Homework 2 Solutions

Problem 5

a-) $G_c(s) = 1$, $d(t) = 0$, $r(t)$ is the unit step function. Then

$$K_p = \lim_{s \to 0} G(s) = \lim_{s \to 0} \frac{100(s+2)}{(s^2-1)} = -200$$

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1-200} = -\frac{1}{199} = -0.005025$$

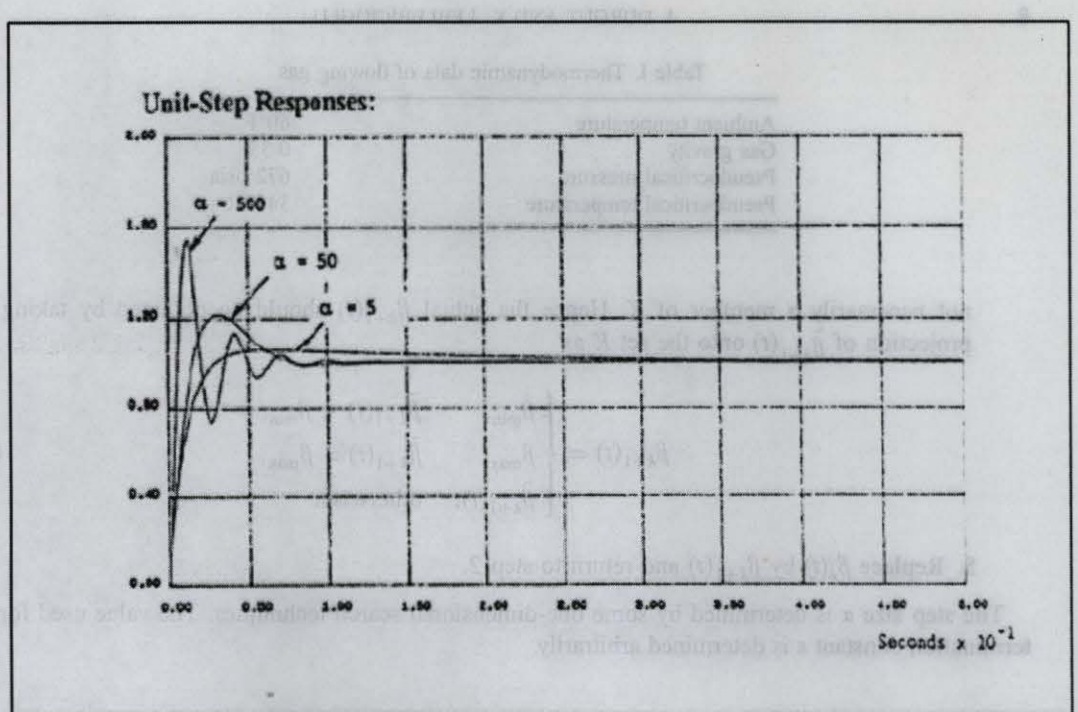
b-) $G_c(s) = \frac{s+\alpha}{s}$, $d(t) = 0$, $r(t)$ is the unit step function.

$$G(s) = \frac{100(s+2)}{(s^2-1)} \cdot \frac{(s+\alpha)}{s}$$

$$\Rightarrow K_p = \lim_{s \to 0} G(s) = \infty \Rightarrow e_{ss} = 0.$$

c-) $d(t) = 0$, $G_c(s) = \frac{s+\alpha}{s}$, $r(t)$ is the unit step function. $y(t)$, $0 \leq t \leq 0.5$ is required together with the corresponding M_p for $\alpha = 5$, 50 and 500. Here, MATLAB should better be used.

Unit-Step Responses:



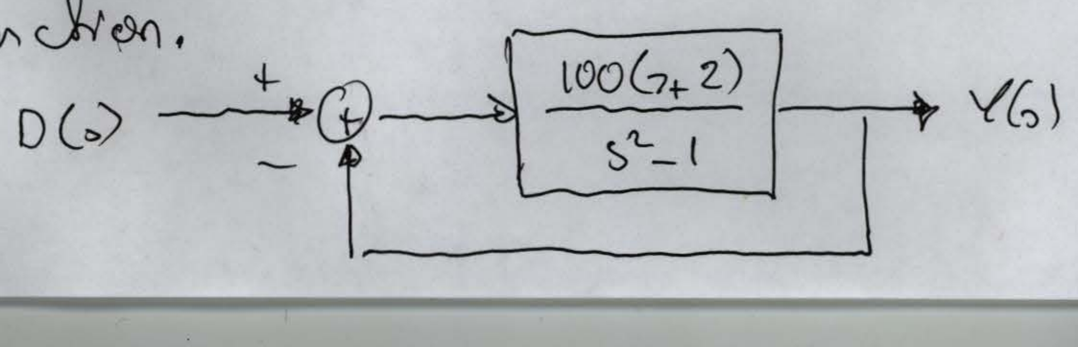
$\alpha = 5$, $M_p \cong 5.6\%$

$\alpha = 50$, $M_p \cong 22\%$

$\alpha = 500$, $M_p \cong 55\%$

As the value of α increases, the maximum overshoot increases because the damping effect of the zero at $s = -\alpha$ becomes less effective.

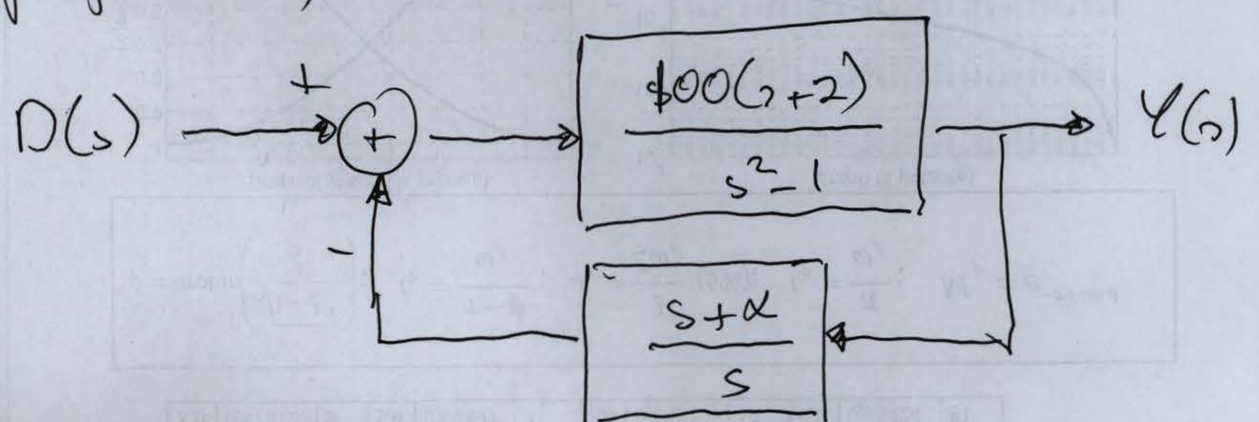
d-) $r(t) = 0$, $G_c(s) = 1$, $d(t)$ is the unit step function.



$$Y(s) = \frac{100(s+2)}{s\left[s^2-1+100(s+2)\right]}$$

$$y_{ss} = \lim_{s\to 0} sY(s) = \frac{200}{199}$$

e-) $r(t)=0$, $G_c(s) = \frac{s+\alpha}{s}$, $d(t)$ is the unit step function, $\alpha = 5, 50$ and 500.



$$Y(s) = \frac{100\,s(s+2)}{s^3+100s^2+(100\alpha+199)s+200\alpha} \cdot \frac{1}{s}$$

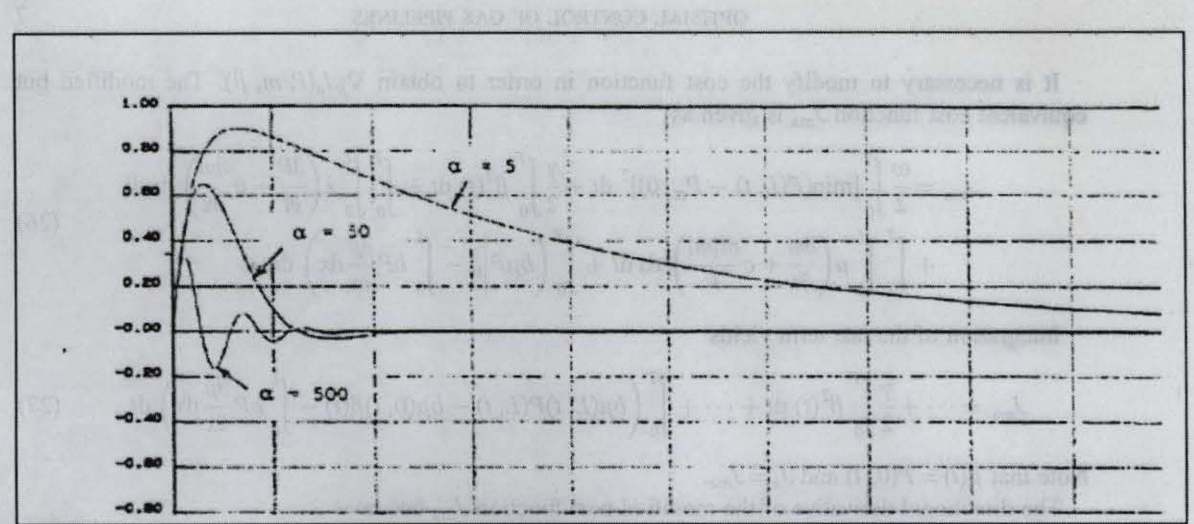
$$y_{ss} = \lim_{s\to 0} sY(s) = 0 \quad \forall\alpha.$$

f-) $r(t)=0$, $G_c(s) = \frac{s+\alpha}{s}$, $\alpha = 5, 50, 500$

$d(t)$ is the unit step function. Here, we should better use MATLAB.

$$\alpha = 5 \Rightarrow Y(s) = \frac{100(s+2)}{s^3+100s^2+699s+1000}$$

Unit-Step Responses:



$$\alpha = 50 \Rightarrow Y(s) = \frac{100(s+2)}{s^3 + 100s^2 + 5199s + 10000}$$

$$\alpha = 500 \Rightarrow Y(s) = \frac{100(s+2)}{s^3 + 100s^2 + 50199s + 100000}$$

g-) As the value of α increases, the output response $y(t)$ due to $r(t)$ becomes more oscillatory and the overshoot is larger. As the value of α increases, the amplitude of the output response $y(t)$ due to $d(t)$ becomes smaller and more oscillatory.

Problem 1 $M_p = 0.2$

$$M_p = 0.2 = \exp\left\{-\frac{\pi\xi}{\sqrt{1-\xi^2}}\right\}$$

$$-\frac{\pi\xi}{\sqrt{1-\xi^2}} = \ln 0.2 = -1.609$$

$$\pi\xi = 1.609\sqrt{1-\xi^2} \Longrightarrow \xi = \sqrt{\frac{(1.609)^2}{\pi^2+(1.609)^2}}$$

$$\Longrightarrow \xi = 0.456.$$

$$t_p = \frac{\pi}{w_d} = \frac{\pi}{w_n\sqrt{1-\xi^2}}.$$

From the figure, it can be read as $t_p(A) = 0.15$

$t_p(B) = 0.20$, $t_p(C) = 0.25$, $t_p(D) = 0.34$, $t_p(E) = 0.44$

$$w_n(A) = \frac{\pi}{t_p(A)\sqrt{1-\xi^2}} = \frac{\pi}{t_p(A) \cdot 0.89} \cong 23.53$$

$$w_n(B) = \frac{\pi}{t_p(B) \cdot 0.89} = \frac{3.53}{0.20} \cong 17.65$$

$$w_n(C) = \frac{3.53}{0.25} = 14.12$$

$$w_n(D) = \frac{3.53}{0.34} \cong 10.38$$

$$\omega_n(E) = \frac{3.53}{0.44} \cong 8.02$$

<u>System A</u> $\Longrightarrow$ $s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 2.(0.456).$

$.(23.53)s + (23.53)^2 = s^2 + 21.459s + 553.661 = 0$

$$s_{1,2} = \frac{-21.459 \mp \sqrt{(21.459)^2 - 4.(553.661)}}{2}$$

$= -10.73 \mp j\,20.94$

<u>System B</u> $s^2 + 2(0.456)(17.65) + (17.65)^2$

$s_{1,2} = -8.0484 \mp j\,15.71$

<u>System C</u> $s^2 + 2(0.456)(14.12) + (14.12)^2$

$s_{1,2} = -6.44 \mp j\,12.57$

<u>System D</u> $s^2 + 2.(0.456)(10.38) + (10.38)^2$

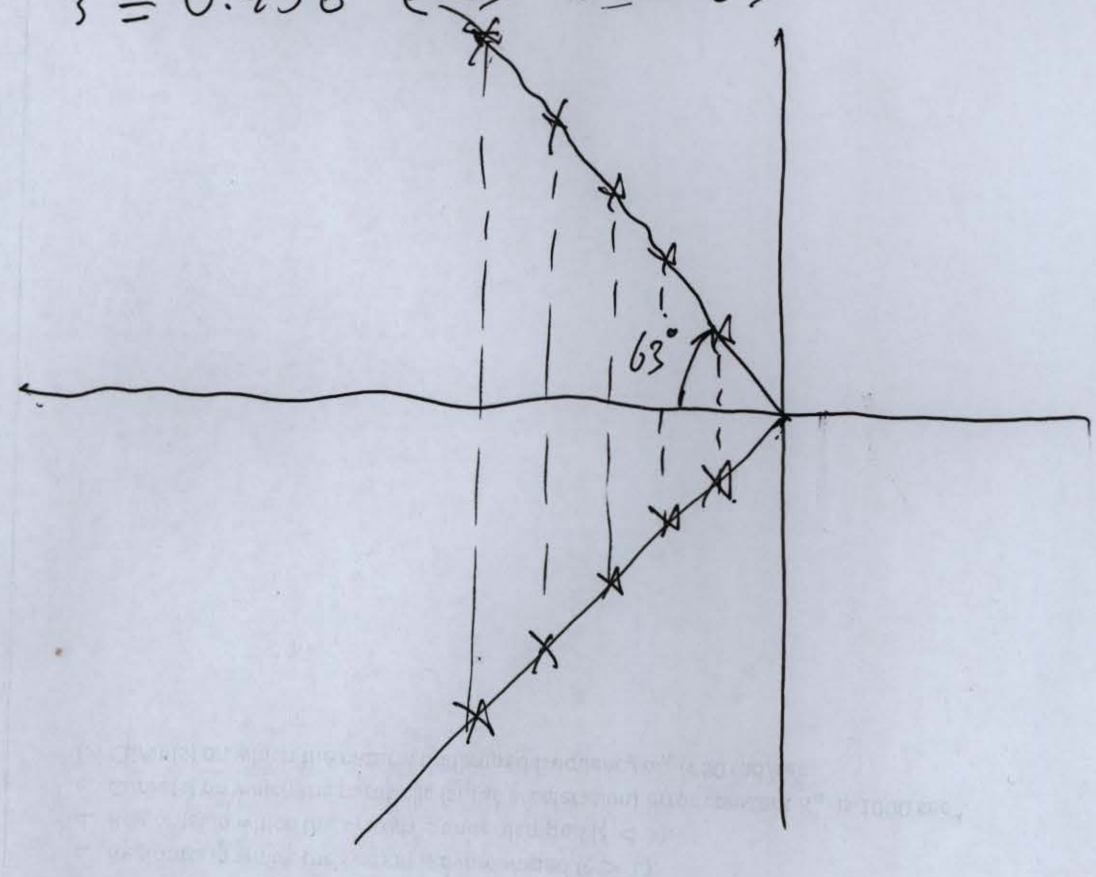
$s_{1,2} = -4.73 \mp j\,9.24$

<u>System E</u> $s^2 + 2(0.456)(8.02) + (8.02)^2$

$s_{1,2} = -3.66 \mp j\,7.14$

If we consider our measurements from the figure are not very accurate, it is obvious that the system poles lie on the lines (in the LHP) $\zeta = 0.456$ in decreasing magnitudes as shown below: ⑦

$\zeta = 0.456 \Longleftrightarrow \theta \cong 63^\circ$



Problem 2 $\quad D(s)=4, \quad G(s)=\dfrac{1}{s+1}$

$$Y(s)=\frac{4}{s+5}\cdot R(s).$$

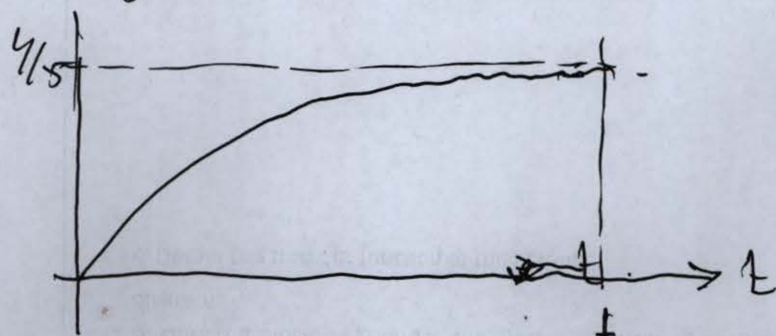
a-) $r(t)$ is the unit step signal

$$Y(s)=\frac{4}{s(s+5)}=\frac{A}{s}+\frac{B}{s+5}=\frac{4/5}{s}+\frac{-4/5}{s+5}.$$

$$\Longrightarrow y(t)=\left\{\frac{4}{5}-\frac{4}{5}e^{-5t}\right\}u(t)$$

$u(t)$ → unit step function

$$=\frac{4}{5}\left(1-e^{-5t}\right)u(t).$$



$y_{ss}=0.2$

No M_p and $t_{ss}=\infty$

b-) MATLAB results are identical to the theoretical results in part (a) (Simulink is used).

c-) $D(s)=4\left(1+\dfrac{1}{5s}\right), \quad G(s)=\dfrac{20s+4}{5s(s+1)}$

$$T(s)=\frac{20s+4}{5s^2+25s+4}=\frac{4s+4/5}{s^2+5s+4/5}.$$

$$Y(s)=\frac{4s+4/5}{s(s^2+5s+4/5)}=\frac{4s+0.8}{s(s+4.8345)(s+0.165)}$$

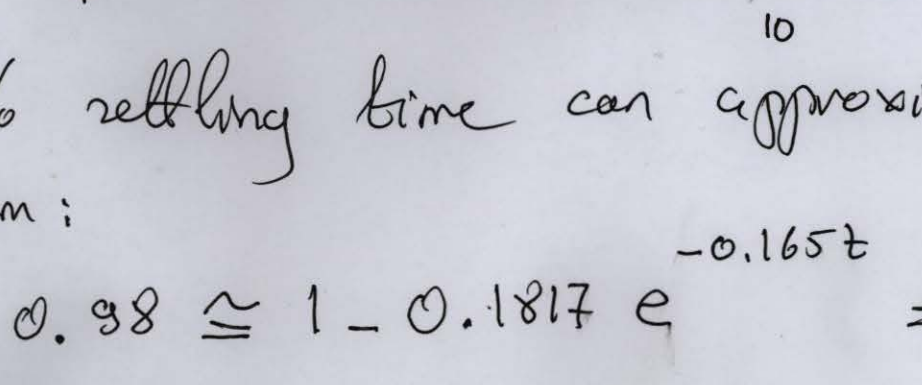
$$= \frac{A}{s} + \frac{B}{s+4.8845} + \frac{C}{s+0.165}$$

$$= \frac{1}{s} + \frac{-0.8212}{s+4.8345} + \frac{-0.1817}{s+0.165}$$

$$y(t) = \left\{ 1 - 0.8212\, e^{-4.8345t} - 0.1817\, e^{-0.165t} \right\} u(t)$$

<u>Remark:</u> Because of some rounding of numbers $1 - 0.81212 - 0.1817 \neq 0$. I will proceed as if this sum $= 0$.

<u>Remark:</u> Response consists of a constant, 1, a very fast decaying real exponent, $e^{-4.8345t}$, and a relatively very slowly decaying exponential $e^{-0.165t}$.



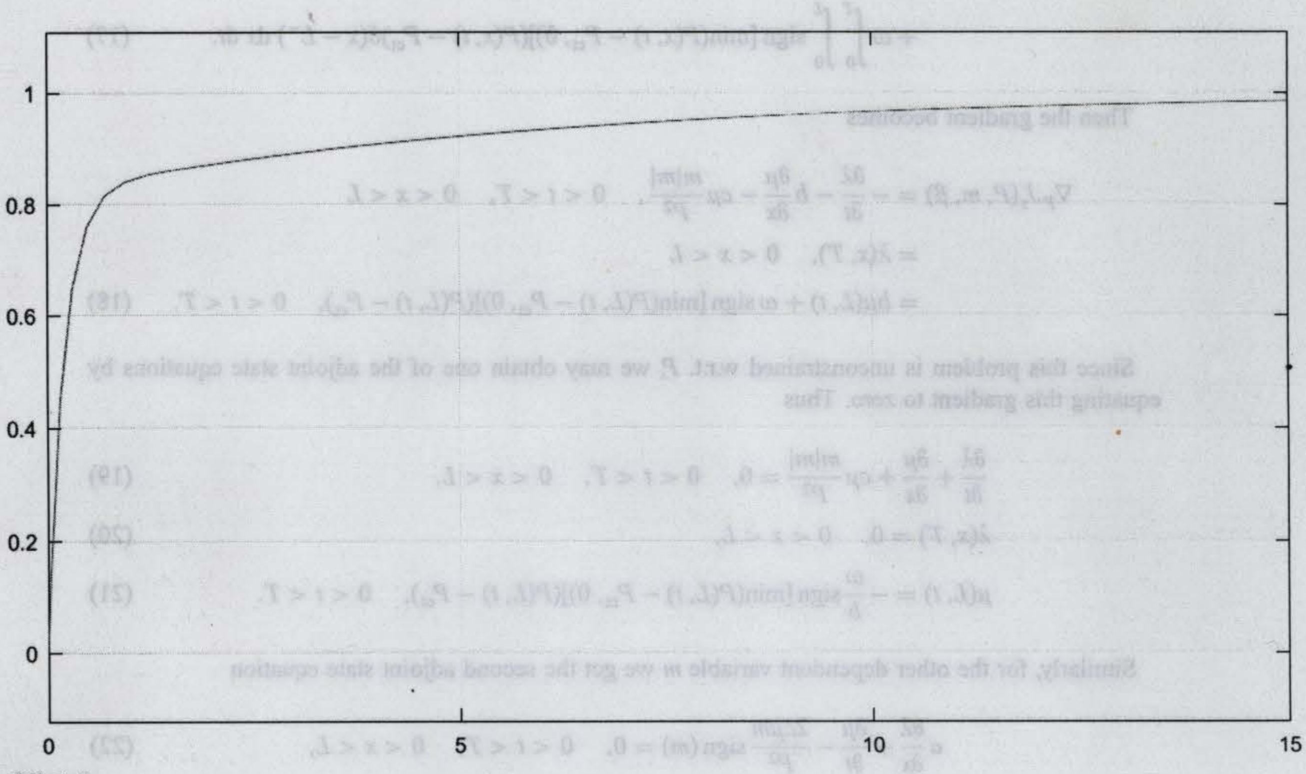
No overshoot.

$y_{ss} = 1.$

2 % settling time can approximately be found from:

$$0.98 \cong 1 - 0.1817\, e^{-0.165t} \Longrightarrow t_s(\%2) \cong 12.8 \text{ sec.}$$

MATLAB(Simulink) output is shown below. As expected it is no different from our analytical expectations.



d-) Now $D(s) = 4\left(1 + \frac{1}{0.4s}\right) = \frac{4s+10}{s}$

$$G(s) = \frac{(4s+10)}{s(s+1)} \Rightarrow T(s) = \frac{4s+10}{s^2+5s+10}$$

$$Y(s) = \frac{4s+10}{s(s^2+5s+10)} = \frac{A}{s} + \frac{Bs+C}{s^2+5s+10}.$$

$$= \frac{1}{s} - \frac{s+1}{s^2+5s+10} =$$

$$= \frac{1}{s} - \frac{s+\frac{5}{2}}{\left(s+\frac{5}{2}\right)^2+\left(10-\frac{25}{4}\right)} - \frac{1-\frac{5}{2}}{\left(s+\frac{5}{2}\right)^2+\left(10-\frac{25}{4}\right)}$$

$$\Rightarrow y(t) = \left\{ 1 - e^{-\frac{5}{2}t} \cdot \cos 1.9365t + 2.91\, e^{-5/2 t} \cdot \sin 1.9365t \right\} \cdot u(t)$$

$y_{ss} = 0$

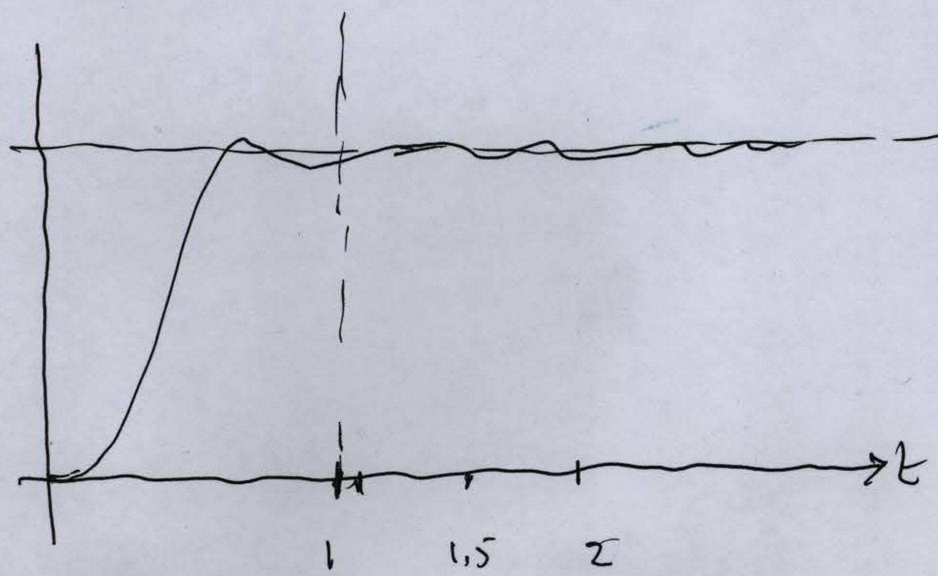
$$2 \cdot \zeta w_n = 5 \Rightarrow \zeta = \frac{5}{2w_n} = \frac{5}{2\sqrt{10}} = 0.7906.$$

$$w_n^2 = 10 \Rightarrow w_n = \sqrt{10}$$

$$M_p = \exp\left\{ -\frac{\pi \zeta}{\sqrt{1-\zeta^2}} \right\} = \exp\left\{ -4.0562 \right\} = 0.0173.$$

(Almost no overshoot).

$$t_s(2\%) \cong \frac{4}{\zeta w_n} = \frac{4}{(0.7906)\sqrt{10}} = 1.6 \text{ sec.}$$



This is the MATLAB (Simulink) output. It is as expected once again.

e.) Let us make the following table for comparison

	M_p	y_{ss}	t_{ss} (2%)
Case 1	—	0.2	∞
Case 2	—	1.0	12.8
Case 3	0.0173	1.0	1.6

System 1 has a too big settling time

System 2 has large settling time.

System 3 has a little overshoot but it has very small settling time.

Problem 3

$$H(s) = 1, \quad T(s) = \frac{Ks+b}{s^2+as+b}$$

a-) $T(s) = \dfrac{G_0(s)}{1+G_0(s)} \Rightarrow G_0(s) = \dfrac{T(s)}{1-T(s)}$

$$\Rightarrow G_0(s) = \frac{Ks+b}{s^2+as+\cancel{b}-Ks-\cancel{b}} = \frac{Ks+b}{s(s+a-K)}$$

b-) Now $G_0(s) = \dfrac{rs+m}{s^2+ns}$

Following conditions are imposed on the system:

i-) e_{ss} (under unit ramp) $= 0.04$

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \cancel{s}\cdot\frac{1}{1+\frac{rs+m}{s^2+ns}}\cdot\frac{1}{s^{\cancel{2}}}$$

$$= \lim_{s\to 0}\frac{1}{s+\frac{rs+m}{s+n}} = \boxed{\frac{n}{m} = 0.04}$$

ii-) $w_n = 5.$

$$T(s) = \frac{G_0(s)}{1+G_0(s)} = \frac{rs+m}{s^2+(n+r)s+m}.$$

$m = w_n^2 = 25 \implies \boxed{m=25}, \boxed{n=1}$

iii-) It is given that the unit step response of our system is

$$y(t) = A - e^{-3t}\left(\delta \cos(wt) + \beta \sin(wt)\right)$$

We know that (in the ideal case), if the system is underdamped, its response will be

$$y(t) = 1 - e^{-\zeta w_n t}\cdot\left(\cos w_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin w_d t\right).$$

$$\implies \zeta w_n = 3 \implies \zeta = \frac{3}{5}$$

$$s^2 + (n+r)s + m = s^2 + (1+r)s + 25$$

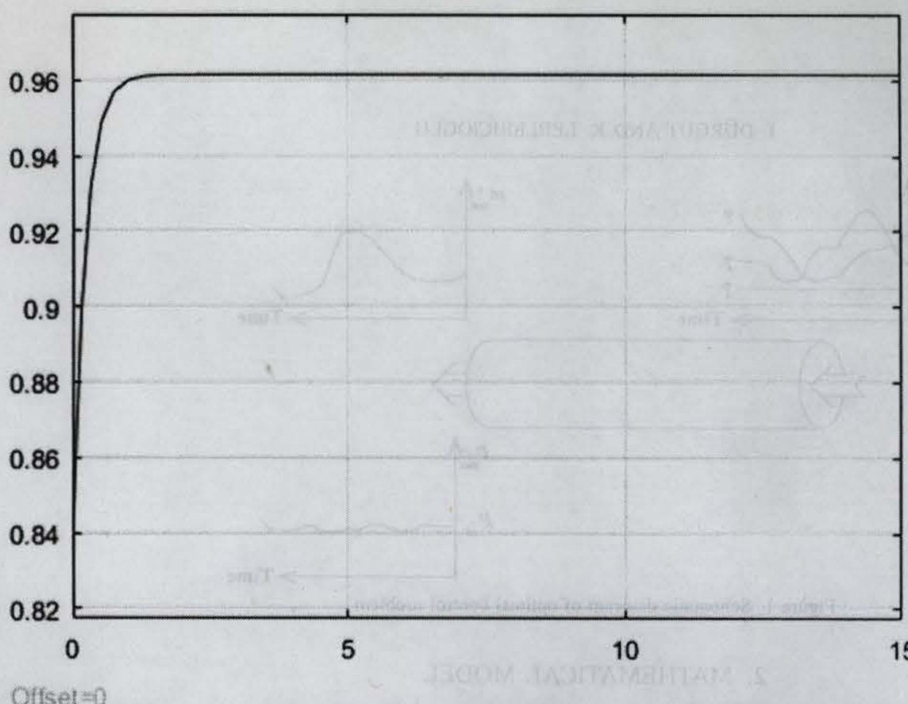
$$= s^2 + 2\zeta w_n s + w_n^2 = s^2 + 2\cdot\frac{3}{\cancel{5}}\cancel{5}\, s + 25$$

$$= s^2 + 6s + 25$$

$$\implies r+1 = 6 \implies \boxed{r=5}$$

c-) $A = 1,\ \delta = 1,\ \beta = \dfrac{\frac{3}{5}}{\sqrt{1-\frac{9}{25}}} = \dfrac{\frac{3}{5}}{\frac{4}{5}} = \dfrac{3}{4}.$

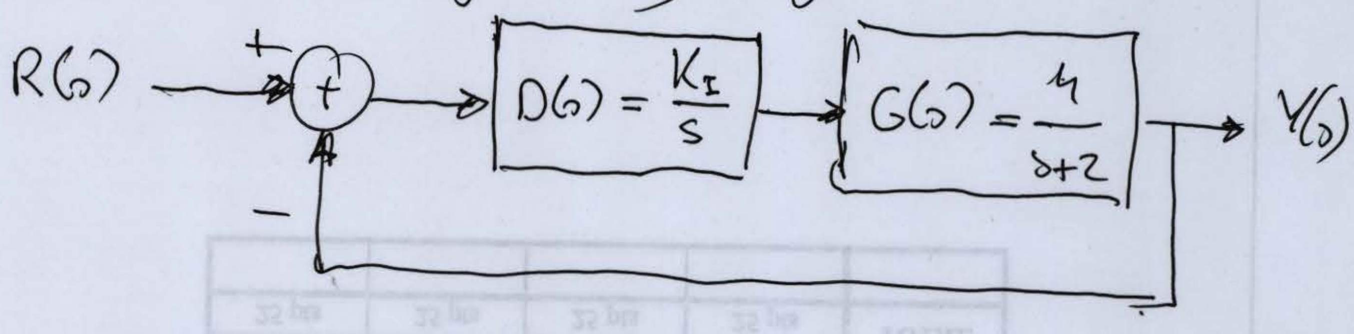
$$M_p = \exp\left\{-\frac{\pi\frac{3}{5}}{\sqrt{1-\frac{9}{25}}}\right\} = \exp\left\{-\frac{3}{4}\pi\right\} \cong 0.1.$$



This is the MATLAB output. Clearly the response is not exactly the response of a standard 2nd order system. We did many approximations and the correct one is seen above.

Problem 4

We consider the following figure:



a-) Suppose $K_I = 1$.

$$D(s)G(s) = \frac{4}{s(s+2)}.$$

$$e_{ss}(\text{unit step}) = \lim_{s \to 0} \cancel{s} \cdot \frac{1}{1 + \frac{4}{s(s+2)}} \cdot \frac{1}{\cancel{s}} = 0$$

(This is expected since Type = 1).

$$e_{ss}(\text{unit ramp}) = \lim_{s \to 0} \cancel{s} \cdot \frac{1}{1 + \frac{4}{s(s+2)}} \cdot \frac{1}{s^{\cancel{2}}}$$

$$= \lim_{s \to 0} \frac{1}{\frac{4}{s+2}} = \frac{1}{2}.$$

$$T(s) = \frac{D(s)G(s)}{1 + D(s)G(s)} = \frac{4}{s^2 + 2s + 4}.$$

$$\omega_n^2 = 4 \Rightarrow \omega_n = 2, \quad 2\zeta\omega_n = 2 \Rightarrow \zeta = \frac{1}{2}.$$

$$M_p = 100 \exp\left\{-\frac{\pi\frac{1}{2}}{\sqrt{1-\frac{1}{4}}}\right\} = 100 \exp\{-1.8138\}$$ (17)

$$= 16.3.$$

b-) $e_{ss}(\text{unit ramp}) \leq 0.125$

$$\frac{1}{2K_I} \leq 0.125 \Longrightarrow \boxed{K_I \geq 4}$$

$$T(s) = \frac{4K_I}{s^2+2s+4K_I}$$

$$w_n^2 = 4K_I \Longrightarrow w_n = 2\sqrt{K_I}$$

$$2\zeta w_n = 2 \Longrightarrow \zeta = \frac{1}{w_n} = \frac{1}{2\sqrt{K_I}}$$

$$M_p = \exp\left\{-\frac{\pi\frac{1}{2\sqrt{K_I}}}{\sqrt{1-\frac{1}{4K_I}}}\right\} = \exp\left\{-\frac{\pi}{\sqrt{4K_I-1}}\right\}.$$

$M_p \downarrow$ iff $\sqrt{4K_I-1}$ $\downarrow$

Minimum value of $\sqrt{4K_I-1} = \sqrt{15}$. which occurs when $K_I = 4$.

c-) $M_p = \exp\left\{-\frac{\pi}{\sqrt{15}}\right\} = 0.443.$

d-) $K_I \uparrow \Rightarrow \dfrac{\pi}{\sqrt{4K_I - 1}} \downarrow .$

$$\frac{\pi}{\sqrt{4K_I-1}} \downarrow \Rightarrow -\frac{\pi}{\sqrt{4K_I-1}} \Uparrow$$

$$-\frac{\pi}{\sqrt{4K_I-1}} \uparrow \Rightarrow \exp\left\{\frac{\pi}{\sqrt{4K_I-1}}\right\} \uparrow$$

$$\exp\left\{-\frac{\pi}{\sqrt{4K_I-1}}\right\} \Uparrow \Rightarrow M_p \uparrow$$