EE 301
Signals and Systems

Department of Electrical and Electronics Engineering
Middle East Technical University

October 14, 2018

Outline

General remarks

- Concepts of signals and systems arise everywhere.
- Tools associated with these concepts is continuously yielding new developments in diverse fields:
  - Imaging, communications, biomedical eng, circuit design, energy distribution systems, speech processing, ...
- In this course, we will regard everything as
  - Either signals (**functions** in math)
  - or systems (**devices** that change the shapes of functions)
- We will learn how to build useful structures for the representation, processing, and analysis of signals by using the mathematical framework of signals and systems.

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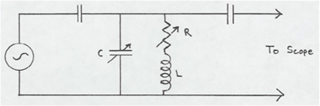
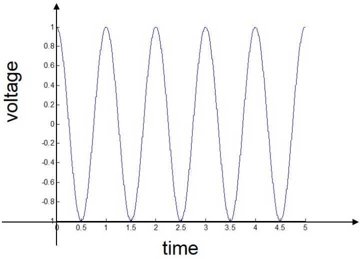
- Defn: A signal is the **variation** of a physical, or non-physical quantity with respect to one or more (independent) variables
- Mathematically, signals are functions of variables, which **contain information** about the behavior of some process or source.

Examples of Signals

Examples of Signals

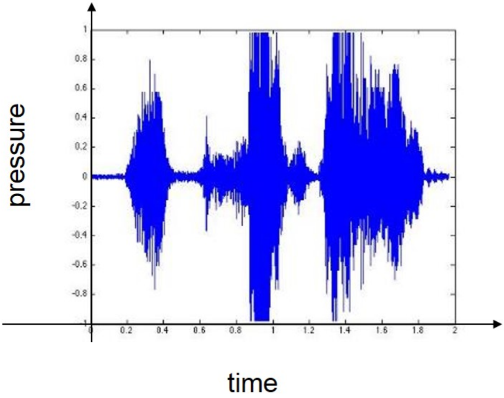
- Electrical signals: voltage variation over time





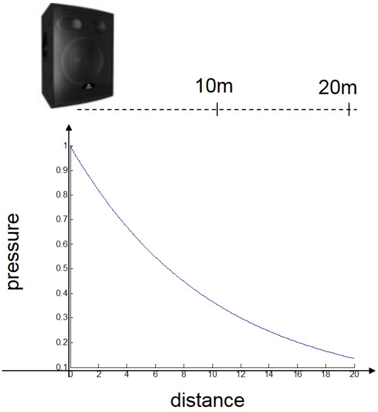
Examples of Signals

- Speech signals: air pressure variation over time



Examples of Signals

- Speech signals: air pressure variation over space



Examples of Signals

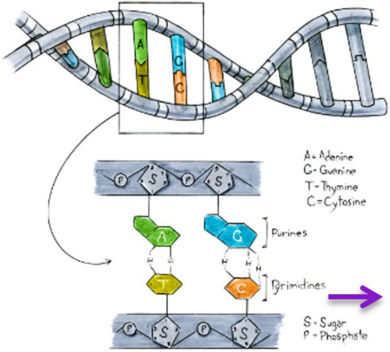
- Optical signals: (reflected) light brightness in space

Examples of Signals

- Electromagnetic signals: Computed tomography (CT) image of abdomen absorption of x-rays in space

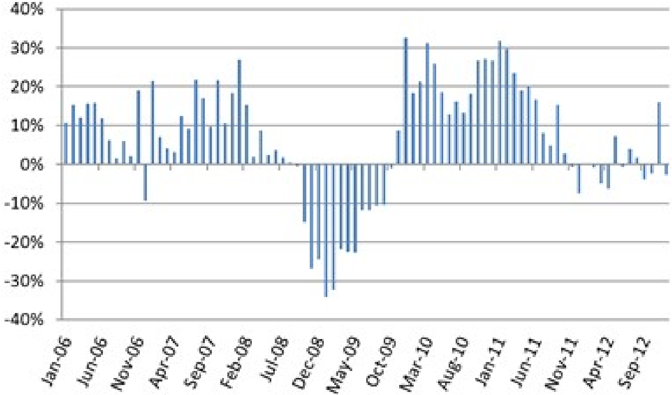
Examples of Signals

- Biological signals: Sequence of bases in a gene



Examples of Signals

- Financial signals: Change in import volume over time



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- Defn: A system is any **process** that results in the **transformation of signals**
- Examples of systems:
- Our goal in this course: Learn **mathematical tools** to analyze and design signal processing systems

How is signal processing useful in real world?

- https://www.youtube.com/watch?v=EErkgr1MWw0
(Search youtube for “What is Signal Processing?”)
- https://www.youtube.com/watch?v=mexN6d8QF9o
(Search youtube for “Signal Processing and Machine Learning”)

Signals as functions of variables

- We will mostly refer to the (independent) variable as time.
  - It can be other things depending on the application (such as space, distance, location index, and so on).
- Independent variables can be 1-D, 2-D,...

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- Independent variables can be 1-D, 2-D,...
  - Ex: an image is ...
  - Ex: a video is ...

Types of Signals

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Continuous-time (CT) signals

- continuous variation

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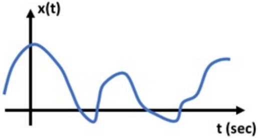

Discrete-time (CT) signals

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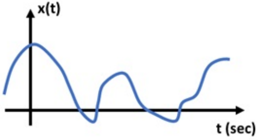

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- variation at specified time instants

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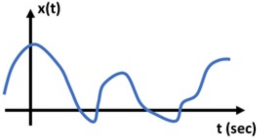

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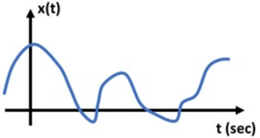
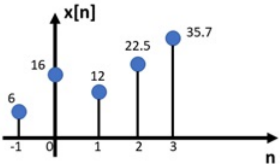


Discrete-time (CT) signals

- variation at specified time instants
- $x[n]$ (sequence of numbers)
- n : discrete-time variable
- $x[n]$ is defined only for integer values of n



Continous-time vs discrete-time

Examples:

- Voltage in time
- # of people in lectures over time
- Blood pressure recorded every half an hour

Remarks on DT signals

- DT signals are undefined at all values other than integer values of n
  $\Rightarrow$ referred as a discrete-time sequence

- Ex: # of people in lectures
  - No meaning for lecture number 1/2

Remarks on CT signals

- Most physical signals are CT but not all.
- A DT signal may arise
  - from a process that is inherently discrete, as in ...
  - from the sampling of a CT signal, as in ...

- Sampling is very important due to digital computers/signal processors (EE 430).

Transformations of the independent variable

We sometimes work with signals after modifying the independent variable (i.e. time axis).

Example

Fast forward

- **Time Shift** : $x(t - t_0)$
- **Time Reversal** : $x(-t)$
- **Time Scaling** : $x(\frac{t}{a})$ or $x(at)$, a: real number

Note that $x(at)$ is always defined for $a \epsilon \mathbb{R}$. The same is not true for $x[an]$.

Example

$x[n]$: a discrete sequence. Which of the following are defined: $x[2n]$, $x[\sqrt{2}n]$, $x[\frac{n}{2}]$?

Example

Find $y(t) = x(2t - 1)$ for the given $x(t)$.

In general, to find $x(at + b)$,

1. First time-shift by ...
2. OR ...

Periodic signals

Definition

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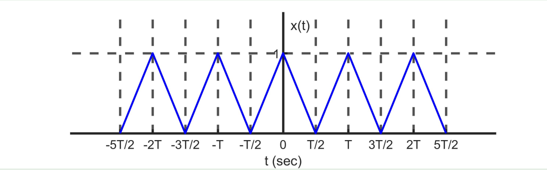
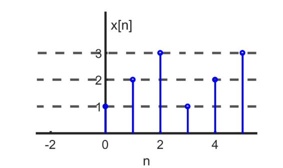
$$x(t) = x(t+T) \text{ holds for all } t$$

Periodicity is defined similarly for DT signals:
$x[n]$ is periodic with N if $x[n] = x[n+N]$ for all n where N is an integer.

Example





Note that if $x(t)$ is periodic with T then it is also periodic with $2T$, $3T$, $4T$,....

- **Fundamental period** T_0 of $x(t)$ (N_0 of $x[n]$): the smallest positive T (N) for which the above equalities hold.

Example (Challenge yourself!)

Prove whether $x_p(t) = \sum_{m=-\infty}^{\infty} x(t - mT)$ is periodic or not, when T: some constant, $x(t)$: arbitrary CT signal.

Example (Challenge yourself!)

If $x(t)$ is periodic with T, how about $x(t + t_0)$ and $x(\frac{t}{a})$? Similarly, if $x[n]$ is periodic with N, how about $x[n + n_0]$ and $x[Mn]$?

Even and Odd Signals

A signal is even if

Even and Odd Signals

A signal is even if $x(-t) = x(t)$ for all t. (DT: $x[-n] = x[n]$)

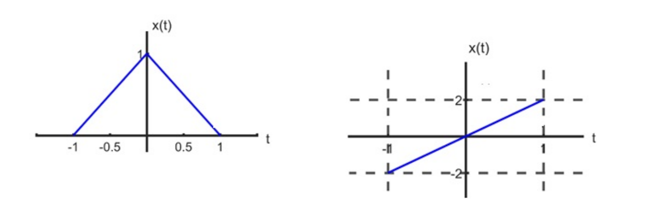
A signal is odd if

Even and Odd Signals

A signal is even if $x(-t) = x(t)$ for all t. (DT: $x[-n] = x[n]$)

A signal is odd if $x(-t) = -x(t)$ for all t. $(x[-n] = -x[n])$

Example



Example (Challenge yourself!)

Can we decompose arbitrary signals into a sum of two signals, one of which is even and the other is odd:

$$x(t) = x_{ev}(t) + x_{od}(t)$$

- Is the decomposition unique?
- Closed-form expression for the decomposition?

Check the book for the solution:

$$x_{ev}(t) = \frac{1}{2}(x(t) + x(-t)), \quad x_{od}(t) = \frac{1}{2}(x(t) - x(-t))$$

These are called even and odd parts of $x(t)$.

CT Complex Exponential & Sinusoidal Signals

[Short digression: Review of complex numbers (Page 71)]

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The CT complex exponential signal is generally of the form

$$x(t) = C\, e^{at}$$

where C and a are in general complex numbers.

1. **Real Exponential Signals:**
   C and a are real numbers
   - If $a > 0$, growing exponential:
   - If $a < 0$, decaying exponential:

2. Periodic Complex Exponential Signals:

a is purely imaginary

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$$x(t) = Ce^{j\omega_0 t} \qquad (\omega_0\text{: radians/second})$$

$\Rightarrow$ Verify periodicity:

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- $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$

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$\Rightarrow$ Verify periodicity:

- Fundamental period $T_0 = \frac{2\pi}{|\omega_0|}$
- $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$ have the same fundamental period.
- Euler's relations:
- Harmonically related complex exponentials:
- Does $x(t)$ have finite average power? finite energy?

3. Sinusoidal Signals:

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- odd signal when $\theta = \frac{\pi}{2} + k\pi$

Example (Challenge yourself!)

How to express $x(t)$ in terms of periodic complex exponentials?

4. General Complex Exponential Signals:
Both C and a are complex.

Let us represent C in polar form as $C = re^{j\theta}$ and a in rectangular form as $a = \alpha + j\omega_0$. Then
$x(t) = Ce^{at} =$

- $\alpha = 0$: both real and imaginary parts are
- $\alpha > 0$: both real and imaginary parts are
- $\alpha < 0$: both real and imaginary parts are

Example

$x(t) = 2e^{(3+j\omega_0)t}$
$x(t) = e^{(-1+j\omega_0)t}$

DT Complex Exponential & Sinusoidal Signals

The DT complex exponential signal is generally of the form

$$x[n] = C\alpha^n \quad \text{or} \quad x[n] = Ce^{\beta n} \; (\alpha = e^{\beta})$$

where C and α are in general complex numbers.

1. **Real Exponential Signals:**
   C and α are reals.

Example

DT real exponential signals with various behaviors

2. Sinusoidal Signals:
If β is purely imaginary (i.e. $|\alpha| = 1$), we obtain $x[n] = Ce^{j\Omega_0 n}$. This is related to sinusoidal signals:

3. General Complex Exponentials (Damped sinusoids):
β is not purely imaginary.

By representing C and α in polar form,

$$\begin{aligned} x[n] &= C\alpha^n = |C|e^{j\theta}(|\alpha|e^{j\Omega_0})^n = |C||\alpha|^n e^{j(\Omega_0 n+\theta)} \\ &= |C||\alpha|^n (\cos(\Omega_0 n + \theta) + j\sin(\Omega_0 n + \theta)) \end{aligned}$$

- $|\alpha| > 1$: both real and imaginary parts are ...
- $|\alpha| < 1$: both real and imaginary parts are ...

Periodicity Properties of DT Complex Exponentials

Although there are many similarities between CT and DT signals, there are also some important differences. One such difference exists between $e^{j\omega_0 t}$ and $e^{j\Omega_0 n}$.

Remember:

- CT signal $e^{j\omega_0 t}$ is

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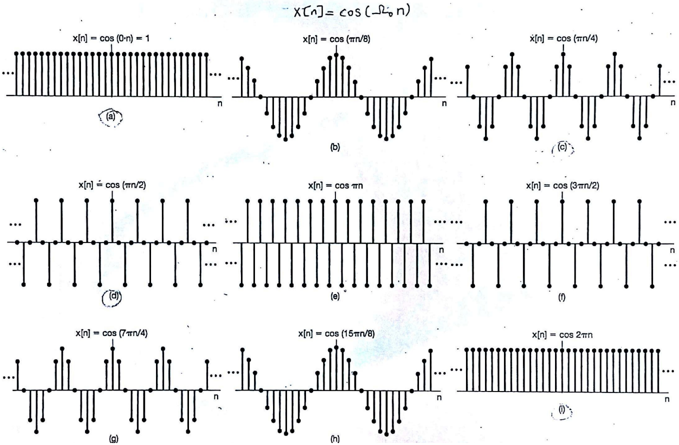
Both of the above properties are different for $e^{j\Omega_0 n}$.

DT sinusoidal sequences for several frequencies



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For $e^{j\Omega_0 n}$:

- As $|\Omega_0|$ increases, the rate of oscillation does not increase continuously.
- $e^{j\Omega_0 n}$ is not even distinct for different Ω_0; identical for $\Omega_0 + 2\pi k$
- Periodicity? $e^{j\Omega_0 n}$ is periodic only if Ω_0 can be written in the form $\Omega_0 = 2\pi \frac{k}{N}$ for some integers $N > 0$ and k.

Example

Is $x[n] = e^{jn}$ periodic? If so, find the fundamental period and frequency.

Example

Is $x[n] = 5\cos(\frac{3\pi}{4}n)$ periodic? If so, find the fundamental period and frequency.

Differences between CT and DT complex exponentials

CT: $e^{j\omega_0 t}$	DT: $e^{j\Omega_0 n}$
Distinct signals for distinct ω_0	Same signals for $\Omega_0 + 2\pi k$
Periodic for any ω_0	Periodic only if $\frac{\Omega_0}{2\pi}$ is a rational number
Fundamental period T_0 $\omega_0 = 0 \;\Rightarrow$ undefined $\omega_0 \neq 0 \;\Rightarrow\; T_0 = \frac{2\pi}{\lvert\omega_0\rvert}$	Fundamental period N_0 $\Omega_0 = 0 \;\Rightarrow$ undefined $\Omega_0 \neq 0 \;\Rightarrow\; N_0 = k\frac{2\pi}{\Omega_0}$ with min. possible k
Fundamental frequency $\lvert\omega_0\rvert$	Fundamental frequency $\frac{2\pi}{N_0}$

DT Unit Impulse & Unit Step Sequences

- DT unit impulse is defined as

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0. \end{cases}$$

- DT unit step is defined as

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0. \end{cases}$$

- The relation between $u[n]$ and $\delta[n]$:
  $\delta[n] =$
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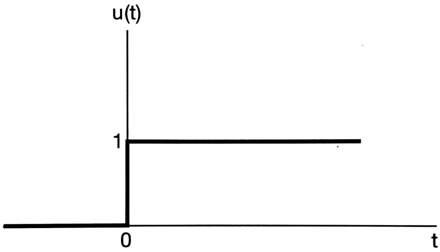
(Sum of Shifted Impulses / Running Sum Interpretations)

Properties of DT Unit Impulse $\delta[n]$

- $$\sum_{n=-\infty}^{\infty} \delta[n] = \qquad , \qquad \sum_{n=n_1}^{n=n_2} \delta[n] =$$
- $x[n]\delta[n] =$
  $(x[n]\delta[n-n_0] = \qquad\qquad)$ (Sampling Prop.)
- $\sum_{n=-\infty}^{\infty} x[n]\delta[n] =$
  $\sum_{n=n_1}^{n=n_2} x[n]\delta[n] =$
  $(\sum_{n=-\infty}^{\infty} x[n]\delta[n_0-n] = \qquad\qquad)$ (Convolution Prop.)

CT Unit Step & Unit Impulse Functions

The CT unit step function: $u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0. \end{cases}$



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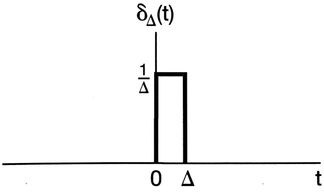
however, this is problematic in the classical sense since $u(t)$ is formally not differentiable at $t = 0$ ("generalized derivative").

For a more formal development, let us define new functions (approximations to the original step & impulse):

That is, more formally, the unit impulse func. is defined as

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$



- Note that $\delta_\Delta(t)$ is a short pulse of duration Δ and with unit area for any value of Δ. $\delta(t)$ should be viewed as an idealization of the short pulse when the duration Δ is very small (for any practical purpose).
- In practical terms, you can think of $\delta(t)$ as any func. of unit area, concentrated very near $t = 0$.

Properties of unit impulse function $\delta(t)$

- It is a signal of unit area vanishing everywhere except at origin.
  $\int_{-\infty}^{\infty} \delta(t)dt =$
  $\int_{t_1}^{t_2} \delta(t)dt =$
- It is the (generalized) derivative of the unit step func.
- For any cont. func. $x(t)$,
  - $x(t)\delta(t) =$
    $x(t)\delta(t - t_0) =$
  - $\int_{-\infty}^{\infty} x(t)\delta(t)dt =$
    $\int_{t_1}^{t_2} x(t)\delta(t)dt =$
    $(\int_{-\infty}^{\infty} x(t)\delta(t_0 - t)dt = \quad)$

Example (Challenge yourself!)

Prove the following:
Running Integral Interpretation : $u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$

Example

Prove the following:
Moving Impulse Interpretation : $u(t) = \int_{0}^{\infty} \delta(t-\tau) d\tau$

Example (Challenge yourself!)

Example 1.7 in the book (square type signal)

Systems

Definition

System: Any process that results in the transformation of signals.

Example

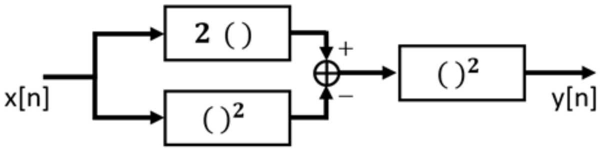
Examples of systems

CT and DT systems:

Interconnection of systems:

Example

$$y[n] = (2x[n] - x[n]^2)^2$$



Memory

Definition

A system is memoryless (instantaneous) if its output at any time instant depends only on the value of the input at that particular instant.

Example

Examples

Causality

Definition

A system is causal if its output at any time depends only on the values of the input at present time and/or in the past.

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- Physical systems that process time-domain signals in real-time can not respond before they are stimulated. In this case, causality is equivalent to realizability.
- Non-causal systems are of interest when the independent variable is not time or offline processing is possible.

Example (Challenge yourself!)

Causal or not:

- $y[n] = x[3n]$
- $y[n] = \sum_{m=-\infty}^{n} \alpha_m x[n-m]$
- $y(t) = x(t)^2$
- $y(t) = \int_{-\infty}^{t} x(\tau) d\tau$
- a memoryless system

Invertibility

Definition

A system is invertible if for distinct inputs, $x_1(t) \neq x_2(t)$, it generates distinct outputs, $y_1(t) \neq y_2(t)$.

If a system is invertible, then a corresponding inverse system exists.

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Example (Challenge yourself!)

Invertible or not: $y(t) = 5x(t - 1/2)$?

Stability

Definition (Bounded Input Bounded Output (BIBO) stab.)

A system is stable if bounded inputs lead to bounded outputs.

Example

Examples

Stability

Definition (Bounded Input Bounded Output (BIBO) stab.)

A system is stable if bounded inputs lead to bounded outputs.

Example

Examples

Example (Challenge yourself!)

Stable or not:

- $y(t) = \frac{x(t)}{x(t-1)+1}$
- $y(t) = e^{\alpha t}x(t)$, $\alpha \in \mathcal{C}$
- $y(t) = \frac{d}{dt}x(t)$

Time Invariance

Definition

A system is time-invariant if a time shift in the input signal causes same amount of time shift in the output signal.

Example

Examples

Time Invariance

Definition

A system is time-invariant if a time shift in the input signal causes same amount of time shift in the output signal.

Example

Examples

Example (Challenge yourself!)

Time-invariant or not:

- $y(t) = \sin(x(t))$
- $y(t) = x(2t)$
- $y(t) = e^t x(t)$
- $y[n] = n\,x[n]$

Linearity

Definition

A linear system has the superposition property:

Example

Examples

Example (Challenge yourself!)

Linear or not:

- $y[n] = 5x[n] + 2$
- $y(t) = x^2(t)$
- $y(t) = \frac{d^n x(t)}{dt^n}$