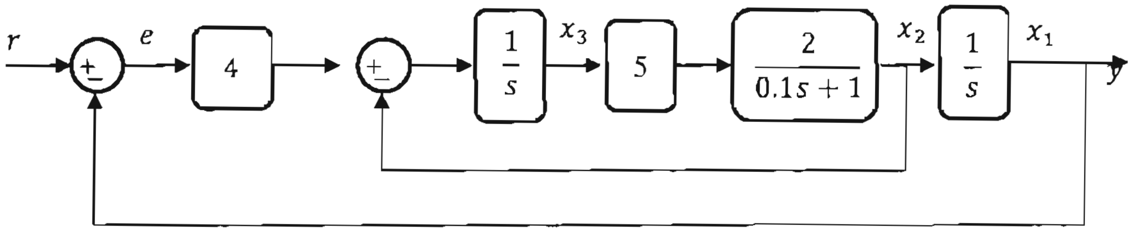
EE 302 – Assignment #3

Given: March 28, 2019; Deadline: April 10, 2019 15:40

There will be a box to drop the assignments in front of D-226. The box will be removed after 15:40.

Q1. Consider the system below given in block diagram form.



(a) Derive the steady-state error for a unit-step reference input,
(b) Derive the steady-state error for a unit-ramp input,
(c) What is the "Type" of the system? Justify your answer.
(d) Suppose x_1, x_2 and x_3 are defined as the "states" of the system. Obtain the state-space representation for this system.
(e) Implement this system in Matlab-Simulink. Simulate for unit-step and unit-ramp reference inputs. Provide the response plots and verify your results in (a) and (b).

Q2. Consider the open-loop "plant" given by the transfer function

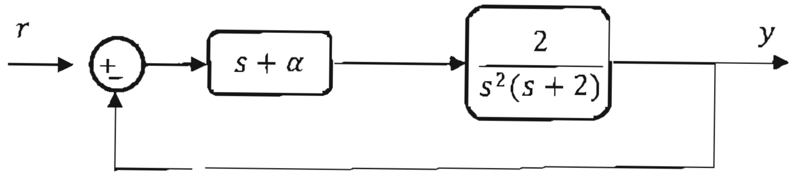
$$G(s) = \frac{1}{s(s+1)(s+2)}$$

The system is to be controlled by a "Proportional" P-controller in a unity negative feedback configuration.

(a) Sketch the block diagram of the overall system,
(b) Determine the full range of K (K>0 as well as K<0) for the closed-loop system to be stable. For what value of K is the system *critically stable*?

Q3. Consider again the system described in Q2. Sketch the Root-Locus of the closed-loop system for all values of K>0. Obtain all relevant information and mark on your sketch.

Q4. Consider the unity negative feedback system below. Plot the root-locus as the design parameter α is varied $\alpha: 0 \to \infty$. Determine the value of α such that the dominant closed-loop poles gives us a damping ratio of 0.5.



EE 302 Spring 2019 – Homework #3 Guide Solutions
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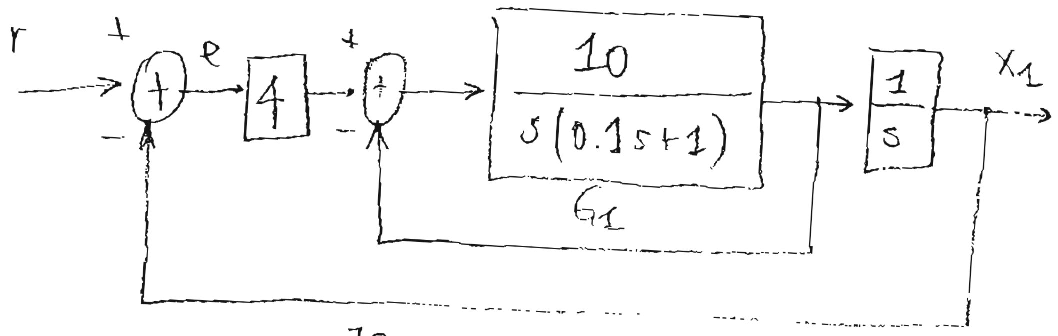
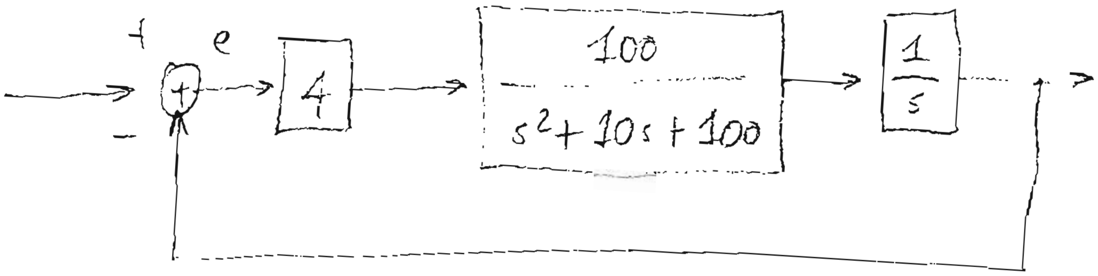
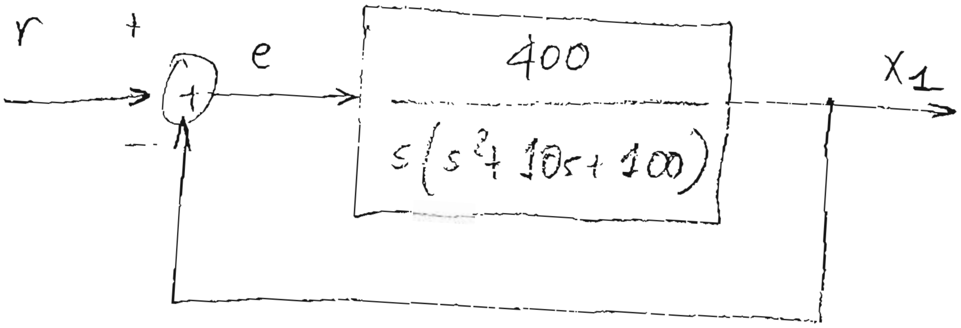
Q1.) (a) Let us simplify the block diagram into a single (outside) negative feedback loop. Combine cascade blocks and eliminate the inside feedback loop



$$\frac{G_1}{1+G_1} = \frac{\dfrac{10}{s(0.1s+1)}}{1+\dfrac{10}{s(0.1s+1)}} = \frac{10}{0.1s^2+s+10} = \frac{100}{s^2+10s+100}$$





This is clearly a Type 1 system. Therefore, the e_{ss} to a unit step response is zero. Alternatively derive $E(s)$ and find $\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$ where $R(s) = \frac{1}{s}$.

(b) let us solve (b) with the ground-up approach
(alternative stated to a.)

$$E(s) = R(s) - \frac{400}{s(s^2+10s+100)} E(s)$$

$$\left[1 + \frac{400}{s(s^2+10s+100)}\right] E(s) = R(s) \longrightarrow E(s) = \frac{1}{1+\frac{400}{s(s^2+10s+100)}} R(s)$$

$$E(s) = \frac{s(s^2+10s+100)}{s^3+10s^2+100s+400} R(s) \quad \text{where } R(s) = \frac{1}{s^2}$$

for unit ramp input.

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \cancel{s} \frac{\cancel{s}(s^2+10s+100)}{s^3+10s^2+100s+400} \frac{1}{s^{\cancel{2}}}$$

$$= \frac{100}{400} = \frac{1}{4}.$$

(c) Answered in (a)

(d) States are given on the block diagram. We need to find $\dot{x}_1, \dot{x}_2, \dot{x}_3$ and put into matrix form

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}\right\}$$

input (.)

Note: State-space representation is in time domain!

let us start with the output. Clearly; $\boxed{y(t) = x_1(t)}$

we have $X_1(s) = \frac{1}{s} X_2(s) \longrightarrow X_2 = sX_1 \longrightarrow \boxed{\dot{x}_1(t) = x_2(t)}$

area code/alan kodu

1.(d) continued

$$X_2(s) = \frac{10}{0.1s+1} \cdot X_3(s) \longrightarrow 0.1\,s\,X_2(s) + X_2(s) = 10\,X_3(s)$$

$$s X_2(s) + 10\,X_2(s) = 100\,X_3(s)$$

$$s X_2(s) = -10\,X_2(s) + 100\,X_3(s)$$

$$\mathcal{L}^{-1}\{\cdot\} \longrightarrow \boxed{\dot{x}_2(t) = -10\,x_2(t) + 100\,x_3(t)}$$

Now

$$X_3(s) = \frac{1}{s}\left[4 \cdot E(s) - X_2(s)\right] = \frac{1}{s}\left[4 \cdot R(s) - 4X_1(s) - X_2(s)\right]$$

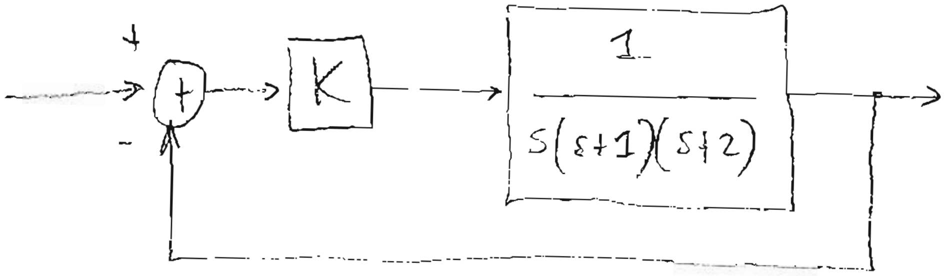
$$\mathcal{L}^{-1}\{\cdot\} \quad s X_3(s) = -4X_1(s) - X_2(s) + 4R(s) \quad \text{input}$$

$$\boxed{\dot{x}_3(t) = -4x_1(t) - x_2(t) + 4r(t)}$$

Put into matrix form: (all variables a function of t)

$$\boxed{\begin{gathered}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & -10 & 100\\ -4 & -1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 4\end{bmatrix} r \\ y = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}\end{gathered}}$$

Q.2) (a) The block diagram of the system:



(b) We will need to use the Routh-Hurwitz criterion.
(Note that we know how to sketch Root-Locus only for $K>0$)

Find the closed loop characteristic Equation:

$$q(s) = 1 + K\frac{1}{s(s+1)(s+2)} = 0$$

$$\longrightarrow (s^2+s)(s+2) + K = 0$$

$$s^3 + 2s^2 + s^2 + 2s + K = 0 \longrightarrow s^3 + 3s^2 + 2s + K = 0$$

Construct the Routh-Array:

$$\begin{array}{c|ccc} s^3 & 1 & 2 & 0 \\ s^2 & 3 & K & \\ s^1 & \frac{6-K}{3} & 0 & \\ s^0 & K & & \end{array}$$

We want a stable system $\longrightarrow$ No closed-loop poles on the right-half plane
$\longrightarrow$ No sign change in 1st column of Routh Array.

Hence we need $K>0$ and $\frac{6-K}{3}>0 \longrightarrow K<6$

$\longrightarrow$ $\boxed{0<K<6}$ for stability.

System is "critically stable" when poles are on the $j\omega$ axis $\longrightarrow$ Sustained oscillation. This happens when $\boxed{K=6}$

Q3) Again consider the characteristic equation

$$q(s) = 1 + K\frac{1}{s(s+1)(s+2)} = 1 + K\frac{N(s)}{D(s)}$$

m = 0

n = 3

polynom orders.

$\#$ of branches $= \max(m, n) = 3$

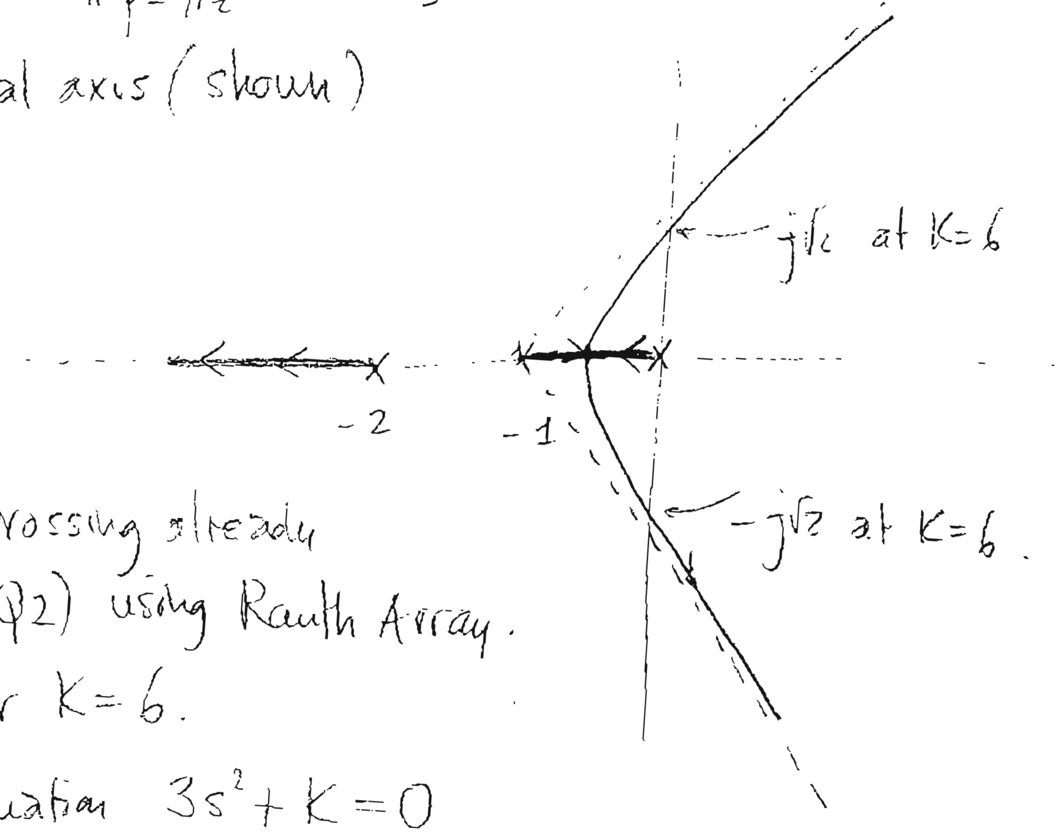
$\#$ of asymptotes $= |m - n| = 3$

Asymptote angles $\phi = \frac{\pm 180}{3}(2l+1)$ → $+60^\circ$, $+180^\circ$, -60°

Centroid (Asymptote intersection with real axis)

$$\sigma_0 = \frac{\sum p - \sum z}{\# p - \# z} = \frac{0-1-2}{3} = -\frac{3}{3} = -1$$

Locus on real axis (shown)



$j\omega$ axis crossing already found in Q2) using Routh Array.

Happens for K = 6.

Auxiliary equation $3s^2 + K = 0$

$$3s^2 + 6 = 0$$

$$s^2 = -2 \longrightarrow \boxed{s_{1,2} = \pm j\sqrt{2}}$$

PS: Also find the Break-away point!

Q4) We need to find the characteristic equation again and put into the standard form

$$q(s) = 1 + \alpha \frac{N(s)}{D(s)}$$

$$q(s) = 1 + (s+\alpha)\frac{2}{s^2(s+2)} = 0$$

$$= s^2(s+2) + 2s + 2\alpha = 0$$

$$= (s^3 + 2s^2 + 2s) + 2\alpha = 0$$ Divide by $(s^3 + 2s^2 + 2s)$

$$= 1 + \alpha \frac{2}{s(\underbrace{s^2 + 2s + 2}_{\text{factorize}})}$$

$\Delta = b^2 - 4ac = 4 - 4(2) = -4$

$$s_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}$$

$$s_{1,2} = \frac{-2 \pm j2}{2} = -1 \pm j$$

$$q(s) = 1 + \alpha \frac{2}{s(s+1+j)(s+1-j)}$$

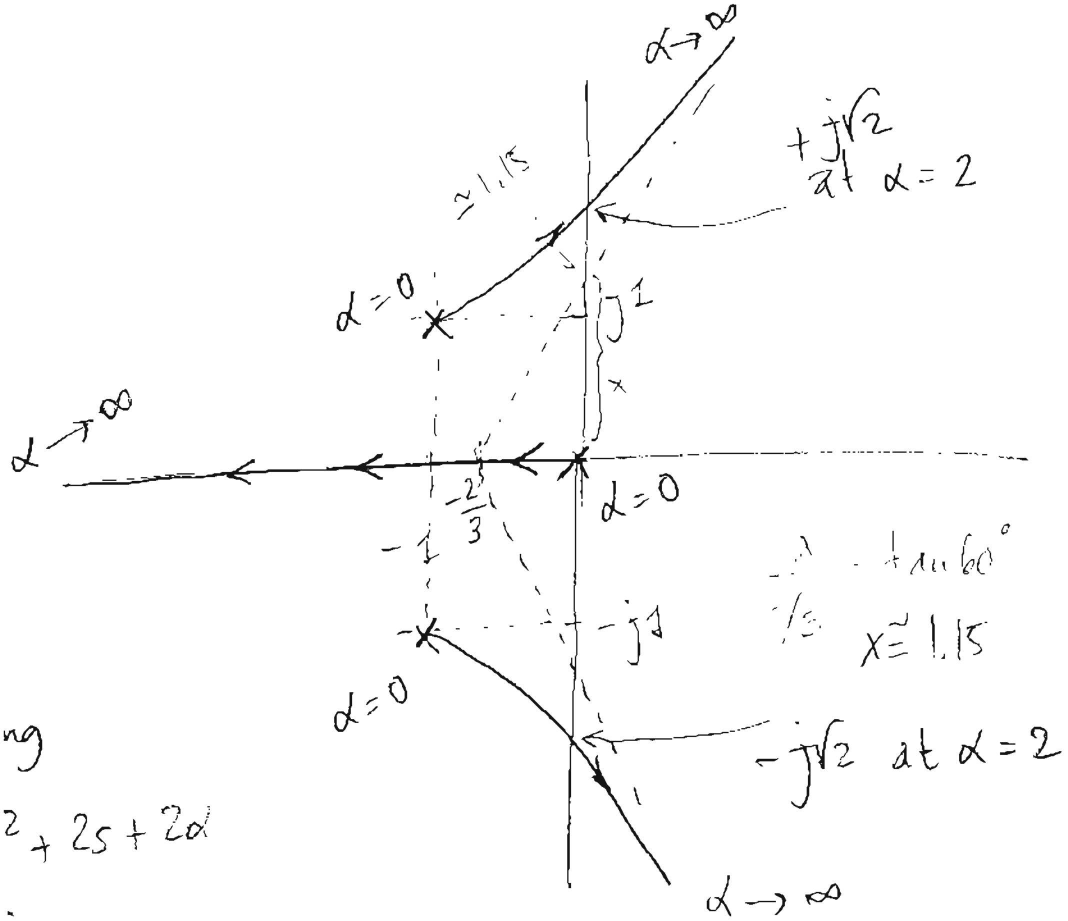
$m = 0$
$n = 3$ again.

# of Branches $= \max(m,n) = 3$
# of asymptotes $= |m - n| = 3$

$$\phi = \frac{\pm 180}{3}(2l+1) \rightarrow \phi = \begin{matrix} +60^\circ \\ -60^\circ \\ +180^\circ \end{matrix}$$

Locus on the real axis (shown)

Centroid $\sigma_0 = \dfrac{\sum p - \sum z}{\# p - \# z} = \dfrac{0 - 1 - 1}{3} = -\dfrac{2}{3}$



jω axis crossing

$q(s) = s^3 + 2s^2 + 2s + 2\alpha$

Routh - Array:

s^3	1	2	0
s^2	2	2α	
s^1	a	0	
s^0	2α		

$$a = \frac{\cancel{2}\cdot 2 - \cancel{2}\alpha}{\cancel{2}} = 2 - \alpha$$

jω axis crossing when

$2 - \alpha = 0 \rightarrow \boxed{\alpha = 2}$

Auxiliary equation:

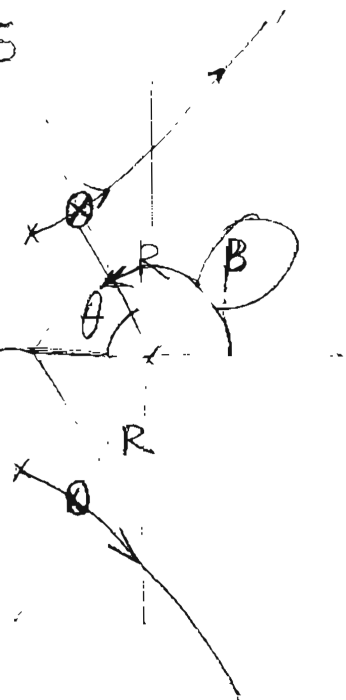
$2s^2 + 4 = 0 \quad s^2 = -2$

$\boxed{s = \pm j\sqrt{2}} \quad \sqrt{2} \simeq 1.41$

Break-away point ?
None expected.

Now let us find the α value for $\xi = 0.5$

$$\xi = 0.5 \longrightarrow \cos\theta = 0.5$$
$$\longrightarrow \theta = 60^\circ = \frac{\pi}{3}$$



For the given damping ratio;
the complex conjugate pole locations are shown.
How can we find them? Pole angle is known

$$s_1 = Re^{j\beta} \quad ; \quad s_2 = Re^{-j\beta}$$

R : unknown but $\beta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

s_1 and s_2 needs to satisfy the characteristic equation

$$q(s) = s^3 + 2s^2 + 2s + 2\alpha = 0$$

$$q(s)\Big|_{s=s_1} = R^3 e^{j3\beta} + 2R^2 e^{j2\beta} + 2Re^{j\beta} + 2\alpha = 0$$

put into $a + jb$ form:

$$q(s)\Big|_{s=s_1} = R^3 \underbrace{e^{j2\pi}}_{1} + 2R^2 e^{j\frac{4\pi}{3}} + 2Re^{j\frac{2\pi}{3}} + 2\alpha = 0$$

this is a complex equation with
Re = 0
Im = 0
α, β unknowns.

$$= R^3 + 2R^2\left(\cos\frac{4\pi}{3} + j\sin\frac{4\pi}{3}\right) + 2R\left(\cos\frac{2\pi}{3} + j\sin\frac{2\pi}{3}\right) + 2\alpha = 0$$

$$= R^3 + 2R^2\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) + 2R\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) + 2\alpha = 0$$

$$\left(R^3 - R^2 - R + 2\alpha\right) + j\left(2R\frac{\sqrt{3}}{2} - 2R^2\frac{\sqrt{3}}{2}\right) = 0$$

$$Im = 0 \longrightarrow 2R\left(\frac{\sqrt{3}}{2} - R\frac{\sqrt{3}}{2}\right) = 0 \quad \boxed{R = 1}$$

$$Re\Big|_{R=1} = 0 \longrightarrow \cancel{1} - \cancel{1} - 1 + 2\alpha = 0$$

$$2\alpha = 1$$

$$\boxed{\alpha = \frac{1}{2}}$$

This result can be verified using Matlab.