Example: (Determination of the $j\omega$ crossing) $\quad G(s)H(s) = s^3+9s^2+11s-21$

$$G(s)H(s) = \frac{K}{(s-1)(s+3)(s+7)} \qquad q(s) = 1+G(s)H(s) = \emptyset$$

$$q(j\omega) = 1+G(j\omega)H(j\omega) = 0 \longrightarrow \frac{1}{(j\omega)^3+9(j\omega)^2+11(j\omega)-21} = -\frac{1}{K}$$

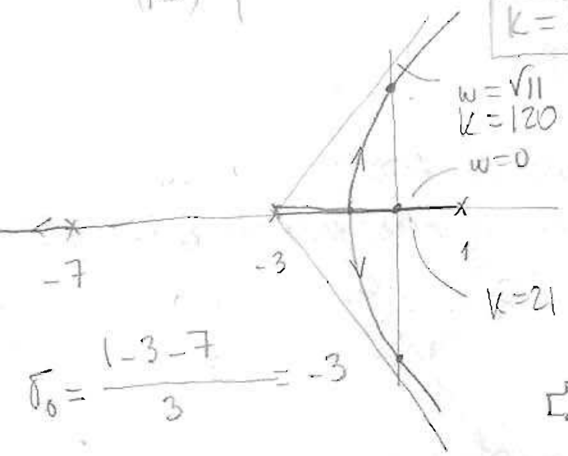
$$-j\omega^3 - 9\omega^2 + j11\omega - 21 + K = \emptyset \longrightarrow \left.\begin{aligned} -9\omega^2 - 21 + K &= \emptyset \quad (Re) \\ -\omega^3 + 11\omega &= \emptyset \quad (Im) \end{aligned}\right\}$$

(Im) Gives us : $\omega(\omega^2 - 11) = 0 \longrightarrow \boxed{\omega = 0}$ or $\boxed{\omega = \sqrt{11}}$

(Re) for $\omega = 0 \rightarrow -21 + K = \emptyset$ $\boxed{K = 21}$ $\qquad$ for $\omega = \sqrt{11} \longrightarrow -9(11) - 21 + K = 0$

$$-99 - 21 + K = 0$$

$$\boxed{K = 120}$$



$$\sigma_0 = \frac{1-3-7}{3} = -3$$

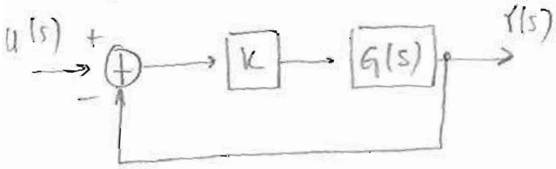
$\longrightarrow$ Also # of asymptotes $\dfrac{\pm 180(2n+1)}{\underbrace{|(n-m)|}_{3 \text{ asymptotes}}} = +60^\circ, -60^\circ, +180^\circ$

$\Rightarrow$ Also tells us system is stable for $\boxed{21 < K < 120}$

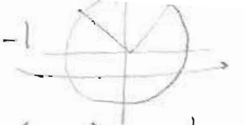
HW: Solve the same example using Routh-Hurwitz test to determine $j\omega$ axis crossing and range of K for stability.

See examples on

Example:



$G(s) = \frac{1}{s^4+1} \Rightarrow s^4 = -1$



$s =$

Rule 1: $n=4$; $m=0$ $\max(m,n) = 4$ branches in locus.

Rule 2: Starting & Ending points: 4 start at poles, all end at infinity (asymptotes)

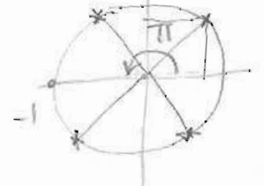
Rule 3: Symmetry.

Rule 4: Asymptotes: $\phi = \frac{\pm 180(2\ell+1)}{|n-m|} = \pm 45(2\ell+1) = +45^\circ, -45^\circ, +135^\circ, -135^\circ$

Rule 5: Centroid (Meeting point): $\sigma_0 = \frac{\sum p_j - \sum z_i}{n-m} = \frac{0}{4} = \emptyset$ Asymptotes meet at origin.

Open loop poles of the system:

$s^4 = -1 = e^{j(2k+1)\pi} \longrightarrow s = e^{j\frac{\pi}{4}(2k+1)}$

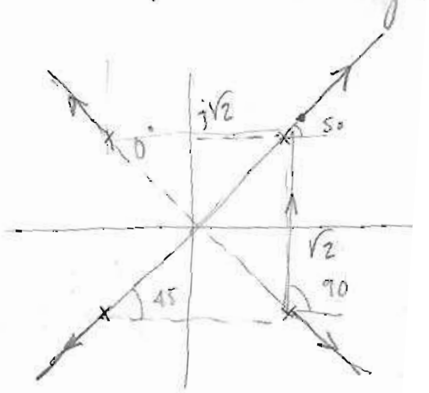


$s_1 = e^{j\pi/4} = \sqrt{2} + j\sqrt{2}$

$s_2 = e^{j3\pi/4} = -\sqrt{2} + j\sqrt{2}$

$s_3 = e^{+j5\pi/4} = -\sqrt{2} - j\sqrt{2}$

$s_4 = e^{j7\pi/4} = +\sqrt{2} - j\sqrt{2}$



Rule 6: No locus on real axis.

Rule 7: No break away points.

Rule 8: Angle of departure from poles: $-0^\circ - 90^\circ - 45^\circ - \phi = \pm 180(2\ell+1)$

$-135^\circ = \pm 180^\circ(2\ell+1)$

$= \pm 180 + 135 = -45^\circ$

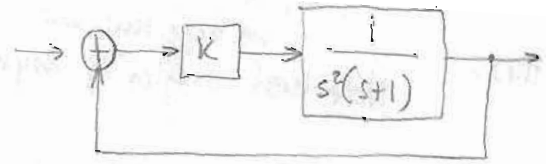
$+45^\circ$

$+135^\circ$

-135°

⇨ System always unstable $\forall K > 0$

Example:



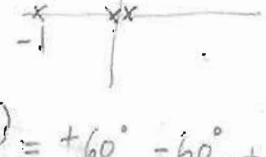
– System is of Type 2

$e_{ss} = 0$ for unit step or unit ramp.

– What about stability?

Construct the root-locus:

Rule 1: $n=3$, $m=0$ $\max(m,n)$: 3 branches of the locus.



Rule 2: Start at three poles, end at infinity.

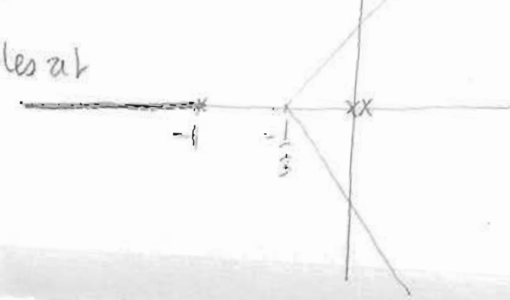
Rule 3: Symmetry

Rule 4: Asymptotes $|n-m|$: 3 asymptotes $\phi = \frac{\pm 180(2\ell+1)}{|n-m|} = +60^\circ, -60^\circ, +180^\circ$

Rule 5: Centroid $\sigma_0 = \frac{0+0+(-1)}{3} = -\frac{1}{3}$

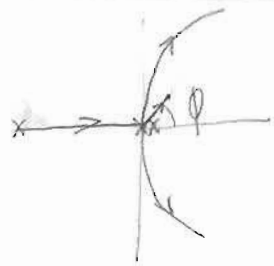
Rule 6: Locus on real axis

(odd num of zeros+poles at right)

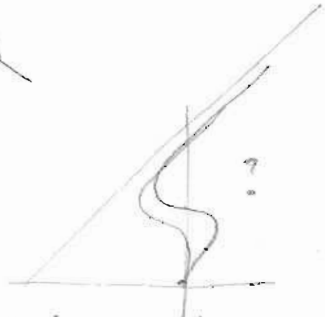


Rule 7: Break away: obvious $s=0$

Rule 9: Angle of departure from poles:



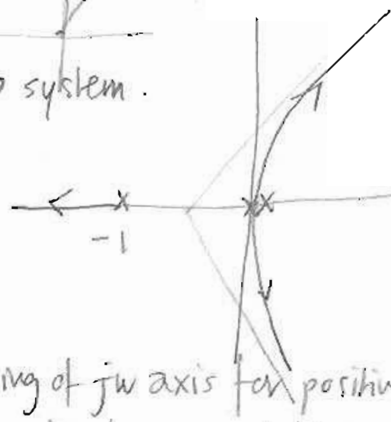
$$-\phi-\phi-0=\pm 180(2l+1)$$
$$-2\phi=\pm 180(2l+1)$$
$$\phi=\pm 90(2l+1)$$
$$\boxed{90^\circ}$$



Rule 8: Is there an intersection with $j\omega$ axis?

Routh Array: Find the denominator of the closed loop system.

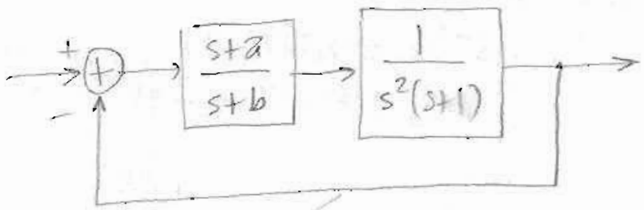
$$q(s)=1+\frac{K}{s^2(s+1)}=\frac{s^3+s^2+K}{s^2(s+1)}=0 \rightarrow s^3+s^2+K=\emptyset$$



s^3	1	0	0
s^2	1	K	0
s^1	-K	0	0
s^0	K		

$\rightarrow$ All zeros only when $K=0$ (no crossing of $j\omega$ axis for positive k)
$K>0$ always two sign changes. System is always unstable

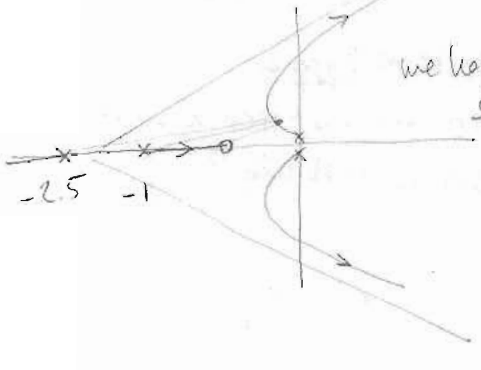
Now: Question: Can this system be made stable by the addition of another block into the system?



We are adding one ~~zero~~ and one pole.

Consider what happens with the centroid:

$$\sigma_0=\frac{0+0+(-1)-b+a}{3}<-\frac{1}{3} \text{ if } b>a$$



we hope to get something like this.

if $b>a$; the centroid will move to the left.
we hope this will create a stable zone.

Note that angles of asymptotes will stay the same.

But what happens with angle of departure?

$$-2\phi-0-0+0=\pm 180(2l+1)$$
$$\phi=\pm 90^\circ \text{ still the same.}$$

Try: $\left.\begin{matrix} a=0.5 \\ b=2.5 \end{matrix}\right\}$ And check with Routh-Hurwitz whether we can create a $j\omega$ crossing.

Closed loop transfer function and characteristic equation:

$$q(s)=0 \rightarrow 1+G(s)H(s)=0$$

$$1+\frac{s+a}{s+b}\cdot\frac{K}{s^2(s+1)}=0 \rightarrow (s+b)s^2(s+1)+(s+a)K=0$$

$$(s+b)(s^3+s^2)+sK+aK=0$$

$$s^4+s^3+bs^3+bs^2+Ks+aK=\emptyset$$

$$s^4+(b+1)s^3+bs^2+Ks+aK=0 \qquad a=0.5,\ b=2.5$$

Routh Array:

s^4	1	2.5	0.5K
s^3	3.5	K	$\emptyset$
s^2	$\frac{(3.5)(2.5)-K}{3.5}$	0.5K	$\emptyset$
s^1	c	$\emptyset$	$\emptyset$
s^0	d = 0.5K		

$$3.5\times2.5=7.85$$

$$c=\frac{\frac{7.85K-K^2}{3.5}-1.75K}{\frac{7.85K-K^2}{3.5}}=\frac{7.85K-K^2-6.125K}{7.85K-K^2}$$

$$=\frac{(1.725-K)\cancel{K}}{(7.85-K)\cancel{K}}=\frac{(1.725-K)}{(7.85-K)}$$

$d=0.5K$

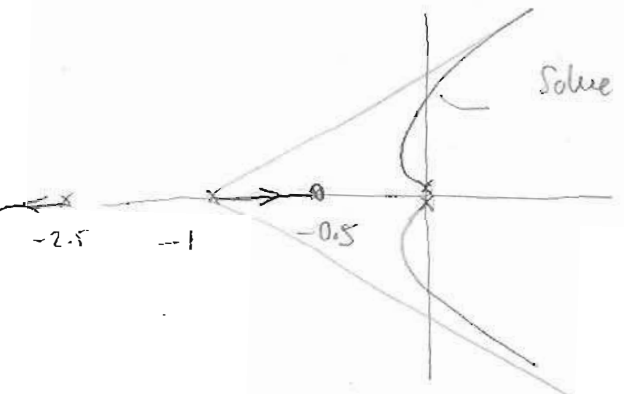
$K<7.85$ & $K<1.725$ or $K>7.85$

From s^2 we have $\frac{7.85-K}{3.5}>0$ $\boxed{K<7.85}$

$7.85-K>0$ ~~$K<7.85$~~ $\boxed{K<1.725}$ or $K>1.725$ & $\boxed{K>7.85}$

Therefore for $\boxed{K<1.725}$ the system has a stable region.

We have designed a stable system by adding $\frac{s+a}{s+b}$ and choosing K appropriately.



Solve $3.5s^2+K=0$

$3.5s^2+1.725=0$

$(K=1.725)$

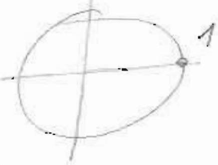
Another interesting question: What happens if $\underline{\bar{K}<0}$.?

We have the "complementary locus" → the poles will again move in the s-plane

roots of $1+\bar{K}G(s)H(s)$ with $\bar{K}<0 \longrightarrow K>0$

$$1-KG(s)H(s)=0$$

$$1-K\frac{R(s)}{P(s)}=0 \longrightarrow \frac{R(s)}{P(s)}=1 \qquad \boxed{\angle\frac{R(s)}{P(s)} = 180^\circ(2\ell)}$$



A new angle condition!

Using the new angle condition, we can derive:

1.) # of branches = max(m,n) same

2.) <u>Rule 2</u>: