Signals and Linear Systems

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Overview

Signals

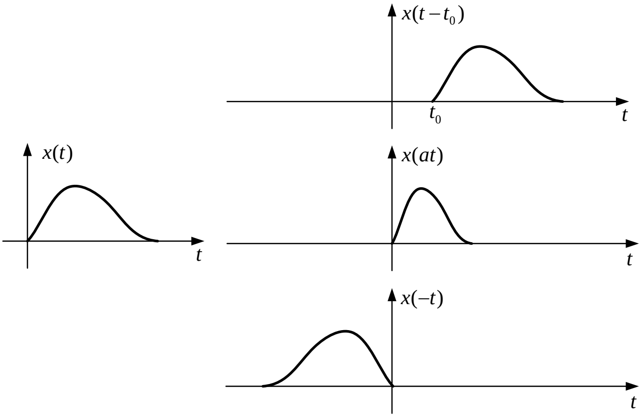

Figure: Time shifting, time scaling, time reversal.

$$x(t) \to x(t - t_0); \quad x(t) \to x(at); \quad x(t) \to x(-t)$$

Classification of Signals

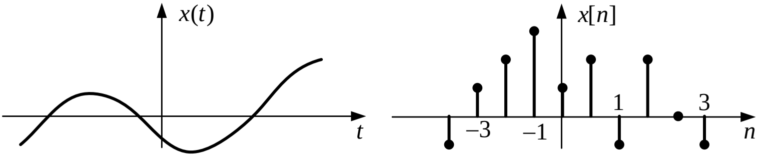

Figure: Continuous-time and discrete-time signals.

$$x(t), t \in \mathbb{R}; \quad x[n], n \in \mathbb{Z}$$

Classification of Signals

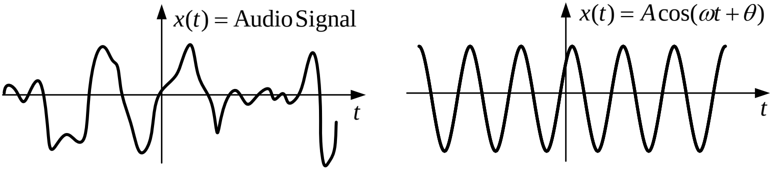

Figure: Random and deterministic signals.

$$x(t,\omega) \in \mathbb{R}, t \in \mathbb{R}, \omega \sim P[\Omega = \omega]; \quad x(t) \in \mathbb{R}, t \in \mathbb{R}$$

Classification of Signals

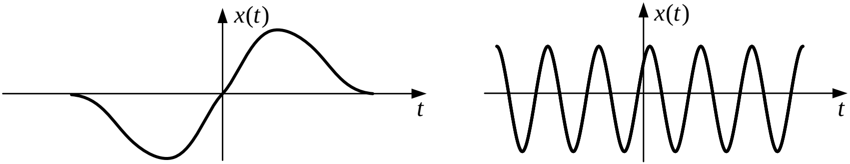

Figure: Nonperiodic and periodic signals.

$$\nexists T_0 : x(t + T_0) = x(t); \quad \exists T_0 : x(t + T_0) = x(t)$$

Classification of Signals

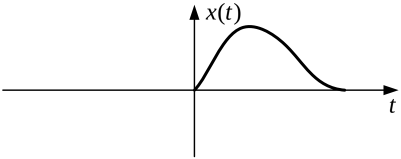

Figure: Causal and noncausal signals.

$$\forall t < 0 : x(t) = 0; \quad \exists t < 0 : x(t) \neq 0$$

Classification of Signals

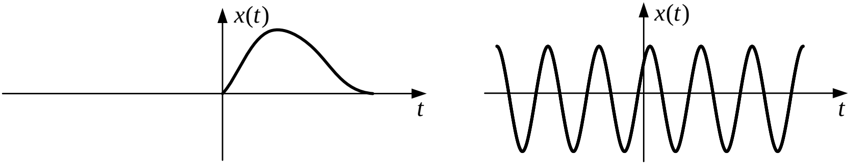

Figure: Energy and power signals.

$$0 < \mathcal{E}_x = \lim_{T\to\infty} \int_{-T/2}^{T/2} |x(t)|^2 dt < \infty; \quad 0 < \mathcal{P}_x = \lim_{T\to\infty} \frac{\int_{-T/2}^{T/2} |x(t)|^2 dt}{T} < \infty$$

Classification of Signals

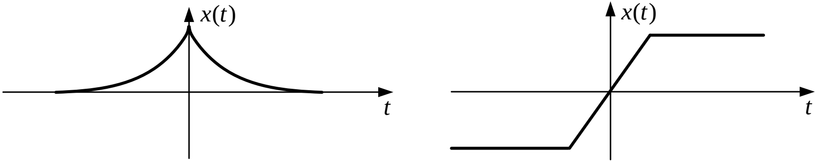

Figure: Even and odd signals.

$$x(t) = x(-t); \quad x(t) = -x(-t)$$

Statement (Even-Odd Decomposition)

Any signal $x(t)$ can be written as the sum of its even and odd parts as $x(t) = x_e(t) + x_o(t)$, where

$$x_e(t) = \frac{x(t) + x(-t)}{2}$$

$$x_o(t) = \frac{x(t) - x(-t)}{2}$$

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Classification of Signals

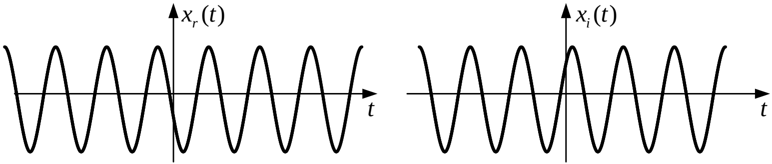

Figure: Real and complex signals.

$$x(t) \in \mathbb{R}; \quad x(t) \in \mathbb{C}$$

$$x_r(t) = A\cos(2\pi f_0 t + \theta); \quad x_i(t) = A\sin(2\pi f_0 t + \theta)$$

$$x(t) = \Re\{x(t)\} + j\Im\{x(t)\} = x_r(t) + jx_i(t)$$

Classification of Signals

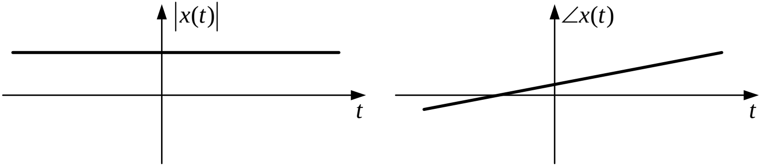

Figure: Real and complex signals.

$$x(t) \in \mathbb{R}; \quad x(t) \in \mathbb{C}$$

$$|x(t)| = |A|; \quad \angle x(t) = 2\pi f_0 t + \theta$$

$$x(t) = |x(t)| e^{j\angle x(t)}$$

Statement (Complex Signal Representation)

For the complex signal $x(t) = x_r(t) + jx_i(t) = \Re\{x(t)\} + j\Im\{x(t)\} = |x(t)|e^{j\angle x(t)}$,

$$x_r(t) = \Re\{x(t)\} = |x(t)|\cos(\angle x(t))$$

$$x_i(t) = \Im\{x(t)\} = |x(t)|\sin(\angle x(t))$$

$$|x(t)| = \sqrt{x_r^2(t) + x_i^2(t)}$$

$$\angle x(t) = \begin{cases} \tan^{-1}(\frac{x_i(t)}{x_r(t)}) & , \quad x_r(t) \geqslant 0, x_i(t) \geqslant 0 \\ \tan^{-1}(\frac{x_i(t)}{x_r(t)}) & , \quad x_r(t) \geqslant 0, x_i(t) < 0 \\ \pi + \tan^{-1}(\frac{x_i(t)}{x_r(t)}) & , \quad x_r(t) < 0, x_i(t) \geqslant 0 \\ -\pi + \tan^{-1}(\frac{x_i(t)}{x_r(t)}) & , \quad x_r(t) < 0, x_i(t) < 0 \end{cases}$$

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Some Important Signals

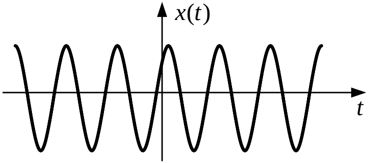


Figure: Sinusoidal signal.

$$x(t) = A\cos(2\pi f_0 t + \theta) = A\cos(2\pi t/T_0 + \theta)$$

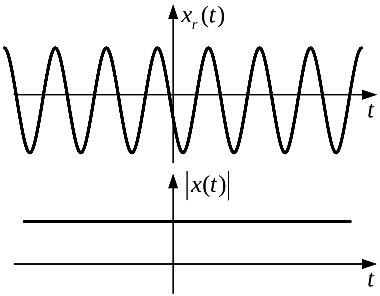


Figure: Complex exponential signal.

$$x(t) = A\cos(2\pi f_0 t + \theta) + jA\sin(2\pi f_0 t + \theta) = Ae^{j(2\pi f_0 t + \theta)}, \quad A \geqslant 0$$

Some Important Signals

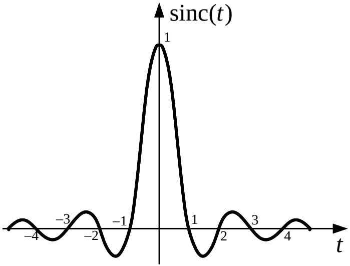


Figure: Sinusoidal signal.

$$\operatorname{sinc}(t) = \begin{cases} \frac{\sin(\pi t)}{\pi t}, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

Some Important Signals

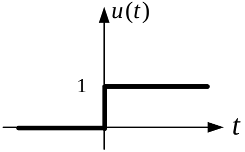


Figure: Step signal

$$u(t) = \begin{cases} 1, t \geq 0 \\ 0, t < 0 \end{cases} .$$

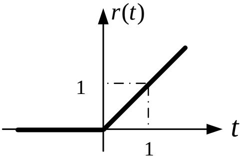


Figure: Ramp signal

$$r(t) = \begin{cases} t, t \geq 0 \\ 0, t < 0 \end{cases} .$$

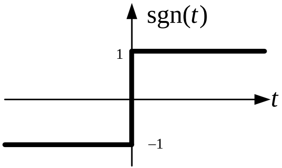


Figure: Sign signal

$$\text{sgn}(t) = \begin{cases} 1, t > 0 \\ 0, t = 0 \\ -1, t < 0 \end{cases} .$$

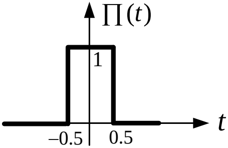
Some Important Signals

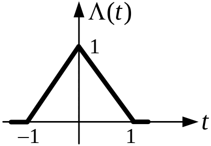
Example (Rectangular signal)

$$\sqcap(t) = u(t+0.5) - u(t-0.5)$$
$$= \begin{cases} 1, & |t| \leqslant 0.5 \\ 0, & |t| > 0.5 \end{cases}$$

Example (Triangle signal)

$$\Lambda(t) = r(t+1) - 2r(t) + r(t-1)$$
$$= \begin{cases} 1-|t|, & |t| \leqslant 1 \\ 0, & |t| > 1 \end{cases}$$





Some Important Signals

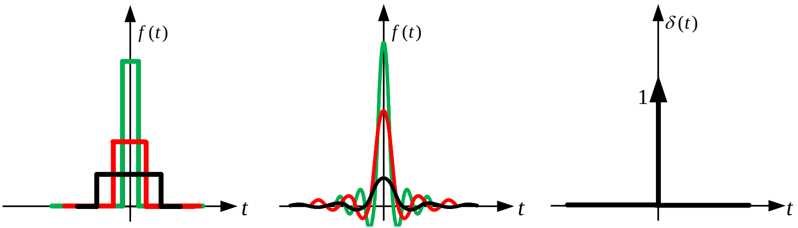


Figure: Unit impulse signal.

$$\delta(t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \sqcap \left(\frac{t}{\epsilon}\right) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \text{sinc}\left(\frac{t}{\epsilon}\right) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

Definition (Convolution)

The convolution of the functions $h(t)$ and $x(t)$ is defined as

$$y(t) = x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Definition (Test Function)

$x(t)$ is called a test function if it is infinitely differentiable and is zero outside a finite interval.

Singular Functions

Definition (Unit Impulse Signal)

The unit impulse function $u_0(t) = \delta(t)$ is defined as the function satisfying

$$\int_{-\infty}^{+\infty} \delta(t)x(t)dt = x(0)$$

for any test function $x(t)$.

Definition (Equal Singular Functions)

Two singular functions $y_1(t)$ and $y_2(t)$ are equal if and only if

$$\int_{-\infty}^{+\infty} y_1(t)x(t)dt = \int_{-\infty}^{+\infty} y_2(t)x(t)dt$$

for any test function $x(t)$.

Singular Functions

Theorem (Properties of Unit Impulse Signal)

The unit impulse function satisfies the following identities

$$
\begin{aligned}
\int_{-\infty}^{+\infty} \delta(t)dt &= 1 \\
\delta(at) &= \frac{1}{|a|}\delta(t), a \neq 0 \\
\delta(t) &= 0, t \neq 0 \\
t\delta(t) &= 0 \\
f(t)\delta(t) &= f(0)\delta(t) \\
f(t) &= \delta(t) * f(t)
\end{aligned}
$$

Example (Sampling property of $\delta(t)$)

If $f(t)$ is a continuous function at $f(0)$, then $f(t)\delta(t) = f(0)\delta(t)$.

$$\int_{-\infty}^{+\infty} \big[f(t)\delta(t)\big] x(t)dt = \int_{-\infty}^{+\infty} \delta(t)\big[f(t)x(t)\big]dt = f(0)x(0)$$

$$= f(0)\int_{-\infty}^{+\infty} \delta(t)x(t)dt = \int_{-\infty}^{+\infty} \big[f(0)\delta(t)\big] x(t)dt$$

Example (Area under $\delta(t)$)

The area under the unit impulse function is 1.

For the test function $x(t) = \frac{1}{t^2+1}$,

$$1 = x(0) = \int_{-\infty}^{+\infty} \delta(t)x(t)dt = \int_{-\infty}^{+\infty} \delta(t)x(0)dt = \int_{-\infty}^{+\infty} \delta(t)dt$$

Singular Functions

Definition (Unit Doublet Signal)

The unit doublet function $u_1(t) = \delta'(t)$ is defined as the function satisfying

$$\int_{-\infty}^{+\infty} \delta'(t)x(t)dt = -x'(0)$$

for any test function $x(t)$.

Definition (Higher-order Impulse Signals)

Generally, $u_n(t) = \delta^{(n)}(t), n \geq 0$ is defined as the function satisfying

$$\int_{-\infty}^{+\infty} \delta^{(n)}(t)x(t)dt = (-1)^n x^{(n)}(0)$$

for any test function $x(t)$.

Singular Functions

Theorem (Convolution with $u_n(t)$)

$u_n(t), n \geq 1$ *satisfies* $x^{(n)}(t) = u_n(t) * x(t)$.

For $n = 1$,

$$u_1(t) * x(t) = \int_{-\infty}^{+\infty} \delta'(\tau)x(t-\tau)d\tau = -\frac{dx(t-\tau)}{d\tau}|_{\tau=0} = x'(t)$$

Theorem (Relation of $\delta'(t)$ and $u_n(t)$)

$u_n(t), n \geq 2$ *relates to* $u_1(t) = \delta'(t)$ *as* $u_n(t) = \underbrace{u_1(t) * u_1(t) * \cdots * u_1(t)}_{n \text{ times}}$.

For $n = 2$,

$$\frac{d^2x(t)}{dt^2} = \frac{d}{dt}(\frac{dx(t)}{dt}) = \frac{d}{dt}(x(t) * u_1(t)) = x(t) * u_1(t) * u_1(t)$$

Singular Functions

Definition (Unit Step Signal)

The unit step function $u_{-1}(t) = u(t)$ is defined as the function satisfying

$$\int_{-\infty}^{+\infty} u(t)x(t)dt = \int_{0}^{+\infty} x(t)dt$$

for any test function $x(t)$.

Definition (Higher-order Step Signals)

Generally, $u_{-n}(t), n \geq 2$ is defined as

$$u_{-n}(t) = \underbrace{u_{-1}(t) * u_{-1}(t) * \cdots * u_{-1}(t)}_{n \text{ times}}$$

Singular Functions

Theorem (Explicit representation of $u_{-n}(t), n \geq 2$)

$u_{-n}(t), n \geq 2$ can be represented as

$$u_{-n}(t) = \frac{t^{n-1}}{(n-1)!} u_{-1}(t)$$

.

For $n = 2$,

$$u_{-2}(t) = u_{-1}(t) * u_{-1}(t) = u(t) * u(t) = tu(t) = r(t)$$

Singular Functions

Theorem (Generalized derivative of $u_n(t), n \in \mathbb{W}$)

Singular functions are related as

$$u_n'(t) = u_{n+1}(t)$$

.

For $n = -1$,

$$u'(t) = u_{-1}'(t) = u_0(t) = \delta(t)$$

For $n = 0$,

$$\delta'(t) = u_0'(t) = u_1(t) = \delta'(t)$$

Singular Functions

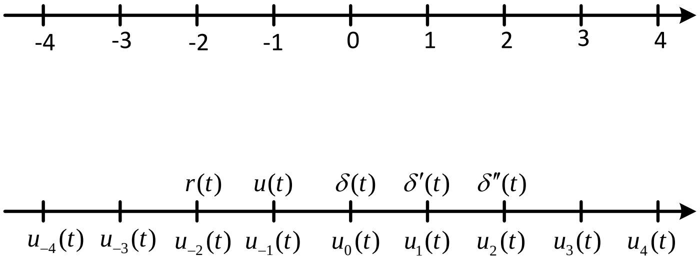
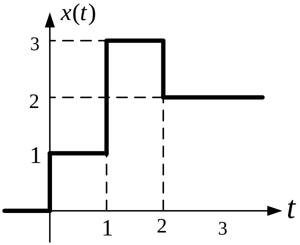


Figure: Singular functions.

Singular Functions

Example (Representation of other signals using the singular signals)

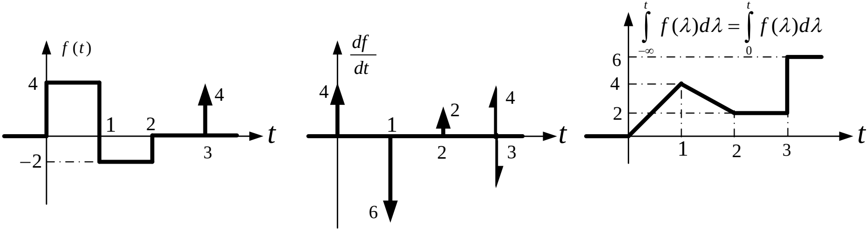
$x(t)$ can be represented by $u(t)$ and its shifted versions as $x(t) = u(t) + 2u(t-1) - u(t-2)$



$$x(t) = \left[u(t) - u(t-1)\right] + 3\left[u(t-1) - u(t-2)\right] + 2u(t-2)$$

Example (Derivative and integral of discontinuous functions)

Singular functions can be used in derivative and integral calculations.



$$f(t) = 4u(t) - 6u(t-1) + 2u(t-2) + 4\delta(t-3)$$

$$\frac{df(t)}{dt} = 4\delta(t) - 6\delta(t-1) + 2\delta(t-2) + 4\delta'(t-3)$$

$$\int_{-\infty}^{t} f(\lambda)d\lambda = 4tu(t) - 6(t-1)u(t-1) + 2(t-2)u(t-2) + 4u(t-3)$$

Example (Simplification using the properties of the singular functions)

$$\cos(t)\delta(t) = \cos(0)\delta(t) = \delta(t)$$

$$\cos(t)\delta(2t-3) = \cos(t)\delta(2(t-\frac{3}{2})) = \frac{1}{2}\delta(t-\frac{3}{2})\cos(t) = \frac{\cos(\frac{3}{2})}{2}\delta(t-\frac{3}{2})$$

$$\int_{-\infty}^{\infty} e^{-t}\delta'(t-1)dt = \int_{-\infty}^{\infty} e^{-u-1}\delta'(u)du = e^{-1}(-1)\frac{de^{-u}}{du}|_{u=0} = e^{-1}$$

Systems

Classification of Signals

Definition (System)

A system is an entity that is excited by an input signal $x(t)$ and, as a result of this excitation, produces an output signal $y(t)$. The output is uniquely defined for any legitimate input by

$$y(t) = \mathcal{T}\{x(t)\}$$

.

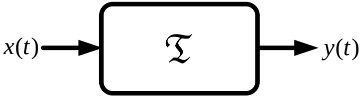


Figure: System block diagram.

Classification of Systems

Definition (Continuous-time System)

For a continuous-time system, both input and output signals are continuous-time signals.

Definition (Discrete-time System)

For a discrete-time system, both input and output signals are discrete-time signals.

Definition (Linear System)

A system $\mathcal{T}$ is linear if and only if, for any two input signals $x_1(t)$ and $x_2(t)$ and for any two scalars α and β, we have,

$$\mathcal{T}\{\alpha x_1(t) + \beta x_2(t)\} = \alpha \mathcal{T}\{x_1(t)\} + \beta \mathcal{T}\{x_2(t)\}$$

.

Definition (Nonlinear System)

A system is nonlinear if it is not linear.

Classification of Systems

Definition (Time-Invariant System)

A system is time-invariant if and only if, for all $x(t)$ and all values of t_0, its response to $x(t - t_0)$ is $y(t - t_0)$, where $y(t)$ is the response of the system to $x(t)$.

Definition (Time-variant System)

A system is time-variant if it is not time-invariant.

Classification of Systems

Definition (Causal System)

A system is causal if its output at any time t_0 depends on the input at times prior to t_0, i.e.,

$$y(t_0) = \mathcal{T}\{x(t) : t \leqslant t_0\}.$$

Definition (Noncausal System)

A system is noncausal if it is not causal.

Classification of Systems

Definition (Stable System)

A system is stable if its output is bounded for any bounded input, i.e.,

$$|x(t)| < B \Rightarrow |y(t)| < M.$$

Definition (Instable System)

A system is instable if it is not stable.

LTI Systems

Statement (Linear Time-Invariant System)

A system is Linear Time-Invariant (LTI) if it is simultaneously linear and time-invariant. An LTI system is completely characterized by its impulse response $h(t) = \mathcal{T}\{\delta(t)\}$.

$$
\begin{aligned}
y(t) &= \mathcal{T}\{x(t)\} \\
&= \mathcal{T}\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\} \\
&= \int_{-\infty}^{\infty} x(\tau)\mathcal{T}\{\delta(t-\tau)\}d\tau \\
&= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
&= x(t) * h(t)
\end{aligned}
$$

LTI System

Statement (Causality of LTI Systems)

An LTI system is causal if and only if $h(t) = 0, t < 0$.

Statement (Stability of LTI Systems)

An LTI system is stable if and only if $\int_{-\infty}^{+\infty} |h(t)| dt < \infty$.

Example (Complex exponential response)

The response of an LTI system $h(t)$ to the exponential input $x(t) = Ae^{j(2\pi f_0 t+\theta)}$ can be obtained by

$$y(t) = AH(f_0)e^{j(2\pi f_0 t+\theta)} = A|H(f_0)|e^{j(2\pi f_0 t+\theta+\angle H(f_0))}$$

, where

$$H(f_0) = |H(f_0)|e^{j\angle H(f_0)} = \int_{-\infty}^{\infty} h(\tau)e^{-j2\pi f_0 \tau} d\tau$$

.

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)Ae^{j(2\pi f_0(t-\tau)+\theta)} d\tau \\ &= Ae^{j(2\pi f_0 t+\theta)} \int_{-\infty}^{\infty} h(\tau)e^{-j2\pi f_0 \tau} d\tau \\ &= A|H(f_0)|e^{j(2\pi f_0 t+\theta+\angle H(f_0))} \end{aligned}$$

Fourier Series

Fourier Series and Its Properties

Definition (Fourier Series)

The periodic signal $x(t + T_0) = x(t)$ can be expanded in terms of the complex exponential $\{e^{j2\pi nt/T_0}\}_{n=-\infty}^{\infty}$ as

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{j2\pi nt/T_0}$$

, where

$$x_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-j2\pi nt/T_0} dt$$

.

Fourier Series and Its Properties

Dirichlet sufficient conditions for existence of the Fourier series are:

1. $x(t)$ is absolutely integrable over its period, i.e., $\int_0^{T_0} |x(t)|dt < \infty$.
2. The number of maxima and minima of $x(t)$ in each period is finite.
3. The number of discontinuities of $x(t)$ in each period is finite.

Fourier Series and Its Properties

1. The quantity $f_0 = 1/T_0$ is called the fundamental frequency of the signal $x(t)$.
2. The frequency of the nth complex exponential signal is nf_0, which is called the nth harmonic.
3. In general, $x_n = |x_n|e^{j\angle x_n}$, where $|x_n|$ gives the magnitude of the nth harmonic and $\angle x_n$ gives its phase.
4. For real signals $x(t) = x^*(t)$, $x_{-n} = x_n^*$.

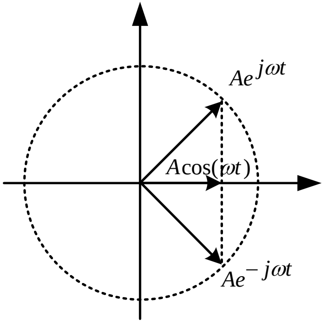
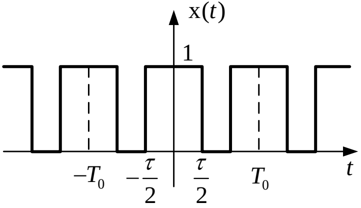


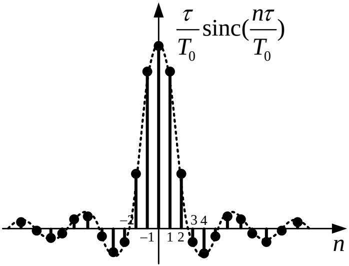
Figure: Positive and negative frequencies. The absolute value of the frequency shows the rate of rotation while its sign indicates the direction of rotation.

Fourier Series and Its Properties

Example (Fourier series of rectangular-pulse train)

$$x(t) = \sum_{n=-\infty}^{\infty} \sqcap\left(\frac{t - nT_0}{\tau}\right) = \sum_{n=-\infty}^{\infty} \frac{\tau}{T_0}\text{sinc}\left(\frac{n\tau}{T_0}\right)e^{jn2\pi t/T_0}$$





Fourier Series and Its Properties

Definition (Trigonometric Fourier Series)

The real periodic signal $x(t + T_0) = x(t)$ can be expanded as

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi nt/T_0) + \sum_{n=1}^{\infty} b_n \sin(2\pi nt/T_0)$$

, where

$$a_n = \frac{2}{T_0} \int_{T_0} x(t) \cos(2\pi nt/T_0) dt$$

and

$$b_n = \frac{2}{T_0} \int_{T_0} x(t) \sin(2\pi nt/T_0) dt$$

.

Fourier Series and Its Properties

1. $x_n = \frac{a_n}{2} - j\frac{b_n}{2}$.
2. For even real periodic signals, $b_n = 0$.
3. For odd real periodic signals, $a_n = 0$.

Example (Response of LTI Systems to Periodic Signals)

The response of an LTI system $h(t)$ to the periodic input $x(t+T_0) = x(t)$ can be obtained by

$$y(t) = \sum_{n=-\infty}^{\infty} x_n H(n/T_0) e^{j2\pi nt/T_0}$$

, where

$$H(f) = |H(f)| e^{j\angle H(f)} = \int_{-\infty}^{+\infty} h(t) e^{-j2\pi ft} dt.$$

$$y(t) = \mathcal{T}\{x(t)\} = \mathcal{T}\{\sum_{n=-\infty}^{\infty} x_n e^{j2\pi nt/T_0}\}$$

$$= \sum_{n=-\infty}^{\infty} x_n \mathcal{T}\{e^{j2\pi nt/T_0}\} = \sum_{n=-\infty}^{\infty} x_n H(n/T_0) e^{j2\pi nt/T_0}$$

Fourier Series and Its Properties

1. If the input to an LTI system is periodic with period T_0, then the output is also periodic with period T_0.
2. The output has a Fourier-series expansion given by $y(t) = \sum_{n=-\infty}^{\infty} y_n e^{\frac{j2\pi nt}{T_0}}$, where $y_n = x_n H(n/T_0)$.
3. An LTI system cannot introduce new frequency components in the output.

Statement (Rayleigh's Relation)

For a periodic signal $x(t+T_0) = x(t)$,

$$\mathcal{P}_x = \frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |x_n|^2$$

.

Fourier Transform

Fourier Transform

Definition (Fourier Transform)

If the Fourier transform of $x(t)$, defined by

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

exists, the original signal can be obtained from its Fourier transform by

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

.

Fourier Transform and Its Properties

Dirichlet sufficient conditions for existence of the Fourier transform are:

1. $x(t)$ is absolutely integrable over the real line, i.e., $\int_{-\infty}^{\infty} |x(t)| dt < \infty$.
2. The number of maxima and minima of $x(t)$ in any finite real interval is finite.
3. The number of discontinuities of $x(t)$ in any finite real interval is finite.

1. $X(f)$ is generally a complex function. Its magnitude $|X(f)|$ and phase $\angle X(f)$ represent the amplitude and phase of various frequency components in $x(t)$.
2. The function $X(f)$ is sometimes referred to as the spectrum of the signal $x(t)$.
3. To denote that $X(f)$ is the Fourier transform of $x(t)$, we frequently employ the notations $X(f) = \mathcal{F}\{x(t)\}$, $x(t) = \mathcal{F}^{-1}\{X(f)\}$, or $x(t) \leftrightarrow X(f)$.

1. For real signals $x(t) = x^*(t)$,

$$
\begin{aligned}
X(-f) &= X^*(f) \\
\Re[X(-f)] &= \Re[X(f)] \\
\Im[X(-f)] &= -\Im[X(f)] \\
|X(-f)| &= |X(f)| \\
\angle X(-f) &= -\angle X(f)
\end{aligned}
$$

2. If $x(t)$ is real and even, $X(f)$ will be real and even.
3. If $x(t)$ is real and odd, $X(f)$ will be imaginary and odd.

Statement (Signal Bandwidth)

We define the bandwidth of a real signal $x(t)$ as the range of positive frequencies contributing strongly in the spectrum of the signal.

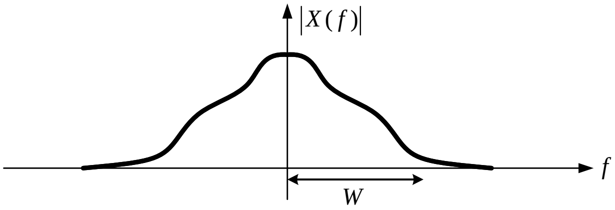
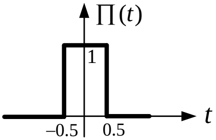


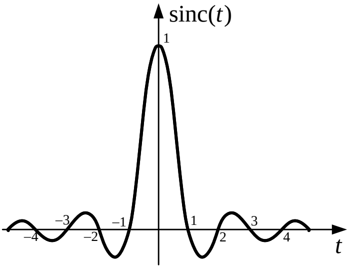
Figure: Bandwidth of a real signal.

Fourier Transform and Its Properties

Example (Fourier transform of $\sqcap(t)$)

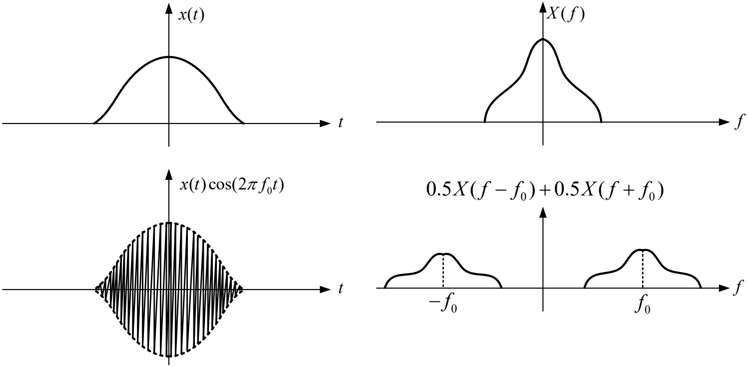
$$\mathcal{F}\{\sqcap(t)\} = \int_{-\infty}^{+\infty} \sqcap(t) e^{-j2\pi ft} dt = \int_{-0.5}^{0.5} e^{-j2\pi ft} dt = \frac{\sin(\pi f)}{\pi f} = \text{sinc}(f)$$





Example (Modulation Property)

$$x(t)\cos(2\pi f_0 t) \leftrightarrow \frac{1}{2}[X(f-f_0)+X(f+f_0)]$$



Fourier Transform and Its Properties

Property	Signal	Fourier
Assumption	$x(t)$	$X(f)$
Assumption	$y(t)$	$Y(f)$
Linearity	$ax(t)+by(t)$	$aX(f)+bY(f)$
Time Shifting	$x(t-t_0)$	$e^{-j2\pi f t_0}X(f)$
Frequency Shifting	$e^{j2\pi f_0 t}x(t)$	$X(f-f_0)$
Time Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X(\frac{f}{a})$
Conjugation	$x^*(t)$	$X^*(-f)$
Convolution	$x(t)*y(t)$	$X(f)Y(f)$
Modulation	$x(t)y(t)$	$X(f)*Y(f)$
Sinusoidal Modulation	$x(t)\cos(2\pi f_0 t)$	$\frac{1}{2}[X(f-f_0)+X(f+f_0)]$
Auto-correlation	$x(t)*x^*(-t)$	$\lvert X(f)\rvert^2$
Time Differentiation	$\frac{dx(t)}{dt}$	$j2\pi f X(f)$
Time Differentiation	$\frac{d^n x(t)}{dt^n}$	$(j2\pi f)^n X(f)$
Frequency Differentiation	$t^n x(t)$	$(\frac{j}{2\pi})^n \frac{d^n X(f)}{df^n}$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{X(f)}{j2\pi f}+\frac{1}{2}X(0)\delta(f)$
Duality	$X(t)$	$x(-f)$
Periodicity	$\sum_{n=-\infty}^{\infty} x_n e^{j2\pi nt/T_0}$	$\sum_{n=-\infty}^{\infty} x_n \delta(f-n/T_0)$

Table: Properties of the Fourier transform.

Fourier Transform and Its Properties

Signal	Fourier
$\delta(t)$	1
1	$\delta(f)$
$\delta(t-t_0)$	$e^{-j2\pi f t_0}$
$\delta^n(t)$	$(j2\pi f)^n$
$e^{j2\pi f_0 t}$	$\delta(f-f_0)$
$\mathrm{sgn}(t)$	$\frac{1}{j\pi f}$
$\frac{1}{t}$	$-j\pi \mathrm{sgn}(f)$
$u(t)$	$\frac{1}{j2\pi f}+\frac{1}{2}\delta(f)$
$\cos(2\pi f_0 t)$	$\frac{1}{2}[\delta(f-f_0)+\delta(f+f_0)]$
$\sin(2\pi f_0 t)$	$\frac{1}{2j}[\delta(f-f_0)-\delta(f+f_0)]$
$\sqcap(t)$	$\mathrm{sinc}(f)$
$\mathrm{sinc}(t)$	$\sqcap(f)$
$\Lambda(t)$	$\mathrm{sinc}^2(f)$
$\mathrm{sinc}^2(t)$	$\Lambda(f)$
$e^{-at}u(t), a>0$	$\frac{1}{j2\pi f+a}$
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t), a>0$	$\frac{1}{(j2\pi f+a)^n}$
$\sum_{n=-\infty}^{\infty}\delta(t-nT_0)$	$\frac{1}{T_0}\sum_{n=-\infty}^{\infty}\delta(f-n/T_0)$

Table: Fourier transform of elementary functions.

Fourier Transform and Its Properties

Statement (Parseval's Relation)

If the Fourier transforms of the signals $x(t)$ and $y(t)$ are denoted by $X(f)$ and $Y(f)$, respectively, then

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df$$

.

Statement (Rayleigh's Relation)

If the Fourier transforms of the signals $x(t)$ is denoted by $X(f)$, then

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

.

Example (LTI Systems)

The output of an LTI system is represented by the convolution integral

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

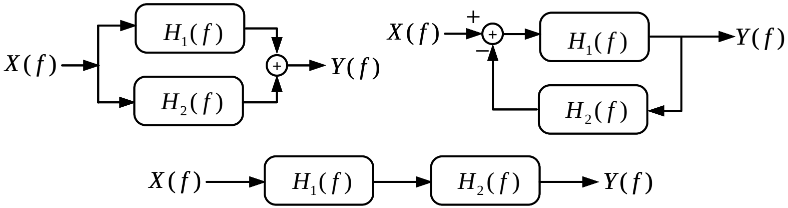
, where $h(t)$ is the impulse response of the LTI system. In the frequency domain,

$$Y(f) = H(f)X(f)$$

, where the frequency response $H(f)$ is the Fourier transform of the impulse response $h(t)$.

Example (Interconnection of LTI systems)

The overall frequency response $H(f)$ of the parallel, feedback, and series interconnection of the LTI systems $H_1(f)$ and $H_2(f)$ is $H_1(f) + H_2(f)$, $H_1(f)/(1 + H_1(f)H_2(f))$, and $H_1(f)H_2(f)$, respectively.



Power and Energy

Power and Energy

Definition (Energy Signal)

The signal $x(t)$ is energy-type if its energy content is nonzero and limited, i.e.,

$$0 < \mathcal{E}_x = \int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$$

.

Definition (Power Signal)

The signal $x(t)$ is power-type if its power content is nonzero and limited, i.e.,

$$0 < \mathcal{P}_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt < \infty$$

.

Power and Energy

1. A signal cannot be both power- and energy-type because $\mathcal{P}_x = 0$ for energy-type signals, and $\mathcal{E}_x = \infty$ for power-type signals.
2. A signal can be neither energy-type nor power-type, e.g., $x(t) = t^2$.

Energy-Type Signals

Definition (Autocorrelation)

For an energy-type signal $x(t)$, we define the autocorrelation function

$$R_x(\tau) = x(\tau) * x^*(-\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt = \int_{-\infty}^{\infty} x(t+\tau)x^*(t)dt$$

.

Energy-Type Signals

1. $\mathcal{F}\{R_x(\tau)\} = |X(f)|^2 = \mathcal{E}_x(f)$, where $\mathcal{E}_x(f)$ is called the energy spectral density of a signal $x(t)$.
2. $\mathcal{E}_x = R_x(0) = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} \mathcal{E}_x(f) df$.
3. If we pass the signal $x(t)$ through an LTI system with the impulse response $h(t)$ and frequency response $H(f)$,

$$\begin{aligned} R_y(\tau) &= \mathcal{F}^{-1}\{|Y(f)|^2\} \\ &= \mathcal{F}^{-1}\{|X(f)|^2 |H(f)|^2\} \\ &= \mathcal{F}^{-1}\{|X(f)|^2\} * \mathcal{F}^{-1}\{|H(f)|^2\} = R_x(\tau) * R_h(\tau) \end{aligned}$$

Energy-Type Signals

Example (Energy of rectangular pulse)

The energy content of $x(t) = A \sqcap \left(\frac{t}{T}\right)$ is $\mathcal{E}_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-T/2}^{T/2} A^2 dt = A^2 T$.

Example (Energy spectral density of rectangular pulse)

The energy spectral density of $x(t) = A \sqcap \left(\frac{t}{T}\right)$ is $\mathcal{E}_x(f) = \left|\mathcal{F}\{A \sqcap \left(\frac{t}{T}\right)\}\right|^2 = T^2 A^2 \text{sinc}^2(Tf)$.

Example (Autocorrelation of rectangular pulse)

The autocorrelation of $x(t) = A \sqcap \left(\frac{t}{T}\right)$ is $\mathcal{R}_x(\tau) = \mathcal{F}^{-1}\{\mathcal{E}_x(f)\} = A^2 T \Lambda\left(\frac{\tau}{T}\right)$.

Power-Type Signals

Definition (Time-Average Autocorrelation)

For a power-type signal $x(t)$, we define the time-average autocorrelation function

$$R_x(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x^*(t-\tau)dt$$

.

Power-Type Signals

1. $\mathcal{S}_x(f) = \mathcal{F}\{R_x(\tau)\}$ is called power-spectral density or the power spectrum of the signal $x(t)$.
2. $\mathcal{P}_x = R_x(0) = \int_{-\infty}^{\infty} \mathcal{S}_x(f)df$.
3. If we pass the signal $x(t)$ through an LTI system with the impulse response $h(t)$ and frequency response $H(f)$,
   $R_y(\tau) = R_x(\tau) * h(\tau) * h^*(-\tau)$ and $S_y(f) = S_x(f)|H(f)|^2$.

Power-Type Signals

Example (Power of periodic signals)

Any periodic signal $x(t) = x(t + T_0)$ is a power-type signal and its power content equals the average power in one period as

$$\begin{aligned}\mathcal{P}_x &= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \lim_{n\to\infty} \frac{1}{nT_0}\int_{-nT_0/2}^{nT_0/2} |x(t)|^2 dt \\ &= \lim_{n\to\infty} \frac{n}{nT_0}\int_{-T_0/2}^{T_0/2} |x(t)|^2 dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} |x(t)|^2 dt\end{aligned}$$

Example (Power of cosine)

The power content of $x(t) = A\cos(2\pi f_0 t + \theta)$ is

$$\mathcal{P}_x = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} A^2\cos^2(2\pi f_0 t + \theta) dt = \frac{A^2}{2}$$

Power-Type Signals

Example (Time-average autocorrelation of periodic signals)

Let the signal $x(t)$ be a periodic signal with the period T_0. Then,

$$R_x(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)x^*(t-\tau)dt$$

.

$$\begin{aligned}
R_x(\tau) &= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)x^*(t-\tau)dt \\
&= \lim_{k\to\infty} \frac{1}{kT_0}\int_{-kT_0/2}^{kT_0/2} x(t)x^*(t-\tau)dt \\
&= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)x^*(t-\tau)dt
\end{aligned}$$

Power-Type Signals

Example (Time-average autocorrelation of periodic signals)

Let the signal $x(t)$ be a periodic signal with the period T_0 and have the Fourier-series coefficients x_n. Then, $R_x(\tau) = \sum_{n=-\infty}^{\infty} |x_n|^2 e^{j2\pi n\tau/T_0}$.

$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} e^{j2\pi(n-m)t/T_0} dt = \delta_{nm}$, which is nonzeros when $n = m$. So,

$$
\begin{aligned}
R_x(\tau) &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)x^*(t-\tau)dt \\
&= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x_n x_m^* e^{j2\pi m\tau/T_0} e^{j2\pi(n-m)t/T_0} dt \\
&= \sum_{n=-\infty}^{\infty} |x_n|^2 e^{j2\pi n\tau/T_0}
\end{aligned}
$$

Hilbert Transform

Hilbert Transform

Definition (Hilbert Transform)

The Hilbert transform of the signal $x(t)$ is a signal $\hat{x}(t)$ whose frequency components lag the frequency components of $x(t)$ by 90°.

1. A delay of $\pi/2$ for $e^{j2\pi f_0 t}$ results in $e^{j(2\pi f_0 t-\pi/2)} = -je^{j2\pi f_0 t}$.
2. A delay of $\pi/2$ for $e^{-j2\pi f_0 t}$ results in $e^{-j(2\pi f_0 t-\pi/2)} = je^{-j2\pi f_0 t}$.

Statement (Hilbert Transform)

Assume that $x(t)$ is real and has no DC component, i.e., $X(0) = 0$. Then,

$$\mathcal{F}\{\hat{x}(t)\} = -jsgn(f)X(f)$$

and

$$\hat{x}(t) = \frac{1}{\pi t} * x(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau$$

Hilbert Transform

1. The Hilbert transform of an even real signal is odd, and the Hilbert transform of an odd real signal is even.
2. Applying the Hilbert-transform operation to a signal twice causes a sign reversal of the signal, i.e., $\hat{\hat{x}}(t) = -x(t)$.
3. Energy content of a signal is equal to the energy content of its Hilbert transform, i.e., $\mathcal{E}_x = \mathcal{E}_{\hat{x}}$.
4. The signal $x(t)$ and its Hilbert transform are orthogonal, i.e.,

$$\int_{-\infty}^{\infty} x(t)\hat{x}(t)dt = 0$$

.

Hilbert Transform

Example (Hilbert transform of a cosine)

$$x(t) = A\cos(2\pi f_0 t + \theta) \leftrightarrow \frac{A}{2}e^{j\theta}\delta(f - f_0) + \frac{A}{2}e^{-j\theta}\delta(f + f_0)$$

$$\hat{x}(t) \leftrightarrow -j\text{sgn}(f)\left[\frac{A}{2}e^{j\theta}\delta(f - f_0) + \frac{A}{2}e^{-j\theta}\delta(f + f_0)\right]$$

$$\hat{x}(t) \leftrightarrow \frac{A}{2j}e^{j\theta}\delta(f - f_0) - \frac{A}{2j}e^{-j\theta}\delta(f + f_0)$$

$$\hat{x}(t) = A\sin(2\pi f_0 t + \theta) \leftrightarrow \frac{A}{2j}e^{j\theta}\delta(f - f_0) - \frac{A}{2j}e^{-j\theta}\delta(f + f_0)$$

Hilbert Transform

Example (Energy of a signal and its Hilbert transform)

$$\mathcal{E}_{\hat{x}} = \int_{-\infty}^{\infty} |\hat{x}(t)|^2 dt = \int_{-\infty}^{\infty} |\mathcal{F}\{\hat{x}(t)\}|^2 df$$
$$= \int_{-\infty}^{\infty} |-j\mathrm{sgn}(f)X(f)|^2 df = \int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} |x(t)|^2 dt = \mathcal{E}_x$$

Example (Orthogonality of a signal and its Hilbert transform)

$$\int_{-\infty}^{\infty} \hat{x}(t)x(t)dt = \int_{-\infty}^{\infty} \hat{x}(t)\left[x^*(t)\right]^* dt =$$
$$\int_{-\infty}^{\infty} -j\mathrm{sgn}(f)X(f)\left[X^*(-f)\right]^* df = \int_{-\infty}^{\infty} -j\mathrm{sgn}(f)X(f)X(-f)df = 0$$

Lowpass and Bandpass Signals

Lowpass and Bandpass Signals

Definition (Lowpass Signal)

A lowpass signal is a signal, whose spectrum is located around the zero frequency.

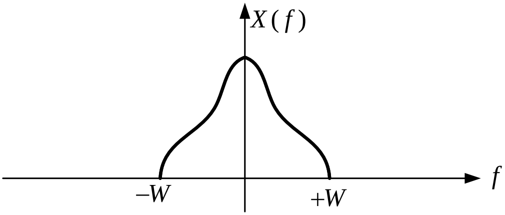


Figure: Spectrum of a lowpass signal.

Definition (Bandpass Signal)

A bandpass signal is a signal with a spectrum far from the zero frequency.

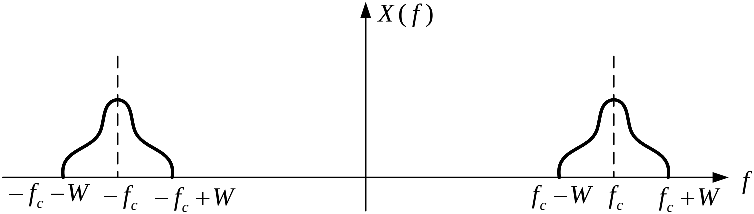


Figure: Spectrum of a bandpass signal.

Lowpass and Bandpass Signals

1. The spectrum of a bandpass signal is usually located around a center frequency f_c, which is much higher than the bandwidth of the signal.
2. The extreme case of a bandpass signal is $x(t) = A\cos(2\pi f_c t + \theta)$, which can be represented by a phasor $x_l = Ae^{j\theta} = x_c + jx_s$, where A, θ, x_c, and x_s are called envelope, phase, in-phase component, and quadrature component, respectively.
3. The original signal $x(t)$ can be reconstructed from its phasor as $x(t) = A\cos(2\pi f_c t + \theta) = x_c\cos(2\pi f_c t) - x_s\sin(2\pi f_c t)$.

Lowpass and Bandpass Signals

Statement (Slowly-varying Lowpass Phasor)

Assume that we have a slowly-varying lowpass phasor $x_l(t) = A(t)e^{j\theta(t)} = x_c(t) + jx_s(t)$, *where* $A(t) \geq 0$, $\theta(t)$, $x_s(t)$, *and* $x_c(t)$ *are slowly-varying signals compared to* f_c. *The real bandpass signal* $x(t) = A(t)\cos(2\pi f_c t + \theta(t))$ *relates to the complex time-varying phasor* $x_l(t)$ *as*

$$
\begin{aligned}
x(t) &= \Re\{x_l(t)e^{j2\pi f_c t}\} = \Re\{A(t)e^{j(2\pi f_c t + \theta(t))}\} \\
&= x_c(t)\cos(2\pi f_c t) - x_s(t)\sin(2\pi f_c t)
\end{aligned}
$$

.

Lowpass and Bandpass Signals

1. $x_l(t) = A(t)e^{j\theta(t)} = x_c(t) + jx_s(t)$ is is called the lowpass equivalent of the bandpass signal $x(t) = A(t)\cos(2\pi f_c t + \theta(t))$.
2. The envelope $|x_l(t)|$ and the phase $\angle x_l(t)$ of the bandpass signal are defined as
$$|x_l(t)| = A(t) = \sqrt{x_c^2(t) + x_s^2(t)}$$
and roughly,
$$\angle x_l(t) = \theta(t) = \tan^{-1}(\frac{x_s(t)}{x_c(t)})$$
3. Obviously, the in-phase and quadrature components satisfy
$$x_c(t) = A(t)\cos(\theta(t))$$
and
$$x_s(t) = A(t)\sin(\theta(t))$$

Lowpass and Bandpass Signals

Example (Spectrum of the bandpass signal)

$$x(t) = \Re\{x_l(t)e^{j2\pi f_c t}\} = \frac{1}{2}\left[x_l(t)e^{j2\pi f_c t} + x_l^*(t)e^{-j2\pi f_c t}\right]$$

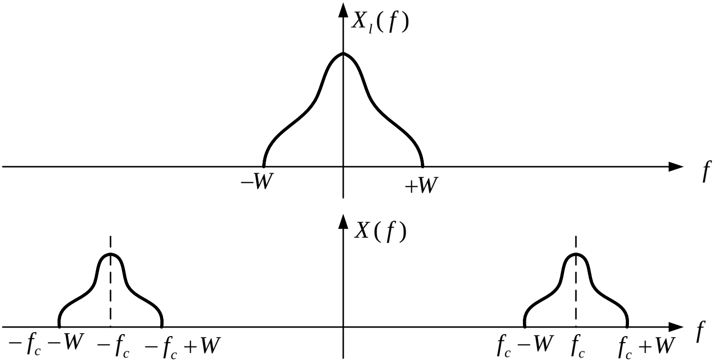
So,

$$X(f) = \frac{1}{2}X_l(f - f_c) + \frac{1}{2}X_l^*(-(f + f_c))$$

Lowpass and Bandpass Signals

Example (Spectrum of the bandpass signal)

$$X(f) = \frac{1}{2}X_l(f - f_c) + \frac{1}{2}X_l^*(-(f + f_c))$$



Example (Spectrum of the lowpass signal)

If the bandwidth of the bandpass signal W is much less than the central frequency f_c, then

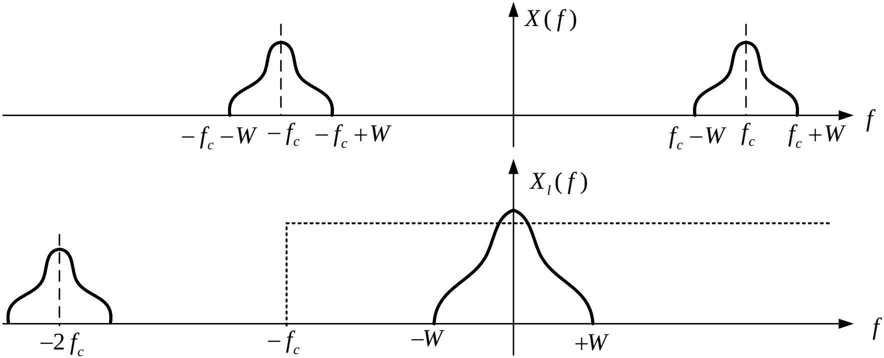
$$X(f) = \frac{1}{2}X_l(f - f_c) + \frac{1}{2}X_l^*(-(f + f_c))$$
$$X(f + f_c) = \frac{1}{2}X_l(f) + \frac{1}{2}X_l^*(-(f + 2f_c))$$
$$X(f + f_c)u(f + f_c) = \frac{1}{2}X_l(f)u(f + f_c) + \frac{1}{2}X_l^*(-(f + 2f_c))u(f + f_c)$$
$$X(f + f_c)u(f + f_c) = \frac{1}{2}X_l(f)$$
$$2X(f + f_c)u(f + f_c) = X_l(f)$$

Lowpass and Bandpass Signals

Example (Spectrum of the lowpass signal)

If the bandwidth of the bandpass signal W is much less than the central frequency f_c, then

$$X_l(f) = 2X(f + f_c)u(f + f_c)$$



Lowpass and Bandpass Signals

Example (Lowpass equivalent of a bandpass signal)

$$\begin{aligned} X_l(f) &= 2X(f+f_c)u(f+f_c) \\ &= 2X(f+f_c)\frac{1+\mathrm{sgn}(f+f_c)}{2} \\ &= 2X(f+f_c)\frac{1-j^2\mathrm{sgn}(f+f_c)}{2} \\ &= X(f+f_c)+j\big[-j\mathrm{sgn}(f+f_c)X(f+f_c)\big] \end{aligned}$$

So,

$$x_l(t) = \big[x(t)+j\hat{x}(t)\big]e^{-j2\pi f_c t}$$

Lowpass and Bandpass Signals

Example (In-phase component of a bandpass signal)

$$x_l(t) = \left[x(t) + j\hat{x}(t)\right] e^{-j2\pi f_c t}$$

So,

$$x_l(t) = \left[x(t) + j\hat{x}(t)\right]\left[\cos(2\pi f_c t) - j\sin(2\pi f_c t)\right]$$

$$x_l(t) = x(t)\cos(2\pi f_c t) + \hat{x}(t)\sin(2\pi f_c t) + j\left[\hat{x}(t)\cos(2\pi f_c t) - x(t)\sin(2\pi f_c t)\right]$$

and,

$$\Re\{x_l(t)\} = x_c(t) = x(t)\cos(2\pi f_c t) + \hat{x}(t)\sin(2\pi f_c t)$$

Example (Quadrature component of a bandpass signal)

$$x_l(t) = \left[x(t) + j\hat{x}(t)\right] e^{-j2\pi f_c t}$$

So,

$$x_l(t) = \left[x(t) + j\hat{x}(t)\right]\left[\cos(2\pi f_c t) - j\sin(2\pi f_c t)\right]$$

$$x_l(t) = x(t)\cos(2\pi f_c t) + \hat{x}(t)\sin(2\pi f_c t) + j\left[\hat{x}(t)\cos(2\pi f_c t) - x(t)\sin(2\pi f_c t)\right]$$

and,

$$\Im\{x_l(t)\} = x_s(t) = \hat{x}(t)\cos(2\pi f_c t) - x(t)\sin(2\pi f_c t)$$

Example (Envelope of a bandpass signal)

$$x_l(t) = \left[x(t) + j\hat{x}(t)\right] e^{-j2\pi f_c t}$$

So,

$$|x_l(t)| = A(t) = \sqrt{x^2(t) + \hat{x}^2(t)}$$

Example (Phase of a bandpass signal)

$$x_l(t) = \left[x(t) + j\hat{x}(t)\right] e^{-j2\pi f_c t}$$

So,

$$x_l(t) = \left[x(t) + j\hat{x}(t)\right]\left[\cos(2\pi f_c t) - j\sin(2\pi f_c t)\right]$$

$$x_l(t) = x(t)\cos(2\pi f_c t) + \hat{x}(t)\sin(2\pi f_c t) + j\left[\hat{x}(t)\cos(2\pi f_c t) - x(t)\sin(2\pi f_c t)\right]$$

and roughly,

$$\angle x_l(t) = \theta(t) = \tan^{-1}\left[\frac{\hat{x}(t)\cos(2\pi f_c t) - x(t)\sin(2\pi f_c t)}{x(t)\cos(2\pi f_c t) + \hat{x}(t)\sin(2\pi f_c t)}\right]$$

Lowpass and Bandpass Signals

Example (Lowpass equivalent of sinusoidal signal)

Lowpass equivalent of the bandpass signal $x(t) = A\cos(2\pi f_c t + \theta)$ is

$$\begin{aligned} x_l(t) &= \left[x(t) + j\hat{x}(t)\right] e^{-j2\pi f_c t} \\ &= \left[A\cos(2\pi f_c t + \theta) + jA\sin(2\pi f_c t + \theta)\right] e^{-j2\pi f_c t} \\ &= Ae^{j(2\pi f_c t + \theta)} e^{-j2\pi f_c t} = Ae^{j\theta} \end{aligned}$$

So, $A(t) = |A|$, $\theta(t) = \theta + u(-A)\pi$, $x_s(t) = A\cos(\theta)$, and $x_s(t) = A\sin(\theta)$.

Lowpass and Bandpass Signals

Example (Lowpass equivalent of sinusoidal signal)

Lowpass equivalent of the bandpass signal $x(t) = \text{sinc}(t)\cos(2\pi f_c t + \frac{\pi}{4})$ can be obtained as

$$x(t) = \text{sinc}(t)\cos(\frac{\pi}{4})\cos(2\pi f_c t) - \text{sinc}(t)\sin(\frac{\pi}{4})\sin(2\pi f_c t)$$

$$x_c(t) = \frac{\sqrt{2}}{2}\text{sinc}(t), \quad x_s(t) = \frac{\sqrt{2}}{2}\text{sinc}(t)$$

$$x_l(t) = x_c(t) + jx_s(t) = \frac{\sqrt{2}}{2}\text{sinc}(t)(1+j) = \text{sinc}(t)e^{j\frac{\pi}{4}}$$

Filters

Lowpass Filter

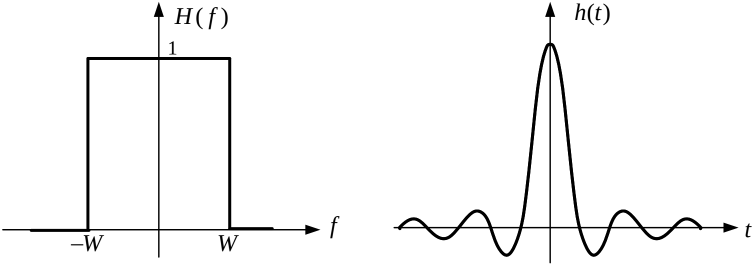


Figure: Ideal LPF frequency response and its impulse response.

$$H(f) = \sqcap(\frac{f}{2W}) \longleftrightarrow h(t) = 2W\,\mathrm{sinc}(2Wt)$$

Lowpass Filter

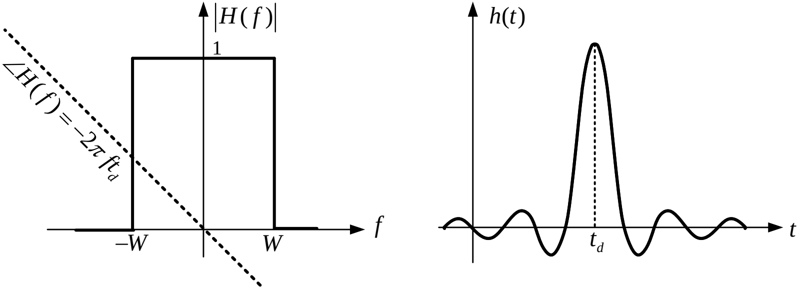


Figure: Linear-phase ideal LPF frequency response and its impulse response.

$$H(f) = \sqcap(\frac{f}{2W})e^{-j2\pi f t_d} \longleftrightarrow h(t) = 2W\,\mathrm{sinc}(2W(t - t_d))$$

Lowpass Filter

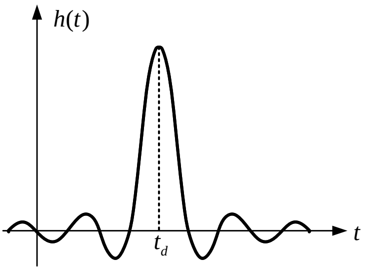


Figure: Truncated LPF impulse response.

$$h(t) = 2W \operatorname{sinc}(2W(t - t_d)) \qquad h_T(t) = 2W \operatorname{sinc}(2W(t - t_d))u(t)$$

Lowpass Filter

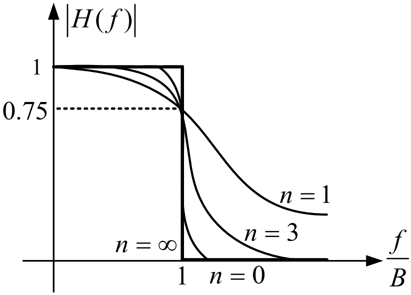


Figure: Butterworth LPF frequency characteristic.

$$|H(f)| = \frac{1}{\sqrt{1+(\frac{f}{B})^{2n}}}$$

Lowpass Filter

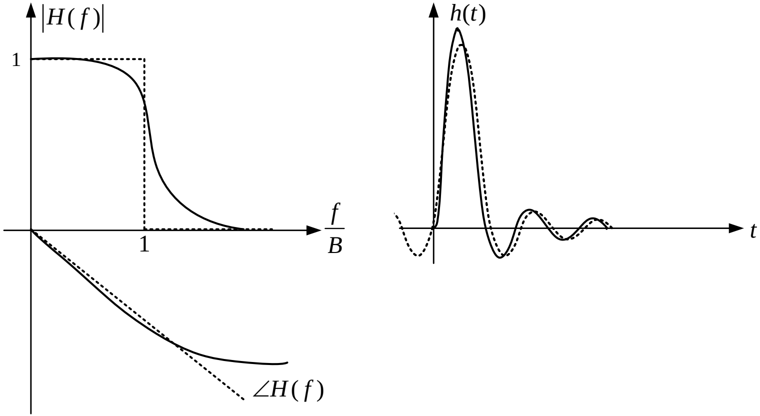


Figure: Comparison of Butterworth and ideal filters.

Basic Filters

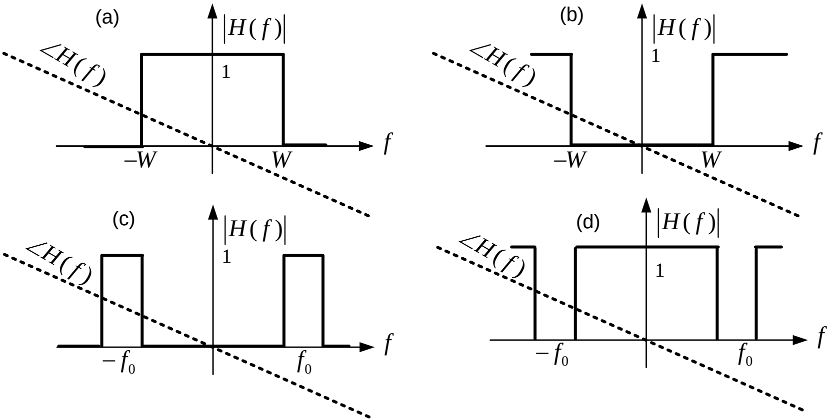


Figure: Basic filters. (a) LPF (b) HPF (c) BPF (d) BSF.

Filter Design

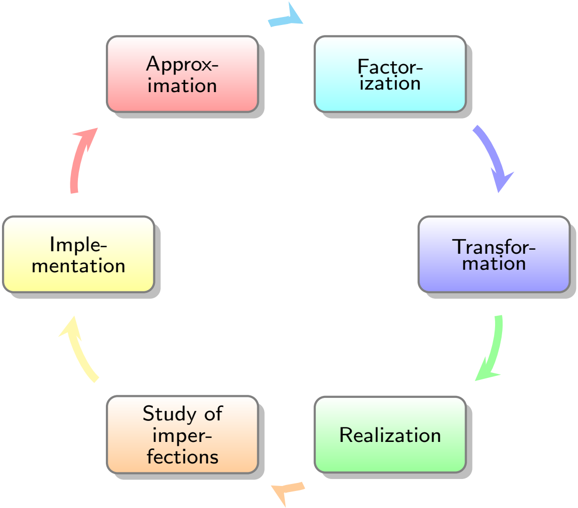


Figure: Design process.

Bandwidth

Bandwidth

Definition (Absolute Bandwidth)

Absolute bandwidth of the signal $x(t)$ is the smallest positive frequency band, where, for frequencies outside it, $|X(f)|$ is zero.

Definition (Half-power Bandwidth)

3-dB or half-power bandwidth of the signal $x(t)$ is the positive frequency band, where, for frequencies outside it, $|X(f)|$ is never greater than $1/\sqrt{2}$ times its maximum value.

Definition (Null-to-null Bandwidth)

Null-to-null or zero-crossing bandwidth is the frequency band, where the band edge frequencies create the first spectrum nulls. For the lowpass signals, the right side edge frequency only creates the null.

Bandwidth

Definition (Power Bandwidth)

Power bandwidth is the positive frequency band in which 49.5% of the total power (or energy) resides.

Definition (RMS Bandwidth)

The Root Mean Square (RMS) bandwidth is defined as $\sqrt{\frac{\int_0^{+\infty} f^2|X(f)|^2 df}{\int_0^{+\infty} |X(f)|^2 df}}$ for the lowpass signal $x(t)$ and $2\sqrt{\frac{\int_0^{+\infty}(f-f_0)^2|X(f)|^2 df}{\int_0^{+\infty} |X(f)|^2 df}}$ for the bandpass signal $x(t)$ centered around f_0.

Bandwidth

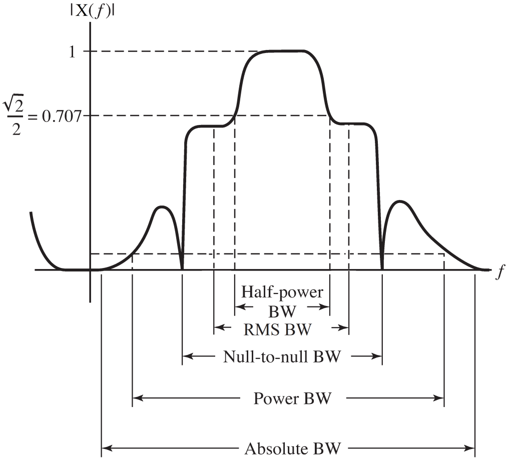


Figure: Various ways to define bandwidth. $X(f)$ may represent the spectrum of a real signal or frequency response of a filter.

Bandwidth

Example (Bandwidth of the rectangular spectrum)

For a signal with the spectrum $X(f) = A \sqcap (\frac{f}{B})$,

$$W_{abs} = \frac{B}{2}$$
$$W_{3db} = \frac{B}{2}$$
$$W_{n2n} = \frac{B}{2}$$
$$\int_0^{W_{pow}} A^2 \, df = W_{pow} A^2 = \frac{49.5}{100} BA^2 \Rightarrow W_{pow} = 0.495B$$
$$W_{rms} = \sqrt{\frac{\int_0^{+\infty} f^2 |X(f)|^2 df}{\int_0^{+\infty} |X(f)|^2 df}} = \sqrt{\frac{\int_0^{\frac{B}{2}} f^2 A^2 df}{\int_0^{\frac{B}{2}} A^2 df}} = \sqrt{\frac{\frac{B^3}{24} A^2}{\frac{B}{2} A^2}} = \frac{B}{\sqrt{12}} = 0.287B$$

The End