Chapter 3

Fourier Series Representation of Periodic Signals

Remember that superposition holds in LTI systems:

$$\begin{aligned} w_k(t) &\xrightarrow{S} z_k(t) \\ x(t) = \sum_{k=-\infty}^{\infty} a_k w_k(t) &\xrightarrow{S} \end{aligned}$$

In the study of LTI systems, it is advantageous to represent signals as <u>linear combinations of basic signals</u> that possess the following two properties:

1.
2.

<u>Complex exponential signals</u> possess both properties.

$\Rightarrow$ **Fourier series & transforms**

In this chapter, we will first discuss the continuous-time Fourier series representation, and then the discrete-time Fourier series representation for periodic signals. Fourier transforms will be covered in later chapters.

3.1 Eigenfunctions of a CT system

We will write $x(t)$ in terms of the special set of functions that are called the <u>eigenfunctions</u> of the system.

- <u>Eigenfunction</u>:

$$\varphi_k(t) \xrightarrow{S}$$

For an LTI system with eigenfunctions ..., $\varphi_{-1}(t), \varphi_0(t), ...$,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \varphi_k(t) \xrightarrow{S} y(t) =$$

$\Rightarrow$ The input-output relationship of the system can be specified by its eigenvalues λ_k, $k = \ldots, -1, 0, 1, \ldots$.

3.1.1 Response of LTI Systems to Complex Exponentials

NEVER EVER FORGET THIS! 6 *Complex exponentials are eigenfunctions of LTI systems:*

$$\varphi(t) = e^{st} \text{ for any complex constant } s$$

Proof:

$\Rightarrow$ e^{st} is an eigenfunction whenever the following is finite:

$$H(s) = \int_{-\infty}^{\infty} e^{-s\tau} h(\tau) d\tau$$

$\Rightarrow$ $H(s)$ is the eigenvalue corresponding to e^{st}:

$$e^{st} \xrightarrow{\text{LTI } S} H(s)\, e^{st}$$

Ex: $x(t) = a_1e^{s_1t} + a_2e^{s_2t} + a_3e^{s_3t}$ is the input to an LTI system. The impulse response of the system is $h(t)$. Determine the output.

3.2 Fourier series representation of CT periodic signals

3.2.1 Periodic signals as sums of complex exponentials

Remember that

- A signal $x(t)$ with period T satisfies

$$x(t) =$$

- Fundamental period T_0:
- Fundamental frequency ω_0 :
- Two basic CT signals periodic with freq. ω_0 (period T_0):

Consider the set of harmonically related complex exponentials : $\varphi_k(t) = e^{jk\omega_0 t} \quad , k = 0, \pm 1, \pm 2, ...$
These signals have a common period T_0. Now consider a linear combination of them.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T_0}t}$$

is also ....

Fourier series representation: For a periodic signal $x(t)$, a representation in the form of

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

is called the Fourier series representation of the signal.

NEVER EVER FORGET THIS! 7 *Fourier series represents a periodic signal as a weighted sum of complex exponentials:*

$$x(t) =$$

- Terms with $k = \pm 1$: Fundamental components
- Terms with $k = \pm N$: N^{th} harmonic components

Ex: Consider a periodic signal whose Fourier series representation is given by $x(t) = \sum_{k=-2}^{2} a_k e^{jk2\pi t}$ with $a_0 = 1$, $a_1 = a_{-1} = 3/4$, $a_2 = a_{-2} = -1/2$.

3.2.2 Determination of Fourier Series Coefficients

Q: Assuming that a given periodic signal can be represented with Fourier series, how can we determine the coefficients a_k?

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}}$$

A:

$$\boxed{a_k = \qquad\qquad\qquad\qquad}$$

- The coefficients $\{a_k\}$ are called the Fourier series coefficients of $x(t)$.
- The limits of integral can be $\int_{t_i}^{t_i+T_0}$.

Ex: Fourier series coefficients of the signal $x(t) = sin(\omega_0 t)$?

Ex: Periodic square wave defined over one period T as follows:

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < \frac{T}{2} \end{cases}.$$

Fourier series coefficients of $x(t)$?

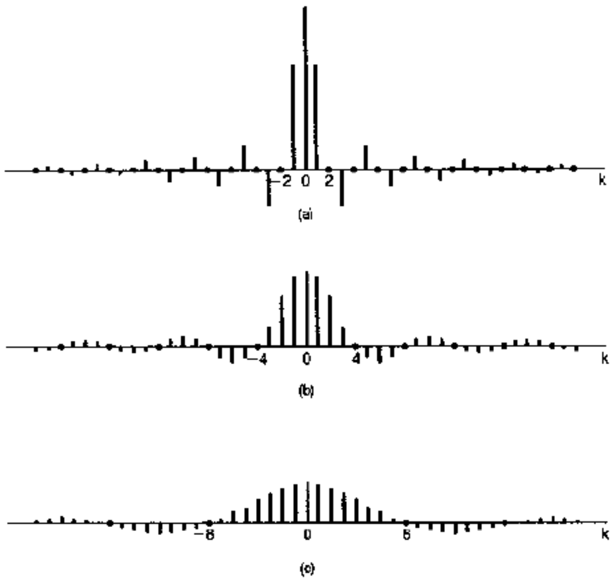

Figure 3.7 Plots of the scaled Fourier series coefficients Ta_k for the periodic square wave with T_1 fixed and for several values of T. (a) $T = 4T_1$; (b) $T = 8T_1$; (c) $T = 16T_1$. The coefficients are regularly spaced samples of the envelope $(2\sin\omega T_1)/\omega$, where the spacing between samples, $2\pi/T$, decreases as T increases.

Figure 3.1: Fourier series coefficients of the periodic square wave

Ex: [Challenge yourself!] Fourier series coefficients of the periodic impulse train?

Ex: [Challenge yourself!] Fourier series coefficients of $x(t) = 1+\sin(\omega_0 t)+2\cos(\omega_0 t)+\cos(2\omega_0 t+\frac{\pi}{4})$?

Summary:

For a periodic CT signal $x(t)$ with a fundamental period $T_0 = \frac{2\pi}{\omega_0}$, the Fourier series representation, if it exists, is

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}}$$

with the Fourier series coefficients $\{a_k\}$ given by

$$\boxed{a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt}$$

where the integration is over any period of the signal.

3.2.3 Existence and Convergence of CT Fourier Series

Dirichlet Conditions: The Fourier Series decomposition of a periodic signal $x(t)$ is possible if the following sufficient conditions hold:

- $x(t)$ is absolutely integrable over any period $\int_0^T |x(t)|dt < \infty$
- $x(t)$ has a finite number of maxima and minima within a period
- $x(t)$ has a finite number of discontinuities within a period.

Almost all physical signals of interest in engineering satisfy the Dirichlet conditions.

Dirichlet conditions guarantee that

$$\lim_{N\to\infty} e_N(t) = 0 \text{ for almost all } t$$

where $e_N(t) = |x(t) - x_N(t)|$ is the error signal with $x_N(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$.

Examples of some pathological signals that do not satisfy Dirichlet conditions:

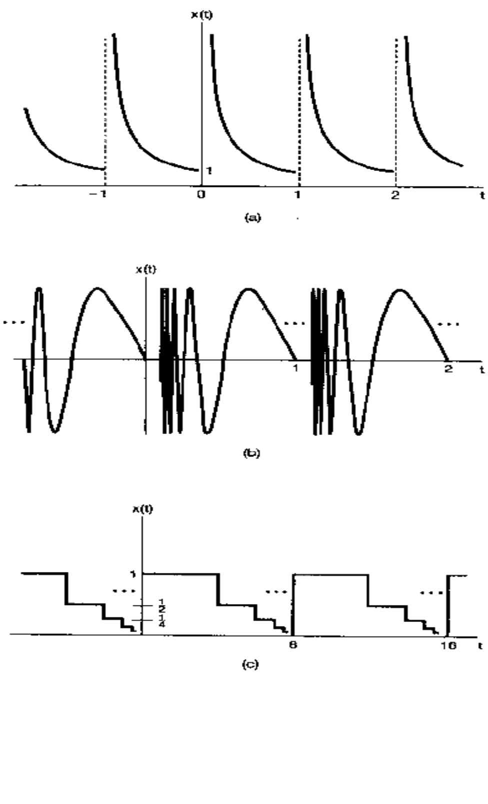

Figure 3.8 Signals that violate the Dirichlet conditions: (a) the signal $x(t) = 1/t$ for $0 < t \leq 1$, a periodic signal with period 1 (this signal violates the first Dirichlet condition); (b) the periodic signal of eq. (3.57), which violates the second Dirichlet condition; (c) a signal periodic with period 8 that violates the third Dirichlet condition [for $0 \leq t < 8$, the value of $x(t)$ decreases by a factor of 2 whenever the distance from t to 8 decreases by a factor of 2; that is, $x(t) = 1$, $0 \leq t < 4$, $x(t) = 1/2$, $4 \leq t < 6$, $x(t) = 1/4$, $6 \leq t < 7$, $x(t) = 1/8$, $7 \leq t < 7.5$, etc.].

Alternative Condition: The Fourier Series decomposition of a periodic signal $x(t)$ is possible if

$$\int_0^T |x(t)|^2 dt < \infty \quad \text{(finite energy over a period)}$$

Finite energy condition guarantees that

$$\lim_{N\to\infty} \int_{T_0} e_N^2(t)dt = 0 \;\; \text{(error energy)}$$

3.3 Properties of CT Fourier series representation

To discuss the CT Fourier series properties, suppose we have two periodic signals $x(t)$ and $y(t)$, both periodic with the same period T_0, and their FS coefficients are a_k and b_k, i.e.

$$x(t) \longleftrightarrow a_k \qquad \text{and} \qquad y(t) \longleftrightarrow b_k.$$

3.3.1 Response through an LTI System

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \xrightarrow{\text{LTI } S} y(t) =$$

Ex: The following signal is input to an LTI system: $x(t) = \sum_{k=-2}^{2} a_k e^{jkt}$, with $a_0 = 1$, $a_1 = a_{-1} = \frac{3}{4}$, $a_2 = a_{-2} = -\frac{1}{2}$. About the system, we know that $H(0) = 0$, $H(1) = H(-1) = 2$, and $H(2) = H(-2) = 3$. What is the response of the system?

3.3.2 Linearity

$ax(t) + by(t) \longleftrightarrow$

3.3.3 Time Shifting

$x(t - t_0) \longleftrightarrow$

3.3.4 Time Reversal

$x(-t) \longleftrightarrow$

3.3.5 Time Scaling

$x(at) \longleftrightarrow$ a is a nonzero real number

3.3.6 Symmetry for real signals

If $x(t)$ is a real periodic signal, $a_k^* = a_{-k}$

Proof:

As a result, we have the following alternative form when $x(t)$ is real:

3.3.7 Even and Odd Signals

- If $x(t)$ is *even*, i.e. $x(t) = x(-t)$, then $a_k =$

  ⇒ In this case, alternative form for the representation:

- If $x(t)$ is *odd*, i.e. $x(t) = -x(-t)$, then

⇒In this case, alternative form for the representation:

3.3.8 Differentiation

$\frac{d}{dt}x(t) \longleftrightarrow$

3.3.9 Multiplication (Modulation) property

$x(t)y(t) \longleftrightarrow ....$

- **Multiplication in time** corresponds to **convolution of FS coefficients**.

3.3.10 Parseval's Relation

$\frac{1}{T_0}\int_0^{T_0} |x(t)|^2 dt =$

Interpretation: Average power in a periodic signal equals to ...

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l}x(t)\\y(t)\end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t)$, $\alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

3.4 Fourier Series Representation of DT Periodic Signals

We have so far discussed the Fourier series representation of CT signals and its importance. We will now extend these results and observations to discrete-time.

3.4.1 Response of DT LTI Systems to Complex Exponentials

- What is the response to a DT complex exponential $x[n] = e^{j\Omega_0 n}$?

$\Rightarrow$ Similar to the CT case, DT complex exponentials are also eigenfunctions of DT LTI systems.

- Response to linear combination of complex exponentials:

$\Rightarrow$ Main motivation for the use of Fourier series and transforms in the analysis of LTI systems

3.4.2 DT Fourier Series Representation

Recall that for continuous-time complex exponentials,

- $x(t) = e^{j\omega_0 t}$ is always periodic with period $T = \frac{2\pi}{|\omega_0|}$
- the set of complex exponentials that are periodic with T are given by

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk\frac{2\pi}{T}t}, \quad \text{for } k = 0, \pm 1, \pm 2, ...$$

For discrete-time complex exponentials,

- $x[n] = e^{j\Omega_0 n}$ is periodic only if Ω_0 is in the form
- the period of $x[n] = e^{j\frac{2\pi}{N}n}$ is

- the set of complex exponentials that are periodic with N are given by

$$\phi_k[n] = e^{jk\frac{2\pi}{N}n}, \quad \text{for } k = 0, \pm 1, \pm 2, ...$$

Difference between CT and DT harmonically related complex exponentials:

- $e^{jk\omega_0 t}$ is always distinct for different k

- Is $e^{jk\frac{2\pi}{N}n}$ with different k distinct?

A more general periodic DT signal with period N:

- A representation of this form is called the **Fourier series** representation of periodic DT signal $x[n]$ with fundamental period N.
- The summation index k varies over a range of N successive integers beginning with an arbitrary value k_0. This is the meaning of the notation $\sum_{k=<N>}$.
- Coefficients a_k:
- Fundamental, N^{th} harmonic, and DC (constant) components of the FS representation are defined in the same way as in the CT FS representation.

Determination of DTFS coefficients

How to obtain a_k's for a given periodic signal $x[n]$?

Existence of DTFS

Unlike the existence of CTFS, the existence of DTFS is worry-free since it involves finite series. Hence DTFS representation exists for *all* finite-valued periodic DT signals.

Let's now summarize the DTFS representation and compare & contrast with CTFS representation:

$$\boxed{x[n] = \sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n}}$$

$$\boxed{a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\frac{2\pi}{N}n}}$$

Ex: Is it intuitive that we only need N FS coefficients to represent a DT periodic signal with fundamental period N?

Ex: [Challenge yourself!] Let's define the vector $\phi_k = \begin{bmatrix} \phi_k[0] \\ \phi_k[1] \\ \vdots \\ \phi_k[N-1] \end{bmatrix}$. Does $\{\phi_1, \phi_2, \ldots, \phi_N\}$ form a basis for periodic signals $x[n]$ with period N, i.e. $\mathbf{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$? Are they orthogonal/orthonormal? Note that another way to determine the DTFS coefficients a_k is to solve a set of N linear equations given by $x[n] = \sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n}$ for $n = 1, \ldots, N$. From the above discussion, there is a unique solution (hence DTFS coefficients exist uniquely.)

Ex: Find FS representation of $x[n] = \cos(\frac{2\pi}{5}n)$.

Ex: [Challenge yourself!] Repeat for $x[n] = \cos(\frac{2\pi}{N}Mn)$ for arbitrary integer M.

Ex: [Challenge yourself!] Repeat for DT periodic square wave with N defined over one period $-\frac{N}{2} - 1 \leq n \leq \frac{N}{2}$ as follows: $x[n] = \begin{cases} 1, & -N_1 \leq n \leq N_1 \\ 0, & \text{otherwise.} \end{cases}$

Ex: (**DTFS and LTI systems**) Consider a DT LTI system with a known frequency response $H(\Omega)$. Find the output when the system is excited by $x[n] = 1 + \sin(\frac{2\pi}{N}n) + \cos(3\frac{2\pi}{N}n)$.

3.5 Properties of DTFS

To discuss the DT Fourier series properties, suppose we have two periodic signals $x[n]$ and $y[n]$, both periodic with the same period N, and their DTFS coefficients are a_k and b_k, i.e.

$$x[n] \longleftrightarrow a_k \qquad \text{and} \qquad y[n] \longleftrightarrow b_k.$$

There are strong similarities between the properties of CTFS and DTFS. This can be seen from the below table that summarizes the properties of DTFS. We will only discuss a few of these properties, including the important ones that are different.

3.5.1 Periodicity

a_k's are periodic with

3.5.2 Differencing

$x[n] - x[n-1] \longleftrightarrow$

3.5.3 Multiplication property

$x[n]y[n] \longleftrightarrow$ ....

- **Multiplication in time** corresponds to **convolution of FS coefficients**.

3.5.4 Parseval's Relation

$\frac{1}{N}\sum_{n=<N>} |x[n]|^2 =$

Interpretation:

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n}x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m}a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$

Figure 3.2: Properties of DT FS.