EE302 HW1 SOLUTIONS

Q1-) Input: y_1
a-) Output: y_3

$$M_{body}\ddot{y}_3 = -k_2(y_3-y_2) - b(\dot{y}_3-\dot{y}_2)$$
$$M_{tire}\ddot{y}_2 = -k_2(y_2-y_3) - b(\dot{y}_2-\dot{y}_3) - k_1(y_2-y_1)$$

$$\Rightarrow \underbrace{\left(M_{body}s^2+bs+k_2\right)}_{\triangleq D_1(s)} Y_3(s) = (bs+k_2)Y_2(s)$$
$$\underbrace{\left(M_{tire}s^2+bs+k_1+k_2\right)}_{\triangleq D_2(s)} Y_2(s) = (bs+k_2)Y_3(s)+k_1Y_1(s)$$

$$\Rightarrow \quad D_1Y_3 = (bs+k_2)Y_2 \quad \Rightarrow \quad D_1D_2Y_3 = (bs+k_2)\left((bs+k_2)Y_3+k_1Y_1\right)$$
$$D_2Y_2 = (bs+k_2)Y_3 + k_1Y_1$$

$$\Rightarrow \left(D_1D_2-(bs+k_2)^2\right)Y_3 = (bs+k_2)k_1Y_1$$

$$\Rightarrow \frac{Y_3(s)}{Y_1(s)} = \frac{k_1(bs+k_2)}{D_1(s)D_2(s)-(bs+k_2)^2}$$

b-) System is 4^{th} order (order of the denominator polynomial

$$x \triangleq \begin{bmatrix} y_3 \\ \dot{y}_3 \\ y_2 \\ \dot{y}_2 \end{bmatrix} \qquad \dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_2}{M_{body}} & -\frac{b}{M_{body}} & \frac{k_2}{M_{body}} & \frac{b}{M_{body}} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{M_{tire}} & \frac{b}{M_{tire}} & -\frac{k_1+k_2}{M_{tire}} & -\frac{b}{M_{tire}} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k_1}{M_{tire}} \end{bmatrix} \underbrace{u}_{y_1}$$

$$\underbrace{y}_{\text{output } y_3} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x$$

Q2-) a-)

$$v_f = K_\theta(\theta_r - \theta_o)$$
$$v_f = L_f \frac{di_f}{dt} + R_f i_f$$
$$v_g = K_g i_f$$
$$v_g - v_b = \underbrace{(L_g + L_m)}_{L_t} \frac{di_m}{dt} + \underbrace{(R_g + R_m)}_{R_t} i_m$$
$$\dot{\theta}_o = \omega_o$$
$$v_b = K_b \omega_o$$
$$\tau_o = K_\tau i_m$$
$$J\dot{\omega}_o = \tau_o - B\omega_o$$

$$\boxed{\begin{array}{l} \theta_r : \text{input} \\ \theta_o : \text{output} \end{array}}$$

$$L_t \triangleq L_g + L_m$$
$$R_t \triangleq R_g + R_m$$

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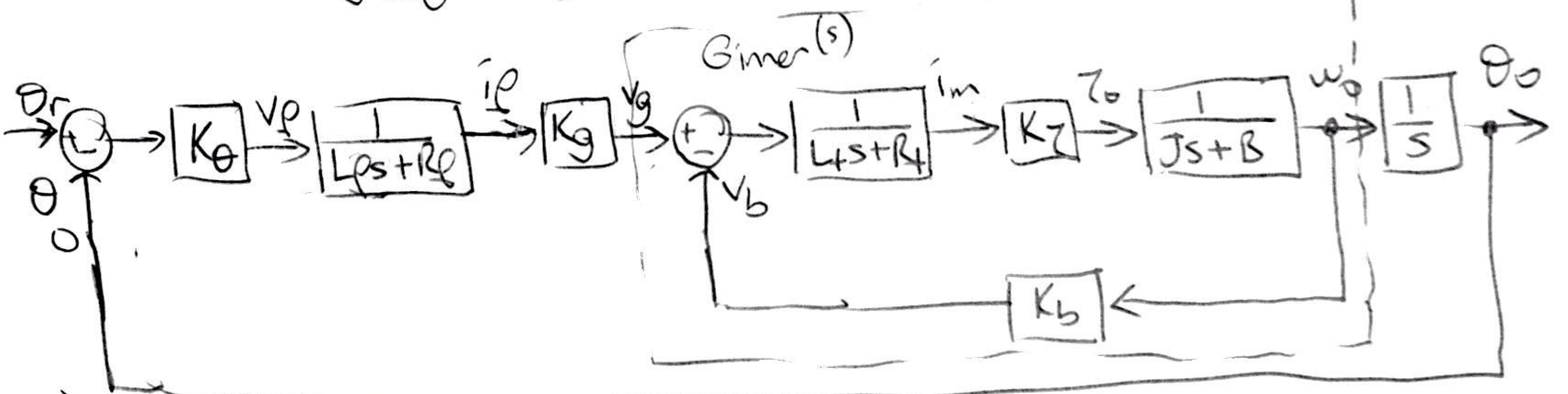


b-)

$$G_{inner}(s) = \frac{\dfrac{K_\tau}{(L_t s + R_t)(Js + B)}}{1 + \dfrac{K_\tau K_b}{(L_t s + R_t)(Js + B)}} = \frac{K_\tau}{(L_t s + R_t)(Js + B) + K_\tau K_b}$$

$$G(s) = \frac{\theta_o(s)}{\theta_r(s)} = \frac{\dfrac{K_\theta K_g K_\tau}{s(L_f s + R_f)\left((L_t s + R_t)(Js + B) + K_\tau K_b\right)}}{1 + \dfrac{K_\theta K_g K_\tau}{s(L_f s + R_f)\left((L_t s + R_t)(Js + B) + K_\tau K_b\right)}}$$

$$K_\theta K_g K_\tau$$

c-) System is 4^{th} order. Hence the minimum # of state variables to use is 4.

d-) i. is not possible since θ_r is input, it cannot be in the state.

ii. is possible.

iii. is not possible since there is no equation to express $\dddot{\theta}_o$ in terms of state variables. Also the output equation cannot be written with this state definition.

iv. is not possible since $\tau_o = K_\tau i_m$. Hence two state variables are linear combinations of each other. Hence the effective number of state variables is 3. We need at least 4 state variables which cannot be written as a linear combination of each other.

As a result, only ii. is suitable for writing a state-space representation for this system.

$$\Rightarrow \quad x \triangleq \begin{bmatrix} i_f & i_m & \theta_o & \dot{\theta}_o \end{bmatrix}^T$$

d-) (Continued)

$$L_f \dot{i}_f = K_\theta(\theta_r - \theta_o) - R_f i_f$$

$$L_t \dot{i}_m = \underbrace{K_g i_f}_{v_g} - \underbrace{K_b w_o}_{v_b} - R_t i_m$$

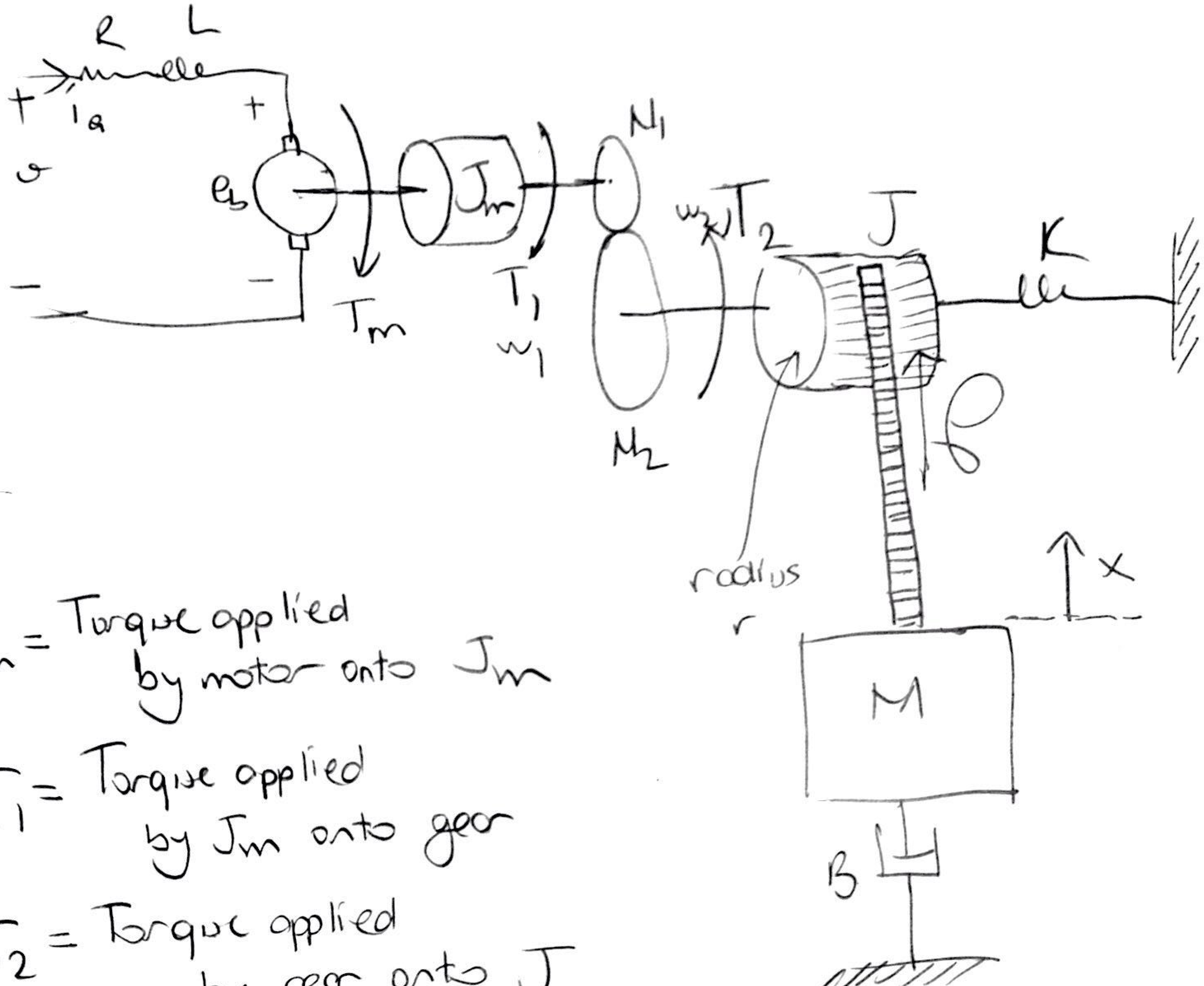
$$\dot{\theta}_o = w_o$$

$$J \dot{w}_o = \underbrace{K_T i_m}_{T_o} - B w_o$$

$$\dot{x} = \begin{bmatrix} -\frac{R_f}{L_f} & 0 & -\frac{K_\theta}{L_f} & 0 \\ \frac{K_g}{L_t} & -\frac{R_t}{L_t} & 0 & -\frac{K_b}{L_t} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{K_T}{J} & 0 & -\frac{B}{J} \end{bmatrix} x + \begin{bmatrix} \frac{K_\theta}{L_f} \\ 0 \\ 0 \\ 0 \end{bmatrix} \underbrace{u}_{\theta_r}$$

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} x$$

Q3-) A more detailed schematic diagram is given below. (5)



T_m = Torque applied by motor onto J_m

T_1 = Torque applied by J_m onto gear

T_2 = Torque applied by gear onto J

f; force applied by J onto ideal gear and M,

a-)

$$v - e_b = L\frac{di_a}{dt} + Ri_a$$

$$T_m = K_m i_a$$

$$e_b = K_b w_1$$

$$\dot{\theta}_1 = w_1$$

$$J_m \dot{\omega}_1 = T_m - T_1$$

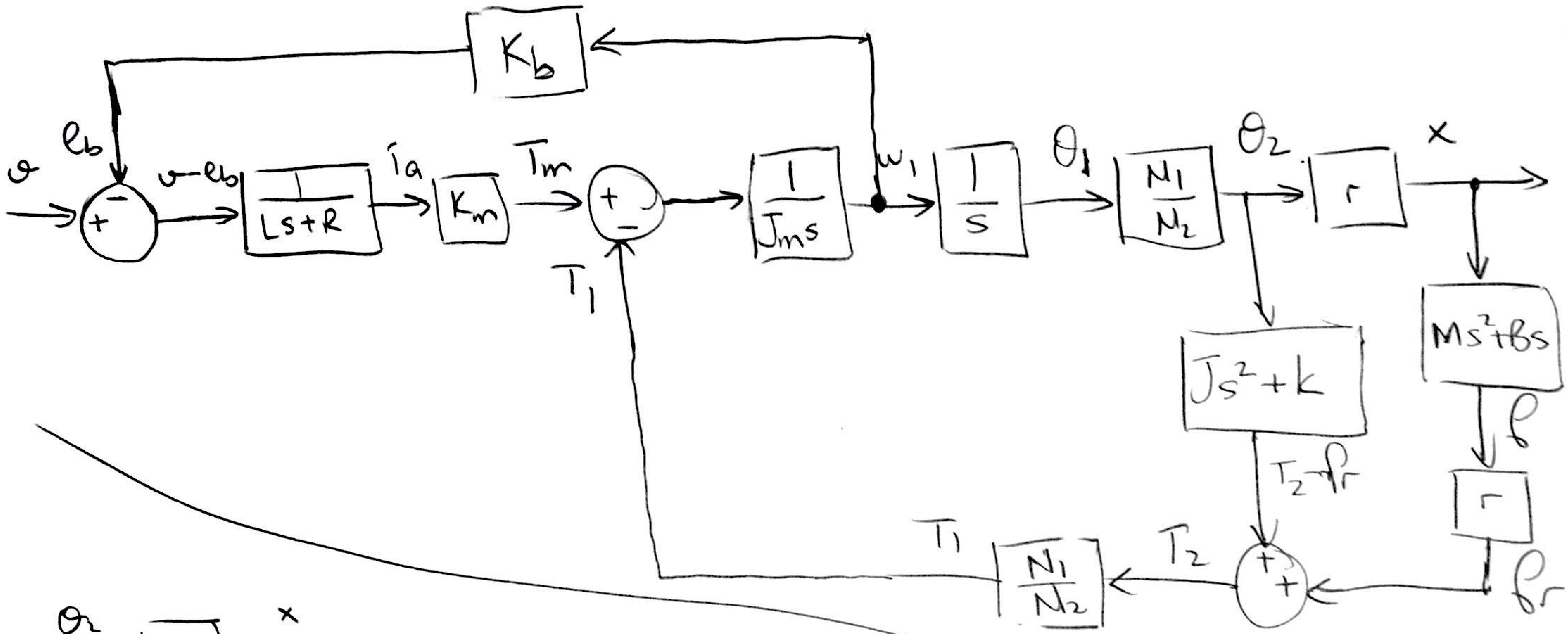
$$T_2 = \frac{N_2}{N_1} T_1 \qquad \dot{\theta}_2 = w_2$$

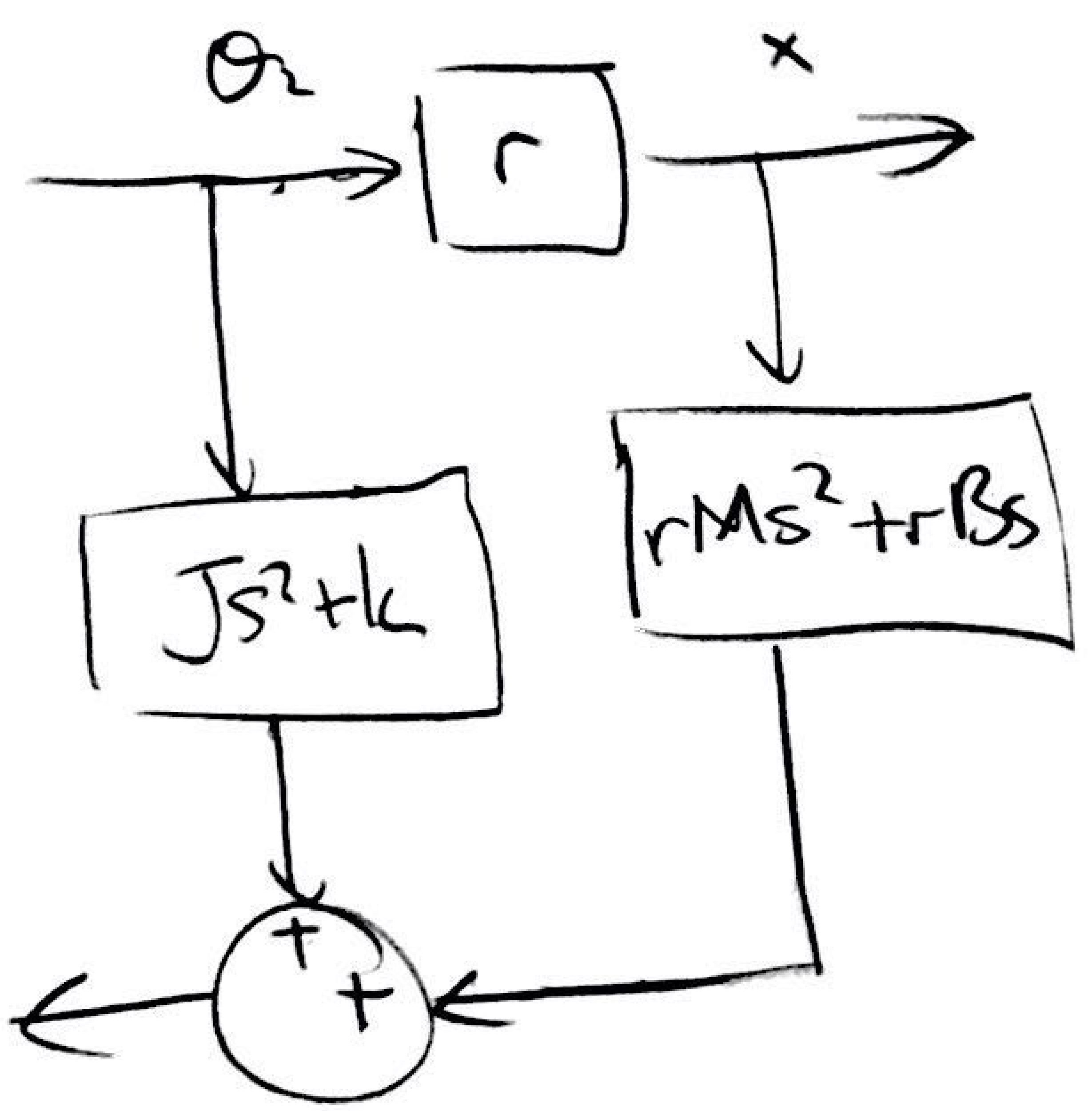
$$w_2 = \frac{N_1}{N_2} w_1 \qquad \theta_2 = \frac{N_1}{N_2}\theta_1$$

$$J\ddot{\theta}_2 = T_2 - fr - k\theta_2$$

$$M\ddot{x} = +f - B\dot{x}$$

$$x = r\theta_2$$





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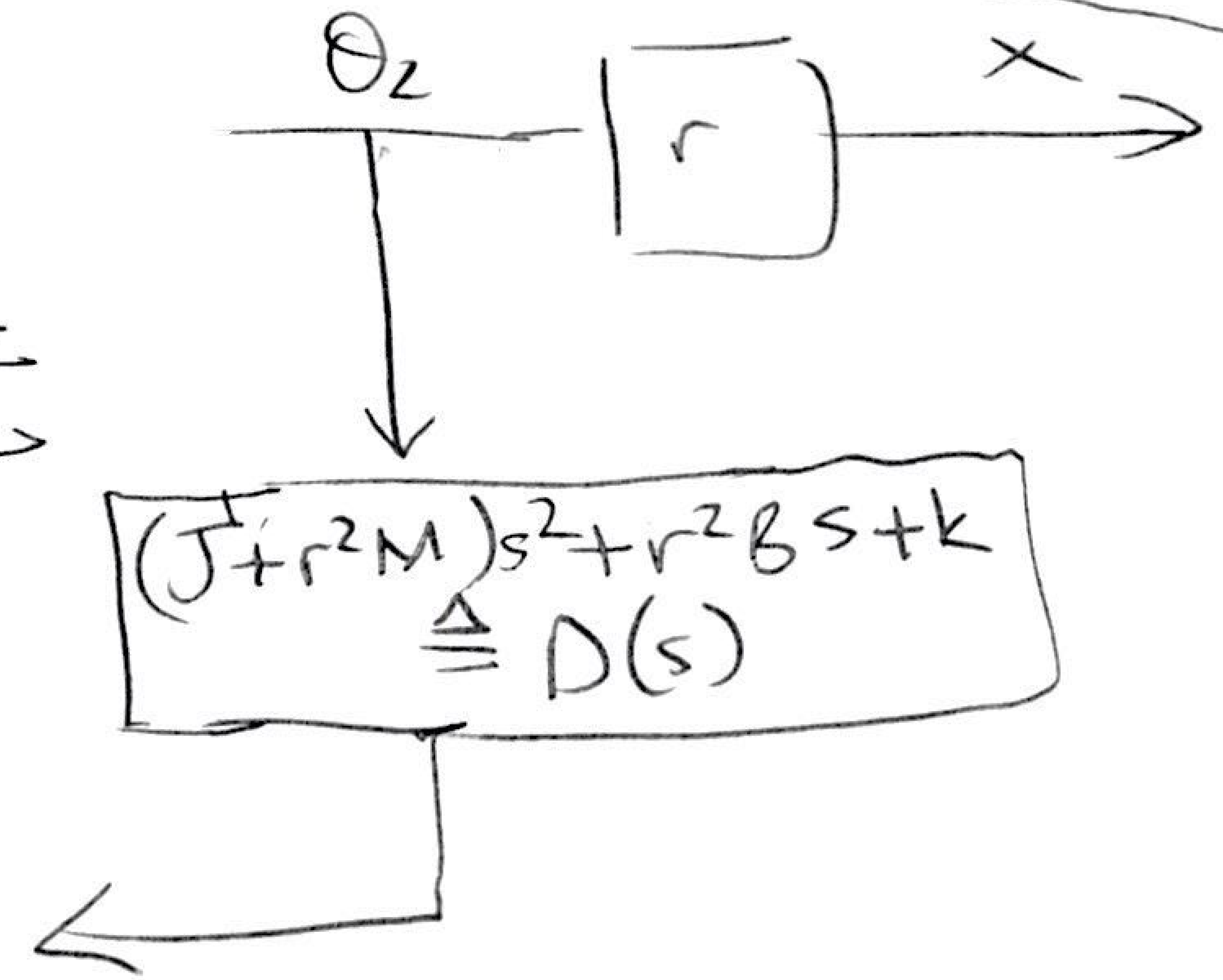
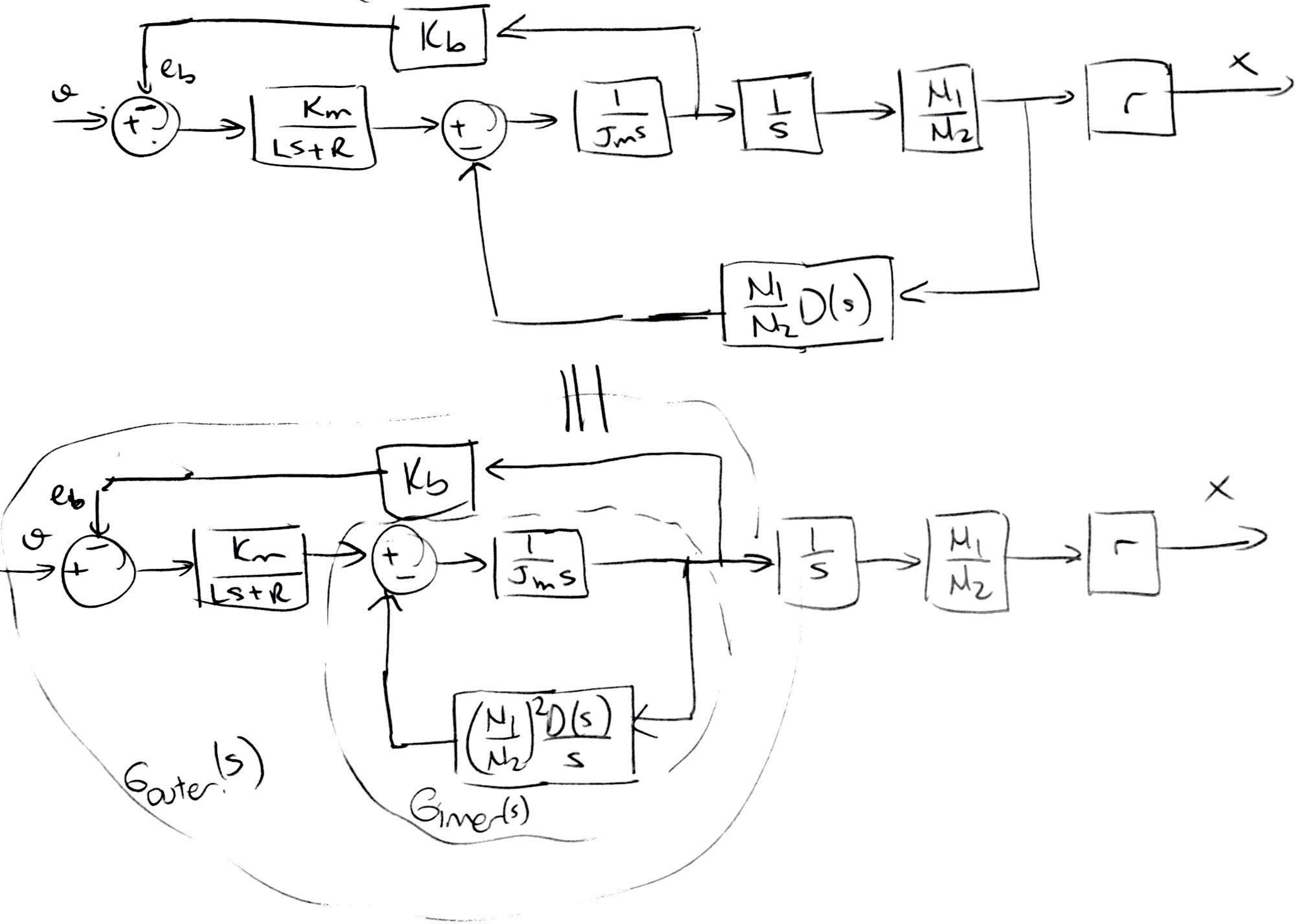


$$D(s) = (J + r^2 M)s^2 + r^2 B s + k$$



$$G_{inner}(s) = \frac{\frac{1}{J_m s}}{1 + \left(\frac{N_1}{N_2}\right)^2 \frac{D(s)}{J_m s^2}} = \frac{s}{J_m s^2 + \left(\frac{N_1}{N_2}\right)^2 D(s)}$$

$$G_{outer}(s) = \frac{\frac{K_m s}{(Ls+R)\left(J_m s^2 + \left(\frac{N_1}{N_2}\right)^2 D(s)\right)}}{1 + \frac{K_b K_m s}{(Ls+R)\left(J_m s^2 + \left(\frac{N_1}{N_2}\right)^2 D(s)\right)}} = \frac{K_m s}{(Ls+R)\left(J_m s^2 + \left(\frac{N_1}{N_2}\right)^2 D(s)\right) + K_b K_m s}$$

$$G(s) = \frac{X(s)}{V(s)} = \frac{(N_1/N_2)\, r K_m}{(Ls+R)\underbrace{\left(J_m s^2 + \left(\frac{N_1}{N_2}\right)^2 D(s)\right)} + K_b K_m s}$$

$$J_m s^2 + \left(\frac{N_1}{N_2}\right)^2 (J + r^2 M) s^2 + \left(\frac{N_1}{N_2}\right)^2 r^2 B s + \left(\frac{N_1}{N_2}\right)^2 k = J_{eq} s^2 + B_{eq} s + K_{eq}$$

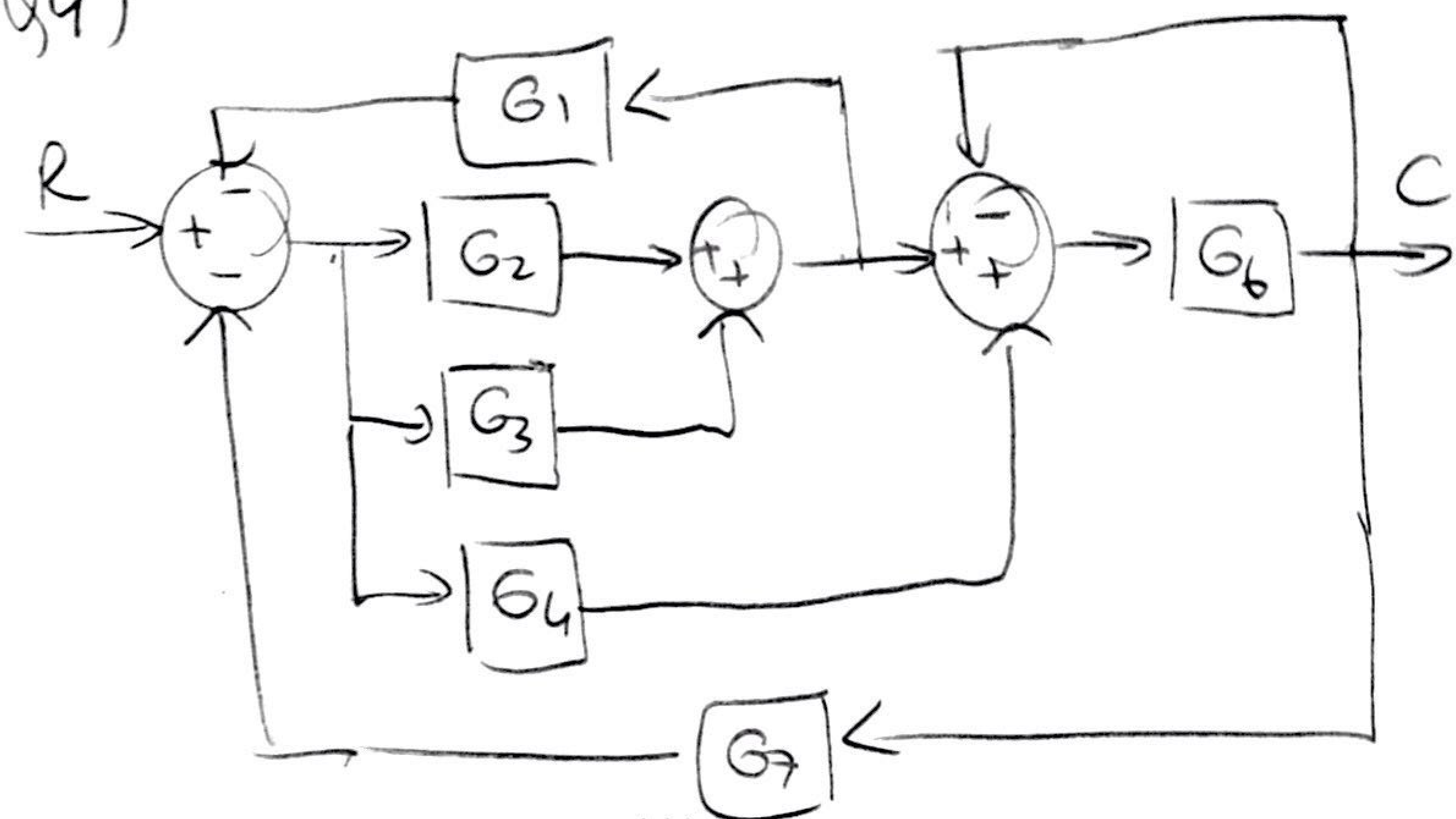
$$\Rightarrow \quad J_{eq} \triangleq J_m + \left(\frac{N_1}{N_2}\right)^2 (J + r^2 M)$$

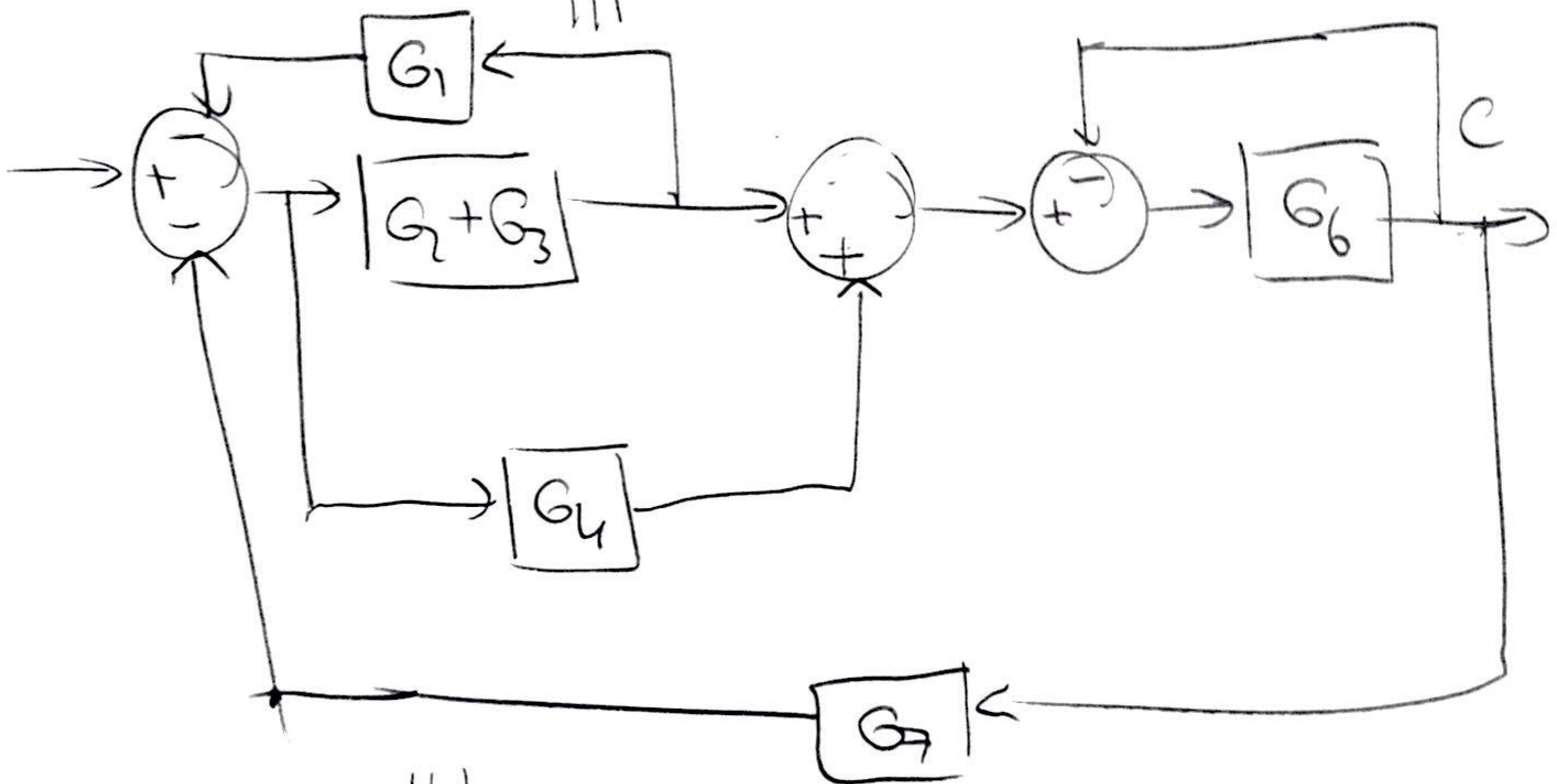
$$B_{eq} = \left(\frac{N_1}{N_2}\right)^2 r^2 B$$

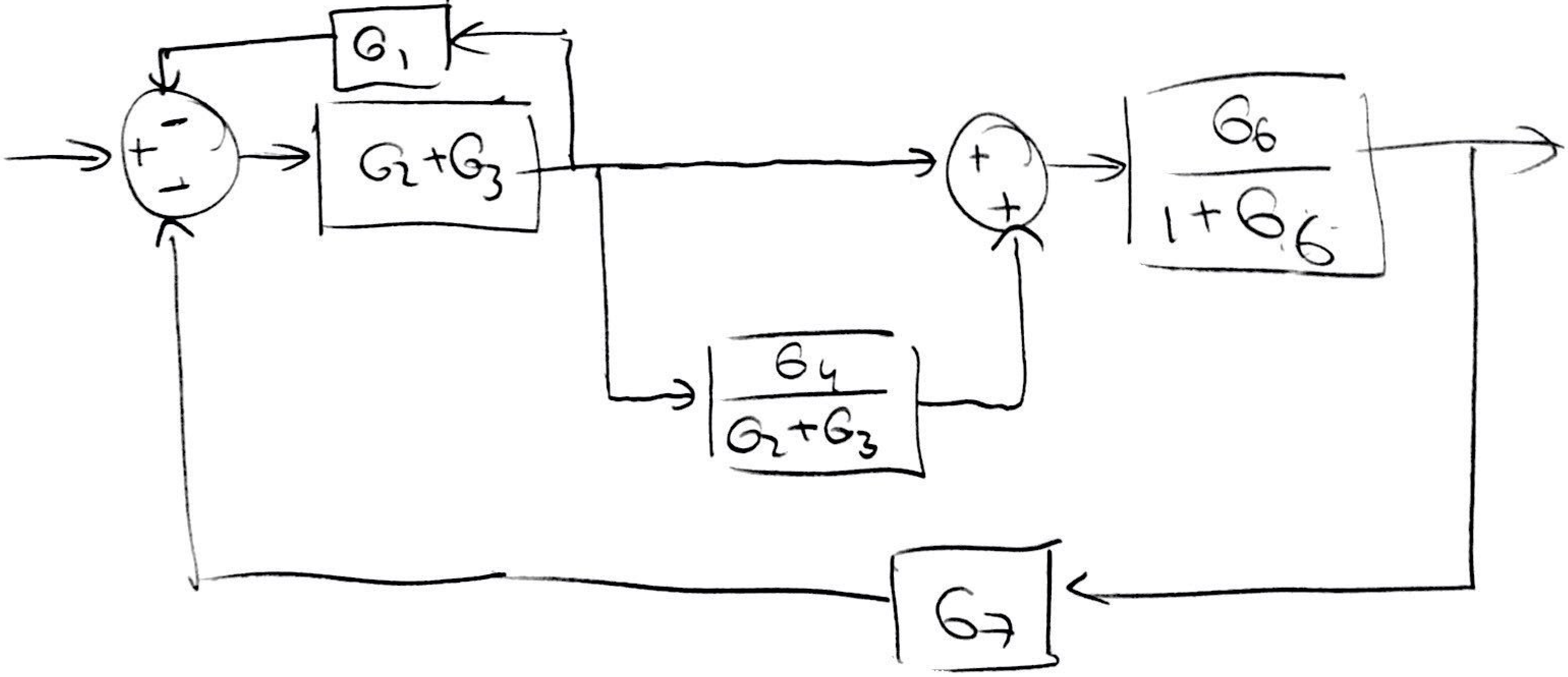
$$K_{eq} = \left(\frac{N_1}{N_2}\right)^2 k$$

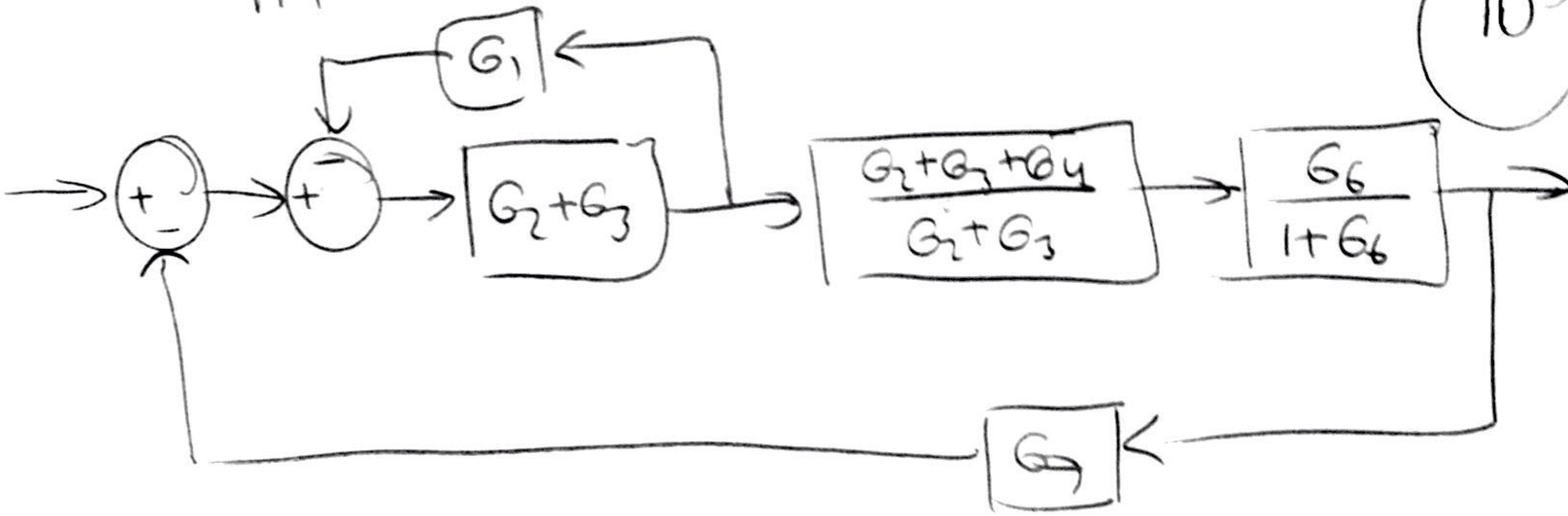
$$\boxed{G(s) = \frac{(N_1/N_2)\, r K_m}{(Ls+R)\left(J_{eq} s^2 + B_{eq} s + K_{eq}\right)}}$$

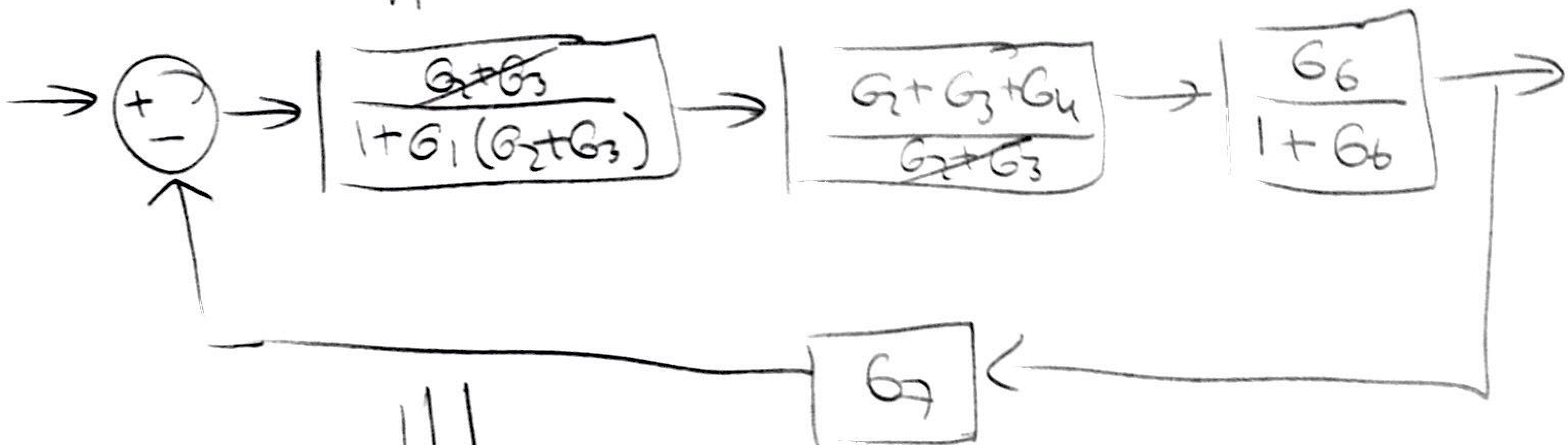
Q4)











$$\frac{\dfrac{G_6(G_2+G_3+G_4)}{(1+G_6)(1+G_1(G_2+G_3))}}{1+\dfrac{G_6G_7(G_2+G_3+G_4)}{(1+G_6)(1+G_1(G_2+G_3))}}$$

$$\frac{G_6(G_2+G_3+G_4)}{(1+G_6)(1+G_1(G_2+G_3))+G_6G_7(G_2+G_3+G_4)}$$