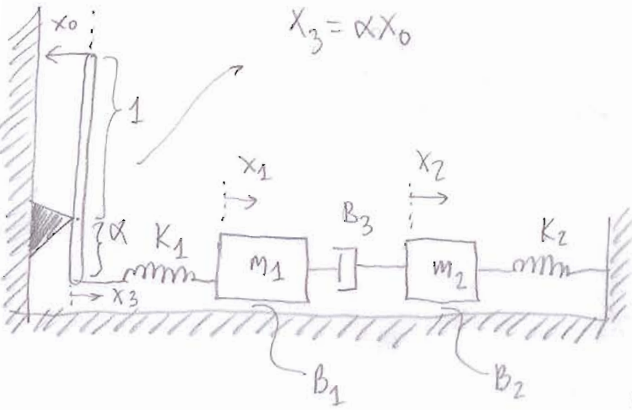
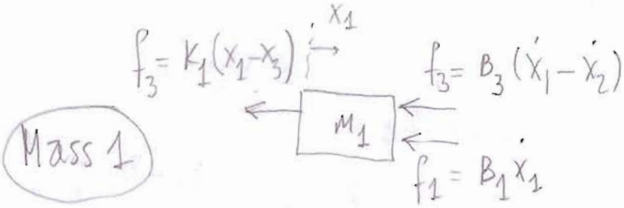
Translational / Rotational Electro_Mechanical Systems
Additional Examples

March 05, 2015
Handout

Example: (Translational Mechanical)



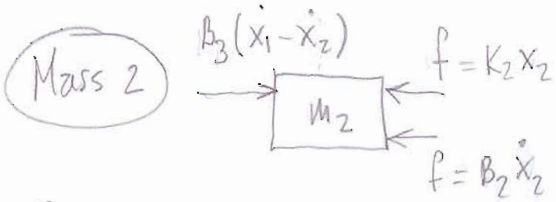
Selection: input: x_0: displacement
output: x_2: displacement



$$m_1 \ddot{x}_1 = -K_1(x_1 - x_3) - B_1\dot{x}_1 - B_3(\dot{x}_1 - \dot{x}_2)$$

$x_3 = \alpha x_0$

$$\boxed{m_1\ddot{x}_1 = -(B_1 + B_3)\dot{x}_1 - K_1 x_1 + K_1 \alpha x_0 + B_3 \dot{x}_2} \quad (*)$$



$$m_2 \ddot{x}_2 = B_3(\dot{x}_1 - \dot{x}_2) - K_2 x_2 - B_2 \dot{x}_2$$

$$\boxed{m_2 \ddot{x}_2 = -(B_2 + B_3)\dot{x}_2 + B_3 \dot{x}_1 - K_2 x_2} \quad (**)$$

Can I obtain the TF Directly? Yes, but a bit tedious:

From $(*)$ $\underbrace{\left(ms^2 + (B_3 + B_1)s + K_1\right)}_{\triangleq D_1(s)} X_1 = K_1 \alpha X_0 + B_3 s X_2 \longrightarrow X_1 = \dfrac{K_1 \alpha}{D_1(s)} X_0 + \dfrac{B_3 s}{D_1(s)} X_2$ — $(***)$

From $(**)$ and $(***)$: $\underbrace{\left(m_2 s^2 + (B_2 + B_3)s + K_2\right)}_{\triangleq D_2(s)} X_2 = B_3 s X_1 = B_3 s \left[\dfrac{K_1 \alpha}{D_1(s)} X_0 + \dfrac{B_3 s}{D_1(s)} X_2\right]$

$$D_1 D_2 X_2 = B_3 K_1 \alpha s X_0 + B_3^2 s^2 X_2 \longrightarrow \left[D_1 D_2 - B_3^2 s^2\right] X_2 = B_3 K_1 \alpha s X_0$$

$$\boxed{T(s) = \frac{X_2(s)}{X_0(s)} = \frac{B_3 K_1 \alpha s}{D_1(s) D_2(s) - B_3^2 s^2}}$$

Can we obtain a state-space representation? Yes, again.
Maybe somewhat easier.

Define our states as $\underline{z}$

$$
\begin{aligned}
z_1 &= x_1 & \dot{z}_1 &= z_2 \\
z_2 &= \dot{x}_1 & \dot{z}_2 &= \ddot{x}_1 = -\frac{K_1}{m_1}x_1 + \frac{K_1\alpha}{m_1}x_0 - \frac{B_3+B_1}{m_1}\dot{x}_1 + \frac{B_3}{m_1}\dot{x}_2 \\
z_3 &= x_2 & & \\
z_4 &= \dot{x}_2 & &= -\frac{K_1}{m_1}z_1 - \frac{B_3+B_1}{m_1}z_2 + \frac{B_3}{m_1}z_4 + \frac{K_1\alpha}{m_1}x_0
\end{aligned}
$$

$$
\begin{aligned}
\dot{z}_3 &= z_4 \\
\dot{z}_4 &= -\frac{K_2}{m_2}x_2 - \frac{B_2+B_3}{m_2}\dot{x}_2 + \frac{B_3}{m_2}\dot{x}_1 \\
&= \frac{B_3}{m_2}z_2 - \frac{K_2}{m_2}z_3 - \frac{B_2+B_3}{m_2}z_4
\end{aligned}
$$

Then we can write:

$$
\dot{\underline{z}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K_1}{m_1} & -\frac{B_1+B_3}{m_1} & 0 & \frac{B_3}{m_1} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{B_3}{m_2} & -\frac{K_2}{m_2} & -\frac{B_2+B_3}{m_2} \end{bmatrix}}_{\underline{\underline{A}}} \underline{z} + \underbrace{\begin{bmatrix} 0 \\ \frac{K_1\alpha}{m_1} \\ 0 \\ 0 \end{bmatrix}}_{\underline{\underline{B}}} x_0
$$

$$
x_2 = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}}_{\underline{\underline{C}}} \underline{z} \qquad (D = 0)
$$

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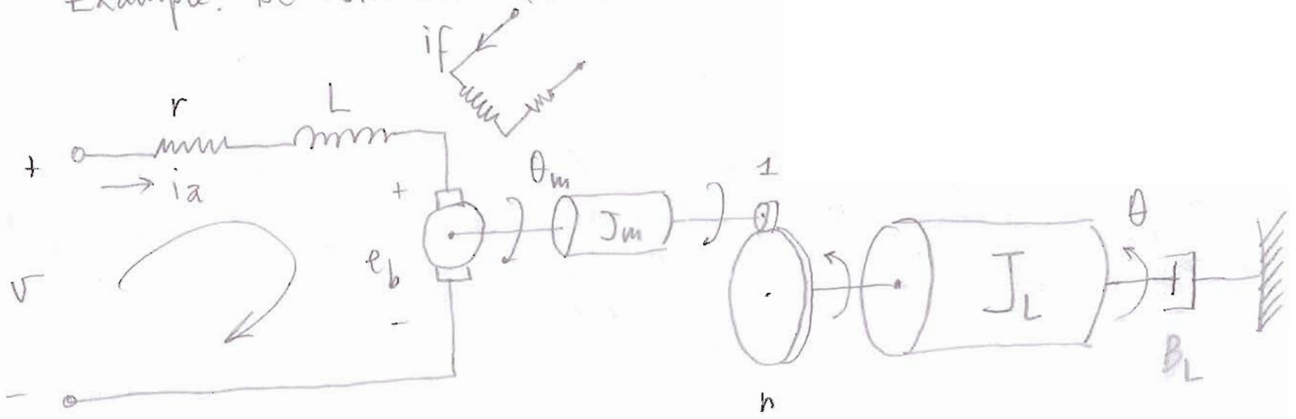
Example: DC Motor with Gear and Load.



Assumption: i_f constant. (Hence motor is armature controlled).

Let $T = K_a i_a$ is the motor torque. Question: Find $\frac{\theta(s)}{V(s)}$

KVL: $-V + r i_a + L\frac{di_a}{dt} + e_b$

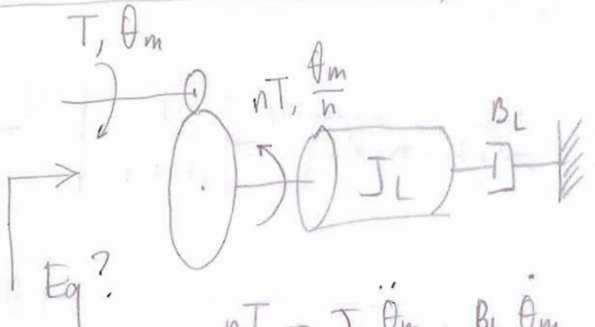
(1) — $\boxed{V - E_b = (sL + r) I_a}$

We also have the Back-EMF of the DC motor

(2) — $\boxed{e_b = K_b w_m}$

$T = (s^2 J_{eq} + s B_{eq})\theta_m$

(3) — $\boxed{T = (s J_{eq} + B_{eq}) w_m}$

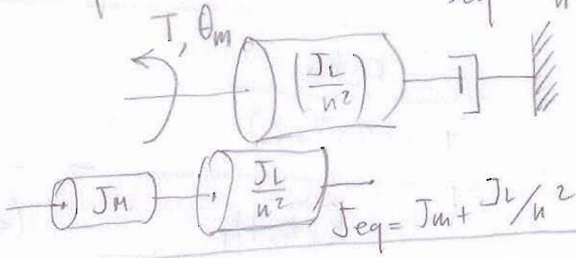


$$nT = J_L \frac{\ddot{\theta}_m}{n} + B_L \frac{\dot{\theta}_m}{n}$$

$$T = \left(\frac{J_L}{n^2}\right)\ddot{\theta}_m + \left(\frac{B_L}{n^2}\right)\dot{\theta}_m$$

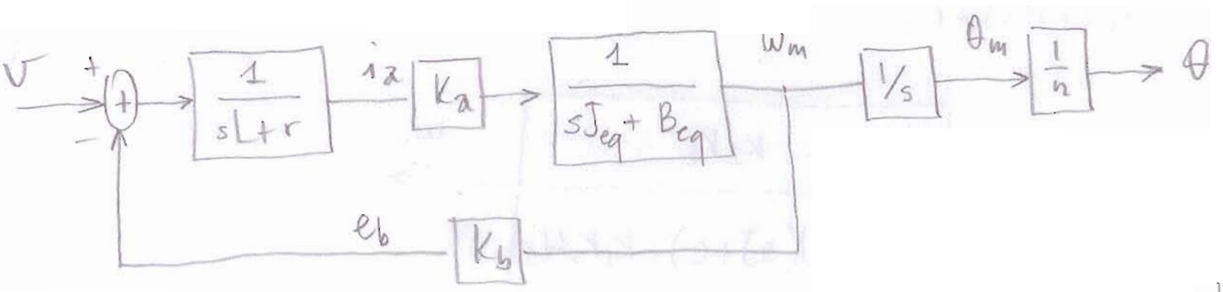
Hence the above system with gear is equivalent to

$B_{eq} = \frac{B_L}{n^2}$



$J_{eq} = J_m + J_L/n^2$

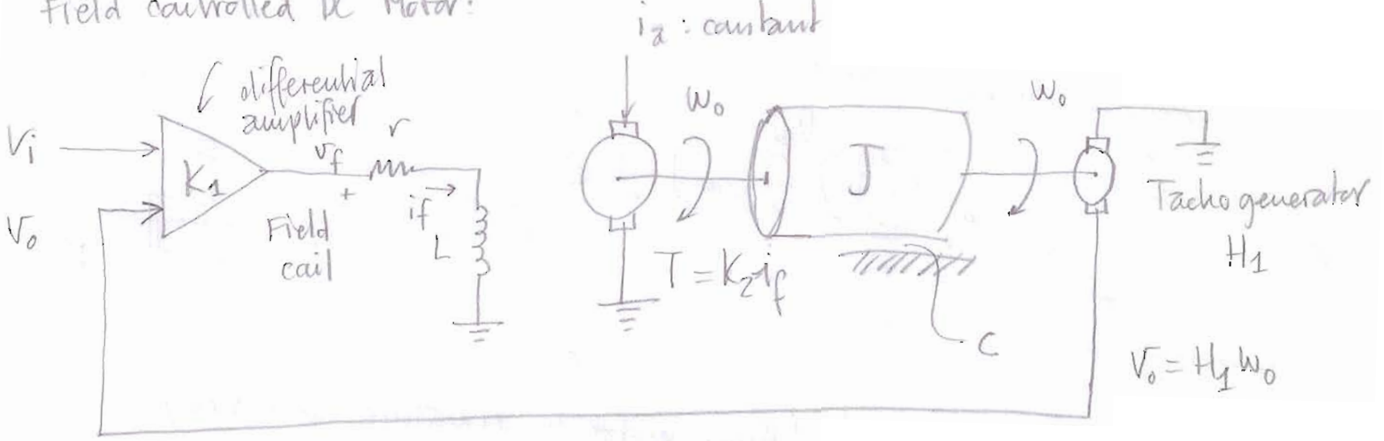
Construct the Block Diagram:



Block diagram simplification $T(s) = \frac{\theta(s)}{V(s)} = \frac{K_a/n}{s(sL+r)(sJ_{eq}+B_{eq}) + K_a K_b s}$

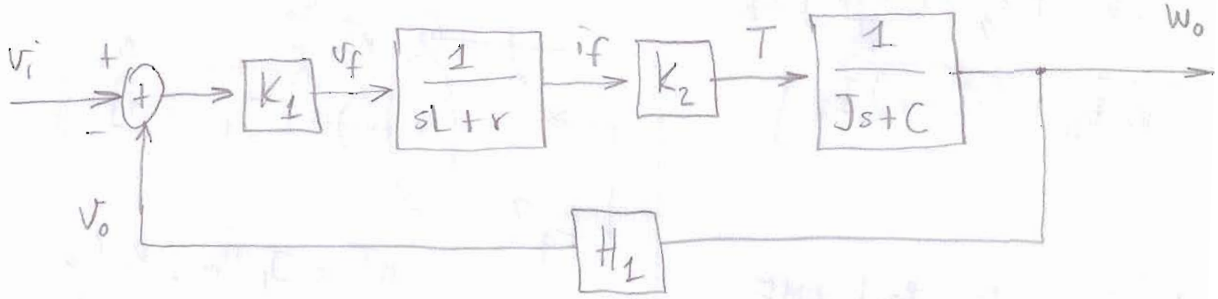
handout-c

Example (Q6 from homework)
Field controlled DC Motor:

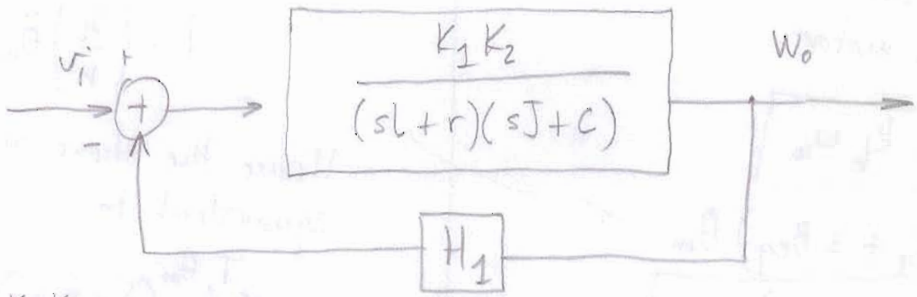


v_i : input
w_o : output

Let us incrementally construct the block diagram.



$\equiv$



$$\frac{G}{1+GH} = \frac{\dfrac{K_1K_2}{(sL+r)(sJ+C)}}{1+\dfrac{K_1K_2H_1}{(sL+r)(sJ+C)}} = \frac{K_1K_2}{(sL+r)(sJ+C)+K_1K_2H_1}$$

$\equiv$