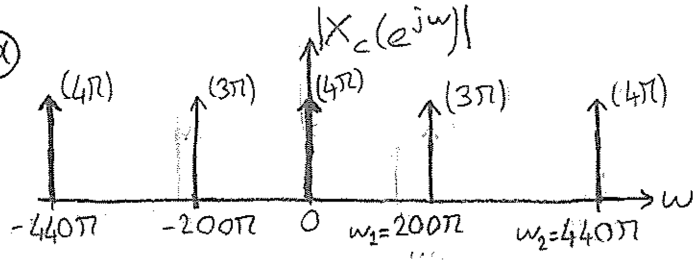
EE 301 HOMEWORK 5 SOLUTIONS

Q1

Input $x_c(t) = 2 + 3 \cdot \cos(w_1 t) + 4 \cdot \sin(w_2 t)$ for $-\infty < t < \infty$

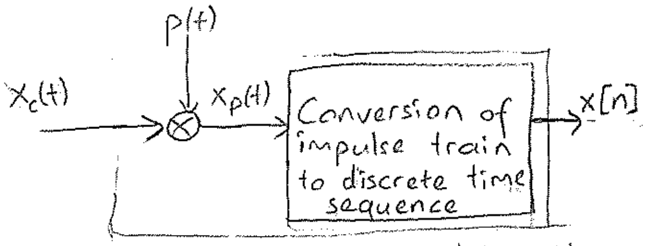
a)

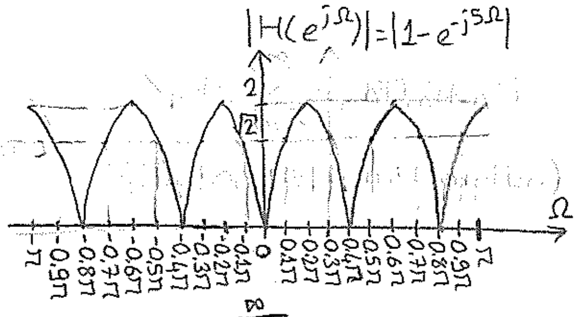


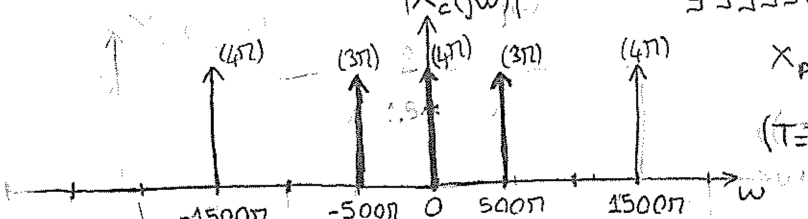
If $w_1 = 200\pi$ and $w_2 = 440\pi$, then $w_M = w_2 = 440\pi$.
To avoid from aliasing, $w_s > 2 w_M = 880\pi$.

$\boxed{\text{Thus, Nyquist rate for } x_c(t) \text{ is strictly greater than } 880\pi.}$

b)

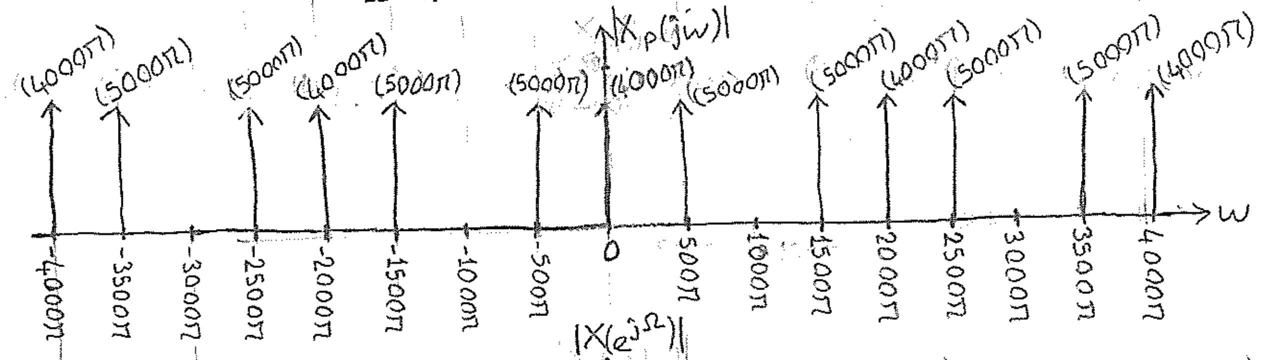


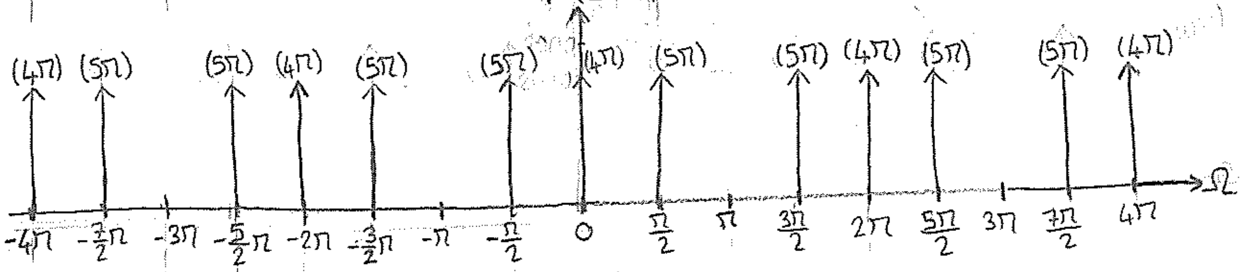




$$x_p(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\, \delta(t - nT)$$

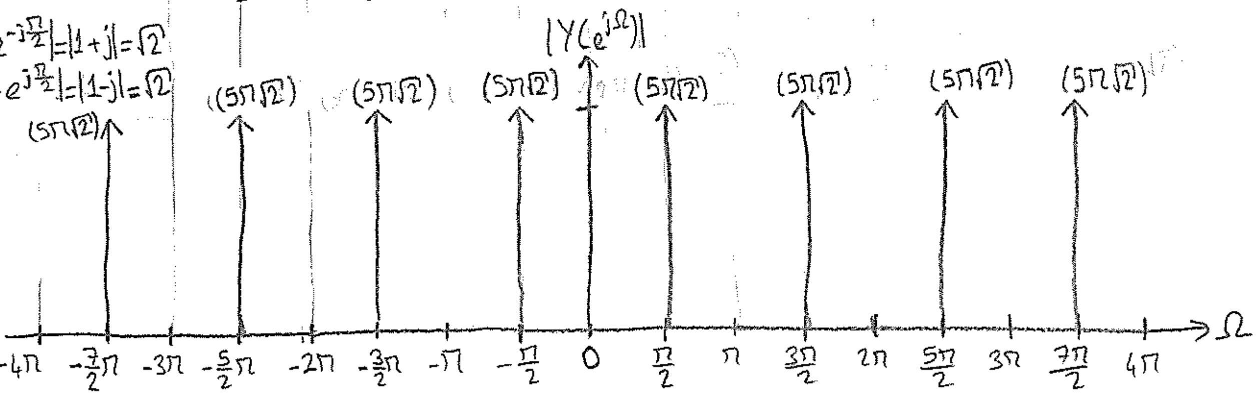
($T = T_1 = 1000$ Hz for this problem)

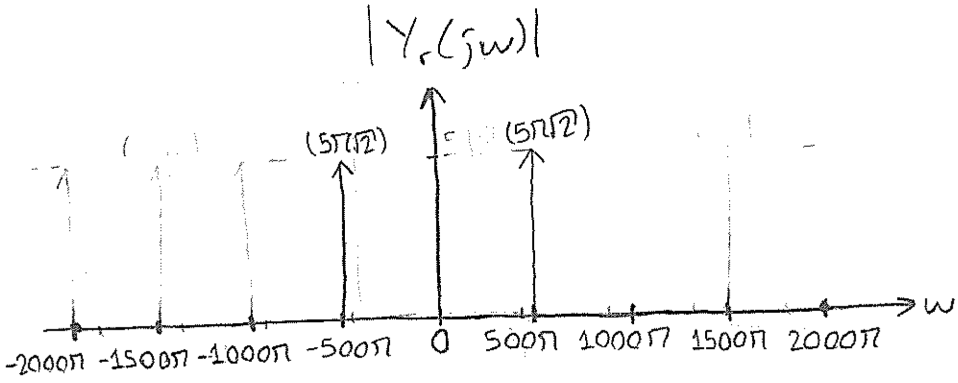




$|H(e^{j\frac{\pi}{2}})| = |1 - e^{-j\frac{5\pi}{2}}| = |1 + j| = \sqrt{2}$

$|H(e^{-j\frac{\pi}{2}})| = |1 - e^{j\frac{5\pi}{2}}| = |1 - j| = \sqrt{2}$





(c) Since $\left[|H(e^{j\Omega})|\right]\Big|_{\Omega=0} = |1-e^{-j0}| = 0$, the term 2 does not exist at the output signal $y_r(t)$.

For the interval $-\pi \leq \Omega \leq \pi$;

$|H(e^{j\Omega})| = 0$ for $\Omega = \underbrace{0, \frac{2\pi}{5}, \frac{4\pi}{5}}_{}$, (0 is trivial solution)

Multiply them with $T_1 = 1000$ (Consider the D/C block in the system)

w_1 and w_2 should be chosen among 400π and 800π.

$$\boxed{w_1 \neq w_2 \text{ and } w_1, w_2 \in \{400\pi, 800\pi\}}$$

Q2 $H(s)=\dfrac{1}{s(s+3)}$

a) * $H(s)=\dfrac{1}{s(s+3)}=\dfrac{A}{s}+\dfrac{B}{s+3}$

where

$A=\left[s.H(s)\right]_{s=0}=\left[\cancel{s}.\dfrac{1}{\cancel{s}(s+3)}\right]_{s=0}=\dfrac{1}{3}$

$B=\left[(s+3)H(s)\right]_{s=-3}=\left[\cancel{(s+3)}\dfrac{1}{s\cancel{(s+3)}}\right]_{s=-3}=-\dfrac{1}{3}$

* Thus, $H(s)=\dfrac{1}{3}\cdot\dfrac{1}{s}+\left(-\dfrac{1}{3}\right)\cdot\dfrac{1}{s+3}$

* The term $\dfrac{1}{s+3}$ has the following possible expressions.

$e^{-3t}u(t)\overset{L}{\longleftrightarrow}\dfrac{1}{s+3}$ ROC: $Re\{s\}>-3$ (1)

$-e^{-3t}u(-t)\overset{L}{\longleftrightarrow}\dfrac{1}{s+3}$ ROC: $Re\{s\}<-3$ (2)

* The term $\dfrac{1}{s}$ has the following possible expressions

$u(t)\overset{L}{\longleftrightarrow}\dfrac{1}{s}$ ROC: $Re\{s\}>0$ (3)

$-u(-t)\overset{L}{\longleftrightarrow}\dfrac{1}{s}$ ROC: $Re\{s\}<0$ (4)

* All possible expressions for the impulse response, $h(t)$,

Expression 1: From (1) and (3), $h_1(t)=\frac{1}{3}u(t)+\left(-\frac{1}{3}\right)\cdot e^{-3t}u(t)$ ROC: $Re\{s\}>0$

Expression 2: From (1) and (4), $h_2(t)=\frac{1}{3}\left[-u(-t)\right]+\left(-\frac{1}{3}\right)\cdot e^{-3t}u(t)$ ROC: $-3<Re\{s\}<0$

Expression 3: From (2) and (3), $h_3(t)=\frac{1}{3}u(t)+\left(-\frac{1}{3}\right)\cdot\left[-e^{-3t}u(-t)\right]$ ROC: Nowhere

Expression 4: From (2) and (4), $h_4(t)=\frac{1}{3}\left[-u(-t)\right]+\left(-\frac{1}{3}\right)\left[-e^{-3t}u(-t)\right]$ ROC: $Re\{s\}<-3$

* If the ROC of a rational system function is the right-half plane to the right of the rightmost pole, then this system is causal. From this definition, Expression 1, $h_1(t)$, is causal whereas the other expressions are noncausal. $\boxed{\text{Notice that only } h_1(t)=0 \text{ for } t<0 \text{ (causal)}}$

* If the ROC of a rational system function is the left-half plane to the left of the leftmost pole, then this system is anticausal. From this definition, only Expression 4, $h_4(t)$ is anticausal. $\boxed{\text{Only } h_4(t)=0 \text{ for } t>0 \text{ (Anticausal)}}$

(b) It is not possible to find an expression for the impulse response $h(t)$ corresponding to a stable system because none of the expressions has an ROC which includes $j\omega$-axis ($Re\{s\}=0$). Therefore, all four expressions of $h(t)$ given in part (a) correspond to unstable systems.

(c) $G(s) = \frac{1}{(s-\alpha)(s+3)}$ where α is real.

* For $G(s)$ to be causal, from the definition in part (a),

$$\boxed{ROC: Re\{s\} > \max(-3, \alpha)}$$

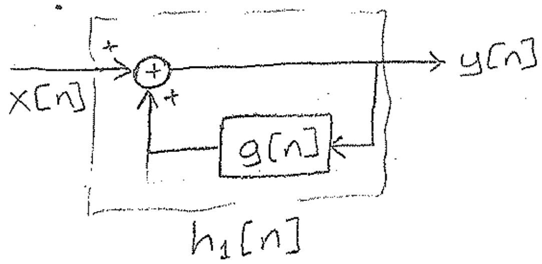
* For $G(s)$ to be stable, from the definition in part (b), ROC should include the $j\omega$-axis; therefore, $\boxed{\alpha < 0}$

* Notice that all poles of a causal system must have negative real parts for the system to be stable.

$G(s)$ already has a pole with negative real part, -3. If the other pole, α, has a negative real part, then the system is stable.

Q3

(a)



$g[n] = a\,\delta[n-1]$

where $a > 1$.

$$y[n] = x[n] + g[n] * y[n]$$
$$= x[n] + a\,\delta[n-1] * y[n]$$
$$= x[n] + a\,y[n-1]$$

$$Y(z) = X(z) + a z^{-1} Y(z)$$

$$\boxed{H_1(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - a z^{-1}}}$$ ROC: $|z| > |a| > 1$ (since the system is causal)

$$\boxed{h_1[n] = a^n u[n] \text{ where } a > 1}$$ (Unstable)

The system that has the impulse response $h_1[n]$ is unstable since its ROC does not include the unit circle ($|z| = 1$).

(b) $$\boxed{H_2(z) = K \cdot \frac{1 - B z^{-1}}{1 - \frac{1}{a} z^{-1}}, \quad \text{ROC}: |z| > \left|\frac{1}{a}\right|}$$

$$H_2(1) = 1 = K \cdot \frac{1 - B}{1 - \frac{1}{a}} \Longrightarrow \boxed{K = \frac{1 - \frac{1}{a}}{1 - B}}$$

(c) $$|H_2(e^{j\Omega})| = \underbrace{\left|\frac{1 - \frac{1}{2}}{1 - \frac{3}{2}}\right|}_{=1} \cdot \left|\frac{1 - \frac{3}{2} e^{-j\Omega}}{1 - \frac{1}{2} e^{-j\Omega}}\right| \Longrightarrow \boxed{\Big[|H_2(e^{j\Omega})|\Big]_{\Omega=0} = \left|\frac{1 - \frac{3}{2} \cdot 1}{1 - \frac{1}{2} \cdot 1}\right| = 1}$$

$$\boxed{\Big[|H_2(e^{j\Omega})|\Big]_{\Omega=\frac{\pi}{2}} = \left|\frac{1 - \frac{3}{2}(-j)}{1 - \frac{1}{2}(-j)}\right| = \sqrt{\frac{13}{5}}}$$

$$\boxed{\Big[|H_2(e^{j\Omega})|\Big]_{\Omega=\pi} = \left|\frac{1 - \frac{3}{2}(-1)}{1 - \frac{1}{2}(-1)}\right| = \frac{5}{3}}$$

$$\boxed{\Big[|H_2(e^{j\Omega})|\Big]_{\Omega=\frac{3\pi}{2}} = \left|\frac{1 - \frac{3}{2}(+j)}{1 - \frac{1}{2}(+j)}\right| = \sqrt{\frac{13}{5}}}$$

(d) If the two systems are cascaded, $H(z)=H_1(z).H_2(z)=?$

$$H(z)=H_1(z).H_2(z)$$

$$H(z)=\frac{1}{1-\alpha z^{-1}}\cdot\frac{1-\beta z^{-1}}{1-\frac{1}{\alpha}z^{-1}} \quad \text{with ROC: } |z|>|\alpha|>1 \text{ if } \beta\neq\alpha$$

For the system with response $H(z)$ to be stable, β should be equal to α for pole zero cancellation.

For $\alpha=\beta$,

$$H(z)=\frac{1}{1-\frac{1}{\alpha}z^{-1}} \quad \text{with ROC: } |z|>\left|\frac{1}{\alpha}\right|$$

(It includes the unit circle, $|z|=1$. Therefore, it is stable if $\alpha=\beta$.)

Q4) $h[n]=\left(\frac{1}{3}\right)^{(n-n_0)}u[n-n_0] \xleftrightarrow{z} H(z)=z^{-n_0}\frac{1}{1-\left(\frac{1}{3}\right)z^{-1}}$, ROC: $|z|>\left|\frac{1}{3}\right|=\frac{1}{3}$

$x[n]=\left(-\frac{1}{2}\right)^n u[n] \xleftrightarrow{z} X(z)=\frac{1}{1-\left(-\frac{1}{2}\right)z^{-1}}$, ROC: $|z|>\left|-\frac{1}{2}\right|=\frac{1}{2}$

a) The ROC of $X(z).H(z)$ is at least the intersection of ROC of $H(z)$ and ROC of $X(z)$. Therefore, the ROC of $X(z).H(z)$ includes $|z|>\frac{1}{2}$. Since the ROC of $X(z).H(z)$ includes the unit circle ($|z|=1$), $\boxed{\text{the system is stable.}}$

$\boxed{\text{The system is causal if } n_0 \geq 0.}$ Notice that $h[n]=0$ for all $n<0$ if the system is causal. However, for $n_0<0$, $h[n]\neq 0$ for some $n<0$.

b) $$Y(z)=X(z).H(z)=\frac{1}{1+\frac{1}{2}z^{-1}}\cdot z^{-n_0}\frac{1}{1-\frac{1}{3}z^{-1}}=z^{-n_0}\left[\frac{A}{1+\frac{1}{2}z^{-1}}+\frac{B}{1-\frac{1}{3}z^{-1}}\right]$$

$$A=\left[\left(1+\frac{1}{2}z^{-1}\right)H(z)\right]\Big|_{z=-\frac{1}{2}}=\frac{1}{1-\frac{1}{3}(-2)}=\frac{3}{5}$$

$$B=\left[\left(1-\frac{1}{3}z^{-1}\right)H(z)\right]\Big|_{z=\frac{1}{3}}=\frac{1}{1+\frac{1}{2}\cdot 3}=\frac{2}{5}$$

$$\boxed{Y(z)=\frac{3}{5}z^{-n_0}\frac{1}{1+\frac{1}{2}z^{-1}}+\frac{2}{5}z^{-n_0}\frac{1}{1-\frac{1}{3}z^{-1}}}$$

$$\boxed{y[n]=\frac{3}{5}\left(-\frac{1}{2}\right)^{n-n_0}u[n-n_0]+\frac{2}{5}\left(\frac{1}{3}\right)^{n-n_0}u[n-n_0]}$$

c) As the ROC of $Y(z)$ includes the unit circle ($|z|=1$), the system is stable. Therefore, Fourier transform of $y[n]$ exists (ROC of $Y(z)$: $|z|>\frac{1}{2}$)

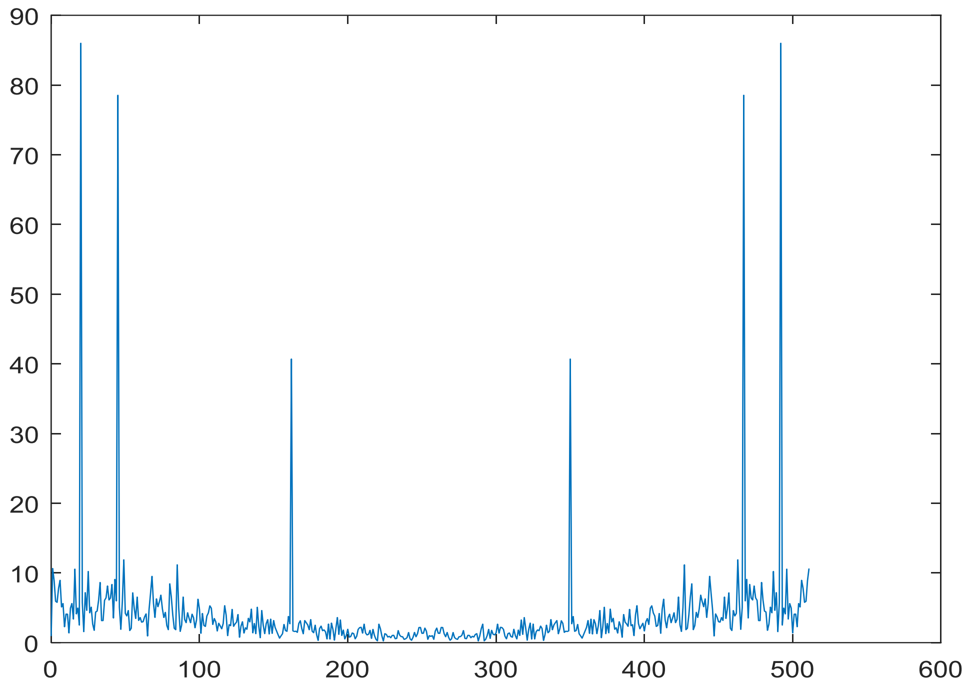
Fourier Transform of $y[n]$

$$y[n] \xleftrightarrow{F} \frac{3}{5}e^{-j\Omega n_0}\frac{1}{1+\frac{1}{2}e^{-j\Omega}}+\frac{2}{5}\cdot e^{-j\Omega n_0}\cdot\frac{1}{1-\frac{1}{3}\cdot e^{-j\Omega}}$$

Q5) PART A

I)

The X[k] vs. k plot from 0 to 511 is given as follows:



II)

The dominant (peak) values of magnitude of X[k]

Y = 85.9878 85.9878 78.5307 78.5307 40.6881 40.6881

The indices of the dominant frequencies in magnitude of X[k]

I = 21 493 46 468 163 351

PART B

I)

The sequence $x[n]$ is real-valued, which yields $x[n] = x^*[n]$.
$X(e^{j\Omega}) = F\{x[n]\} = F\{x^*[n]\} = X^*(e^{-j\Omega})$.

Real parts of are even, i. e., $Re\{X(e^{j\Omega})\} = Re\{X^*(e^{-j\Omega})\}$.

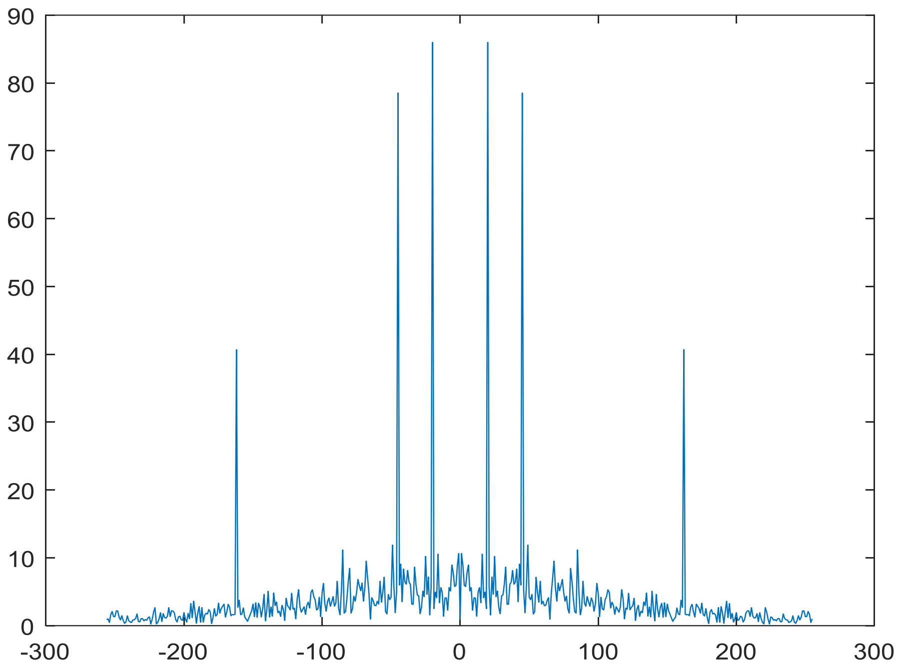
Imaginal parts of are odd, i. e., $Im\{X(e^{j\Omega})\} = -Im\{X^*(e^{-j\Omega})\}$.

$$|X(e^{j\Omega})|^2 = \left|\left(Re\{X(e^{j\Omega})\}\right)^2 + \left(Im\{X(e^{j\Omega})\}\right)^2\right| = \left|\left(Re\{X^*(e^{-j\Omega})\}\right)^2 + \left(-Im\{X^*(e^{-j\Omega})\}\right)^2\right|$$

$$|X^*(e^{-j\Omega})|^2 = |(Re\{X^*(e^{-j\Omega})\})^2 + (Im\{X^*(e^{-j\Omega})\})^2|$$

From the last two equations, $|X(e^{j\Omega})| = |X^*(e^{-j\Omega})|$.

The X[k] vs. k plot from -256 to 255 is given as follows:



The dominant (peak) values of magnitude of X[k]

Y = 85.9878 85.9878 78.5307 78.5307 40.6881 40.6881

The indices of the dominant frequencies in magnitude of X[k]

I = 237 277 212 302 95 419

After shifting the indices as the fftshifted signal starts from -256, all the indices are symmetric.

I = -20 20 -45 45 -162 162

II) We can hear 3 dominant frequencies the corresponding indices of which are 20, 45, 165.

III) We have already obtained $f_s = 48000$ from the audioread function. Recall from part (A), [xn,Fs]=audioread('hw5audio.wav'). N=512.

From the given formula, $f_k = \frac{kf_s}{N}$,

- Index $k = 20$ corresponds to the frequency $f_k = \frac{20\times48000}{512} = 1875\ Hz.$
- Index $k = 45$ corresponds to the frequency $f_k = \frac{45\times48000}{512} = 4218.75\ Hz.$
- Index $k = 162$ corresponds to the frequency $f_k = \frac{162\times48000}{512} = 15187.5\ Hz.$

MATLAB CODE

```
clear all
close all
clc

k=[0:511]';
[xn, Fs]=audioread('hw5audio.wav')

%%%PART A

%%PART I
%length of x[n] is 500
length(xn)

%12 zero padding to reach 512-length sequence
xn1=[xn' zeros(1,12)]';

%X[k]=DFT of x[n]
Xk=fft(xn1)

%Magnitude of X[k]
MagXk=abs(Xk)

%Plot of Magnitude of X[k] vs. k
plot(k, MagXk')

%%PART II
%Find the index of dominant frequency
MagX=MagXk;
%the number of peaks
maxN=6;

%The dominant values of  Magnitude of X[k]
Y=zeros(1,maxN);
%The index of the dominant values of  Magnitude of
X[k]
I=zeros(1,maxN);

%Find the first N maximum number of peaks
for i=1:maxN
    [Y(i), I(i)]=max(MagX)
    MagX(I(i))=0;
end
```

```
%%%PART B

%%PART I

Xk=fftshift(Xk)

%Magnitude of X[k]
MagXk=abs(Xk)

%Plot of Magnitude of X[k] vs. k
plot(k-256, MagXk')

%Find the index of dominant frequency
MagX=MagXk;
%the number of peaks
maxN=6;

%The dominant values of  Magnitude of X[k]
Y=zeros(1,maxN);
%The indices of the dominant values of  Magnitude of
X[k]
I=zeros(1,maxN);

%Find the first N maximum number of peaks
for i=1:maxN
    [Y(i), I(i)]=max(MagX)
    MagX(I(i))=0;
end

%Due to fftshift we should also shift the indices by
(256+1)
I=I-(256+1)
%The indices of the dominant values of  Magnitude of
X[k] are symmetric
%with respect to origin, 0.
```