MATHEMATICAL QUESTIONS

Question 1

A vestigial-sideband modulation system is depicted in Fig. 1. The bandwidth of the message signal $m(t)$ is W, and the transfer function of the bandpass filter is shown in the figure.

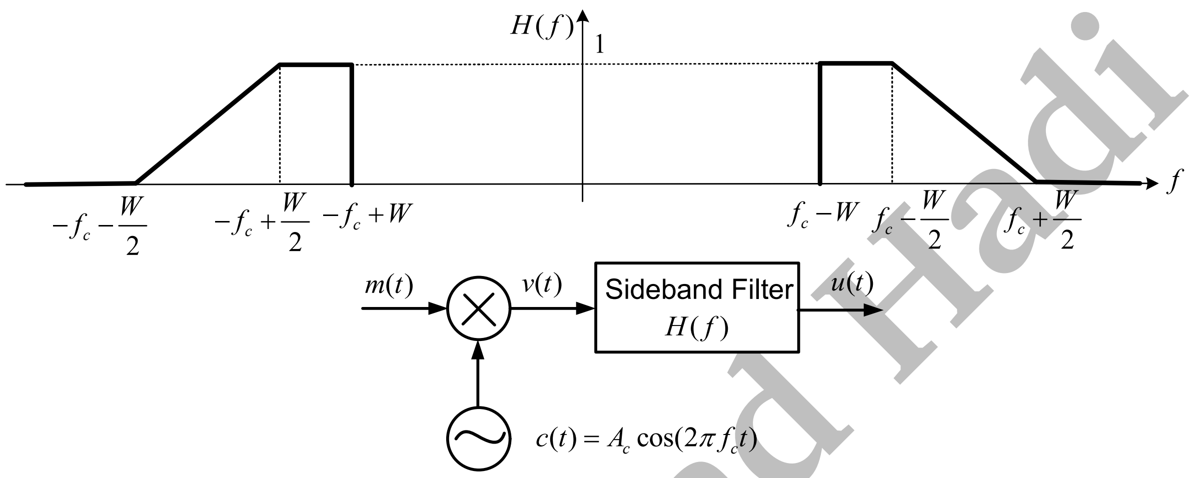

Figure 1: A VSB modulation system.

(a) Determine $h_l(t)$, the lowpass equivalent of $h(t)$, where $h(t)$ represents the impulse response of the bandpass filter.

The low pass equivalent transfer function of the system is

$$H_l(f) = 2u(f+f_c)H(f+fc) = 2\begin{cases} 1, & -W < f \leq \frac{-W}{2} \\ \frac{-1}{W}f + \frac{1}{2} & |f| < \frac{W}{2} \end{cases} = \begin{cases} 2, & -W < f \leq \frac{-W}{2} \\ \frac{-2}{W}f + 1 & |f| < \frac{W}{2} \end{cases}$$

Taking the inverse Fourier transform, we obtain

$$h_l(t) = \mathcal{F}^{-1}[H_l(f)] = \int_{-W}^{\frac{W}{2}} H_l(f)e^{j2\pi ft}df = 2\int_{-W}^{\frac{-W}{2}} e^{j2\pi ft}df + 2\int_{\frac{-W}{2}}^{\frac{W}{2}} (\frac{-1}{W}f + \frac{1}{2})e^{j2\pi ft}df =$$

$$\frac{-2}{W}(\frac{1}{j2\pi t}fe^{j2\pi ft} + \frac{1}{4\pi^2t^2}e^{j2\pi ft})\Big|_{\frac{-W}{2}}^{\frac{W}{2}} + \frac{1}{j2\pi t}e^{j2\pi ft}\Big|_{\frac{-W}{2}}^{\frac{W}{2}} + \frac{2}{j2\pi t}e^{j2\pi ft}\Big|_{-W}^{\frac{-W}{2}} =$$

$$\frac{-1}{j\pi t}e^{-j2\pi Wt} + \frac{-j}{\pi^2t^2W}\sin(\pi Wt) = \frac{-j}{\pi t}[\text{sinc}(Wt) - e^{-j2\pi Wt}]$$

Question 1 continued on next page...

(b) Derive an expression for the modulated signal $u(t)$.

Since $M(f)$ is nonzero in $f \in [-W, W]$, it is clear that

$$\mathcal{F}[m(t) * \tfrac{1}{j\pi t}e^{-j2\pi Wt}] = -M(f)\,\mathrm{sgn}(f+W) = -M(f) \Longrightarrow m(t) * \tfrac{1}{j\pi t}e^{-j2\pi Wt} = -m(t)$$

Further, if $U_l(f) = V_l(f)H_l(f)$, then

$$U(f) = V_b(f)H_b(f) = \frac{1}{2}\big[V_l(f-f_c) + V_l^*(-(f+f_c))\big]\frac{1}{2}\big[H_l(f-f_c) + H_l^*(-(f+f_c))\big]$$

$$U(f) = \frac{1}{4}\big[V_l(f-f_c)H_l(f-f_c) + V_l^*(-(f+f_c))H_l^*(-(f+f_c))\big]$$

$$U(f) = \frac{1}{2}\frac{1}{2}\big[U_l(f-f_c) + U_l^*(-(f+f_c))\big] = \frac{1}{2}U_b(f)$$

So, if we pass $v_l(t)$ through the LTI system having the impulse response $h_l(t)$ to generate the output $u_l(t) = v_l(t)*h_l(t)$, then $u(t) = 0.5u_b(t) = 0.5\Re\{u_l(t)e^{j2\pi f_c t}\}$. So, the modulated signal can be represented as

$$u(t) = \tfrac{1}{2}\Re\{u_l(t)e^{j2\pi f_c t}\} = \tfrac{1}{2}\mathcal{R}\{\big(A_c m(t) * h_l(t)\big)e^{j2\pi f_c t}\}$$

As shown in (a), $h_l(t) = \frac{-j}{\pi t}[\mathrm{sinc}(Wt) - e^{-j2\pi Wt}]$. So,

$$A_c m(t) * h_l(t) = A_c m(t) * \big(\frac{-j}{\pi t}[\mathrm{sinc}(Wt) - e^{-j2\pi Wt}]\big)$$

$$A_c m(t) * h_l(t) = A_c m(t) * \frac{-j}{\pi t}\,\mathrm{sinc}(Wt) - A_c m(t) * \frac{1}{j\pi t}e^{-j2\pi Wt}$$

$$A_c m(t) * h_l(t) = A_c m(t) * \frac{-j}{\pi t}\,\mathrm{sinc}(Wt) + A_c m(t)$$

$$\Re\{\big(A_c m(t) * h_l(t)\big)e^{j2\pi f_c t}\} = A_c\big[m(t) * \frac{\mathrm{sinc}(Wt)}{\pi t}\big]\sin(2\pi f_c t) + A_c m(t)\cos(2\pi f_c t)$$

Therefore,

$$u(t) = \frac{1}{2}\Re\{\big(A_c m(t) * h_l(t)\big)e^{j2\pi f_c t}\} = \frac{A_c}{2}\big[m(t) * \frac{\mathrm{sinc}(Wt)}{\pi t}\big]\sin(2\pi f_c t) + \frac{A_c}{2}m(t)\cos(2\pi f_c t)$$

Question 2

Follow the steps below to show the power of the FM signal $u(t) = A_c\cos(2\pi f_c t + \phi(t))$ is $\frac{A_c^2}{2}$.

(a) Write the power expression for the FM signal and show that the power equals $P = \frac{A_c^2}{2} + I$, where

$$I = \lim_{T\to\infty}\frac{A_c^2}{2T}\int_{-T/2}^{T/2}\cos(4\pi f_c t + 2\phi(t))dt$$

Question 2 continued on next page...

$$
\begin{aligned}
P &= \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} A_c^2 \cos^2(2\pi f_c t + \phi(t))dt \\
&= \lim_{T\to\infty} \frac{A_c^2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1+\cos(4\pi f_c t + 2\phi(t))}{2}dt \\
&= \lim_{T\to\infty} \frac{A_c^2 T}{2T} + \lim_{T\to\infty} \frac{A_c^2}{2T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(4\pi f_c t + 2\phi(t))dt \\
&= \frac{A_c^2}{2} + I
\end{aligned}
$$

(b) Show that

$$
I_\infty = \int_{-\infty}^{\infty} \cos(4\pi f_c t + 2\phi(t))dt
$$

relates to the Fourier transforms $\mathcal{F}\{e^{j2\phi(t)}\}$ and $\mathcal{F}\{e^{-j2\phi(t)}\}$ at the frequencies $-2f_c$ and $2f_c$, respectively.

$$
\begin{aligned}
I_\infty &= \int_{-\infty}^{\infty} \cos(4\pi f_c t + 2\phi(t))dt \\
&= \int_{-\infty}^{\infty} \frac{e^{j(2\phi(t)+4\pi f_c t)} + e^{-j(2\phi(t)+4\pi f_c t)}}{2}dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty} e^{j2\phi(t)}e^{-j2\pi(-2f_c)t}dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-j2\phi(t)}e^{-j2\pi(2f_c)t}dt \\
&= \frac{1}{2}\mathcal{F}\{e^{j2\phi(t)}\}|_{f\to -2f_c} + \frac{1}{2}\mathcal{F}\{e^{-j2\phi(t)}\}|_{f\to 2f_c}
\end{aligned}
$$

(c) Use Taylor series expansion to show that I_∞ depends to the Fourier transforms $\mathcal{F}\{\phi^n(t)\}, n \in \mathbb{W}$ at the frequency $\pm 2f_c$.

$$
\begin{aligned}
I_\infty &= \int_{-\infty}^{\infty} \cos(4\pi f_c t + 2\phi(t))dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty} e^{j2\phi(t)}e^{-j2\pi(-2f_c)t}dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-j2\phi(t)}e^{-j2\pi(2f_c)t}dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty}[1 + \frac{j2\phi(t)}{1!} + \frac{(j2\phi(t))^2}{2!} + \cdots]e^{-j2\pi(-2f_c)t}dt \\
&+ \frac{1}{2}\int_{-\infty}^{\infty}[1 + \frac{-j2\phi(t)}{1!} + \frac{(-j2\phi(t))^2}{2!} + \cdots]e^{-j2\pi(2f_c)t}dt \\
&= \frac{1}{2}[\mathcal{F}\{1\}|_{f\to-2f_c} + \mathcal{F}\{\frac{j2\phi(t)}{1!}\}|_{f\to-2f_c} + \mathcal{F}\{\frac{(j2\phi(t))^2}{2!}\}|_{f\to-2f_c} + \cdots] \\
&+ \frac{1}{2}[\mathcal{F}\{1\}|_{f\to2f_c} + \mathcal{F}\{\frac{-j2\phi(t)}{1!}\}|_{f\to2f_c} + \mathcal{F}\{\frac{(-j2\phi(t))^2}{2!}\}|_{f\to2f_c} + \cdots] \\
&= \frac{1}{2}[\mathcal{F}\{1\}|_{f\to-2f_c} + \frac{j2}{1!}\mathcal{F}\{\phi(t)\}|_{f\to-2f_c} + \frac{(j2)^2}{2!}\mathcal{F}\{\phi^2(t)\}|_{f\to-2f_c} + \cdots] \\
&+ \frac{1}{2}[\mathcal{F}\{1\}|_{f\to2f_c} + \frac{-j2}{1!}\mathcal{F}\{\phi(t)\}|_{f\to2f_c} + \frac{(-j2)^2}{2!}\mathcal{F}\{\phi^2(t)\}|_{f\to2f_c} + \cdots] \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(j2)^n}{n!}\mathcal{F}\{\phi^n(t)\}|_{f\to-2f_c} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-j2)^n}{n!}\mathcal{F}\{\phi^n(t)\}|_{f\to2f_c}
\end{aligned}
$$

(d) *Show that if $f_c \gg W$, where W is the bandwidth of the message-related phase $\phi(t)$, $I_\infty \approx 0$.*

Assuming a bandwidth of W for the message-related phase, $\phi(t)$, the bandwidth of signal $\phi^n(t)$ is smaller than nW. Therefore,

For small n, we have $nW < 2f_c$, so $\mathcal{F}\{\phi^n(t)\}_{f\to2f_c} = \mathcal{F}\{\phi^n(t)\}_{f\to-2f_c} = 0$.
For large n, we have $\frac{2^n}{n!} \approx 0$, so $\frac{2^n}{n!}\mathcal{F}\{\phi^n(t)\}_{f\to2f_c} = \frac{2^n}{n!}\mathcal{F}\{\phi^n(t)\}_{f\to-2f_c} \approx 0$.

Therefore, we conclude that $I_\infty \approx 0$.

(e) *Show that the power is approximately equal to $\frac{A_c^2}{2}$.*

Since $I_\infty \approx 0$, we have

$$
I = \lim_{T\to\infty}\frac{A_c^2}{2T}\int_{-T/2}^{T/2}\cos(4\pi f_c t + 2\phi(t))dt = \lim_{T\to\infty}\frac{A_c^2 I_\infty}{2T} \approx 0
$$

Finally,

$$
P = \frac{A_c^2}{2} + I \approx \frac{A_c^2}{2}
$$

.

Question 3 continued on next page. . .

Question 3

Find the spectrum of the narrowband FM signal

$$u(t) = A_c \cos(2\pi f_c t) - A_c \big[2\pi k_f \int_{-\infty}^{t} m(\tau) d\tau\big] \sin(2\pi f_c t)$$

and narrowband PM signal

$$u(t) = A_c \cos(2\pi f_c t) - A_c k_p m(t) \sin(2\pi f_c t)$$

in terms of the message spectrum $M(f)$.

We know that

$$\mathcal{F}\{2\pi k_f \int_{-\infty}^{t} m(\tau) d\tau\} = \frac{2\pi k_f M(f)}{j2\pi f} + \frac{2\pi k_f}{2} M(0)\delta(f) = \frac{k_f M(f)}{jf}$$

by the integral property of the Fourier transform. Note that the message is assumed to have no DC component so $M(0) = 0$. Now,

$$\begin{aligned}
U(f) &= \mathcal{F}\{A_c \cos(2\pi f_c t)\} - \mathcal{F}\{2\pi k_f \int_{-\infty}^{t} m(\tau) d\tau\} * \mathcal{F}\{A_c \sin(2\pi f_c t)\} \\
&= \frac{A_c}{2}(\delta(f - f_c) + \delta(f + f_c)) - \frac{k_f M(f)}{jf} * \big[\frac{A_c}{2j}(\delta(f - f_c) - \delta(f + f_c))\big] \\
&= \frac{A_c}{2}(\delta(f - f_c) + \delta(f + f_c)) - \frac{A_c k_f M(f - f_c)}{2j^2(f - f_c)} + \frac{A_c k_f M(f + f_c)}{2j^2(f + f_c)} \\
&= \frac{A_c}{2}(\delta(f - f_c) + \delta(f + f_c)) + \frac{A_c k_f M(f - f_c)}{2(f - f_c)} - \frac{A_c k_f M(f + f_c)}{2(f + f_c)}
\end{aligned}$$

The spectrum of the narrowband PM signal is as follows

$$U(f) = \mathcal{F}\{A_c \cos(2\pi f_c t)\} - \mathcal{F}\{A_c k_p m(t)\} * \mathcal{F}\{\sin(2\pi f_c t)\}$$

$$= \tfrac{A_c}{2}(\delta(f - f_c) + \delta(f + f_c)) - \tfrac{A_c k_p M(f)}{2j} * (\delta(f - f_c) - \delta(f + f_c))$$

$$= \tfrac{A_c}{2}(\delta(f - f_c) + \delta(f + f_c)) - \tfrac{A_c k_p M(f - f_c)}{2j} + \tfrac{A_c k_p M(f + f_c)}{2j}$$

Question 4

The cross-correlation of the power signals $w(t)$ and $v(t)$ is defined as $R_{vw}(\tau) = \langle v(t) w^*(t - \tau)\rangle$, where the time average operator

$$\langle x(t)\rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) dt$$

.

Question 4 continued on next page...

(a) Show that $R_{vw}(\tau) = R^*_{wv}(-\tau)$.

$$R_{vw}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} v(t)w^*(t-\tau)dt$$

Using the variable change $t - \tau = p$,

$$\begin{aligned}R_{vw}(\tau) &= \lim_{T\to\infty} \frac{1}{T}\int_{\frac{-T}{2}-\tau}^{\frac{T}{2}-\tau} v(p+\tau)w^*(p)dp\\ &= \lim_{T\to\infty} \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} w^*(p)v(p+\tau)dp\\ &= \lim_{T\to\infty} \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} w^*(t)v(t+\tau)dt = R^*_{wv}(-\tau)\end{aligned}$$

(b) Prove that $|R_{vw}(\tau)|^2 \leqslant P_vP_w$.

Based on Cauchy-Schwarz-Inequality,

$$\Big|\int_{\frac{-T}{2}}^{\frac{T}{2}} v(t)w^*(t-\tau)dt\Big|^2 \leq \int_{\frac{-T}{2}}^{\frac{T}{2}} |v(t)|^2dt \int_{\frac{-T}{2}}^{\frac{T}{2}} |w^*(t-\tau)|^2dt$$

$$\Big|\frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} v(t)w^*(t-\tau)dt\Big|^2 \leq \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} |v(t)|^2dt \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} |w^*(t-\tau)|^2dt$$

When $T \to \infty$,

$$\Big|\frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} v(t)w^*(t-\tau)dt\Big|^2 \leq \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} |v(t)|^2dt \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} |w(t)|^2dt$$

$$|R_{vw}(\tau)|^2 \leqslant P_vP_w$$

Question 5

Fig. 2 shows the block diagram of an FM to AM demodulator and the schematic of its practical implementation, which is called balanced FM demodulator or FM discriminator.

(a) Find the output of the FM to AM converter block if its frequency response is $H(f) = j[V_0 + k(f - f_c), \quad |f - f_c| < 0.5B_c$, where B_c denotes the bandwidth of the input FM modulated signal $u(t) = A_c \cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau)$. Note that the FM to AM impulse response is real and therefore, $H(-f) = H^*(f)$.

The lowpass equivalent of the filter is

$$H_l(f) = 2H(f + f_c)u(f + f_c) = j2(V_0 + kf), \quad |f| < 0.5B_c$$

Question 5 continued on next page...

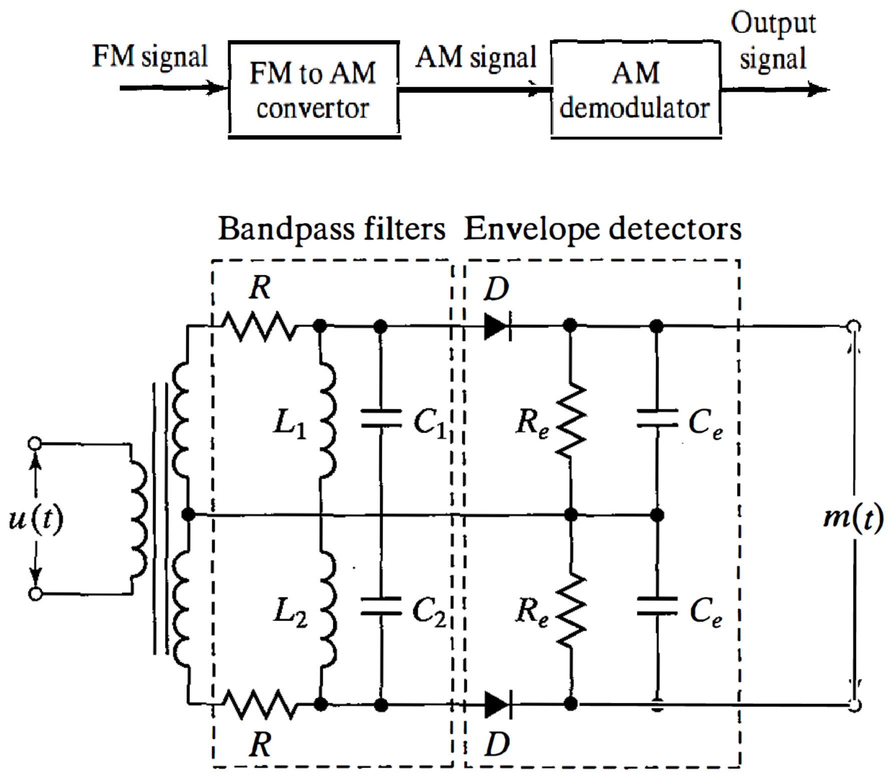

Figure 2: Balanced FM demodulator.

If $V_l(f)$ is the lowpass equivalent of the output of the FM to AM converter,

$$V_l(f) = \frac{1}{2}H_l(f)U_l(f) = j(V_0 + kf)U_l(f)$$

Note that the spectrum of the $U_l(f)$ occupies the frequency band $[-B_c, B_c]$. In time-domain,

$$v_l(t) = jV_0u_l(t) + \frac{k}{2\pi}u_l'(t), \quad u_l(t) = A_c\cos(\phi(t)) + jA_c\sin(\phi(t)), \quad \phi(t) = 2\pi k_f\int_{-\infty}^{t} m(\tau)d\tau$$

$$v_l(t) = jV_0A_c\cos(\phi(t)) - V_0A_c\sin(\phi(t)) - \frac{k}{2\pi}A_c\phi'(t)\sin(\phi(t)) + j\frac{k}{2\pi}A_c\phi'(t)\cos(\phi(t))$$

Question 5 continued on next page...

Now,

$$\begin{aligned}
v(t) =&\Re\{v_l(t)e^{j2\pi f_c t}\}\\
=&\big[-V_0A_c\sin(\phi(t))-\frac{k}{2\pi}A_c\phi'(t)\sin(\phi(t))\big]\cos(2\pi f_c t)\\
&\quad-\big[V_0A_c\cos(\phi(t))+\frac{k}{2\pi}A_c\phi'(t)\cos(\phi(t))\big]\sin(2\pi f_c t)\\
=&-V_0A_c\sin(2\pi f_c t+\phi(t))-\frac{k}{2\pi}A_c\phi'(t)\sin(2\pi f_c t+\phi(t))\\
=&-\big[V_0+\frac{k}{2\pi}\phi'(t)\big]A_c\sin(2\pi f_c t+\phi(t))\\
=&\big[V_0+\frac{k}{2\pi}\phi'(t)\big]A_c\cos(2\pi f_c t+\phi(t)+\frac{\pi}{2})\\
=&\big[V_0+kk_f m(t)\big]A_c\cos(2\pi f_c t+\phi(t)+\frac{\pi}{2})
\end{aligned}$$

(b) Explain how the shown schematic implements the FM to AM demodulator? Why are there two filters and two envelope detectors in the schematic?

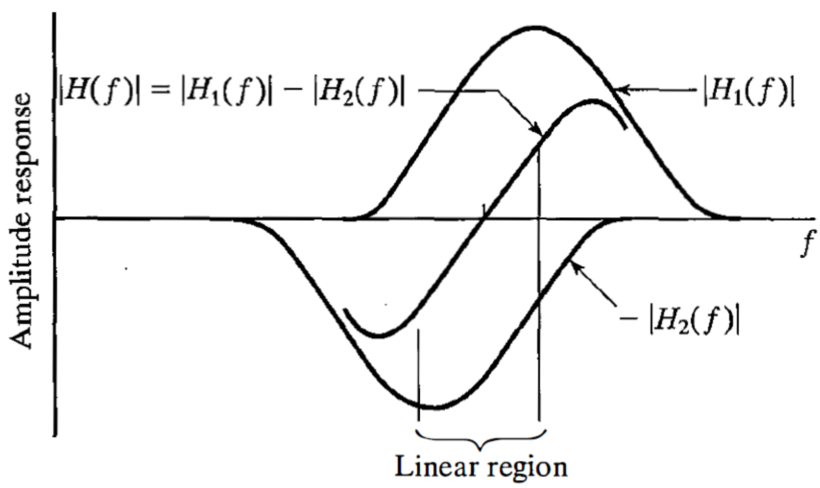

Figure 3: Differentiator in a balanced FM demodulator.

The balanced discriminator consists of two parts. The left part implements a differentiator while the right part is an envelope detector. The differentiator uses the linear part of the frequency response of an RLC circuit. As shown in Fig. 3, the up and low RLC parts of the differentiator circuit are tuned at two approperate frequencies f_1 and f_2 such that their overall reponse extends the linear region of the filter. This allows the differentiator to work on FM signals with wider bandwidth.

SOFTWARE QUESTIONS

Question 6

Develop a MATLAB/Python code that plots the cross-correlation $R_{vw}(\tau)$ of two power signals $v(t)$ and $w(t)$. Illustrate the output of the code for sample power signals.

Here is a sample implementation.

```
%%
clear;
close all;
clc;
x=0:0.01:8*pi;
[row,column]=size(x);
%defining power signals
v=sin(x);
w=cos(x);
%computing cross-correlation
Rvw=zeros(1,599);
for i=-299:1:299
shiftedw=conj(circshift(w,i));
accumulation=v.*shiftedw;
Rvw(i+300)=1/(column)*sum(accumulation);
end
%plotting cross-correlation of v(t) and w(t)
j=-2.99:0.01:2.99;
figure;
scatter(j,Rvw);
title('cross-correltion of v(t) and w(t)');
xlabel('\tau')
ylabel('R_{vw}(\tau)')
grid on;
grid minor;
box on;
%%
```

Figs. 4-6 illustrate the auto- and cross-correlation of two power signals, $\sin(t)$ and $\cos(t)$.

Question 6 continued on next page...

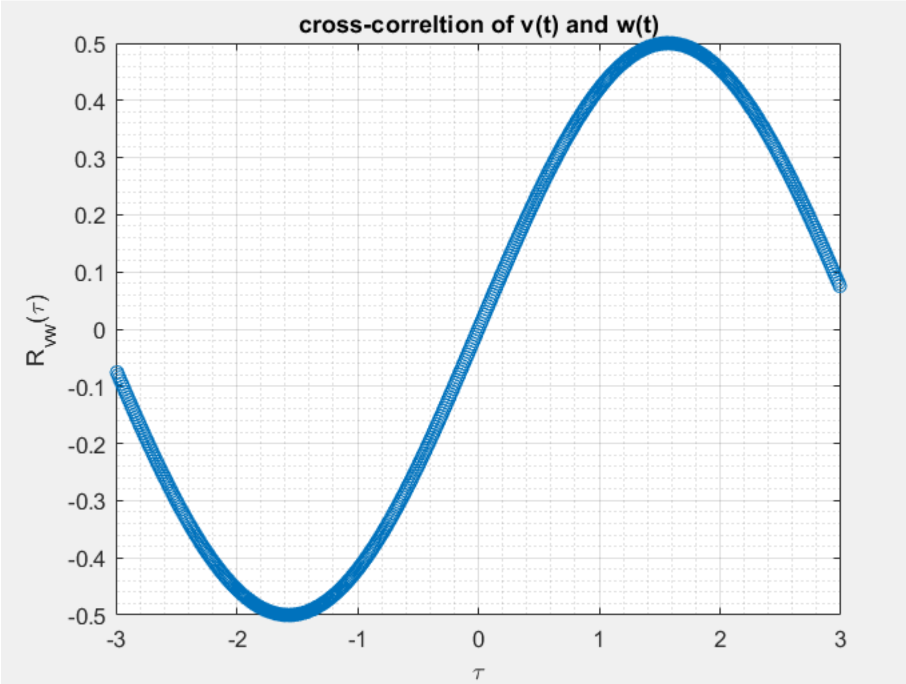

Figure 4: cross-correlation of $\sin(t)$ and $\cos(t)$.

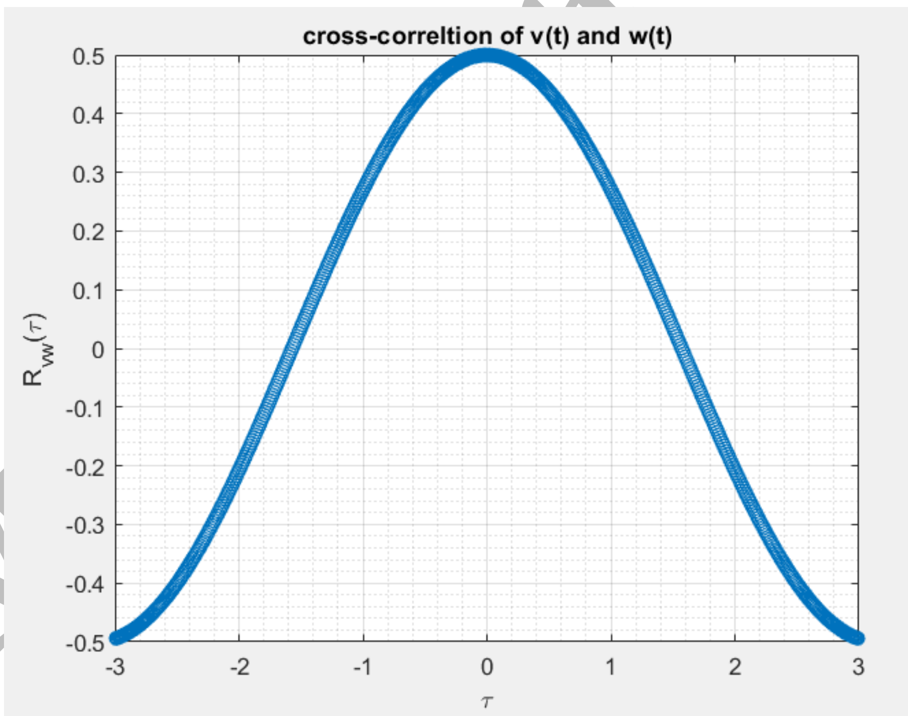

Figure 5: Auto-correlation of $\sin(t)$

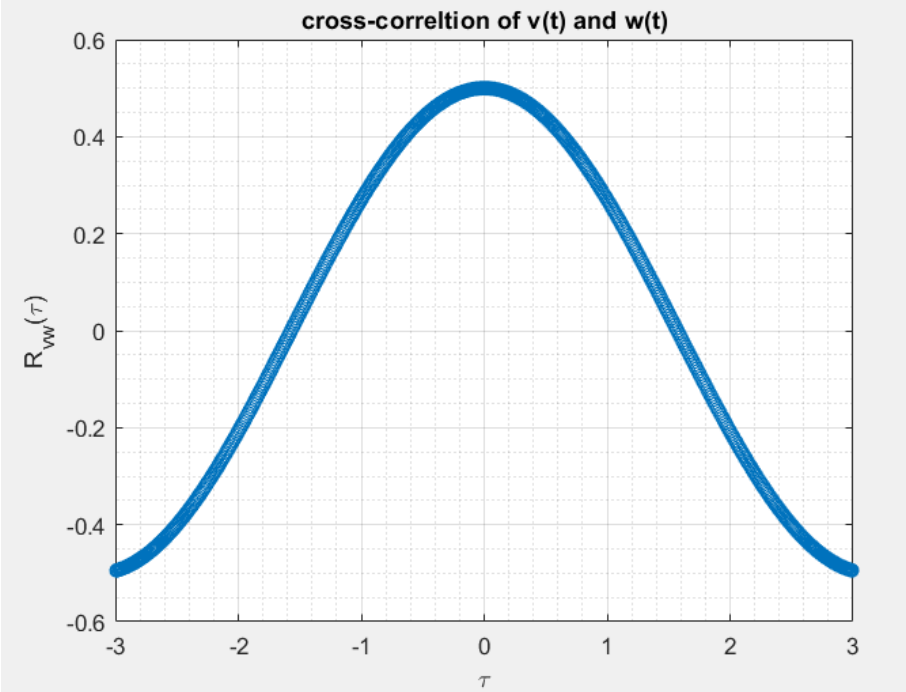

Figure 6: Auto-correlation of $\cos(t)$.

BONUS QUESTIONS

Question 7

Two power signals of $v(t)$ and $w(t)$ are called uncorrelated if their cross-correlation $R_{vw}(\tau) = 0, \forall\tau$. Show that for these uncorrelated signals, the power P_z of $z(t) = v(t) + w(t)$ equals $P_z = P_v + P_w$.

Define $z(t) = v(t) + w(t)$, so

$$\begin{aligned} P_z &= \langle z(t)z(t)^* \rangle = \langle (v(t) + w(t))(v(t) + w(t))^* \rangle \\ &= \langle v(t)v(t)^* \rangle + \langle w(t)w(t)^* \rangle + \langle v(t)w(t)^* \rangle + \langle v(t)^*w(t) \rangle \end{aligned}$$

$\langle v(t)w(t)^* \rangle$ and $\langle v(t)^*w(t) \rangle$ are zero because cross-correlation of $v(t)$ and $w(t)$ is zero and they are uncorrelated. So,

$$P_z = \langle v(t)v(t)^* \rangle + \langle w(t)w(t)^* \rangle = P_v + P_w$$

Question 8

Return your answers by filling the LaTeXtemplate of the assignment.