HW5 SOLUTIONS

$x \equiv$ do not care

Q1-) a-) $G(s) = C(sI-A)^{-1}B = \dfrac{C\,\mathrm{adj}(sI-A)B}{\det(sI-A)}$

$$= \frac{1}{\det(sI-A)}\begin{bmatrix}1 & 0 & 1\end{bmatrix}\underbrace{\begin{bmatrix}x & a(s) & x\\ x & x & x\\ x & b(s) & x\end{bmatrix}}_{\mathrm{adj}(sI-A)}\begin{bmatrix}0\\1\\0\end{bmatrix} = \frac{a(s)+b(s)}{\det(sI-A)}$$

$$sI-A = \begin{bmatrix}s & +2 & 0\\ 0 & s & -1\\ -1 & 1 & s\end{bmatrix} \qquad \det(sI-A) = s\begin{vmatrix}s & -1\\ 1 & s\end{vmatrix} - 2\begin{vmatrix}0 & -1\\ -1 & s\end{vmatrix}$$

$$= s(s^2+1) + 2$$

$$= s^3+s+2$$

$$\left.\begin{aligned} a(s) &= -\begin{vmatrix}2 & 0\\ 1 & s\end{vmatrix} = -2s \\ b(s) &= -\begin{vmatrix}s & 2\\ -1 & 1\end{vmatrix} = -(s+2) = -s-2\end{aligned}\right\} \Rightarrow a(s)+b(s) = -3s-2$$

$$\Rightarrow \boxed{G(s) = \frac{-3s-2}{s^3+s+2}}$$

b-) $Q = \begin{bmatrix}B & AB & A^2B\end{bmatrix} = \begin{bmatrix}0 & -2 & 0\\ 1 & 0 & -1\\ 0 & -1 & -2\end{bmatrix}$ $\qquad \det(Q) = -(-2)\begin{vmatrix}1 & -1\\ 0 & -2\end{vmatrix}$

$$= 2(-2) = -4 \neq 0$$

$\Rightarrow$ system is controllable.

c-) $A_{CL} = A - BK = \begin{bmatrix}0 & -2 & 0\\ 0 & 0 & 1\\ 1 & -1 & 0\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}k_1 & k_2 & k_3\end{bmatrix}$

$$= \begin{bmatrix}0 & -2 & 0\\ 0 & 0 & 1\\ 1 & -1 & 0\end{bmatrix} - \begin{bmatrix}0 & 0 & 0\\ k_1 & k_2 & k_3\\ 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & -2 & 0\\ -k_1 & -k_2 & -(k_3-1)\\ 1 & -1 & 0\end{bmatrix}$$

$$\Rightarrow sI - A_{CL} = \begin{bmatrix}s & 2 & 0\\ k_1 & s+k_2 & k_3-1\\ -1 & 1 & s\end{bmatrix}$$

$$\det(sI - A_{CL}) = s \begin{vmatrix} s+k_2 & k_3-1 \\ 1 & s \end{vmatrix} - 2 \begin{vmatrix} k_1 & k_3-1 \\ -1 & s \end{vmatrix}$$

$$= s\left(s^2 + k_2 s - (k_3-1)\right) - 2\left(k_1 s + k_3 - 1\right)$$

$$= s^3 + k_2 s^2 - (k_3-1)s - 2k_1 s - 2(k_3-1)$$

$$= s^3 + k_2 s^2 + (1 - k_3 - 2k_1)s - 2(k_3-1)$$

Desired char. polynomial is

$$d_{desired}(s) = (s+1)\left((s+1)^2+1\right)$$

$$= (s+1)(s^2+2s+2) = s^3 + 2s^2 + 2s + s^2 + 2s + 2$$

$$= s^3 + 3s^2 + 4s + 2$$

$\Rightarrow$ $k_2 = 3$

$1 - k_3 - 2k_1 = 4 \Rightarrow 1 - 2k_1 = 4 \Rightarrow k_1 = -3/2$

$-\cancel{2}(k_3 - 1) = \cancel{2} \Rightarrow k_3 = 0$

$\Rightarrow u = r - Kx = r + \dfrac{3}{2}x_1 - 3x_2$ $\leftarrow$ state feedback rule

Q2 a-) Write the controllability matrix

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}$$

$\Rightarrow \det(\mathcal{C}) = -2 - (-2) = 0 \Rightarrow$ The system is not completely controllable.

b-) Closed Loop A matrix is given as

$$A_c = A - BK = \begin{bmatrix} -4 & 2 \\ 2 & -4 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

$$= \begin{bmatrix} -4-k_1 & 2-k_2 \\ 2-k_1 & -4-k_2 \end{bmatrix}$$

$\Rightarrow$ Characteristic eqn. for the CL system is given as

$$\det(sI - A_c) = 0.$$

$$sI - A_c = \begin{bmatrix} s+k_1+4 & k_2-2 \\ k_1-2 & s+k_2+4 \end{bmatrix}$$

$$\Rightarrow \det(sI - A_c) = s^2 + (k_1+k_2+8)s + (k_1+4)(k_2+4) - (k_1-2)(k_2-2)$$

$$= s^2 + (k_1+k_2+8)s + \cancel{k_1k_2} + 4k_1 + 4k_2 + 16 - \cancel{k_1k_2} + 2k_1 + 2k_2 - 4$$

$$= s^2 + (k_1+k_2+8)s + (6k_1+6k_2+12)$$

In order to place the CL poles at -4 and -4 we need to have

$\Rightarrow$

$\det(sI - A_c) = (s+4)^2 = s^2 + 8s + 16$

$\Rightarrow \quad (k_1 + k_2 + 8) = 8 \Rightarrow k_1 + k_2 = 0$

$6k_1 + 6k_2 + 12 = 16 \Rightarrow k_1 + k_2 = 2/3$

These eqn's conot be satisfied simultoneously.

$\Rightarrow$ We conot place the CL poles at $s = -4, -4$.

Since the system is not completely controllable, this is natural.

c-) To place the CL poles at $s = -4$ and $s = -6$ we need to have

$\det(sI - A_c) = (s+4)(s+6) = s^2 + 10s + 24.$

$\Rightarrow \quad k_1 + k_2 + 8 = 10 \Rightarrow k_1 + k_2 = 2$

$6k_1 + 6k_2 + 12 = 24 \Rightarrow k_1 + k_2 = 2$

Since the eqn's are satisfied simultoneously we can plece the CL poles at $s = -4$ and $s = -6$. For any K vector satisfying $k_1 + k_2 = 2$.

e.g's $K = [0 \quad 2]$

$K = [2 \quad 0]$

$K = [1 \quad 1]$ etc.

Q3 a-) Calculate the controllability matrix

$$Q = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$\det(Q) = 4 \neq 0 \Rightarrow$ System is controllable.

b-) Calculate the observability matrix

$$W = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & -20 & -40 \\ +40 & -40 & +20 \\ +80 & 20 & 80 \end{bmatrix}$$

$$\det(W) = -40 \begin{vmatrix} -20 & -40 \\ 20 & 80 \end{vmatrix} + 80 \begin{vmatrix} -20 & -40 \\ -40 & 20 \end{vmatrix}$$

$$= -40(-1600+800) + 80(-400-1600)$$

$$= -40(-800) + 80(-2000)$$

$$= +32000 - 160000 = -128000 \neq 0$$

$\Rightarrow$ System is observable.

c-) Let us calculate the TF of the OL system!

$$G(s) = C(sI-A)^{-1}B = \frac{C\,\mathrm{adj}(sI-A)B}{\det(sI-A)} = \frac{N(s)}{D(s)}$$

$$N(s) = C\,\mathrm{adj}(sI-A)\,B = \begin{bmatrix} 0 & -20 & -40 \end{bmatrix} \underbrace{\begin{bmatrix} x & x & x \\ a_{21} & x & x \\ a_{31} & x & x \end{bmatrix}}_{\mathrm{adj}(sI-A)} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

don't care

$$sI-A = \begin{bmatrix} s & 0 & -1 \\ 2 & s & 1 \\ 0 & -1 & s \end{bmatrix}$$

$$a_{21} = -\begin{vmatrix} 2 & 1 \\ 0 & s \end{vmatrix} = -2s$$

$$a_{31} = +\begin{vmatrix} 2 & s \\ 0 & -1 \end{vmatrix} = -2$$

$$N(s) = -20\,a_{21} - 40\,a_{31}$$

$$= -20(-2s) - 40(-2) = 40s + 80$$

$$N(s) = 40(s+2)$$

$\Rightarrow$ Open Loop System has a zero at $s=-2$

The CL system should be second order

$\Rightarrow$ One of the CL poles should cancel the zero at $s=-2$.

$\Rightarrow$ Desired Closed Loop poles should satisfy

$$D(s) = (s+2)(s^2 + 2\zeta w_n s + w_n^2) = 0$$

where ζ and ω_n are to be selected to have

$t_s = 1$ second $\omega_d = 2$ rad/sec.

$$t_s = \frac{4}{\zeta\omega_n} = 1 \Rightarrow \zeta\omega_n = 4$$

$$\sqrt{1-\zeta^2}\,\omega_n = 2 \Rightarrow (1-\zeta^2)\omega_n^2 = 4$$

$$\Rightarrow \omega_n^2 - \zeta^2\omega_n^2 = 4$$

$$\Rightarrow \omega_n^2 - 16 = 4$$

$$\Rightarrow \omega_n^2 = 20 \Rightarrow \omega_n = 2\sqrt{5}$$

$\Rightarrow$ Desired Closed Loop Characteristic Polynomial is

$$D(s) = (s+2)(s^2+8s+20)$$

$$= s^3 + 8s^2 + 20s + 2s^2 + 16s + 40$$

$$= s^3 + 10s^2 + 36s + 40 \quad \Big\} \text{ Desired Polynomial.}$$

Closed Loop A matrix is

$$A_C = A - BK = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 & k_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -k_1 & -k_2 & 1-k_3 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$sI - A_c = \begin{bmatrix} s+k_1 & k_2 & k_3-1 \\ +2 & s & 1 \\ 0 & -1 & s \end{bmatrix}$$

$$\det\left(sI - A_c\right) = -(-1)\begin{vmatrix} s+k_1 & k_3-1 \\ 2 & 1 \end{vmatrix} + s\begin{vmatrix} s+k_1 & k_2 \\ 2 & s \end{vmatrix}$$

$$= s+k_1 - 2(k_3-1) + s(s^2+k_1 s - 2k_2)$$

$$= s^3 + k_1 s^2 + (1-2k_2)s + k_1 - 2(k_3-1)$$

$$= s^3 + 10s^2 + 36s + 40$$

$$\Rightarrow \quad k_1 = 10$$

$$1 - 2k_2 = 36 \Rightarrow k_2 = \frac{-35}{2}$$

$$k_1 - 2(k_3-1) = 40 \Rightarrow 10 - 2(k_3-1) = 40$$

$$\Rightarrow \quad -30 = 2(k_3-1)$$

$$\Rightarrow k_3 - 1 = -15$$

$$\Rightarrow k_3 = -14$$

$$\Rightarrow K = \begin{bmatrix} 10 & -\frac{35}{2} & -14 \end{bmatrix}$$

d-) The Closed Loop System Has the SS representation

$$\dot{x} = \underbrace{(A - BK)}_{A_c} x + \underbrace{B}_{B_c} u$$

$$y = \underbrace{C}_{C_c} x$$

$$A_c = A - BK = \begin{bmatrix} -10 & +\frac{35}{2} & 15 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$C_c = C = \begin{bmatrix} 0 & -20 & -40 \end{bmatrix}$$

Check Observability using observability matrix;

$$W = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 \; + & -20 & -40 \\ 40 \; - & -40 & 20 \\ -320 \; + & 720 & 640 \end{bmatrix}$$

$$\det(W) = -40 \begin{vmatrix} -20 & -40 \\ 720 & 640 \end{vmatrix} - 320 \begin{vmatrix} -20 & -40 \\ -40 & 20 \end{vmatrix}$$

$$= -40(-12800 + 28800) - 320(-400 - 1600)$$

$$= -4000\left[(-128 + 288) + 8(-4 - 16)\right]$$

$$= -4000\left[160 - 160\right] = 0 \Rightarrow$$ The Closed Loop system is unobservable.

Due to the <u>pole-zero cancellation</u>, CL system is unobservable