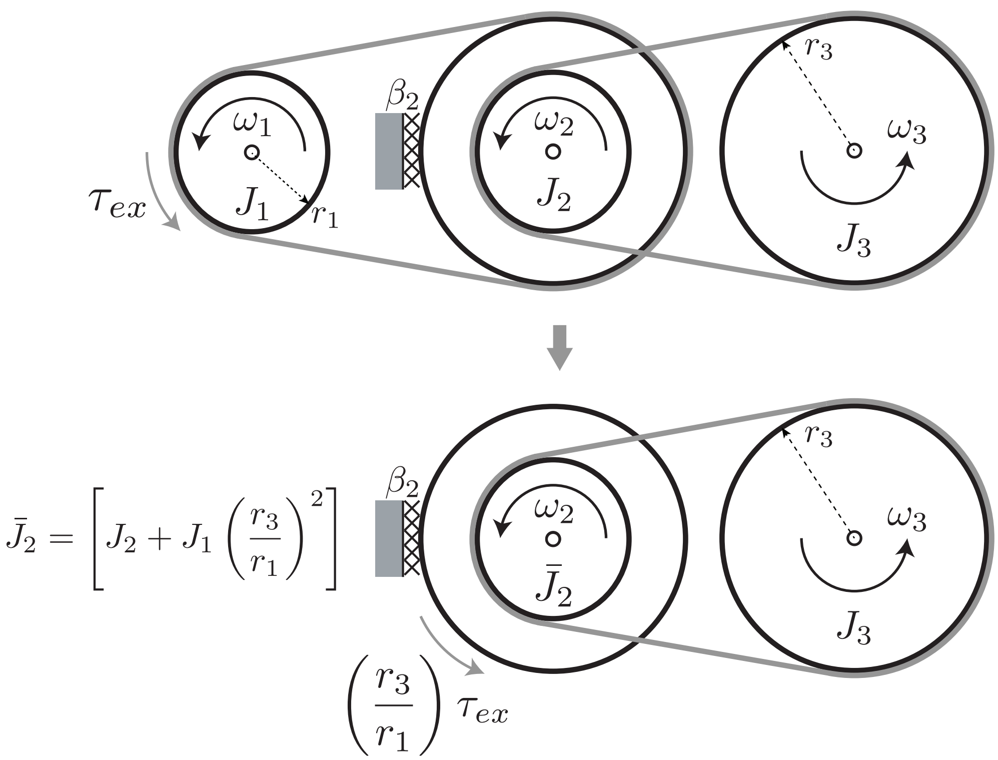
EE302 - Assignment 1 - Solutions

March 11, 2019

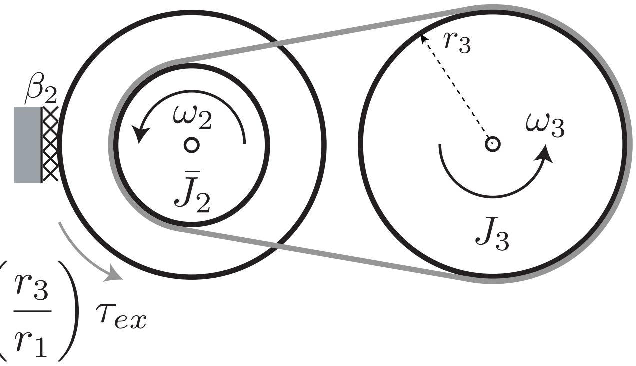
1. (a) Let's solve this problem using the concept of reflected inertia, damping, and torque.

    If we reflect the variables and parameters of first pullet to the second pulley we obtain



Now if we reflect the variables and parameters of the modified second pulley to the the third pulley we obtain

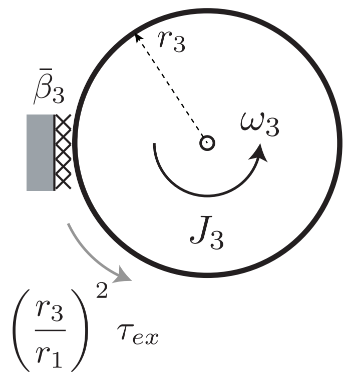
$$\bar{J}_2 = \left[J_2 + J_1 \left(\frac{r_3}{r_1} \right)^2 \right]$$



$$\bar{J}_3 = \left[J_3 + \bar{J}_2 \left(\frac{r_3}{r_1} \right)^2 \right]$$

$$= \left[J_3 + J_2 \left(\frac{r_3}{r_1} \right)^2 + J_1 \left(\frac{r_3}{r_1} \right)^4 \right]$$

$$\bar{\beta}_3 = \left[\beta_2 \left(\frac{r_3}{r_1} \right)^2 \right]$$

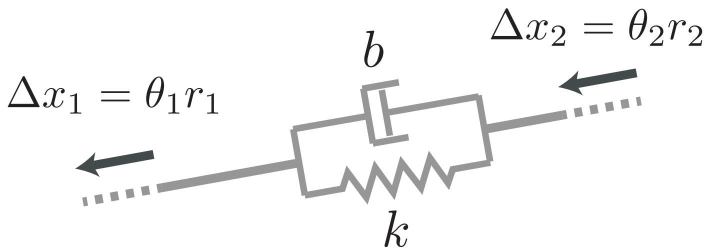


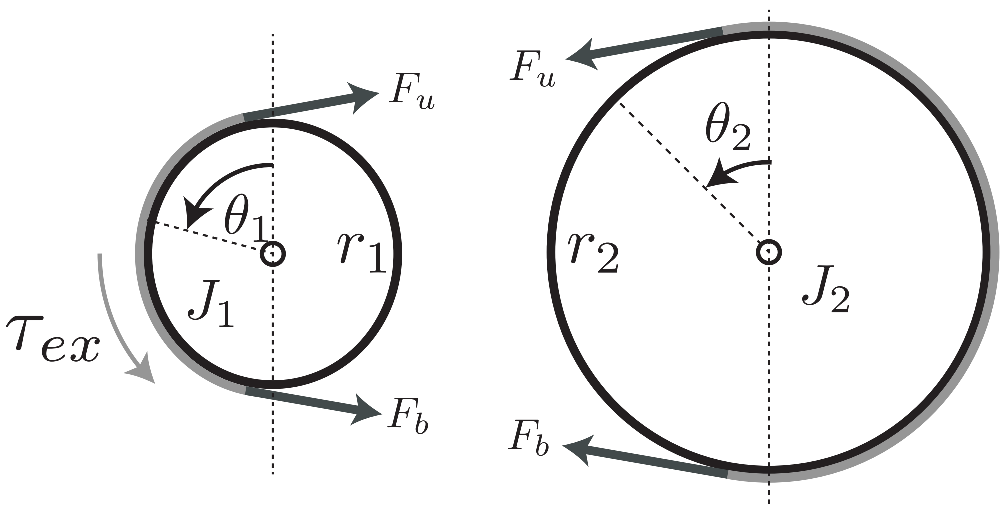
Hence, ode and transfer function of the system can be computed as

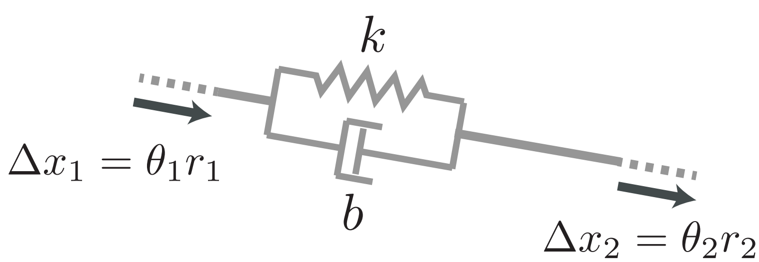
$$\bar{J}_3 \dot{\omega}_3 + \bar{\beta}_3 \omega_3 = \left(\frac{r_3}{r_1} \right)^2 \tau_{ex} \quad \rightarrow \quad \dot{y} + \frac{\bar{\beta}_3}{\bar{J}_3} = \frac{\left(\frac{r_3}{r_1} \right)^2}{\bar{J}_3} u$$

$$\frac{Y(s)}{U(s)} = \frac{\left(\frac{r_3}{r_1} \right)^2 \frac{1}{\bar{J}_3}}{s + \frac{\bar{\beta}_3}{\bar{J}_3}}$$

(b) We assume that when $[\theta_1\ \theta_2\ \dot{\theta}_1\ \dot{\theta}_2] = [0\ 0\ 0\ 0]$ the mechanism is at rest condition. Then let's draw the free-body diagrams







Let's first write the spring force relations

$$
\begin{aligned}
F_u &= k\left(\Delta x_1 - \Delta x_2\right) + b\frac{d}{dt}\left(\Delta x_1 - \Delta x_2\right) \\
&= k\left(r_1\theta_1 - r_2\theta_2\right) + b\left(r_1\dot{\theta}_1 - r_2\dot{\theta}_2\right) \\
&= kr_1\theta_1 - kr_2\theta_2 + br_1\dot{\theta}_1 - br_2\dot{\theta}_2 \\
F_b &= k\left(-\Delta x_1 + \Delta x_2\right) + b\frac{d}{dt}\left(-\Delta x_1 + \Delta x_2\right) \\
&= k\left(-r_1\theta_1 + r_2\theta_2\right) + b\left(-r_1\dot{\theta}_1 + r_2\dot{\theta}_2\right) \\
&= -kr_1\theta_1 + kr_2\theta_2 - br_1\dot{\theta}_1 + br_2\dot{\theta}_2
\end{aligned}
$$

Now let's write the equations of motion of the individual bodies

$$
\begin{aligned}
J_1\ddot{\theta}_1 &= \tau_{ex} - F_u r_1 + F_b r_1 \\
&= \tau_{ex} + \left(-kr_1^2\theta_1 + kr_2r_1\theta_2 - br_1^2\dot{\theta}_1 + br_2r_1\dot{\theta}_2\right) + \left(-kr_1^2\theta_1 + kr_2r_1\theta_2 - br_1^2\dot{\theta}_1 + br_2r_1\dot{\theta}_2\right) \\
&= \tau_{ex} - 2kr_1^2\theta_1 + 2kr_2r_1\theta_2 - 2br_1^2\dot{\theta}_1 + 2br_2r_1\dot{\theta}_2 \\
J_2\ddot{\theta}_2 &= F_u r_2 - F_b r_2 \\
&= \left(kr_1r_2\theta_1 - kr_2^2\theta_2 + br_1r_2\dot{\theta}_1 - br_2^2\dot{\theta}_2\right) + \left(kr_1r_2\theta_1 - kr_2^2\theta_2 + br_1r_2\dot{\theta}_1 - br_2^2\dot{\theta}_2\right) \\
&= 2kr_1r_2\theta_1 - 2kr_2^2\theta_2 + 2br_1r_2\dot{\theta}_1 - 2br_2^2\dot{\theta}_2
\end{aligned}
$$

i. Let $\mathbf{x} = \left[\theta_1 \ \dot{\theta}_1 \ \theta_2 \ \dot{\theta}_2\right]^T$, then we can find a state-space representation

$$
\begin{aligned}
\dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{2kr_1^2}{J_1} & -\frac{2br_1^2}{J_1} & \frac{2kr_1r_2}{J_1} & \frac{2br_1r_2}{J_1} \\ 0 & 0 & 0 & 1 \\ \frac{2kr_1r_2}{J_2} & \frac{2br_1r_2}{J_2} & -\frac{2kr_2^2}{J_2} & -\frac{2br_2^2}{J_2} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \frac{1}{J_1} \\ 0 \\ 0 \end{bmatrix} u \\
y &= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x}
\end{aligned}
$$

ii. Let's take the Laplace transform of the derived diffferential equations

$$
\begin{aligned}
\left[J_1s^2 + 2br_1^2s + 2kr_1^2\right]\Theta_1(s) &= U(s) + \left[2br_2r_1s + 2kr_2r_1\right]Y(s) \\
\left[J_2s^2 + 2br_2^2s + 2kr_2^2\right]Y(s) &= \left[2kr_1r_2 + 2br_1r_2\right]\Theta_1(s)
\end{aligned}
$$

In order to simplify the expression let

$$
B_1 = 2br_1^2\ ,\ K_1 = 2kr_1^2\ ,\ B_2 = 2br_2^2\ ,\ K_2 = 2kr_2^2\ ,\ B_{12} = 2br_2r_1\ ,\ K_{12} = 2kr_1r_2
$$

Then we can have

$$
\begin{aligned}
&\left[J_1s^2 + B_1s + K_1\right]\Theta_1(s) = U(s) + \left[B_{12}s + K_{12}\right]Y(s) \\
&\left[J_2s^2 + B_2s + K_2\right]Y(s) = \left[B_{12}s + K_{12}\right]\Theta_1(s) \quad \rightarrow \quad \Theta_1(s) = \frac{\left[J_2s^2 + B_2s + K_2\right]}{\left[B_{12}s + K_{12}\right]}Y(s) \\
&Y(s)\left\{\left[J_1s^2 + B_1s + K_1\right]\frac{\left[J_2s^2 + B_2s + K_2\right]}{\left[B_{12}s + K_{12}\right]} - \left[B_{12}s + K_{12}\right]\right\} = U(s) \\
&Y(s)\left\{\left[J_1s^2 + B_1s + K_1\right]\frac{\left[J_2s^2 + B_2s + K_2\right]}{\left[B_{12}s + K_{12}\right]} - \left[B_{12}s + K_{12}\right]\right\} = U(s)
\end{aligned}
$$

Let $\frac{Y(s)}{U(s)} = \frac{N(s)}{D(s)}$, then

$$
\begin{aligned}
D(s) =& J_1J_2s^4 + (B_1J_2 + B_2J_1)s^3 + (-B_{12}^2 + B_1B_2 + J_1K_2 + J_2K_1)s^2 \\
&+ (B_1K_2 + B_2K_1 - 2B_{12}K_{12})s + (-K_{12}^2 + K_1K_2) \\
D(s) =& J_1J_2s^4 + (B_1J_2 + B_2J_1)s^3 + (J_1K_2 + J_2K_1)s^2 + 0 + 0
\end{aligned}
$$

Finally the transfer function can be computed as

$$
\frac{Y(s)}{U(s)} = \frac{B_{12}s + K_{12}}{J_1J_2s^4 + (B_1J_2 + B_2J_1)s^3 + (J_1K_2 + J_2K_1)s^2}
$$

iii. The state-space representation with the given coefficients take the form

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 100 & 0 & 0 \\ -50 & -5 & 100 & 10 \\ 0 & 0 & 0 & 1 \\ 10 & 1 & -20 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x}$$

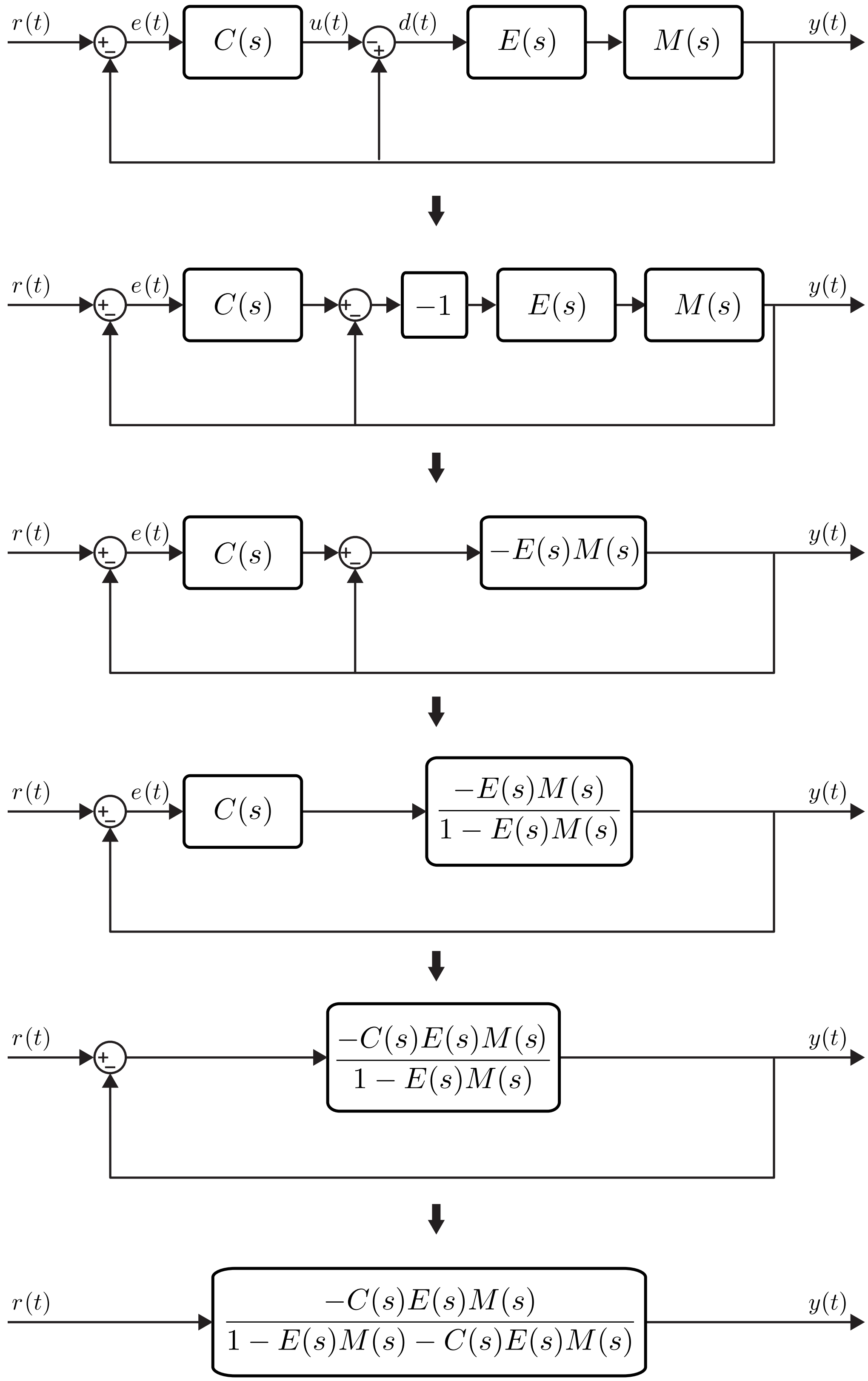
Transfer function with the given coefficients take the form

$$G(s) = \frac{Y(s)}{U(s)} = \frac{100(s+10)}{s^4 + 7s^3 + 70s^2}$$

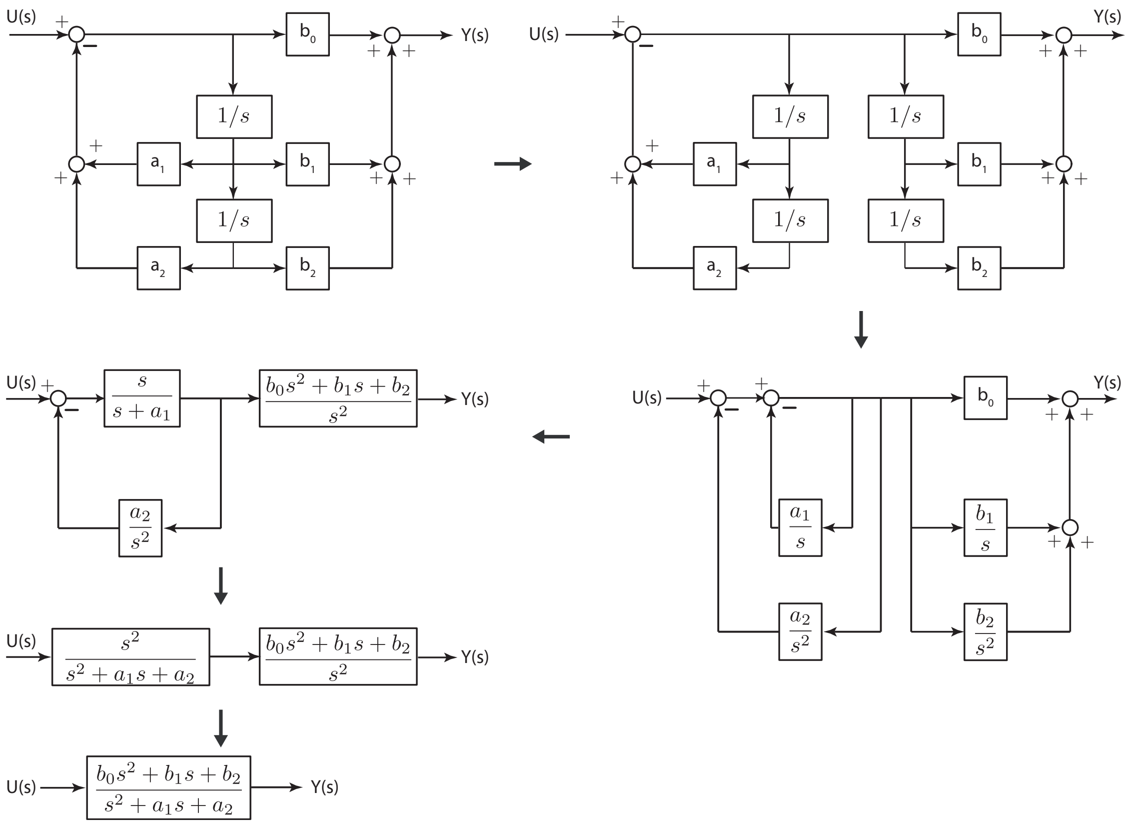
A sample MATLAB code piece which converts the state-space form to the transfer function form (in terms of numerator and denumerator coefficients) is provided below. It is clear that, the computed coefficients match the previous ones.

```
>> r1 = 0.05;
>> r2 = 0.1;
>> J1 = 0.01;
>> J2 = 0.1;
>> k = 100;
>> b = 10;
>>
>> A(1,:) = [0 1 0 0];
>> A(2,:) = [-2*k*r1^2/J1 -2*b*r1^2/J1 2*k*r1*r2/J1 2*b*r1*r2/J1];
>> A(3,:) = [0 0 0 1];
>> A(4,:) = [2*k*r1*r2/J2  2*b*r1*r2/J2 -2*k*r2^2/J2 -2*b*r2^2/J2];
>> A

A =

         0    1.0000         0         0
  -50.0000   -5.0000  100.0000   10.0000
         0         0         0    1.0000
   10.0000    1.0000  -20.0000   -2.0000

>> B = [0 ; 100 ; 0 ; 0];
>> C = [0 0  1 0];
>> D = 0;
>>
>> [num,denum] = ss2tf(A,B,C,D)

num =

   1.0e+03 *

         0         0         0    0.1000    1.0000


denum =

    1.0000    7.0000   70.0000    0.0000    0.0000

>> |
```

2. (a) Solution of the block-diagram simplification process can be seen below



(b) Solution of the block-diagram simplification process can be seen below



(c) Solution of the block-diagram simplification process can be seen below