**Q1.** Consider the following polynomial

$$D(s) = s^4 - 2s^3 + (K+1)s^2 + 2Ks + K$$

**a.** Find the range of the parameter $-\infty < K < \infty$ for the polynomial to have no root, 1 root, 2 roots, 3 roots and 4 roots in the open right half plane (ORHP) using the Routh array. Fill in the following table with your results. Note that some of the ranges might be empty set $\emptyset$.

| # of roots in ORHP | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Range of $-\infty < K < \infty$ | $\phi$ | $-\infty < K < -1$ | $\infty > K > 0$ | $-1 < K < -\frac{1}{2}$, $-\frac{1}{2} < K < 0$ | $\phi$ |

# of sign chages = # of poles w/ Re{s}>0

**b.** Using the Routh array, find the $K$ value(s) for which $D(s)$ has roots on the imaginary axis.

**c.** At the $K$ value(s) for which $D(s)$ has roots on the imaginary-axis, find all of the roots of $D(s)$.

| | | | |
|---|---|---|---|
| $s^4$ | $1$ | $(K+1)$ | $K$ |
| $s^3$ | $-2$ | $2K$ | $0$ |
| $s^2$ | $2K+1$ | $K$ | $0$ |
| $s^1$ | $\frac{4K^2+4K}{2K+1}$ | $0$ | |
| $1$ | $K$ | | |

$\frac{-2K-2-2K}{-2} = 2K+1$

$\frac{-2K}{-2} = K$

First Colum: $1$, $-2$, $2K+1$, $\frac{4K(K+1)}{2K+1}$, $K$

at least 1 sign chage exit from 1 to −2.

To have only one sign change,

$2K+1<0$ $\boxed{K<-1/2}$ $\boxed{K<0}$ → intersection is $\boxed{-\infty<K<-1}$

| | $-1$ | $-1/2$ | $0$ | |
|---|---|---|---|---|
| $4K(K+1)$ | $-$ | $-$ | $+$ | |
| $2K+1$ | $-$ | $+$ | $+$ | |

$\boxed{-\frac{1}{2}<K<0 \cup -\infty<K<-1}$

To have 4 sign choge

$2K+1>0$ $\boxed{K>-1/2}$, $\boxed{K>0}$

$\frac{4K(K+1)}{2K+1} < 0$

| | $-1$ | $-1/2$ | $0$ | |
|---|---|---|---|---|
| $4K(K+1)$ | $+$ | $-$ | $-$ | $+$ |
| $2K+1$ | $-$ | $-$ | $+$ | $+$ |

$\boxed{-\infty<K<-1 \cup -\frac{1}{2}<K<0}$

no region satisfying all of them

To have 2 sign chage

$2K+1>0$

$\frac{4K(K+1)}{2K+1}>0$

$K>0$

$\boxed{K>0}$

To have 3 sign chage

$2K+1<0$

$K<0$

$\frac{4K(K+1)}{2K+1}>0$

$\boxed{-1<K<-\frac{1}{2}}$

and $2K+1>0$

$K<0$

$\frac{4K(K+1)}{2K+1}<0$

$\boxed{-\frac{1}{2}<K<0}$

b) To have roots on $j\omega$ axis

$s$ row is $0$.

$$\frac{4K(K+1)}{2K+1} = 0 \qquad K = -1$$

Auxiliary polyn

$-s^2 - 1 = 0$

$s^2 + 1 = 0$

$$\boxed{\boxed{s = \pm j} \quad K = -1}$$

if $K = 0$

Auxiliary poly

$s^2 = 0$

$s = 0$ (both on imaginary and real axis)

5/5

c) $s^4 - 2s^3 - 2s - 1 \;\big|\; s^2 + 1$

$\quad\; s^2 - 2s - 1$

$-s^4 + s^2$

$-2s^3 - s^2 - 2s - 1$

$--2s^3 - 2s$

$-s^2 - 1$

$-1^2 - 1$

$0$

5/5

$(s^2+1)(s^2 - 2s - 1) = D(s)$

$4-(4 \cdot 1 \cdot 1)$

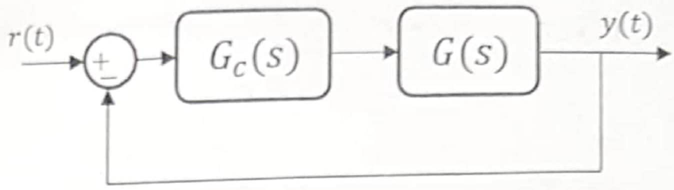
$\Delta = 4 + 4 = 8$

$$\boxed{s_{1,2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}}$$

$$\boxed{s_{3,4} = \pm j}$$

25 K=0

**Q2.** Consider the control system given on the right where the plant




$$G(s) = \frac{1}{(s+2)(s+4)}$$

is controlled using a controller with the transfer function $G_c(s)$. The aim of the controller is to make the damped natural frequency of the closed loop system 1rad/s and damping ratio to be $\sqrt{2}/2$.

**a.** Find the required dominant closed-loop pole positions.

**b.** Which one of the controllers below is suitable for satisfying the design requirements above?

- **i.** P-Controller
- **ii.** PD-Controller
- **iii.** PI-Controller

Write the transfer function of each controller and explain your reasoning by drawing roughly the root loci for each case with respect to a suitable parameter. In each root locus, do not forget to show the asymptotes. Details such as exact $j\omega$-crossings, break-away/in points etc. are not required.

**c.** Find the parameters of the controller you chose in part-b which satisfies the requirements. Explain your reasoning clearly.

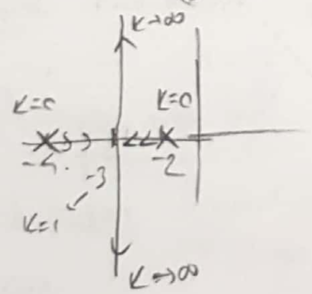
a) $w_d = 1$ rad/sec $\quad \xi = \frac{\sqrt{2}}{2} < 1$

$w_d = w_n\sqrt{1-\xi^2} \rightarrow w_n = \sqrt{2}$

$S_{1,2} = -\xi w_n \pm \underbrace{w_n\sqrt{\xi^2-1}}_{w_d} = -1 \pm 1j$ ✓ 3

$s_1 = -1+j$, $s_2 = -1-j$ } Dominant poles of closed loop system.

b) i) OLTF $= \frac{K}{(s+2)(s+4)}$ $\quad$ CLTF $= \frac{K}{s^2+6s+8+K}$ $\quad$ 3




2 asymptotes $\pm 90°$ $\quad \frac{-4-2}{2} = -3 = \sigma_0$

Until $K=1$, poles getting closer ✓ and $K\to\infty$ from $K=1$ no change at real axis.

$\sigma_0 = -3 \neq -1$, it does not satisfy our req.

ii) OLTF $= \frac{K+K_d s}{(s+2)(s+4)} = \frac{K_d(\frac{K}{K_d}+s)}{(s+2)(s+4)} = \frac{K_d(s+\alpha)}{(s+2)(s+4)}$ $\quad \frac{K}{K_d} = \alpha$

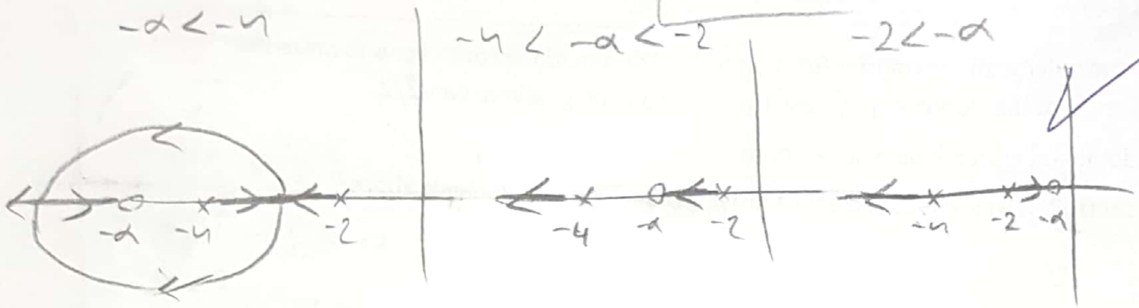
CLTF $= \frac{K_d(s+\alpha)}{s^2+(6+K_d)s+8+\alpha K_d}$ .

a new zero

$$\text{OLTF} = \frac{K_d(s+\alpha)}{(s+2)(s+4)}$$

$\boxed{\begin{matrix}1 \text{ asymptotes} \\ \phi = 180^\circ\end{matrix}}$

Root locus wrt $K_d$




In any case, I cannot reach $-1 \pm j$ case

It does not satisfy (3)

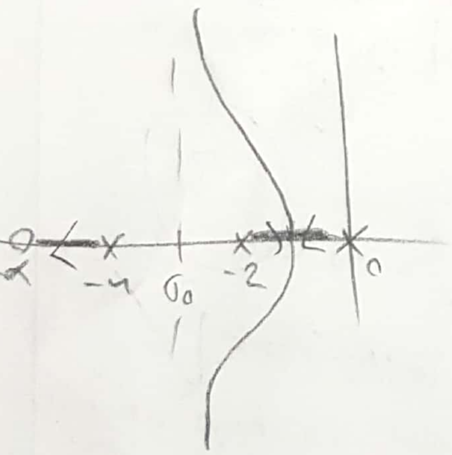
iii)
$$\frac{K + \frac{K_I}{s}}{(s+2)(s+4)} = \frac{sK + K_I}{s(s+2)(s+4)} = \frac{K\left(s + \frac{K_I}{K}\right)}{s(s+2)(s+4)} = \frac{K(s+\alpha)}{s(s+2)(s+4)} = \text{OLTF}$$

a new zero and pole

$$\text{CLTF} = \frac{K(s+\alpha)}{s^3 + 6s^2 + (8+K)s + \underbrace{\alpha K}_{K_I}}$$

2 asymptotes $\pm 90^\circ$

$$\sigma_a = \frac{-6+\alpha}{2} = \frac{\alpha}{2} - 3$$




I can reach $-1 \pm j$ condition by changing $K$ and $\frac{K_I}{K}$ ratio. It can satisfy.

→ ch. eqn must be satisfied by $-1 \pm j$

c) $s^3 + 6s^2 + (8+K)s + K_I = 0$   $s = -1 + j = \sqrt{2}\angle 135^\circ$

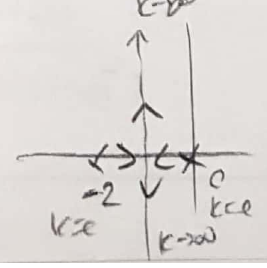
$$2\sqrt{2}\angle 135\times 3^\circ + 12\angle 270^\circ + 8\sqrt{2}\angle 135^\circ + K\sqrt{2}\angle 135^\circ + K_I = 0$$

$$(2+2j) - 12j - 8 + 8j - K + Kj + K_I = 0$$

$-6 - K + K_I = 0$ (Imaginary part = 0)

$-2 + K = 0$ (Real part = 0)

$\boxed{\begin{matrix}K = 2 \\ K_I = 8\end{matrix}}$ $\Big\}$ $\alpha = 4$



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Q3. Consider a unity (negative) feedback control system with open-loop transfer function $KG(s)$ where $K$ is a non-negative scalar and $G(s)$ is given as

$$G(s) = \frac{(s+7)}{s(s+1)(s+9)} = \frac{(s+7)}{s^3 + 10s^2 + 9s}$$

a. Draw the root-locus for $K \geq 0$ for the closed loop poles with all the details. You need to show where you expect and how to calculate the break-away/in points, but you do not have to calculate them numerically.

b. Show the dominating closed-loop pole or poles of the closed loop system on the root-locus.

c. Where should the dominant closed loop poles be located for the closed-loop system to have settling-time (5%) $t_s = 3$ seconds. Explain your reasoning.

d. Find the value of $K \geq 0$ for which the closed-loop system has settling-time (5%) $t_s = 3$ seconds.

e. Assuming that the root-locus for this system never intersects the asymptotes, how should you choose $K$ to minimize the settling time for this system? Comment also on how much you can reduce the settling-time by choosing $K$.

a)

→ # of branches = max(3,1) = 3

→ # of asymptotes = |3−1| = 2

→ $\phi = \frac{\pm 180(2k+1)}{2} = \pm 90°$ (Angle of asymptotes)

→ line of root locus $(-9,-7) \cup (-1,0)$

→ $\sigma_a = \frac{0-1-9-(-7)}{2} = \frac{-3}{2}$ (intersection of asymptotes)

→ break away-in

$$\left.\frac{d\,G(s)}{ds}\right|_{s=s_0} = 0$$

I expect it between −1 and 0.

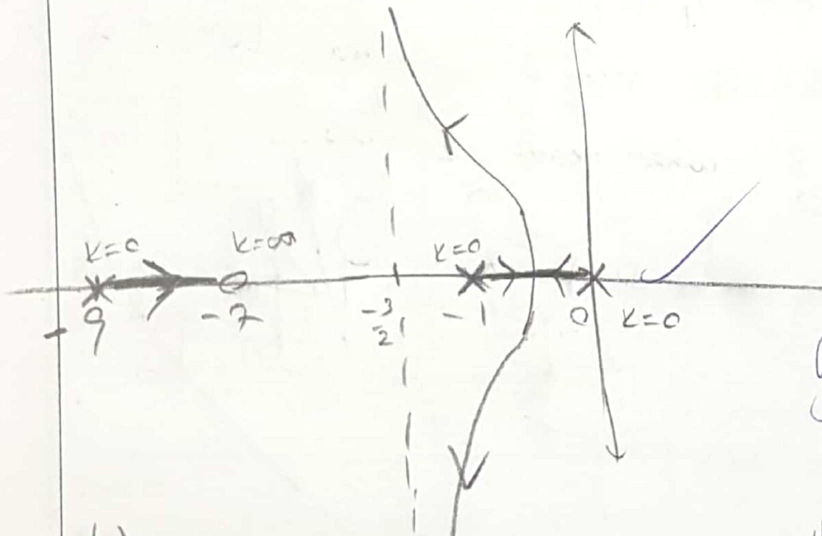
→ No $j\omega$ crossing.

| | | |
|---|---|---|
| $s^3$ | 1 | $9+K$ |
| $s^2$ | 10 | $7K$ |
| $s$ | $\frac{90+3K}{10}$ | |
| 1 | | |

$K = -30$

$10s^2 = 210$

$s$ is real, not on $j\omega$.

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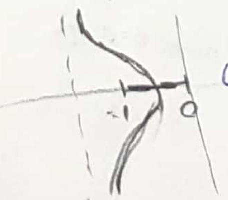
b) Dominating closed loop poles are at the right of the asymptotes. They are on locus.




4/4

c) $t_s = \frac{3}{\tau} = \frac{3}{\zeta\omega_n} = 3$

$\zeta\omega_n = 1$

$\omega_n^2 = \zeta^2\omega_n^2 - \omega_d^2$

$\omega_d^2 = 1 - \omega_n^2$   $\omega_d = \sqrt{1-\omega_n^2}$


Real part of two complex conjugate poles at $-\zeta\omega_n = -1$.

4/4

$|G(s)| = \frac{1}{K}$

Ch. eqn $q(s) = s^3 + 10s^2 + (9+K)s + 7K$

$s = -1 \pm jw_d$

$(-1+w_d j)^3 + 10(-1+w_d j)^2 + (9+K)w_d j + 7K = 0$

Real part $= 0$ | $w_d = --$
Im part $= 0$ | $K = --$

No time to solve it.

2/4

e) To minimize settling time, $\xi w_n$ should be larger. Therefore I choose $K$ as large as possible. But it is limited with $\xi w_n = \frac{-3}{2}$ when occurs $K \to \infty$.

$t_{s,min} = ?$

3/4

$(-1+w_d j)^2 ((9+w_d j)$

$(1 + w_d^2 - 2w_d j)(9 + w_d j) + (9+K)w_d j + 7K = 0$

$-9 + 9w_d^2 + 2w_d^2 + 7K = 0$

$-18w_d j + w_d j + w_d^3 j + (9+K)w_d j = 0$

$-18 + 1 + w_d^2 + 9 + K = 0$

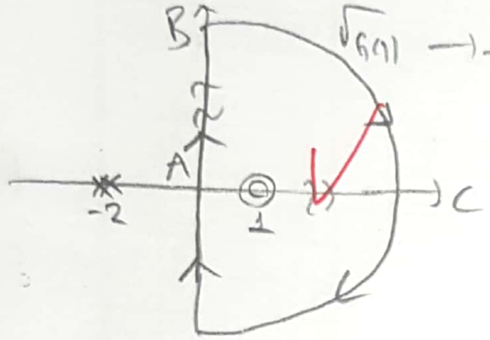
**Q4.** Consider a unity (negative) feedback control system with open-loop transfer function given as

$$KG(s) = K\frac{(s-1)^2}{(s+2)^2}.$$

**a.** Draw the Nyquist plot for the system. Show and explain the analysis you made for drawing each part of the plot.

**Hint:** $\alpha^2 - 13\alpha + 4 \approx (\alpha - 12.7)(\alpha - 0.3)$

**b.** Make a stability analysis for the closed loop system using your Nyquist plot and find the range of $-\infty \le K \le \infty$ for stability.




$\Gamma_{G(s)}$ → I pick that contour that contains all right half plane.

A → B $\quad s = j\omega \quad \omega: 0 \to \infty$

$$G(s) = \frac{(j\omega-1)^2}{(j\omega+2)^2} = \frac{-\omega^2 - 2\omega j + 1}{-\omega^2 + 4\omega j + 4}$$

$$G(s) = \frac{(1-\omega^2-2\omega j)}{4-\omega^2+j4\omega}\frac{(4-\omega^2-j4\omega)}{(4-\omega^2-j4\omega)} = \frac{\omega^4 - 13\omega^2 + 4 - j(12\omega - 6\omega^3)}{(4-\omega^2)^2 + 16\omega^2}$$

| $\omega$ | $0$ | $\sqrt{3/10}$ | $\sqrt{2}$ | $\sqrt{12.7}$ |
|---|---|---|---|---|
| Re | $\frac{1}{4}$ | $0$ | $-0.5$ | $0$ |
| Im | $0$ | A | $0$ | B |

Re = 0 $\quad (\omega^2 - 12.7)(\omega^2 - 0.3) = 0$

$\boxed{\omega = \sqrt{0.3}}$

$\omega = \sqrt{12.7}$

Im = 0 $\quad 6\omega(2-\omega^2) = 0$

$\boxed{\omega = \sqrt{2}}$

$\omega \to \infty \quad$ Re $= 1 \quad$ Im $\to 0$

Re = 0 $\quad \omega = \sqrt{0.3}$

$$Im = \frac{6\sqrt{\frac{3}{10}}\left(2 - \frac{3}{10}\right)}{\left(4-\frac{3}{10}\right)^2 + 16 \cdot \frac{3}{10}} = \frac{\frac{6\sqrt{3}}{\sqrt{10}}\left(\frac{17}{10}\right)}{\left(\frac{37}{10}\right)^2 + \frac{48}{10}} = A$$

Im = 0

$$Re = \frac{4 - 13 \cdot 2 + 4}{4+32} = \frac{-26+8}{36} = \frac{-18}{36} = -0.5$$

$|B| > |A|$

Re = 0 $\quad \omega = \sqrt{12.7}$

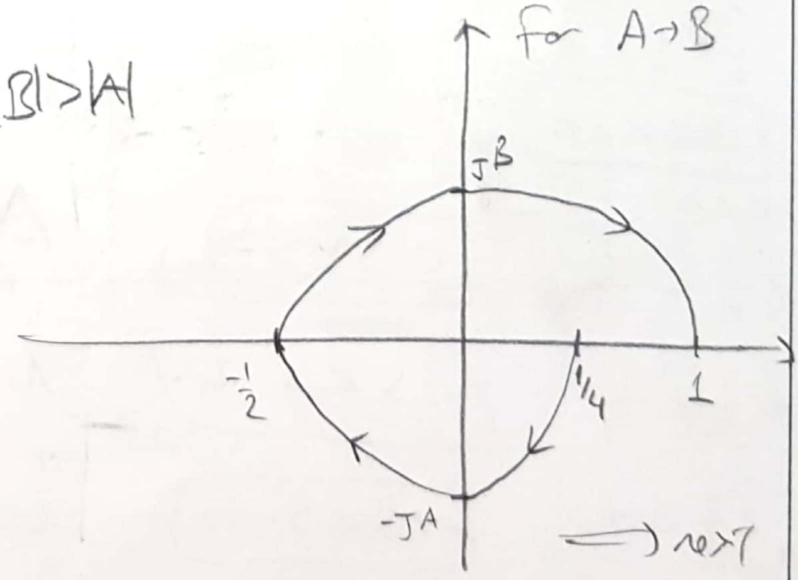
$$Im = -j\frac{6\sqrt{12.7}\,(2-12.7)}{(4-12.7)^2 + 12.7} = jB$$

$\omega \to \infty \quad$ Re $= 1$

Im $\to 0$

$6\omega^3 - 12\omega$

$-\sqrt{2} \quad 0 \quad \sqrt{2}$




→ next

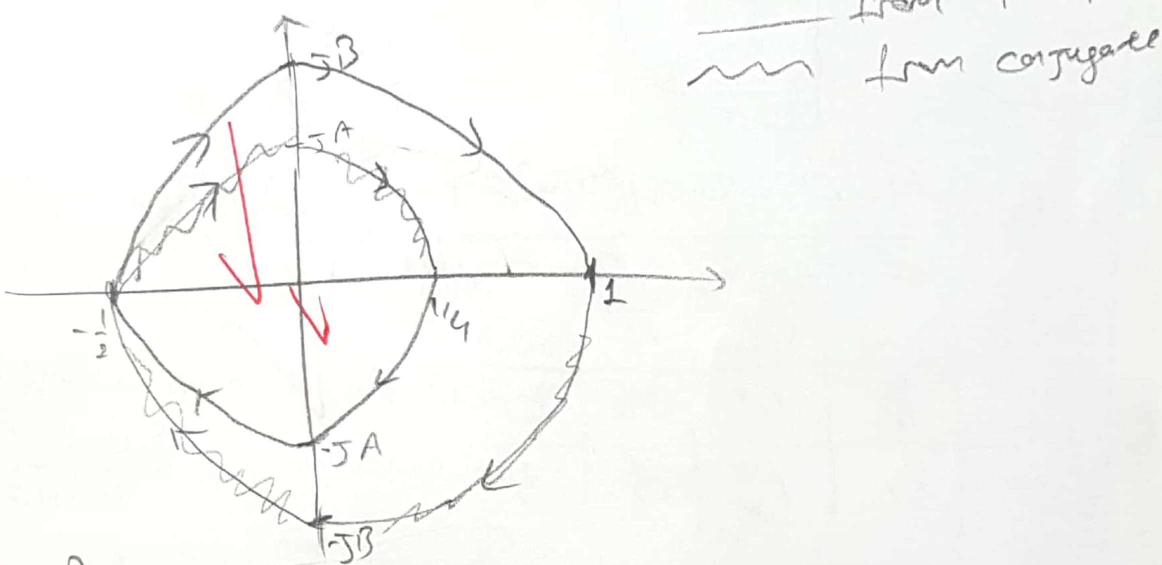
$B \to C \qquad \varepsilon \to 0$

$\frac{1}{\varepsilon} e^{j\theta} \qquad \theta : \frac{\pi}{2} \to 0$

$\underline{\frac{1}{\varepsilon}}$
very large

$G(s) \cong \underline{1} = \lim_{s \to \infty} \frac{(s-1)^2}{(s+2)^2} = 1$ this means in $B \to C$, contour stays at $\underline{1}$.

Remaining part is conjugate of first part.




b) $N = P_{CL} - P_{OL}$ , $P_{OL} = 0$

$K > 0$

1) $-\infty < -\frac{1}{K} < -\frac{1}{2}$ $\boxed{0 < K < 2}$ $N = 0$ $P_{CL} = 0$ , stable

2) $-\frac{1}{2} < -\frac{1}{K} < 0$

$\frac{1}{2} < \frac{1}{K} < 0$ $N = 2$ $P_{CL} = 2$ unstable w/ 2 poles

$\boxed{2 < K < \infty}$

3) $K < 0$

$-\infty < \frac{1}{K} < -1$ $\boxed{0 > K > -1}$ $N = 0$ $P_{CL} = 0$ stable

$-1 < \frac{1}{K} < -\frac{1}{4}$ $\boxed{-1 > K > -4}$ $N = 1$ $P_{CL} = 1$ unstable w/ 1 pole

$-\frac{1}{4} < \frac{1}{K} < 0$ $\boxed{-4 > K > -\infty}$ $N = 2$ $P_{CL} = 2$ unstable w/ 2 poles.