Discrete Mathematics

Session VI

Generating Functions

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Partitions of Integers: Definition

Definition 1. A ***partition*** of a positive integer n is an expression of n as a sum of a number of positive integers where the order of summands is irrelevant.

The number of partitions of a positive integer n is denoted by $p(n)$. A summand in a partition is also called a ***part*** of the partition.

Example 1. Find $p(1), p(2), p(3), p(4)$, and $p(5)$.

Solution. We have

$$
\begin{aligned}
&1 = 1,\\
&2 = 2 = 1 + 1,\\
&3 = 3 = 2 + 1 = 1 + 1 + 1,\\
&4 = 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1,\\
&5 = 5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1\\
&\quad = 1 + 1 + 1 + 1 + 1.
\end{aligned}
$$

Thus, $p(1) = 1$, $p(2) = 2$, $p(3) = 3$, $p(4) = 5$, and $p(5) = 7$.

To find the number of partitions of a given positive integer n, we should determine the number of ways that a number of 1's, a number of 2's, a number of 3's, and so on can be selected such that the sum of the selected numbers is n.

Partitions of Integers: Euler's Function

The number of partitions of n is the coefficient of x^n in the following function which is known as ***Euler's function***.

$$\begin{aligned}\mathcal{E}(x) &= (1+x+x^2+\cdots)(1+x^2+x^4+\cdots)(1+x^3+x^6+\cdots)\cdots \\ &= \frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^3}\cdots \\ &= \prod_{i=1}^{\infty}\frac{1}{1-x^i} = \sum_{i=0}^{\infty} p(i)x^i\end{aligned}$$

We define $p(0) = 1$. Thus, $\mathcal{E}(x)$ is the generating function for the sequence $\{p(n)\}_{n=0}^{\infty}$.

Note that $p(n)$ is also the number of nonnegative integer solutions to the equation

$$x_1 + 2x_2 + 3x_3 + \cdots + nx_n = n.$$

Unfortunately, the generating function $\mathcal{E}(x)$ does not lead to a closed formula for $p(n)$. It, however, helps us study the properties of integer partitions more systematically.

Example 2. Find the generation function for $p_d(n)$, the number of partitions of a positive integer n with distinct summands.

Solution.

$$\prod_{i=1}^{\infty}(1+x^i).$$

Partitions of Integers: Examples

Example 3. Let $p_o(n)$ be the number of partitions of a positive integer n with odd summands. Prove that $p_o(n) = p_d(n)$.

Solution. The generating function for $p_d(n)$ is

$$P_d(x) = (1+x)(1+x^2)(1+x^3)(1+x^4)\cdots$$

$$= \frac{1-x^2}{1-x}\frac{1-x^4}{1-x^2}\frac{1-x^6}{1-x^3}\frac{1-x^8}{1-x^4}\cdots$$

$$= \frac{1}{1-x}\frac{1}{1-x^3}\frac{1}{1-x^5}\frac{1}{1-x^7}\cdots$$

$$= (1+x+x^2+\cdots)(1+x^3+x^6+\cdots)(1+x^5+x^{10}+\cdots)\cdots$$

$$= p_o(x)$$

Another argument. Assume that that d_i's are distinct and $n = d_1 + d_2 + \cdots + d_k$. It can be written as $n = 2^{\alpha_1}O_1 + 2^{\alpha_2}O_2 + \cdots + 2^{\alpha_k}O_k$ where O_i's are odd integers. If we group odd numbers, we have

$$n = (2^{\alpha_1} + \cdots + 2^{\alpha_{r_1}}) \cdot O_{i_1} + \left(2^{\beta_1} + \cdots + 2^{\beta_{r_2}}\right) \cdot O_{i_2} + \cdots + \left(2^{\gamma_1} + \cdots + 2^{\gamma_{r_j}}\right) \cdot O_{i_j},$$

where $1 \le i_1 < i_2 < \ldots < i_j \le k$ and α_i's are distinct, β_i's are distinct, and so on. So, $n = x_{i_1}O_{i_1} + x_{i_2}O_{i_2} + \cdots + x_{i_j}O_{i_j}$ with positive integers x_{i_s}'s, $s = 1, 2, \ldots, j$.

Partitions of Integers: Examples (Ctd.)

Example 4. Prove that the number of partitions of a positive integer n with no part equal to 1 is $p(n) - p(n-1)$.

Solution. Let $f(x)$ be the generating function for the partitions of a positive integer n with no part equal to 1. We have

$$\begin{aligned} f(x) &= (1 + x^2 + x^4 + \cdots)(1 + x^3 + x^6 + \cdots)(1 + x^4 + x^8 + \cdots)\cdots \\ &= \frac{1}{1-x^2}\frac{1}{1-x^3}\frac{1}{1-x^4}\cdots = \frac{1-x}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^3}\frac{1}{1-x^4}\cdots \\ &= (1-x)\mathcal{E}(x) = \sum_{i=0}^{\infty} p(i)x^i - \sum_{i=0}^{\infty} p(i)x^{i+1} \\ &= \sum_{i=0}^{\infty} p(i)x^i - \sum_{i=1}^{\infty} p(i-1)x^i = p(0) + \sum_{i=1}^{\infty} (p(i) - p(i-1))x^i \end{aligned}$$

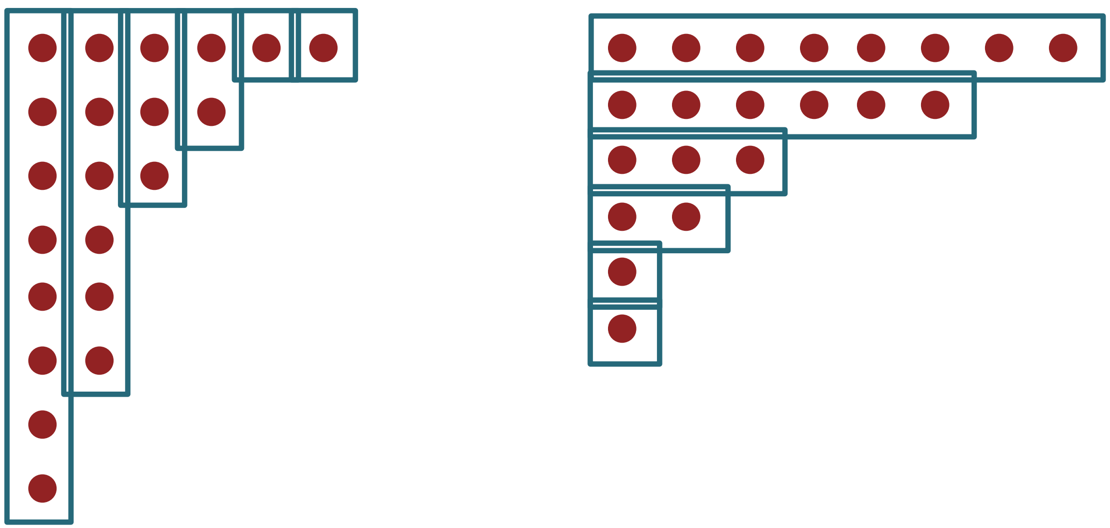
Thus, the coefficient of x^n in $f(x)$ equals $p(n) - p(n-1)$.

Another argument. Adding a part 1 to any partition of $n-1$ yields a partition of n with at least one part equal to 1. Conversely, omitting a part 1 from every partition of n with at least one part equal to 1 yields a partition of $n-1$. Therefore, there is a one-to-one correspondence between the two sets.

The Ferrers Diagram

A **Ferrers diagram** (**graph**) is a way for representing a partition of an integer using rows of dots where the number of dots in a row does not increases as we go to the row below it. The number of rows equals the number of parts of the partition, and the number of dots in a row represents the corresponding part.

The following are Ferrers diagrams for two of the partitions of 21. The figure at the left represents the partition $6+4+3+2+2+2+1+1$ and the one at the right represents $8+6+3+2+1+1$. The graph at the right is to be the ***transposition*** of the graph at the left and vice versa, because one can be obtained from the other by interchanging rows and columns.



Can we deduce that the number of partitions with the largest part k equals the number of partitions with k parts?

Partitions $p_k(n)$ and $p_{\leq k}(n)$

Let $p_k(n)$ be the number of partitions of n whose largest part is k. For example, $p_2(5) = 2$, because the partitions of 5 with the largest part 2 are $2 + 2 + 1$ and $2 + 1 + 1 + 1$. Moreover, let $p_{\leq k}(n)$ be the number of partitions of n where no part is larger than k. The generating function for $\{p_{\leq k}(n)\}_{n=0}^{\infty}$ is

$$\frac{1}{(1-x)(1-x^2)\cdots(1-x^k)}.$$

What is the generating function for the sequence $\{p_k(n)\}_{n=0}^{\infty}$? The coefficient of x^n in the generating function for the sequence $\{p_k(n)\}_{n=0}^{\infty}$ is $p_{\leq k}(n-k)$, that is the coefficient of x^{n-k}, in the generating function for $\{p_{\leq k}(n)\}_{n=0}^{\infty}$. Thus, the generating function of $\{p(n,k)\}_{n=0}^{\infty}$ is

$$\frac{x^k}{(1-x)(1-x^2)\cdots(1-x^k)}.$$

We have $p(n) = \sum_{k=1}^{\infty} p_k(n)$. Thus,

$$\mathcal{E}(x) = \sum_{n=0}^{\infty} p(n)x^n = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} p_k(n)\, x^n = \sum_{k=1}^{\infty}\sum_{n=0}^{\infty} p_k(n)\, x^n = \sum_{k=1}^{\infty} \frac{x^k}{(1-x)(1-x^2)\cdots(1-x^k)}.$$

Therefore,

$$\mathcal{E}(x) = \prod_{i=1}^{\infty} \frac{1}{(1-x^i)} = \sum_{k=1}^{\infty} \frac{x^k}{(1-x)(1-x^2)\cdots(1-x^k)}.$$

Some Recurrence Relations

Let $p_k(n)$ be the number of partitions of n whose largest part is k. We have

$$p_k(n) = p_{k-1}(n-1) + p_k(n-k).$$

The first term is the number of partitions of n with exactly one part equal to k and the second term is the number of those with more than one part equal to k. The equality $p_1(n) = p_n(n) = 1$ holds for all positive integers n. Moreover, $p_k(n) = 0$ for all $k > n$, and $p_k(0) = 0$. For example,

$$p_2(5) = p_1(4) + p_2(3) = 1 + p_1(2) + p_2(1) = 1 + 1 + 0 = 2.$$

Let $p_{\leq k}(n)$ be the number of partitions of n with no part greater than k. It holds that

$$p_{\leq k}(n) = p_{\leq k-1}(n) + p_{\leq k}(n-k).$$

It is immediate that

$$p(n) = p_{\leq n}(n).$$

We have $p_{\leq 1}(n) = p_{\leq k}(1) = p_{\leq k}(0) = 1$ and $p_{\leq k}(n) = p_{\leq n}(n)$ for $k > n$.

For example,

$$\begin{aligned} p_{\leq 3}(5) &= p_{\leq 2}(5) + p_{\leq 3}(2) \\ &= p_{\leq 1}(5) + p_{\leq 2}(3) + p_{\leq 2}(2) \\ &= 1 + p_{\leq 1}(3) + p_{\leq 2}(1) + p_{\leq 1}(2) + p_{\leq 2}(0) \\ &= 1 + 1 + 1 + 1 + 1 = 5 \end{aligned}$$

Balls and Containers

Now, let's get pack to the problem of putting (to distribute) m objects (balls) into n containers $(m \geq n.)$ We have already established the results summarized in the following table.

Objects Are Distinct	**Containers Are Distinct**	**Some Containers May Be Empty**	**Number of Distributions**
Yes	Yes	Yes	n^m
Yes	Yes	No	$n!\,S(m,n)$
Yes	No	Yes	$\sum_{i=1}^{n} S(m,i)$
Yes	No	No	$S(m,n)$
No	Yes	Yes	$\binom{n+m-1}{m}$
No	Yes	No	$\binom{m-1}{n-1}$

Balls and Containers (Ctd.)

Let $p(n, k)$ be the number of partitions of n with k parts. We know that

$$p(n, k) = p_k(n).$$

Consider the problem of balls and containers where the balls are indistinguishable and the containers are indistinguishable as well. The results are summarized in the following table.

Objects Are Distinct	Containers Are Distinct	Some Containers May Be Empty	Number of Distributions
No	No	Yes	$p(m)$ for $m = n$ $\sum_{i=1}^{n} p(m, i)$ for $m > n$
No	No	No	$p(m, n)$

Exponential Generating Functions

For each $n \in \mathbb{Z}^+$,

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i = \sum_{i=0}^{n} P(n,i)\frac{x^i}{i!}.$$

The coefficient of x^i is the combination of i of n, that is, $C(n,i)$. In other words, $(1+x)^n$ is the (ordinary) generating function for the sequence

$$C(n,0), C(n,1), \dots, C(n,n), 0, 0, 0, \dots .$$

The coefficient of $\frac{x^i}{i!}$ is the permutation of i of n, that is, $P(n,i)$.

Definition 2. For a sequence $\{a_i\}_{i=0}^{\infty}$ of real numbers, the function

$$a_0 + a_1 x + a_2\frac{x^2}{2!} + a_3\frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}$$

is called the ***exponential generating function*** for (of) the sequence.

As $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty}\frac{x^i}{i!}$, the function e^x is the exponential generating function for the sequence $1, 1, 1, 1, \dots$.

Applications

In how many ways can one order 4 of the letters of ENGINE?

We have solved this problem using elementary techniques.

E E N N	$\frac{4!}{2!\,2!}$	E G N N	$\frac{4!}{2!}$
E E G N	$\frac{4!}{2!}$	E I N N	$\frac{4!}{2!}$
E E I N	$\frac{4!}{2!}$	G I N N	$\frac{4!}{2!}$
E E G I	$\frac{4!}{2!}$	E I G N	$4!$

Now, c der followi xpression.

$$\overset{G}{\boxed{(1+x)}}\overset{I}{\boxed{(1+x)}}\overset{E}{\boxed{\left(1+x+\frac{x^2}{2!}\right)}}\overset{N}{\boxed{\left(1+x+\frac{x^2}{2!}\right)}} = (1+x)^2\left(1+x+\frac{x^2}{2!}\right).$$

The coefficient of $\frac{x^4}{4!}$ is the answer (the coefficient of x^4 multiplied by $4!$.)

Applications

Consider the Maclaurin series of e^x and e^{-x}.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

Thus,

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \qquad \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

Example 5. A ship carries 48 flags, 12 each of the colors red, white, blue, and black. Twelve of these flags are placed on a vertical pole in order to communicate a signal to other ships. How many of these signals use an even number of blue flags and an odd number of black flags?

Solution. The answer is the coefficient of $\frac{x^{12}}{12!}$ in the following exponential generating function

$$f(x) = \left(1 + x + \frac{x^2}{2!} + \cdots\right)^2 \left(1 + \frac{x^2}{2!} + \frac{x^4}{2!} + \cdots\right)\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

$$= (e^x)^2 \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^x - e^{-x}}{2}\right) = \left(\frac{1}{4}\right)(e^{4x} - 1) = \left(\frac{1}{4}\right)\sum_{i=1}^{\infty} \frac{(4x)^i}{i!}$$

The coefficient of $\frac{x^{12}}{12!}$ is $\frac{4^{12}}{4} = 4^{11}$.

Textbook: Ralph P. Grimaldi, Discrete and Combinatorial Mathematics

Do exercises of Chapter 9 as homework and upload your solutions via Moodle (follow the instructions on the page of the TA of this course.)