EE 301
Fourier Series

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Outline

Motivation for Fourier Series and Transforms

Remember that superposition holds in LTI systems:

$$w_k(t) \xrightarrow{S} z_k(t)$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k w_k(t) \xrightarrow{S}$$

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In the study of LTI systems, it is advantageous to represent signals as linear combinations of basic signals that possess the following two properties:

1. These basic signals, $w_k(t)$, can be used to construct a broad class of signals.
2. The response, $z_k(t)$, of an LTI system to each of these basic signals is simple.

Complex exponential signals possess both properties
$\Rightarrow$ **Fourier series & transforms**

Eigenfunctions of a CT system

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$$x(t) = \sum_{k=-\infty}^{\infty} a_k \varphi_k(t) \xrightarrow{S} y(t) = \sum_{k=-\infty}^{\infty} a_k \lambda_k \varphi_k(t)$$

$\Rightarrow$ The input-output relationship of the system can be specified by its eigenvalues λ_k, $k = \ldots, -1, 0, 1, 2, \ldots$.

Response of LTI Systems to Complex Exponentials

Never ever forget this!

Complex exponentials are eigenfunctions of LTI systems:

$$\varphi(t) = e^{st} \text{ for any complex constant } s$$

Proof:

$\Rightarrow$ e^{st} is an eigenfunction whenever the following is finite:

$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

$\Rightarrow$ $H(s)$ is the eigenvalue corresponding to e^{st}:

$$e^{st} \xrightarrow{\text{LTI } S}$$

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Example

$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$ is the input to an LTI system. Suppose the impulse response of the system, $h(t)$, is given. How can you determine the output?

Periodic signals as sums of complex exponentials

Remember that

- A signal $x(t)$ with period T satisfies

$$x(t) =$$

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- Two basic CT signals periodic w freq. ω_0 (period T_0):

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- Fundamental period T_0: Minimum positive T satisfying above relation
- Fundamental frequency ω_0 : $\omega_0 = \frac{2\pi}{T_0}$
- Two basic CT signals periodic w freq. ω_0 (period T_0): $\cos(\omega_0 t)$ and $e^{j\omega_0 t}$

- Consider the set of harmonically related complex exponentials : $\varphi_k(t) = e^{jk\omega_0 t} \quad , k = 0, \pm 1, \pm 2, ...$ These signals have a common period T_0.
- Now consider a linear combination of them.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T_0}t}$$

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$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T_0}t}$$

is also periodic with T_0.

Fourier series representation

For a periodic signal $x(t)$, a representation in the form of

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

is called the Fourier series representation of the signal.

Fourier series representation

Never ever forget this!

Fourier series represents a periodic signal as a weighted sum of complex exponentials:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

- Terms with $k = \pm 1$: Fundamental components
- Terms with $k = \pm N$: N^{th} harmonic components

Example

Consider a periodic signal whose Fourier series representation is given by $x(t) = \sum_{k=-2}^{2} a_k e^{jk2\pi t}$ with $a_0 = 1$, $a_1 = a_{-1} = 3/4$, $a_2 = a_{-2} = -1/2$.

Determination of Fourier Series Coefficients

Q: Assuming that a given periodic signal can be represented with Fourier series, how can we determine the coefficients a_k?

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}}$$

Determination of Fourier Series Coefficients

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$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}}$$

A:

$$\boxed{a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}dt}$$

- The coefficients $\{a_k\}$ are called the Fourier series coefficients of $x(t)$.
- The limits of integral can be $\int_{t_i}^{t_i+T_0}$.

Example

Fourier series coefficients of the signal $x(t) = sin(\omega_0 t)$?

Example

Periodic square wave defined over one period T as follows:

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < \frac{T}{2} \end{cases}.$$

Fourier series coefficients of $x(t)$?

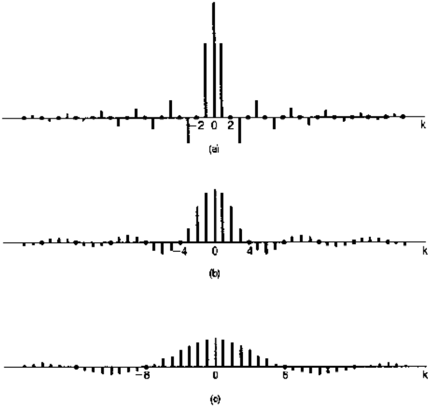

Figure 3.7 Plots of the scaled Fourier series coefficients Ta_k for the periodic square wave with T_1 fixed and for several values of T. (*a*) $T = 4T_1$; (*b*) $T = 8T_1$; (*c*) $T = 16T_1$. The coefficients are regularly spaced samples of the envelope $(2 \sin \omega T_1)/\omega$, where the spacing between samples, $2\pi/T$, decreases as T increases.

Figure: Fourier series coefficients of the periodic square wave

Example (Challenge yourself!)

Fourier series coefficients of the periodic impulse train?

Example (Challenge yourself!)

Fourier series coefficients of
$x(t) = 1 + \sin(\omega_0 t) + 2\cos(\omega_0 t) + \cos(2\omega_0 t + \frac{\pi}{4})$?

What did we learn about Fourier series so far?

For a **periodic CT signal** $x(t)$ with a fundamental period $T_0 = \frac{2\pi}{\omega_0}$, the Fourier series representation, if it exists, is

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}}$$

with the Fourier series coefficients $\{a_k\}$ given by

$$\boxed{a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt}$$

where the integration is over any period of the signal.

Existence and Convergence of the Fourier Series

Dirichlet Conditions: The Fourier Series decomposition of a periodic signal $x(t)$ is possible if the following sufficient conditions hold:

- $x(t)$ is absolutely integrable over any period $\int_0^T |x(t)|dt < \infty$
- $x(t)$ has a finite number of maxima and minima within a period
- $x(t)$ has a finite number of discontinuities within a period.

Almost all physical signals of interest in engineering satisfy the Dirichlet conditions.

Examples of some pathological signals that do not satisfy Dirichlet conditions:

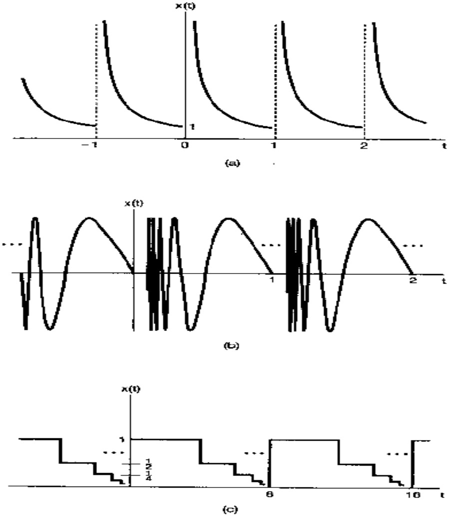

Figure 3.8 Signals that violate the Dirichlet conditions: (a) the signal $x(t) = 1/t$ for $0 < t \leq 1$, a periodic signal with period 1 (this signal violates the first Dirichlet condition); (b) the periodic signal of eq. (3.57), which violates the second Dirichlet condition; (c) a signal periodic with period 8 that violates the third Dirichlet condition [for $0 \leq t < 8$, the value of $x(t)$ decreases by a factor of 2 whenever the distance from t to 8 decreases by a factor of 2; that is, $x(t) = 1$, $0 \leq t < 4$, $x(t) = 1/2$, $4 \leq t < 6$, $x(t) = 1/4$, $6 \leq t < 7$, $x(t) = 1/8$, $7 \leq t < 7.5$, etc.].

Dirichlet conditions guarantee that

$$\lim_{N\to\infty} e_N(t) = 0 \text{ for almost all } t$$

where $e_N(t) = |x(t) - x_N(t)|$ is the error signal with $x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$.

Alternative Condition: The Fourier Series decomposition of a periodic signal $x(t)$ is possible if

$$\int_0^T |x(t)|^2 dt < \infty \quad \text{(finite energy over a period)}$$

Finite energy condition guarantees that

$$\lim_{N \to \infty} \int_{T_0} e_N^2(t) dt = 0 \quad \text{(error energy)}$$

Properties of CT Fourier series representation

P.1 Symmetry with real signals:

If $x(t)$ is a real periodic signal, $a_k^* = a_{-k}$
Proof:

As a result, we have the following alternative form when $x(t)$ is real:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} =$$

P.2 Response through an LTI System:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \xrightarrow{\text{LTI } S} y(t) =$$

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$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \xrightarrow{\text{LTI } S} y(t) = \sum_{k=-\infty}^{\infty} a_k H(k\omega_0) e^{jk\omega_0 t}$$

Example

The following signal is input to an LTI system:
$x(t) = \sum_{k=-2}^{2} a_k e^{jkt}$, with $a_0 = 1$, $a_1 = a_{-1} = \frac{3}{4}$, $a_2 = a_{-2} = -\frac{1}{2}$. About the system, we know that $H(0) = 0$, $H(1) = H(-1) = 2$, and $H(2) = H(-2) = 3$. What is the response of the system?

P.3 Even and Odd Signals:

- If $x(t)$ is even, i.e. $x(t) = x(-t)$, then $a_k =$

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  $\Rightarrow$In this case, alternative form for the representation:

P.4 Manipulation of Signals:

Time shifting:

Time reversal:

Differentiation:

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right) a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{Ev}\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{Od}\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\mathcal{Re}\{a_k\}$ $j\mathcal{Im}\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$