Chapter 2

Linear Time-Invariant Systems

Systems that are both linear and time-invariant are called *linear time-invariant (LTI) systems.* An important objective of this course is to provide a detailed understanding of LTI systems, including their properties and tools to analyze them, and introduce some very important applications that exploits this understanding.

The reason why we pay this much attention to LTI systems are as follows:

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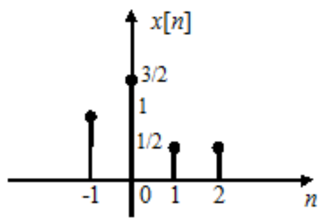
NEVER EVER FORGET THIS! 1 *An LTI system is uniquely described by its response to a unit impulse, i.e. impulse response.*

2.1 DT LTI Systems: The convolution sum

2.1.1 Representation of DT Signals in terms of Impulses

Any DT signal can be written as the sum of ....

Ex: $x[n] = ...$



In general: $\boxed{x[n] = \quad}$

Interpretation: Any DT signal can be represented as a weighted sum of shifted impulses $\delta[n-k]$, where the weights are determined by the signal $x[n]$.

2.1.2 Characterization of LTI Systems in Terms of Impulse Response

NEVER EVER FORGET THIS! 2 *One and only one thing that changes from one LTI system to another is its response to the unit impulse.*

Consider a DT LTI system whose output to a unit impulse $\delta[n]$ is $h[n]$:
$\Rightarrow h[n]$: *impulse response of the system*

What is the response to an arbitrary input $x[n]$?

Response to an LTI system:

$$\boxed{y[n] = \qquad\qquad\qquad\qquad\qquad\qquad}$$

Interpretation: Response of the LTI system, $y[n]$, is the weighted sum of shifted unit impulse responses $h[n-k]$, whose weights are determined by the input signal $x[n]$.

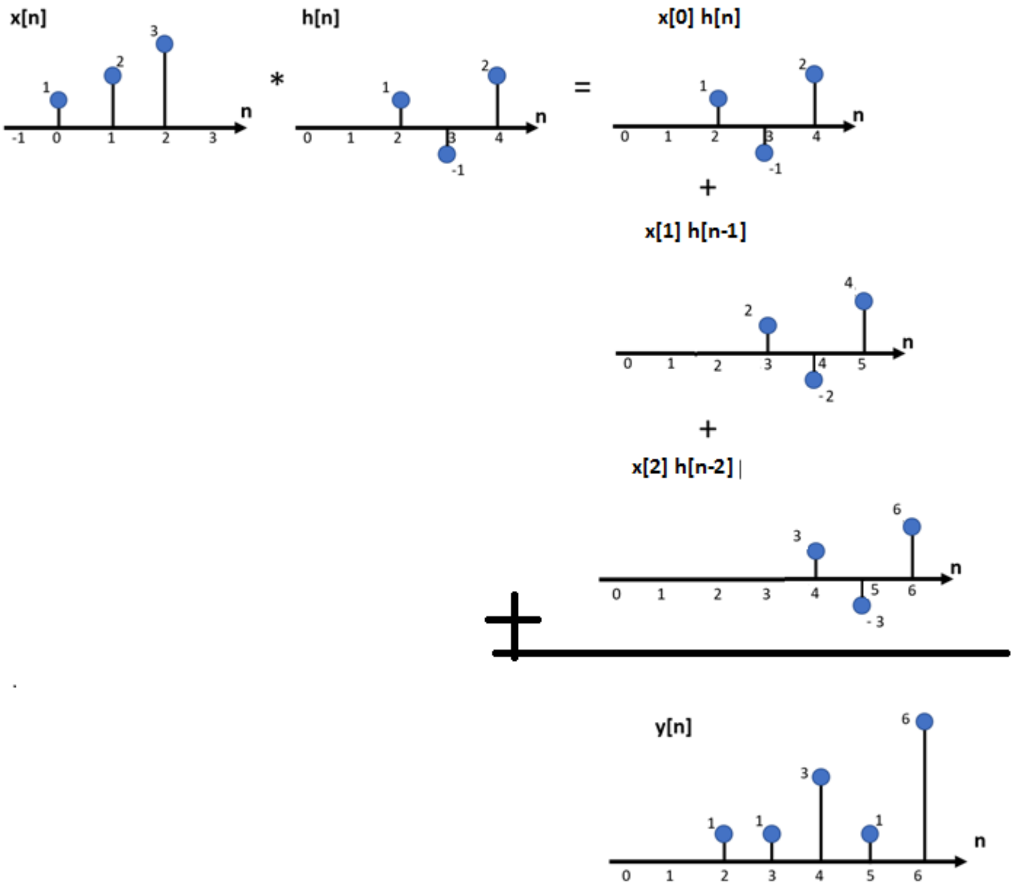
NEVER EVER FORGET THIS! 3 *Response of the* **LTI system** *is given by the* <u>*convolution*</u> *of the input signal with the impulse response of the system.*

Note that asterisk $*$ denotes the discrete *convolution* operation. Let $x[n]$ and $v[n]$ be two DT signals. Then their convolution is defined as

$$x[n] * v[n] = \sum_{k=-\infty}^{\infty} x[k]v[n-k]$$

Ex: Computing convolution using two approaches
Approach #1: (suitable for short length signals)

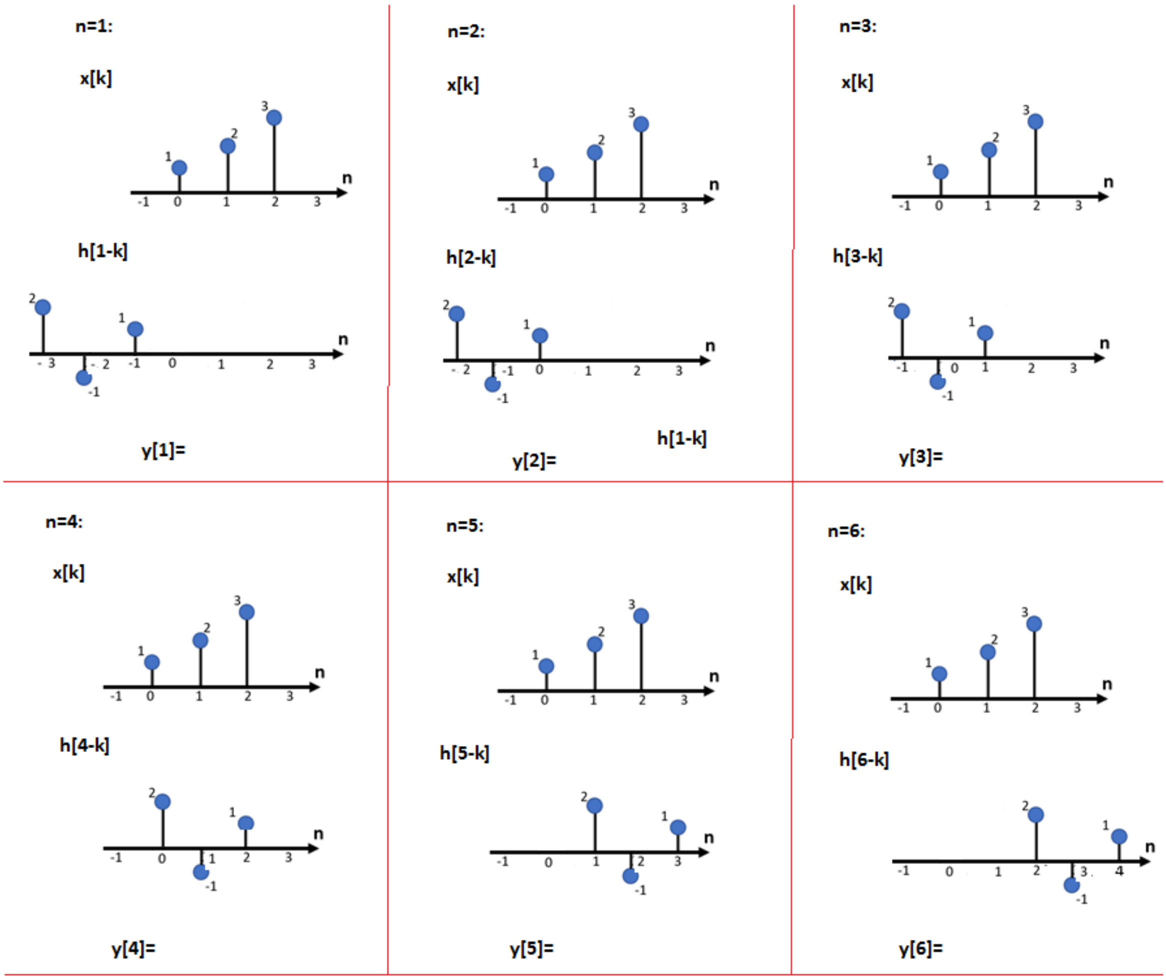
$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = ... + x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + ...$$

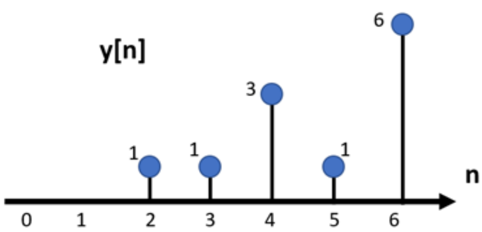


Approach #2: (sliding window method)

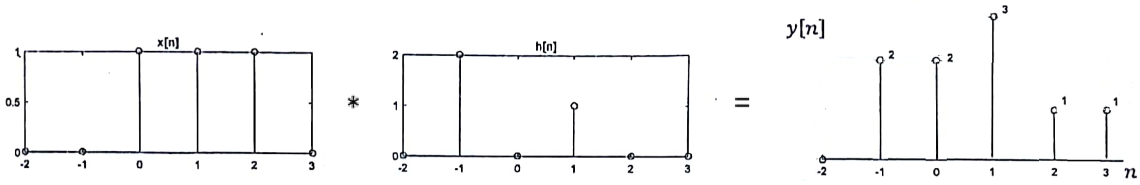
$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

1. View $x[k]$ and $h[n-k]$ as functions of k with n fixed (for example, $n = n_0$)
2. Multiply the sequence $h[n_0 - k]$ with $x[k]$ for all values of k, and sum the resulting sequence over k
3. This gives the output value at $n = n_0$. Repeat this for all n. This will be equivalent to *sliding* the sequence $h[n-k]$ over $x[k]$.





Ex: [Challenge yourself!] Compute the following convolution:



Ex: Input signal: $x[n] = \alpha^n u[n]$, $0 < \alpha < 1$
Impulse response: $h[n] = u[n]$
Response of the LTI system?

Ex: [Challenge yourself!] TRUE or FALSE?

- If $y[n] = x[n] * h[n]$, is it true that $y[2n] = x[2n] * h[2n]$

- If $y[n] = x[n] * h[n]$, is it true that $y[-n] = x[-n] * h[-n]$

2.2 CT LTI Systems : The convolution integral

2.2.1 Representation of CT Signals in terms of Impulses

- Remember the basic properties of the unit impulse:

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau =$$

- OR, equivalently, consider a staircase approximation to a CT signal $x(t)$ and express it in terms of $\delta_{\Delta}(t)$:

$$\boxed{x(t) =}$$

Interpretation: Any CT signal can be represented as a weighted "sum" of shifted impulses $\delta(t-\tau)$, where the weights are determined by the signal $x(t)$.

2.2.2 Characterization of LTI Systems in Terms of Impulse Response

NEVER EVER FORGET THIS! 4 *One and only one thing that changes from one LTI system to another is* its response to the unit impulse.

To understand how, consider a CT LTI system whose output to $\delta_\Delta(t)$ is $h_\Delta(t)$.
What is the response to an arbitrary input $x(t)$?

Response of an LTI system:

$$\boxed{y(t) = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

Interpretation: Response of the LTI system, $y(t)$, is the weighted "sum" of shifted unit impulse responses $h(t-\tau)$, whose weights are determined by the input signal $x(t)$.

NEVER EVER FORGET THIS! 5 *Response of the* **LTI system** *is given by the* convolution *of the input signal with the unit impulse response.*

Note that asterisk $*$ denotes the continuous *convolution* operation. Let $x(t)$ and $v(t)$ be two CT signals. Then their convolution is defined as

$$x(t) * v(t) = \int_{-\infty}^{\infty} x(\tau)v(t-\tau)d\tau$$

Ex: Input signal: $x(t) = u(t) - u(t-1)$
Impulse response: $h(t) = 2[u(t+1) - u(t-2)]$
Response of the LTI system?

Ex: [Challenge yourself!] Input signal: $x(t) = u(t) - u(t-1)$
Impulse response: $h(t) = u(t)$
Response of the LTI system?

Ex: [Challenge yourself!]

- $x(t) = e^{-\alpha t}u(t)$, $\alpha > 0$, $h(t) = u(t)$, $x(t) * h(t) =?$

- $x(t) = 2(1-t)$ if $0 < t < 1$, and zero elsewhere, $h(t) = u(t) - u(t-1)$, $x(t) * h(t) =?$

2.2.3 Don't get confused!

- $\int_{\tau=-\infty}^{\infty} \delta(\tau) d\tau =$
- $\int_{\tau=-\infty}^{\infty} \delta(t-\tau) d\tau =$
- $x(t)\delta(t) =$
- $x(t)\delta(t-t_0) =$
- $\int_{\tau=-\infty}^{\infty} x(\tau)\delta(t-\tau) d\tau =$
- $\int_{\tau=-\infty}^{\infty} x(\tau)\delta(t-t_0-\tau) d\tau =$
- $x(t) * \delta(t) =$
- $x(t) * \delta(t-t_0) =$

2.3 Properties of LTI Systems

P.0 Impulse Response (NEVER EVER FORGET THIS!):

The behavior of an LTI system is completely and uniquely determined by its impulse response.
DT: $y[n] =$

CT: $y(t) =$

Ex: Consider a DT system whose response to the unit impulse is $h[n] = \begin{cases} 1, & \text{if } n = 0, 1 \\ 0, & \text{otherwise} \end{cases}$

- What is the input-output relationship of the LTI system that has this impulse response?

- Can you provide input-output relationship of another DT system that is *not LTI*, but has the *same* response to the unit impulse?

P.1 Commutative Property

$x[n] * h[n] =$ $(x(t) * h(t) =$ $)$

Proof:

Interpretation:

P.2 Associative Property

$[x(t) * h_1(t)] * h_2(t) =$ (also applies to DT convolution)

Proof: Exercise

Interpretation:

P.3 Distributive Property

$x(t) * [h_1(t) + h_2(t)] =$
Proof: Exercise
Interpretation:

Ex: [Challenge yourself!] $x[n] = \delta[n-5]$, $h[n] = \delta[n] + \delta[n-1]$, $x[n] * h[n] =?$

Ex: [Challenge yourself!] $x[n] = \left(\frac{1}{3}\right)^n u[n] + \delta[n]$, $h[n] = u[n]$, $x[n] * h[n] =?$

P.4 LTI Systems Without Memory

An LTI system is *memoryless* if and only if

$$h[n] =$$

Proof:

Ex:

P.5 Causality for LTI Systems

An LTI system is *causal* if and only if

$$h[n] =$$

Proof:

P.6 Stability for LTI Systems

An LTI system is *stable* if and only if

$$h[n]$$

Proof:

Ex: Delay system: $y[n] = x[n - n_0]$ where n_0 is some constant. Stable?

Ex: Integrator: $y(t) = \int_{-\infty}^{t} x(\tau) d\tau$. Stable?

P.7 Unit-Step Response

Unit-Step Response: Response of a system to the unit step signal

Can we obtain impulse response from step response, or vice versa?
DT:

CT:

Hence the unit step response also fully characterizes an LTI system.

Ex: [Challenge yourself!]

- Prove that $x(t) * h(t) = \left(\frac{d}{dt}x(t)\right) * g(t)$ where $g(t)$ is the unit step response given by $g(t) = \int_{-\infty}^{t} h(\tau)d\tau$.
- Apply this result to $x(t) = u(t) - u(t - T_1)$ and $h(t) = u(t) - u(t - T_2)$ to obtain $x(t) * h(t)$.

2.4 LTI Systems Described by Differential and Difference Equations

Input-output relationship of many physical systems can be described by *linear differential or difference equations* with constant coefficients (LDECC).
Ex: (CT)

Ex: (DT)

We now introduce some of the basic ideas involved in solving LDECC and later we will learn more powerful tools (Fourier, Laplace and Z Transforms).

Note that the difference (differential) equation by itself does not specify a unique solution $y[n]$ to the input $x[n]$. Hence auxiliary conditions have to specified to completely determine the output signal $y[n]$ ($y(t)$).

2.4.1 Continuous-time case: Causal LTI systems described by differential equations

A system described by a general Nth order LDECC

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

is a **causal LTI system** under the condition of **initial rest**.

Initial rest condition:

Ex: Consider the following system:

$$\frac{d^2 y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$$

under the condition of initial rest.

1. Block diagram representation of the system?
2. What is the impulse response of this LTI system?
3. (Exercise) What is the response of this LTI system to the input $x(t) = e^{3t}u(t)$?

See Problem 2.56 from Oppenheim to better understand the approach in the general case.

Solution Cont'ed:

2.4.2 Discrete-time case: Causal LTI systems described by difference equations

A system described by a general Nth order LDECC

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

is a **causal LTI system** under the condition of **initial rest**.

<u>Initial rest condition:</u>

Ex:
Consider a system whose input-output relationship is

$$y[n] - \frac{1}{4}y[n-2] = x[n]$$

The system is initially at rest.

1. Block diagram representation of the system?
2. What is the impulse response of this LTI system?

See Problem 2.54 & 2.55 from Oppenheim to better understand the approach in the general case.

Solution Cont'ed: