Middle East Technical University
Department of Electrical and Electronics Engineering
EE301: Signals and Systems I

Solutions for Homework 2
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1. (a) Two DT signals are given by

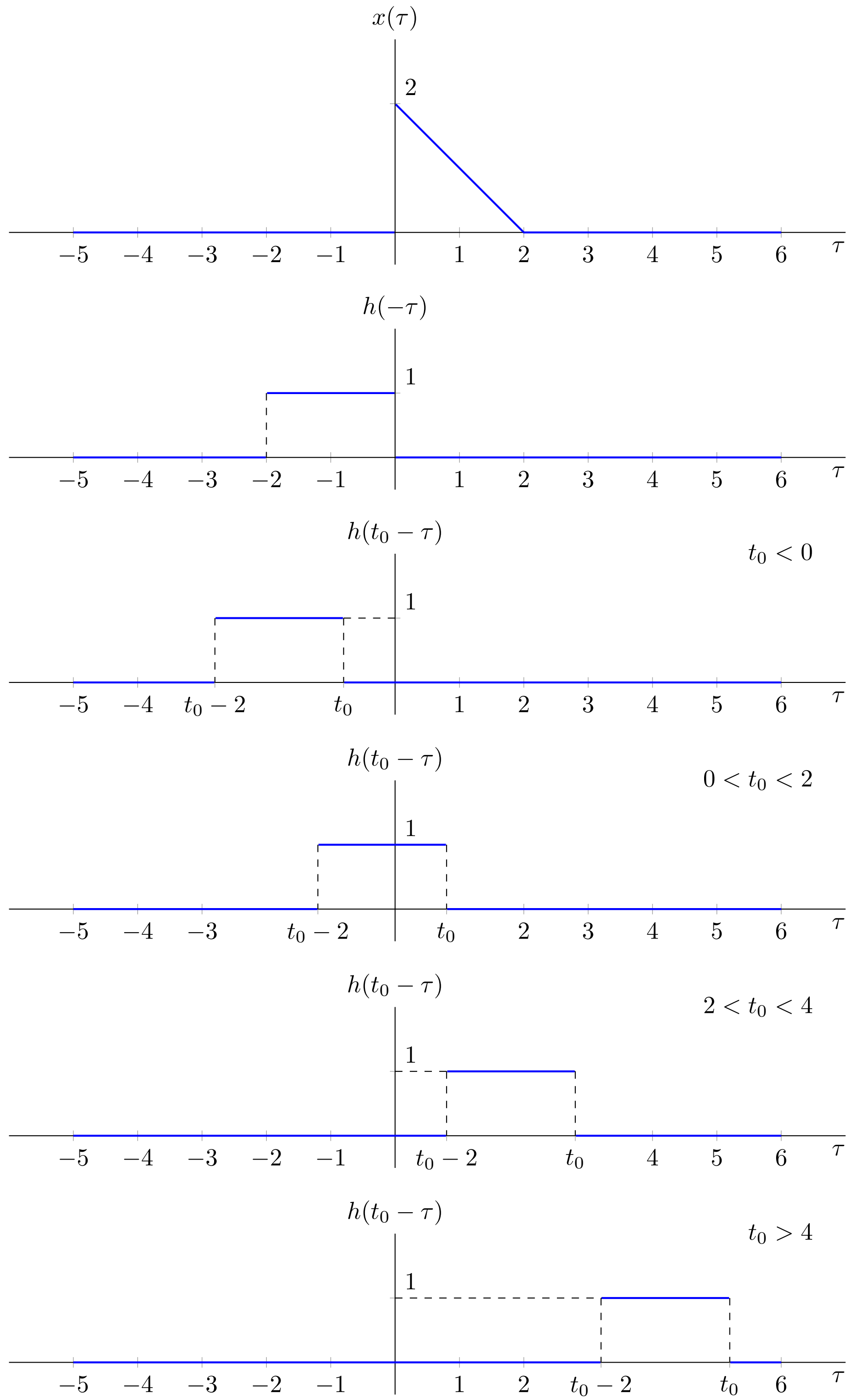
$$x[n] = \left(\frac{1}{2}\right)^n u[n-1]$$

$$w[n] = u[n+1]$$

and the convolution of these signals is computed as follows:

$$\begin{aligned}
x[n] * w[n] &= \sum_{k=-\infty}^{\infty} x[k]\, w[n-k] \\
&= \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k u[k-1]\, u[n-k+1] \\
&= \sum_{k=1}^{n+1} \left(\frac{1}{2}\right)^k, \qquad && n \geqslant 0 \\
&= \sum_{k=0}^{n+1} \left(\frac{1}{2}\right)^k - 1, && n \geqslant 0 \\
&= \frac{1-\left(\frac{1}{2}\right)^{n+2}}{1-\frac{1}{2}} - 1, && n \geqslant 0 \\
&= u[n] - \left(\frac{1}{2}\right)^{n+1} u[n]
\end{aligned}$$

(b)	Two DT signals are given by $$x[n] = \left(\frac{1}{2}\right)^n (u[n-1] - u[n-100]) + \delta[n]$$ $$w[n] = u[2n] = u[n]$$ and the convolution of these signals is computed as follows: $$x[n] * w[n] = \underbrace{\left(\frac{1}{2}\right)^n (u[n-1] - u[n-100]) * u[n]}_{\sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k (u[k-1] - u[k-100])\, u[n-k]} + \overbrace{\underbrace{\delta[n] * u[n]}_{u[n]}}^{\text{from linearity property } \downarrow}$$ $$= \sum_{k=1}^{99} \left(\frac{1}{2}\right)^k u[n-k]$$ $$= \begin{cases} \sum_{k=1}^{99} \left(\frac{1}{2}\right)^k = 2 - 2^{-99}, & n > 99 \\ \sum_{k=1}^{n} \left(\frac{1}{2}\right)^k = 2 - 2^{-n}, & 1 \leqslant n \leqslant 99 \\ 0, & n < 1 \end{cases}$$ Then, the result can be expressed as below. $$x[n] * w[n] = \begin{cases} 2 - 2^{-99}, & n > 99 \\ 2 - 2^{-n}, & 1 \leqslant n \leqslant 99 \\ 1, & n = 0 \\ 0, & n < 0 \end{cases}$$ $$x[n] * w[n] = u[n] + u[n-1] - 2^{-n} (u[n-1] - u[n-100]) - 2^{-99}\, u[n-100]$$
(c)	The input to a CT LTI system is given by $$x(t) = \begin{cases} 2 - t, & 0 < t < 2 \\ 0, & \text{otherwise} \end{cases}$$ and the impulse response of the system is $$h(t) = u(t) - u(t-2).$$ For $t = t_0$, the response of the system is computed as follows: $$y(t_0) = \int_{-\infty}^{\infty} x(\tau)\, h(t_0 - \tau)\, \mathrm{d}\tau$$



We have sketched $x(\tau)$, $h(-\tau)$, $h(t_0-\tau)$ for different t_0's. According to these plots, the cases of the overlap between $x(\tau)$ and $h(t_0-\tau)$ are observed as follows:

- For $t_0 < 0$, there is no overlap.
- For $0 < t_0 < 2$, there is overlap from 0 to t_0.
- For $2 < t_0 < 4$, there is overlap from $t_0 - 2$ to 2.
- For $t_0 > 4$, there is no overlap.

Considering these intervals, we can compute the convolution integral as follows:

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, \mathrm{d}\tau \\
y(t) &= 0, && t < 0 \\
y(t) &= \int_{0}^{t} (2-\tau)\, \mathrm{d}\tau, && 0 < t < 2 \\
&= \left(2\tau - \frac{1}{2}\tau^2\right)\Big|_0^t, && 0 < t < 2 \\
&= 2\,t - \frac{1}{2}\,t^2, && 0 < t < 2 \\
y(t) &= \int_{t-2}^{2} (2-\tau)\, \mathrm{d}\tau, && 2 < t < 4 \\
&= \left(2\tau - \frac{1}{2}\tau^2\right)\Big|_{t-2}^{2}, && 2 < t < 4 \\
&= 4 - \frac{1}{2}\,4 - 2\,(t-2) + \frac{1}{2}\,(t-2)^2, && 2 < t < 4 \\
&= 8 - 4\,t + \frac{1}{2}\,t^2, && 2 < t < 4 \\
y(t) &= 0, && t > 4
\end{aligned}
$$

Thus, the response of the system to the input $x(t)$ is found as below.

$$
y(t) = \begin{cases}
0, & t < 0 \\
\frac{1}{2}\,t\,(4-t) = 2\,t - \frac{1}{2}\,t^2, & 0 < t < 2 \\
\frac{1}{2}(t-4)^2 = 8 - 4\,t + \frac{1}{2}\,t^2, & 2 < t < 4 \\
0, & 4 < t
\end{cases}
$$

2.	(a)	A system has the input-output relationship is given by

$$y[n] = x[n] - \frac{1}{2}\, x[n-2].$$

- If the system possesses the superposition property, then it is linear.

  Let the response of the system for $x_1[n]$ and $x_2[n]$ be

$$y_1[n] = x_1[n] - \frac{1}{2}\, x_1[n-2]$$
$$y_2[n] = x_2[n] - \frac{1}{2}\, x_2[n-2]$$

  then find the response to the input $x_3[n] = a\, x_1[n] + b\, x_2[n]$, where a and b are complex constants.

$$\begin{aligned} y_3[n] &= x_3[n] - \frac{1}{2}\, x_3[n-2] \\ &= a\, x_1[n] + b\, x_2[n] - \frac{1}{2}\left(a\, x_1[n-2] + b\, x_2[n-2]\right) \\ &= a \underbrace{\left(x_1[n] - \frac{1}{2}\, x_1[n-2]\right)}_{y_1[n]} + b \underbrace{\left(x_2[n] - \frac{1}{2}\, x_2[n-2]\right)}_{y_2[n]} \\ &= a\, y_1[n] + b\, y_2[n] \end{aligned}$$

  System 2 is **linear** since it possesses the superposition property.

- If a time shift in the input signal results in an identical time shift in the output signal, then the system is time-invariant.

  Let $x_2[n] = x_1[n-n_0]$, then express the response to input $x_2[n]$ in terms of $x_1[n]$.

$$y_2[n] = x_2[n] - \frac{1}{2}\, x_2[n-2] = x_1[n-n_0] - \frac{1}{2}\, x_1[n-n_0-2]$$

  Using the given input-output relationship, we can find $y_1[n-n_0]$ as follows:

$$\begin{aligned} y_1[n-n_0] &= x_1[n-n_0] - \frac{1}{2}\, x_1[n-n_0-2] \\ &= y_2[n] \end{aligned}$$

  When $x_2[n] = x_1[n-n_0]$, we observe that $y_2[n] = y_1[n-n_0]$. Hence, the system is **time-invariant**.

Yes, the system is LTI since it is both linear and time-invariant. The difference equation is not recursive, namely the system does not require memorizing any previous values of the output. Thus, we do not need to have initially at rest condition to prove our result.

The output will be equal to the impulse response of the system when the input is $\delta[n]$. The impulse response of the system is found as

$$h[n] = \delta[n] - \frac{1}{2}\, \delta[n-2],$$

which has 2 nonzero values. Therefore, it is **FIR**.

(b) A system has the input-output relationship is given by

$$y[n] = x[n] - \frac{1}{2}\,y[n-2].$$

Yes, this system is LTI since it is initially at rest, i.e., $x[n] = 0$ and $y[n] = 0$ for $n < n_0$.

The impulse response can be found either in two ways.

- **1st Approach:** When $x[n] = \delta[n]$, we know that $y[n] = 0$ for $n < 0$ due to the condition of initial rest. Using this initial condition, we list the successive values $y[n]$ as follows:

n	-2	-1	0	1	2	3	4	5	6	7	8
$x[n]$	0	0	1	0	0	0	0	0	0	0	0
$y[n-2]$	0	0	0	0	1	0	$-\frac{1}{2}$	0	$\frac{1}{4}$	0	$-\frac{1}{8}$
$y[n] = x[n] - \frac{1}{2}\,y[n-2]$	0	0	1	0	$-\frac{1}{2}$	0	$\frac{1}{4}$	0	$-\frac{1}{8}$	0	$\frac{1}{16}$

According these values, the impulse response is

$$h[n] = \begin{cases} \left(-\frac{1}{2}\right)^{n/2}, & \text{even } n \\ 0, & \text{odd } n \end{cases}$$

for $n \geqslant 0$. Due to initial rest condition, $h[n] = 0$ for $n < 0$.

Thus, $h[n]$ can be written in a compact form by using $j = \sqrt{-1}$ as follows:

$$h[n] = \frac{1}{2}\left[\left(j\sqrt{\frac{1}{2}}\right)^n + \left(-j\sqrt{\frac{1}{2}}\right)^n\right] u[n]$$

- **2nd Approach:** Using the initial condition, we obtain the impulse response

$$h[n] = h_h[n] + h_p[n],$$

where $h_h[n]$ denotes the homogeneous solution and $h_p[n]$ denotes the particular solution.

For $n < 0$, $h[n] = 0$ due to initial rest condition.

For $n \geqslant 0$ and $x[n] = \delta[n]$, we have

$$h[n] + \frac{1}{2}\,h[n-2] = \delta[n].$$

The particular solution is zero since $x[n] = 0$ for $n > 0$. Therefore, $h[n] = h_h[n]$, and we only need to find the homogeneous solution.

For $n > 0$, the difference equation becomes

$$h[n] + \frac{1}{2}\,h[n-2] = 0.$$

Let $h[n] = z^n$, and find the characteristic equation.

$$z^n + \frac{1}{2}\,z^{n-2} = 0 \implies 1 + \frac{1}{2}\,z^{-2} = 0$$

The roots of the characteristic equation are

$$z_{1,2} = \pm j\sqrt{\frac{1}{2}}.$$

The homogeneous solution is

$$h[n] = A\left(j\sqrt{\frac{1}{2}}\right)^n + B\left(-j\sqrt{\frac{1}{2}}\right)^n u[n].$$

Owing to the initial rest condition, the system is causal, i.e., $h[n] = 0$ for $n < 0$. We can use this property to find A and B.

$$\left.\begin{array}{ll} n=0: & h[0] + \frac{1}{2}h[-2] = 1 \implies h[0] = 1 = A + B \\ n=1: & h[1] + \frac{1}{2}h[-1] = 0 \implies h[1] = 0 = A - B \end{array}\right\} A = B = \frac{1}{2}$$

Thus, the impulse response is found as as follows:

$$h[n] = \frac{1}{2}\left[\left(j\sqrt{\frac{1}{2}}\right)^n + \left(-j\sqrt{\frac{1}{2}}\right)^n\right] u[n]$$

The system is **IIR** since its impulse response has infinite duration, namely infinite number of nonzero values.

3. (a) The impulse response of System 1 is given as

$$h_1[n] = u[n+2]$$

System 1 is **not memoryless** since $h_1[n] \neq 0$ for $n \neq 0$, e.g. $h_1[1] = 1$.

System 1 is **not causal** since $h_1[n] \neq 0$ for $n < 0$, e.g. $h_1[-1] = 1$.

System 1 is **not stable** since $h_1[n]$ is not absolutely summable, i.e.,

$$\sum_{k=-\infty}^{\infty} |h_1[k]| = \sum_{k=-\infty}^{\infty} |u[k+2]| = \sum_{k=-2}^{\infty} 1 = \infty.$$

(b)	The input-output relationship of System 2 is given by $$y[n] = \sum_{m=-3}^{-1} w[n-m]$$ • If System 2 possesses the superposition property, then it is linear. Let the response of System 2 for $w_1[n]$ and $w_2[n]$ be $$y_1[n] = \sum_{m=-3}^{-1} w_1[n-m]$$ $$y_2[n] = \sum_{m=-3}^{-1} w_2[n-m]$$ then find the response to the input $w_3[n] = a\,w_1[n] + b\,w_2[n]$, where a and b are complex constants. $$\begin{aligned} y_3[n] &= \sum_{m=-3}^{-1} w_3[n-m] \\ &= \sum_{m=-3}^{-1} (a\,w_1[n-m] + b\,w_2[n-m]) \\ &= a\underbrace{\sum_{m=-3}^{-1} w_1[n-m]}_{y_1[n]} + b\underbrace{\sum_{m=-3}^{-1} w_2[n-m]}_{y_2[n]} \\ &= a\,y_1[n] + b\,y_2[n] \end{aligned}$$ System 2 is **linear** since it possesses the superposition property. • If a time shift in the input signal results in an identical time shift in the output signal, then System 2 is time-invariant. Let $w_2[n] = w_1[n-n_0]$, then express the response to input $w_2[n]$ in terms of $w_1[n]$. $$y_2[n] = \sum_{m=-3}^{-1} w_2[n-m] = \sum_{m=-3}^{-1} w_1[n-n_0-m]$$ Using the given input-output relationship, we can find $y_1[n-n_0]$ as follows: $$\begin{aligned} y_1[n-n_0] &= \sum_{m=-3}^{-1} w_1[n-n_0-m] \\ &= y_2[n] \end{aligned}$$ When $w_2[n] = w_1[n-n_0]$, we observe that $y_2[n] = y_1[n-n_0]$. Hence, System 2 is **time-invariant**. Yes, System 2 is LTI since it is both linear and time-invariant.

(c)	To find the impulse response of the overall system, let $x[n] = \delta[n]$, and find $y[n]$. When $x[n] = \delta[n]$, $w[n] = u[n+2]$. Then, $$y[n] = \sum_{m=-3}^{-1} w[n-m] = \sum_{m=-3}^{-1} u[n-m+2] = u[n+5] + u[n+4] + u[n+3] = h[n].$$ Simplifying the answer, we can write $$h[n] = \delta[n+5] + 2\,\delta[n+4] + 3\,u[n+3].$$
(d)	For the overall system, the input-output relationship as a difference equation is found as follows: $$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] \\ &= \sum_{k=-\infty}^{\infty} x[k]\,\left(\delta[n-k+5] + 2\,\delta[n-k+4] + 3\,u[n-k+3]\right) \\ &= x[n+5] + 2\,x[n+4] + 3\sum_{k=-\infty}^{n+3} x[k] \end{aligned}$$ According to the last term of the RHS, we deduce that the difference equation is recursive. Let's write $y[n-1]$ and $y[n]$ as follows: $$y[n-1] = x[n+4] + 2\,x[n+3] + 3\sum_{k=-\infty}^{n+2} x[k]$$ $$y[n] = x[n+5] + 2\,x[n+4] + 3\,x[n+3] + 3\sum_{k=-\infty}^{n+2} x[k]$$ If we subtract these equations, we obtain the difference equation. $$\begin{aligned} y[n] - y[n-1] &= x[n+5] + x[n+4] + x[n+3] \\ y[n] &= x[n+5] + x[n+4] + x[n+3] + y[n-1] \end{aligned}$$
(e)	The overall system is **LTI** since both System 1 and System 2 are LTI. The overall system is **not memoryless** since $h[n] \neq 0$ for $n \neq 0$, e.g. $h[1] = 3$. The overall system is **not causal** since $h[n] \neq 0$ for $n < 0$, e.g. $h[-1] = 3$. The overall system is **not stable** since $h[n]$ is not absolutely summable, i.e., $$\sum_{k=-\infty}^{\infty} \lvert h_1[k]\rvert = \sum_{k=-\infty}^{\infty} \lvert\delta[k+5] + 2\,\delta[k+4] + 3\,u[k+3]\rvert = 1 + 2 + 3\sum_{k=-3}^{\infty} 1 = \infty.$$

4. (a) The input-output relationship of a system is given by

$$y(t) = \int_{-\infty}^{t} e^{-2(t-\tau-1)} x(\tau)\,\mathrm{d}\tau$$

An LTI system is both linear and time-invariant.

- If we show that the given system possesses the superposition property, we verify that the system is linear.

  Let the response to $x_1(t)$ and $x_2(t)$ be

$$y_1(t) = \int_{-\infty}^{t} e^{-2(t-\tau-1)} x_1(\tau)\,\mathrm{d}\tau$$

$$y_2(t) = \int_{-\infty}^{t} e^{-2(t-\tau-1)} x_2(\tau)\,\mathrm{d}\tau$$

  then find the response to the input $x_3(t) = a\,x_1(t) + b\,x_2(t)$, where a and b are complex constants.

$$\begin{aligned}
y_3(t) &= \int_{-\infty}^{t} e^{-2(t-\tau-1)} x_3(\tau)\,\mathrm{d}\tau \\
&= \int_{-\infty}^{t} e^{-2(t-\tau-1)} \left[a\,x_1(\tau) + b\,x_2(\tau)\right]\mathrm{d}\tau \\
&= a\underbrace{\int_{-\infty}^{t} e^{-2(t-\tau-1)} x_1(\tau)\,\mathrm{d}\tau}_{y_1(t)} + b\underbrace{\int_{-\infty}^{t} e^{-2(t-\tau-1)} x_2(\tau)\,\mathrm{d}\tau}_{y_2(t)} \\
&= a\,y_1(t) + b\,y_2(t)
\end{aligned}$$

  We have shown that the given system possesses the superposition property. Hence, the given system is verified to be linear.

- If we show that a time shift in the input signal results in an identical time shift in the output signal for the given system, we verify that the system is time-invariant.

  Let $x_2(t) = x_1(t - t_0)$, then express the response to input $x_2(t)$ in terms of $x_1(t)$.

$$y_2(t) = \int_{-\infty}^{t} e^{-2(t-\tau-1)} x_2(\tau)\,\mathrm{d}\tau = \int_{-\infty}^{t} e^{-2(t-\tau-1)} x_1(\tau - t_0)\,\mathrm{d}\tau$$

  Using the given input-output relationship, we can find $y_1(t - t_0)$ as follows:

$$\begin{aligned}
y_1(t - t_0) &= \int_{-\infty}^{t-t_0} e^{-2(t-t_0-\tau-1)} x_1(\tau)\,\mathrm{d}\tau \\
&= \int_{-\infty}^{t} e^{-2(t-\lambda-1)} x_1(\lambda - t_0)\,\mathrm{d}\lambda \qquad (\text{for } \lambda \triangleq \tau + t_0) \\
&= y_2(t)
\end{aligned}$$

	For $x_2(t) = x_1(t - t_0)$, we have shown that $y_2(t) = y_1(t - t_0)$. Hence, the given system is verified to be time-invariant.
(b)	We know that the response to a unit impulse is the impulse response of the system. Let the input $x(t) = \delta(t)$, then find the output $y(t) = h(t)$. $$h(t) = \int_{-\infty}^{t} e^{-2(t-\tau-1)}\delta(\tau)\,\mathrm{d}\tau = e^{-2(t-1)} \int_{-\infty}^{t} \delta(\tau)\,\mathrm{d}\tau = \begin{cases} e^{-2(t-1)}, & t > 0 \\ 0, & \text{otherwise} \end{cases}$$ The impulse response of this system is $$h(t) = e^{-2(t-1)}u(t)$$
(c)	$$x(t) = \begin{cases} 2, & 0 < t < 1 \\ -1, & -1 < t < 0 \\ 0, & \text{otherwise} \end{cases}$$ Let a dummy signal $v(t)$ be $$v(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$$ then find the response of given system to $v(t)$. $$\begin{aligned} w(t) &= h(t) * v(t) \\ &= \int_{-\infty}^{\infty} v(\tau)\,h(t-\tau)\,\mathrm{d}\tau \\ &= \int_{0}^{1} e^{-2(t-\tau-1)}u(t-\tau)\,\mathrm{d}\tau \end{aligned}$$ We can compute this integral by specifying the intervals of t. For $t < 0$, $u(t-\tau) = 0$ between the limits of integral. Hence, we obtain $$w(t) = 0.$$ For $0 < t < 1$, $u(t-\tau) = 0$ when $\tau > t$. Hence, the response of the system corresponding to this interval is given as below. $$\begin{aligned} w(t) &= \int_{0}^{t} e^{-2(t-\tau-1)}\,\mathrm{d}\tau \\ &= \frac{1}{2}\left(e^{-2(t-\tau-1)}\right)\Big|_{\tau=0}^{t} \\ &= \frac{e^2 - e^{-2(t-1)}}{2} \end{aligned}$$

	For $t \geqslant 1$, $u(t-\tau) = 1$ between the limits of integral. Then, the response of the system corresponding to this interval is given as below. $$\begin{aligned} w(t) &= \int_0^1 e^{-2(t-\tau-1)}\,\mathrm{d}\tau \\ &= \frac{1}{2}\left(e^{-2(t-\tau-1)}\right)\Big\|_{\tau=0}^{1} \\ &= \frac{e^{-2(t-2)} - e^{-2(t-1)}}{2} \end{aligned}$$ Combining these results, we can express the response $w(t)$ as follows: $$w(t) = \begin{cases} 0, & t < 0 \\ \dfrac{e^2 - e^{-2(t-1)}}{2}, & 0 < t < 1 \\ \dfrac{e^{-2(t-2)} - e^{-2(t-1)}}{2}, & 1 < t \end{cases}$$ The given input signal $x(t)$ can be expressed in terms of $v(t)$ as follows: $$x(t) = 2\,v(t) - v(t+1)$$ Since the given system is LTI, the response of the system to $x(t)$ can be expressed in terms of $w(t)$ $$y(t) = 2\,w(t) - w(t+1),$$ where $$w(t+1) = \begin{cases} 0, & t < -1 \\ \dfrac{e^2 - e^{-2t}}{2}, & -1 < t < 0 \\ \dfrac{e^{-2(t-1)} - e^{-2t}}{2}, & 0 < t \end{cases}$$ then, the response is found as below. $$y(t) = \begin{cases} 0, & t < -1 \\ \dfrac{e^{-2t} - e^2}{2}, & -1 < t < 0 \\ \dfrac{e^{-2t} - 3\,e^{-2(t-1)} + 2\,e^2}{2} = e^2 + \dfrac{e^{-2t}}{2}\left(1 - 3\,e^2\right), & 0 < t < 1 \\ \dfrac{e^{-2t} - 3\,e^{-2(t-1)} + 2\,e^{-2(t-2)}}{2} = \dfrac{e^{-2t}}{2}\left(1 - 3\,e^2 + 2\,e^4\right), & 1 < t \end{cases}$$
(d)	The system is **not memoryless** since $h(t) \neq 0$ for $t \neq 0$. The system is **causal** since $h(t) = 0$ for $t < 0$. The system is **stable** since $h(t)$ is absolutely integrable, i.e., $$\int_{-\infty}^{\infty} \|h(\tau)\|\,\mathrm{d}\tau = \int_0^{\infty} \left\|e^{-2(\tau-1)}\right\|\,\mathrm{d}\tau = \int_0^{\infty} e^{-2(\tau-1)}\,\mathrm{d}\tau = \frac{1}{2}\,e^2\left(e^{-2\tau}\right)\Big\|_{\infty}^{0} = \frac{e^2}{2} < \infty.$$

(e)

$$y(t) = \int_{t-4}^{t+4} e^{-2(\tau-t)} x(\tau-1)\,\mathrm{d}\tau + x(t-1)$$

For $x(t) = \delta(t)$, the term $\delta(\tau-1)$ makes the integral zero if $\tau = 1$ is outside the interval of integral. Thus, the integral is nonzero only for $-3 < t < 5$.

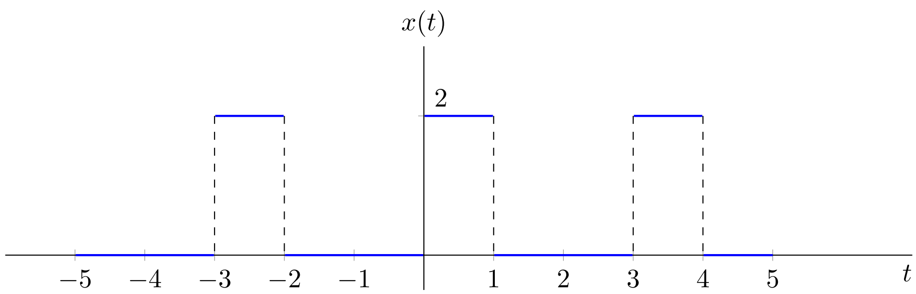
$$h(t) = \int_{t-4}^{t+4} e^{-2(\tau-t)} \delta(\tau-1)\,\mathrm{d}\tau + \delta(t-1) = \begin{cases} e^{-2(1-t)} + \delta(t-1), & -3 < t < 5 \\ \delta(t-1), & \text{otherwise} \end{cases}$$

The impulse response of this system is

$$h(t) = e^{-2(1-t)}\left[u(t+3) - u(t-5)\right] + \delta(t-1).$$

5. (a) A periodic signal with fundamental period $T_0 = 3$ is given as

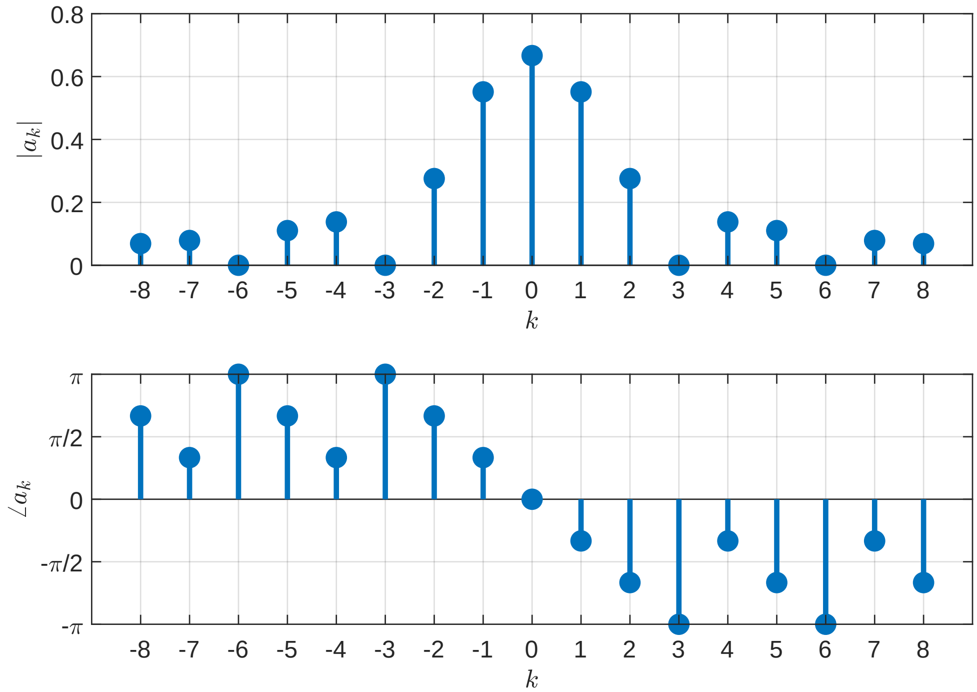
$$x(t) = \begin{cases} 2, & 0 < t < 1 \\ 0, & 1 < t < 3 \end{cases}$$



(b) The CTFS coefficients of $x(t)$ are

$$\begin{aligned}
a_k &= \frac{1}{T_0} \int_{\langle T_0 \rangle} x(t) e^{-jk\frac{2\pi}{T_0}t}\,\mathrm{d}t \\
&= \frac{1}{3} \int_0^1 2\, e^{-jk\frac{2\pi}{3}t}\,\mathrm{d}t \\
&= \frac{2}{3} \cdot \frac{-3}{j2\pi k} \left(e^{-jk\frac{2\pi}{3}t} \right) \Big|_0^1 \\
&= \frac{2}{\pi k} \cdot \frac{1}{j2} \left(1 - e^{-jk\frac{2\pi}{3}} \right) \\
&= \frac{2}{\pi k} \cdot \frac{e^{-jk\frac{\pi}{3}}}{j2} \left(e^{jk\frac{\pi}{3}} - e^{-jk\frac{\pi}{3}} \right) \\
&= \frac{2\, e^{-jk\frac{\pi}{3}}}{\pi k} \sin\left(k\frac{\pi}{3} \right) \\
&= \frac{2}{3}\, e^{-jk\frac{\pi}{3}} \operatorname{sinc}\left(\frac{k}{3} \right) \qquad \left(\operatorname{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta} \right)
\end{aligned}$$

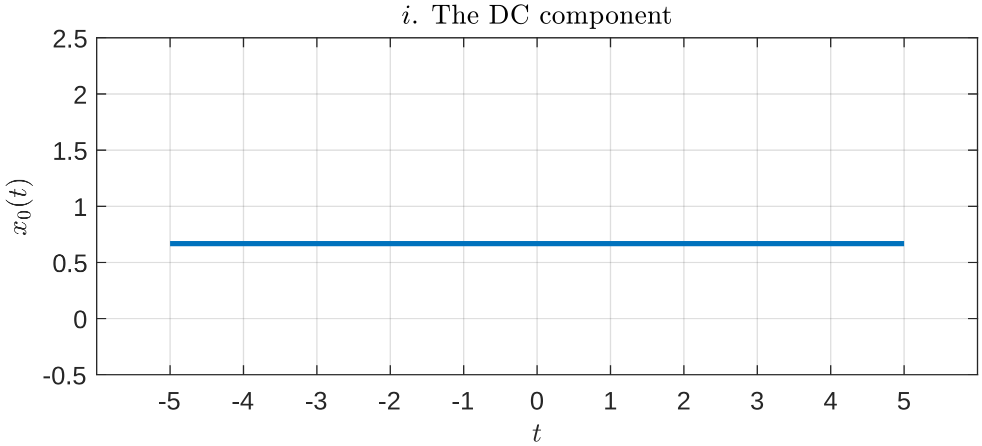
(c) The magnitude and phase of a_k are shown below.



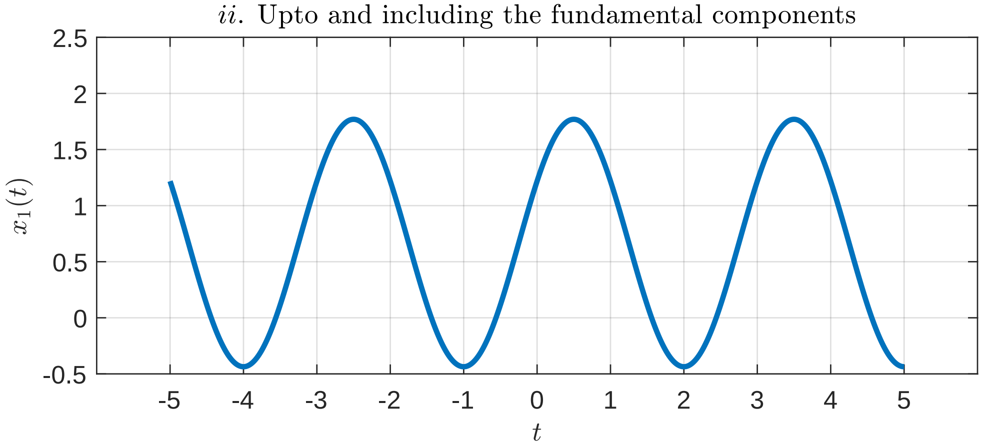
(d) The approximation to $x(t)$ is given as follows:

$$x_M(t) = \sum_{k=-M}^{M} a_k e^{jk\omega_0 t}$$

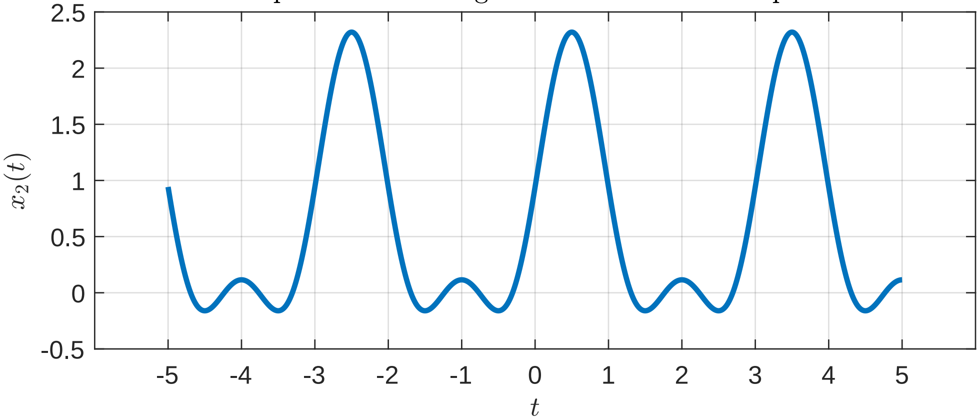
i. The DC component



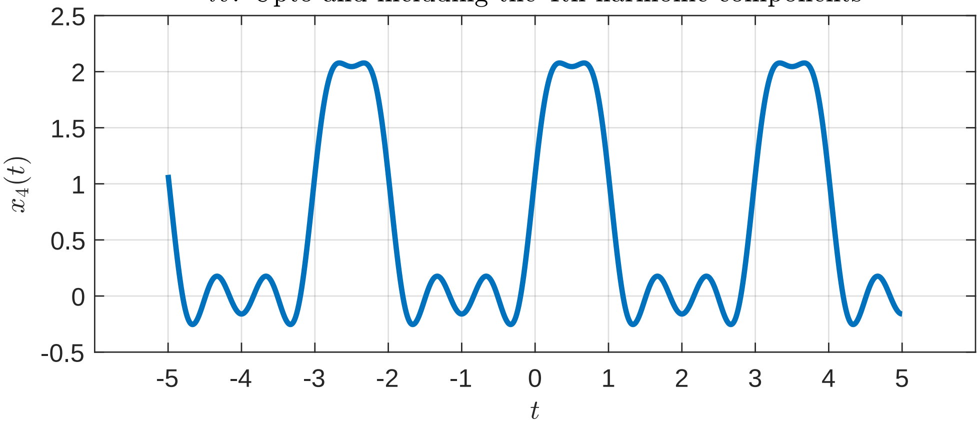
ii. Upto and including the fundamental components



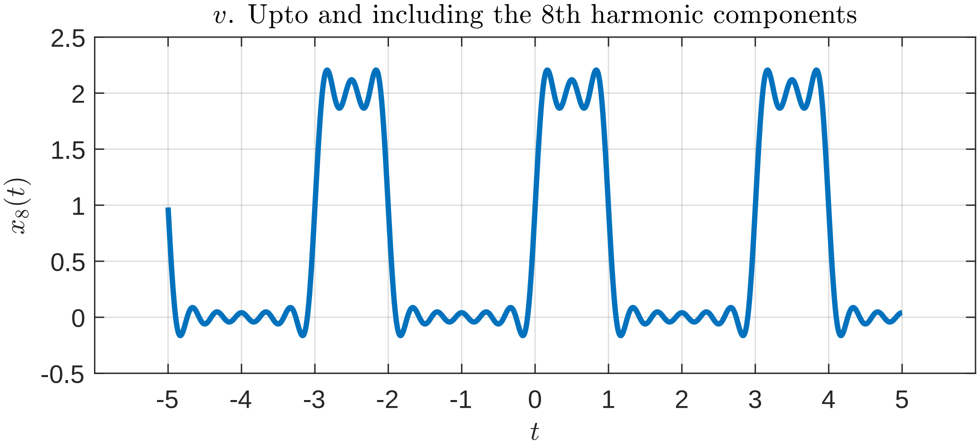
$iii.$ Upto and including the 2nd harmonic components



$iv.$ Upto and including the 4th harmonic components



$v.$ Upto and including the 8th harmonic components



No, $x_M(t)$ is not equal to $x(t)$ for some finite M. For any finite M, there are always ripples on $x_M(t)$. As M increases, the ripples of the approximated signal become compressed toward the point where the original signal has discontinuity, with the peak amplitude of the ripples remaining constant independently of M. This overshoot problem at discontinuities is known as the Gibbs phenomenon.

To plot the approximation $x_M(t)$ in MATLAB, the sample code is given below.

```
clc;clear;close all
% discretization of continuous time
dt = 0.01; % the difference between successive time samples
t = -5:dt:5; % the time samples from the interval [-5, 5]

To = 3; % the fundamental period
ak = @(k)((2/To)*exp(-1j*k*pi/To)*sinc(k/To)); % the CTFS

% computation of the approximation x_M(t), which includes
% up to including the Mth harmonic components
M = 8; % M is in the set {0, 1, 2, 4, 8}
xM = zeros(1,numel(t));
for k = -M:M
    xM = xM+ak(k)*(exp(1j*k*2*pi/To*t));
end

% visualization of the approximation x_M(t)
plot(t,real(xM),'LineWidth',2);grid on;
xlabel('$$t$$','Interpreter','latex');
ylabel(['$$x_' num2str(M) '(t)$$'],'Interpreter','latex');
set(gca,'XLim',[-6 6],'YLim',[-0.5 2.5],'XTick',-5:5);
```

6. (a) Suppose $y(t) = x(t) * h(t)$. Let

$$w(t) = u(t) * v(t),$$

where $u(t) = x(2t)$ and $v(t) = h(2t)$.

$$w(t) = \int_{-\infty}^{\infty} u(\tau)\, v(t-\tau)\, \mathrm{d}\tau = \int_{-\infty}^{\infty} x(2\tau)\, h(2t-2\tau)\, \mathrm{d}\tau \overset{\lambda \triangleq 2\tau}{=} \frac{1}{2} \underbrace{\int_{-\infty}^{\infty} x(\lambda)\, h(2t-\lambda)\, \mathrm{d}\lambda}_{y(2t)} = \frac{1}{2}\, y(2t)$$

Therefore, the result is **not equal** to $y(2t)$.

(b) Suppose $y[n] = x[n] * h[n]$. Let

$$w[n] = u[n] * v[n],$$

where $u[n] = x[2n]$ and $v[n] = h[2n]$.

$$w[n] = \sum_{k=-\infty}^{\infty} u[k]\, v[n-k] = \sum_{k=-\infty}^{\infty} x[2k]\, h[2n-2k] \overset{\ell \triangleq 2k}{=} \sum_{\substack{\ell=-\infty \\ \text{even } \ell}}^{\infty} x[\ell]\, h[2n-\ell]$$

However, we have

$$y[2n] = \sum_{\ell=-\infty}^{\infty} x[\ell]\, h[2n-\ell],$$

where the summation consists of not only the even samples, but also the odd samples. Therefore, the result is **not equal** to $y[2n]$.