MIDDLE EAST TECHNICAL UNIVERSITY
DEPARTMENT OF ELECTRICAL AND ELECTRONICS ENGINEERING
EE301: SIGNALS AND SYSTEMS I

Solutions for Homework 3
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1. (a) The rectangular pulse train $x(t)$ with period $T = 4$ is given as

$$x(t) = \sum_{m=-\infty}^{\infty} g(t+4m)$$

where one period of $x(t)$ is given below.

$$g(t) = \begin{cases} 1, & -1 \leqslant t \leqslant 1 \\ 0, & 1 < t < 3 \end{cases}$$

Let the FS coefficients of $x(t)$ be X_k. Then, we compute X_k's for $k = 0$ and $k \neq 0$, as follows:

$$X_0 = \frac{1}{T}\int_{\langle T\rangle} x(t)\,\mathrm{d}t = \frac{1}{4}\int_{-1}^{1} \mathrm{d}t = \frac{1}{4}\int_{-2d}^{2d} \mathrm{d}t = \frac{1}{4}\left(2d - (-2d)\right) = d$$

$$X_k = \frac{1}{T}\int_{\langle T\rangle} x(t)\,e^{-jk\omega_0 t}\,\mathrm{d}t = \frac{1}{4}\int_{-2d}^{2d} e^{-jk\frac{\pi}{2}t}\,\mathrm{d}t = \frac{1}{jk2\pi}\left(e^{jk\pi d} - e^{-jk\pi d}\right) = \frac{\sin(\pi k d)}{\pi k}$$

$$X_k = \begin{cases} d, & k = 0 \\ \dfrac{\sin(\pi k d)}{\pi k}, & \text{otherwise} \end{cases}$$

Then, we obtain

$$X_k = \begin{cases} \dfrac{1}{2}, & k = 0 \\ \dfrac{\sin\left(\frac{\pi}{2}k\right)}{\pi k}, & \text{otherwise} \end{cases}$$

by substituting d.

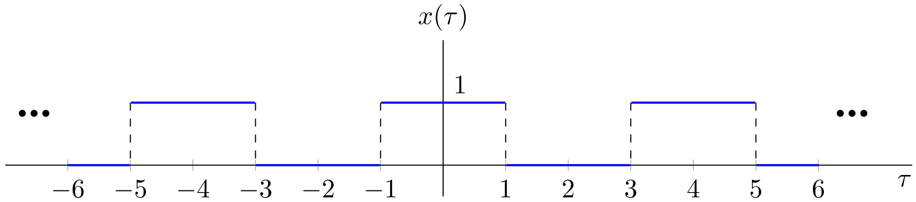
To simplify the expression of X_k, we can use the identity $\operatorname{sinc}(\theta) = \dfrac{\sin(\pi\theta)}{\pi\theta}$ and obtain

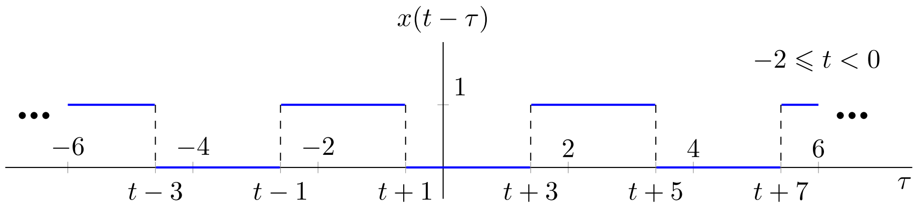
$$X_k = \frac{1}{2}\operatorname{sinc}\left(\frac{k}{2}\right), \qquad k \in \mathbb{Z}.$$

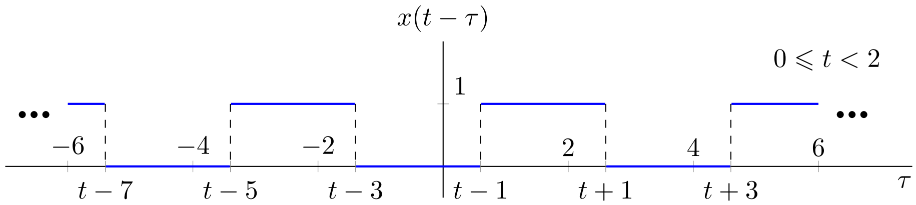
(b) We have

$$z(t) = \int_{0}^{4} x(\tau)\,x(t-\tau)\,\mathrm{d} = \int_{-2}^{2} x(\tau)\,x(t-\tau)\,\mathrm{d}\tau$$

since $x(t)$ is periodic. Before computing the convolution integral, we can sketch $x(\tau)$ and $x(t-\tau)$ for different t's as below.







According to these plots, for $\tau \in [-2, 2]$, the convolution integral can be computed w.r.t. the cases of the overlap between $x(\tau)$ and $x(t-\tau)$ as follows:

- For $-2 \leqslant t < 0$, there is overlap from $\tau = -1$ to $\tau = t+1$, and we have

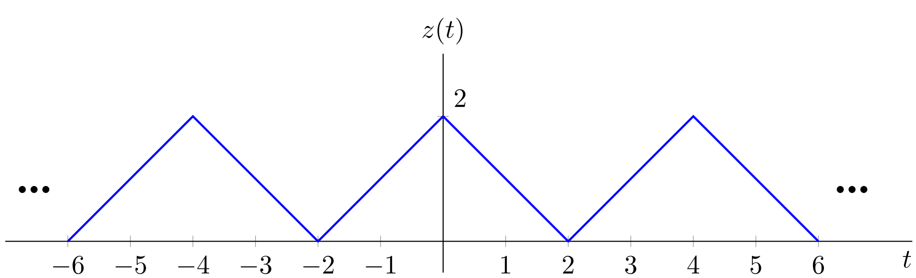
$$z(t) = \int_{-2}^{2} x(\tau)\, x(t-\tau)\, \mathrm{d}\tau = \int_{-1}^{t+1} \mathrm{d}\tau = (t+1) - (-1) = t + 2.$$

- For $0 \leqslant t < 2$, there is overlap from $\tau = t-1$ to $\tau = 1$, and we have

$$z(t) = \int_{-2}^{2} x(\tau)\, x(t-\tau)\, \mathrm{d}\tau = \int_{t-1}^{1} \mathrm{d}\tau = 1 - (t-1) = 2 - t.$$

Thus, one period of $z(t)$ is found as below.

$$z(t) = \begin{cases} t+2, & -2 \leqslant t < 0 \\ 2-t, & 0 \leqslant t < 2 \end{cases}$$



Using the *convolution in time* property of the CTFS representation, we can express the FS coefficients of $z(t)$ as below.

$$Z_k = T\, X_k\, X_k = 4\, X_k^2, \qquad k \in \mathbb{Z}$$

	(c)	$$y(t)=\frac{d}{dt}z(t)=\begin{cases}1, & -2<t<0\\ -1, & 0<t<2\end{cases}$$ We can express $y(t)$ in terms of $x(t)$. To do so, we have several expressions such as $$y(t)=2\,x(t+1)-1$$ or $$y(t)=x(t)*[\delta(t-3)-\delta(t-1)]=x(t-3)-x(t-1).$$
	(d)	Using $y(t)=x(t-3)-x(t-1)$ and *time shifting* property of the CTFS representation, we can find the FS coefficients Y_k of $y(t)$ as below. $$\begin{aligned} x(t-1) &\overset{\text{CTFS}}{\longleftrightarrow} X_k\,e^{-jk\frac{\pi}{2}}\\ x(t-3) &\overset{\text{CTFS}}{\longleftrightarrow} X_k\,e^{-jk\frac{3\pi}{2}}=X_k\,e^{jk\frac{\pi}{2}}\\ y(t) &\overset{\text{CTFS}}{\longleftrightarrow} Y_k=X_k\,e^{jk\frac{\pi}{2}}-X_k\,e^{-jk\frac{\pi}{2}}=j2\,X_k\,\sin\left(\frac{\pi}{2}k\right)\end{aligned}$$ $$Y_k=j\ \mathrm{sinc}\left(\frac{k}{2}\right)\ \sin\left(\frac{\pi}{2}k\right),\qquad k\in\mathbb{Z}$$
	(e)	Using the *differentiation* property of the CTFS representation, we obtain $$Y_k=jk\omega_0\,Z_k=jk\frac{\pi}{2}\,Z_k\implies Z_k=\frac{2}{jk\pi}\,Y_k,\qquad k\in\mathbb{Z}.$$ Then, we can find Z_k as $$Z_k=\frac{2}{jk\pi}\,j\ \mathrm{sinc}\left(\frac{k}{2}\right)\ \sin\left(\frac{\pi}{2}k\right)=\left(\mathrm{sinc}\left(\frac{k}{2}\right)\right)^2,\qquad k\in\mathbb{Z}.$$
	(f)	In part (b), we have found Z_k as $$Z_k=4\,X_k^2=4\left(\frac{1}{2}\,\mathrm{sinc}\left(\frac{k}{2}\right)\right)^2=\left(\mathrm{sinc}\left(\frac{k}{2}\right)\right)^2,\qquad k\in\mathbb{Z}.$$ Thus, Z_k's found in part (b) and (e) are equal.

2.	(a)	$$f[k]=\sum_{n=0}^{N-1}e^{j\frac{2\pi}{N}kn}=\begin{cases}\displaystyle\sum_{n=0}^{N-1}e^{j\frac{2\pi}{N}\ell Nn}=\sum_{n=0}^{N-1}e^{j2\pi\ell n}=\sum_{n=0}^{N-1}1=N, & k=\ell N\text{ for }\ell\in\mathbb{Z}\\ \displaystyle\sum_{n=0}^{N-1}e^{j\frac{2\pi}{N}kn}=\frac{1-e^{j\frac{2\pi}{N}kN}}{1-e^{j\frac{2\pi}{N}k}}=\frac{1-e^{j2\pi k}}{1-e^{j\frac{2\pi}{N}k}}=0, & \text{otherwise}\end{cases}$$
	(b)	$$f[k]=\sum_{n=M}^{M+N-1}e^{j\frac{2\pi}{N}kn}\overset{m\triangleq n-M}{=}\sum_{m=0}^{N-1}e^{j\frac{2\pi}{N}k(m+M)}=e^{j\frac{2\pi}{N}kM}\sum_{m=0}^{N-1}e^{j\frac{2\pi}{N}km}=\begin{cases}N, & k=\ell N\text{ for }\ell\in\mathbb{Z}\\ 0, & \text{otherwise}\end{cases}$$ where M is an integer.

3.	(a)	The given LTI system has the impulse response $$h(t) = e^{-t}u(t).$$ The system is **causal** since $h(t) = 0$ for $t < 0$. The system is **stable** since $h(t)$ is absolutely integrable, i.e., $$\int_{-\infty}^{\infty} \lvert h(\tau)\rvert \,\mathrm{d}\tau = \int_{0}^{\infty} \lvert e^{-\tau}\rvert \,\mathrm{d}\tau = \int_{0}^{\infty} e^{-\tau} \,\mathrm{d}\tau = \left(e^{-\tau}\right)\Big\vert_{\infty}^{0} = 1 < \infty.$$ The given input $x(t) = (j)^t = e^{j\frac{\pi}{2}t}$ is a complex exponential with $s = j\dfrac{\pi}{2}$, which is an eigenfunction of CT LTI systems. Then, the eigenvalue associated with the eigenfunction $e^{j\frac{\pi}{2}t}$ is $$H(s) = \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\,\mathrm{d}\tau\Big\vert_{s=j\frac{\pi}{2}} = \int_{-\infty}^{\infty} h(\tau)\, e^{-j\frac{\pi}{2}\tau}\,\mathrm{d}\tau = \int_{0}^{\infty} e^{-\tau}\, e^{-j\frac{\pi}{2}\tau}\,\mathrm{d}\tau = \left(\frac{e^{-\tau}}{1+j\frac{\pi}{2}}\right)\Bigg\vert_{\infty}^{0} = \frac{2}{2+j\pi}$$ and the output $y(t)$ is $$y(t) = H\left(j\frac{\pi}{2}\right) e^{j\frac{\pi}{2}t} = \frac{2}{2+j\pi}\, e^{j\frac{\pi}{2}t} = \frac{2}{2+j\pi}\,(j)^t.$$
	(b)	The impulse response of the LTI system is $$h[n] = 2^{-n}\,u[n+1].$$ The system is **not causal** since $h[n] \neq 0$ for $n < 0$, e.g. $h[-1] = 2$. The system is **stable** since $h[n]$ is absolutely summable, i.e., $$\sum_{k=-\infty}^{\infty} \lvert h[k]\rvert = \sum_{k=-\infty}^{\infty} \left\lvert 2^{-k}\,u[k+1]\right\rvert = \sum_{k=-1}^{\infty} 2^{-k} = \frac{2}{1-2^{-1}} = 4 < \infty.$$ The given input $x[n] = (j)^n = e^{j\frac{\pi}{2}n}$ is a complex exponential sequence with $z = j = e^{j\frac{\pi}{2}}$, which is an eigenfunction of DT LTI systems. By evaluating $H(z)$ at $z = j$, we can compute the eigenvalue associated with the eigenfunction j^n. $$H(z) = \sum_{k=-\infty}^{\infty} h[k]\, z^{-k}\Big\vert_{z=j} = \sum_{k=-\infty}^{\infty} h[k]\, j^{-k} = \sum_{k=-1}^{\infty} \left(\frac{1}{j2}\right)^{k} = \frac{j2}{1-\frac{1}{j2}} = \frac{4}{1-j2} = 0.8 + j1.6$$ Then, the output $y[n]$ is $$y(t) = H(j)\,(j)^n = (0.8 + j1.6)\,(j)^n.$$

4.	(a)	*i.*

$$x(t) = \sin(2\,t) + \cos(3\,t)$$

$\sin(2\,t)$ is periodic with period $\dfrac{2\pi}{2} = \pi$ and $\cos(3\,t)$ is periodic with period $\dfrac{2\pi}{3}$.
$\therefore \sin(2\,t) + \cos(3\,t)$ is **periodic** with period $T = \mathrm{LCM}\left(\pi, \dfrac{2\pi}{3}\right) = 2\pi$.

One can use Euler's relation, $e^{j\theta} = \cos(\theta) + j\sin(\theta)$, to express $x(t)$ in terms of complex exponentials as

$$x(t) = \overbrace{\frac{1}{j2}\,e^{j2\,t} - \frac{1}{j2}\,e^{-j2\,t}}^{\sin(2\,t)} + \overbrace{\frac{1}{2}\,e^{j3\,t} + \frac{1}{2}\,e^{-j3\,t}}^{\cos(3\,t)},$$

which is equal to the FS representation of $x(t)$. For $T = 2\pi$, we have

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\,e^{jk\frac{2\pi}{T}t} = \sum_{k=-\infty}^{\infty} a_k\,e^{jkt}.$$

$$a_k = \begin{cases} -\dfrac{1}{j2}, & k = -2 \\ \dfrac{1}{j2}, & k = 2 \\ \dfrac{1}{2}, & k = \pm 3 \\ 0, & \text{otherwise} \end{cases}$$

ii.

$$x(t) = \sin\left(\frac{\pi}{2}\,t\right) + \cos\left(\frac{\pi}{3}\,t\right)$$

$\sin\left(\dfrac{\pi}{2}\,t\right)$ is periodic with period $\dfrac{2\pi}{\pi/2} = 4$ and $\cos\left(\dfrac{\pi}{3}\,t\right)$ is periodic with period $\dfrac{2\pi}{\pi/3} = 6$.
$\therefore \sin\left(\dfrac{\pi}{2}\,t\right) + \cos\left(\dfrac{\pi}{3}\,t\right)$ is **periodic** with period $T = \mathrm{LCM}(4, 6) = 12$.

For $T = 12$, we can express $x(t)$ as below.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\,e^{jk\frac{2\pi}{T}t} = \sum_{k=-\infty}^{\infty} a_k\,e^{jk\frac{\pi}{6}t} = \overbrace{\frac{1}{j2}\,e^{j\frac{\pi}{2}t} - \frac{1}{j2}\,e^{-j\frac{\pi}{2}t}}^{\sin\left(\frac{\pi}{2}t\right)} + \overbrace{\frac{1}{2}\,e^{j\frac{\pi}{3}t} + \frac{1}{2}\,e^{-j\frac{\pi}{3}t}}^{\cos\left(\frac{\pi}{3}t\right)}$$

Then, the CTFS coefficients are given as follows:

$$a_k = \begin{cases} \dfrac{1}{2}, & k = \pm 2 \\ -\dfrac{1}{j2}, & k = -3 \\ \dfrac{1}{j2}, & k = 3 \\ 0, & \text{otherwise} \end{cases}$$

iii.

$$x(t) = \sin(2\,t) + \cos\left(\frac{\pi}{3}\,t\right)$$

$\sin(2\,t)$ is periodic with period $\dfrac{2\pi}{2} = \pi$ and $\cos\left(\dfrac{\pi}{3}\,t\right)$ is periodic with period $\dfrac{2\pi}{\pi/3} = 6$.

The signal $x(t)$ is **not periodic** since $\dfrac{6}{\pi}$ is irrational and we cannot find nonzero integers k, m such that $\pi\,k = 6\,m$ holds.

(b) *i.*

$$x[n] = \sin(2\,n) + \cos(3\,n)$$

The signal $\sin(\omega_0\,n)$ with $\omega_0 = 2$ is not periodic since $\dfrac{\omega_0}{2\pi} = \dfrac{1}{\pi}$ is irrational. The signal $\cos(\omega_0\,n)$ with $\omega_0 = 3$ is not periodic since $\dfrac{\omega_0}{2\pi} = \dfrac{3}{2\pi}$ is irrational. Therefore, $x[n]$ is **not periodic**.

ii.

$$x[n] = \sin\left(\frac{\pi}{2}\,n\right) + \cos\left(\frac{\pi}{3}\,n\right)$$

$\sin\left(\dfrac{\pi}{2}\,n\right)$ is periodic with period $\dfrac{2\pi}{\pi/2} = 4$ and $\cos\left(\dfrac{\pi}{3}\,n\right)$ is periodic with period $\dfrac{2\pi}{\pi/3} = 6$.

$\therefore \sin\left(\dfrac{\pi}{2}\,n\right) + \cos\left(\dfrac{\pi}{3}\,n\right)$ is **periodic** with period $N = \mathrm{LCM}(4, 6) = 12$.

For $N = 12$, we can express $x[n]$ as below.

$$x[n] = \sum_{k=\langle N\rangle} a_k\, e^{jk\frac{2\pi}{N}n} = \sum_{k=-6}^{5} a_k\, e^{jk\frac{\pi}{6}n} = \overbrace{\frac{1}{j2}\,e^{j\frac{\pi}{2}n} - \frac{1}{j2}\,e^{-j\frac{\pi}{2}n}}^{\sin\left(\frac{\pi}{2}n\right)} + \overbrace{\frac{1}{2}\,e^{j\frac{\pi}{3}n} + \frac{1}{2}\,e^{-j\frac{\pi}{3}n}}^{\cos\left(\frac{\pi}{3}n\right)}$$

Then, the DTFS coefficients are given as follows:

$$a_k = \begin{cases} \dfrac{1}{2}, & k = \pm 2 + 12\ell \text{ for } \ell \in \mathbb{Z} \\ -\dfrac{1}{j2}, & k = -3 + 12\ell \text{ for } \ell \in \mathbb{Z} \\ \dfrac{1}{j2}, & k = 3 + 12\ell \text{ for } \ell \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$$

iii.

$$x[n] = \sin(2\,n) + \cos\left(\frac{\pi}{3}\,n\right)$$

As seen from part (i), $\sin(2\,n)$ is not periodic. Hence, $x[n]$ is **not periodic**.

5. (a) A rectangular pulse train is given as

$$x[n] = \sum_{m=-\infty}^{\infty} A \text{ rect}\left(\frac{n+mN}{2N_1+1}\right),$$

where the pulse duration is less than the period, i.e., $2N_1 + 1 < N$ and

$$\text{rect}\left(\frac{n}{2N_1+1}\right) = \begin{cases} 1, & |n| \leqslant N_1 \\ 0, & \text{otherwise} \end{cases}$$

is the rectangular pulse. The *duty ratio* of the pulse train is

$$d = \frac{2N_1+1}{N}.$$

The FS coefficients a_k of $x[n]$ can be given in terms of A, N and d as below.

$$a_k = \begin{cases} \dfrac{A}{N}\,(2N1+1) = A\,d, & k = \ell N \text{ for } \ell \in \mathbb{Z} \\ \dfrac{A}{N}\,\dfrac{\sin\left(\dfrac{\pi}{N}k(2N_1+1)\right)}{\sin\left(\dfrac{\pi}{N}k\right)} = \dfrac{A}{N}\,\dfrac{\sin(\pi k d)}{\sin\left(\dfrac{\pi}{N}k\right)}, & \text{otherwise} \end{cases}$$

See Example 3.12 from Oppenheim to compute the FS coefficients a_k of $x[n]$.

(b) If $x[n]$ is real-valued, then $x^*[n] = x[n]$. Using $x^*[n] = x[n]$, we obtain

$$a_k^* = \left(\frac{1}{N}\sum_{n=\langle N\rangle} x[n]\,e^{-jk\frac{2\pi}{N}n}\right)^* = \frac{1}{N}\sum_{n=\langle N\rangle} x^*[n]\,e^{jk\frac{2\pi}{N}n} = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\,e^{jk\frac{2\pi}{N}n} = a_{-k},$$

which shows that *the DTFS coefficients a_k are conjugate symmetric when $x[n]$ is real-valued.*

If $x[n]$ is even, then $x[-n] = x[n]$. Using $x[-n] = x[n]$, we obtain

$$a_{-k} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\,e^{jk\frac{2\pi}{N}n} \overset{\ell \triangleq -n}{=} \frac{1}{N}\sum_{\ell=1-N}^{0} x[-\ell]\,e^{-jk\frac{2\pi}{N}\ell} = \frac{1}{N}\sum_{\ell=1-N}^{0} x[\ell]\,e^{-jk\frac{2\pi}{N}\ell} = a_k,$$

which shows that *the DTFS coefficients a_k are even when $x[n]$ is even.*

$$\left.\begin{array}{rl} \text{real } x[n] \implies & a_k^* = a_{-k} \\ \text{even } x[n] \implies & a_{-k} = a_k \end{array}\right\} \implies a_k^* = a_{-k} = a_k \text{ (real-valued and even DTFS coefficients)}$$

Thus, the DTFS coefficients a_k are real-valued and even, if $x[n]$ is real-valued and even.

(c) If b_k's are purely imaginary, $y[n]$ must be real-valued and odd. However, we cannot obtain an odd signal $y[n] = x[n-n_0] + c$ for any $n_0 \in \mathbb{Z}$ and $c \in \mathbb{R}$.

By utilizing the DTFS properties, we determine b_k's from a_k's as follows:

$$\begin{array}{rcll} x[n] & \overset{\text{DTFS}}{\longleftrightarrow} & a_k & \\ x[n-n_0] & \overset{\text{DTFS}}{\longleftrightarrow} & e^{-jk\frac{2\pi}{N}n_0}\,a_k & \text{(Time shifting)} \\ y[n] = x[n-n_0] + c & \overset{\text{DTFS}}{\longleftrightarrow} & b_k = \begin{cases} a_0 + c, & k = 0 \\ e^{-jk\frac{2\pi}{N}n_0}\,a_k, & \text{otherwise} \end{cases} & \text{(Linearity)} \end{array}$$

Let's consider the CT counterpart of given problem. A rectangular pulse train is given as

$$x(t) = \sum_{m=-\infty}^{\infty} A \text{ rect}\left(\frac{t+mT}{2T_1}\right),$$

where the pulse duration is less than the period, i.e., $2T_1 < T$ and

$$\text{rect}\left(\frac{t}{2T_1}\right) = \begin{cases} 1, & |t| \leqslant T_1 \\ 0, & \text{otherwise} \end{cases}$$

is the rectangular pulse. The *duty ratio* of the pulse train is $d = \dfrac{2T_1}{T}$.

Letting the duty ratio $d = 1/2$, we will find c and t_0 such that the shifted signal, $y(t) = x(t-t_0) + c$, has purely imaginary DTFS coefficients.

The given signal $x(t)$ is even, but $y(t) = x(t-t_0)+c$ becomes odd with $t_0 = T_1$ and $c = -A/2$. By utilizing the CTFS properties, we can show that the FS coefficients Y_k of $y(t)$ are

$$Y_k = \begin{cases} 0, & k = 0 \\ e^{-jk\frac{2\pi}{T}T_1} X_k = e^{-jk\frac{\pi}{2}} X_k, & \text{otherwise} \end{cases}$$

where X_k's are the FS coefficients of $x(t)$. According to the result found in Q1, X_k's are real-valued, and they become zero when k is even. Hence, Y_k's are purely imaginary.

6. A periodic sequence $x[n]$ is defined as

$$x[n] = \sum_{m=-\infty}^{\infty} g[n+Nm],$$

where

$$g[n] = \begin{cases} 1, & |n| \leqslant 2 \\ 0, & \text{otherwise} \end{cases}$$

and $N > 5$.

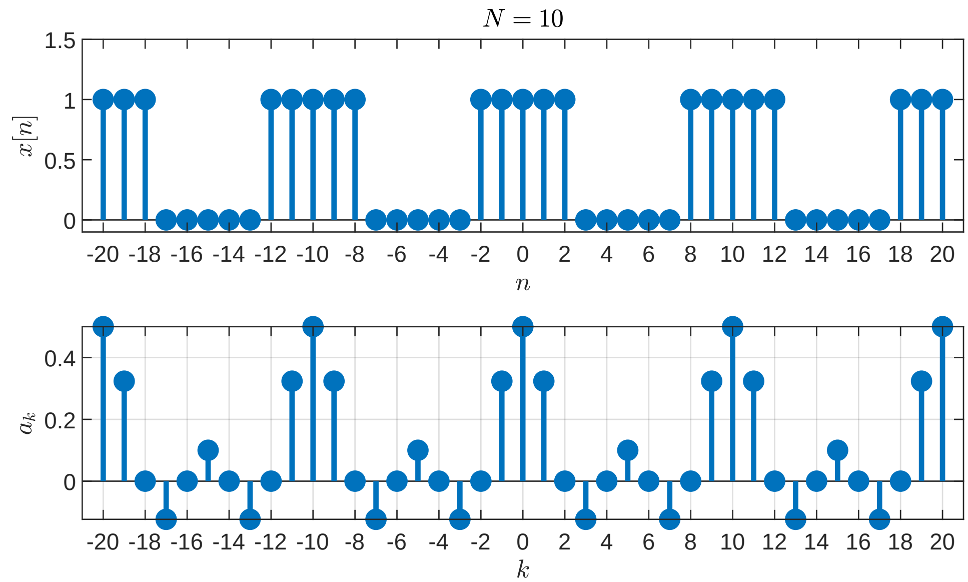
i. To compute the DTFS coefficients a_k of $x[n]$ directly from DTFS analysis equation, the MATLAB function code is given below.

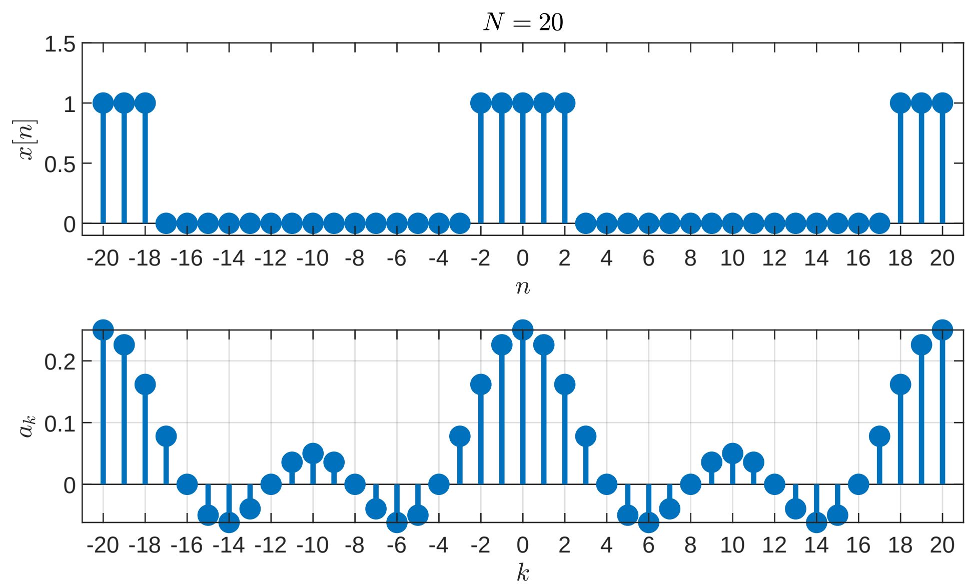
```
function ak = ee301hw3q6dtfs(N)
%% i. directly from DTFS analysis equation
if N<6
    disp('N should be greater than 5.')
    ak = 0;
    return;
end
None = 2;
ak = zeros(1,N);
for k=0:N-1
    for n=-None:None
        ak(k+1) = ak(k+1)+(1/N)*exp(-1j*k*(2*pi/N)*n);
    end
end

% The DTFS coefficients ak should be real-valued since x[n] is
% real-valued and even.
ak = real(ak); % The imaginary parts come from the rounding error.
ak(abs(ak)<1e-16) = 0; % eliminate the rounding error
```

ii. To compute the DTFS coefficients a_k of $x[n]$ by using the result found in part (a) of Q5, the MATLAB function code is given below.

```
function ak = ee301hw3q6dtfsq5parta(N)
%% ii. using the result found in part (a) of Q5
if N<6
    disp('N should be greater than 5.')
    ak = 0;
    return;
end
None = 2;
d = (2*None+1)/N;
A = 1;
ak = [(A*d), (A*sin(pi*(1:N-1)*d)./sin(pi*(1:N-1)/N)/N)];
ak(abs(ak)<1e-16) = 0; % eliminate the rounding error
```

(a) For $N = 10$ and $N = 20$, the plots of $x[n]$ and the DTFS coefficients a_k are given below.





To get these plots, the MATLAB function code is given below.

```
function ee301hw3q6parta(N)
%% obtain x[n]
n = -20:20; % sample index

% g[n]: rectangular pulse
None = 2;
gn = @(n) 1*(abs(n)<=None); % g[n]=1 if |n|<= None

% x[n]: rectangular pulse train
xn = zeros(1,numel(n));
for m=-2:2
    xn = xn+gn(n+m*N);
end

%% obtain the DTFS coefficients ak
akone = ee301hw3q6dtfsq5parta(N); % ak for k=0,1,...,N-1
k = -20:20;
ak = zeros(1,numel(k));
for i=k
    ak(i+21) = akone(1+mod(i,N));
end

% visualization of x[n] and the DTFS coefficients ak
subplot(211)
stem(n,xn,'filled','LineWidth',2);
xlabel('$$n$$','Interpreter','latex');
ylabel('$$x[n]$$','Interpreter','latex');
title(['$$N=' num2str(N) '$$'],'Interpreter','latex');
set(gca,'XLim',[-21 21],'YLim',[-0.1 1.5],'XTick',-20:2:20);

subplot(212)
stem(k,ak,'filled','LineWidth',2);grid on;
xlabel('$$k$$','Interpreter','latex');
ylabel('$$a_k$$','Interpreter','latex');
set(gca,'XLim',[-21 21],'XTick',-20:2:20);
```

(b) In MATLAB, we can find the Fast Fourier Transform (FFT) of $x[n]$ by using `fft` command. Let the FFT of $x[n]$ be

$$X_k = \text{fft}(x[n]),$$

where $n \in [0, N-1]$.

For $N = 10$ and $N = 20$, we have found X_k's and compared them with a_k's, where $k \in [0, N-1]$. Then, we have observed that there is a relation between X_k and a_k as follows:

$$a_k = \frac{1}{N} X_k$$

Tables showing X_k, X_k/N and a_k versus $k \in [0, N-1]$ are given on the next page with the MATLAB code used to obtain these tables.

N=10

k	Xk	XkOverN	ak
0	5	0.5	0.5
1	3.2361	0.32361	0.32361
2	0	0	0
3	-1.2361	-0.12361	-0.12361
4	0	0	0
5	1	0.1	0.1
6	0	0	0
7	-1.2361	-0.12361	-0.12361
8	0	0	0
9	3.2361	0.32361	0.32361

N=20

k	Xk	XkOverN	ak
0	5	0.25	0.25
1	4.5201	0.22601	0.22601
2	3.2361	0.1618	0.1618
3	1.5575	0.077877	0.077877
4	0	0	0
5	-1	-0.05	-0.05
6	-1.2361	-0.061803	-0.061803
7	-0.7936	-0.03968	-0.03968
8	0	0	0
9	0.71592	0.035796	0.035796
10	1	0.05	0.05
11	0.71592	0.035796	0.035796
12	0	0	0
13	-0.7936	-0.03968	-0.03968
14	-1.2361	-0.061803	-0.061803
15	-1	-0.05	-0.05
16	0	0	0
17	1.5575	0.077877	0.077877
18	3.2361	0.1618	0.1618
19	4.5201	0.22601	0.22601

```
function ee301hw3q6partb(N)
%% obtain fft(x[n])
n = 0:N-1; % sample index

% g[n]: rectangular pulse
None = 2;
gn = @(n) 1*(abs(n)<=None); % g[n]=1 if |n|<= None

% x[n]: rectangular pulse train
xn = zeros(1,numel(n));
for m=-2:2
    xn = xn+gn(n+m*N);
end

% The FFT of x[n] should be real-valued since x[n] is real-valued and
% even.
Xk = fft(xn);
Xk = real(Xk); % The imaginary parts come from the rounding error.

%% obtain the DTFS coefficients ak
ak = ee301hw3q6dtfsq5parta(N); % ak for k=0,1,...,N-1
k = 0:N-1;

%% display results
k = k(:); Xk = Xk(:); XkOverN = Xk(:)/N; ak = ak(:); % set columns
T = table(k, Xk, XkOverN, ak);
disp(T);
```