In the Name of God

Communication Systems (25751-4)

Problem Set 02

Fall Semester 1401-02

Department of Electrical Engineering

Sharif University of Technology

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Due on Aban 3, 1400 at 18:00

($*$) starred problems are optional and have a bonus mark!

1 The Hilbert Transform

Find the Hilbert transform of following signals:

1. $x_1(t) = 2\text{sinc}(2t)$
2. $x_2(t) = A\sin(2\pi f_0 t + \theta)$
3. $x_3(t) = \delta'(t)$
4. $x_4(t) = A\Pi(\frac{t}{T}) = A[u(t+\frac{T}{2}) - u(t-\frac{T}{2})]$
5. $x_5(t) = \frac{1}{a^2+t^2}$

2 Hilbert Transform Properties

1. Show that:
$$\mathcal{F}\{\widehat{\tfrac{d}{dt}x(t)}\} = 2\pi|f|\mathcal{F}\{x(t)\}.$$
2. Let $x(t)$ represent a bandpass signal and $m(t)$ denote a lowpass signal with nonoverlapping spectra. Show that the Hilbert transform of $c(t) = m(t)x(t)$ is equal to $\hat{c}(t) = m(t)\hat{x}(t)$.

3 Generalized Hilbert Transform

We have seen that the Hilbert transform introduces a 90 $^\circ$ phase shift in the components of a signal and the transfer function of a quadrature filter can be written as

$$H(f) = \begin{cases} e^{-j\frac{\pi}{2}} & f > 0 \\ 0 & f = 0 \\ e^{j\frac{\pi}{2}} & f < 0 \end{cases}$$

We can generalize this concept to a new transform that introduces a phase shift of θ in the frequency components of a signal, by introducing

$$H_\theta(f) = \begin{cases} e^{-j\theta} & f > 0 \\ 0 & f = 0 \\ e^{j\theta} & f < 0 \end{cases}$$

and denoting the result of this transform by $x_\theta(t)$, i.e., $X_\theta(f) = X(f)H_\theta(f)$, where $X_\theta(f)$ denotes the Fourier transform of $x_\theta(t)$. Throughout this problem, assume that the signal x(t) does not contain any DC components.

1. Find $h_\theta(t)$, the impulse response of the filter representing this transform.
2. Show that $x_\theta(t)$ is a linear combination of x(t) and its Hilbert transform.
3. Show that if x(t) is an energy-type signal, $x_\theta(t)$ will also be an energy-type signal and its energy content will be equal to the energy content of x(t).

4 Attenuation and Amplification

Consider the following system:

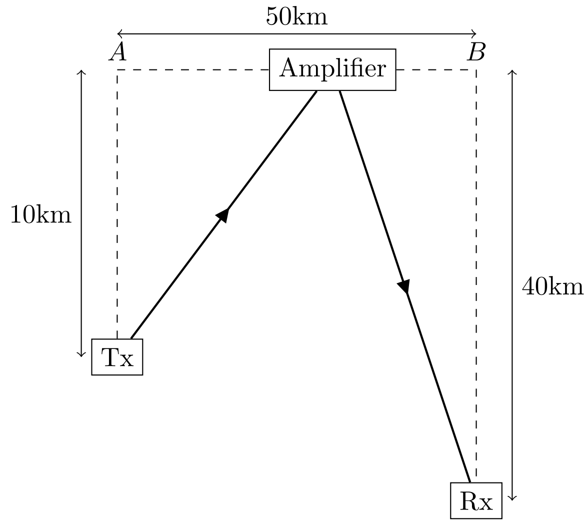

Figure 1: Problem 4

The transmitter sends a message with power $P_{\text{in}} = 5$ dBm. An amplifier amplifies this message on its way. The location of the amplifier must be on the line segment AB, but you can adjust it. Note that the cable connecting the transmitter to the amplifier, same as the cable connecting the amplifier to the receiver, is along the straight line that connects them.The minimum input power of the receiver is $P_{\text{out}} = 2$ dBm. Assume that the cable attenuation is $\alpha = 1\frac{\text{dB}}{\text{km}}$. Find the location and minimum gain of the amplifier in these cases:

1. The minimum input power of amplifier is -10 dBm.
2. The minimum input power of amplifier is -5 dBm.

5 RMS Bandwidth and RMS Duration

The root mean-square (RMS) bandwidth of a low-pass signal of finite energy is defined by:

$$W_{\text{RMS}} = \left[\frac{\int_{-\infty}^{\infty} f^2 \left|G(f)\right|^2 df}{\int_{-\infty}^{\infty} \left|G(f)\right|^2 df}\right]^{\frac{1}{2}}$$

where $\left|G(f)\right|^2$is the energy spectral density of the signal. Correspondingly, the root mean-square (RMS) duration of the signal is defined by:

$$T_{\text{RMS}} = \left[\frac{\int_{-\infty}^{\infty} t^2 \left|g(t)\right|^2 dt}{\int_{-\infty}^{\infty} \left|g(t)\right|^2 dt}\right]^{\frac{1}{2}}$$

1. Show that:

$$T_{\text{RMS}} W_{\text{RMS}} \geq \frac{1}{4\pi}.$$

   Assume that $|g(t)| \to 0$ faster than $\frac{1}{\sqrt{|t|}}$ as $|t| \to \infty$.

2. Consider a Gaussian pulse defined by:

$$g(t) = e^{-\pi t^2}.$$

   Show that, for this signal, the above inequality holds with equality:

$$T_{\text{RMS}} W_{\text{RMS}} = \frac{1}{4\pi}.$$

   Hint: Use Cauchy–Schwarz's inequality and the fact that for any complex number c, $c + c^* = 2\mathbf{Re}\{c\} \leq 2|c|$ to show that for any two complex functions $g_1(t), g_2(t)$, we have:

$$\left\{ \int_{-\infty}^{\infty} [g_1^*(t) g_2(t) + g_1(t) g_2^*(t)] dt \right\}^2 \leq 4 \int_{-\infty}^{\infty} |g_1(t)|^2 \, dt \int_{-\infty}^{\infty} |g_2(t)|^2 \, dt.$$

   and then set $g_1(t) = t g(t)$ and $g_2(t) = \frac{dg(t)}{dt}$.