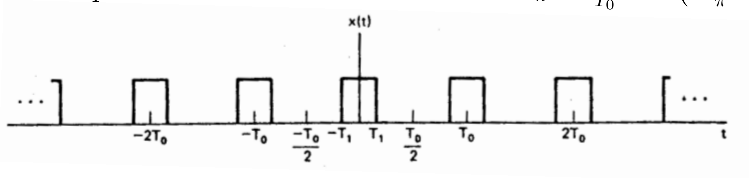
Chapter 4

The CT Fourier Transform

Fourier Series: Representation of periodic signals as sums of complex exponentials
Fourier Transform: Extension of the above idea to aperiodic signals

4.1 Representation of aperiodic signals: Fourier transform

An aperiodic signal can be viewed as the limit of a periodic signal when the period becomes arbitrarily large.
Recall that the periodic square wave has Fourier coefficients $a_k = \frac{2T_1}{T_0}\mathrm{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right)$



Inspect the Fourier coefficients as T_0 increases (T_1 fixed):

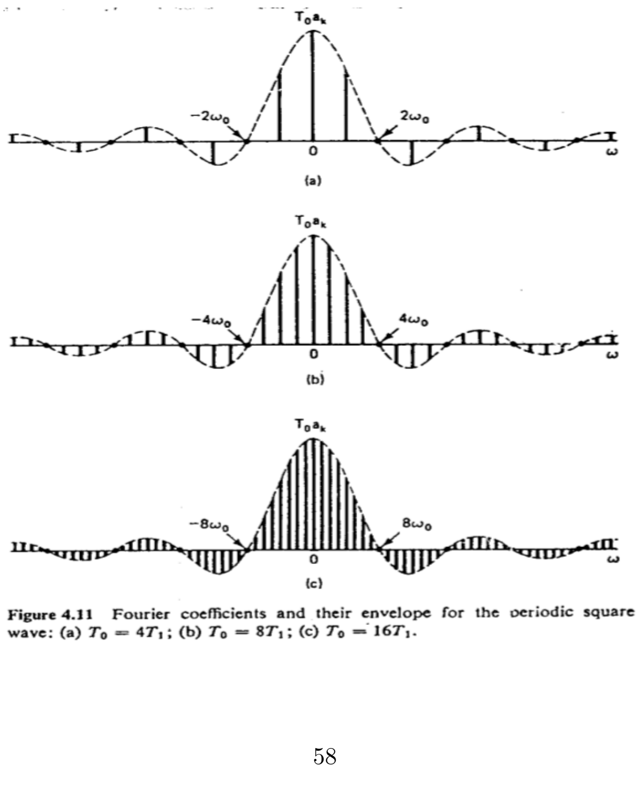


Figure 4.11 Fourier coefficients and their envelope for the periodic square wave: (a) $T_0 = 4T_1$; (b) $T_0 = 8T_1$; (c) $T_0 = 16T_1$.

- $T_0 a_k$ are samples of a continuous envelope: $T_0 a_k =$
- As period $T_0 \uparrow$,
- As $T_0 \to \infty$,

This example illustrates the basic idea behind Fourier Transform.

4.1.1 Formal Development of the Fourier Transform

Let's now formally derive the Fourier Transform Representation. For this, consider an aperiodic signal $x(t)$ and a periodic signal $\tilde{x}(t)$ made out of $x(t)$:

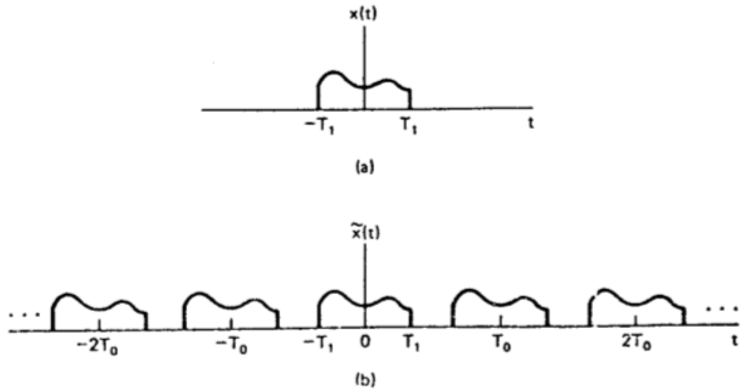


Figure 4.12 (a) Aperiodic signal $x(t)$; (b) periodic signal $\tilde{x}(t)$, constructed to be equal to $x(t)$ over one period.

$$x(t) = \tag{4.1}$$

with

$$X(j\omega) = \tag{4.2}$$

Interpretation:

- Eqn. (4.2): **Fourier transform** of the signal $x(t)$
  Eqn. (4.1):
- Eqn. (4.1):
  Eqn. (4.2):

4.1.2 Fourier series versus Fourier transform

Representation of signals as linear combinations of complex exponentials

- **Periodic signals:**

- **Aperiodic signals:**

  $\Rightarrow X(j\omega)$: spectrum of $x(t)$

Ex: $x(t) = e^{-at}u(t)$, where $a > 0$. Find $X(j\omega)$.

Ex: $X(j\omega) = \sum_i \frac{\alpha_i}{a_i + j\omega}$, with $Re\{a_i\} > 0$. Find $x(t)$.

Ex: Rectangular pulse $x(t) = \frac{1}{2T_1}\text{rect}\left(\frac{\text{t}}{2\text{T}_1}\right)$.

- Find $X(j\omega)$.
- FT of $\delta(t)$?

Ex: $X(j\omega) = \frac{1}{2\omega_1}\text{rect}\left(\frac{\omega}{2\omega_1}\right)$. Find $x(t)$.

Ex: [Challenge yourself!] $x(t) = e^{-a|t|}$, where $a > 0$. Find $X(j\omega)$.

4.2 Convergence of Fourier Transforms

We derived the FT pair for aperiodic signals with finite duration. In fact, FT and its inverse are valid for an extremely broad class of signals (possibly infinite duration).
Convergence means that $x(t)$ can be written as $\dfrac{1}{2\pi}\int_{\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$.

Like periodic signals, a set of sufficient conditions called **Dirichlet conditions** exists:

- $x(t)$ is absolutely integrable: $\int_{-\infty}^{\infty} |x(t)|dt < \infty$
- $x(t)$ has a finite number of maxima and minima within any finite interval
- $x(t)$ has a finite number of discontinuities within any finite interval. Each discontinuity must also be finite.

Alternative Condition: Another sufficient condition for existence and convergence of FT of $x(t)$:

- $x(t)$ has finite energy (i.e. square integrable)

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$$

Ex: [Challenge yourself!] Show that the examples we discuss satisfy these sufficient conditions.

4.3 Fourier Transform of Periodic Signals

We will now develop Fourier Transform for periodic signals. Hence both periodic and aperiodic signals can be studied within a unified context using Fourier transform.
Consider a signal $x(t)$ with Fourier Transform $X(j\omega)$ that is a single impulse of strength 2π at $\omega = \omega_0$:

More generally, consider a linear combination of impulses equally spaced in frequency:

$$\mathcal{F}\{\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\} = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$

Hence we can construct the FT of a periodic signal directly from its FS representation.
Ex: $x(t) = cos(\omega_0 t)$

Ex: $x(t) = sin(\omega_0 t)$

Ex: The periodic square wave (again!)

Ex: Periodic impulse train $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$

4.4 Properties of the Fourier Transform

If $X(j\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$, then $x(t)$ and $X(j\omega)$ form a Fourier transform pair:

$$x(t) \longleftrightarrow X(j\omega)$$

4.4.1 Linearity

$x_1(t) \longleftrightarrow X_1(j\omega)$
$x_2(t) \longleftrightarrow X_2(j\omega)$
$ax_1(t) + bx_2(t) \longleftrightarrow$

4.4.2 Time Shift

$x(t) \longleftrightarrow X(j\omega)$
$x(t - t_0) \longleftrightarrow$

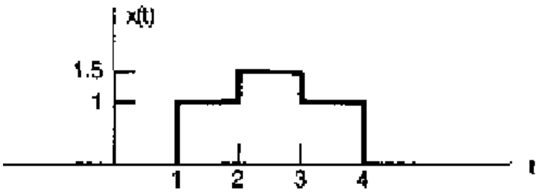
4.4.3 Scaling

$x(t) \longleftrightarrow X(j\omega)$
$x(at) \longleftrightarrow$ a is a nonzero real number

Note that time and frequency scales are inversely proportional. When one expands, the other

contracts:
Contracting in time $\Longleftrightarrow$ Expanding in frequency
Ex: $x(-t) \longleftrightarrow$

Ex: Fourier Transform of



4.4.4 Conjugation and Conjugate Symmetry

$x(t) \longleftrightarrow X(j\omega)$
$x^*(t) \longleftrightarrow$

If $x(t)$ is **real**,

- $x(t) = x^*(t) \Longleftrightarrow$
- Magnitude and phase of its FT:

- Real and imaginary parts of its FT:

If $x(t)$ is both **real and even**, $X(j\omega)$ is ...

If $x(t)$ is both **real and odd**, $X(j\omega)$ is ...

Ex: [Challenge yourself!] Remember that any real signal $x(t)$ can be broken down into a sum of even and odd parts: $x(t) = x_e(t) + x_o(t)$. Using some of the above properties show that
$x_e(t) \longleftrightarrow Re\{X(j\omega)\}$ and $x_o(t) \longleftrightarrow j \cdot Im\{X(j\omega)\}$

4.4.5 Differentiation and Integration

$x(t) \longleftrightarrow X(j\omega)$
$\frac{d}{dt}x(t) \longleftrightarrow$

Integration has an extra term in addition to $X(j\omega)/j\omega$:
$\int_{-\infty}^{t} x(\tau)d\tau \longleftrightarrow \frac{1}{j\omega}X(j\omega) + ...$

Ex: Fourier Transform of Unit step signal $u(t)$?

Ex: $x(t) = t$ if $t \in [-1, 1]$ and zero otherwise. $X(j\omega)$=?

4.4.6 Duality Principle

For any transform pair, there is a **dual pair** with the time and freq. variables interchanged. Suppose the following holds: $g(\xi) = \int_{-\infty}^{\infty} f(u)e^{-j\xi u}du$

Ex: In an exercise, you showed that $e^{-a|t|} \longleftrightarrow \frac{2a}{a^2+\omega^2}$ (for $a > 0$). Find $\mathcal{F}\{\frac{1}{1+t^2}\}$.

Ex: [Challenge yourself!] Duality principle can be applied to derive additional properties:

- Frequency shift:
- Frequency differentiation:
- Frequency integration:

4.4.7 Parseval's Relation

$x(t) \longleftrightarrow X(j\omega) \quad \Longrightarrow \quad \int_{-\infty}^{\infty} |x(t)|^2 dt = ...$
Proof:

Interpretation:

- Energies in the time and frequency domains are ...
- Total energy in the signal can be determined either
  - from $x(t)$ by integrating its energy per unit time ($|x(t)|^2$) over all time
  - or from $X(j\omega)$ by integrating its energy per unit frequency ($\frac{|X(j\omega)|^2}{2\pi}$) over all frequencies.
- $|X(j\omega)|^2$: *energy-density spectrum* of signal $x(t)$

4.4.8 Convolution Property

$x_1(t) * x_2(t) \longleftrightarrow ....$

- Convolution in time domain $\longleftrightarrow$ multiplication in the frequency domain
- We can analyze LTI systems also in the frequency domain.

  - $\mathcal{F}\{h(t)\} = H(j\omega)$ is called the **frequency response** of the LTI system. Like impulse response, $H(j\omega)$ completely characterizes an LTI system.
  - Note that frequency response cannot be defined for every LTI system. However, all *stable* LTI systems of practical interest has a well-defined $H(j\omega)$. (Why?)
  - To analyze *unstable* LTI systems, we will later develop a generalization of the CT Fourier transform - *the Laplace transform.*

Ex: Consider an LTI system with $h(t) = \delta(t - t_0)$. Find its frequency response $H(j\omega)$ and the output's FT in terms of the input's FT.

Ex: Consider an LTI system described by a differential equation.

- Find its frequency response.
- Find its impulse response.

Ex: (Cont'ed)

Ex: The system in the earlier example is a causal lowpass filter, which can be easily implemented with an RC circuit. In general, frequency selective filtering is accomplished with an LTI system whose frequency response $H(j\omega)$ passes desired range of frequencies and significantly attenuates frequencies outside that range. Consider also the *ideal low-pass filter* with $H(j\omega) = \begin{cases} 1, |\omega| < \omega_c \\ 0, \text{otherwise.} \end{cases}$.

- What is its impulse response?
- What are the challenges involved in implementing this filter?
- Consider an input signal $x(t) = \sin(\omega_i t)$. What is the corresponding output?

Ex: [Challenge yourself!] Find the impulse response of the LTI system described by the following differential equation: $\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$.

Ex: [Challenge yourself!] Prove or disprove the following: Convolution of two sinc functions is always a sinc function.

Ex: [Challenge yourself!] Consider an LTI system with $h(t) = e^{-at}u(t)$ and an input $x(t) = e^{-bt}u(t)$ where $a > 0$ and $b > 0$. While the output can be computed in the time domain via convolution, instead first find the output in the frequency domain and then in the time-domain.

4.4.9 Multiplication (Modulation) property

$x_1(t) \cdot x_2(t) \longleftrightarrow ....$

- **Multiplication in one domain** corresponds to **convolution in the other domain** (by duality).
- Multiplication of one signal by another is interpreted as using one signal to scale (**modulate**) the amplitude of the other.

Proof: Exercise (use duality or the definition of FT)

Ex: $y(t) = x(t)e^{j\omega_0 t}$. Find $Y(j\omega)$ in terms of the spectrum of $x(t)$.

Ex: $y(t) = x(t)cos(\omega_0 t)$. Find $Y(j\omega)$ in terms of the spectrum of $x(t)$.

Ex: [Challenge yourself!] Find FT of $x(t) = \dfrac{\sin(t)\sin(\frac{t}{2})}{\pi t^2}$.

4.4.10 Table of properties of CT FT

The following table from the textbook summarizes all properties.

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t)+by(t)$	$aX(j\omega)+bY(j\omega)$
4.3.2	Time Shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t)*y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t}x(t)dt$	$\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega)=X^*(-j\omega) \\ \mathcal{R}e\{X(j\omega)\}=\mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\}=-\mathcal{I}m\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert=\lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega)=-\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t)=\mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t)=\mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\mathcal{R}e\{X(j\omega)\}$ $j\mathcal{I}m\{X(j\omega)\}$
4.3.7	Parseval's Relation for Aperiodic Signals	$\int_{-\infty}^{+\infty}\lvert x(t)\rvert^2dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\lvert X(j\omega)\rvert^2d\omega$	

4.4.11 Table of basic signals and their CT FT and FS

The following table from the textbook summarizes the CT FT of some basic signals (and their CT FS if the signal is periodic) .

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \ \mathcal{R}e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \ \mathcal{R}e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t)$, $\mathcal{R}e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

4.5 Some applications of Fourier transform

4.5.1 Amplitude Modulation (AM) in Communication Systems

Modulation: The process of *embedding* an information-bearing signal into a second signal (carrier signal) by means of scaling the amplitude of one signal with another. Modulation produces a signal whose spectrum is centered around the carrier signal's frequency.

Reason: To change it into a form suitable for effective transmission over the channel
For example,

-
-
-

As an example, consider long-distance transmission of voice signals over microwave or satellite links (i.e. telephone). Voice signals are in the frequency range 0.3-3 kHz, whereas microwave communication links require signals in the range 300 MHz-300 GHz, and satellite links operate in the frequency range 100 MHz-40 GHz.

Sinusoidal AM: Modulation of signals by means of a sinusoidal carrier around a desired frequency band

Sinusoidal carrier $c(t) = \cos(\omega_c t + \theta_c)$

Ex: (Effect of sinusoidal AM on the spectrum)

Demodulation: Process of *recovering* the original signal from the modulated signal.

Synchronous demodulation: Carrier frequency and phase is perfectly known at the receiver.

To summarize, synchronous demodulation is achieved in two steps :

1. modulation with the same carrier frequency
2. low-pass filter

In summary,

Ex: [Challenge yourself!] There are other modulation methods discussed in your book. For example, consider modulation of a real signal $x(t)$ with a complex exponential carrier, i.e. $c(t) = e^{j(\omega_c t + \theta_c)}$. Resulting spectrum? How to implement modulation and demodulation steps?

4.5.2 Frequency Division Multiplexing (FDM)

Question: What if *many* users want to transmit at the same time over the same communication medium? (as in phone systems)

Answer:

Question: How to recover the original signals back after transmission?

1st step:

2nd step:

4.5.3 Single Sideband Modulation (SSB)

Sinusoidal AM: Inefficient ....

Single sideband modulation: Removes the redundancy at the cost of a more complicated modulation system

Ex: [Challenge yourself!] How to extract lower sidebands without a filtering approach? (Answer in your book)