Chapter 6: Discrete Fourier Transform, DFT

DFT is a Fourier Transform where the time-domain signal and its Fourier Transform are both finite-length sequences. Hence it is a unitary transformation from a set of numbers to another set of numbers. This is a great flexibility in comparison to other forms of Fourier Transform in Digital Processing. Furthermore it is possible to process infinite length sequences (or signals) using block-based approaches with the finite-length representation of DFT.

DFT is defined for only finite-length sequences x[n], which are limited to $0 \le n \le N-1$. Note that this is just a convention and it is always possible to handle any finite length sequence which is not confined to the aforementioned interval through time shifting. One critical fact about DFT is that it depends on a length parameter N. Hence whenever we mention about DFT, we should consider its length and therefore N-point DFT. DFT result may change significantly depending on the DFT length, N.

While DFT has its own foundation, its roots are based on the Discrete Fourier Series (DFS) and DTFT. In fact, DFT can be obtained from DFS and DTFT in two ways.

DFT- DFS Relation: DFT coefficients are just one period of DFS coefficients for $0 \le k \le N-1$.

DFT- DTFT Relation: N-point DFT can be obtained from DTFT by sampling with a sampling interval, $\frac{2\pi}{N}$, in frequency i.e.,

$$X[k] = X\left(e^{j\Omega}\right)\Big|_{\Omega=\frac{2\pi}{N}k}, \quad 0 \le k \le N-1$$

Since we sample DTFT $X(e^{j\Omega})$ in frequency, we have a periodic repetition in time domain which may cause aliasing in time. Hence sampling interval and N should be selected appropriately in order to avoid aliasing.

The analysis and synthesis equations for N-point DFT are given as,

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad (1)$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn} = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-kn} \quad (2)$$

DFT, DFS and DTFT Relationships

Consider a finite-length sequence x[n] defined in 0≤n≤N-1. It is possible to obtain a periodic sequence $\tilde{x}[n]$ as follows,

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN], \; and \qquad (3)$$

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1 \\ 0, & otherwise \end{cases} \qquad (4)$$

$$\tilde{x}[n] = x[((n))_N] = x[(n \; MODULO \, N)] \qquad (5)$$

Note that the periodic sequence has DFS analysis and synthesis equations as,

$$\tilde{X}[k] = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn}, \qquad -\infty < k < \infty$$

$$\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{X}[k] e^{j\frac{2\pi}{N}kn}, \qquad -\infty < n < \infty$$

Both of the summations in DFS equations are finite and $\tilde{x}[n]$, and $\tilde{X}[k]$ are periodic infinite length sequences with a period of N.

Using (4), we can write,

$$X\left(e^{j\Omega}\right) = \sum_{n=0}^{N-1} x[n] e^{-j\Omega n} = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\Omega n}$$

$$X[k] = X\left(e^{j\Omega}\right)\Big|_{\Omega=\frac{2\pi}{N}k}, \qquad 0 \le k \le N-1$$

$$\tilde{X}[k] = \frac{1}{N} X[k]$$

Above equation shows the relationship between DFS, DFT and the DTFT.

Sampling the DTFT

This part shows that sampling the DTFT corresponds to repetition in time by N. Let x[n] be a N-point finite sequence with DTFT, $X(e^{j\Omega})$,

$$X\left(e^{j\Omega}\right) = \sum_{m=-\infty}^{\infty} x[n] e^{-j\Omega m}$$

Let Y[k] be the samples from DTFT,

$$Y[k] = X\left(e^{j\Omega}\right)\Big|_{\Omega=\frac{2\pi}{N}k} = \sum_{m=-\infty}^{\infty} x[m] e^{-j\frac{2\pi}{N}km} \qquad (6)$$

$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} Y[k] e^{j\frac{2\pi}{N}kn} \qquad (7)$$

Combining (6) and (7), we obtain

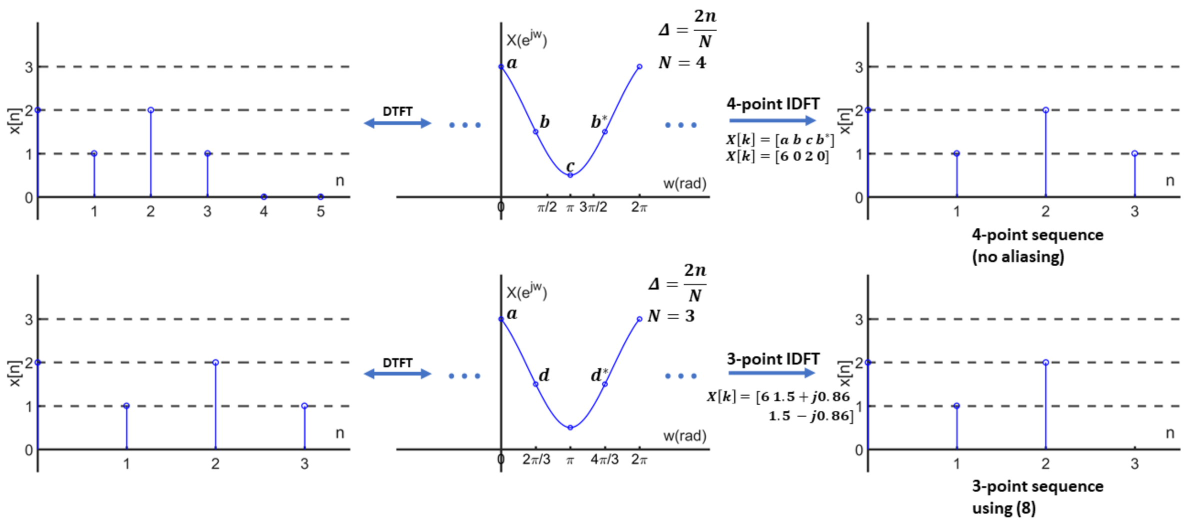
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]\left[\frac{1}{N}\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(n-m)}\right]$$

Note that

$$\frac{1}{N}\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(n-m)} = \begin{cases} 1, & for\ n-m=rN,\ r\ \text{int} \\ 0, & otherwise \end{cases}$$

Then

$$y[n] = \sum_{r=-\infty}^{\infty} x[n-rN] \qquad (8)$$



Ex:

Let x[n]=δ[n]+δ[n-1]+δ[n-2]. Its DTFT is given as

$$X\left(e^{j\Omega}\right) = e^{-j\Omega}\left(1+2\cos(\Omega)\right)$$

Let N=3, then DFT samples from $X(e^{j\Omega})$ at Ω=2π/3k are X[0]=3, X[1]=0, X[2]=0. Inverse DFT returns x[n]=δ[n]+δ[n-1]+δ[n-2] which is the original sequence and there is no time-aliasing.

If N=2 is selected then the DFT samples from $X(e^{j\Omega})$ at Ω=2π/2k=πk are X[0]=3, X[1]=1.

2-point inverse DFT of this sequence is

$$\hat{x}[n] = 2\delta[n] + \delta[n-1]$$

This corresponds to time-aliasing and can be easily seen from equation (8)

Linearity

If the two finite length sequences $x_1[n]$ and $x_2[n]$ are linearly combined, i.e.,

$$x_3[n] = ax_1[n] + bx_2[n], \text{ then the DFT}$$
$$X_3[k] = aX_1[k] + bX_2[k]$$

If $x_1[n]$ and $x_2[n]$ have different lengths, N_1 and N_2 ($N_2 > N_1$) respectively, then $x_1[n]$ should be padded by N_2-N_1 zeros to have the same length for both sequences.

Ex: Linearity + zero padding