EE 301
Linear Time-Invariant Systems

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Outline

Linear Time-Invariant (LTI) Systems

Systems that are both linear and time-invariant

Why our focus?

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- Many physical processes are LTI.
- There are powerful mathematical tools to study/analyze/design LTI systems.

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Why our focus?

- Many physical processes are LTI.
- There are powerful mathematical tools to study/analyze/design LTI systems.

Never ever forget this!

An LTI system is uniquely described by its response to a unit impulse, i.e. **impulse response**.

Representation of DT Signals in terms of ....?....

Any DT signal can be written as the sum of

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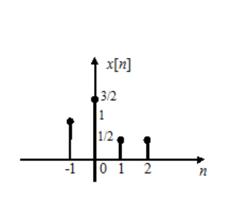
Any DT signal can be written as the sum of IMPULSES.

Example

$x[n] = \ldots$



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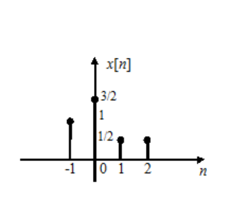

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Interpretation:

$$\boxed{x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]}$$

Interpretation: Any DT signal can be represented as a weighted sum of shifted impulses $\delta[n-k]$, where the weights are determined by the signal $x[n]$.

Characterization of LTI Systems in Terms of Impulse Response

Never ever forget this!

One and only one thing that changes from one LTI system to another is its response to the unit impulse.

Consider a DT LTI system whose output to a unit impulse $\delta[n]$ is $h[n]$: $\Rightarrow h[n]$: impulse response of the system

What is the response to an arbitrary input $x[n]$?

Response of an LTI system (Convolution Sum)

$$y[n] =$$

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$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n] = (x * h)[n]$$

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]}$$

Interpretation:

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]}$$

Interpretation: Response of the LTI system, $y[n]$, is the weighted sum of shifted unit impulse responses $h[n-k]$, whose weights are determined by the input signal $x[n]$.

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Never ever forget this!

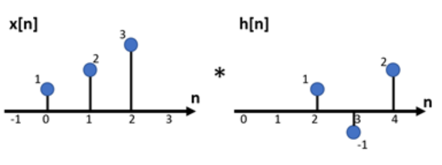
Response of the **LTI system** is given by the convolution of the input signal with the impulse response of the system.

Note that asterisk $*$ denotes the discrete convolution operation. Let $x[n]$ and $v[n]$ be two DT signals. Then their convolution is defined as

$$x[n] * v[n] = \sum_{k=-\infty}^{\infty} x[k]v[n-k]$$

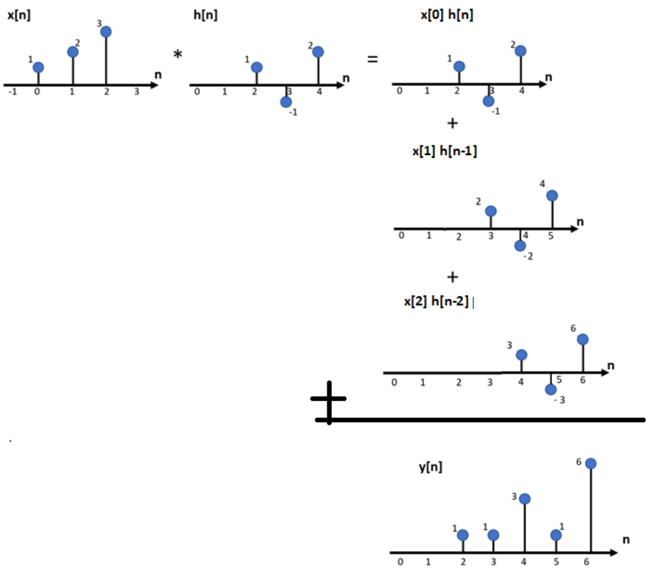
Example

Computing convolution using two approaches



Approach #1: (suitable for short length signals)

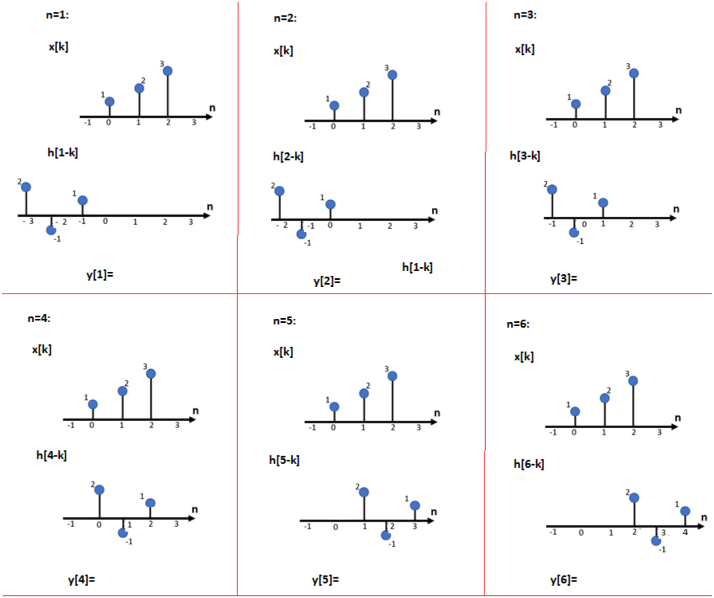
$$y[n] = x[n]*h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = ...+x[0]h[n]+x[1]h[n-1]+$$


x[n]
h[n]
x[0] h[n]
x[1] h[n-1]
x[2] h[n-2]
y[n]
n

Approach #2: (sliding window method)

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

1. View $x[k]$ and $h[n-k]$ as functions of k with n fixed (for example, $n = n_0$)
2. Multiply the sequence $h[n_0 - k]$ with $x[k]$ for all values of k, and sum the resulting sequence over k
3. This gives the output value at $n = n_0$. Repeat this for all n. This will be equivalent to *sliding* the sequence $h[n-k]$ over $x[k]$.


n=1:
x[k]
h[1-k]
y[1]=
n=2:
x[k]
h[2-k]
h[1-k]
y[2]=
n=3:
x[k]
h[3-k]
y[3]=
n=4:
x[k]
h[4-k]
y[4]=
n=5:
x[k]
h[5-k]
y[5]=
n=6:
x[k]
h[6-k]
y[6]=

Example

Input signal: $x[n] = \alpha^n u[n]$, $0 < \alpha < 1$
Impulse response: $h[n] = u[n]$
Response of the LTI system?

Example (Challenge yourself!)

TRUE or FALSE?

- If $y[n] = x[n] * h[n]$, is it true that $y[2n] = x[2n] * h[2n]$
- If $y[n] = x[n] * h[n]$, is it true that $y[-n] = x[-n] * h[-n]$

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- Remember the basic properties of the unit impulse:

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau =$$

Representation of CT signals in terms of impulses

- Remember the basic properties of the unit impulse:

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t)$$

- OR, equivalently, consider a staircase approximation to a CT signal $x(t)$ and express it in terms of $\delta_\Delta(t)$:

$$\boxed{x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau}$$

Interpretation:

$$\boxed{x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau}$$

Interpretation: Any CT signal can be represented as a weighted "sum" of shifted impulses $\delta(t-\tau)$, where the weights are determined by the signal $x(t)$.

Characterization of LTI Systems Using Impulse Resp.

Never ever forget this!

One and only one thing that changes from one LTI system to another is its response to the unit impulse.

To understand how, consider a CT LTI system whose output to $\delta_\Delta(t)$ is $h_\Delta(t)$.
What is the response to an arbitrary input $x(t)$?

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$$y(t) =$$

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Note that asterisk $*$ denotes the continuous *convolution* operation. Let $x(t)$ and $v(t)$ be two CT signals. Then their convolution is defined as

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Example

Input signal: $x(t) = u(t) - u(t-1)$
Impulse response: $h(t) = u(t)$
Response of the LTI system?

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Input signal: $x(t) = u(t) - u(t-1)$
Impulse response: $h(t) = 2[u(t+1) - u(t-2)]$
Response of the LTI system?

Example (Challenge yourself!)

- $x(t) = e^{-\alpha t}u(t)$, $\alpha > 0$, $h(t) = u(t)$, $x(t) * h(t) =?$
- $x(t) = 2(1-t)$ if $0 < t < 1$, and zero elsewhere, $h(t) = u(t) - u(t-1)$, $x(t) * h(t) =?$

Don't get confused!

- $\int_{\tau=-\infty}^{\infty} \delta(\tau) d\tau =$
- $\int_{\tau=-\infty}^{\infty} \delta(t-\tau) d\tau =$
- $x(t)\delta(t) =$
- $x(t)\delta(t-t_0) =$
- $\int_{\tau=-\infty}^{\infty} x(\tau)\delta(t-\tau) d\tau =$
- $\int_{\tau=-\infty}^{\infty} x(\tau)\delta(t-t_0-\tau) d\tau =$
- $x(t) * \delta(t) =$
- $x(t) * \delta(t-t_0) =$

Properties of LTI Systems

P.0 Impulse Response:

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The behavior of an LTI system is completely and uniquely determined by its *impulse response*.

DT: $y[n] =$

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Example

Consider a DT system whose response to the unit impulse

$$h[n] = \begin{cases} 1, & \text{if } n = 0, 1 \\ 0, & \text{otherwise} \end{cases}$$

a) What is the input-output behavior of the LTI system that has this impulse response?

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a) What is the input-output behavior of the LTI system that has this impulse response?

b) Provide input-output behavior of another DT system that is not LTI, but has the same response to the unit impulse?

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$x[n] * h[n] =$

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Interpretation:

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$[x(t) * h_1(t)] * h_2(t) =$

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$[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$

Proof: Exercise
Interpretation:

Properties of LTI Systems

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P.2 Associative Property

$[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$

Proof: Exercise
Interpretation: It does not matter in which order we convolve signals.

Properties of LTI Systems

P.3 Distributive Property

$x(t) * [h_1(t) + h_2(t)] =$

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$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)]$$

Proof: Exercise

Interpretation:

Properties of LTI Systems

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$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)]$$

Proof: Exercise

Interpretation: Convolution distributes over addition.

Example (Challenge yourself!)

$x[n] = \delta[n-5]$, $h[n] = \delta[n] + \delta[n-1]$, $x[n] * h[n] = ?$

Example (Challenge yourself!)

$x[n] = \left(\frac{1}{3}\right)^n u[n] + \delta[n]$, $h[n] = u[n]$, $x[n] * h[n] = ?$

Properties of LTI Systems

P.4 LTI Systems Without Memory

An LTI system is *memoryless* if and only if

$$h[n] =$$

Proof:

Example

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Proof:

Example

Identity system with $h[n] = \delta[n]$

Properties of LTI Systems

P.5 Causality for LTI Systems

An LTI system is *causal* if and only if

$$h[n] =$$

Proof:

Properties of LTI Systems

P.6 Stability for LTI Systems

An LTI system is *stable* if and only if

$$h[n]$$

Proof:

Example

Delay system: $y[n] = x[n - n_0]$ where n_0 is some constant. Stable?

Example

Integrator: $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$. Stable?

Properties of LTI Systems

P.7 Unit-Step Response

Unit-Step Response: Response of a system to the unit step signal

Can we obtain impulse response from step response, or vice versa?
DT:
CT:

$\Rightarrow$

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Unit-Step Response: Response of a system to the unit step signal

Can we obtain impulse response from step response, or vice versa?
DT:
CT:

$\Rightarrow$ The unit step response also fully characterizes an LTI system.

Example (Challenge yourself!)

- Prove that $x(t) * h(t) = \left(\frac{d}{dt}x(t)\right) * g(t)$ where $g(t)$ is the unit step response given by $g(t) = \int_{-\infty}^{t} h(\tau)d\tau$.
- Apply this result to $x(t) = u(t) - u(t - T_1)$ and $h(t) = u(t) - u(t - T_2)$ to obtain $x(t) * h(t)$.

LTI Systems Described by Differential and Difference Equations

Input-output relationship of many physical systems can be described by linear differential or difference equations with constant coefficients (LDECC).

Example

CT and DT examples

- We now introduce some of the basic ideas involved in solving LDECC and later we will learn additional tools (Fourier, Laplace and Z Transforms).
- The difference (differential) equation by itself does not specify a unique solution $y[n]$ to the input $x[n]$. Auxiliary conditions have to specified to completely determine the output signal $y[n]$ ($y(t)$).

Causal LTI systems described by differential equations

A system described by a general Nth order LDECC

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

is a causal LTI system under the condition of initial rest.

Initial rest condition:

Causal LTI systems described by differential equations

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Initial rest condition: If $x(t) = 0$ for $t < t_0$,

$$y(t) = 0 \text{ for } t < t_0.$$

This also implies that

$$\frac{dy(t)}{dt} = \ldots = \frac{d^{N-1}y(t)}{dt^{N-1}} = 0 \text{ for } t < t_0.$$

Example

Consider the following system:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$$

under the condition of initial rest.

1. Block diagram representation of the system?
2. What is the impulse response of this LTI system?
3. (Exercise) What is the response of this LTI system to the input $x(t) = e^{3t}u(t)$?

See Problem 2.56 from Oppenheim to better understand the approach in the general case.

Causal LTI systems described by difference equations

A system described by a general Nth order LDECC

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

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Initial rest condition: If $x[n] = 0$ for $n < n_0$,

$$y[n] = 0 \text{ for } n < n_0.$$

Example

Consider a system whose input-output relationship is

$$y[n] - \frac{1}{4}y[n-2] = x[n]$$

The system is initially at rest.

1. Block diagram representation of the system?
2. What is the impulse response of this LTI system?

See Problem 2.54 & 2.55 from Oppenheim to better understand the approach in the general case.