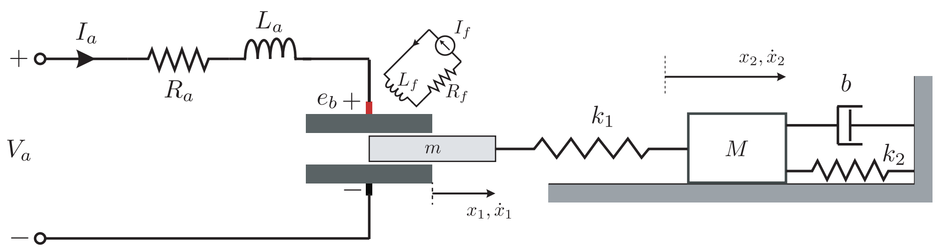
EE302 - Feedback Systems - Assignment 2 - Solutions
Department of Electrical and Electronics Engineering, METU

March 12, 2019

1. .



(a) Input–output of the system is the are $u(t) = V_a(t)$, and $y(t) = x_2(t)$ respectively.
Electrical part of the system is a first order dynamical system (RL Circuit). The mechanical sub-system has two "independent" masses connected to each other via a spring, and thus this part has 2 degrees-of-freedom, hence is 4^{th} order (since we need positional variables). Since the electro-mechanical variable conversions are static (memoryless), whole system is 5^{th} order.

Since the system has an order of 5, let the state variables be $\mathbf{z} = [I_a \ \dot{x}_1 \ x_1 \ \dot{x}_2 \ x_2]$. Based on these state definitions we can write the equations that govern the dynamics of the electrical sub-part as

$$
\begin{aligned}
L_a \dot{i}_a &= V_a - R_a I_a - e_b \\
L_a \dot{z}_1 &= R_a z_1 + u - K_b \dot{x}_1 \\
\dot{z}_1 &= -\frac{R_a}{L_a} z_1 + \frac{1}{L_a} u - \frac{K_b}{L_a} z_2
\end{aligned}
$$

If we write the equations of motion based on a free body diagram of the little mass (motor pin), we obtain

$$
\begin{aligned}
m\ddot{x}_1 &= F_{motor} + F_{k_1} \\
m\ddot{x}_1 &= K_A I_a + k_1(x_2 - x_1) \\
\dot{z}_2 &= \frac{K_a}{m} z_1 - \frac{k_1}{m} z_3 + \frac{k_1}{m} z_5 \\
\dot{z}_3 &= z_2
\end{aligned}
$$

Finally, we can write the equations of motion of the second mass as

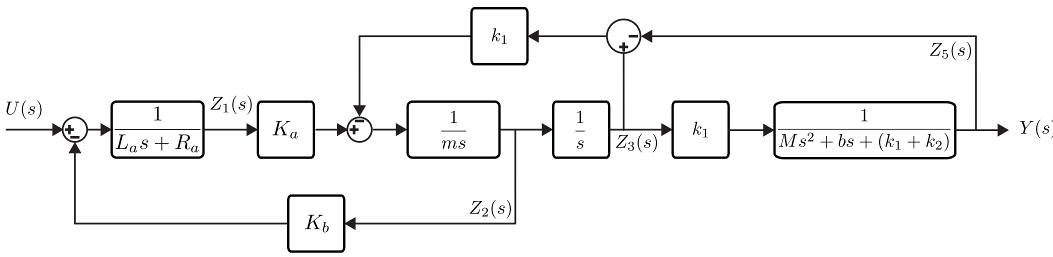
$$
\begin{aligned}
m\ddot{x}_2 &= k_1(x_1 - x_2) - k_2 x_2 - b\dot{x}_2 \\
\dot{z}_4 &= \frac{k_1}{M} z_3 - \frac{k_1 + k_2}{M} z_5 - \frac{b}{M} z_4 \\
\dot{z}_5 &= z_4
\end{aligned}
$$

Finally, we can collect all equations in state-space form as

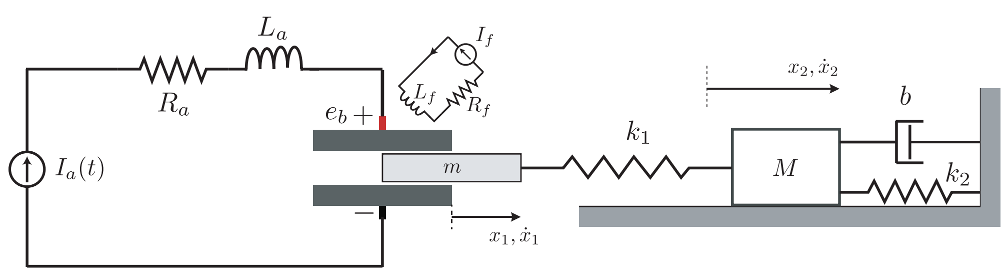
$$\dot{\mathbf{x}} = \begin{bmatrix} -\frac{R_a}{L_a} & -\frac{K_b}{L_a} & 0 & 0 & 0 \\ \frac{K_a}{m} & 0 & -\frac{k_1}{m} & 0 & \frac{k_1}{m} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{k_1}{M} & -\frac{b}{M} & -\frac{k_1+k_2}{M} \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{1}{L_a} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}$$

(b) A detailed (fully causal) block diagram can be constructed as given below



(c) In current based motor control policy, the armature current (suplied by a variable current source) becomes the input of the system $u(t) = I_a(t)$. Thus $I_a(t)$ is not a state anymore, which reduces the dimension of state-space by 1. Also, feedback coming from the back emf voltage does not affect the "dynamics" since this back-emf voltage is suppressed by the high-bandwith current source.

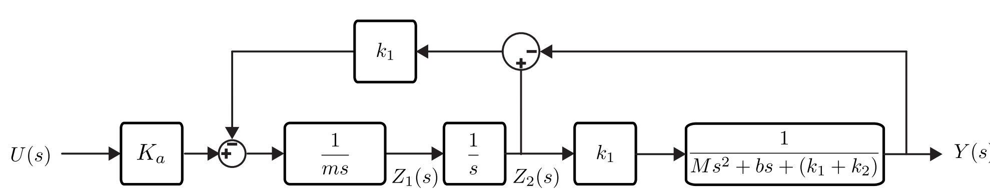


(d) State-space of the new system has the following states $\mathbf{z} = [\dot{x}_1 \; x_1 \; \dot{x}_2 \; x_2]$. Based on this definition we can simply update the state-space representation as

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & -\frac{k_1}{m} & 0 & \frac{k_1}{m} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{k_1}{M} & -\frac{b}{M} & -\frac{k_1+k_2}{M} \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{K_a}{m} \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

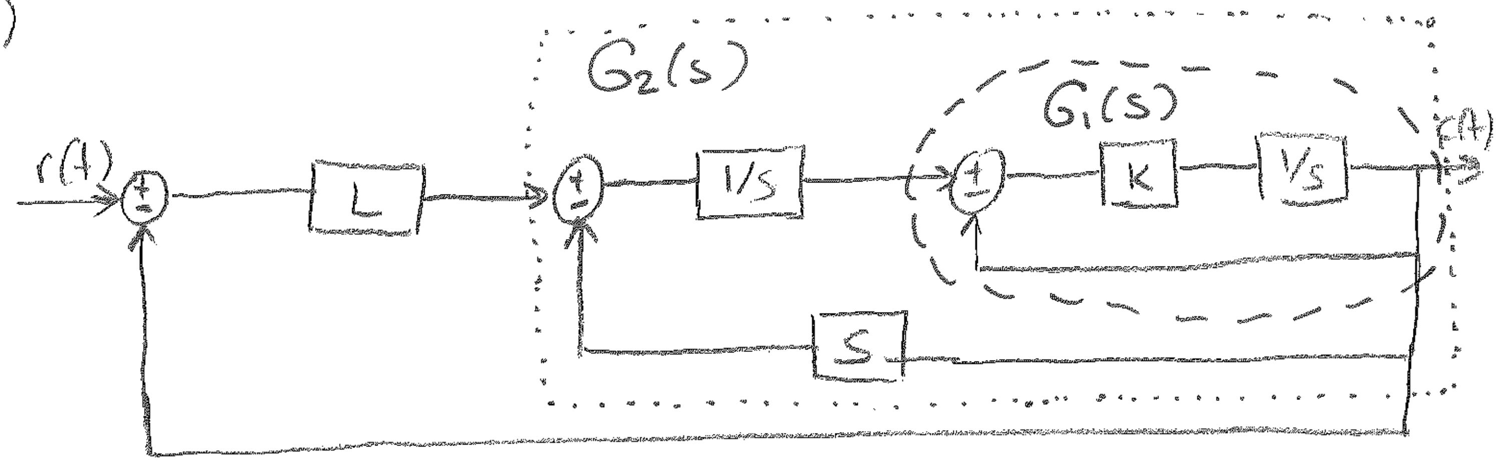
$$y = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}$$

(e) Block-diagram of the new system can be updated as



Q2) Let us modify the block diagram in order to make simplifications easier.

a)



$$G_1(s) = \frac{K/s}{1+K/s} = \frac{K}{s+K}$$

$$G_2(s) = \frac{G_1(s)/s}{1+\frac{G_1(s)}{\cancel{s}}\cdot\cancel{s}} = \frac{K/(s^2+Ks)}{1+\frac{K}{s+K}} = \frac{K}{s^2+2Ks}$$

$$G_{CL}(s) = \frac{C(s)}{R(s)} = \frac{L\cdot G_2(s)}{1+L\,G_2(s)} = \frac{KL/(s^2+2Ks)}{1+\frac{KL}{s^2+2Ks}} = \frac{KL}{s^2+2Ks+KL}$$

b) $$G_{CL}(s) = \frac{KL}{s^2+2Ks+KL} = \frac{w_n^2}{s^2+2\zeta w_n s+w_n^2}$$

$$\left.\begin{array}{l} w_n^2 = KL \Rightarrow w_n = \sqrt{KL} \\ \zeta w_n = K \end{array}\right\} \quad \zeta = \frac{K}{\sqrt{KL}} = \sqrt{\frac{K}{L}}$$

For critically damped response $\zeta = 1$

$$\zeta = \sqrt{\frac{K}{L}} = 1 \Rightarrow K = L$$

c) $\zeta = \sqrt{\frac{K}{L}}$, If $L > K$ then $\zeta < 1$: Underdamped, two complex conj. poles

$$s_{1,2} = -\zeta w_n \mp \sqrt{1-\zeta^2}\,w_n j = -K \mp j\sqrt{\frac{L-K}{L}}\cdot\sqrt{KL} = -K \mp j\sqrt{(L-K)K}$$

If L increases, real part of the poles does not change, imaginary part increases.

d) $L\uparrow \quad \zeta\downarrow \quad M_p\uparrow$, $\quad t_s = \frac{3}{\zeta w_n} = \frac{3}{K}$: t_s does not depend on L does not change.

$t_r = \frac{\pi-\phi}{w_d}$: $\phi\uparrow$ to $\frac{\pi}{2}$, $w_d\uparrow \Rightarrow t_r\downarrow$

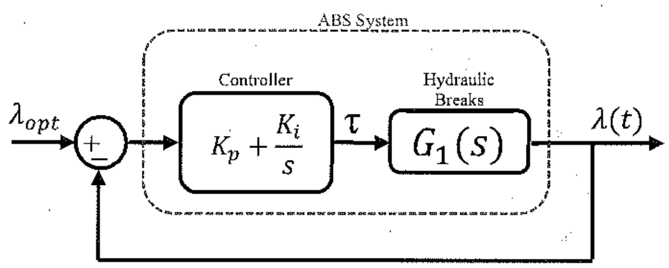
Q3. (Braking Bad) An Anti-lock Braking System (ABS) is used to prevent wheel lock and optimize breaking performance in various vehicles. For this purpose, the system monitors one quantity called the **longitudinal slip**, denoted by **λ(t)**, which is the relative motion between a tire and the road surface it is moving on. The equation which relates λ(t) with the angular velocity of the wheel $\boldsymbol{\omega(t)}$, and the longitudinal speed of the car $\boldsymbol{v}$ is given as $\boldsymbol{\lambda(t)} = \frac{v(t)-r\omega(t)}{v(t)}$, where **r** is the wheel radius. Notice that, if **λ(t) =0**, then there is no breaking; if **λ(t)= 1**, then the wheels are locked. In an ABS system, a dynamic torque **τ(t)** is applied to hydraulic breaking systems to reach the optimal slip value $\boldsymbol{\lambda_{opt}} \approx 0.15$. This is done by a controller in a feed-back loop. Assume the relationship between the torque **τ(t)** and the slip **λ(t)**, is given by the following differential equation.

$$\dot{\lambda}(t) = -\frac{r^2 C_0}{v_0 J}\lambda(t) + \frac{r}{v_0 J}\tau(t),$$

where r, C_0, v_0, m, J are constant parameters.

a) Find the transfer function $G_1(s) = \frac{\lambda(s)}{\tau(s)}$ for $r = 1$, $v_0 = 10$, $C_0 = 50$, $J = 0.1$.

b) Consider the configuration below.



By using the transfer function found in part (a) as $G_1(s)$, find the parameters of the controller (K_p and K_i) for the system to have a settling time (%5) of 0.06 seconds and percent overshoot of 10% to a unit step input *(assume that the effect of zeros of the transfer function on the step response can be neglected).*

a) $\dot{\lambda}(t) = -\frac{1\times 50}{10\times 0.1}\lambda(t) + \frac{1}{10\times 0.1}\tau(t) = -50\lambda(t) + \tau(t)$

$s.\lambda(s) = -50\lambda(s) + \tau(s) \Rightarrow G_1(s) = \frac{1}{s+50}$

b) $G_{CL}(s) = \frac{\frac{K_p s + K_i}{s}\cdot G_1(s)}{1 + \frac{K_p s + K_i}{s}\cdot G_1(s)} = \frac{K_p s + K_i}{s^2 + (50+K_p)s + K_i} = \frac{N(s)}{D(s)}$

Let's focus on D(s) (see: Hint),

$D(s) = s^2 + (50+K_p)s + K_i = s^2 + 2\zeta w_n s + w_n^2$ → $K_i = w_n^2$ .... (1)
→ $2\zeta w_n = 50 + K_p$ ... (2)

Given Info: $t_s = \frac{6}{100} = \frac{3}{\zeta w_n} \Rightarrow \zeta w_n = 50$

$M_p = 10\% \Rightarrow \zeta \approx 0.6$

$w_n = \frac{250}{3} = 83.\overline{3}$

$K_i = w_n^2 = (83.\overline{3})^2$

$\zeta w_n = 50$ & from(2) $\Rightarrow K_p = 50$