Discrete Mathematics

Session XIII

# Induction and Inductive Definitions

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# Introduction

In the previous session, we introduced the ***weak*** and ***strong*** forms of the ***mathematical induction***, also known as induction on ***positive integers***.

$$\big(\alpha(1) \wedge \forall k \in \mathbb{Z}^+ .\big(\alpha(k) \longrightarrow \alpha(k+1)\big)\big) \Rightarrow_{u} \forall n \in \mathbb{Z}^+ . \alpha(n).$$

$$\forall k \in \mathbb{Z}^+ .\Big(\big(\forall m \in \mathbb{Z}^+ .\big(m < k \longrightarrow \alpha(m)\big)\big) \longrightarrow \alpha(k)\Big) \Rightarrow \forall n \in \mathbb{Z}^+ . \alpha(n).$$

We also extended the proof principle of mathematical induction to properties of the elements of sets other than positive integers.

$$\big(\alpha(n_0) \wedge \forall k \in \mathbb{Z}^{\geq n_0} .\big(\alpha(k) \longrightarrow \alpha(k+1)\big)\big) \Rightarrow \forall n \in \mathbb{Z}^{\geq n_0} . \alpha(n).$$

$$\forall k \in \mathbb{Z}^{\geq n_0} .\Big(\big(\forall m \in \mathbb{Z}^{\geq n_0} .\big(m < k \longrightarrow \alpha(m)\big)\big) \longrightarrow \alpha(k)\Big) \Rightarrow \forall n \in \mathbb{Z}^{\geq n_0} . \alpha(n).$$

In general, to prove that the predicate $\alpha$ is true of all elements of a set $A$, that is, $\forall x \in A . \alpha(x)$. One may take the following steps:

1. Find a function $f : A \longrightarrow \mathbb{Z}^+$.
2. Define $\beta(n) \stackrel{\text{def}}{=} \forall x \in A . \big(f(x) = n \longrightarrow \alpha(x)\big)$.
3. Prove that $\forall n \in \mathbb{Z}^+ . \beta(n)$.

In this session, we introduce ***inductive definitions***.

It is also explained how inductive definitions lead to the general proof principle of ***rule induction***.

# Inductive Definitions

Numbers are abstract entities. No one has ever seen a number. The following are the numerals that denote positive integers.

**1, 2, 3, …**  **one, two, three, …**  **I, II, III, …**  **,** **,** **, …**

But, how can one define positive integers?

Consider the following rules.

1. **one** is a positive integer. (One)
2. If $a$ is a positive integer, then $\mathbf{succ}(a)$ is a positive integer. (Successor)

Thus, **one**, **succ**(**one**), **succ**(**succ**(**one**)), **succ**(**succ**(**succ**(**one**))), and so on are positive integers.

The set of positive integers is defined to be the ***smallest set closed under the rules*** One and Successor. In fact, everything that is obtained from a finite number of applications of the rules One and Successor is a positive integer. Moreover, every positive integer is obtained from a finite number of applications of the rules One and Successor.

The rules One and Successor can also be written as follows:

Axiom

$$\frac{}{\textbf{one} \quad \text{posint}} \; (\text{One})$$

Proper Rule

$$\frac{a \quad \text{posint}}{\mathbf{succ}(a) \quad \text{posint}} \; (\text{Successor})$$

# Inductive Definitions (Ctd.)

A set is said to be ***defined inductively*** if it is defined as the smallest set closed under a collection of rules.

In fact, inductive definitions define a so-called ***judgment form*** representing a set. In this way, every instance of the judgment form will have a proof, that is, a finite number of applications of the rules. Here is a proof of "**succ**(**succ**(**succ**(**succ**(**one**)))) posint"

$$\frac{}{\textbf{one}\quad\text{posint}}\ (\text{One}) \qquad \frac{a\quad\text{posint}}{\textbf{succ}(a)\quad\text{posint}}\ (\text{Successor})$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{\textbf{one}\quad\text{posint}}}{\textbf{succ}(\textbf{one})\quad\text{posint}}}{\textbf{succ}(\textbf{succ}(\textbf{one}))\quad\text{posint}}}{\textbf{succ}(\textbf{succ}(\textbf{succ}(\textbf{one})))\quad\text{posint}}}{\textbf{succ}(\textbf{succ}(\textbf{succ}(\textbf{succ}(\textbf{one}))))\quad\text{posint}}$$

Forward (Bottom-Up) Derivation

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{\textbf{one}\quad\text{posint}}}{\textbf{succ}(\textbf{one})\quad\text{posint}}}{\textbf{succ}(\textbf{succ}(\textbf{one}))\quad\text{posint}}}{\textbf{succ}(\textbf{succ}(\textbf{succ}(\textbf{one})))\quad\text{posint}}}{\textbf{succ}(\textbf{succ}(\textbf{succ}(\textbf{succ}(\textbf{one}))))\quad\text{posint}}$$

Backward (Top-Down) Derivation

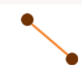
# Inductive Definitions (Ctd.)

The set of binary trees can be defined inductively by the following rules.

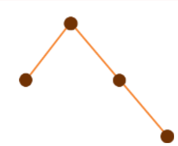
$$\frac{}{\textbf{empty}\ \ \text{btree}}\ (\text{E-Tree}) \qquad \frac{a\ \ \text{btree} \qquad b\ \ \text{btree}}{\textbf{node}(a, b)\ \ \text{btree}}\ (\text{N-Tree})$$

The following are sample derivations.

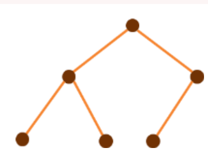
$$\dfrac{\dfrac{}{\textbf{empty}\ \ \text{btree}} \qquad \dfrac{\dfrac{}{\textbf{empty}\ \ \text{btree}} \qquad \dfrac{}{\textbf{empty}\ \ \text{btree}}}{\textbf{node}(\textbf{empty}, \textbf{empty})\ \ \text{btree}}}{\textbf{node}(\textbf{empty}, \textbf{node}(\textbf{empty}, \textbf{empty}))\ \ \text{btree}}$$



$$\dfrac{\dfrac{\dfrac{}{\textbf{empty}\ \ \text{btree}} \qquad \dfrac{}{\textbf{empty}\ \ \text{btree}}}{\textbf{node}(\textbf{empty}, \textbf{empty})\ \ \text{btree}} \qquad \dfrac{\dfrac{}{\textbf{empty}\ \ \text{btree}} \qquad \dfrac{\dfrac{}{\textbf{empty}\ \ \text{btree}} \qquad \dfrac{}{\textbf{empty}\ \ \text{btree}}}{\textbf{node}(\textbf{empty}, \textbf{empty})\ \ \text{btree}}}{\textbf{node}(\textbf{empty}, \textbf{node}(\textbf{empty}, \textbf{empty}))\ \ \text{btree}}}{\textbf{node}(\textbf{node}(\textbf{empty}, \textbf{empty}), \textbf{node}(\textbf{empty}, \textbf{node}(\textbf{empty}, \textbf{empty})))\ \ \text{btree}}$$



$$\textbf{node}(\textbf{node}(\textbf{node}(\textbf{empty}, \textbf{empty}), \textbf{node}(\textbf{empty}, \textbf{empty})), \textbf{node}(\textbf{node}(\textbf{empty}, \textbf{empty}), \textbf{empty}))$$



# The Proof Principle of Induction

A set is ***inductively defined*** if it is represented by the ***strongest judgment form***, ***assertion***, closed under a collection of ***rules***. In fact, the set is defined as the smallest set closed under the rules defining that set.

This gives rise to the ***proof principle of rule Induction***.

The principle states that to show that a predicate $\beta$ holds of all derivable judgments (from the given collection of rules) $a\ J$, it is enough to show that $\beta$ is ***closed under*** the rules defining the judgment form $J$.

The predicate $\beta$ is said to be ***closed*** under the rule

$$\frac{a_1\ J \quad a_2\ J \quad \cdots \quad a_k\ J}{a\ J}$$

if $\beta(a)$ is true whenever $\beta(a_1)$, $\beta(a_2)$, …, and $\beta(a_k)$ are true. The assumptions $\beta(a_1)$, $\beta(a_2)$, …, and $\beta(a_k)$ are called the ***induction hypotheses***, and $\beta(a)$ is called the ***induction conclusion.***

For example, a predicate $\alpha$ holds of all derivable judgments "$a$ posint", iff

1. $\alpha(\mathbf{one})$ holds, and
2. for every $a$, if $\alpha(a)$ holds, then $\alpha(\mathbf{succ}(a))$ holds.

$$\frac{}{\textbf{one} \quad \text{posint}}\ (\text{One})$$

$$\frac{a \quad \text{posint}}{\mathbf{succ}(a) \quad \text{posint}}\ (\text{Successor})$$

# The Proof Principle of Induction (Ctd.)

We define the judgment forms "$a$ nat", "$(a,b,c)$ sumnat", "$(a,b)$ leqnat", "$(t,n)$ numnodes", "$(t,n)$ numleaves", and "$(t,n)$ numinodes" as follows (***iterative*** definitions):

$$\frac{}{\mathbf{zero}\ \text{nat}} \qquad \frac{a\ \text{nat}}{\mathbf{succ}(a)\ \text{nat}}$$

$$\frac{a\ \text{nat}}{(\mathbf{zero}, a, a)\ \text{sumnat}} \qquad \frac{(a,b,c)\ \text{sumnat}}{\big(\mathbf{succ}(a), b, \mathbf{succ}(c)\big)\ \text{sumnat}}$$

$$\frac{a\ \text{nat}}{(\mathbf{zero}, a)\ \text{leqnat}} \qquad \frac{(a,b)\ \text{leqnat}}{\big(\mathbf{succ}(a), \mathbf{succ}(b)\big)\ \text{leqnat}}$$

$$\frac{}{(\mathbf{empty}, \mathbf{zero})\ \text{numnodes}}$$

$$\frac{(t_1,n_1)\ \text{numnodes} \qquad (t_2,n_2)\ \text{numnodes} \qquad (n_1,n_2,n)\ \text{sumnat}}{\big(\mathbf{node}(t_1,t_2), \mathbf{succ}(n)\big)\ \text{numnodes}}$$

$$\frac{}{(\mathbf{empty}, \mathbf{zero})\ \text{numleaves}}$$

$$\frac{}{\big(\mathbf{node}(\mathbf{empty}, \mathbf{empty}), \mathbf{succ}(\mathbf{zero})\big)\ \text{numleaves}}$$

$$\frac{(t_1,n_1)\ \text{numleaves} \qquad (t_2,n_2)\ \text{numleaves} \qquad (n_1,n_2,n)\ \text{sumnat} \qquad (\mathbf{succ}(\mathbf{zero}), n)\ \text{leqnat}}{(\mathbf{node}(t_1,t_2), n)\ \text{numleaves}}$$

$$\frac{t\ \text{btree} \qquad (t,n)\ \text{numnodes} \qquad (t,n_1)\ \text{numleaves} \qquad (n_1,n_2,n)\ \text{sumnat}}{(t,n_2)\ \text{numinodes}}$$

$$\frac{}{\mathbf{zero}\ \mathsf{nat}} \qquad \frac{a\ \mathsf{nat}}{\mathbf{succ}(a)\ \mathsf{nat}}$$

$$\frac{a\ \mathsf{nat}}{(\mathbf{zero}, a)\ \mathsf{leqnat}} \qquad \frac{(a,b)\ \mathsf{leqnat}}{(\mathbf{succ}(a), \mathbf{succ}(b))\ \mathsf{leqnat}}$$

# The Proof Principle of Induction (Ctd.)

**Example 1.** Prove that for all judgments "$a$ nat", "$b$ nat", and "$c$ nat", if "$(a,b)$ leqnat" and "$(b,c)$ leqnat", then "$(a,c)$ leqnat".

**Solution.** We use induction on "$a$ nat". That is we show that the predicate $\beta$ defined below is true for all derivable judgments "$a$ nat" (universal quantification on all other judgments is implicit, for the sake of simplicity.)

$$\beta(a) \stackrel{\text{def}}{=} (a\ \text{nat} \wedge b\ \text{nat} \wedge c\ \text{nat} \wedge (a,b)\ \text{leqnat} \wedge (b,c)\ \text{leqnat}) \longrightarrow (a,c)\ \text{leqnat}.$$

Thus, we must show that $\beta$ is closed under the rules defining the judgment form "$a$ nat". For the axiom $\frac{}{\mathbf{zero}\ nat}$, we must show that $\beta(\mathbf{zero})$ holds. That is, the following is true.

$$(\mathbf{zero}\ \text{nat} \wedge b\ \text{nat} \wedge c\ \text{nat}) \longrightarrow \Big(\big((\mathbf{zero}, b)\ \text{leqnat} \wedge (b,c)\ \text{leqnat}\big) \longrightarrow (\mathbf{zero}, c)\ \text{leqnat}\Big).$$

This is immediate because "$(\mathbf{zero}, c)$ leqnat" is true (derivable) whenever "$c$ nat" is true (derivable). To show that $\beta$ is closed under the rule $\frac{a\ nat}{\mathbf{succ}(a)\ nat}$, we must show that $\beta(\mathbf{succ}(a))$ is true if $\beta(a)$ is true.

$$\beta(\mathbf{succ}(a)) \stackrel{\text{def}}{=} \left( \begin{array}{c} \boxed{\mathbf{succ}(a)\ \text{nat}} \wedge \boxed{b\ \text{nat}} \wedge \boxed{c\ \text{nat}} \\ \wedge \\ \boxed{(\mathbf{succ}(a), b)\ \text{leqnat}} \wedge \boxed{(b,c)\ \text{leqnat}} \end{array} \right) \longrightarrow \boxed{(\mathbf{succ}(a), c)\ \text{leqnat}}$$

$a$ nat

$b = \mathbf{succ}(b')$

$c = \mathbf{succ}(c')$

$b'$ nat

$c'$ nat

$\beta(a)$

$$(a\ \text{nat} \wedge b'\ \text{nat} \wedge c'\ \text{nat} \wedge (a,b')\ \text{leqnat} \wedge (b',c')\ \text{leqnat}) \longrightarrow (a,c')\ \text{leqnat}.$$

$$\frac{}{\textbf{empty}\ \ \mathsf{btree}}\ (\text{E-Tree}) \qquad \frac{a\ \ \mathsf{btree} \qquad b\ \ \mathsf{btree}}{\textbf{node}(a,b)\ \ \mathsf{btree}}\ (\text{N-Tree})$$

$$\frac{}{(\textbf{empty},\textbf{zero})\ \ \mathsf{numnodes}}$$

$$\frac{(t_1,n_1)\ \ \mathsf{numnodes} \qquad (t_2,n_2)\ \ \mathsf{numnodes} \qquad (n_1,n_2,n)\ \ \mathsf{sumnat}}{(\textbf{node}(t_1,t_2),\textbf{succ}(n))\ \ \mathsf{numnodes}}$$

$$\frac{}{(\textbf{empty},\textbf{zero})\ \ \mathsf{numleaves}}$$

$$\frac{}{(\textbf{node}(\textbf{empty},\textbf{empty}),\textbf{succ}(\textbf{zero}))\ \ \mathsf{numleaves}}$$

$$\frac{(t_1,n_1)\ \ \mathsf{numleaves} \qquad (t_2,n_2)\ \ \mathsf{numleaves} \qquad (n_1,n_2,n)\ \ \mathsf{sumnat} \qquad (\textbf{succ}(\textbf{zero}),n)\ \ \mathsf{leqnat}}{(\textbf{node}(t_1,t_2),n)\ \ \mathsf{numleaves}}$$

$$\frac{t\ \ \mathsf{btree} \qquad (t,n)\ \ \mathsf{numnodes} \qquad (t,n_1)\ \ \mathsf{numleaves} \qquad (n_1,n_2,n)\ \ \mathsf{sumnat}}{(t,n_2)\ \ \mathsf{numinodes}}$$

$$\beta(t) \stackrel{\text{def}}{=} t\ \ \mathrm{btree} \longrightarrow \Big(\big((t,m)\ \ \mathrm{numleaves} \wedge (t,n)\ \ \mathrm{numinodes}\big) \longrightarrow (m,\textbf{succ}(n))\ \ \mathrm{leqnat}\Big)$$

We must show that $\beta$ is closed under the rules E-Tree and N-Tree. For E-Tree, we must directly prove that $\beta(\textbf{empty})$ is true. That is, the following formula is true.

$$\textbf{empty}\ \ \mathrm{btree} \longrightarrow \Big(\big((\textbf{empty},m)\ \ \mathrm{numleaves} \wedge (\textbf{empty},n)\ \ \mathrm{numinode}\ \ldots\ \mathrm{nat}\Big).$$

zero

zero

zero

$n_1 = r_1 - m_1$
$n_2 = r_2 - m_2$
$n = r - m$
$r = r_1 + r_2 + 1$

For N-Tree, we must prove that $\beta(a) \wedge \beta(b) \longrightarrow \beta\big(\textbf{node}(a,b)\big)$ is tru

$$\textbf{node}(a,b)\ \ \mathrm{btree} \longrightarrow \Big(\big((\textbf{node}(a,b),m)\ \ \mathrm{numleaves} \wedge (\textbf{node}(a,b),n)\ \ \mathrm{numinodes}\big) \longrightarrow \boxed{(m,\textbf{succ}(n))\ \ \mathrm{leqnat}}\Big)$$

$a$ btree $\wedge$ $b$ btree

$$\beta(a):\quad \Big(\boxed{\big((a,m_1)\ \ \mathrm{numleaves}} \wedge \boxed{(a,n_1)\ \ \mathrm{numinodes}\big)} \longrightarrow \boxed{(m_1,\textbf{succ}(n_1))\ \ \mathrm{leqnat}}\Big)$$

$$\beta(b):\quad \Big(\boxed{\big((b,m_2)\ \ \mathrm{numleaves}} \wedge \boxed{(b,n_2)\ \ \mathrm{numinodes}\big)} \longrightarrow \boxed{(m_2,\textbf{succ}(n_2))\ \ \mathrm{leqnat}}\Big)$$

$m = m_1 + m_2$

or

$m_1$ and $m_2$ are **zero** and $m$ is $\textbf{succ}(\textbf{zero})$

$m_1 \leq n_1 + 1$
$m_2 \leq n_2 + 1$

**Textbook: Ralph P. Grimaldi, Discrete and Combinatorial Mathematics**

**Do exercises of Chapter 4 as homework and upload your solutions via Moodle (follow the instructions on the page of the TA of this course.)**