2j}$ o.w. $a_k=0$

		iii)	iii) Like, last 2 part $2T=2\pi k$ $\quad \frac{\pi}{3}T = 2\pi k$ $T=\pi k$ $\quad T = 6k$ It is not periodic.
	b)	i)	b) i) if periodic $x[n]=x[n+N]$ $\sin(2n)+\cos(3n)=\sin(2n+2N)+\cos(3n+3N)$ $2N=2\pi k$ $\quad 3N=2\pi k$ $N=\pi k$ $\quad N=\frac{2\pi}{3}k$ N is integer. Then, this signal is not periodic
		ii)	ii) $\sin(\frac{\pi}{2}n)+\cos(\frac{\pi}{3}n)=\sin(\frac{\pi}{2}n+\frac{\pi}{2}N)+\cos(\frac{\pi}{3}n+\frac{\pi}{3}N)$ $\frac{\pi}{2}N=2\pi k$ $\quad \frac{\pi}{3}N=2\pi k$ $N=4k$ $\quad N=6k$ GCD(4, 6) = 12, periodic Then, $\boxed{N_0=12}$ $\sin(\frac{\pi}{2}n)+\cos(\frac{\pi}{3}n)=\frac{e^{j\frac{\pi}{2}n}-e^{-j\frac{\pi}{2}n}}{2j}+\frac{e^{j\frac{\pi}{3}n}+e^{-j\frac{\pi}{3}n}}{2}$ $\Omega_0=\frac{2\pi}{N}=\frac{\pi}{6}$ $a_2=a_{-2}=\frac{1}{2}$, $a_{-3}=-\frac{1}{2j}$, $a_3=\frac{1}{2j}$ o.w, $a_k=0$
		iii)	$\sin(2n)+\cos(\frac{\pi}{3}n)=\sin(2n+2N)+\cos(\frac{\pi}{3}n+\frac{\pi}{3}N)$ $2N=2\pi k$ $N=\pi k$ It is not periodic since N is not integer.

5) a)



$$a_k = \frac{1}{N}\sum_{\langle N\rangle} x[n]\, e^{-\frac{j2\pi nk}{N}}$$

we can use 1 period of $x[n]$ to find a_k

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} A\,\mathrm{rect}\left(\frac{n}{2N_1+1}\right) e^{-\frac{j2\pi}{N}nk}$$

$$a_k = \frac{1}{N}\sum_{n=0}^{N_1} A\, e^{-\frac{j2\pi}{N}nk} + \frac{1}{N}\sum_{n=1}^{N_1} A\, e^{j\frac{2\pi}{N}nk}$$

$$= \frac{A}{N}\cdot\frac{1-\left(e^{-\frac{j2\pi}{N}k}\right)^{N_1+1}}{1-e^{-\frac{j2\pi}{N}k}} + \frac{A}{N}\cdot\frac{1-\left(e^{\frac{j2\pi}{N}k}\right)^{N_1+1}}{1-e^{\frac{j2\pi}{N}k}} - \frac{A}{N}$$

$$= \frac{A}{N}\left(\frac{e^{\frac{j2\pi}{N}k} - e^{-\frac{j2\pi}{N}k\cdot N_1}}{e^{\frac{j2\pi k}{N}}-1} + \frac{\left(e^{\frac{j2\pi k}{N}}\right)^{N_1+1}-1}{e^{\frac{j2\pi}{N}k}-1} - 1\right)$$

$$= \frac{A}{N}\left(\frac{\left(e^{\frac{j2\pi}{N}k}\right)^{N_1+1} - e^{-\frac{j2\pi}{N}k\cdot N_1}}{e^{\frac{j2\pi k}{N}}-1}\right) = \frac{A}{N}\, e^{\frac{j\pi k}{N}}\, \frac{\left(e^{\left(\frac{j2\pi k}{N}\right)N_1+\frac{1}{2}} - e^{\left(-\frac{j2\pi k}{N}\right)N_1+\frac{1}{2}}\right)}{e^{\frac{j2\pi k}{N}}-1}$$

$$e^{\frac{j\pi k}{N}}\left(e^{\frac{j\pi k}{N}} - e^{-\frac{j\pi k}{N}}\right)$$

$$= \frac{A}{N}\,\frac{\sin\left(\frac{2\pi k}{N}\left(N_1+\frac{1}{2}\right)\right)}{\sin\left(\frac{\pi k}{N}\right)} = \frac{A}{N}\,\frac{\sin(\pi k d)}{\sin\left(\frac{\pi k}{N}\right)}$$

$$\boxed{a_k = \frac{A}{N}\,\frac{\sin(\pi k d)}{\sin\left(\frac{\pi k}{N}\right)}}$$

	b)	$a_k = a_{-k}^*$ $a_k^* = \frac{A}{N} \frac{\sin(-\pi k d)}{\sin(\frac{-\pi k}{N})} = \frac{A}{N} \frac{\sin(\pi k d)}{\sin(\frac{\pi k}{N})} = a_k$
	c)	$a_0 = Ad$ c depends only a_0 and on odd function a_0 is zero $a_0 + c = 0$ $\boxed{c = -Ad}$ $\boxed{c = \frac{-A}{2}}$ $b_k = -b_{-k}$ -> odd function $b_k = a_k e^{-jk\frac{2\pi}{N}n_0}$ $-b_{-k} = -a_{-k} e^{jk\frac{2\pi}{N}n_0}$ a_k is even $a_k = a_{-k}$ $\cancel{a_k} e^{-jk\frac{2\pi}{N}n_0} = -\cancel{a_k} e^{jk\frac{2\pi}{N}n_0}$ $e^{jk\frac{2\pi}{N}n_0} + e^{-jk\frac{2\pi}{N}n_0} = 0$ $2\cos\left(\underbrace{k\frac{2\pi}{N}n_0}\right) = 0$ $\frac{2\cancel{\pi}}{N} n_0 = \frac{\cancel{\pi}}{2}$ $\boxed{n_0 = \frac{N}{4}}$

6)

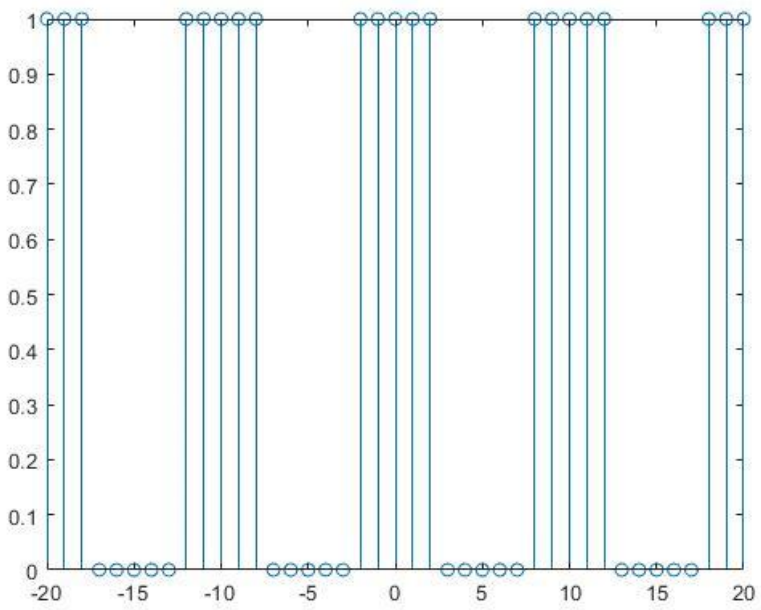


Figure 1: x[n] for N = 10

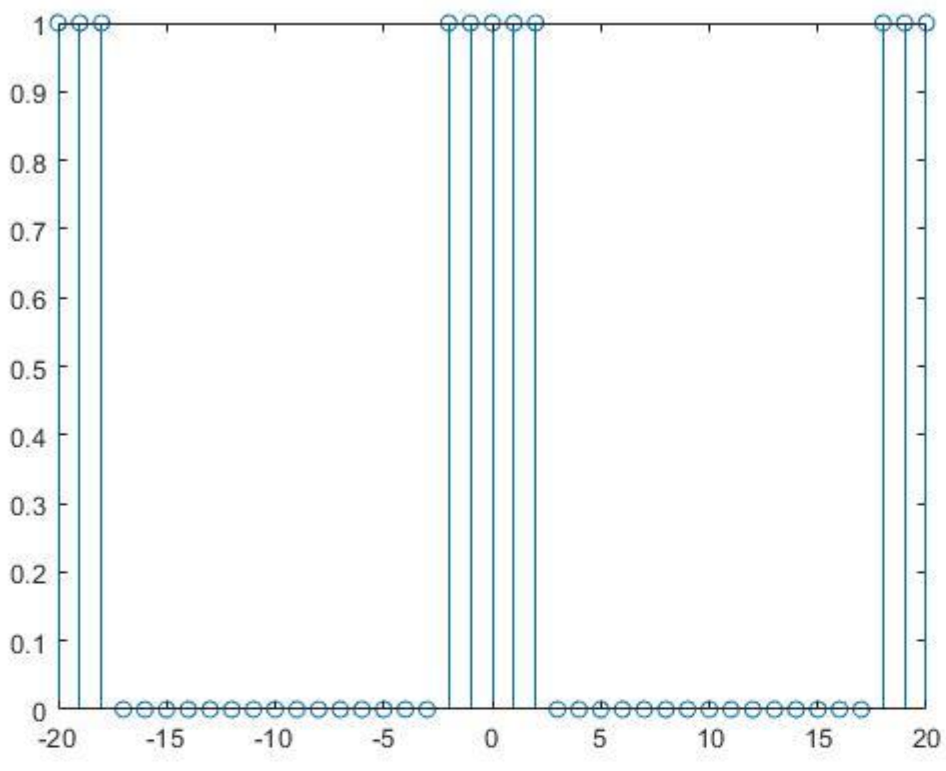


Figure 2: x[n] for N = 20

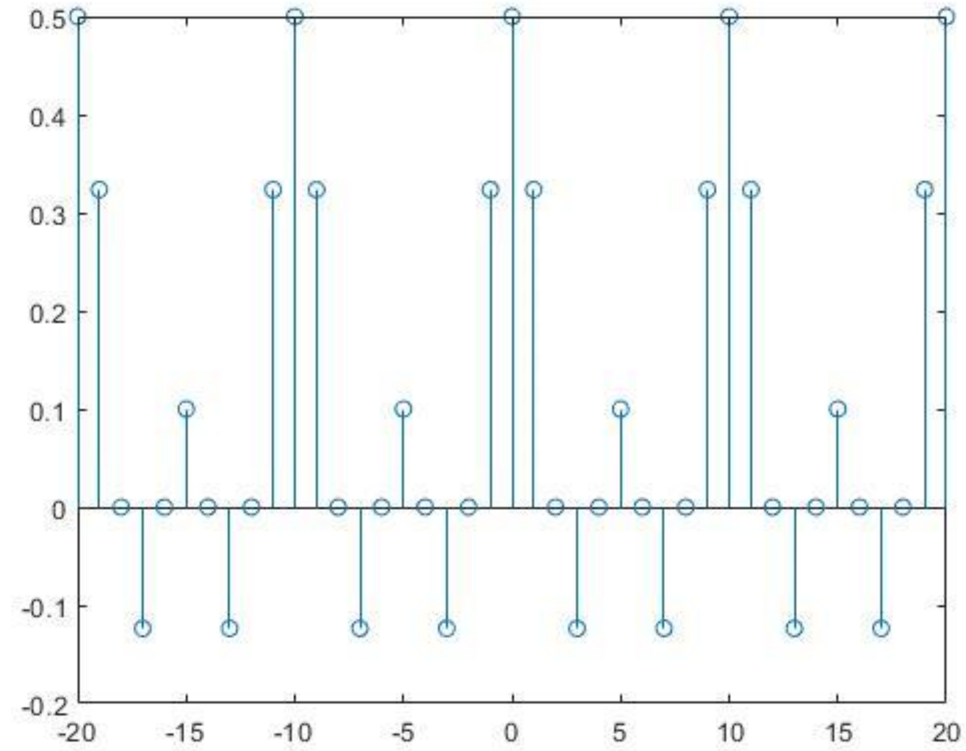


Figure 3: ak for N = 10

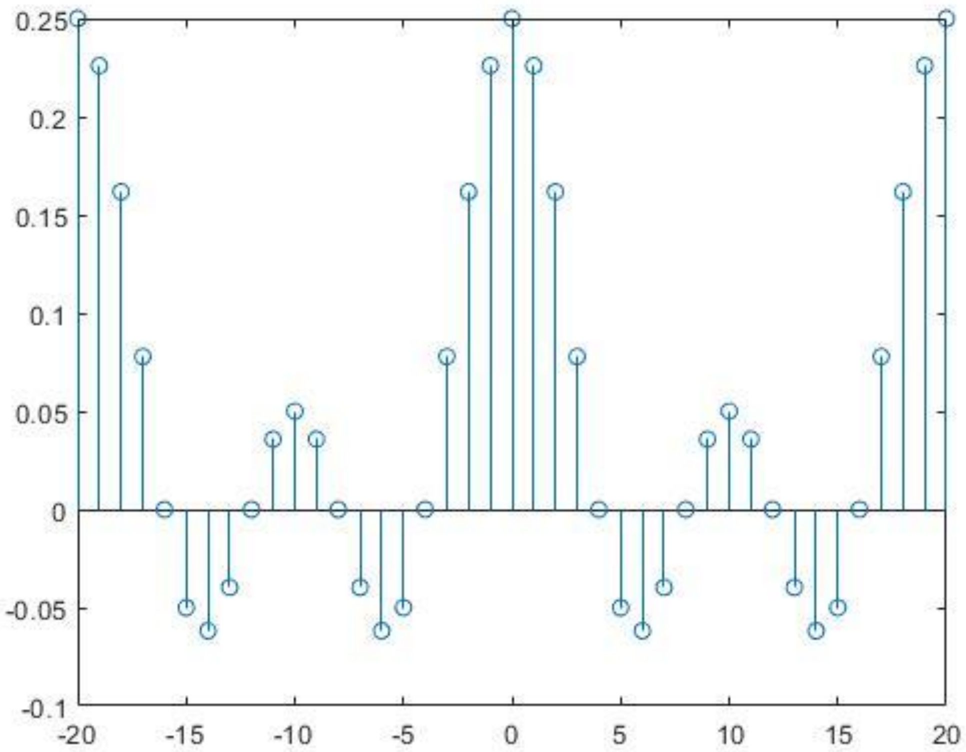


Figure 4: ak for N = 20

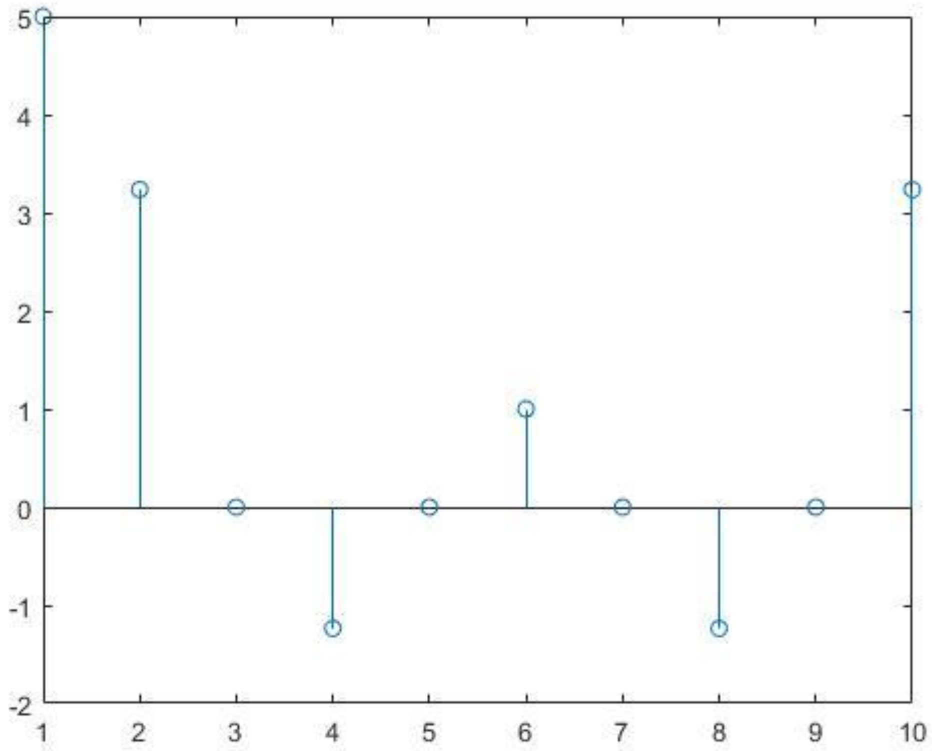


Figure 5: fft for N = 10

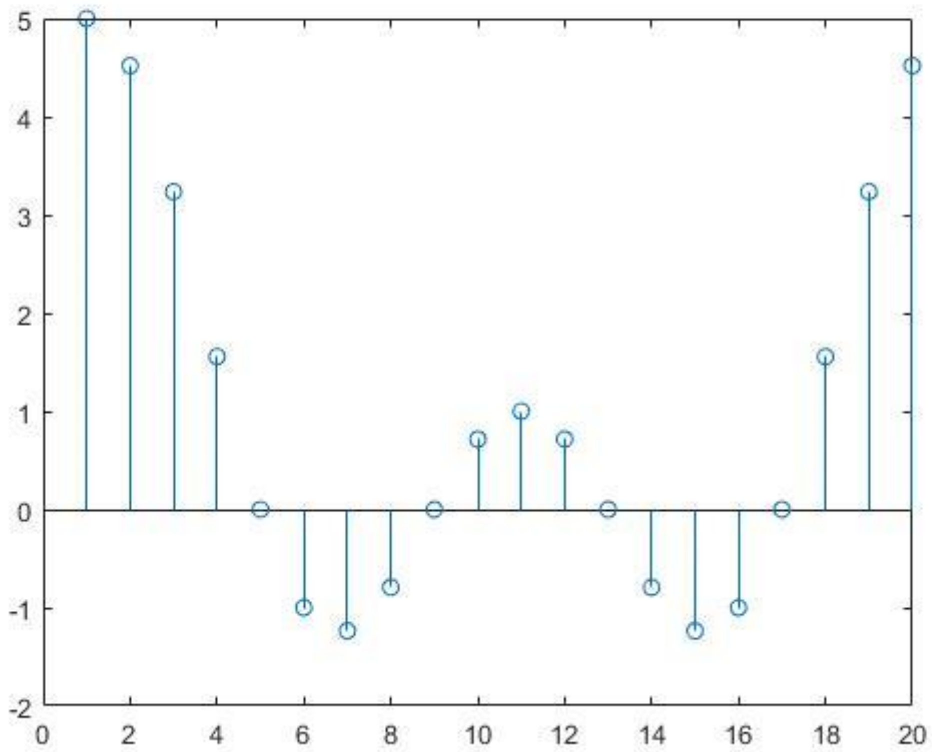


Figure 6: fft for N = 20