EE302 Spring 2018 Homework 3 – Guide Solution
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Q1.) a) $q(s) = s^3 + 20s^2 + 10s + 400$

s^3	1	10	0
s^2	20	400	0
s^1	$a=-10$	0	
s^0	$b=400$		

$$a = \frac{20.10 - 400}{20} = -\frac{20\cancel{0}}{2\cancel{0}} = -10$$

There are two sign changes in the first column. System unstable with 2 right half plane (RHP) poles.

b) $q(s) = s^5 + 3s^4 + 2s^3 + 6s^2 + 3s + 1$

s^5	1	2	3	0
s^4	3	6	1	0
s^3	ε	$8/3$	0	
s^2	$6-8/\varepsilon$	1	0	
s^1	$\frac{8}{3}$	0		
s^0	1			

$\frac{3.2-6}{3} = 0$! zero first column, replace with ε

$$\frac{6\varepsilon - 8}{\varepsilon} = 6 - \frac{8}{\varepsilon} < 0$$

$$\frac{\left(6-\frac{8}{\varepsilon}\right)\frac{8}{3} - \varepsilon}{6 - \frac{8}{\varepsilon}} = \frac{8}{3} - \frac{\varepsilon}{\frac{6\varepsilon-8}{\varepsilon}} = \frac{8}{3} - \underbrace{\frac{\varepsilon^2}{6\varepsilon - 8}}_{\simeq -8} \simeq \frac{8}{3}$$

very small positive number.

for $\varepsilon > 0 \longrightarrow 6 - \frac{8}{\varepsilon} < 0 \longrightarrow$ 2 sign changes (very large positive $\frac{8}{\varepsilon}$)

for $\varepsilon < 0 \longrightarrow 6 - \frac{8}{\varepsilon} > 0 \longrightarrow$ 2 sign changes. (very large negative $\frac{8}{\varepsilon}$)

$\longrightarrow$ System unstable with 2 RHP poles.

Alternatively you may try to multiply $q(s)$ by known factors $(s+\alpha)$ and attempt to get rid of zero first column.

Q.1. c) $q(s) = s^4 - s^3 + 2s^2 - 4s - 8$

sign changes in $q(s)$ immediately concludes system is unstable.

Routh array:

s^4	1	2	-8
s^3	-1	-4	0
s^2	-2	-8	0
s^1	~~0~~ -4	~~0~~ 0	
s^0	-8		

$$\frac{-1.2-(-4)}{-1} = \frac{+2}{-1} = -2$$

$$\frac{-2(-4)-(-1)(-8)}{-2} = \frac{8-8}{-2} = 0 \,!$$

We have a zero row on s^1.

Auxiliary polynomial

$r(s) = -2s^2 - 8$

$\frac{dr(s)}{ds} = -4s$ replace zero row by this coefficient and continue with Routh array.

→ One sign change in the first column. System has one unstable pole in R.H.P

But zero row indicates that we also have complex-conjugate poles on the $j\omega$ axis.

→ Use the auxiliary polynomial to solve for them:

$r(s) = -2s^2 - 8 = 0 \rightarrow 2s^2 = -8$

$s^2 = -4$

$\boxed{s = \pm j2}$

Q.1. d) $q(s) = s^5 + 2s^4 + 16s^3 + 32s^2 + 100s + 200$

s^5	1	16	100	0
s^4	2	32	200	0
s^3	~~0~~ 8	~~0~~ 64	~~0~~ 0	
s^2	16	200	0	
s^1	-36	0		
s^0	200			

$$\left.\begin{array}{l}\dfrac{2.16 - 32}{2} = 0! \\ \dfrac{2.100 - 200}{2} = 0!\end{array}\right\} \text{zero row.}$$

Auxiliary polynomial:

$$r(s) = 2s^4 + 32s^2 + 200$$

$$\frac{dr(s)}{ds} = 8s^3 + 64s$$

replace the zero row with these coefficients.

$$\frac{\cancel{8}.32 - 2(\cancel{64})^{8}}{\cancel{8}} = 32 - 16 = 16$$

$$\frac{\cancel{16}^{2}.64 - \cancel{8}(\cancel{200})^{100}}{\cancel{16}\,\cancel{2}} = 64 - 100 = -36$$

Two sign changes in the first column. System unstable

We are not asked to find the poles causing the symmetry but solving $r(s) = 0$ (e.g. using Matlab roots(.) command) gives:

$$\left.\begin{array}{l} s_{1,2} = -1 \pm j3 \\ s_{3,4} = 1 \pm j3 \end{array}\right\} \text{symmetry around the } j\omega \text{ axis with two complex conjugate poles.}$$

Q2. $G(s)H(s) = \dfrac{K(s+2)}{s(1+Ts)(1+2s)}$ stability regions in T-K plane are requested.

Routh-Hurwitz can be used to obtain these regions. First calculate the closed-loop characteristic Equation.

$$q(s) = 1 + \frac{K(s+2)}{s(1+Ts)(1+2s)} = 0$$

$$(s+Ts^2)(1+2s) + Ks + 2K$$
$$s + 2s^2 + Ts^2 + 2Ts^3 + Ks + 2K = 0$$
$$q(s) = 2Ts^3 + (2+T)s^2 + (K+1)s + 2K = 0$$

Construct the Routh-Array:

$$\begin{array}{c|c|cc}
s^3 & 2T & K+1 & 0 \\
s^2 & 2+T & 2K & 0 \\
\hline
s^1 & a & 0 & \\
s^0 & 2K & &
\end{array}$$

$$a = \frac{(2+T)(K+1) - 4KT}{2+T}$$

For stability, we need no sign change on first column. This can happen in two ways: all terms positive (Case I) and all terms negative (Case II)

Case I: All terms in first column positive.

This requires $\left.\begin{array}{l} 2T > 0 \rightarrow T > 0 \\ 2+T > 0 \rightarrow T > -2 \end{array}\right\} \boxed{T > 0}$

$2K > 0 \rightarrow \boxed{K > 0}$

We also need $a > 0$

$a>0 \rightarrow \frac{(2+T)(K+1)-4KT}{2+T}>0$ but we know we need $2+T>0$

$\longrightarrow (2+T)(K+1)-4KT>0$

$2K+2+TK+T-4TK>0$

$2K+2+T(1-3K)>0$

$2K+2>T(3K-1)$

If: $(3K-1)>0 \rightarrow K>\frac{1}{3}$
then $\boxed{T<\frac{2K+2}{3K-1}}$

If $(3K-1)<0 \rightarrow K<\frac{1}{3}$
$\boxed{T>\frac{2K+2}{3K-1}}$

Sketch $T=\frac{2K+2}{3K-1}$ boundary

* $3K-1>0$

$\lim_{K\to(\frac{1}{3})^+} T=+\infty$ if $K\to+\infty$ $\lim_{K\to+\infty} T\simeq\frac{2\cancel{K}}{3\cancel{K}}=\frac{2}{3}$

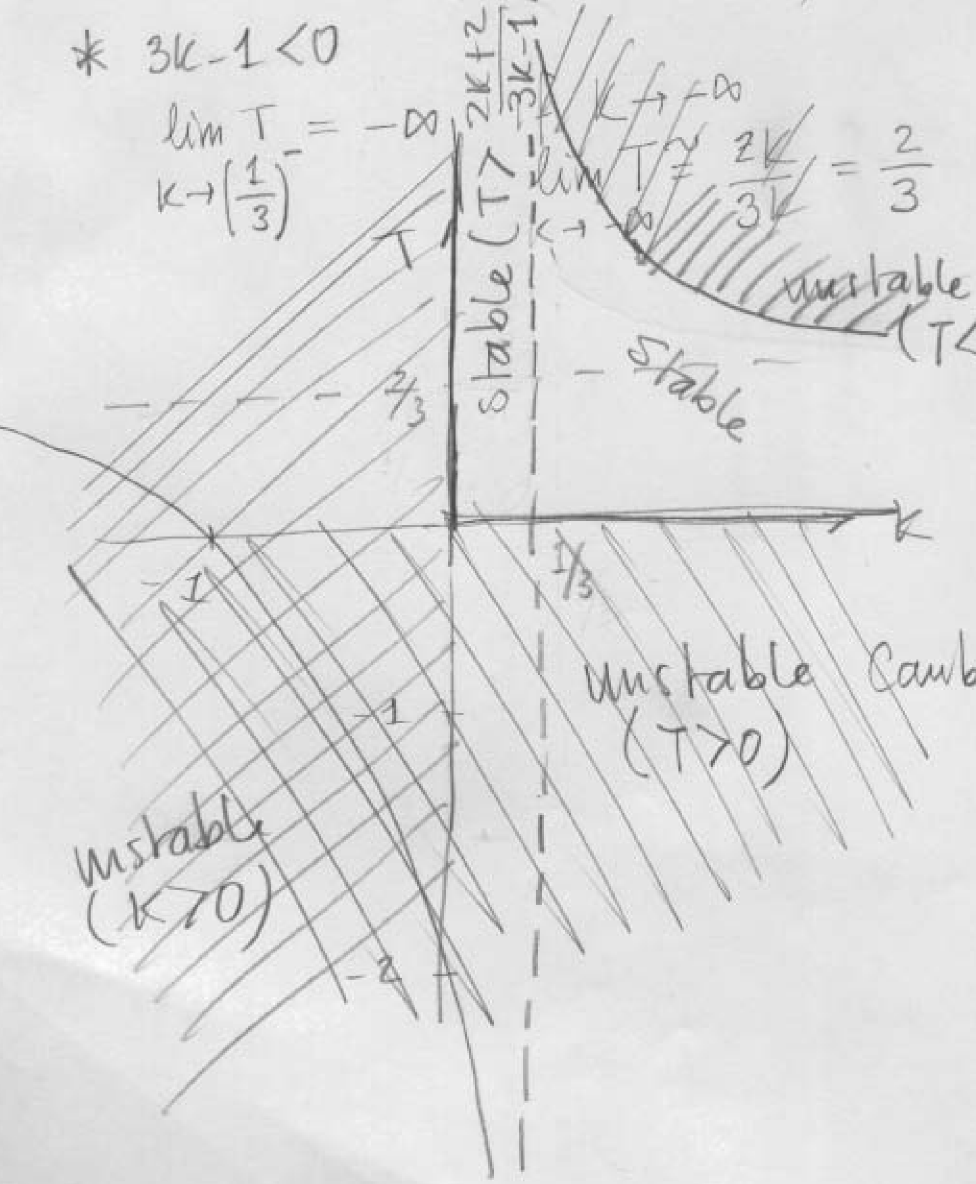
* $3K-1<0$

$\lim_{K\to(\frac{1}{3})^-} T=-\infty$

Axis intersection:

$K=0 \rightarrow T=-2$

$T=0 \quad 2K+2=0$
$K=-1$



Combine $\left.\begin{matrix}T>0\\K>0\end{matrix}\right\}$ only the 1st quadrant

Now let us check the boundaries: (including internal asymptote)

1) $K=0 \quad q(s) = 2Ts^3+(2+T)s^2+s = s(2Ts^2+(2+T)s+1)$

one pole at the origin. stability boundary.

2.) $K=\frac{1}{3} \quad q(s) = 2Ts^3+(2+T)s^2+\underbrace{\left(\frac{1}{3}+1\right)}_{\frac{4}{3}}s+\frac{2}{3}$ with $T>0$

$$\begin{array}{c|ccc} s^3 & 2T & \frac{4}{3} & 0 \\ s^2 & 2+T & 2/3 & 0 \\ \hline s^1 & a & 0 & \\ s^0 & 2/3 & & \end{array}$$

$$a = \frac{\frac{4}{3}(2+T)-\frac{4T}{3}}{4T/3} = \frac{\frac{8}{3}+\cancel{\frac{4T}{3}}-\cancel{\frac{4T}{3}}}{4T/3}$$

$\left.\begin{array}{l} a>0 \\ 2T>0 \\ 2+T>0 \end{array}\right\}$ system stable.

3.) $T=0 \quad q(s) = 2s^2+(K+1)s+2K=0$ with $K>0$

$$\begin{array}{c|cc} s^2 & 2 & 2K \\ s^1 & K+1 & 0 \\ \hline s^0 & 2K & \end{array}$$

$\left.\begin{array}{l} 2>0 \\ K+1>0 \\ 2K>0 \end{array}\right\}$ system stable but of reduced order. (2^{nd} order)

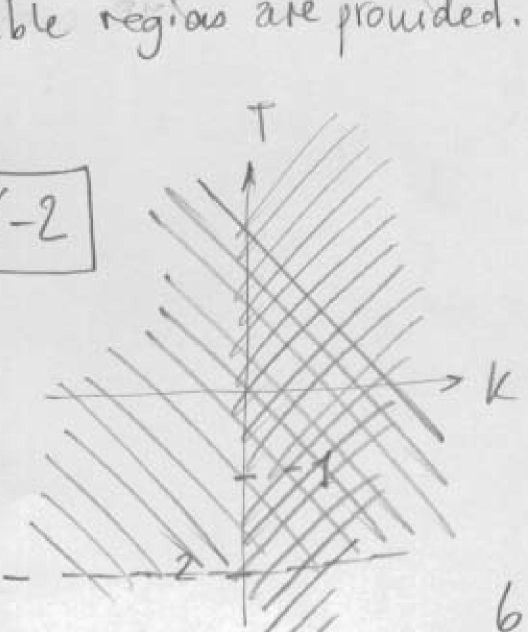
Case II: Now let us consider first column all negative and see whether any extra stable regions are provided.

Original Routh array: on page 4.

We need $\left.\begin{array}{l} 2T<0 \rightarrow T<0 \\ 2+T<0 \rightarrow T<-2 \end{array}\right\}$ $\boxed{T<-2}$

$2K<0 \rightarrow \boxed{K<0}$

We also need $a<0$

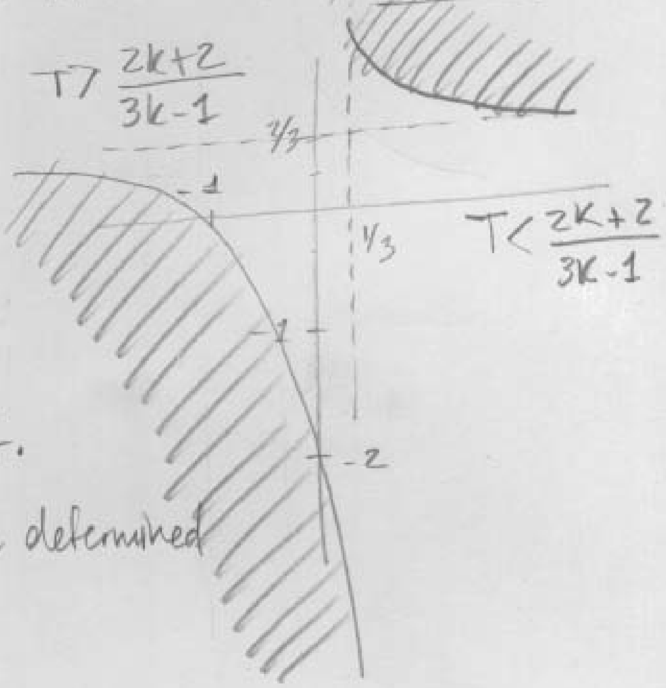


$$a<0 \rightarrow \frac{(2+T)(K+1)-4KT}{2+T}<0$$

But we already need $2+T<0$ so, we additionally need

$$(2+T)(K+1)-4KT>0$$

This is the same condition as we had in Case I.



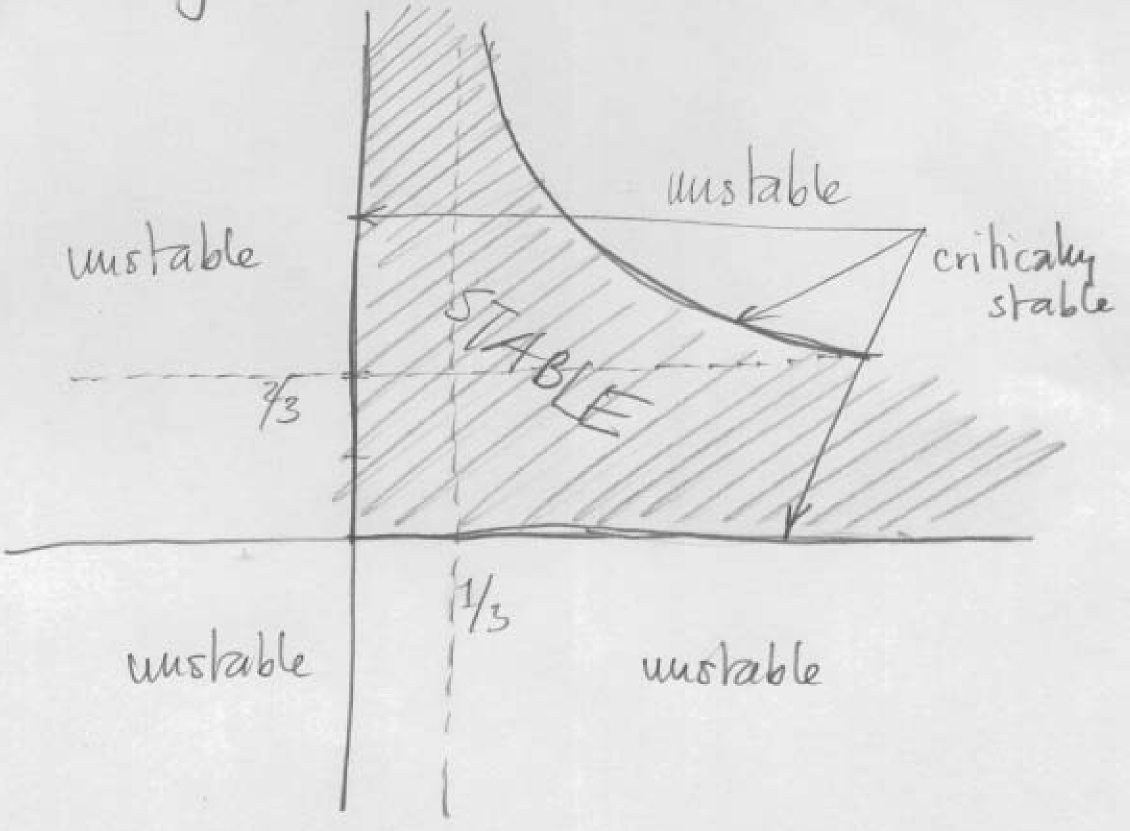
When we combine this with $T<-2$

$K<0$

The overlap is non-existent.

→ No other stable region are determined from Case II.

The stable region is illustrated again below



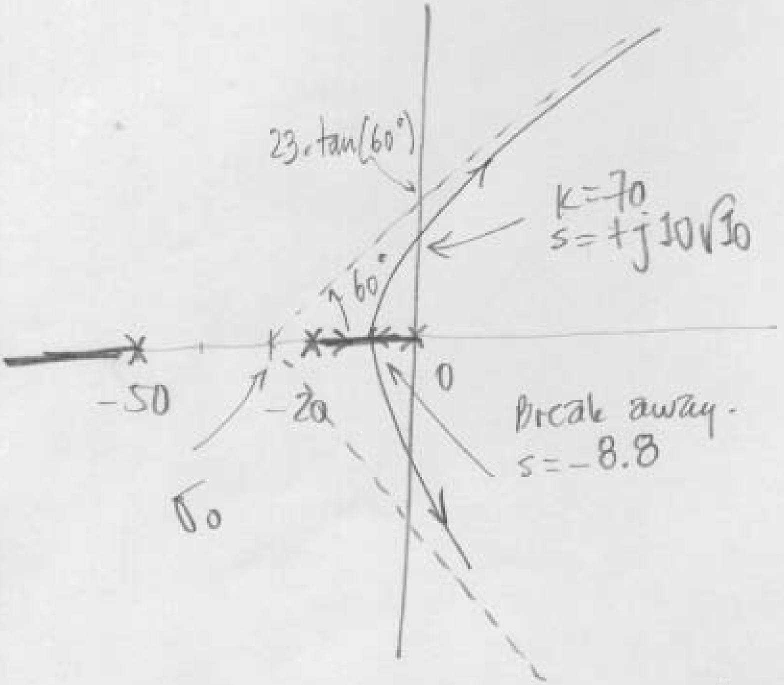
Q.3 a) $G(s) = \dfrac{K}{s(1+0.02s)(1+0.05s)}$ for $K>0$

We want to sketch the root locus. But careful about the denominator:

$$G(s) = \frac{K}{s(0.02)(s+50)(0.05)(s+20)} = \frac{1000K}{s(s+50)(s+20)}$$

let $\bar{K} = 1000K$

$$q(s) = 1 + \bar{K}\frac{1}{s(s+50)(s+20)}$$



Rule 1: # of branches: $\max(m,n) = 3$

Rule 2: Branches start at the poles and end up at the three asymptotes.

Rule 3: root-locus is symmetric wrt the real line.

Rule 4: asymptotes: $|m-n| = 3$

$$\phi = \frac{\pm 180}{3}(2l+1) = \begin{matrix} +60 \\ -60 \\ +180^\circ \end{matrix}$$

Rule 5: Centroid (real axis intersection of asymptotes)

$$\sigma_0 = \frac{\sum p - \sum z}{\#p - \#z} = \frac{0-20-50-(0)}{3-0} = -\frac{70}{3} \cong -23$$

$\boxed{\text{Rule 6: Locus on real line}}$

Rule 7: Break away point: Between -20 and 0.

$$\frac{d}{ds}\left[\frac{1}{s(s+50)(s+20)}\right] = \frac{d}{ds}\left(\frac{1}{s^3+70s^2+1000s}\right) = \frac{-3s^2-140s-1000}{(s^3+70s^2+1000s)^2}$$

$(s^2+50s)(s+20)$ $\quad \left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}$

Solve $-3s^2-140s-1000 = 0$ → $s = -37.9$, $\boxed{s = -8.8}$

let $q(s)\big|_{s=-8.8} = 0 \longrightarrow$ Find $\bar{K}$ and then K.

(Q3.a) Continued

$$q(-8.8) = 1 + \bar{K} \frac{1}{(-8.8)(-8.8+50)(-8.8+20)} = 0 \rightarrow \bar{K} \cong 4060$$

$$\rightarrow \boxed{K \cong 4.06}$$

Rule 8:

For $j\omega$ axis crossing:

$$q(s) = 1 + \bar{K} \frac{1}{s(s+50)(s+20)} = 0$$

$$(s^2+50s)(s+20) + \bar{K} = 0$$

$$s^3 + \underbrace{20s^2 + 50s^2}_{70s^2} + 1000s + \bar{K} = 0$$

Routh-Array:

s^3	1	1000	0
s^2	70	$\bar{K}$	0
s^1	$\frac{7\times10^4 - \bar{K}}{70}$	0	
s^0	$\bar{K}$		

$j\omega$ axis crossing at $7\times10^4 - \bar{K} = 0$

$$\bar{K} = 7\times10^4 \longrightarrow \boxed{K = 70}$$

Auxiliary equation $r(s) = 70s^2 + \bar{K} = 0$

$$70s^2 + 7\times10^4 = 0$$

$$s^2 = -\frac{70000}{70} = 1000 \longrightarrow \boxed{s = \pm j10\sqrt{10}}$$

Q.3 b) $G(s) = \dfrac{K}{s(s+1)(s+2)(s+5)}$ for $K > 0$

$$q(s) = 1 + K\,\frac{1}{s(s+1)(s+2)(s+5)}$$

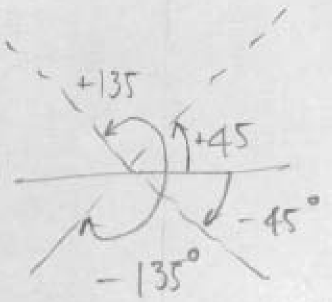
Rule 1: # of branches $= \max(m, n) = 4$

Rule 2: 4 branches start at OL poles. No open-loop zeros. (all will go to asymptotes)

Rule 3: Symmetry ✓

Rule 4: # of asymptotes $= |m-n| = 4$

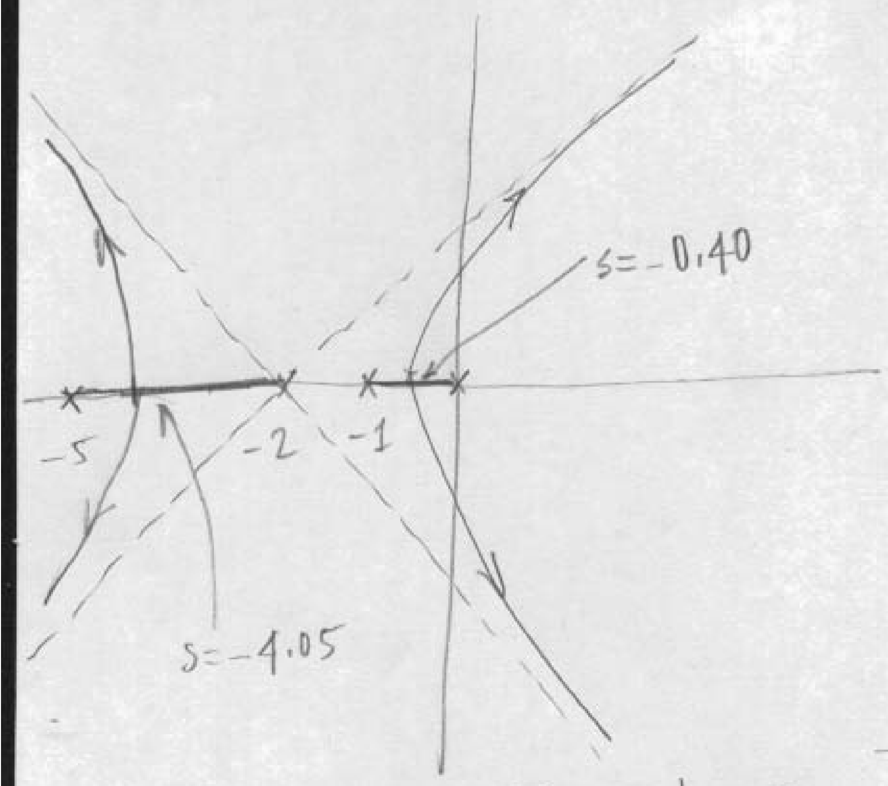
Angles of asymptotes: $\phi = \dfrac{\pm 180}{4}(2\ell+1) = \pm 45(2\ell+1)$



Rule 5: Centroid:

$$\sigma_0 = \frac{\sum p - \sum z}{\#p - \#z} = \frac{-1-2-5}{4} = \frac{-8}{4} = -2$$

Rule 6: Locus on the real line: shown.



Rule 7: Break-away points.

$$\frac{d}{ds}\left(\frac{1}{s^4+8s^3+17s^2+10s}\right) = 0$$

$$= \frac{-4s^3-24s^2-34s-10}{(\dots)^2} = 0$$

→ Using roots(.) in Matlab

$s_1 = -4.05$ ✓

$\boxed{s_2 = -1.54 \times}$

$\boxed{s_3 = -0.40}$

Put these s into $q(s) = 0$ to get corresponding K values.

Rule 8: $j\omega$ axis crossing

$$q(s) = 1 + K\frac{1}{s(s+1)(s+2)(s+3)} = 0$$

$$q(s) = s(s+1)(s+2)(s+3) + K$$
$$= s^4 + 8s^3 + 17s^2 + 10s + K$$

Routh array:

s^4	1	17	K
s^3	8	10	0
s^2	5.75	K	0
s^1	a	0	
s^0	K		

$$\frac{\cancel{8}.17-10}{\cancel{8}} = 17 - \frac{10}{8} = 15.75$$

$$\frac{(15.75)(10) - 8K}{15.75}$$

$j\omega$ axis crossing when $a = 0$

$$8K = (15.75)(10)$$
$$\boxed{K \cong 19.69}$$

The corresponding poles are found from the auxiliary polynomial equation

$$17.75s^2 + K = 0$$
$$15.75s^2 + 19.69 = 0 \rightarrow s^2 = -\frac{19.69}{15.75} \cong -1.25$$

$$\boxed{s \cong \pm j1.12}$$

Q.3 c) $G(s) = \dfrac{10(s+\alpha)(s+3)}{s(s^2-1)}$ $\quad \alpha > 0$

First find the characteristic polynomial:

$$q(s) = 1 + \frac{10(s+\alpha)(s+3)}{s(s^2-1)} = \frac{s(s^2-1) + 10(s+\alpha)(s+3)}{s(s^2-1)} = 0$$

numerator of

$$\longrightarrow q(s) = s^3 - s + 10(s^2 + 3s + \alpha s + 3\alpha) = 0$$

$$q(s) = s^3 - s + 10s^2 + 30s + 10\alpha s + 30\alpha = 0$$

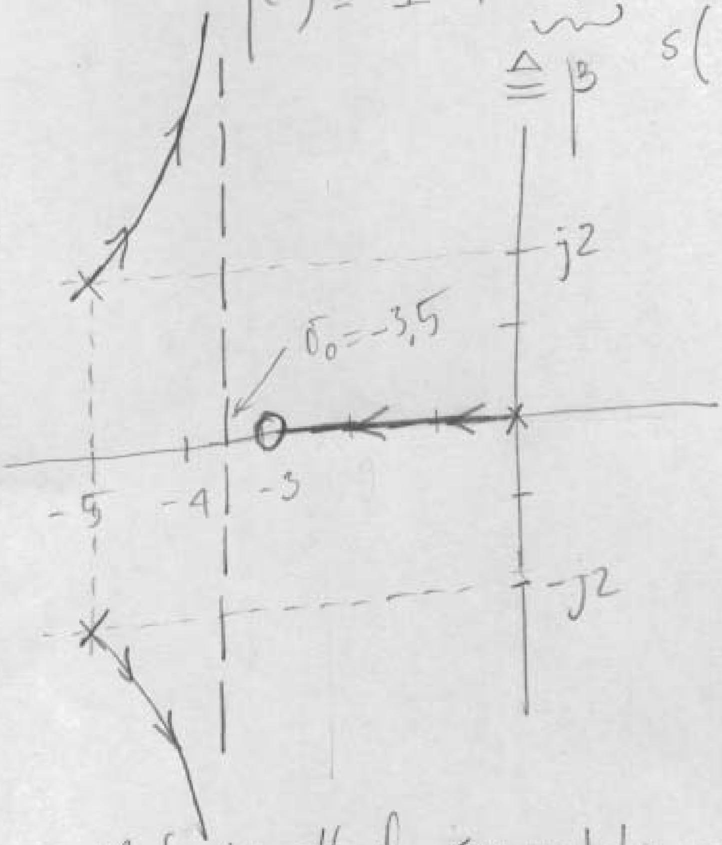
$$q(s) = s^3 + 10s^2 + 29s + 10\alpha(s+3) = 0$$

Put into standard form $q(s) = 1 + K\dfrac{N(s)}{D(s)}$

$$q(s) = 1 + \underbrace{10\alpha}_{\triangleq \beta} \frac{s+3}{s(\underbrace{s^2+10s+29}_{})}$$

$s_1 = -5 + j2$

$s_2 = -5 - j2$



Rule 1: # of branches $= \max(m, n) = 3$

Rule 2: 3 branches start at the OL poles. One end up at OL zero, others at asymptotes.

Rule 3: Symmetry.

Rule 4: # of asymptotes $= |m - n| = 2$

Angles $\phi = \dfrac{\pm 180}{2}(2l+1)$ $\quad +90°$, $-90°$

Rule 5: Centroid

$$\sigma_0 = \frac{\Sigma p - \Sigma z}{\#p - \#z} = \frac{0 - 5 + \cancel{j2} - 5 - \cancel{j2} - (-3)}{3-1} = \frac{-10+3}{2} = -\frac{7}{2}$$

Rule 6: Locus on the real-line: shown.

Rule 7: We do not expect any break-in/break-away points.

Rule 8: $j\omega$ axis crossing: We do not expect any because of the asymptote location. If branches cross to the RHP, they need to come back requiring at least 4 $j\omega$ axis crossing. That requires an auxiliary polynomial of order 4. Which is not possible since $q(s)$ is of order 3 only.

Without solving for the angle of departure from the complex conjugate poles; the simplest sketch is shown.

Note: Note that the sketch is for $\beta > 0$. Since there are no critical points for which we find β; we do not need to compute the corresponding α. But note that we move on the branches $\alpha = \frac{\beta}{10}$ (10 times slower) with α parameter.