Middle East Technical University
Department of Electrical and Electronics Engineering
EE301: Signals and Systems I

Solutions for Homework 4
December 22, 2018
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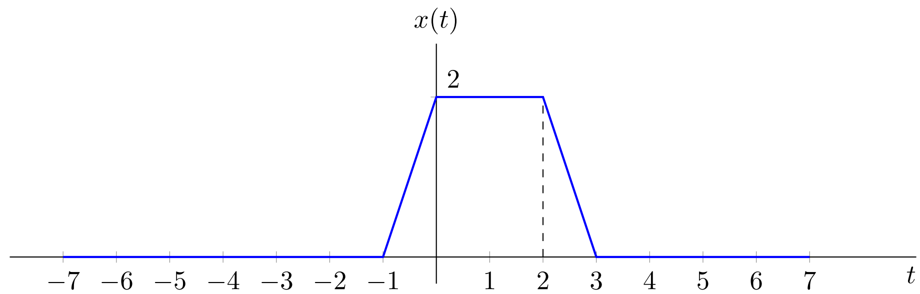
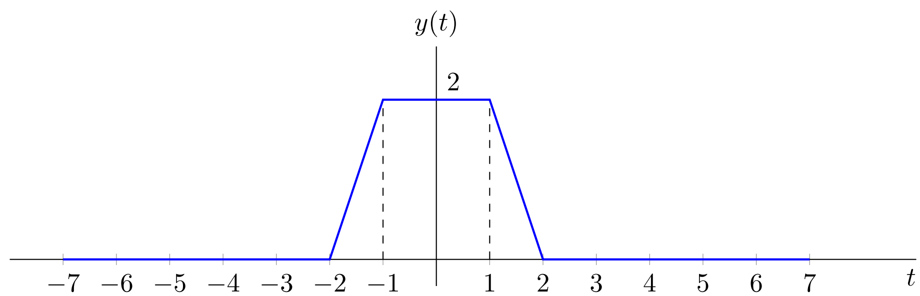
1. (a)



Let $y(t) = x(t+1)$, then note that $y(t)$ is real and even as shown below.



By the time shifting property of the Fourier transform

$$y(t) = x(t+1) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega) = e^{j\omega} X(j\omega)$$

Let $X(j\omega) = A(j\omega)e^{j\Theta(j\omega)}$, then

$$\angle Y(j\omega) = \omega + \Theta(j\omega).$$

We know that $Y(j\omega)$ is real and even since $y(t)$ is real and even. Thus, $\angle Y(j\omega) = 0$ and $\Theta(j\omega) = -\omega$.

(b) By using the Fourier transform equation,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,\mathrm{d}t$$

we can find $X(0)$ as below.

$$X(j0) = \int_{-\infty}^{\infty} x(t)\,\mathrm{d}t = 6$$

(c) By using the inverse Fourier transform equation,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,\mathrm{d}\omega$$

we can find the area under $X(j\omega)$ as below.

$$\int_{-\infty}^{\infty} X(j\omega)\,\mathrm{d}\omega = 2\pi\, x(0) = 4\pi$$

(d)

Let $Y(j\omega) = \dfrac{\sin\omega}{\omega} e^{j3\omega}$, then $y(t) = \mathcal{F}^{-1}\{Y(j\omega)\}$ is a shifted rectangular pulse signal.

$$\frac{1}{2}\left[u(t+1) - u(t-1)\right] \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \frac{\sin\omega}{\omega}$$

$$y(t) = \frac{1}{2}\left[u(t+4) - u(t+2)\right] \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \frac{\sin\omega}{\omega} e^{j3\omega} \quad \text{(Time shifting)}$$

$$y(t) = \begin{cases} \dfrac{1}{2}, & -4 \leqslant t \leqslant -2 \\ 0, & \text{elsewhere} \end{cases}$$

By the convolution property of the Fourier transform,

$$g(t) = x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau) y(t-\tau)\,\mathrm{d}\tau \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad G(j\omega) = X(j\omega)\,Y(j\omega) = X(j\omega)\,\frac{\sin\omega}{\omega} e^{j3\omega}$$

From part (c), it is obvious that

$$\int_{-\infty}^{\infty} X(j\omega) \frac{\sin\omega}{\omega} e^{j\omega}\,\mathrm{d}\omega = \int_{-\infty}^{\infty} G(j\omega)\,\mathrm{d}\omega = 2\pi g(0) = 2\pi \int_{-\infty}^{\infty} x(\tau) y(-\tau)\,\mathrm{d}\tau = \pi.$$

(e)

By using Parseval's relation,

$$\int_{-\infty}^{\infty} |x(t)|^2\,\mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2\,\mathrm{d}\omega$$

we can find the area under $|X(j\omega)|^2$ as below.

$$\int_{-\infty}^{\infty} |X(j\omega)|^2\,\mathrm{d}\omega = 2\pi \int_{-\infty}^{\infty} |x(t)|^2\,\mathrm{d}t = \frac{64\pi}{3}$$

(f)

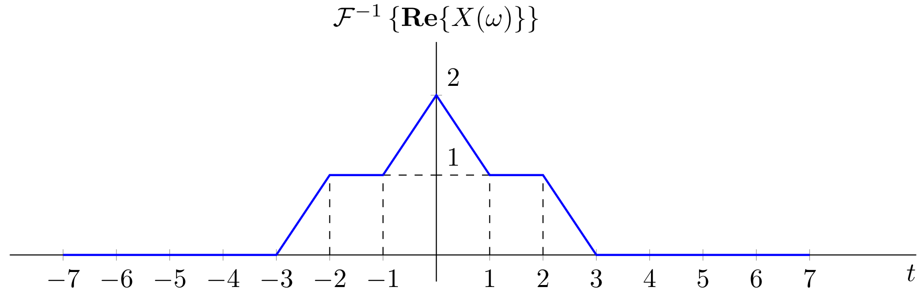
Since $x(t)$ is a real signal, we know that

$$\mathcal{E}\nu(x(t)) \overset{\mathcal{F}}{\longleftrightarrow} \mathbf{Re}\{X(j\omega)\}.$$

Thus, we get

$$\mathcal{F}^{-1}\left\{\mathbf{Re}\{X(j\omega)\}\right\} = \mathcal{E}\nu(x(t)) = \frac{1}{2}\left(x(t) + x(-t)\right).$$

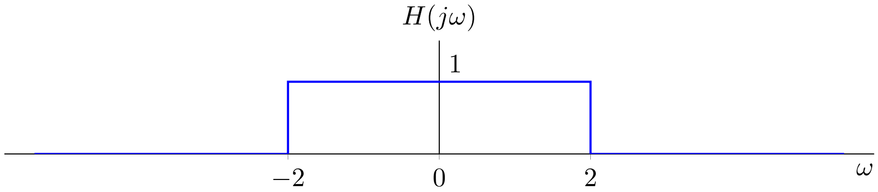
$\mathcal{F}^{-1}\left\{\mathbf{Re}\{X(\omega)\}\right\}$



2.

(a)

$$h(t) = \frac{\sin(2t)}{\pi t} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad H(j\omega) = \begin{cases} 1, & |\omega| < 2 \\ 0, & |\omega| > 2 \end{cases}$$



$$x_1(t) = \sin(3t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X_1(j\omega) = \frac{\pi}{j}\left[\delta(\omega - 3) - \delta(\omega + 3)\right]$$

$$Y_1(j\omega) = X_1(j\omega)\, H(j\omega) = 0 \implies y_1(t) = x_1(t) * h(t) = 0$$

(b)

$$x_2(t) = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \cos\left(\frac{k}{2}t\right) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X_2(j\omega) = \pi \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \left[\delta\left(\omega - \frac{k}{2}\right) + \delta\left(\omega + \frac{k}{2}\right)\right]$$

$$Y_2(j\omega) = X_2(j\omega)\, H(j\omega) = \pi \sum_{k=0}^{3} \left(\frac{1}{2}\right)^k \left[\delta\left(\omega - \frac{k}{2}\right) + \delta\left(\omega + \frac{k}{2}\right)\right]$$

$$y_2(t) = x_2(t) * h(t) = \sum_{k=0}^{3} \left(\frac{1}{2}\right)^k \cos\left(\frac{k}{2}t\right) = 1 + \frac{1}{2}\cos\left(\frac{1}{2}t\right) + \frac{1}{4}\cos(t) + \frac{1}{8}\cos\left(\frac{3}{2}t\right)$$

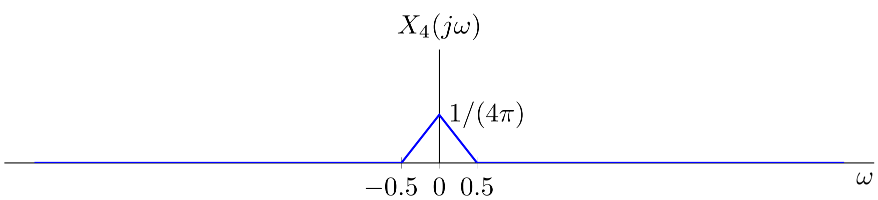
(c)

$$x_3(t) = \frac{\sin\big(3(t-1)\big)}{\pi(t-1)} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X_3(j\omega) = \begin{cases} e^{-j\omega}, & |\omega| < 3 \\ 0, & |\omega| > 3 \end{cases}$$

$$Y_3(j\omega) = X_3(j\omega)\, H(j\omega) = e^{-j\omega}\, H(j\omega) \implies y_3(t) = x_3(t) * h(t) = \frac{\sin\big(2(t-1)\big)}{\pi(t-1)}$$

(d)

$$x_4(t) = \left[\frac{\sin(t/4)}{\pi t}\right]^2 \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X_4(j\omega)$$



$$Y_4(j\omega) = X_4(j\omega)\, H(j\omega) = X_4(j\omega) \implies y_4(t) = x_4(t) * h(t) = x_4(t) = \left[\frac{\sin(t/4)}{\pi t}\right]^2$$

3. (a) To obtain

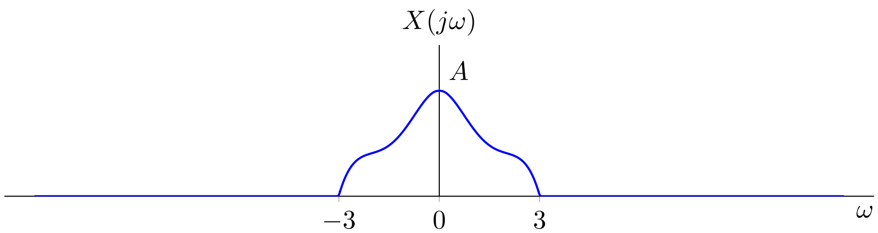
$$y(t) = \Big[x(t)\,\cos^2(4t)\Big] * \frac{\sin\Big(2(t-1)\Big)}{\pi(t-1)},$$

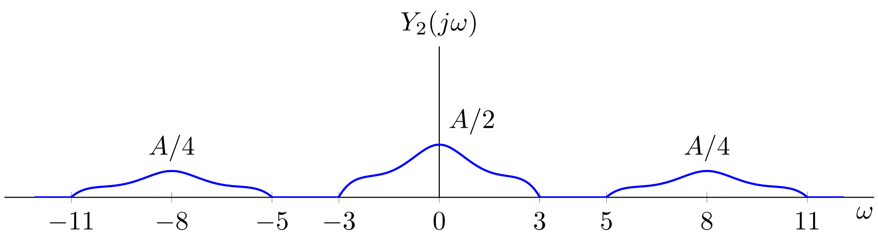
we can follow the steps given below.

$$y_1(t) = \cos^2(4t) = \frac{1+\cos(8t)}{2} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad Y_1(j\omega) = \pi\delta(\omega) + \frac{\pi}{2}\delta(\omega-8) + \frac{\pi}{2}\delta(\omega+8)$$

$$y_2(t) = x(t)\,y_1(t) = x(t)\,\cos^2(4t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad Y_2(j\omega) = \frac{1}{2\pi}\left[X(j\omega) * Y_1(j\omega)\right]$$

$$Y_2(j\omega) = \frac{1}{2}X(j\omega) + \frac{1}{4}X(j(\omega-8)) + \frac{1}{4}X(j(\omega+8))$$





$$g(t) = \frac{\sin\Big(2(t-1)\Big)}{\pi(t-1)} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad G(j\omega) = \begin{cases} e^{-j\omega}, & |\omega| < 2 \\ 0, & |\omega| > 2 \end{cases}$$

$$y(t) = y_2(t) * g(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad Y(j\omega) = Y_2(j\omega)\,G(j\omega) = \begin{cases} e^{-j\omega}\,Y_2(j\omega), & |\omega| < 2 \\ 0, & |\omega| > 2 \end{cases}$$

We observe that $Y(j\omega)$ can be expressed as

$$Y(j\omega) = \frac{1}{2}X(j\omega)\,G(j\omega),$$

which implies

$$y(t) = \frac{1}{2}\left[x(t) * g(t)\right] = x(t) * h(t),$$

where

$$h(t) = \frac{1}{2}\,g(t) = \frac{\sin\Big(2(t-1)\Big)}{2\pi(t-1)}$$

is the impulse response of the LTI system used to obtain $y(t)$ from $x(t)$.

(b) For $X(j\omega) = u(\omega+3) - u(\omega-3)$,

$$y(t) = h(t) = \frac{\sin\Big(2(t-1)\Big)}{2\pi(t-1)},$$

since $Y(j\omega) = X(j\omega)\,H(j\omega) = H(j\omega)$.

Duality provide us with the following transform pairs.

$$g(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad F(j\omega) \Longrightarrow F(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad 2\pi\,g(-j\omega)$$

To use

$$e^{-t}u(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \frac{1}{1+j\omega}$$

and duality, let

$$g(-j\omega) = \frac{1}{1+j\omega},$$

then obtain the transform pair given below.

$$\frac{1}{1-jt} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad 2\pi\, e^{-\omega}\, u(\omega)$$

By using the multiplication property of Fourier transform, we get

$$w(t) = \frac{y(t)}{1-jt} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad W(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\theta)\, 2\pi\, e^{-(\omega-\theta)}\, u(\omega-\theta)\, \mathrm{d}\theta = \int_{-\infty}^{\infty} \frac{y(t)}{1-jt}\, e^{-j\omega t}\, \mathrm{d}t$$

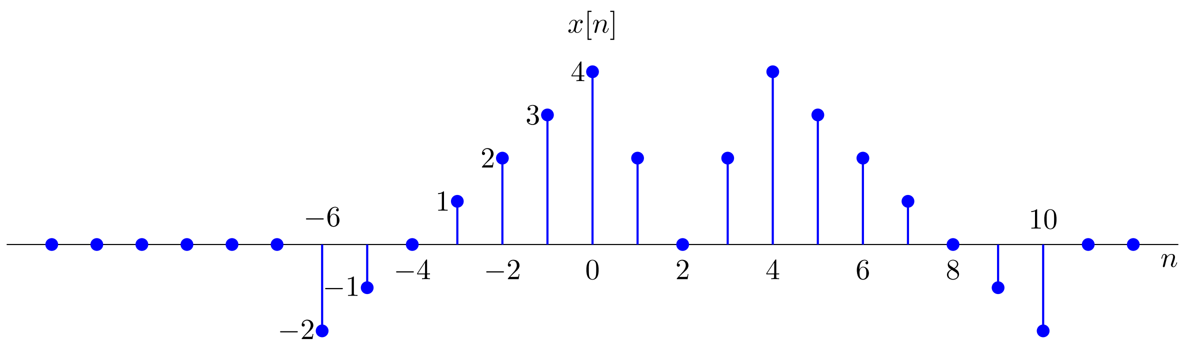
and

$$W(j1) = \int_{-\infty}^{\infty} Y(j\theta)\, e^{-(1-\theta)}\, u(1-\theta)\, \mathrm{d}\theta = \int_{-\infty}^{\infty} \frac{y(t)}{1-jt}\, e^{-jt}\, \mathrm{d}t.$$

$$W(j1) = \int_{-\infty}^{\infty} \frac{e^{-j\theta}}{2}\, [u(\theta+2) - u(\theta-2)]\, e^{-(1-\theta)} u(1-\theta)\, \mathrm{d}\theta = \frac{e^{-1}}{2}\int_{-2}^{1} e^{(1-j)\theta}\, \mathrm{d}\theta = \frac{e^{-j} - e^{j2-3}}{2(1-j)}$$

$$= 0.362 - j0.081$$

4. (a)



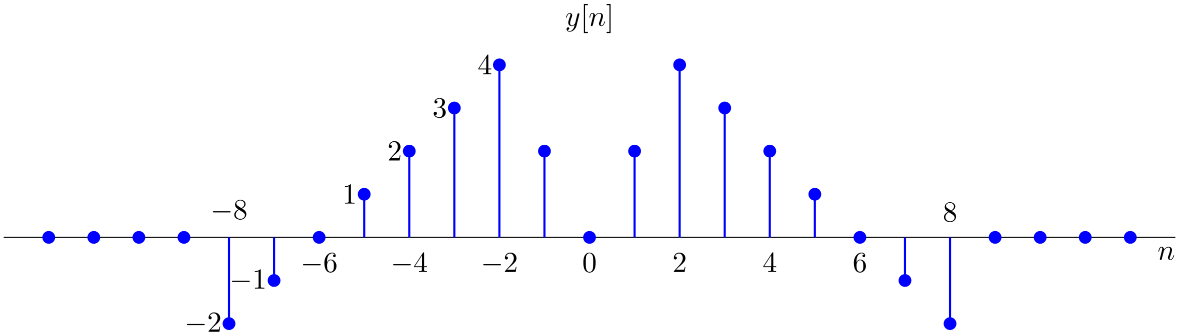
By using the Fourier transform equation,

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

we can find $X(e^{j0})$ as below.

$$X(e^{j0}) \sum_{n=-\infty}^{\infty} x[n] = 18$$

(b) Let $y[n] = x[n+2]$, then note that $y[n]$ is real and even as shown below.



By the time shifting property of the Fourier transform, we get

$$y[n] = x[n+2] \stackrel{\mathcal{F}}{\longleftrightarrow} Y(e^{j\Omega}) = e^{j2\Omega} X(e^{j\Omega})$$

and

$$\angle Y(e^{j\Omega}) = 2\Omega + \angle X(e^{j\Omega}).$$

We know that $Y(e^{j\Omega})$ is real and even since $y[n]$ is real and even. Thus, $\angle Y(e^{j\Omega}) = 0$ and $\angle X(e^{j\Omega}) = -2\Omega$.

(c) By using the inverse Fourier transform equation,

$$x[n] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(e^{j\Omega})\, e^{j\Omega n}\, d\Omega$$

we can evaluate the desired integral as below.

$$\int_{-\pi}^{\pi} X(e^{j\Omega})\, d\Omega = 2\pi\, x[0] = 8\pi$$

(d)

$$X(e^{j\pi}) = \sum_{n=-\infty}^{\infty} x[n]\,(-1)^n = -2$$

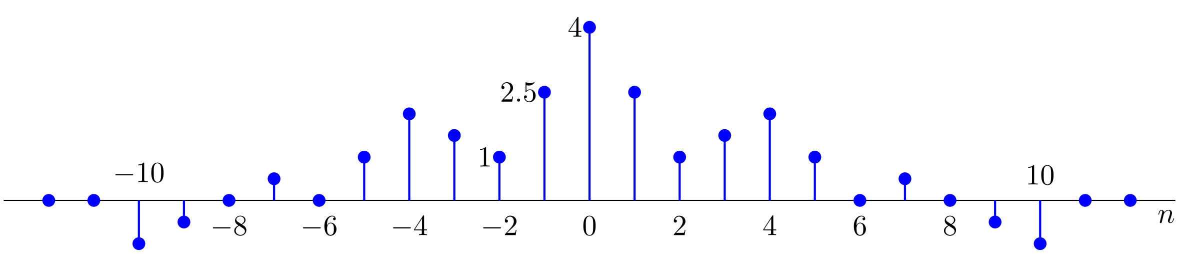
(e) Since $x[n]$ is a real signal, we know that

$$\mathcal{E}\nu(x[n]) \stackrel{\mathcal{F}}{\longleftrightarrow} \mathbf{Re}\{X(e^{j\Omega})\}.$$

Therefore we get

$$\mathcal{F}^{-1}\left\{\mathbf{Re}\{X(e^{j\Omega})\}\right\} = \mathcal{E}\nu(x[n]) = \frac{1}{2}\left(x[n] + x[-n]\right).$$

$$\mathcal{F}^{-1}\left\{\mathbf{Re}\{X(e^{j\Omega})\}\right\}$$



(f) By using Parseval's relation,

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} \left| X(e^{j\Omega}) \right|^2 \mathrm{d}\Omega$$

we can evaluate the desired integral as below.

$$\int_{-\pi}^{\pi} \left| X(e^{j\Omega}) \right|^2 \mathrm{d}\Omega = 2\pi \sum_{n=-\infty}^{\infty} |x[n]|^2 = 156\pi$$

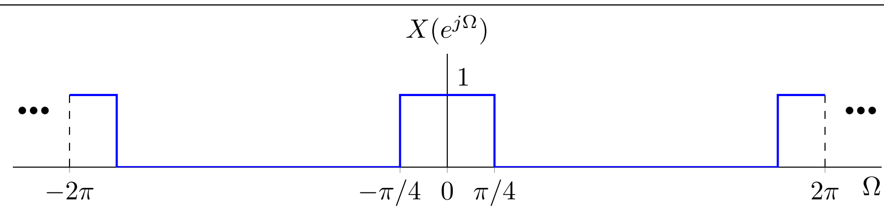
(g) By the differentiation in frequency property of the Fourier transform

$$n\, x[n] \overset{\mathcal{F}}{\longleftrightarrow} j \frac{d}{d\Omega} X(e^{j\Omega})$$

and Parseval's relation, we can evaluate the desired integral

$$\int_{-\pi}^{\pi} \left| \frac{d}{d\Omega} X(e^{j\Omega}) \right|^2 \mathrm{d}\Omega = 2\pi \sum_{n=-\infty}^{\infty} |n\, x[n]|^2 = 2796\pi \qquad \text{(remember that } |j\alpha| = |\alpha|\text{)}$$

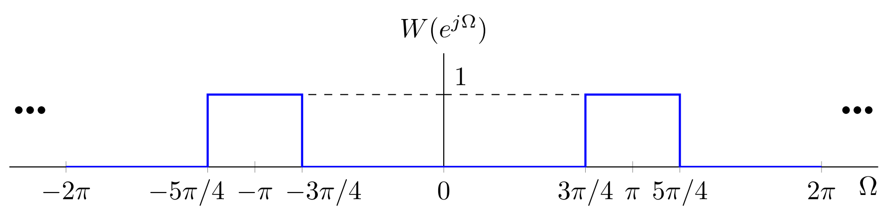
5. (a)



i.

$$p[n] = \cos(\pi n) = \frac{1}{2} \left(e^{j\pi n} + e^{-j\pi n} \right) = e^{j\pi n}$$

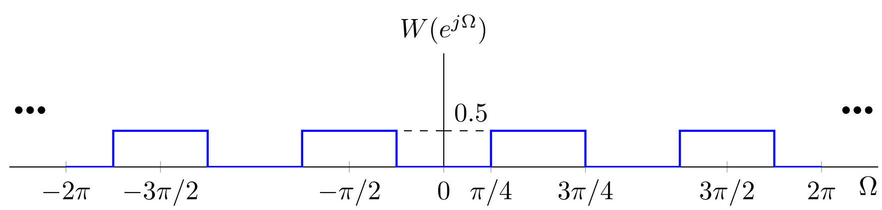
$$w[n] = x[n]\, p[n] = e^{j\pi n}\, x[n] \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad W(e^{j\Omega}) = X(e^{j(\Omega-\pi)})$$



ii.

$$p[n] = \cos\left(\frac{\pi}{2} n\right) = \frac{1}{2} \left(e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n} \right)$$

$$w[n] = \frac{1}{2} \left(e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n} \right) x[n] \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad W(e^{j\Omega}) = \frac{1}{2} \left(X(e^{j(\Omega-\frac{\pi}{2})}) + X(e^{j(\Omega+\frac{\pi}{2})}) \right)$$



iii.

$$p[n] = \sum_{k=-\infty}^{\infty} \delta[n-4k] \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad P(e^{j\Omega}) = \frac{\pi}{2} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - \frac{\pi}{2}k\right)$$

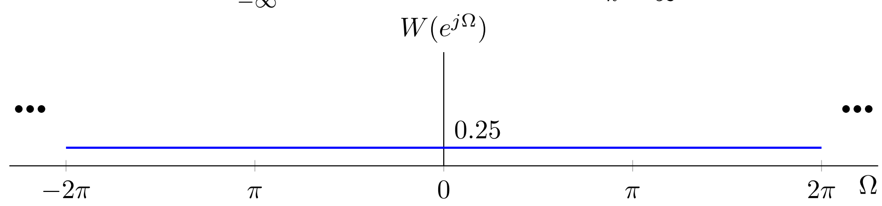
$$w[n] = x[n]\,p[n] \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad W(e^{j\Omega}) = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(e^{j\theta})\,P(e^{j(\Omega-\theta)})\,\mathrm{d}\theta$$

Using

$$\tilde{X}(e^{j\Omega}) = \begin{cases} X(e^{j\Omega}), & |\Omega| \leqslant \pi \\ 0, & \text{otherwise} \end{cases}$$

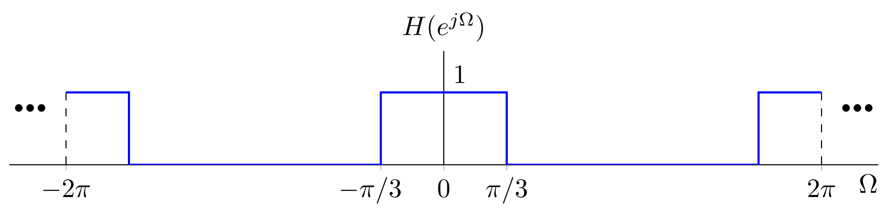
we can convert the periodic convolution to the aperiodic convolution as below.

$$W(e^{j\Omega}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{X}(e^{j\theta})\,P(e^{j(\Omega-\theta)})\,\mathrm{d}\theta = \frac{1}{4} \sum_{k=-\infty}^{\infty} \tilde{X}\left(e^{j(\Omega - \frac{\pi}{2}k)}\right)$$



(b)

$$h[n] = \frac{\sin\left(\frac{\pi}{3}n\right)}{\pi\,n} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad H(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} \left[u\left(\Omega + \frac{\pi}{3} - 2\pi k\right) - u\left(\Omega - \frac{\pi}{3} - 2\pi k\right)\right]$$



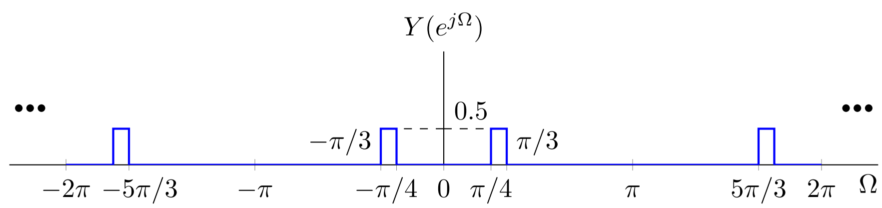
By the convolution of the Fourier transform,

$$y[n] = w[n] * h[n] \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad Y(e^{j\Omega}) = W(e^{j\Omega})\,H(e^{j\Omega})$$

i.

$$Y(e^{j\Omega}) = W(e^{j\Omega})\,H(e^{j\Omega}) = 0 \implies y[n] = 0$$

ii.



$$y[n] = \frac{1}{2}\left(h[n] - \frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n}\right) = \frac{\sin\left(\frac{\pi}{3}n\right)}{2\pi n} - \frac{\sin\left(\frac{\pi}{4}n\right)}{2\pi n}$$

iii.

$$Y(e^{j\Omega}) = W(e^{j\Omega})\,H(e^{j\Omega}) = \frac{1}{4}\,H(e^{j\Omega}) \implies y[n] = \frac{1}{4}\,h[n] = \frac{\sin\left(\frac{\pi}{3}n\right)}{4\pi n}$$

6.	For the Discrete Fourier Transform (DFT), we have the following equations. $$\text{Analysis equation:} \quad X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}, \qquad 0 \leqslant k \leqslant N-1$$ $$\text{Synthesis equation:} \quad x[n] = \frac{1}{N}\sum_{n=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}, \quad 0 \leqslant n \leqslant N-1$$ Thus, we can find $x[n]$ as below. $$x[n] = \frac{1}{N}\sum_{n=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn} = \frac{1}{N}\left(e^{j\Theta}e^{j\frac{2\pi}{N}n} + e^{-j\Theta}\overbrace{e^{j\frac{2\pi}{N}n(N-1)}}^{e^{-j\frac{2\pi}{N}n}}\right) = \frac{2}{N}\cos\left(\frac{2\pi}{N}n + \Theta\right), \quad 0 \leqslant n \leqslant N-1$$ Yes, $x[n]$ is **real**.