معادلات قابل تبدیل به معادلات همگن

$$y' = \frac{ax + by + c}{ax' + by' + c'}
\begin{cases}
\begin{vmatrix} a & b \\ a' & b' \end{vmatrix} = 0 & \begin{cases} u = ax + by \\ u = ax' + by' \end{cases} \\[2em]
\begin{vmatrix} a & b \\ a' & b' \end{vmatrix} \neq 0 & \begin{cases} x = X + \alpha \,, y = Y + \beta \\ \text{حل دستگاه و یافتن } \beta \text{ و } \alpha \end{cases}
\end{cases}$$

حبیب پور

مثال -

$$(3+2x+4y)y' = 1+x+2y$$

$$y' = \frac{x+2y+1}{2x+4y+3}$$

$$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$$

$$u = x+2y$$

$$u' = 1+2y'$$

$$y' = \frac{u'-1}{2}$$

$$u' = \frac{4u+5}{2u+3}$$

$$\frac{du}{dx} = \frac{4u+5}{2u+3}$$

$$\frac{2u+3}{4u+5}du = dx$$

حبیب پور

$$\int \frac{2u+3}{4u+5}\,du = \int dx$$

$$\frac{1}{2}\int \frac{4u+6}{4u+5}\,du = \int dx$$

$$\frac{1}{2}\int \frac{(4u+5)+1}{4u+5}\,du = \int dx$$

$$\frac{1}{2}\left[\int du + \int \frac{1}{4u+5}\right] = x + c$$

$$\frac{1}{2}\left[u + \frac{1}{4}\ln(4u+5)\right] = x + c$$

$$\frac{x+2y}{2} + \frac{1}{8}\ln(4x+8y+5) = x + c$$

حبیب پور

مثال)

$$y' = \frac{2x+y-1}{x-y-2}$$

$$\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -2-1 = -3 \neq 0$$

$$\begin{cases} 2x+y=1 \\ x-y \quad =2 \end{cases}$$

$$x=1\,,y=-1$$

$$\alpha=1\,,\beta=-1$$

$$x=X+1\,,y=Y-1$$

$$Y' = \frac{2X+Y}{X-Y}$$

$$Y' = \frac{2+\frac{Y}{X}}{1-\frac{Y}{X}}$$

$$u=\frac{Y}{X}$$

$$Y' = u+u'X$$

حبیب پور

$$u + u'X = \frac{2+u}{1-u}$$

$$Xu' = -\frac{u^2+2}{u-1}$$

$$\frac{u-1}{u^2+2}\,du = -\frac{dX}{X}$$

$$\int \frac{u-1}{u^2+2}\,du = -\int \frac{dX}{X}$$

$$\int \frac{u}{u^2+2}\,du - \int \frac{1}{u^2+2} = -\int \frac{dX}{X}$$

$$\frac{1}{2}\ln(u^2+2) - \frac{\sqrt{2}}{2}\, arctang\, \frac{\sqrt{2}}{2}u = -\ln X + \ln c$$

$$u = \frac{Y}{X}$$

$$Y = y+1\,, X = x-1$$

حبیب پور

معادلات خطی مرتبه اول

$$y' + p(x)y = q(x)$$

$$\mu(x) = e^{\int p(x)dx} \qquad y = \frac{1}{\mu(x)}\left[\int \mu(x)q(x)dx + c\right]$$

$$y = e^{-\int p(x)dx}\left[\int e^{\int p(x)dx} q(x)dx\right]$$

حبیب پور

مثال)

$$y' = (sin^2 x - y)\cos x$$

$$y' = sin^2 x \cos x - y\cos x$$

$$y' + \underbrace{\cos(x)}_{p(x)}\, y = \underbrace{sin^2 x \cos x}_{q(x)}$$

$$\mu = e^{\int \cos x \; dx} = e^{\sin x}$$

$$y = e^{-\sin x}\left[\int sin^2 x \cos x \, e^{\sin x} dx\right]$$

حبیب پور

مثال)

$$y = e^{-\sin x}\left[\int sin^2 x \cos x \, e^{\sin x} dx\right]$$

$$\sin x = u$$

$$\cos x \; dx = du$$

$$y = e^{-u}\left[\int u^2 e^u \mathrm{du} + \mathrm{c}\right] = e^{-u}[u^2 e^u - 2ue^u + 2e^u + c]$$

$$y = u^2 - 2u + 2 + ce^{-u} = sin^2 x - 2\sin x + 2 + ce^{-\sin x}$$

مشتق		انتگرال
u^2	$+$	e^u
$2u$	$-$	e^u
2	$+$	e^u
0		e^u

حبیب پور

مثال -

$$y' = \frac{y + \ln x}{x} \qquad y' \underbrace{- \frac{y}{x}}_{p(x)} = \underbrace{\frac{\ln x}{x}}_{q(x)}$$

$$\mu = e^{\int \frac{-1}{x}\,dx} = e^{-\ln x} = \frac{1}{x} \qquad y = x\left[\int \overbrace{\frac{1}{x} \times \frac{\ln x}{x}}^{\frac{\ln x}{x^2}}\,dx + c\right]$$

$$\ln x = u \qquad \frac{1}{x}dx = du \qquad x = e^u$$

حبیب پور

مثال -

$$y = e^{u}\left[\int e^{-u}\,u\;du + c\right] = e^{u}[-ue^{-u} - e^{-u} + c] = -u - 1 + ce^{u}$$

$$y = -\ln x - 1 + cx$$

حبیب پور

معادلات کامل

$$M(x,y)dx + N(x,y)dy = 0$$

$M_y = N_x \longleftarrow$ شرط کامل بودن

$$\int M(x,y)dx \qquad \int N(x,y)dy$$

$c =$ عبارت های مشترک +عبارت های غیر مشترک

حبیب پور

مثال/

$$(3x^2 + y\cos x)dx - (4y^3 - \sin x)dy = 0$$

$$(3x^2 + y\cos x)dx + (\sin x - 4y^3)dy = 0 \qquad M_y = \cos x = N_x$$

$$\int M(x,y)dx = \int (3x^2 + y\cos x)dx = x^3 + ysin\, x$$

$$\int N(x,y)dy = \int (\sin x - 4y^3)dy = ysin\, x - y^4$$

$$x^3 - y^4 + ysin\, x = \mathrm{c}$$

حبیب پور

مثال)

$$(e^x \sin y - 2y\sin x)dx + (e^x \cos y + 2\cos x)dy = 0$$

$$M_y = e^x \cos y - 2\sin x = N_x$$

$$\int M(x,y)dx = \int (e^x \sin y - 2y\sin x)dx = e^x \sin y + 2y\cos x$$

$$\int N(x,y)dy = \int (e^x \cos y + 2\cos x)dy = e^x \sin y + 2y\cos x$$

$$e^x \sin y + 2y\cos x = c$$

حبیب پور

فاکتور انتگرال

$$M_y \neq N_x$$

$$* = \frac{M_y - N_x}{N} \longrightarrow$$ برحسب x $$\longrightarrow \mu = e^{\int * \, dx}$$

$$* = \frac{M_y - N_x}{-M} \longrightarrow$$ برحسب y $$\longrightarrow \mu = e^{\int * \, dy}$$

$$* = \frac{M_y - N_x}{yN - xM} \longrightarrow$$ برحسب xy $$\longrightarrow \mu = e^{\int * \, d(xy)}$$

حبیب پور

فاکتور انتگرال

فاکتور انتگرال بر حسب $Z(x, y)$ باش

بر حسب $Z(x, y)$ $\longrightarrow$ $\mu = e^{\int * \, dz}$ $\longrightarrow$ $* = \dfrac{M_y - N_x}{NZ_x - MZ_y}$

با ضرب $x^{\alpha} y^{\beta}$ در معادله و با اعمال شرط کامل بودن مقدار α, β را میابیم

حبیب پور

مثال)

$$(x^2 + 2x + y)dx + (3x^2 - x)dy = 0$$

$$M_y = 1 \neq N_x = 6xy - 1 \qquad M_y - N_x = 2 - 6xy$$

$$\frac{M_y - N_x}{N} = \frac{2 - 6xy}{3x^2y - x} = \frac{2(1 - 3xy)}{-x(1 - 3xy)} = \frac{-2}{x}$$

$$\mu = e^{\int \frac{-2}{x}} = e^{-lnx} = \frac{1}{x^2}$$

حبیب پور

مثال)

$$\left(1+\frac{2}{x}+\frac{y}{x^2}\right)dx+\left(3y-\frac{1}{x}\right)dy=0$$

$$\int\left(1+\frac{2}{x}+\frac{y}{x^2}\right)dx=x+2\ln x\ -\frac{y}{x}$$

$$\int\left(3y-\frac{1}{x}\right)dy=\frac{3y^2}{2}\ -\frac{y}{x}$$

$$x+2\ln x+\frac{3y^2}{2}-\frac{y}{x}=c$$

حبیب پور

مثال)

$$(y+2xy^2)dx-(2x+3x^2y)dy=0$$

$$M_y=1+4xy\neq N_x=-2-6xy \qquad M_y-N_x=3+10xy$$

$$\frac{M_y-N_x}{N}=\frac{3+10xy}{-2x-3x^2y} \qquad \frac{M_y-N_x}{-M}=\frac{3+10xy}{y+2xy^2}$$

$$\frac{M_y-N_x}{yN-xM}=\frac{3+10xy}{-3xy-5x^2y^2}=\frac{3+10xy}{-xy(3+5xy)}$$

حبیب پور

مثال)

$$\mu = e^{\int \frac{3+10xy}{-xy(3+5xy)}\,d(xy)} = e^{\int \frac{(3+5xy)+5xy}{-xy(3+5xy)} d(xy)}$$

$$= e^{\int \left(\frac{-1}{xy} - \frac{5}{3+5xy}\right)\,d(xy)}$$

$$\mu = e^{\int \frac{-d(xy)}{xy} - \int \frac{5}{3+5xy} d(xy)} = e^{-\ln xy - \ln(3+5xy)} = \frac{1}{xy(3+5xy)}$$

حبیب پور

مثال)

$$\left(\frac{y+2xy^2}{xy(3+5xy)}\right)dx-\left(\frac{2x+3x^2y}{xy(3+5xy)}\right)dy=0$$

$$\left(\frac{1+2xy}{x(3+5xy)}\right)dx+\left(\frac{-2-3xy}{y(3+5xy)}\right)dy=0$$

$$\int\left(\frac{1+2xy}{x(3+5xy)}\right)dx=\frac{1}{3}\left[\int\frac{3+5xy+xy}{x(3+5xy)}dx\right]=\frac{1}{3}\left[\ln x+\frac{1}{5}\ln(3+5xy\right]$$

$$=\frac{1}{3}lnx+\frac{1}{15}\ln(3+5xy)$$

$$\int\left(\frac{-2-3xy}{y(3+5xy)}\right)dy=\frac{2}{3}\ln y+\frac{1}{15}\ln(3+5xy)$$

$$\frac{1}{3}lnx+\frac{2}{3}\ln y+\frac{1}{15}\ln(3+5xy)=c$$

حبیب پور

مثال)

$$(y+2xy^2)dx-(2x+3x^2y)dy=0 \qquad \times x^{\alpha}y^{\beta}$$

$$\left(x^{\alpha}y^{\beta+1}+2x^{\alpha+1}y^{\beta+2}\right)dx+\left(-2x^{\alpha+1}y^{\beta}-3x^{\alpha+2}y^{\beta+1}\right)dy=0$$

$$M_y=(\beta+1)x^{\alpha}y^{\beta}+2(\beta+2)x^{\alpha+1}y^{\beta+1}$$

$$N_x=-2(\alpha+1)x^{\alpha}y^{\beta}-3(\alpha+2)x^{\alpha+1}y^{\beta+1}$$

$$\alpha=4\,,\beta=-11$$

حبیب پور

مثال)

$$(x^4y^{-10}+2x^5y^{-9})dx+(-2x^5y^{-11}-3x^6y^{-10})dy=0$$

$$\int(x^4y^{-10}+2x^5y^{-9})dx=\frac{x^5y^{-10}}{5}+\frac{x^6y^{-9}}{3}$$

$$\int(-2x^5y^{-11}-3x^6y^{-10})dy=\frac{x^5y^{-10}}{5}+\frac{x^6y^{-9}}{3}$$

$$\frac{x^5y^{-10}}{5}+\frac{x^6y^{-9}}{3}=\mathrm{c}$$

حبیب پور

مثال) نشان دهید معادله ی زیر دارای فاکتور انتگرال ساز بر حسب x^2+y^2 است.
سپس آن را حل کنید.

$$(x^2+y^2-x)dx-ydy=0$$

$$\frac{M_y-N_x}{NZ_x-MZ_y}=\frac{2y}{-2xy-(x^2+y^2-x)2y}=\frac{-1}{x^2+y^2}$$

$$\mu=e^{\int\frac{-1}{x^2+y^2}\,\mathrm{dz}}=e^{\int\frac{-1}{z}\,\mathrm{dz}}=e^{-\ln z}=\frac{1}{z}=\frac{1}{x^2+y^2}$$

حبیب پور

مثال)

$$\left(1-\frac{x}{x^2+y^2}\right)dx+\left(\frac{-y}{x^2+y^2}\right)dy=0$$

$$\int\left(1-\frac{x}{x^2+y^2}\right)dx=x-\frac{1}{2}\ln(x^2+y^2)$$

$$\int\left(\frac{-y}{x^2+y^2}\right)dy=-\frac{1}{2}\ln(x^2+y^2)$$

$$x-\frac{1}{2}\ln(x^2+y^2)=\mathrm{c}$$

حبیب پور

معادله برنولی

$$y' + p(x)y = q(x)y^n \xrightarrow{\quad u = y^{1-n} \quad}$$ معادله خطی مرتبه اول

$$y' + p(x)y = q(x) + R(x)y^2 \xrightarrow{\quad y_1 \quad} y = y_1 + \frac{1}{u}$$

$$\downarrow$$

معادله خطی مرتبه اول

حبیب پور

مثال)

$$y^2y' + 2xy^3 = x$$

$$y' + 2xy = xy^{-2} \quad \begin{cases} u = y^3 \\ u' = 3y'y^2 \end{cases}$$

$$\frac{u'}{3} + 2xu = x \qquad u' + 6\mathrm{xu} = 3\mathrm{x} \qquad \mu = e^{3x^2}$$

$$u = e^{-3x^2}\left[\int e^{3x^2}3x\,dx + c\right] = e^{-3x^2}\left[\frac{1}{2}e^{3x^2} + c\right] = \frac{1}{2} + ce^{-3x^2}$$

$$v = 3x^2, v' = 6xdx$$

$$y = \sqrt[3]{\frac{1}{2} + ce^{-3x^2}}$$

حبیب پور

مثال)

$$xdy - [y + xy^3(1 + \ln x)]dx = 0$$

$$y' - \frac{y}{x} = y^3(1 + \ln x) \qquad u = y^{-2} \qquad u' = -2y^{-3}y'$$

$$u' + \frac{2u}{x} = 2(1 + \ln x) \qquad \mu = e^{\int \frac{2}{x}dx} = x^2$$

$$u = x^{-2}\left[\int 2x^2 + 2x^2(\ln x)dx + c\right] = \frac{2}{3}x\ln x + \frac{4}{9}x + cx^{-2}$$

$$y = \sqrt{\frac{1}{\frac{2}{3}x\ln x + \frac{4}{9}x + cx^{-2}}}$$

حبیب پور

مثال)

$$y' + 2x^{-2} = 2x^{-1}y - y^2 \qquad y_1 = \frac{2}{x}$$

$$y = \frac{2}{x} + \frac{1}{u} \qquad y' = \frac{-2}{x^2} - \frac{u'}{u^2} \qquad u' - \frac{2}{x}u = 1$$

$$\mu = e^{\int \frac{-2}{x}dx} = \frac{1}{x^2} \qquad u = x^2\left[\int \frac{1}{x^2}dx + c\right] = x^2[\frac{-1}{x} + c] = -x + cx^2$$

$$y - \frac{2}{x} = \frac{1}{-x + cx^2} \qquad y = \frac{2}{x} + \frac{1}{-x + cx^2}$$

حبیب پور

$y = xy' + f(y')$ معادله کلرو

$x = f(y, y')$

$y = f(x, y')$

$y'' = f(x, y')$

$y' = p \qquad y'' = p'$

$y'' = f(y, y') \qquad y' = p\,, y'' = pp' \qquad p' = \dfrac{dp}{dy}$

معادله فاقد متغیر مستقل

حبیب پور

مثال)

$$y = xy' + (y')^2 \qquad y' = p$$

$$y = xp + (p)^2 \qquad p = p + xp' + 2pp'$$

$$p'(x+2p) = 0 \quad \begin{cases} p' = 0 \qquad p = c \qquad y = cx \\ y' = \dfrac{-x}{2} \qquad y = \dfrac{-x^2}{4} \end{cases}$$

حبیب پور

مثال)

$$(y^2+1)y''-2y(y')^2=0$$

$$(y^2+1)pp'-2yp^2=0 \qquad p\left[(y^2+1)\frac{dp}{dy}-2yp\right]=0$$

$$p=0 \qquad y=c_1 \qquad (y^2+1)\frac{dp}{dy}-2yp=0$$

$$\frac{dp}{p}=\frac{2y}{y^2+1} \qquad \ln p=\ln(y^2+1)+\ln c_2 \qquad y'=c_2\,(y^2+1)$$

$$\frac{dy}{y^2+1}=c_2\,dx \qquad tg^{-1}y=c_2x+c_3 \qquad y=tg\,(c_2x+c_3)$$

حبیب پور

مسیرهای متعامد و پوش منحنی

- ابتدا C را حذف می‌کنیم
- y' را به $\frac{-1}{y'}$
- معادله حاصل را حل می‌کنیم

مثال) مسیرهای قائم خانواده خم هایی که در معادله زیر صدق می کنند را بدست آورید

$$xy'y + y'y^2 + x^2 = 0$$

$$\frac{-xy}{y'} - \frac{y^2}{y'} + x^2 = 0 \qquad \frac{-1}{y'}(xy + y^2) = -x^2 \qquad y' = \frac{xy + y^2}{x^2}$$

$$y' = \frac{y}{x} + \left(\frac{y}{x}\right)^2 \begin{cases} u = \dfrac{y}{x} \\ y' = u + u'x \end{cases} \qquad x\frac{du}{dx} = u^2 \qquad \frac{dx}{x} = \frac{du}{u^2}$$

$$\frac{-1}{u} = \ln x + \ln c \qquad -\frac{y}{x} = \frac{1}{\ln cx} \qquad y = \frac{-x}{\ln cx}$$

مثال) مسیر های قائم خانواده خم هایی که در معادله زیر صدق می کنند را بدست آورید

$$x^2 + y^2 = 2cx$$

$$2x + 2yy' = 2c \qquad \frac{2x + 2yy'}{2} = c \qquad x^2 + y^2 = 2x^2 + 2xyy'$$

$$x^2 + y^2 = 2x^2 + 2xy\frac{-1}{y'} \qquad y' = \frac{2xy}{x^2 - y^2} = \frac{2\left(\frac{y}{x}\right)}{1 - \left(\frac{y}{x}\right)^2}$$

$$xu' = \frac{u^3 + u}{1 - u^2} \qquad \frac{1 - u^2}{u(u^2 + 1)}du = \frac{dx}{x}$$

مثال)

$$\ln u - \ln(u^2+1) = \ln cx$$

$$\ln\left(\frac{u}{u^2+1}\right) = \ln cx$$

$$\frac{u}{u^2+1} = cx$$

$$u = \frac{y}{x}$$

پوش منحنی

ابتدا c را حذف می کنیم (با مساوی wtv
قرار دان مشتق نسبت به c

در معادله اولیه قرار می دهیم و y را تنها می
کنیم

مثال) پوش خانواده دوایر زیر را بیابید .

$(x-c)^2+y^2=1$ $\quad -2(x-c)=0$ $\quad x=c$ $\quad y^2=1$

مثال) پوش خانواده منحنی زیر را بیابید .

$y=ce^x+\frac{1}{c}$ $\quad e^x-\frac{1}{c^2}=0$ $\quad c^2=e^{-x}$

$$y=e^{\frac{-x}{2}}.e^x+\frac{1}{e^{\frac{-x}{2}}}$$

معادلات مرتبه دوم

- ضرایب ثابت
  - همگن: $ay'' + by' + cy = 0$
  - ناهمگن: $ay'' + by' + cy = g(x)$
- ضرایب غیر ثابت
  - همگن: $y'' + f_1(x)y' + f_2(x)y = 0$
  - ناهمگن: $y'' + f_1(x)y' + f_2(x)y = g(x)$

معادلات مرتبه دوم ضرایب ثابت - همگن

$$y'' = r^2 \qquad y' = r \qquad y = 1 \qquad y^{(n)} = r^n$$

$$ar^2 + br + c = 0$$

$r_1 , r_2 \in R , r_1 \neq r_2$ $\qquad$ $y = c_1\underbrace{e^{r_1x}}_{y_1} + c_2\underbrace{e^{r_2x}}_{y_2}$

$r_1 , r_2 \in R , r_1 = r_2$ $\qquad$ $y = c_1\underline{e^{r_1x}} + c_2\underline{xe^{r_1x}}$

$r_1 , r_2 = \alpha \pm i\beta \in C$ $\qquad$ $y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

مثال)

$$4y'' + 12y' + 9y = 0 \qquad 4r^2 + 12r + 9 = 0$$

$$r = \frac{3}{2} \longrightarrow y = c_1 e^{\frac{3}{2}x} + c_2 x e^{\frac{3}{2}x}$$

مثال)

$$(D^3 + D^2 - D - 1)y = 0 \qquad r^3 + r^2 - r - 1 = 0$$

$$r_1, r_2 = -1, r_3 = 1 \qquad y = c_1 e^{-x} + c_2 x e^{-x} + c_3 e^x$$

معادلات مرتبه دوم ضرایب غیر ثابت – همگن

$$y'' + f_1(x)y' + f_2(x)y = 0 \qquad y_1$$

$$f_1(x) + xf_2(x) = 0 \longrightarrow y_1 = x$$

$$\lambda^2 + \lambda f_1(x) + f_2(x) = 0 \longrightarrow y_1 = e^{\lambda x}$$

$$y_2 = vy_1 \qquad v = \int \frac{e^{-\int f_1(x)dx}}{(y_1)^2}dx$$

مثال)

$$y_1 = \sin x^2 \ , x > 0$$

$$xy'' - y' + 4x^3 y = 0$$

$$v = \int \frac{e^{-\int p(x)dx}}{(y_1)^2}\, dx$$

$$y'' - \frac{y'}{x} + 4x^2 y = 0$$

$$v = \int \frac{e^{-\int -\frac{1}{x}dx}}{(y_1)^2}\, dx \ ,$$

$$v = \int \frac{e^{lnx}}{sin^2 x^2}\, dx$$

$$v = \int \frac{x}{sin^2 x^2}\, dx = \frac{1}{2}\int \frac{2x}{sin^2 x^2} dx$$

$$x^2 = u \ , 2xdx = du$$

$$\frac{1}{2}\int\frac{du}{sin^2u}=-\frac{1}{2}\cot u=-\frac{1}{2}\cot x^2$$

$$\frac{1}{sin^2}=-(1+cot^2x)$$

$$y_2=-\frac{1}{2}sinx^2\times cotx^2=-\frac{1}{2}\ sinx^2\times\frac{cos^2}{sin^2}$$

$$y=c_1y_1+c_2y_2$$

مثال)

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0$$

$$f_1(x) = -\frac{2}{x}$$

$$f_2(x) = \frac{2}{x^2}$$

$$f_1(x) + xf_2(x) = 0 \rightarrow y_1 = x$$

$$v = \int \frac{e^{-\int p(x)dx}}{(y_1)^2}\, dx = \int \frac{e^{\int \frac{2}{x}dx}}{x^2}\, dx = \int \frac{e^{2lnx}}{x^2}\, dx$$

$$v = \int dx = x \longrightarrow$$

$$y_2 = vy_1 = x \times x = x^2$$

$$y = c_1x + c_2x^2$$

معادلات مرتبه دوم ناهمگن

ابتدا معادله همگن متناظر را حل می کنیم $\longleftarrow y_h$

$$y = y_h + y_p$$

روش ضرایب نامعین

روش تغییر پارامتر

معادلات مرتبه دوم نا همگن – تغییر پارامتر

$$y_h = c_1 y_1 + c_2 y_2$$

$$Y_1 = -y_1 \int \frac{y_2 g(x)}{w(y_1, y_2)} dx \qquad Y_2 = -y_2 \int \frac{y_1 g(x)}{w(y_1, y_2)} dx$$

$$y_p = Y_1 + Y_2 \qquad y = y_h + y_p$$

$$w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ {y_1}' & {y_2}' \end{vmatrix}$$

مثال)

$$y'' + y = \tan x$$

$$y'' + y = 0 \qquad r^2 + 1 = 0 \qquad r = \pm i$$

$$y_h = c_1 \underbrace{\cos x}_{y_1} + c_2 \underbrace{\sin x}_{y_2} \qquad w(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$Y_1 = -\cos x \int \frac{sin^2 x}{\cos x} dx = -\cos x \int \frac{1 - cos^2 x}{\cos x} dx$$

$$= -\cos x \left[\ln |\sec x + \tan x| - \sin x\right]$$

$$Y_2 = \sin x \int \frac{\cos x\ .\tan x}{1} dx = \sin x \int \sin x\, dx = -\sin x \cos x$$

$$y = c_1 \cos x + c_2 \sin x - \sin x \cos x$$
$$- \cos x \left[\ln|\sec x + \tan x| - \sin x\right]$$

معادلات مرتبه دوم نا همگن – ضرایب نا معین

$g(x)$	y_p
e^{ax}	$x^k A e^{ax}$
$\sin ax , \cos ax$	$x^k (\mathrm{A} \sin ax + B \cos ax)$
$a_n x^n + \cdots + a_0$	$x^k (A_n x^n + \cdots + A_0)$
$e^{ax} (a_n x^n + \cdots + a_0) \sin \beta x , \cos \beta x$	$x^k e^{ax} [(A_n x^n + \cdots + A_0) \cos \beta x$ $+ (B_n x^n + \cdots + B_0) \sin \beta x$

معادلات مرتبه دوم نا همگن – ضرایب نا معین

مثال) فرم کلی جواب خصوصی معادله زیر را بنویسید

$$D^2(D+1)(D^2+4)^2(D^2-2D+2)^3y$$
$$= cos2x + x^2e^x \cos x + 2\cosh x + x\, sin2x + x^2$$

D	**0**	**-1**	$\pm 2i$	$1 \pm i$
k	2	1	2	3

معادلات مرتبه دوم ناهمگن – ضرایب نامعین

$$\underline{(cos2x + x\ sin2x)} + \underline{x^2 e^x \cos x} + \underline{e^x} + \underline{e^{-x}} + \underline{x^2}$$

$$y_{p_1} = x^2[(A_1 x + A_2)\cos 2x + (A_3 x + A_4)\sin 2x]$$

$$y_{p_2} = x^3 e^x[(B_2 x^2 + B_1 x + B_0)\cos x + (C_2 x^2 + C_1 x + C_0)\sin x]$$

$$y_{p_3} = \mathrm{D}e^x \qquad y_{p_4} = \mathrm{F}xe^{-x}$$

$$y_{p_5} = x^2(E_2 x^2 + E_1 x + E_0)$$

$$y = y_{p_1} + y_{p_2} + y_{p_3} + y_{p_4} + y_{p_5}$$

معادلات مرتبه دوم نا همگن – ضرایب نا معین

مثال) جواب معادله زیر را بنویسید

$$y'' + 4y = 2\cos 2x - \sin 2x$$

$r^2 + 4 = 0 \qquad r = \pm 2i \qquad k = 1 \qquad y_h = C_1 \cos 2x + C_2 \sin 2x$

$y_p = x(A\cos 2x + B\sin 2x) \qquad y', y'' = \cdots \qquad A = \frac{1}{4}, B = \frac{1}{2}$

$$y_p = x\left(\frac{1}{4}\cos 2x + \frac{1}{2}\sin 2x\right)$$

$$y = C_1 \cos 2x + C_2 \sin 2x + x\left(\frac{1}{4}\cos 2x + \frac{1}{2}\sin 2x\right)$$

معادلات مرتبه دوم نا همگن – ضرایب نا معین

مثال) جواب معادله زیر را بنویسید

$$y'' + 4y' + 4y = xe^{-2x}$$

$$r^2 + 4r + 4 = 0 \qquad r = -2 \qquad y_h = c_1 e^{-2x} + c_2 x e^{-2x}$$

$$y_p = x^2 e^{-2x}(Ax + B) \qquad A = \frac{1}{6}, B = 0$$

$$y = c_1 e^{-2x} + c_2 x e^{-2x} + \frac{x^3}{6} e^{-2x}$$

مثال) الف) معادله ای بنویسید که جواب های آن به صورت زیر باشد.

ب) با فرض $g(x) = e^x$ جواب خصوصی آن را بیابید

$$\underbrace{e^x, xe^x}_{1}, \underbrace{e^{-x}\cos 2x\,, e^{-x}\sin 2x}_{-1 \pm 2i}$$

$$(r-1)^2(r^2+2r+5)=0 \qquad r^4+2r^2-8r+5=0$$

$$y^{(4)}+2y''-8y'+5y=e^x \qquad y_p = x^2Ae^x \qquad A=\frac{1}{16}$$

$$y=(c_1+c_2x)e^x+e^{-x}(c_3\cos 2x+c_4\sin 2x)+\frac{x^2}{16}e^x$$

معادله کوشی اویلر

$$x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

$$x = e^t \qquad t = \ln x$$

$$x^n y^{(n)} = D(D-1)(D-2)\ldots\big(D-(n-1)\big)y$$

$$x^2 y'' = D(D-1)y$$

معادله کوشی اویلر

مثال) جواب معادله زیر را بنویسید

$$x^2y'' - 2xy' + 2y = \ln x - \ln x^2$$

$$D(D-1)y - 2\mathrm{D}\mathrm{y} + 2\mathrm{y} = \mathrm{t} - 2\mathrm{t} = -\mathrm{t}$$

$$D^2y - 3Dy + 2y = -t$$

$$r^2 - 3r + 2 = 0 \begin{cases} r = 1 \\ r = 2 \end{cases}$$

$$y_h = c_1e^t + c_2e^{2t}$$

$$y_p = At + B \qquad A = \frac{-1}{2}, B = \frac{-3}{4}$$

$$y = c_1e^t + c_2e^{2t} - \frac{t}{2} - \frac{3}{4}$$

عملگر معکوس

$$(D^n + a_{n-1}D^{n-1} + \cdots + a_0)y = f(x)$$

$$\varphi(D)y = f(x) \qquad\qquad y = \frac{1}{\varphi(D)}(f(x))$$

$$1 - f(x) = c_k x^k + \cdots + c_1 x + c_0$$

$\frac{1}{\varphi(D)}$ را محاسبه کرده و بر f تاثیر میدهیم

$$2 - \frac{1}{\varphi(D)}(e^{ax} f(x)) = e^{ax}\frac{1}{\varphi(D+a)}(f(x))$$

عملگر معکوس

$$3-f(x)=e^{ax} \longrightarrow \frac{1}{\varphi(D)}(e^{ax})=\frac{1}{\varphi(a)}\times e^{ax}$$

$$\varphi(a)=\varphi'(a)=\cdots=\varphi^{(p-1)}(a)=0\,,\varphi^{(p)}\neq 0$$

$$\frac{1}{\varphi(D)}(e^{ax})=\frac{x^{p}}{\varphi^{(p)}\,(a)}\times e^{ax}$$

عملگر معکوس

$$\frac{1}{\varphi(D)}\underbrace{\begin{pmatrix}\sin\beta x\\ \cos\beta x\end{pmatrix}}_{f(x)} \longrightarrow D^2=-\beta^2$$

$$\big\downarrow$$

$$f(x)=e^{i\beta x}$$

$f(x) = cos \longrightarrow$ قسمت حقیقی

$f(x) = sin \longrightarrow$ قسمت موهومی

عملگر معکوس

$$y'' - y' - 2y = 36xe^{2x}$$

$$(D^2 - D - 2)y = 36xe^{2x}$$

$$y = \frac{1}{D^2 - D - 2} 36xe^{2x}$$

$$y = e^{2x} \frac{1}{(D+2)^2 - (D+2) - 2} 36\mathrm{x}$$

$$y = e^{2x} \frac{1}{D^2 + 4D + 4 - D - 2 - 2} 36\mathrm{x}$$

$$= 36e^{2x} \frac{1}{D^2 + 3D} \mathrm{x}$$

عملگر معکوس

$$= 36e^{2x}.\frac{1}{3}\left(\frac{1}{D}-\frac{1}{D+3}\right)(x) = 12e^{2x}\left[\frac{1}{D}(x)-\frac{1}{3}\left(\frac{1}{1+\frac{D}{3}}\right)x\right]$$

$$x = 12e^{2x}\left[\frac{x^2}{2}(x)-\frac{1}{3}\left(1-\frac{D}{3}\right)x\right]$$

$$y = 12e^{2x}\left[\frac{x^2}{2}-\frac{x}{3}+\frac{1}{9}\right]$$

$$y = 6e^{2x}x^2-4xe^{2x}+\frac{4}{3}e^{2x}$$

عملگر معکوس

$$y''' - 3y' = 9x^2$$

$$(D^3 - 3D)y = 9x^2$$

$$y = \frac{1}{(D^3 - 3D)} 9x^2 \qquad y = \frac{1}{D(D^2 - 3)} (9x^2)$$

$$y = -\frac{1}{3}\left(\frac{1}{D} - \frac{D}{(D^2 - 3)}\right)(9x^2)$$

$$y = -\frac{1}{3}\left(\frac{1}{D} + \frac{1}{3}\frac{D}{\left(1 - \frac{D^2}{3}\right)}\right)(9x^2)$$

$$\begin{array}{r|l} D & 1 - \frac{D^2}{3} \\ \cline{2-2} -D + \frac{D^3}{3} & D \\ \hline +\frac{D^3}{3} & \end{array}$$

$$= \frac{-1}{3}(3x^3) - \frac{1}{9}(D)(9x^2) = -x^3 - 2x$$

عملگر معکوس

$$y'' + 2y' + 5y = e^{-x} \sin x$$

$$(D^2 + 2D + 5)y = e^{-x} \sin x$$

$$y = \frac{1}{(D^2 + 2D + 5)} e^{-x} \sin x$$

$$y = e^{-x} \frac{1}{(D-1)^2 + 2(D-1) + 5} \sin x$$

$$= e^{-x} \frac{1}{D^2 - 2D + 1 + 2D - 2 + 5} \sin x$$

عملگر معکوس

$$y = e^{-x}\frac{1}{(D)^2+4}\sin x$$

$$y = e^{-x}\frac{1}{-1+4}\sin x = \frac{e^{-x}\sin x}{3}$$

عملگر معکوس

$$y'' + 4y = \cos 2x$$

$$(D^2 + 4)y = \cos 2x \;\rightarrow y = \frac{1}{D^2 + 4}(\cos 2\,x) \qquad D^2 = -4 \qquad \times$$

$$y = \frac{1}{D^2 + 4}(\,e^{2ix}\,) \qquad \varphi(2i) = 0\,, \varphi'(2i) \neq 0$$

$$y = \frac{x}{2D}(\,e^{2ix}\,) = \frac{i}{i} \times \frac{x}{4i}\; e^{2ix} = -\frac{ix}{4} e^{2ix}$$

$$= \frac{-ix}{4}(\cos 2x + i \sin 2x)$$

$$= \frac{-ix}{4}\cos 2x + \frac{x}{4}\sin 2x$$

همگرایی سری ها

$$\sum a_n(x-x_0)^n$$

در $x=\alpha$ همگرا ست، هرگاه:

$$\lim_{x\to\infty}\sum_{n=0}^{m} a_n(\alpha-x_0)^n$$ موجود باشد.

در $x=\alpha$ همگرای مطلق است، هرگاه:

$$\sum |a_n|\ |\alpha-x_0|^n$$ موجود باشد.

همگرایی سری ها

آزمون نسبت برای همگرایی مطلق

$$L = \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right|$$

$L < 1$ همگرا

$L > 1$ واگرا

$L = 1$ ×

آزمون ریشه

$$L = \lim_{n\to\infty} \sqrt[n]{|u_n|}$$

$L < 1$ همگرا

$L > 1$ واگرا

$L = 1$ ×

همگرایی سری ها

$$\sum \frac{(-1)^n 2^n (n-3)^n}{n(n+1)}$$

$$\left|\frac{(-1)^{n+1}\, 2^{n+1}\, (x-3)^{n+1}\, n(n+1)}{(-1)^n\, 2^n\, (x-3)^n (n+1)(n+2)}\right| = \lim_{n\to\infty}\left|2(x-3)\frac{n}{n+1}\right|$$

$$|x-3| < \frac{1}{2} \to \frac{5}{2} < x < \frac{7}{2} \qquad |x-3| = \frac{1}{2}$$

$$\sum \frac{(-1)^n\, 2^n \left(\frac{1}{2}\right)^n}{n(n+1)} \qquad \sum \frac{(-1)^n}{n(n+1)} \qquad \left[\frac{5}{2}, \frac{7}{2}\right]$$

همگرایی سری ها

$$\sum \frac{n}{2^n} x^n$$

$$L = \lim_{n\to\infty} \left|\frac{(n+1)x^{n+1}2^n}{nx^n 2^{n+1}}\right| = \left|\frac{x}{2}\right|$$

$$\left|\frac{x}{2}\right| < 1 \ \rightarrow |x| < 2 \rightarrow -2 < x < 2$$

در نقطه $x = 2$

$$\sum \frac{n}{2^n} .\ 2^n = \sum n$$ واگرا

در نقطه $x = -2$

$$\sum \frac{n}{2^n} .\ (-2)^n = \sum (-1)^n n$$ واگرا

حل معادله حول نقطه عادی

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{x} = \sum_{n=0}^{\infty} (1-x)^n$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

حل معادله حول نقطه عادی

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^{n}}{n!}$$

$$\tan x = x + \frac{x^{3}}{3} + \frac{2x^{5}}{5} + \cdots$$

$$\tanh x = x - \frac{x^{3}}{3} + \frac{2x^{5}}{5} + \cdots$$

$$tan^{-1}x = \sum_{n=0}^{\infty} \frac{(-1)^{n} x^{2n+1}}{2n+1}$$

حل معادله حول نقطه عادی

$$y'' + p(x)y' + q(x)y = g(x)$$

$p(x)$ در x_0 تحلیلی باشد $\longleftrightarrow$ x_0 نقطه عادی

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad y' = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}$$

حل معادله حول نقطه عادی

$$y=\sum_{n=0}^{\infty}a_n(x-x_0)^n \left\{ \begin{array}{l} a_0(x-x_0)^{o}+\sum\limits_{n=1}^{\infty}a_n(x-x_0)^n \\ \\ \sum\limits_{n=k}^{\infty}a_{n-k}(x-x_0)^{n-k} \end{array} \right.$$

حل معادله حول نقطه عادی

حول ۰ $\quad y(1) = 0 \quad xy' - 3y - 3 = 0$

$$y = \sum_{n=0}^{\infty} a_n (x)^n \qquad y' = \sum_{n=1}^{\infty} n a_n (x)^{n-1}$$

$$x \sum_{n=1}^{\infty} n a_n (x)^{n-1} - 3 \sum_{n=0}^{\infty} a_n (x)^n - 3 = 0$$

حل معادله حول نقطه عادی

$$\sum_{n=1}^{\infty} na_n(x)^n - 3\sum_{n=0}^{\infty} a_n(x)^n - 3 = 0$$

$$\sum_{n=1}^{\infty} na_n(x)^n - 3a_0 - 3\sum_{n=1}^{\infty} a_n(x)^n - 3 = 0$$

$$-3a_0 - 3 = 0 \qquad a_0 = -1$$

حل معادله حول نقطه عادی

$$(n-3)a_n = 0$$

$$\sum_{n=1}^{\infty}(n-3)a_n\,(x)^n = 0$$

$$a_n = 0$$

a_n دلخواه

$$n-3 = 0$$

$$n = 3$$

$$0 = -1 + a_3$$

$$y = -1 + x^3$$

$$y = -1 + a_3 x^3$$

$$1 = a_3$$

حل معادله حول نقطه عادی

$y'' - y = 0$ حول ۰

$$y = \sum_{n=0}^{\infty} a_n (x)^n \qquad y' = \sum_{n=1}^{\infty} n a_n (x)^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x)^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n (x)^{n-2} - \sum_{n=0}^{\infty} a_n (x)^n = 0$$

حل معادله حول نقطه عادی

$$\sum_{n=2}^{\infty} n(n-1)a_n(x)^{n-2} - \sum_{n=0}^{\infty} a_n(x)^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x)^n - \sum_{n=0}^{\infty} a_n(x)^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - a_n]\,(x)^n = 0$$

حل معادله حول نقطه عادی

$$(n+2)(n+1)a_{n+2} - a_n = 0 \qquad n \geq 0$$

$$a_{n+2} = \frac{a_n}{(n+1)(n+2)} \qquad n \geq 0$$

$$a_2 = \frac{a_0}{2} \qquad a_4 = \frac{a_0}{24} \qquad a_{2n} = \frac{a_0}{(2n)!}$$

$$a_3 = \frac{a_1}{6} \qquad a_5 = \frac{a_1}{5!} \qquad a_{2n+1} = \frac{a_1}{(2n+1)!}$$

حل معادله حول نقطه تکین منظم

$$y'' + p(x)y' + q(x)y = g(x)$$

$$\lim_{x\to x_0}(x-x_0)p(x) = p_0 \quad , \quad \lim_{x\to x_0}(x-x_0)^2 q(x) = q_0$$

$$r(r-1) + p_0 r + q_0 = 0$$

$$r_1 - r_2 \notin Z \, , r_1 \neq r_2$$

$$y_1 = \sum_{n=0}^{0} a_n (x-x_0)^{n+r_1}$$

$$y_2 = \sum b_n (x-x_0)^{n+r_2}$$

$$y = c_1 y_1 + c_2 y_2$$

حل معادله حول نقطه تکین منظم

$r_1 - r_2 \notin Z \, , r_1 \neq r_2$

$$y_1 = \sum a_n (x - x_0)^{n+r_1} \;\; , \; y_2 = k y_1 \ln x + \sum_{n=1} b_n (x - x_0)^{n+r_2}$$

$r_1 = r_2$

$$y_1 = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r_1} \qquad y_2 = y_1(x) \ln x + \sum_{n=0}^{\infty} b_n (x)^{n+r_2}$$

حل معادله حول نقطه تکین منظم

$$4xy'' + 2(1-x)y' - y = 0$$

$$\lim_{x\to 0}(x-0)\frac{2(1-x)}{4x} = \frac{1}{2}p_0 \qquad \lim_{x\to 0} x^2\frac{-1}{4x} = 0\ q_0$$

$$\to r(r-1) + \frac{1}{2}r = 0 \qquad r^2 - \frac{r}{2} = 0 \qquad r_{1=0} \qquad r_2 = \frac{1}{2}$$

$$y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1} \qquad y_2 = \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

حل معادله حول نقطه تکین منظم

$$y'_1 = \sum (n + r_1) a_n x^{n+r_1-1}$$

$$y''_1 = \sum (n + r_1)(n + r_1 - 1) a_n x^{n+r_1-2}$$

$$4x \sum (n + r_1)(n + r_1 - 1) a_n x^{n+r_1-2} + 2(1 - x) \sum (n + r_1) a_n x^{n+r_1-1}$$

$$- \sum a_n x^{n+r_1} = 0$$

$$4 \sum (n + r_1)(n + r_1 - 1) a_n x^{n+r_1-2} + 2 \sum (n + r_1) a_n x^{n+r_1-1}$$

$$-2 \sum (n + r_1) a_n x^{n+r_1-1} - \sum a_n x^{n+r_1} = 0$$

حل معادله حول نقطه تکین منظم

$$\to 4\sum_{n=-1}^{\infty}(n+r_1+1)(n+r_1)a_{n+1}x^{n+r_1}+2\sum_{n=-1}^{\infty}(n+r_1+1)a_{n+1}x^{n+r_1}$$

$$-2\sum(n+r_1)a_n x^{n+r_1}-\sum a_n x^{n+1}=0$$

$$\sum\left[[4(n+r_1+1)(n+r_1)+2(n+r_1+1)]a_{n+1}[-2(n+r_1)-1]a_n\right]x^{n+r_1}=0$$

$$2(n+r_1+1)(2n+2r_1+2)a_{n+1}=(2((n+r_1)+1)a_n$$

$$a_{n+1}=\frac{2n+2r_1+1}{2(n+r_1+1)(2n+2r_1+1)}a_n \to a_{n+1}=\frac{a_n}{2(n+r_1+1)}$$

$$a_{n+1}=\frac{a_n}{2n+2}$$

$$\to a_1=\frac{a_0}{2\times 1}\quad , \qquad a_2=\frac{a_1}{2\times 2}=\frac{a_0}{2\times 2^2}$$

حل معادله حول نقطه تکین منظم

$$a_3 = \frac{a_2}{2 \times 3} = \frac{a_0}{3! \times 2^3}$$

$$a_n = \frac{a_0}{2^n n!}$$

$$y_2 = \sum_{n=0} b_n x^{n+r_2}$$

$$b_{n+1} = \frac{b_n}{2(n + \frac{3}{2})}$$

$$b_1 = \frac{b_0}{3 \times 1}$$

$$b_2 = \frac{b_1}{5} = \frac{b_0}{5 \times 3 \times 1}$$

$$b_n = \frac{b_0}{(2n + 1) \dots \times 3 \times 1}$$

حل معادله حول نقطه تکین منظم

$$x^2y'' - (2x + 2x^2)y' + (x^2 + 2x + 2)y = 0 \qquad x_0 = 0$$

$$p_0 = -2, q_0 = 2 \qquad r(r-1) - 2r + 2 = 0$$

$$r_1 = 2\,, r_2 = 1 \qquad y = \sum_{n=0} a_n x^{n+r}$$

$$(r-1)(r-2)a_0x^r + [(r-1)ra_1 + (2-2r)a_0]x^{r+1} +$$

حل معادله حول نقطه تکین منظم

$$+\sum_{n=2}[(n+r-1)(n+r-2)a_n-(4-2(n+r)a_{n-1}+a_{n-2}]x^{n+r}$$

$$r=2 \qquad a_1=a_0=1 \qquad a_n=\frac{2na_{n-1}-a_{n-2}}{n(n+1)},n\geq 2$$

$$a_2=\frac{4a_1-a_0}{2\times 3}=\frac{1}{2} \qquad a_2=\frac{1}{6}=\frac{1}{3!} \qquad \cdots \qquad a_n=\frac{1}{n!}$$

$$y_1=\sum_{n=0}a_nx^{n+2}=x^2\sum_{n=0}\frac{x^n}{n!}=x^2e^x$$

حل معادله حول نقطه تکین منظم

$$y_2 = y_1 \int \frac{e^{\int\left(\frac{2}{x}+x\right)dx}}{y_1^2}\,dx = x^2 e^x \int \frac{x^2 e^{2x}}{x^4 e^{2x}}\,dx = -x e^x$$

معادله بسل

$$x^2y'' + xy' + (x^2 - v^2)y = 0$$

v

$$y = c_1J_v + c_2J_{-v}$$ غیر صحیح

$$y = c_1J_v + c_2Y_v$$ صحیح

$$J_v = x^v \sum_{n=0} \frac{(-1)^n x^{2n}}{2^{2n+v} n!\, \Gamma(n + v + 1)}$$

تابع بسل نوع اول

Y_v

تابع بسل نوع دوم

معادله بسل

معادلات قابل تبدیل به معادله بسل

$$x^2y'' + xy' + (\lambda^2x^2 - v^2)y = 0$$

$$z = \lambda x \,,\, y = c_1J_v(\lambda x) + c_2Y_v(\lambda x)$$

$$4x^2y'' + 4xy' + (x - v^2)y = 0$$

$$z = x^{\frac{1}{2}} \,,\, y = c_1J_v(x^{\frac{1}{2}}x) + c_2Y_v(x^{\frac{1}{2}}x)$$

$$x^2y'' + xy' + 4(x - v^2)y = 0$$

$$z = x^2 \,,\, y = c_1J_v(x^2x) + c_2Y_v(x^2x)$$

معادله بسل

$$x^2y'' + (1+2v)y' + xy = 0$$

$$y = x^{-v}z$$

معادله بسل

$$[x^{v}J_{v}(x)]' = x^{v}J_{v-1}(x)$$

$$[x^{-v}J_{v}(x)]' = -x^{-v}J_{v+1}(x)$$

$$\int x^{v}J_{v-1}(x)\,dx = x^{v}J_{v}(x)$$

$$\int x^{-v}J_{v+1}(x)\,dx = -x^{-v}J_{v}(x)$$

$$J_{v-1}(x) + J_{v+1}(x) = \frac{2v}{x}J_{v}(x)$$

$$J_{v-1}(x) - J_{v+1}(x) = 2J_{v}'(x)$$

معادله بسل

اثبات ۱

$$x^v J_v(x) = \sum \frac{(-1)^n x^{2n+2v}}{2^{2n+v} n!\,\Gamma(n+v+1)}$$

$$\Gamma(n+1) = n\Gamma(n)$$

$$[x^v J_v(x)]' = \sum \frac{(-1)^n (2n+2v) x^{2n+2v-1}}{2^{2n+v} n!\,(n+v)\Gamma(n+v)}$$

$$= x^v \sum \frac{(-1)^n x^{2n+v-1}}{2^{2n+v-1} n!\,\Gamma(n+v)} = x^v J_{v-1}(x)$$

معادله بسل

J_4 را برحسب J_0 و J_1 بنویسید

$$J_{v-1}(x)+J_{v+1}(x)=\frac{2v}{x}J_v(x)$$

$v=1 \quad J_0(x)+J_2(x)=\frac{2}{x}J_1(x) \qquad J_2(x)=\frac{2}{x}J_1(x)-J_0(x)$

$v=2 \quad J_3(x)=\frac{4}{x}J_2(x)-J_1(x)$

$v=3 \quad J_4=\frac{48}{x^3}J_1(x)+\left(-\frac{24}{x^2}+1\right)J_0(x)-\frac{6}{x}J_1(x)$

معادله بسل

$$x^2y'' + (1-2v)xy' + (a^2b^2x^{2a} + v^2 - a^2c^2)y = 0$$

$$y = x^v u\,, z = bx^a$$

$$y' = vx^{v-1}u + x^v u' \qquad y'' = v(v-1)x^{v-2}u + 2vx^{v-1}u' + x^v u''$$

$$z = bx^a \qquad \frac{dz}{dx} = abx^{a-1} \qquad \frac{du}{dx} = \frac{du}{dz} \times \frac{dz}{dx} = \frac{du}{dz} \times abx^{a-1}$$

$$u'' = a(a-1)bx^{a-2} + a^2b^2x^{2a-2}\frac{d^2u}{dx^2}$$

معادله بسل

$$z^2\frac{d^2u}{dz^2}+z\frac{du}{dz}+(z^2-c^2)u=0$$

$$u=c_1J_c(z)+c_2Y_c(z) \qquad y=x^v u\,, z=bx^a$$

$$y=c_1x^vJ_c(bx^a)+c_2x^vY_c(bx^a)$$

معادله لژاندر

$$(1-x^2)y'' - 2xy' + v(v-1)y = 0$$

$$y = a_1 p_n(x) + a_2 Q_n(x)$$

$$p_n = \sum \frac{(-1)^n (2n-2k)!\, x^{n-2k}}{2^n k!\,(n-k)!\,(n-2k)!}$$

معادله لژاندر

$$p_n = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n$$

$$p_0 = 1\,, p_1 = x\,, p_2 = \frac{3}{2}x^2 - \frac{1}{2}$$

$$c_n = \frac{2n+1}{2}\int_{-1}^{1} p_n f(x)$$

$$f(x) = \sum c_n p_n$$

معادله لژاندر

$$f(x) = 5x^3 - 3x^2 - x - 10$$

$$p_0 = 1\,, p_1 = x\,, p_2 = \frac{3}{2}x^2 - \frac{1}{2}$$

$$c_0 = \frac{1}{2}\int_{-1}^{1} p_0 f(x) = -2 \qquad c_1 = 2 \qquad c_2 = -2$$

$$f(x) = \sum c_n p_n = -2 + 2x + \cdots$$

معادله لژاندر

$$f(x)=\begin{cases} 0 & -1\le x\le 0 \\ x & 0\le x\le 1 \end{cases}$$

$$p_0=1\,,p_1=x\,,p_2=\frac{3}{2}x^2-\frac{1}{2}$$

$$c_0=\frac{1}{2}\int_{-1}^{1}p_0 f(x)=\int_{0}^{1}p_0 f(x)=\frac{1}{4}\qquad c_1=\frac{1}{2}\qquad c_2=\frac{5}{16}$$

$$f(x)=\sum c_n p_n=\frac{1}{4}+\frac{1}{2}x+\cdots$$

لاپلاس

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f\,(t)\,dt$$

$$L\{t\} = \int_0^\infty e^{-st} t\ \,dt = \lim_{h\to\infty} \int_0^h e^{-st} t\,dt = \lim_{h\to\infty} \left[\frac{-t}{s} e^{-st} - \frac{1}{s^2} e^{-st}\right]_0^h$$

$$= \frac{1}{s^2}$$

$$L(c_1 f_1 + c_2 f_2) = c_1 L(f_1) + c_2 L(f_2)$$

لاپلاس

$$\Gamma(p+1)=\int_0^\infty e^{-t}t^p\,dt \qquad p>-1$$

$$\Gamma(p+1)=\mathrm{p}\Gamma(p)$$

$$\Gamma(1)=1 \qquad \Gamma(n+1)=\mathrm{n}! \qquad n\in N$$

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

لاپلاس

$f(t)$	$F(s)$
k	$\frac{k}{s}$
$t^p \; p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
$t^n \; n \in N$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$

$f(t)$	$F(s)$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{at}t^n \; n \in N$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at}t^p \; p > -1$	$\frac{\Gamma(p+1)}{(s-a)^{p+1}}$
$e^{at}\sinh bt$	$\frac{b}{(s-a)^2-b^2}$
$e^{at}\cosh bt$	$\frac{s-a}{(s-a)^2-b^2}$

لاپلاس

$$F(s) = \frac{2s-3}{s^2+2s+10}$$

$$L^{-1}[F(s)] = L^{-1}\left[\frac{2s-3}{(s+1)^2+9}\right] = L^{-1}\left[\frac{(2s+2)-5}{(s+1)^2+9}\right]$$

$$= L^{-1}\left[\frac{2(s+1)}{(s+1)^2+9}\right] - L^{-1}\left[\frac{5}{(s+1)^2+9}\right]$$

$$= 2e^{-t}\cos 3t - \frac{5}{3}e^{-t}\sin 3t$$

لاپلاس

$$F(s) = \frac{2s + 2}{s^2 + 2s + 5}$$

$$L^{-1}[F(s)] = L^{-1}\left[\frac{2(s+1)}{(s+1)^2+4}\right] = 2e^{-t}\cos 2t$$

مشتق از لاپلاس

$$F'(s) = L(-tf(t))$$

$$F^{(n)}(s) = L((-1)^n t^n f(t))$$

$$f(t) = te^{-2t}\cos t$$

$$L(te^{-2t}\cos t) = -L(-\underbrace{te^{-2t}\cos t}_{g(t)}) \quad = -(L(g(t))'$$

$$= -\left(\frac{s+2}{(s+2)^2+1}\right)'$$

مشتق از لاپلاس

$$f(t) = t^2 \sin at$$

$$L(t^2 \underbrace{\sin at}_{g(t)}) = (L(\sin at))'' \quad = \left(\frac{a}{s^2 + a^2}\right)''$$

لاپلاس

$$F(s) = \ln\left(\frac{s+1}{s}\right)$$

$$F(s) = \ln\left(\frac{s+1}{s}\right) = \ln(s+1) - \ln s$$

$$F'(s) = \frac{1}{s+1} - \frac{1}{s} = L(-tf(t)) \xrightarrow{L^{-1}} -tf(t) = e^{-t} - 1$$

$$f(t) = \frac{1-e^{-t}}{t}$$

لاپلاس انتگرال

$$L\left(\int_0^t f(T)dT\right) = \frac{F(s)}{S}$$

$$L\left(\int_0^t e^{2t}\cosh 5t\,dt\right)$$

$$= \frac{\frac{s-2}{(s-2)^2-25}}{s} = \frac{s-2}{s[(s-2)^2-25]}$$

لاپلاس انتگرال

$$I=\int_0^\infty e^{-3t}\,t\underbrace{\int_0^t \sin 2u\,du\,dt}_{g(t)} = L(g(t)) \qquad s=3$$

$$= L\left(t\underbrace{\int_0^t \sin 2u\,du}_{h(t)}\right) = -L(h(t))' = -\left(\frac{\frac{2}{s^2+4}}{s}\right)' = \frac{2(3s^2+4)}{[s(s^2+4)]^2}$$

$$\xrightarrow{s=3} \quad I=\frac{62}{39\times 39}$$

لاپلاس تابع متناوب

$f(t)$ متناوب با دوره تناوب T

$$L\big(f(t)\big)=\frac{\int_0^T e^{-st}f(t)dt}{1-e^{-sT}}$$

$f(t)=|\cos t|$ $\quad T=\pi$

بازگشتی

$$L\big(f(t)\big)=\frac{\int_0^{\pi} e^{-st}|cos\ t|\ dt}{1-e^{-s\pi}}=\frac{\int_0^{\frac{\pi}{2}} e^{-st}\cos t\, dt-\int_{\frac{\pi}{2}}^{\pi} e^{-st}cos\ t\ dt}{1-e^{-s\pi}}$$

تابع پله‌ای

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases} \qquad c \geq 0$$

$$L[u_c(t)f(t-c)] = e^{-cs}F(s)$$

$$L^{-1}[e^{-cs}F(s)] = u_c(t)f(t-c)$$

تابع پله‌ای

$$f(t) = \begin{cases} t & 0 \le t < 2 \\ 2 & 2 \le t < 5 \\ 7 - t & 5 \le t < 7 \\ 0 & t \ge 7 \end{cases}$$

$$f(t) = t + u_2(t)(2 - t) + u_5(t)(-t + 5) + u_7(t)(t - 7)$$

تابع پله‌ای

$$f(t) = \begin{cases} \sin t & 0 \le t < \pi \\ \sin t + \cos t & t \ge \pi \end{cases}$$

$$f(t) = \sin t + u_\pi(t)\cos t \qquad \cos t = -\cos(t-\pi)$$

$$F(s) = L[\sin t] + L[u_\pi(t)\cos t] = \frac{1}{s^2+1} - e^{-\pi s}\frac{s}{s^2+1}$$

لاپلاس مشتق

$$L[f^n(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{n-1}(0)$$

$$L[f'(t)] = sL[f(t)] - f(0)$$

$$L[tf'(t)] = -L[f'(t)]'$$

لاپلاس مشتق

$$f(s) = \frac{(s-2)e^{-s}}{s^2-4s+3}$$

$$L^{-1}\left[\frac{(s-2)e^{-s}}{s^2-4s+3}\right] =$$

$4-1$

$$L^{-1}\left[\frac{(s-2)e^{-s}}{(s-2)^2-1}\right] = u_1(t)e^{2(t-1)}cosh(t-1)$$

لاپلاس مشتق

$$ty'' + (1-t)y' + 2y = 0$$

$$L(ty'') + L(y') - L(ty') + 2L(y) = 0$$

$$-[L(y'')]' + L(y') + [L(y')]' + 2L(y) = 0$$

$$-\frac{d}{ds}L(y'') + L(y') + \frac{d}{ds}L(y') + 2L(y) = 0$$

$$-\frac{d}{ds}(s^2y(s) - sy(0) - y'(0)) + (sy(s) - y(0)) + 2L(y)$$
$$+\frac{d}{ds}\left(sy(s) - y(0)\right) + 2L(y) = 0$$

لاپلاس مشتق

$$\rightarrow \frac{y'(s)}{y(s)} = \frac{s-3}{-s(s-1)}$$

$$\ln(y(s)) = \int \frac{s-3}{-s(s-1)} = ln\left(\frac{c(s-1)^2}{s^3}\right)$$

$$y(s) = \frac{c(s-1)^2}{s^3} = c\left(\frac{s^2-2s+1}{s^3}\right) = c\left(\frac{1}{s} - \frac{2}{s^2} + \frac{1}{s^3}\right)$$

$$y(t) = c\left(1 - 2t + \frac{t^2}{2}\right)$$

لاپلاس مشتق

$$2y' - 4y = f(t) \qquad , \qquad y(0) = 0$$

$$f(t) = \begin{cases} t & 0 < t < \pi \\ 0 & \pi < t < 2\pi \\ \sin t & t > 2\pi \end{cases}$$

$$2y' - 4y = t - u_{\pi}(t)\, t + u_{2\pi}(t) \sin t$$

$$2y' - 4y = t - u_{\pi}(t)\, [(t - \pi) + \pi] + u_{2\pi}(t) \sin(t - 2\pi)$$

$$\sin(\mathrm{t} - 2\pi) = -\sin(2\pi - t) = \sin t$$

لاپلاس مشتق

$$2y' - 4y = t - u_{\pi}(t)\,[(t-\pi)+\pi] + u_{2\pi}(t)\sin(t-2\pi)$$

$$2sy(s) - 2y(0) - 4y(s) = \frac{1}{s^2} - \frac{e^{-\pi s}}{s^2} - \frac{e^{-\pi s}\pi}{s} + \frac{e^{-2\pi s}}{s^2+1}$$

$$y(s) = \frac{1}{2s^2(s-2)} - e^{-\pi s}\left(\frac{1}{2s^2(s-2)} + \frac{\pi}{2s(s-2)}\right) + e^{-2\pi s}(\frac{1}{2(s-2)(s^2+1)}$$

ادامه حل تمرین

انتگرال پیچشی (کانولوشن)

$$L(f(t).g(t)) \neq l\big(f(t)\big)L(g(t))$$

$$f * g = \int_0^t f(t-T)g(T)dT$$

$$f * g = g * f \qquad 0 * f = f * 0 = 0 \qquad f * (g+h) = f * g + f * h$$

$$L(f * g) = F(s) \times G(s) \qquad L^{-1}\big(F(s) \times G(s)\big) = f * g$$

انتگرال پیچش (کانولوشن)

$$f(t) = \int_0^t \underbrace{\sin(t-T)}_{g(t-T)} \underbrace{\cos T}_{h(T)} \, dT$$

$$F(S) = G(S) \times H(S)$$

$$F(S) = \frac{1}{s^2+1} \times \frac{s}{s^2+1}$$

انتگرال پیچشی (کانولوشن)

$$f(x)=\int_0^x\int_0^t \sqrt{t-u}\ \times\ u^{\frac{5}{2}}\,du\,dt$$

$$F(S)=\frac{L\left(\int_0^t \sqrt{t-u}\ \times\ u^{\frac{5}{2}}\,du\right)}{s}=\frac{\frac{\Gamma(\frac{3}{2})}{s^{\frac{3}{2}}}\times\frac{\Gamma(\frac{7}{2})}{s^{\frac{7}{2}}}}{s}=\frac{15\pi}{16s^{6}}$$

$$\Gamma\left(\frac{3}{2}\right)=\Gamma\left(\frac{1}{2}+1\right)=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2}$$

$$\Gamma\left(\frac{7}{2}\right)=\Gamma\left(\frac{5}{2}+1\right)=\frac{5}{2}\Gamma\left(\frac{5}{2}\right)=\frac{5}{2}.\frac{3}{2}\Gamma\left(\frac{3}{2}\right)=\frac{15\sqrt{\pi}}{8}$$

معادلات انتگرالی

$$y(t) = t^2 - e^{2t}\int_0^t y(x)e^{-2x}dx + \int_0^t y(t-x)e^{-2x}dx$$

$$y(t) = t^2 - \int_0^t y(x)e^{2(t-x)}dx + \int_0^t y(t-x)e^{-2x}dx$$

$$y(s) = \frac{2}{s^3} - y(s).\frac{1}{s-2} + y(s).\frac{1}{s+2}$$

$$y(s)\left[\frac{s^2}{s^2-4}\right] = \frac{2}{s^3} \qquad y(s) = \frac{2s^2-8}{s^5} = \frac{2}{s^3} - \frac{8}{s^5}$$

$$y(t) = t^2 - \frac{t^4}{3}$$

$$L\left(\frac{f(t)}{t}\right) = \int_{s}^{\infty} F(s)$$

دلتای دیراک

$$\int_{-\infty}^{x} f(t)\delta(t-t_0)dt = f(t_0)$$

$$L\big(\delta(t-t_0)\big) = e^{-st_0}$$

$$L\big(f(t)\delta(t-t_0)\big) = f(t_0)$$

$$y'' + 4y = f(t) + \delta(t - \pi) \qquad f(t) = \begin{cases} t & 0 \le t < 1 \\ 1 & t \ge 1 \end{cases}$$

$y(0) = 0$ و $y'(0) = 1$

$$y'' + 4y = t - u_1(t)(t - 1) + \delta(t - \pi)$$

$$y(s) = \frac{1}{s^2 + 4} + \frac{1 - e^{-s}}{s^2(s^2 + 4)} + \frac{e^{-\pi s}}{s^2 + 4}$$

ادامه حل تمرین

حل دستگاه

$$X' = AX + f(t)$$

$$1)\ X' = AX$$

محاسبه مقادیر ویژه

$$\det(A - \lambda I) = 0$$

محاسبه بردار ویژه:

مقادیر ویژه ها متمایز باشند.

$$Av = \lambda v$$

$$X_h = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} + \cdots + c_n v_n e^{\lambda_n t}$$

حل دستگاه

λ_i ۲ بار تکرار شده باشد

$$c_i v_i e^{\lambda_i t} + c_i' e^{\lambda_i t}(v_i t + w)$$

$$(A - \lambda_i I)w = v_i$$

جواب خصوصی

$$X_p = \Psi \int \Psi^{-1} f(t)$$

$$\Psi = \left(e^{\lambda_i t} v_i\right)$$

حل دستگاه

$$x_1' = 2x_1 + 3x_2 + 2e^{2t}$$

$$x_2' = x_1 + 4x_2 + 3e^{2t}$$

$$x_1' - 2x_1 - 3x_2 = 2e^{2t}$$

$$x_2' - x_1 - 4x_2 = 3e^{2t}$$

$$X' = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} X + \begin{bmatrix} 2e^{2t} \\ 3e^{2t} \end{bmatrix}$$

$$det(A - \lambda I) = 0$$

$$det\left(\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\begin{vmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 6\lambda + 5 = 0$$

$\lambda = 1$ و 5

حل دستگاه

$\lambda = 1$ $\qquad$ $Av = \lambda v$

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \begin{bmatrix} 2v_1 + 3v_2 \\ v_1 + 4v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$2v_1 + 3v_2 = v_1 \qquad v_1 = -3v_2$$

$$v_1 + 4v_2 = v_2 \qquad v = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

حل دستگاه

$\lambda = 5$ $\quad\quad$ $Av = \lambda v$

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \begin{bmatrix} 2v_1 + 3v_2 \\ v_1 + 4v_2 \end{bmatrix} = \begin{bmatrix} 5v_1 \\ 5v_2 \end{bmatrix}$$

$$2v_1 + 3v_2 = 5v_1 \qquad v_1 = v_2 \qquad v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$v_1 + 4v_2 = 5v_2$$

$$X_h = c_1 e^{t}\begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2 e^{5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

حل دستگاه

$$\psi = \begin{bmatrix} -3e^{t} & e^{5t} \\ e^{t} & e^{5t} \end{bmatrix} \qquad \psi^{-1} = \begin{bmatrix} \dfrac{-e^{-t}}{4} & \dfrac{e^{-t}}{4} \\ \dfrac{e^{-5t}}{4} & \dfrac{3e^{-5t}}{4} \end{bmatrix}$$

$$X_p = \Psi \int \Psi^{-1} f(t) dt$$

$$X_p = \begin{bmatrix} -3e^{t} & e^{5t} \\ e^{t} & e^{5t} \end{bmatrix} \int \begin{bmatrix} \dfrac{-e^{-t}}{4} & \dfrac{e^{-t}}{4} \\ \dfrac{e^{-5t}}{4} & \dfrac{3e^{-5t}}{4} \end{bmatrix} \begin{bmatrix} 2e^{2t} \\ 3e^{2t} \end{bmatrix} dt$$

حل دستگاه

$$X_p = \begin{bmatrix} -3e^{t} & e^{5t} \\ e^{t} & e^{5t} \end{bmatrix} \int \begin{bmatrix} \frac{1}{4}e^{t} \\ \frac{11}{4}e^{-3t} \end{bmatrix} dt$$

$$X_p = \begin{bmatrix} -3e^{t} & e^{5t} \\ e^{t} & e^{5t} \end{bmatrix} \begin{bmatrix} \frac{1}{4}e^{t} \\ -\frac{11}{12}e^{-3t} \end{bmatrix} = \begin{bmatrix} \frac{-5}{3}e^{2t} \\ \frac{-7}{6}e^{2t} \end{bmatrix}$$

حل دستگاه

$$X' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} X + \begin{bmatrix} t \\ 1 + e^t \end{bmatrix}$$

$$det(A - \lambda I) = 0$$

$$det\left(\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\begin{vmatrix} 1 - \lambda & -1 \\ 1 & 3 - \lambda \end{vmatrix} = 0$$

$$\lambda^2 - 4\lambda + 4 = 0 \qquad \lambda = 2$$

حل دستگاه

$\lambda = 2$ $\quad Av = \lambda v$

$$\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \begin{bmatrix} v_1 - v_2 \\ v_1 + 3v_2 \end{bmatrix} = \begin{bmatrix} 2v_1 \\ 2v_2 \end{bmatrix}$$

$v_1 - v_2 = 2v_1$ $\quad -v_1 = v_2$

$v_1 + 3v_2 = 2v_2$ $\quad v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

حل دستگاه

$$(A-2I)w = v_i$$

$$\left(\begin{bmatrix}1 & -1\\ 1 & 3\end{bmatrix} - \begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\right)\begin{bmatrix}w_1\\ w_2\end{bmatrix} = \begin{bmatrix}1\\ -1\end{bmatrix}$$

$$-w_1 - w_2 = 1 \qquad w_2 = -1 - w_1 \qquad w = \begin{bmatrix}1\\ -2\end{bmatrix}$$

$$X_h = c_1 e^{2t}\begin{bmatrix}1\\ -1\end{bmatrix} + c_2 e^{2t}\left(\begin{bmatrix}1\\ -1\end{bmatrix}t + \begin{bmatrix}1\\ -2\end{bmatrix}\right)$$

حل دستگاه

$$\psi = \begin{bmatrix} e^{2t} & te^{2t} + e^{2t} \\ -e^{2t} & -te^{2t} - 2e^{2t} \end{bmatrix} \quad \psi^{-1} = \begin{bmatrix} te^{-2t} + 2e^{-2t} & te^{-2t} + e^{-2t} \\ -e^{-2t} & -e^{-2t} \end{bmatrix}$$

$$X_p = \Psi \int \Psi^{-1} f(t) dt$$

$$X_p = \begin{bmatrix} e^{2t} + \frac{7}{4}(1-t)e^{2t} - e^{t} - \frac{3t}{4} - \frac{3}{4} \\ -2e^{2t} + \frac{7}{4}te^{2t} + \frac{t}{4} \end{bmatrix}$$

حل دستگاه

$$X' = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix} X \qquad det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & -1 \\ 0 & 1 & 3-\lambda \end{vmatrix} = 0$$

$$(-1)^2(1-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = 0 \qquad (1-\lambda)(\lambda-2)^2 = 0$$

$$\lambda = 1,2,2$$

حل دستگاه

$\lambda = 1$ $\quad Av = \lambda v$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \qquad \begin{bmatrix} v_1 - v_3 \\ v_2 - v_3 \\ v_2 + 3v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$v_1 - v_3 = v_1 \qquad v_3 = 0$$

$$v_2 - v_3 = v_2 \qquad v_1 = 1 \qquad v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$v_2 + 3v_3 = v_3 \qquad v_2 = 0$$

حل دستگاه

$$\lambda = 2 \qquad Av = \lambda v$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2v_1 \\ 2v_2 \\ 2v_3 \end{bmatrix} \qquad \begin{bmatrix} v_1 - v_3 \\ v_2 - v_3 \\ v_2 + 3v_3 \end{bmatrix} = \begin{bmatrix} 2v_1 \\ 2v_2 \\ 2v_3 \end{bmatrix}$$

$$v_1 - v_3 = 2v_1 \qquad v_3 = 1$$

$$v_2 - v_3 = 2v_2 \qquad v_1 = -1 \qquad v = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$v_2 + 3v_3 = 2v_3 \qquad v_2 = -1$$

حل دستگاه

$$(A-\lambda_i I)w = v_i$$

$$\left(\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -1\\ 0 & 1 & 3\end{bmatrix} - \begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}\right)\begin{bmatrix}w_1\\ w_2\\ w_3\end{bmatrix} = \begin{bmatrix}-1\\ -1\\ 1\end{bmatrix}$$

$$w = \begin{bmatrix}1\\ 1\\ 0\end{bmatrix} \qquad X_h = e^{t}\begin{bmatrix}1\\ 0\\ 0\end{bmatrix} + e^{2t}\begin{bmatrix}-1\\ -1\\ 1\end{bmatrix} + e^{2t}\left(\begin{bmatrix}-1\\ -1\\ 1\end{bmatrix}t + \begin{bmatrix}1\\ 1\\ 0\end{bmatrix}\right)$$

حل دستگاه

$$x_1' + x_1 - x_2 = 1$$

$$x_2' + 2x_1 - x_2 = \cot t$$

$$x_1' = -x_1 + x_2 + 1$$

$$x_2' = -2x_1 + x_2 + \cot t$$

$$X' = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} X + \begin{bmatrix} 1 \\ \cot t \end{bmatrix}$$

$$det(A - \lambda I) = 0$$

$$det\left(\begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\begin{vmatrix} -1-\lambda & 1 \\ -2 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0 \qquad \lambda = \pm i$$

حل دستگاه

$\lambda = i$ $\qquad\qquad$ $Av = \lambda v$

$$\begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \begin{bmatrix} -v_1 + v_2 \\ -2v_1 + v_2 \end{bmatrix} = \begin{bmatrix} iv_1 \\ iv_2 \end{bmatrix}$$

$$-v_1 + v_2 = iv_1 \qquad (1+i)v_1 = v_2$$

$$-2v_1 + v_2 = iv_2 \qquad (1-i)v_2 = 2v_2$$

$$v = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$$

حل دستگاه

$$e^{it}\begin{bmatrix} 1 \\ 1+i \end{bmatrix} = (\cos t + i\sin t)\begin{bmatrix} 1 \\ 1+i \end{bmatrix} = \begin{bmatrix} \cos t + i\sin t \\ \cos t - \sin t + i(\cos t + \sin t) \end{bmatrix}$$

$$= \begin{bmatrix} \cos t \\ \cos t - \sin t \end{bmatrix} + i\begin{bmatrix} \sin t \\ \cos t + \sin t \end{bmatrix}$$

$$X_h = c_1\begin{bmatrix} \cos t \\ \cos t - \sin t \end{bmatrix} + c_2\begin{bmatrix} \sin t \\ \cos t + \sin t \end{bmatrix}$$

حل دستگاه

$$\psi = \begin{bmatrix} \cos t & \sin t \\ \cos t - \sin t & \cos t + \sin t \end{bmatrix}$$

$$\psi^{-1} = \begin{bmatrix} \cos t + \sin t & -\sin t \\ \sin t - \cos t & \cos t \end{bmatrix}$$

$$X_p = \Psi \int \Psi^{-1} f(t)dt$$

$$X_p = \begin{bmatrix} \cos t & \sin t \\ \cos t - \sin t & \cos t + \sin t \end{bmatrix} \begin{bmatrix} -\cos t \\ -\sin t + \ln(\csc t - \cot t) \end{bmatrix}$$

حل دستگاه

$$\begin{cases} y' + 2y + 6\int_0^t x(t)\,dt = -2u_0(t) \\ y' + x' + x = 0 \end{cases}$$

$$x(0) = 6\,,\, y(0) = -5$$

$$\begin{cases} sy(s) - y(0) + 2y(s) + \frac{6}{s}x(s) = -\frac{2}{s} \\ sy(s) - y(0) + sx(s) - x(0) + x(s) = 0 \end{cases}$$

حل دستگاه

$$\begin{cases} sy(s) + 5 + 2y(s) + \dfrac{6}{s}x(s) = -\dfrac{2}{s} \\ sy(s) + 5 + sx(s) - 6 + x(s) = 0 \end{cases}$$

$$\begin{cases} (s^2+2s)y(s) + 6x(s) = -2 - 5s \\ sy(s) + (s+1)x(s) = 1 \end{cases}$$

حل دستگاه

$$y(s)=\frac{\begin{vmatrix}-2-5s & 6\\ 1 & s+1\end{vmatrix}}{\begin{vmatrix}s^2+2s & 6\\ s & s+1\end{vmatrix}}=-\frac{5s^2+7s+8}{s^3+3s^2-4s}$$

$$=\frac{5s^2+7s+8}{s(s-1)(s+4)}=\frac{2}{s}-\frac{4}{s-1}-\frac{3}{s+4}$$

L^{-1}

$$y(t)=2-4e^{t}-3e^{-4t}$$

حل دستگاه

$$x(s)=\frac{\begin{vmatrix} s^2+2s & -2-5s \\ s & 1 \end{vmatrix}}{\begin{vmatrix} s^2+2s & 6 \\ s & s+1 \end{vmatrix}}=-\frac{7s^2+4s}{s^3+3s^2-4s}$$

$$=\frac{s(7s+4)}{s(s-1)(s+4)}=\frac{\frac{11}{5}}{s-1}+\frac{\frac{24}{5}}{s+4}$$

L^{-1}

$$x(t)=\frac{11}{5}e^{t}+\frac{24}{5}e^{-4t}$$