Chapter 8

The Z-transform

In the previous chapter, we discussed the Laplace transform, which is a generalization of the CT Fourier transform. We will now discuss the z-transform, which constitutes a generalization of the DT Fourier transform. Note that these generalizations of CTFT and DTFT are important for the following reasons:

- They provide additional tools and insights for signals and systems compared to those we gained from Fourier transform.
- They are applicable to important contexts that Fourier tools are not applicable or less useful, for example,
  - analysis/design of feedback systems (topic of EE302)
  - investigation of the stability of systems (topic of EE302)
  - filter design (topic of EE430)

8.1 Generalized eigenfunctions of LTI systems and the z-transform

- Response of a DT LTI system to a complex exponential z^n:

$\Rightarrow$ Complex exponentials of the general form $z^n = (re^{j\Omega_0})^n$ are *eigenfunctions* of DT LTI systems with *eigenvalues* given by $H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$ where $h[n]$ is the impulse response of the system.

- The eigenvalue expression provides the definition of the z-transform of a signal $x[n]$ where z is an arbitrary complex number:

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}}$$

We will again use shorthand notations for the z transform of a signal $x[n]$:

$$X(z) = \mathcal{Z}\{x[n]\} \qquad \text{and} \qquad x[n] \longleftrightarrow X(z)$$

- Notice that if DTFT of $x[n]$ exists, z-transform reduces to the DTFT when $z =$
$X(z)|_{z=} \qquad =$

- There is a more general relationship between the z-transform and the DTFT:

The above relation indicates that $X(z)$ converges (exists) whenever

  - This occurs when
  - and thus for a given $x[n]$, convergence of $X(z)$ will depend on

8.2 Region of Convergence (ROC)

Region of Convergence (ROC): The range of r values for which the z-transform $X(z)$ converges. (Note that $r = |z|$.)

Ex: $x[n] = a^n u[n]$, where a is any complex number. Find $X(z)$ and its ROC.

Note that if the ROC of $X(z)$ contains the unit circle ($r = |z| = 1$), then the DTFT of $x[n]$ ...

Ex: Previous example with $a = \frac{1}{2}$ and $a = \frac{3}{2}$. What can you say about $X(e^{j\Omega})$ in these cases?

Ex: $x[n] = -a^n u[-n-1]$. Find $X(z)$ and its ROC.

Notes :

- It is helpful to remember the following frequently used signal and z transform pairs :

$$
\begin{aligned}
a^n u[n] &\longleftrightarrow \frac{z}{z-a}, \quad ROC: |z| > |a| \\
-a^n u[-n-1] &\longleftrightarrow \frac{z}{z-a}, \quad ROC: |z| < |a|
\end{aligned}
$$

- The specification of the z transform requires both
  - the algebraic expression for $X(z)$
  - and the associated ROC.

  Warning:

Ex: $x[n] = (\frac{1}{3})^n u[n] + (\frac{1}{2})^n u[n]$. Find $X(z)$ and its ROC.

Ex: [Challenge yourself!] $x[n] = -(\frac{1}{2})^n u[-n-1] + (\frac{1}{3})^n u[n]$. Find $X(z)$ and its ROC.

8.3 Properties of ROC

Property 1 : The ROC of $X(z)$ depends only on $|z| = r$ and therefore consists of a **ring** in the z-plane centered at the origin.

Why?:

- The relation indicates that the convergence/existence of $X(z)$ requires the convergence/existence of , which happens when $x[n]r^{-n}$ is absolutely summable, i.e. $\sum_{-\infty}^{\infty} |x[n]r^{-n}| = \sum_{-\infty}^{\infty} |x[n]||r^{-n} < \infty$.
- Thus, ROC of $X(z)$ depends only on the magnitude $r = |z|$ and **not** on the angle Ω of z.

Property 2 : For rational $X(z)$, the ROC does **not** contain any poles.

Why?: At a pole, $X(z)$ is ...

Remarks:

- Rational $X(z)$ means $X(z)$ is a ratio of polynomials of z.
- A pole of $X(z)$ is a root of the denominator and a zero of $X(z)$ is a root of the numerator.

Property 3 : If $x[n]$ is of **finite duration**, then the ROC is the entire z-plane except possibly $z = 0$ and/or $z = \infty$.

Why?: A finite duration sequence has only a finite number of nonzero samples, e.g.

- If $0 < N_1 < N_2$, then ...
- If $N_1 < 0$ and $N_2 > 0$, then ...
- If $N_1 < N_2 < 0$, then ...

<u>**Property 4 :**</u> If $x[n]$ is a **right-sided** sequence, then the ROC is the outside of a circle centered at the origin (excluding possibly $z = \infty$).

Why?: For a right-sided sequence, $x[n] = 0$ for

Suppose some r_0 is in the ROC of $X(z)$:

- $N_1 > 0$: For $r_1 > r_0$, we have
- $N_1 < 0$: There is an additional finite sum coming from negative n (i.e. $\sum_{N_1}^{0} x[n] r_1^{-n} e^{-j\Omega n}$). This will not cause a problem for convergence except possibly at $z = \infty$. Hence $X(z)$ is

<u>**Property 5 :**</u> If $x[n]$ is a **left-sided** sequence, then the ROC is the inside of a circle centered at the origin (excluding possibly $z = 0$).

Why?: Reason is similar to the previous property.

<u>**Property 6 :**</u> If $x[n]$ is **two-sided**, then the ROC is either a ring or empty.

Why?: A two-sided sequence =

Ex: $x[n] = \begin{cases} 2^n, & n < 0 \\ (\frac{1}{3})^n, & n = 0, 2, 4, .. \\ 0, & n = 1, 3, 5, ... \end{cases}$ Find $X(z)$ and its ROC.

Property 7 : If $X(z)$ is **rational**, then its ROC is bounded by the poles or extends to infinity.

Why?:

- A signal $x[n]$ with rational $X(z)$ consist of a linear combination of exponentials $\alpha^n u[n]$ or $-\alpha^n u[-n-1]$ which have ROCs bounded by their poles.
- The ROC of $X(z)$ of the linear combination of exponentials thus is the intersection of ROCs bounded by poles (unless there is zero-pole cancellation).

Ex:

Ex: (Cont'ed)

8.4 Inversion of Z transforms

Given $X(z)$ and ROC, how can we determine $x[n]$?

$$x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$$

where the integration is around a circle that is in the ROC. Hence the inverse z transform expression contains integration around a circular contour on the z-plane, and is typically difficult to compute and will not be applied in this course. However, there are a number of alternative procedures for obtaining a sequence from its z transform and associated ROC.

For rational $X(z)$, an alternative way is to perform **partial-fraction expansion** for $X(z)$, and then to recognize the sequence associated with each term in the expansion.

Ex: $X(z) = \dfrac{3z}{(2-z)(2z-1)}$, ROC:$\frac{1}{2} < |z| < 2$. Find $x[n]$.

Ex: [Challenge yourself!] Same $X(z)$ with ROC: $|z| > 2$. Find $x[n]$.

Ex: [Challenge yourself!] Same $X(z)$ with ROC: $|z| < 1/2$. Find $x[n]$.

Another method is to use the **power-series expansion** of $X(z)$ and determine $x[n]$ by inspection.

Ex: $X(z) = 4z^2 + 2 + 3z^{-1}$, ROC: $0 < |z| < \infty$. Find $x[n]$.

Power series expansion is particularly useful for non-rational $X(z)$.

Ex: $X(z) = \log(1 + z^{-1})$, ROC: $|z| > 1$. Find $x[n]$.

Remember the Taylor series expansion of a function $f(x)$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Applying Taylor series expansion to $\log(1 + x)$ with $a = 0$ gives the following:

$$\log(1+x) = 0 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{2}{3!}x^3 \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}, \quad |x| < 1$$

Then

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^{-n}, \quad |z| > 1$$

By inspection, the inverse z-transform of this: $x[n] = \begin{cases} \frac{(-1)^{n+1}}{n}, & \text{if } n \geq 1 \\ 0, & \text{otherwise} \end{cases}$

8.5 Properties of Z transform

To discuss the z transform properties, we use the same convenient shorthand notation we used for Fourier transform properties. In other words, to indicate the pairing of a signal and its z transform, we use :

- $x[n] \longleftrightarrow X(z)$

We sometimes also refer to a z transform with the following notation :

- $X(z) = \mathcal{Z}\{x[n]\},\ ROC : |z| > a$

For the following properties, we consider two signals $x[n]$ and $y[n]$ with corresponding z transforms $X(z)$ and $Y(z)$, and ROCs R_x and R_y, respectively, i.e.

$$x[n] \longleftrightarrow X(z),\ ROC = R_x$$

$$y[n] \longleftrightarrow Y(z),\ ROC = R_y$$

8.5.1 Linearity

$ax[n] + by[n] \longleftrightarrow aX(z) + bY(z), \quad ROC \supseteq R_x \cap R_y$ (pole zero cancellations may occur)

8.5.2 Time Shift

$x[n - n_0] \longleftrightarrow X(z)z^{-n_0},\ ROC = R_x$ with possible inclusion or deletion of $z = 0$ or $z = \infty$.

Ex: $x_1[n] = -\delta[n+1] + \delta[n]$ and $x_2[n] = x_1[n-1]$. Find the z transforms and associated ROCs.

8.5.3 Scaling in the z domain

$z_0^n x[n] \longleftrightarrow X(\frac{z}{z_0}),\ ROC = |z_0|R_x = \{z : \left|\frac{z}{z_0}\right| \in R_x\}$

This property corresponds to frequency shifting property for $z_0 = e^{j\Omega_0}$:
$e^{j\Omega_0 n}x[n] \longleftrightarrow$

Ex: [Challenge yourself!] $R_x :\ \frac{1}{2} < |z| < 5$ and $|z_0| = 3$. Find $|z_0|R_x$.

8.5.4 Time Reversal

$x[-n] \longleftrightarrow X(\frac{1}{z}),\ ROC = \frac{1}{R_x} = \{z : \frac{1}{z} \in R_x\}$

Ex: [Challenge yourself!] $R_x :\ \frac{1}{2} < |z| < 5$. Find $\frac{1}{R_x}$.

8.5.5 Conjugation

$x^*[n] \longleftrightarrow X^*(z^*),\ ROC = R_x$

If $x[n]$ is real, (i.e. $x[n] = x^*[n]$)

- $X(z) = X^*(z^*)$
- thus, if $X(z)$ has a pole (zero) at $z = z_0$, then it must have another pole (zero) at the complex conjugate point $z = z_0^*$.

Ex: [Challenge yourself!] Consider $X(z) = A(z)\dfrac{z-b}{z-a}$. Find $X^*(z^*)$ and its poles. Show that if $X(z) = X^*(z^*)$, then the poles and zeros must appear in complex conjugate pairs.

8.5.6 Convolution

$x[n] * y[n] \longleftrightarrow X(z)Y(z), \quad ROC \supseteq R_x \cap R_y$ (pole zero cancellations may occur)

Ex: Consider $x_1[n]$ and $x_2[n]$ plotted below and find z transform of $x_1[n] * x_2[n]$.

8.5.7 Differentiation in z-domain

$nx[n] \longleftrightarrow -z\dfrac{dX(z)}{dz}, \; ROC = R_x$ (with possible inclusion of $z = 0$)

Ex: [Challenge yourself!] $X(z) = \dfrac{az^{-1}}{(1 - az^{-}1)^2}, \;\; |z| > a$. Determine $x[n]$.

8.5.8 The initial value theorem

If $x[n]$ is causal, i.e. $x[n] = 0$ for $n < 0$, then

$$x[0] = \lim_{z\to\infty} X(z)$$

8.5.9 Table of Z transform properties and some common z transform pairs

The following tables from the textbook summarize z transform properties and common pairs.

TABLE 10.1 PROPERTIES OF THE z-TRANSFORM

Section	Property	Signal	z-Transform	ROC
		$x[n]$ $x_1[n]$ $x_2[n]$	$X(z)$ $X_1(z)$ $X_2(z)$	R R_1 R_2
10.5.1	Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2
10.5.2	Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R, except for the possible addition or deletion of the origin
10.5.3	Scaling in the z-domain	$e^{j\omega_0 n}x[n]$ $z_0^n x[n]$ $a^n x[n]$	$X(e^{-j\omega_0}z)$ $X\left(\frac{z}{z_0}\right)$ $X(a^{-1}z)$	R z_0R Scaled version of R (i.e., $\lvert a\rvert R$ = the set of points $\{\lvert a\rvert z\}$ for z in R)
10.5.4	Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1}, where z is in R)
10.5.5	Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in R)
10.5.6	Conjugation	$x^*[n]$	$X^*(z^*)$	R
10.5.7	Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of R_1 and R_2
10.5.7	First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $\lvert z\rvert > 0$
10.5.7	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - z^{-1}}X(z)$	At least the intersection of R and $\lvert z\rvert > 1$
10.5.8	Differentiation in the z-domain	$nx[n]$	$-z\frac{dX(z)}{dz}$	R
10.5.9		Initial Value Theorem If $x[n] = 0$ for $n < 0$, then $x[0] = \lim_{z\to\infty} X(z)$		

TABLE 10.2 SOME COMMON z-TRANSFORM PAIRS

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z, except 0 (if $m>0$) or ∞ (if $m<0$)
5. $\alpha^n u[n]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
6. $-\alpha^n u[-n-1]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
7. $n\alpha^n u[n]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
8. $-n\alpha^n u[-n-1]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
9. $[\cos\omega_0 n]u[n]$	$\frac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\frac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\frac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\frac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

8.6 LTI Systems and Z-transform

The z-transform plays an important role in the analysis and design of DT LTI systems:

- By convolution property,

- $H(z) = \mathcal{Z}\{h[n]\}$: **transfer function** (system function) of the LTI system
  We can characterize an LTI system by $H(z)$ and its ROC. If unit circle $\in$ ROC, then $H(z)$ can be evaluated at $z = e^{j\Omega}$ (i.e. $|z| = 1$), and hence the frequency response of the system can be obtained:
  $$\mathcal{F}\{h[n]\} = H(e^{j\Omega}) =$$

- Also note that, as discussed before, $H(z)$ is the eigenvalue corresponding to the input eigenfunction z^n.

For LTI systems, many system properties can be directly determined from the ROC, and the poles and zeros of $H(z)$, as is discussed below.

8.6.1 Causality

Property 1 : A DT LTI system is **causal** if and only if the ROC of $H(z)$ is outside of a circle, including infinity.

Why?

Property 2 : A DT LTI system with rational $H(z)$ is **causal** if and only if

- order($a(z)$)$\leq$ order($b(z)$) where $H(z) = \dfrac{a(z)}{b(z)}$.
- the ROC of $H(z)$ is the outside of the outermost pole.

Why?

8.6.2 Stability

Property 1 : A DT LTI system is **stable** if and only if the ROC of $H(z)$ contains the unit circle, i.e. $|z| = 1$.

Why?

Property 2 : A causal DT LTI system with rational $H(z)$ is **stable** if and only if all poles of $H(z)$ lie inside the unit circle (i.e. all poles have magnitudes smaller than 1.)

Why?

Ex: An LTI system satisfies the following difference equation:

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1].$$

What are the possible $h[n]$'s for this system?

Ex: [Challenge yourself!] Consider a stable and causal LTI system with impulse response $h[n]$ and rational system function $H(z)$. The following information is given:

- $H(z)$ contains a pole at $z = \frac{1}{2}$ and a zero somewhere on the unit circle. The precise number and locations of other poles and zeros are unknown.

Are the following statements *true, false or not verifiable*? Justify your answers.

1. $\mathcal{F}\{(\frac{1}{2})^n h[n]\}$ converges.
2. $H(e^{j\Omega}) = 0$ for some Ω.
3. $h[n]$ has finite duration.
4. $h[n]$ is real.
5. $g[n] = n(h[n] * h[n])$ is the impulse response of a causal system.