EE 301
The CT Fourier Transform

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Outline

The Fourier Transform

Fourier Series:

The Fourier Transform

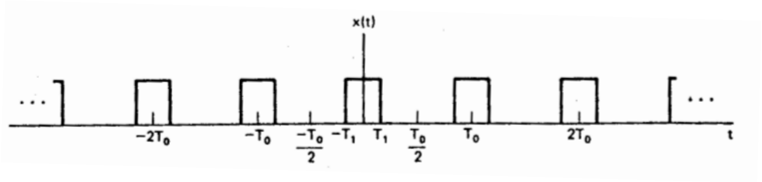
Fourier Series: Representation of periodic signals as sums of complex exponentials

Fourier Transform: Extension of the above idea to aperiodic signals

Representation of aperiodic signals: Fourier transform

An aperiodic signal can be viewed as the limit of a periodic signal when the period becomes arbitrarily large.

Recall that the periodic square wave has Fourier coefficients $a_k = \frac{2T_1}{T_0} \mathrm{sinc}\left(\frac{\mathrm{k}\omega_0 \mathrm{T}_1}{\pi}\right)$



Inspect the Fourier coefficients as T_0 increases (T_1 fixed):

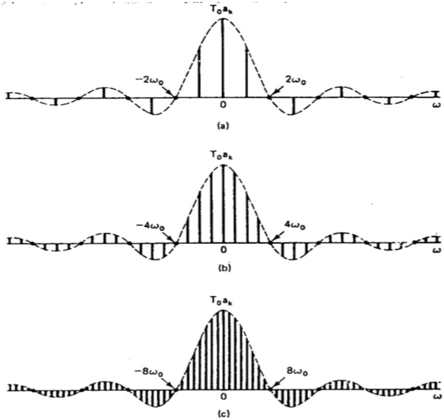


Figure 4.11 Fourier coefficients and their envelope for the periodic square wave: (a) $T_0 = 4T_1$; (b) $T_0 = 8T_1$; (c) $T_0 = 16T_1$.

- $T_0 a_k$ are samples of a continuous envelope: $T_0 a_k = 2T_1 \mathrm{sinc}\left(\frac{\omega \mathrm{T}_1}{\pi}\right)\big|_{\omega=\mathrm{k}\omega_0}$
- As period $T_0 \uparrow$,

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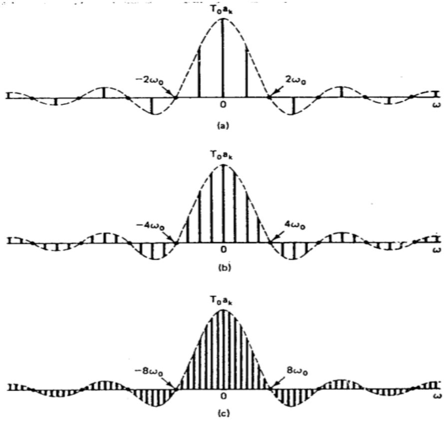

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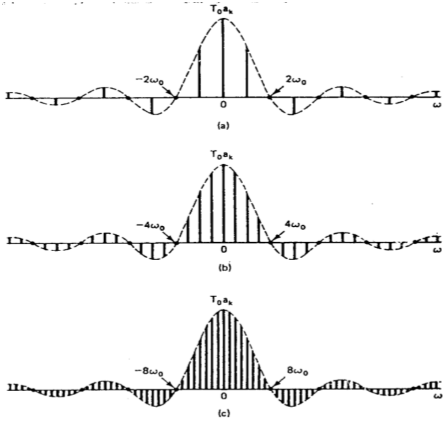

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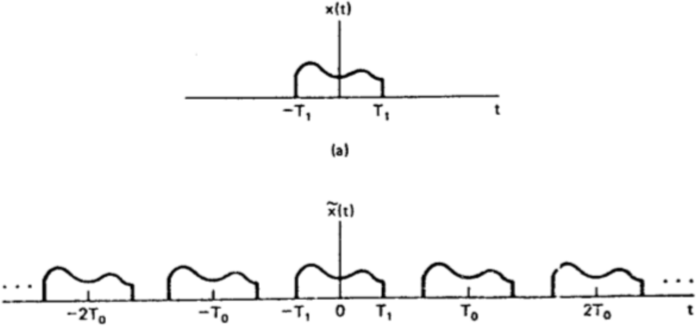
- As period $T_0 \uparrow$, the envelope is sampled with closer spacing.
- As $T_0 \to \infty$, the periodic square wave becomes a single pulse (hence aperiodic) and FS coefficients approach the envelope function.

Formal development of the Fourier transform

This example illustrates the basic idea of the Fourier Transform.

Let's now formally derive the Fourier Transform Representation. For this, consider an aperiodic signal $x(t)$ and a periodic signal $\tilde{x}(t)$ made out of $x(t)$:



The Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \tag{1}$$

with

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \tag{2}$$

Interpretation:

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Interpretation:

- Eqn. (2): Fourier transform of the signal $x(t)$
  Eqn. (1): Inverse Fourier transform

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Interpretation:

- Eqn. (2): Fourier transform of the signal $x(t)$
  Eqn. (1): Inverse Fourier transform
- Eqn. (1): FT representation of an aperiodic signal
  Eqn. (2): Coefficients

Fourier series versus Fourier transform

Representation of signals as linear combinations of complex exponentials

- Periodic signals:

Fourier series versus Fourier transform

Representation of signals as linear combinations of complex exponentials

- Periodic signals: A discrete set of frequencies $k\omega_0$ and weights a_k
- Aperiodic signals:

Fourier series versus Fourier transform

Representation of signals as linear combinations of complex exponentials

- Periodic signals: A <u>discrete</u> set of frequencies $k\omega_0$ and weights a_k
- Aperiodic signals: <u>Continuum</u> of frequencies, ω, and weights $X(j\omega)\frac{d\omega}{2\pi}$

$\Rightarrow$ $X(j\omega)$: spectrum of $x(t)$

Example

$x(t) = e^{-at}u(t)$, where $a > 0$. Find $X(j\omega)$.

Example

$X(j\omega) = \sum_i \frac{\alpha_i}{a_i + j\omega}$, with $Re\{a_i\} > 0$. Find $x(t)$.

Example

Rectangular pulse $x(t) = \frac{1}{2T_1}\mathrm{rect}\left(\frac{\mathrm{t}}{2\mathrm{T}_1}\right)$.

- Find $X(j\omega)$.
- FT of $\delta(t)$?

Example

$X(j\omega) = \frac{1}{2\omega_1}\mathrm{rect}\left(\frac{\omega}{2\omega_1}\right)$. Find $x(t)$.

Example (Challenge yourself!)

$x(t) = e^{-a|t|}$, where $a > 0$. Find $X(j\omega)$.

Convergence of Fourier Transforms

We derived the FT pair for aperiodic signals with finite duration. In fact, FT and its inverse are valid for an extremely broad class of signals (possibly infinite duration).

Convergence means $x(t)$ can be written as $\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$.

Like periodic signals, a set of sufficient conditions called Dirichlet conditions exists:

- $x(t)$ is absolutely integrable: $\int_{-\infty}^{\infty} |x(t)|dt < \infty$
- $x(t)$ has a finite number of maxima and minima within any finite interval
- $x(t)$ has a finite number of discontinuities within any finite interval. Each discontinuity must also be finite.

Alternative Condition: Another sufficient condition for existence and convergence of FT of $x(t)$:

- $x(t)$ has finite energy (i.e. square integrable)

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$$

Example (Challenge yourself!)

Show that the examples we discuss satisfy these sufficient conditions.

Fourier Transform of Periodic Signals

We will now develop Fourier Transform for periodic signals. Hence both periodic and aperiodic signals can be studied within a unified context using Fourier transform.

- Consider a signal $x(t)$ with Fourier Transform $X(j\omega)$ that is a single impulse of strength 2π at $\omega = \omega_0$:
- More generally, consider a linear combination of impulses equally spaced in frequency:

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- More generally, consider a linear combination of impulses equally spaced in frequency:

$$x(t) = \sum_{-\infty}^{\infty} a_k e^{jk\omega_0 t} \longleftrightarrow X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

Example

$x(t) = cos(\omega_0 t)$

Example

$x(t) = sin(\omega_0 t)$

Example

The periodic square wave (again!)

Example

Periodic impulse train $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$

Properties of the Fourier Transform

If $X(j\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$, then $x(t)$ and $X(j\omega)$ form a Fourier transform pair:

$$x(t) \longleftrightarrow X(j\omega)$$

P.1 Linearity:
$x_1(t) \longleftrightarrow X_1(j\omega)$
$x_2(t) \longleftrightarrow X_2(j\omega)$
$ax_1(t) + bx_2(t) \longleftrightarrow$

$$x(t) \longleftrightarrow X(j\omega)$$
$$x(t - t_0) \longleftrightarrow$$