Discrete Mathematics

Session IV

Extensions of the Principle of Inclusion and Exclusion

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Recapitulation

The principle of inclusion and exclusion helps us solve some counting problems that cannot be solved directly using the elementary techniques based on the principles of sum and product.

Given a set S and t conditions $c_1, c_2, \ldots, c_t$, the number of elements of S that satisfy none of the t conditions is noted $\overline{N}$ or $N(\bar{c}_1\bar{c}_2 \ldots \bar{c}_t)$ and is obtained from

$$N - \sum_{1 \le i \le t} N(c_i) + \sum_{1 \le i < j \le t} N(c_i c_j) - \sum_{1 \le i < j < k \le t} N(c_i c_j c_k) + \cdots + (-1)^t N(c_1 c_2 \ldots c_t).$$

The following notation is used to simplify the formula.

$$S_0 = N$$

$$S_k = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le t} N(c_{i_1} c_{i_2} \ldots c_{i_k})$$

Thus, we can write $\overline{N} = N(\bar{c}_1\bar{c}_2 \ldots \bar{c}_t) = S_0 - S_1 + S_2 - \cdots + (-1)^t S_t$.

The application of the principle requires a careful decision on the set S and the conditions $c_1, c_2, \ldots, c_t$.

In this session, we shall generalize the principle so that it can be used in solving a larger class of counting problems.

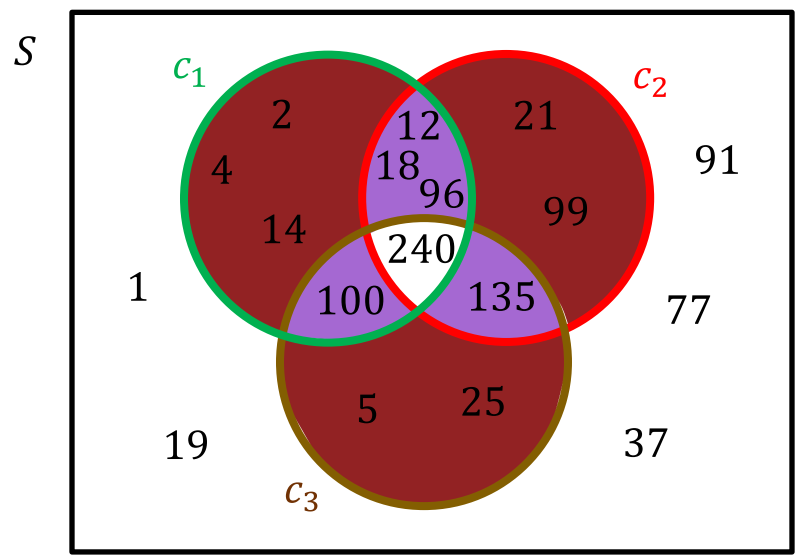
A Generalization

Let S be a finite set and $c_1, c_2, \ldots, c_t$ be t conditions; each element of S satisfies none or some of the elements of S.

Let E_m denote **the number of elements of S that satisfy exactly m of the t conditions**.

In the following figure, dark red regions contain the elements of S that satisfy exactly one of the three conditions c_1, c_2, and c_3. They are $2, 4, 14, 21, 99, 5$, and 25 ($E_1 = 7$.)

What about E_2? It is the number of elements in purple regions, that is, $E_2 = 5$.



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The Case $t = 3$

E_m: The number of elements of S that satisfy exactly m of the t conditions

$$E_0 = \bar{N} = S_0 - S_1 + S_2 - S_3 = S_0 - \binom{1}{1} S_1 + \binom{2}{2} S_2 - \binom{3}{3} S_3$$

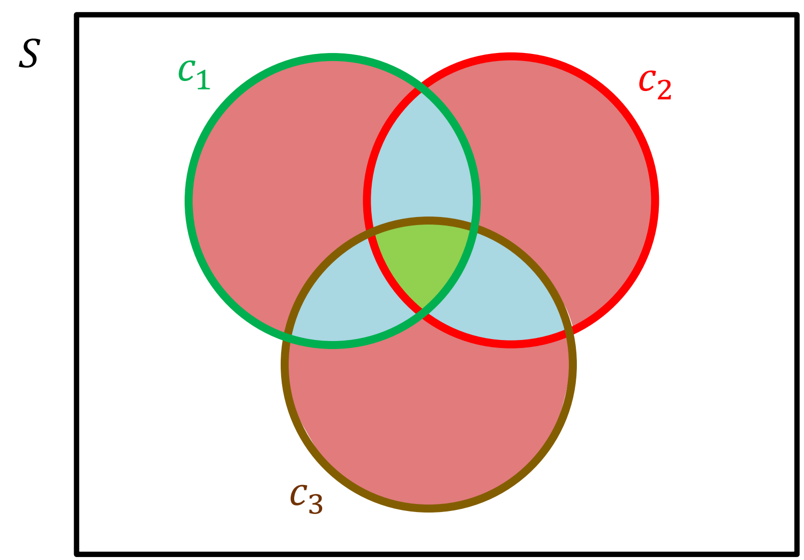
$$E_1 = N(c_1) + N(c_2) + N(c_3) - 2N(c_1c_2) - 2N(c_1c_3) - 2N(c_2c_3) + 3N(c_1c_2c_3)$$

$$= S_1 - 2S_2 + 3S_3 = S_1 - \binom{2}{1} S_2 + \left(\binom{2}{1}\binom{3}{2} - \binom{3}{1}\right) S_3 = S_1 - \binom{2}{1} S_2 + \binom{3}{2} S_3$$

$$E_2 = N(c_1c_2) + N(c_1c_3) + N(c_2c_3) - 3N(c_1c_2c_3)$$

$$= S_2 - 3S_3 = S_2 - \binom{3}{2} S_3$$

$$E_3 = N(c_1c_2c_3) = S_3$$



The Case $t = 4$

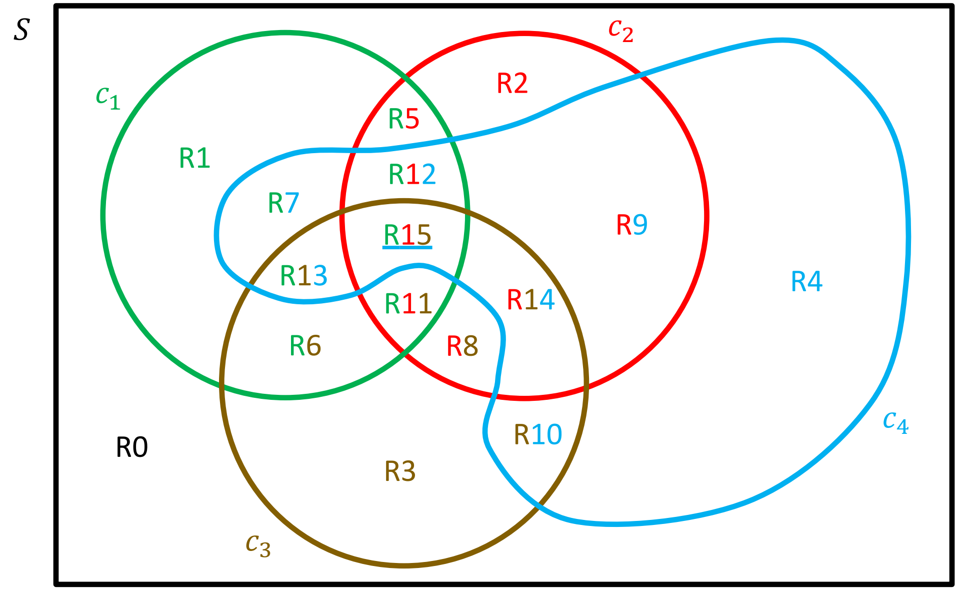
E_0: R0

E_1: R1+R2+R3+R4

E_2: R5+R6+R7+R8+R9+R10

E_3: R11+R12+R13+R14

E_4: R15



$$E_0 = S_0 - S_1 + S_2 - S_3 + S_4 = S_0 - \binom{1}{1} S_1 + \binom{2}{2} S_2 - \binom{3}{3} S_3 + \binom{4}{4} S_4$$

$$E_1 = S_1 - 2S_2 + 3S_3 - 4S_4 = S_1 - \binom{2}{1} S_2 + \binom{3}{2} S_3 - \binom{4}{3} S_4$$

$$E_2 = S_2 - 3S_3 + 6S_4 = S_2 - \binom{3}{1} S_3 + \binom{4}{2} S_4$$

$$E_3 = S_3 - 4S_4 = S_3 - \binom{4}{1} S_4$$

$$E_4 = S_4$$

A Generalized Principle

Theorem. Assume that S is a finite set and that $c_1, c_2, \dots, c_t$ are t conditions. Then, for each $0 \leq m \leq t$, the number of elements of S that satisfy ***exactly*** m of the t conditions is given by

$$E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} - \cdots + (-1)^{t-m} \binom{t}{t-m} S_t.$$

Proof. Let x be an element of S. We consider three cases: (1) x satisfies fewer than m conditions, (2) x satisfies exactly m of the conditions, and (3) x satisfies r of the conditions where $m < r \leq t$. In each case, we show that the number of times that x is counted in the left side of the equality is equal to the number of times it is counted in the right side of the equality. In the first case, x is not counted in the left side. Such an element is not counted in the right side either. In the second case, x is counted once in E_m. In the right side of the equality, it is counted once in S_m (in the summand corresponding to the conditions it satisfies) and is not counted in S_{m+1}, S_{m+2},…, S_t. Now consider the third case. Because x satisfies more than m conditions, it is not counted in E_m. In the right side of the equality, it is counted $\binom{r}{m}$ times in S_m, the number of ways one can select m of the r conditions that x satisfies. Similarly, it is counted $\binom{r}{m+1}$ times in S_{m+1}, $\binom{r}{m+2}$ times in S_{m+2}, …, and $\binom{r}{r}$ times in S_r. The element is not counted in S_{r+1}, S_{r+2}, …, S_t. Thus, the number of times that x is counted in the right side of the equality is

$$\binom{r}{m} - \binom{m+1}{1}\binom{r}{m+1} + \binom{m+2}{2}\binom{r}{m+2} - \cdots + (-1)^{r-m}\binom{r}{r-m}\binom{r}{r}.$$

Proof (Ctd.)

Thus, the number of times that x is counted in the right side of the equality is

$$\binom{r}{m} - \binom{m+1}{1}\binom{r}{m+1} + \binom{m+2}{2}\binom{r}{m+2} - \cdots + (-1)^{r-m}\binom{r}{r-m}\binom{r}{r}.$$

It can be written as

$$\sum_{k=0}^{r-m}(-1)^k\binom{m+k}{k}\binom{r}{m+k} = \sum_{k=0}^{r-m}(-1)^k\frac{(m+k)!}{k!\,m!}\cdot\frac{r!}{(m+k)!\,(r-m-k)!}$$

$$= \sum_{k=0}^{r-m}(-1)^k\frac{r!}{m!\,(r-m)!}\cdot\frac{(r-m)!}{k!\,(r-m-k)!}$$

$$= \sum_{k=0}^{r-m}(-1)^k\binom{r}{m}\binom{r-m}{k}$$

$$= \binom{r}{m}\sum_{k=0}^{r-m}(-1)^k\binom{r-m}{k}$$

$$= \binom{r}{m}(1-1)^{r-m} = 0$$

Application of the Generalized Principle

Example 1. For a function $f: A \to B$, $f(A)$ denotes the range of f. Determine the number of functions f from $A = \{1,2,3,4,5,6,7,8,9,10\}$ to $B = \{1,2,3,4,5,6,7\}$ such that $|f(A)| = 3$. How many functions have at most 3 elements in their range?

Solution. Let S be the set of all functions f from A to B and define the conditions c_i as $i \notin f(A)$ where $i = 1,2,\dots,7$. A function f with $|f(A)| = 3$ satisfies exactly 4 of the 7 conditions. Thus, we should calculate E_4. By the generalized principle, we have

$$E_4 = S_4 - \binom{5}{1} S_5 + \binom{6}{2} S_6 - \binom{7}{3} S_7$$

$$= \binom{7}{4} 3^{10} - \binom{5}{1}\binom{7}{5} 2^{10} + \binom{6}{2}\binom{7}{6} 1^{10} - \binom{7}{3}\binom{7}{7} 0^{10} = 1{,}959{,}300.$$

The second part of the problem can be stated as how many functions $f: A \to B$ satisfy $|f(A)| \leq 3$? Thus, the answer is $E_4 + E_5 + E_6 + E_7$ or equivalently $N - E_0 - E_1 - E_2 - E_3$.

$$E_5 = S_5 - \binom{6}{1} S_6 + \binom{7}{2} S_7 = \binom{7}{5} 2^{10} - \binom{6}{1}\binom{7}{6} 1^{10} + \binom{7}{2}\binom{7}{7} 0^{10} = 21{,}462$$

$$E_6 = S_6 - \binom{7}{1} S_7 = \binom{7}{6} 1^{10} + \binom{7}{1}\binom{7}{7} 0^{10} = 7$$

$$E_7 = 0$$

$$E_4 + E_5 + E_6 + E_7 = 1{,}980{,}769$$

Application of the Generalized Principle (Ctd.)

Example 2. If 13 cards are dealt from a standard deck of 52, what is the probability that these 13 cards include (a) at least one card from each suit? (b) exactly one void (for example, no clubs)? (c) exactly two voids?

Solution. Let S be the set of all possible ways to deal 13 cards from a standard deck of 52. Evidently, $N = |S| = \binom{52}{13}$. Consider also the conditions c_1: the hand is void in clubs (♣), c_2: the hand is void in diamonds (♦), c_3: the hand is void in hearts (♥), and c_4: the hand is void in spades (♠). The answer to (a) is

$$\frac{1}{\binom{52}{13}} \cdot N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = \frac{1}{\binom{52}{13}} \cdot \left(\binom{52}{13} - \binom{4}{1}\binom{39}{13} + \binom{4}{2}\binom{26}{13} - \binom{4}{3}\binom{13}{13}\right).$$

The probability that the hand includes exactly one void, the answer to (b), is

$$\frac{1}{\binom{52}{13}} \cdot E_1 = \frac{1}{\binom{52}{13}} \cdot \left(\binom{4}{1}\binom{39}{13} - \binom{2}{1}\binom{4}{2}\binom{26}{13} + \binom{3}{2}\binom{4}{3}\binom{13}{13}\right).$$

The answer to (c) is

$$\frac{1}{\binom{52}{13}} \cdot E_2 = \frac{1}{\binom{52}{13}} \cdot \left(\binom{4}{2}\binom{26}{13} - \binom{3}{1}\binom{4}{3}\binom{13}{13}\right).$$

Another Generalization of the Principle

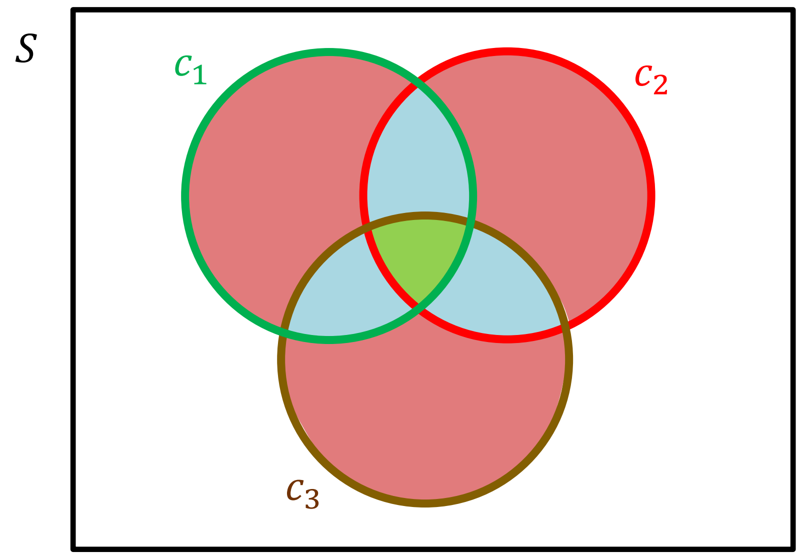
Let L_m denote **the number of elements of S that satisfy at least m of the t conditions**.

It is immediate that $E_m \leq L_m$. Moreover, $L_m \leq S_m$.

The following equalities hold.

a) $L_t = E_t = S_t$

b) $L_{m-1} = E_{m-1} + L_m$



Thus,

$$L_{t-1} = E_{t-1} + L_t$$
$$= S_{t-1} - \binom{t}{1} S_t + S_t$$
$$= S_{t-1} - \binom{t-1}{t-2} S_t.$$

$$L_{t-2} = E_{t-2} + L_{t-1} = S_{t-2} - \binom{t-1}{1} S_{t-1} + \binom{t}{2} S_t + S_{t-1} - \binom{t-1}{t-2} S_t$$
$$= S_{t-2} - \binom{t-2}{t-3} S_{t-1} + \binom{t-1}{t-3} S_t$$

Another Generalization of the Principle (ctd.)

Theorem. Assume that S is a finite set and that $c_1, c_2, \ldots, c_t$ are t conditions. Then, for each $0 \le m \le t$, the number of elements of S that satisfy ***at least*** m of the t conditions is given by

$$L_m = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} - \cdots + (-1)^{t-m} \binom{t-1}{m-1} S_t.$$

Proof. To the reader.

Example 3. Determine the number of functions f from $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ to $B = \{1, 2, 3, 4, 5, 6, 7\}$ such that $|f(A)| \le 3$.

Solution. Let S be the set of all functions f from A to B and define the conditions c_i as $i \notin f(A)$ where $i = 1, 2, \ldots, 7$. A function f with $|f(A)| \le 3$ satisfies at least 4 of the 7 conditions. Thus, we should calculate L_4. By the generalized principle, we have

$$\begin{aligned} L_4 &= S_4 - \binom{4}{3} S_5 + \binom{5}{3} S_6 - \binom{6}{3} S_7 \\ &= \binom{7}{4} 3^{10} - \binom{4}{3}\binom{7}{5} 2^{10} + \binom{5}{3}\binom{7}{6} 1^{10} - \binom{6}{3} \cdot 0 \\ &= 1{,}980{,}769 \end{aligned}$$

Derangement: Nothing Is in Its Right Place

Derangement is a permutation in which no object appears in its normal place. Equivalently, derangements of elements of a finite set A are functions $f: A \to A$ that have no fixed points. A fixed point of a function $f: A \to A$ is an element a of A such that $f(a) = a$.

The derangements of $1, 2, 3, 4$ are 2341, 2413, 2341, 2143, 4321, 4123, 4312, 3142, and 3412.

The number of derangements of n distinct objects is denoted by d_n. Let S be the set of all permutations of $1, 2, \ldots, n$. Assume also that c_i, $1 \leq i \leq n$, is the condition that i is in the i^{th} position (when considered from left in a row.) Then, $d_n = N(\bar{c}_1\bar{c}_2 \ldots \bar{c}_n)$. Thus,

$$d_n = n! - \binom{n}{1}(n-1)! + \binom{n}{1}(n-1)! - \cdots + (-1)^n(n-n)!$$

$$= n! - \frac{n!}{1!} + \frac{n!}{2!} - \cdots + (-1)^n\frac{n!}{n!}$$

$$= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n\frac{1}{n!}\right) = n!\sum_{i=0}^{n}\frac{(-1)^i}{i!}$$

The Maclaurin series for the exponential function is given by $e^x = \sum_{i=0}^{\infty}\frac{x^n}{n!}$, and thus, $e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots$. As a result, $d_n \approx n!\, e^{-1} = 0.36788 \cdot n!$. For example, $d_4 = 9 \approx 0.36788 \cdot 24 = 8.83$.

Derangement (Ctd.)

Example 4. a) In how many ways can the integers $1, 2, \dots, n$ be arranged in a line so that none of the patterns $12, 23, 34, \dots, (n-1)n$ occurs? b) Show that the result in part (a) equals $d_n + d_{n-1}$.

Solution. Let S be the set of all permutations of $1, 2, \dots, n$, and c_i, $1 \le i \le n-1$, be the condition that the pattern $i(i+1)$ occurs in the permutation. The answer to part (a) is thus $N(\bar{c}_1\bar{c}_2 \dots \bar{c}_{n-1})$, which is equal to

$$n! - \binom{n-1}{1}(n-1)! + \binom{n-1}{2}(n-2)! - \cdots + (-1)^{n-1}\binom{n-1}{n-1}1!.$$

The answer to part (a) can be written as

$$\begin{aligned}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}(n-k)! &= \sum_{k=0}^{n-1}(-1)^k\frac{(n-1)!\,(n-k)!}{k!\,(n-k-1)!}\\ &= \sum_{k=0}^{n-1}(-1)^k\frac{(n-1)!\,(n-k)}{k!} = \sum_{k=0}^{n-1}(-1)^k\frac{(n-1)!\,(n-k)}{k!}\\ &= \sum_{k=0}^{n-1}(-1)^k\frac{n!}{k!} + \sum_{k=1}^{n-1}(-1)^{k-1}\frac{(n-1)!}{(k-1)!}\\ &= d_n - (-1)^n + d_{n-1} - (-1)^{n-1} = d_n + d_{n-1}.\end{aligned}$$

Textbook: Ralph P. Grimaldi, Discrete and Combinatorial Mathematics

Do exercises of Chapter 8 as homework and upload your solutions via Moodle (follow the instructions on the page of the TA of this course.)