Chapter 7

Laplace Transform

In the preceding chapters, we have seen that Fourier series and transform representations are very useful in the study of many problems involving signals and LTI systems. This is mainly due to the fact that Fourier tools represent signals as a linear combination of complex exponentials of the form $e^{j\omega t}$ in CT and $e^{j\Omega n}$ in DT, which are eigenfunctions of LTI systems.

However, the eigenfunction property also applies to more general complex exponentials of the form e^{st} in CT (and z^n in DT, as we will see later) with s arbitrary complex number. This observation leads to the generalization of the CT Fourier transform, which is known as the Laplace transform. In the next chapter, we will also discuss the z-transform, which constitutes a generalization of the DT Fourier transform.

As we will see, the Laplace and z transform have many of the beautiful properties of the Fourier transform but also provide additional tools and insights. In particular, for signals for which the Fourier transforms do not exist, the Laplace or z transform may converge/exist. For this reason, these transforms can be very useful in the stability analysis of systems.

This chapter discusses the Laplace transform and the next chapter will discuss the z-transform.

7.1 The Laplace transform as a generalization of the CTFT

- Remember the response of a CT LTI system to a complex exponential e^{st} (where s is an arbitrary complex number):

$\Rightarrow$ Complex exponentials of the general form e^{st} are *eigenfunctions* of CT LTI systems with *eigenvalues* given by $H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$ where $h(t)$ is the impulse response of the system.

- The eigenvalue expression provides the definition of **the Laplace transform** of a signal $x(t)$ where s is an arbitrary complex number:

$$\boxed{X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt}$$

We will again use shorthand notations for the Laplace transform of a signal $x(t)$:

$$X(s) = \mathcal{L}\{x(t)\} \qquad \text{and} \qquad x(t) \longleftrightarrow X(s)$$

- Notice that using $s = j\omega$ (i.e. purely imaginary s), we developed the Fourier series and transform. We now generalize the same idea for arbitrary complex s, that is for $s = \sigma + j\omega$.
- Note that if CT Fourier transform of $x(t)$ exists, then CTFT of $x(t)$ can be obtained from its Laplace transform $X(s)$ by setting $s = j\omega$:

$$X(s)|_{s=} \qquad =$$

- There is a more general relationship between the Laplace transform and the CTFT:

Hence the Laplace transform of $x(t)$ can be interpreted as the Fourier transform of $x(t)$ after multiplication with a real exponential signal.

- The above relation also indicates that $X(s)$ converges (exists) whenever

– and thus for a given $x(t)$, convergence of $X(s)$ only depends on

7.2 Region of Convergence (ROC)

Region of Convergence (ROC): The range of s values for which the Laplace transform $X(s)$ converges

Ex: $x(t) = e^{-at}u(t)$, where a is any complex number. Find $X(s)$ and its ROC.

Note that if the ROC of $X(s)$ contains the imaginary axis (jw-axis), i.e. $\sigma = Re\{s\} = 0$, then the CTFT of $x(t)$ ...

Ex: [Challenge yourself!] Previous example with $a = \frac{1}{2}j$ and $a = -\frac{1}{2}$. What can you say about $\mathcal{F}\{x(t)\}$ in these cases?

Ex: [Challenge yourself!] Compare $\mathcal{L}\{u(t)\}$ with $\mathcal{F}\{u(t)\}$. Is it true that $U(s)|_{s=j\omega} = \mathcal{F}\{u(t)\}$? Why or why not?

Ex: $x(t) = -e^{-at}u(-t)$. Find $X(s)$ and its ROC.

Some remarks:

- It is helpful to remember the following frequently used signal and Laplace transform pairs :

$$\begin{aligned} e^{-at}u(t) &\longleftrightarrow \frac{1}{s+a}, \quad ROC: Re\{s\} > -a \\ -e^{-at}u(-t) &\longleftrightarrow \frac{1}{s+a}, \quad ROC: Re\{s\} < -a \end{aligned}$$

- As in z-transform, the specification of the Laplace transform requires both
  - the algebraic expression for $X(s)$
  - and the associated ROC.

  Warning: ROC is necessary for *uniquely* determining the time-domain signal $x(t)$ from $X(s)$.

Ex: [Challenge yourself!] $x(t) = e^{-2t}u(t) + e^{t}u(t)$. Find $X(s)$ and its ROC.

7.3 Properties of ROC

Property 1 : The ROC of $X(s)$ depends only on $Re\{s\}$ and therefore consists of **strips** parallel to the jw-axis in the s-plane.

Why?:

- The relation $X(s) = \mathcal{F}\{x(t)e^{-\sigma t}\}$ indicates that the convergence/existence of $X(s)$ requires the convergence/existence of , which happens when $x(t)e^{-\sigma t}$ is absolutely integrable, i.e. $\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}| = \int_{-\infty}^{\infty} |x(t)|e^{-\sigma t} < \infty$.
- Thus, ROC of $X(s)$ depends only on the real part $\sigma = Re\{s\}$ and **not** on the imaginary part ω of s.

Property 2 : For a rational $X(s)$, the ROC does **not** contain any poles.

Why?: At a pole, $X(s)$ is ...

Property 3 : For a **finite-duration** signal $x(t)$, if there is at least one value for which $X(s)$ exists, then the ROC is the entire s-plane.

Why?:

Property 4 : If $x(t)$ is a **right-sided** (left-sided) signal, then the ROC is to the right (left) of a vertical line.

Why?: For a right-sided signal, $x(t) = 0$ for $t < T_1$:

Property 5 : If $x(t)$ is **two-sided** (i.e. infinite-extent signal), then the ROC is either a strip in the s-plane or empty.

Why?: A two-sided sequence $x(t) =$

Property 6 : If $X(s)$ is **rational**, then its ROC is bounded by the poles or extends to infinity.

Why?:

- A signal $x(t)$ with rational $X(s)$ consists of a linear combination of signals in the form of $e^{-at}u(t)$ and $-e^{-at}u(-t)$ which have ROCs bounded by their poles.
- The ROC of $X(s)$ for the linear combination of these exponentials thus is the intersection of ROCs bounded by poles (unless there is zero-pole cancellation).

7.4 Inversion of the Laplace transform

Given $X(s)$ and ROC, how can we determine $x(t)$?

$$x(t) = \frac{1}{2\pi j} \oint_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$

where the integration is over a vertical line that is in the ROC (i.e. $Re\{s\} = \sigma$ is in ROC). Hence the inverse Laplace transform expression contains a contour integration on the s-plane, and is typically difficult to compute and will not be applied in this course. However, there are a number of alternative procedures for obtaining a signal from its Laplace transform and associated ROC.

For rational $X(s)$, an alternative way is to perform **partial-fraction expansion** for $X(s)$, and then to recognize the signal associated with each term in the expansion.
Ex: $X(s) = \dfrac{2}{s^2 - a^2}$, ROC: $Re\{s\} > a$ where $a > 0$. Find $x(t)$.

7.5 Properties of Laplace transform

To discuss the Laplace transform properties, we use a similar shorthand notation we used earlier. To indicate the pairing of a signal and its Laplace transform, we use :

- $x(t) \longleftrightarrow X(s)$

We sometimes also refer to a Laplace transform with the following notation :

- $X(s) = \mathcal{L}\{x(s)\}$

For the following properties, we consider two signals $x(t)$ and $y(t)$ with corresponding Laplace transforms $X(s)$ and $Y(s)$, and ROCs R_x and R_y, respectively, i.e.

$$x(t) \longleftrightarrow X(s),\ ROC = R_x$$

$$y(t) \longleftrightarrow Y(s),\ ROC = R_y$$

7.5.1 Linearity

$ax(t) + by(t) \longleftrightarrow aX(s) + bY(s), \quad ROC \supseteq R_x \cap R_y$ (pole zero cancellations may occur)

Ex: [Challenge yourself!] Consider the linear combination for $x(t) = y(t)$ and with constants $a = -b$. What is its Laplace transform and ROC?

Ex: [Challenge yourself!] $X(s) = \dfrac{1}{s^2+5} + \dfrac{1}{s-2}$, ROC=$R_x$
$Y(s) = \dfrac{1}{s^4+6} - \dfrac{1}{s-2}$, ROC=$R_y$.
Consider the Laplace transform of $z(t) = x(t) + y(t)$. What can you say about its ROC?

7.5.2 Time-shifting

$x(t - t_0) \longleftrightarrow X(s)e^{-s\,t_0},\ ROC = R_x$

7.5.3 Shifting in the s-domain

$x(t)e^{s_0 t} \longleftrightarrow X(s-s_0),\ ROC = R_x + Re\{s_0\} = \{s : Re\{s\} - Re\{s_0\} \in R_x\}$

This property corresponds to frequency shifting property for $s_0 = j\omega_0$:
$x(t)e^{j\omega_0 t} \longleftrightarrow X(s-j\omega_0),\ ROC : R_x$

7.5.4 Time scaling

$x(at) \longleftrightarrow \dfrac{1}{|a|}X\left(\dfrac{s}{a}\right),\ ROC = aR_x$

Note this special case: $x(-t) \longleftrightarrow X(-s),\ ROC = -R_x$

Ex: [Challenge yourself!] Suppose ROC of $X(s)$ is given by $r_1 < Re\{s\} < r_2$. Find the ROC of $\dfrac{1}{2}X\left(\dfrac{s}{2}\right)$.

7.5.5 Conjugation

$x^*(t) \longleftrightarrow X^*(s^*),\ ROC = R_x$

If $x(t)$ is real, (i.e. $x(t) = x^*(t)$)

- $X(s) = X^*(s^*)$
- thus, if $X(s)$ has a pole (zero) at $s = a$, then it must have another pole (zero) at the complex conjugate point $s = a^*$.

7.5.6 Convolution

$x(t) * y(t) \longleftrightarrow X(s)Y(s),\quad ROC \supseteq R_x \cap R_y$ (pole zero cancellations may occur)

7.5.7 Time differentiation

$\dfrac{d\,x(t)}{dt} \longleftrightarrow sX(s),\ ROC \supseteq R_x$

7.5.8 Differentiation in the s-domain

$-tx(t) \longleftrightarrow \dfrac{dX(s)}{ds}, \; ROC = R_x$

7.5.9 Time integration

$\int_{-\infty}^{t} x(\tau)d\tau \longleftrightarrow \dfrac{X(s)}{s}, \; ROC \supseteq R_x$

7.5.10 The initial and final value theorems

If $x(t)$ is causal (i.e. $x(t) = 0$ for $t < 0$), contains no impulses or higher order singularities at $t = 0$, and has a finite limit as $t \to \infty$, then

$$\begin{aligned} \text{Initial:} \quad & x(0^+) = \lim_{s\to\infty} sX(s) \\ \text{Final:} \quad & \lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) \end{aligned}$$

7.5.11 Table of Laplace transform properties and some common transform pairs

7.6 LTI Systems and the Laplace transform

The Laplace transform plays an important role in the analysis and design of CT LTI systems:

- By convolution property, $y(t) = x(t) * h(t) \longleftrightarrow Y(s) = H(s)X(s), \quad ROC$
- $H(s) = \mathcal{L}\{h(t)\}$: **transfer function** (system function) of the LTI system
  We can characterize an LTI system by $H(s)$ and its ROC. If jw-axis $\in$ ROC, then $H(s)$ can be evaluated at $s = j\omega$ (i.e. $Re\{s\} = 0$), and hence the frequency response of the system can be obtained:
  $$\mathcal{F}\{h(t)\} = H(s)|_{s=j\omega}$$
- Also note that, as discussed before, $H(s)$ is the eigenvalue corresponding to the input eigenfunction e^{st}.

TABLE 9.1 PROPERTIES OF THE LAPLACE TRANSFORM

Section	Property	Signal	Laplace Transform	ROC
		$x(t)$ $x_1(t)$ $x_2(t)$	$X(s)$ $X_1(s)$ $X_2(s)$	R R_1 R_2
9.5.1	Linearity	$ax_1(t)+bx_2(t)$	$aX_1(s)+bX_2(s)$	At least $R_1 \cap R_2$
9.5.2	Time shifting	$x(t-t_0)$	$e^{-st_0}X(s)$	R
9.5.3	Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s-s_0)$	Shifted version of R (i.e., s is in the ROC if $s-s_0$ is in R)
9.5.4	Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
9.5.5	Conjugation	$x^*(t)$	$X^*(s^*)$	R
9.5.6	Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
9.5.7	Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
9.5.8	Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
9.5.9	Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\mathcal{Re}\{s\} > 0\}$

Initial- and Final-Value Theorems

9.5.10 If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \longrightarrow \infty$, then

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

For LTI systems, many system properties can be directly determined from the ROC, and the poles and zeros of $H(s)$, as is discussed below.

7.6.1 Causality

Property 1 : For a **causal** CT LTI system, the ROC of $H(s)$ is to the right of a vertical line. (The converse is not true)

Why?

Ex: Consider the LTI systems with the following impulse responses and determine whether they are causal or not from the ROC of their transfer functions:

a) $h_1(t) = e^{-t}u(t)$

TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\Re e\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\Re e\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\Re e\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} < -\alpha$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
12	$[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
13	$[e^{-\alpha t}\cos\omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
14	$[e^{-\alpha t}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$

b) $h_2(t) = h_1(t+1)$

Property 2 : A CT LTI system with rational $H(s)$ is **causal** if and only if the ROC is to the right of the vertical line bounded by its right-most pole.

Why?

Ex: Consider $H(s) = \dfrac{-2}{s^2 - 1}$ with ROC: $-1 < Re\{s\} < 1$. Causal system?

7.6.2 Stability

Property 1 : A CT LTI system is **stable** if and only if the ROC of $H(s)$ contains the jw-axis, i.e. $Re\{s\} = 0$.

Why?

Property 2 : A causal DT LTI system with rational $H(s)$ is **stable** if and only if all poles of $H(s)$ lie in the left-half of the s-plane (i.e. all poles have negative real parts.)

Why?

Ex: An LTI system has the following transfer (system) function: $H(s) = \dfrac{s-1}{(s+1)(s-2-2j)}$, but its ROC is unknown. Comment on the possible ROCs and the system properties they lead.

Ex: [Challenge yourself!] Consider a stable and causal LTI system with impulse response $h(t)$ and rational system function $H(s)$. The following information is given:

- $H(s)$ has a pole at $s = -2$.
- $H(s)$ does not have a zero at origin.

The precise number and locations of other poles and zeros are unknown. Are the following statements *true, false or not verifiable*? Justify your answers.

1. $\mathcal{F}\{h(t)e^{3t}\}$ exists.
2. $\int_{-\infty}^{\infty} h(t)dt = 0$.
3. $t.h(t)$ is the impulse response of a causal and stable system.
4. $\dfrac{dh(t)}{dt}$ contains at least one pole in its Laplace transform.
5. $h(t)$ has finite duration.
6. $H(s) = H(-s)$.