Chapter 1

Signals and Systems

The concepts of signals and systems arise in all fields of electrical engineering. The tools associated with these concepts are continuously yielding new developments in diverse areas of science and technology such as communications, imaging, biomedicine, circuit design, energy distribution systems, robotics, speech processing and so on.
In this course, we will regard everything as either **signals** (functions in math) or **systems** (devices that change the shapes of functions). We will learn how to build useful structures for the representation, processing, and analysis of signals by using the **mathematical framework** of signals and systems. This mathematical framework contains several transformations, which we will discuss in detail in this course.
There is a common mathematical language for describing signals and systems. In this chapter, we will first introduce these mathematical descriptions and representations. In the next chapters, we will develop the mathematical framework.

1.1 Signals

Definition 1 *A* **signal** *is ...*

- Mathemetically, signals are **functions** of variables, which .... about the behavior of some process or source.

Ex: Examples of signals

We will discuss the notion of a system later in more detail. Here we briefly mention it since it is directly related to the notion of a signal.

Definition 2 *A* **system** *is ...*

- Physically a system is something that "takes in" one or more input signals and "produces" one or more output signals.

Ex: Examples of systems:

Our goal in this course is to learn mathematical tools to analyze and design signal processing systems.

- https://www.youtube.com/watch?v=EErkgr1MWw0
- https://www.youtube.com/watch?v=mexN6d8QF9o

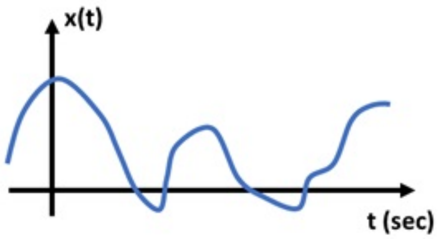
Returning back to the notion of signals, remember that a signal is a function of a (independent) variable.

- We typically refer to this variable as *time*, although it can be other things (such as space, distance, location index, and so on) depending on the application.
- Independent variable can be 1-D, 2-D, or, in general, N-dimensional.
  Ex: An image is ...
  Ex: A video is ...

Independent variables of signals are either continuous or discrete. As a result of this, we consider two types of signals:

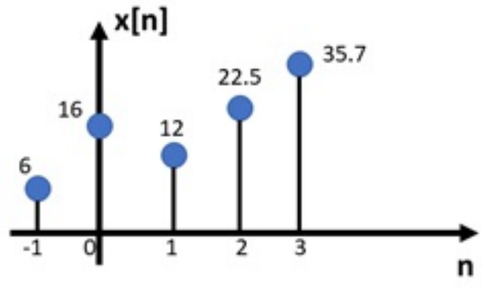
<u>Continuous-time (CT) signals</u>

- continuous variation
-
-
-



<u>Discrete-time (CT) signals</u>

- variation at specified time instants
-
-
-



Examples:

- Voltage in time
- # of people in lectures over time
- Blood pressure recorded every half an hour

Note the following points for DT and CT signals:

- DT signals are **undefined** at all values other than **integer values of** n
  $\Rightarrow$ referred as a *discrete-time sequence*
- Ex: # of people in lectures
  - No meaning for lecture number 1/2
- Most physical signals are CT, but not all.
- A DT signal may arise
  - from a process that is inherently discrete, as in ......

– from the sampling of a CT signal, as in ......

In this course, we will treat CT and DT signals separately but in parallel in order to see the similarities and differences in their treatment more clearly.

1.1.1 Transformations of the independent variable

We sometimes work with signals after modifying the independent variable (i.e. time axis).
Ex: (Fast Forward)

- **Time Shift** : $x(t - t_0)$

- **Time Reversal** : $x(-t)$

- **Time Scaling** : $x(\frac{t}{a})$ or $x(at)$, a is a real number
  Ex:

Note that $x(at)$ is always defined for $a \epsilon \mathbb{R}$. The same is not true for $x[an]$.

Ex:

$x[n]$: a discrete sequence. Which of the following are defined:

$x[2n]$

$x[\sqrt{2}n]$

$x[\frac{n}{2}]$

Ex: Find $y(t) = x(2t - 1)$ for the given $x(t)$.

In general, to find $x(at+b)$,

1. First time-shift by ...
2. OR ...

1.2 Some important properties of signals

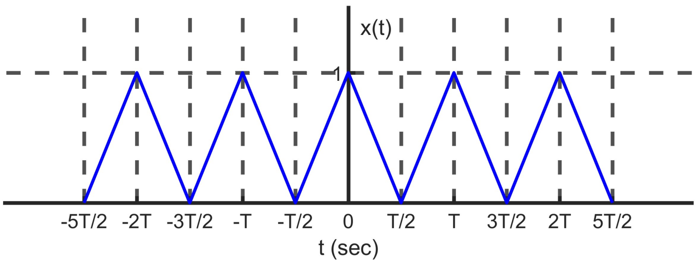
1.2.1 Periodic signals

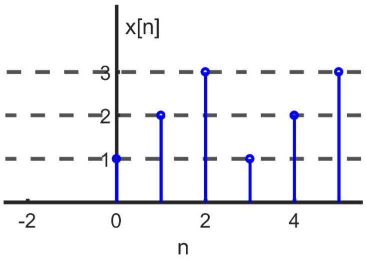
A CT signal $x(t)$ is said to be **periodic** if it repeats itself with a period of T, i.e.

$$x(t) =$$

Periodicity is defined similarly for DT signals:
$x[n]$ is periodic with N if $x[n] = x[n+N]$ for all n where N is an **integer**.

Ex:





Note that if $x(t)$ is periodic with T then it is also periodic with $2T$, $3T$, $4T$,....

- **Fundamental period** T_0 of $x(t)$ (N_0 of $x[n]$): the smallest positive T (N) for which the above equalities hold.

Ex: [Challenge yourself!] Prove whether $x_p(t) = \sum_{m=-\infty}^{\infty} x(t - mT)$ is periodic or not, when T: some constant, $x(t)$: arbitrary CT signal.

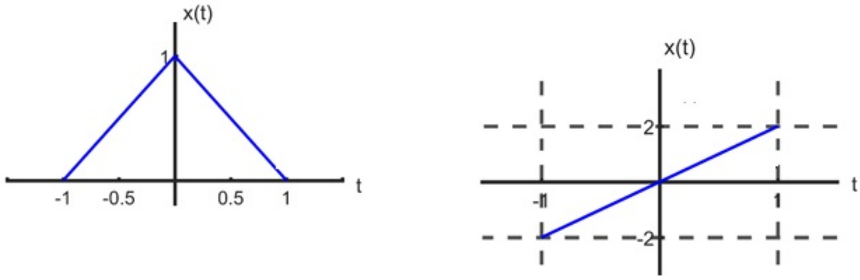
Ex: [Challenge yourself!] If $x(t)$ is periodic with T, how about $x(t + t_0)$ and $x(\frac{t}{a})$? Similarly, if $x[n]$ is periodic with N, how about $x[n + n_0]$ and $x[Mn]$?

1.2.2 Even and Odd Signals

A CT signal is **even** (symmetric) if (DT:

A CT signal is **odd** (symmetric) if (DT:

Ex:



Ex: [Challenge yourself!] Can we decompose arbitrary signals into a sum of two signals, one of which is even and the other is odd:

$$x(t) = x_{ev}(t) + x_{od}(t)$$

- Is the decomposition unique?
- Closed-form expression for the decomposition?

Check the book for the solution:

$$x_{ev}(t) = \frac{1}{2}(x(t) + x(-t)), \quad x_{od}(t) = \frac{1}{2}(x(t) - x(-t))$$

These are called **even and odd parts** of $x(t)$. (This decomposition is used in the derivation of fast algorithms for transforms such as FFT.)

1.3 Basic CT signals

Short digression: Brief review of complex numbers (Page 71 in the book)

1.3.1 CT Complex Exponential & Sinusoidal Signals

The CT *complex exponential signal* is generally of the form

$$x(t) =$$

where C and a are in general complex numbers.

Real Exponential Signals

If C and a are real numbers, real exponential signals are obtained.

- If $a > 0$, growing exponential:

- If $a < 0$, decaying exponential:

Periodic Complex Exponential Signals

If a is purely imaginary, periodic complex exponentials are obtained:

$$x(t) =$$

$\Rightarrow$ Verify periodicity:

Some remarks:

- Fundamental period $T_0 =$
- $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$ have the same fundamental period.

- Euler's relations:

- Harmonically related complex exponentials:

- Does $x(t)$ have finite average power? finite energy?

Sinusoidal Signals

$x(t) =$

- even signal when $\theta =$
- odd signal when $\theta =$

Ex: [Challenge yourself!] How to express $x(t)$ in terms of periodic complex exponentials?

General Complex Exponential Signals

Both C and a are complex.
Let us represent C in polar form as $C = re^{j\theta}$ and a in rectangular form as $a = \alpha + j\omega_0$. Then $x(t) = Ce^{at} =$

- $\alpha = 0:$ both real and imaginary parts are ...
- $\alpha > 0:$ both real and imaginary parts are ...
- $\alpha < 0:$ both real and imaginary parts are ...

Ex: $x(t) = 2e^{(3+j\omega_0)t}$ **Ex:** $x(t) = e^{(-1+j\omega_0)t}$

1.3.2 DT Complex Exponential & Sinusoidal Signals

The DT complex exponential signal is generally of the form

$$x[n] = C\alpha^n \quad \text{or} \quad x[n] = Ce^{\beta n} \ (\alpha = e^{\beta})$$

where C and α are in general complex numbers.

Real Exponential Signals

If C and a are real numbers, real exponential signals are obtained with various behavior.
Ex:

Sinusoidal Signals

If β is purely <u>imaginary</u> (i.e. $|\alpha| = 1$), DT sinusoids are obtained.
Consider $x[n] = Ce^{j\Omega_0 n}$.

General Complex Exponential Signals (Damped sinusoids)

β is <u>not</u> purely imaginary.
By representing C and α in polar form,

$$\begin{aligned} x[n] &= C\alpha^n = |C|e^{j\theta}(|\alpha|e^{j\Omega_0})^n = |C||\alpha|^n e^{j(\Omega_0 n+\theta)} \\ &= |C||\alpha|^n \left(\cos(\Omega_0 n+\theta) + j\sin(\Omega_0 n+\theta)\right) \end{aligned}$$

- $|\alpha| > 1$: both real and imaginary parts are ...
- $|\alpha| < 1$: both real and imaginary parts are ...

Periodicity Properties of DT Complex Exponential Signals

Although there are many similarities between CT and DT signals, there are also some important differences. One such difference exists between $e^{j\omega_0 t}$ and $e^{j\Omega_0 n}$.
Remember:

- CT signal $e^{j\omega_0 t}$ is periodic for any ω_0.
- As $|\omega_0|$ increases, the rate of oscillation (frequency) increases.

Both of the above properties are different for $e^{j\Omega_0 n}$:

- As $|\Omega_0|$ increases, the rate of oscillation does *not* increase continuously.
- $e^{j\Omega_0 n}$ is *not* even distinct for different Ω_0; <u>identical</u> for $\Omega_0 + 2\pi k$

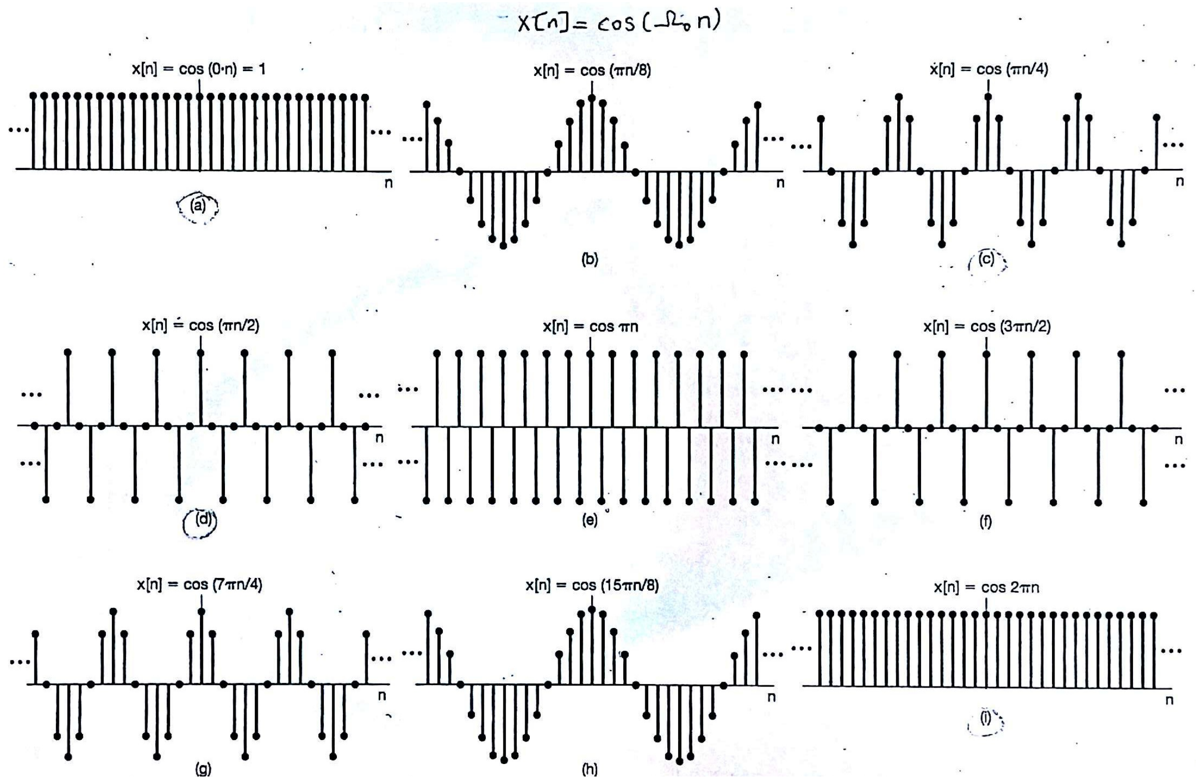


Figure 1.1: DT sinusoidal sequences for several frequencies (taken from texbook)

- $e^{j\Omega_0 n}$ is periodic only if Ω_0 can be written in the form $\Omega_0 = 2\pi\frac{k}{N}$ for some integers $N > 0$ and k.

Ex: Is $x[n] = e^{jn}$ periodic? If so, find the fundamental period and frequency.

Ex: Is $x[n] = 5\cos(\frac{3\pi}{4}n)$ periodic? If so, find the fundamental period and frequency.

Differences between CT and DT complex exponentials can be summarized as follows:

CT: $e^{j\omega_0 t}$	**DT:** $e^{j\Omega_0 n}$
Distinct signals for distinct ω_0	Same signals for $\Omega_0 + 2\pi k$
Periodic for any ω_0	Periodic only if $\frac{\Omega_0}{2\pi}$ is a rational number
Fundamental period T_0 $\omega_0 = 0 \;\Rightarrow$ undefined $\omega_0 \neq 0 \;\Rightarrow\; T_0 = \frac{2\pi}{\lvert\omega_0\rvert}$	Fundamental period N_0 $\Omega_0 = 0 \;\Rightarrow$ undefined $\Omega_0 \neq 0 \;\Rightarrow\; N_0 = k\frac{2\pi}{\Omega_0}$ with min. possible k
Fundamental frequency $\lvert\omega_0\rvert$	Fundamental frequency $\frac{2\pi}{N_0}$

1.3.3 DT Unit Impulse and Unit Step Sequences

DT Unit Impulse

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0. \end{cases}$$

DT Unit Step

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0. \end{cases}$$

The relation between $u[n]$ and $\delta[n]$:

$\delta[n] =$

$u[n] =$

(Sum of Shifted Impulses / Running Sum Interpretations)

Properties of DT Unit Impulse $\delta[n]$

- $$\sum_{n=-\infty}^{\infty} \delta[n] = \qquad , \qquad \sum_{n=n_1}^{n=n_2} \delta[n] =$$
- $x[n]\delta[n] =$

  $(x[n]\delta[n-n_0] =$ $\qquad\qquad\qquad$) (Sampling Prop.)
- $\sum_{n=-\infty}^{\infty} x[n]\delta[n] =$

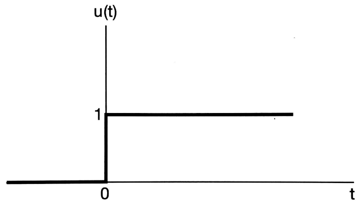
$\sum_{n=n_1}^{n=n_2} x[n]\delta[n] =$

$(\sum_{n=-\infty}^{\infty} x[n]\delta[n_0 - n] =$) (Convolution Prop.)

1.3.4 CT Unit Impulse and Unit Step Functions

CT Unit Step

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0. \end{cases}$$



CT Unit Impulse

The CT unit impulse $\delta(t)$ is related to the unit step by:

$$u(t) =$$

This suggests

$$\delta(t) =$$

However, this is problematic in the classical sense since $u(t)$ is formally not differentiable at $t = 0$ ("generalized derivative").
For a more formal development, let us define new functions (approximations to the original step & impulse):

That is, more formally, the unit impulse func. is defined as

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$

- Note that $\delta_\Delta(t)$ is a short pulse of duration Δ and with unit area for any value of Δ. $\delta(t)$ should be viewed as an *idealization* of the short pulse when the duration Δ is very small (for any practical purpose).

- In practical terms, you can think of $\delta(t)$ as any func. of unit area, concentrated very near $t = 0$.

Properties of unit impulse function $\delta(t)$

- It is a signal of unit area vanishing everywhere except at origin.
  $\int_{-\infty}^{\infty} \delta(t)dt =$

  $\int_{t_1}^{t_2} \delta(t)dt =$

- It is the (generalized) derivative of the unit step func.
- For any cont. func. $x(t)$,
  - $\star$ $x(t)\delta(t) =$

    $x(t)\delta(t - t_0) =$
  - $\star$ $\int_{-\infty}^{\infty} x(t)\delta(t)dt =$

    $\int_{t_1}^{t_2} x(t)\delta(t)dt =$

    $(\int_{-\infty}^{\infty} x(t)\delta(t_0 - t)dt = \qquad)$

Ex: [Challenge yourself!] Prove the following:
Running Integral Interpretation : $u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$

Ex: Prove the following:
Moving Impulse Interpretation : $u(t) = \int_{0}^{\infty} \delta(t - \tau)d\tau$

Ex: [Challenge yourself!] Example 1.7 in the book (square type signal)

1.4 Systems

Definition 3 *A* **system** *is any process that results in the transformation of signals.*

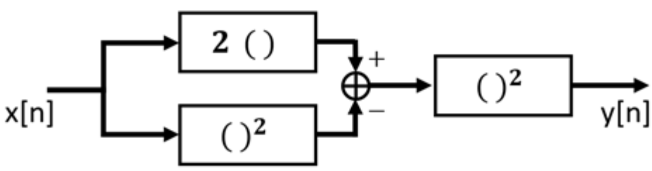
Ex:

CT and DT systems:

Interconnection of systems:

Ex:

$$y[n] = (2x[n] - x[n]^2)^2$$



1.4.1 Basic System Properties

Memory

A system is **memoryless** (instantaneous) if the output at any time instant depends only on the value of the input at that particular instant.
Ex:

Causality

A system is **causal** if its output at any time depends only on the values of the input at present time and in the past.

Ex:

Note that physical systems that process time-domain signals in real-time can not respond before they are stimulated. Hence, in this case, causality is equivalent to realizability.

Non-causal systems are of interest when the independent variable is not time or offline processing is possible.

Ex: [Challenge yourself!] Which of the following are causal:

- $y[n] = x[3n]$
- $y[n] = \sum_{m=-\infty}^{n} \alpha_m x[n-m]$
- $y(t) = x(t)^2$
- $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$
- a memoryless system

Invertibility

A system is **invertible** if for distinct inputs, $x_1(t) \neq x_2(t)$, it generates distinct outputs, $y_1(t) \neq y_2(t)$.
If a system is invertible, then a corresponding inverse system exists.
Ex:

Ex: [Challenge yourself!] Is the following system invertible: $y(t) = 5x(t - 1/2)$?

Stability (Bounded Input Bounded Output (BIBO) Stability)

A system is **stable** if bounded inputs lead to bounded outputs.

Ex:

Ex: [Challenge yourself!] Which of the following are stable:

- $y(t) = \frac{x(t)}{x(t-1)+1}$
- $y(t) = e^{\alpha t}x(t)$, $\alpha \in \mathcal{C}$
- $y(t) = \frac{d}{dt}x(t)$

Time Invariance

A system is **time-invariant** if a time shift in the input signal causes same amount of time shift in the output signal.

Ex:

Ex: [Challenge yourself!] Time-invariant or not:

- $y(t) = \sin(x(t))$
- $y(t) = x(2t)$
- $y(t) = e^t\, x(t)$
- $y[n] = n\, x[n]$

Linearity

A **linear** system has the superposition property:

Ex:

Ex: [Challenge yourself!] Linear or not:

- $y[n] = 5x[n] + 2$
- $y(t) = x^2(t)$
- $y(t) = \frac{d^n x(t)}{dt^n}$