MATHEMATICAL QUESTIONS

Question 1

First-order hold (FOH) performs a linear interpolation to generate the sampled signal

$$y(t) = \sum_{k=-\infty}^{\infty} x(kT_s)\Lambda(\frac{t-kT_s}{T_s})$$

from the band-limited signal $x(t)$ with bandwidth W.

(a) Find the spectrum of the sampled signal $Y(f)$.

We know that $\mathcal{F}[\Lambda(\frac{t}{T_s})] = T_s\text{sinc}^2(T_sf)$. So,

$$y(t) = \sum_{k=-\infty}^{\infty} x(kT_s)\Lambda(\frac{t-kT_s}{T_s}) = \Lambda(\frac{t}{T_s}) * \sum_{k=-\infty}^{\infty} x(kT_s)\delta(t-kT_s) = \Lambda(\frac{t}{T_s}) * x_\delta(t)$$

$$\mathcal{F}[y(t)] = Y(f) = \mathcal{F}[\Lambda(\frac{t}{T_s})]\frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(f-\frac{k}{T_s})$$

$$Y(f) = \text{sinc}^2(T_sf)\sum_{k=-\infty}^{\infty} X(f-\frac{k}{T_s})$$

(b) Propose a condition on the sampling period T_s for the perfect reconstruction of the original signal $x(t)$ from the sampled signal $y(t)$.

To avoid aliasing, Nyquist condition should be held, i.e. $\frac{1}{T_s} > 2W$.

(c) Obtain an expression for the reconstruction filter $H(f)$.

Clearly,

$$X(f) = X_p(f)H(f), \quad X_p(f) = P(f)X_\delta(f)$$

$$H(f) = \frac{1}{\mathcal{F}[\Lambda(\frac{t}{T_s})]}T_s \sqcap (\frac{f}{2W'})$$

, which yields

$$H(f) = \frac{\sqcap(\frac{f}{2W'})}{\text{sinc}^2(T_sf)}$$

, where

$$W < W' < \frac{1}{T_s} - W$$

Question 2

The frequency-domain sampling theorem says that if $x(t)$ is a time-limited signal such that $x(t) = 0$ for $|t| \geq T$, then $X(f)$ is completely determined by its sample values $X(nf_0)$ with $f_0 \leq 1/2T$. Prove this theorem.

$$X_\delta(f) = \sum_{n=-\infty}^{\infty} X(nf_0)\delta(f - nf_0) = X(f)\sum_{n=-\infty}^{\infty}\delta(f - nf_0)$$

Taking the inverse Fourier transform,

$$x_\delta(t) = \mathcal{F}^{-1}\{X_\delta(f)\} = x(t) * [\frac{1}{f_0}\sum_{n=-\infty}^{\infty}\delta(t - \frac{n}{f_0})] = \frac{1}{f_0}\sum_{n=-\infty}^{\infty} x(t - \frac{n}{f_0})$$

Now, if $T \leq \frac{1}{f_0} - T \Rightarrow f_0 \leq \frac{1}{2T}$,

$$x(t) = x_\delta(t) f_0 \text{rect}(\frac{t}{2T'}), \quad T \leq T' \leq \frac{1}{f_0} - T$$

Or equivalently,

$$X(f) = X_\delta(f) * \left[2T' f_0 \text{sinc}(2T' f)\right]$$

Question 3

The analog signal $x(t)$ which takes the values of $[-x_m, +x_m]$ uniformly passes a uniform midrise quantizer with $N = 2^\nu$ levels. Find the SQNR of the quantizer.

If the interval $[-x_m, +x_m]$ is covered by the even number of quantization levels $N = 2^\nu$, the quantization width equals $\Delta = \frac{2x_m}{2^v}$. The quantization interval $[-x_m + k\Delta, -x_m + (k+1)\Delta]$ is mapped to the quantized value $-x_m + k\Delta + \Delta/2$. Hence,

$$E\{(X - Q(X))^2\} = \int_{-x_m}^{x_m} (x - Q(x))^2 f_X(x) dx$$
$$= \sum_{k=0}^{2^\nu - 1}\int_{-x_m + k\Delta}^{-x_m + (k+1)\Delta} [x - (-x_m + k\Delta + \Delta/2)]^2 \frac{1}{2x_m} dx$$
$$= \sum_{k=0}^{2^\nu - 1}\int_{-x_m + k\Delta}^{-x_m + (k+1)\Delta} [x + x_m - k\Delta - \Delta/2]^2 \frac{1}{2x_m} dx$$
$$= \frac{1}{2x_m}\sum_{k=0}^{2^\nu - 1}\int_{-\Delta/2}^{\Delta/2} u^2 du = \frac{2^\nu}{2x_m}\int_{-\Delta/2}^{\Delta/2} u^2 du = \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} u^2 du = \frac{1}{\Delta}\frac{2\Delta^3}{24} = \frac{\Delta^2}{12}$$

$$E\{X^2\} = \int_{-x_m}^{x_m} x^2 f_X(x) dx = \int_{-x_m}^{x_m} x^2 \frac{1}{2x_m} dx = \frac{1}{2x_m} 2\frac{1}{3} x_m^3 = \frac{x_m^2}{3} = \frac{\Delta^2}{12} 2^{2\nu}$$

$$\text{SQNR} = \frac{E\{X^2\}}{E\{(X - Q(X))^2\}} = \frac{\frac{\Delta^2}{12} 2^{2\nu}}{\frac{\Delta^2}{12}} = 2^{2v}$$

Question 4

The lowpass signal $x(t)$ with a bandwidth of W is sampled with a sampling interval of T_s and the signal

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT_s)p(t - nT_s)$$

is reconstructed from the samples, where $p(t)$ is an arbitrary-shaped pulse (not necessarily time limited to the interval $[0, T_s]$).

(a) Find the Fourier transform of $x_p(t)$.

We have

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT_s)p(t - nT_s) = p(t) * \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s) = p(t) * x_\delta(t)$$

Applying the Fourier Transform to both sides and using the convolution property, we get

$$X_p(f) = P(f)X_\delta(f) = \frac{P(f)}{T_s} \sum_{n=-\infty}^{\infty} X(f - \frac{n}{T_s})$$

(b) Find the conditions for perfect reconstruction of $x(t)$ from $x_p(t)$.

In order to avoid aliasing $\frac{1}{T_s} > 2W$. Furthermore, the spectrum $P(f)$ should be invertible for $|f| < W$.

(c) Determine the required reconstruction filter.

Considering that the conditions of perfect reconstruction are satisfied, we have

$$X(f) = X_p(f)H(f) = X_p(f)[\frac{T_s}{P(f)} \sqcap (\frac{f}{2W'})]$$

, where $W \leq W' \leq \frac{1}{T_s} - W$.

Question 5

The lowpass signal $x(t)$ with a bandwidth of W is sampled at the Nyquist rate and the signal

$$y(t) = \sum_{n=-\infty}^{\infty} (-1)^n x(nT_s)\delta(t - nT_s)$$

is generated.

Question 5 continued on next page...

(a) Find the Fourier transform of $y(t)$.

$$y(t) = \sum_{n=-\infty}^{\infty} (-1)^n x(nT_s)\delta(t-nT_s) = x(t)\sum_{n=-\infty}^{\infty} (-1)^n \delta(t-nT_s)$$

$$= x(t)[\sum_{n=-\infty}^{\infty} \delta(t-2nT_s) - \sum_{n=-\infty}^{\infty} \delta(t-T_s-2nT_s)\,]$$

$$\Rightarrow Y(f) = \frac{1}{2T_s} X(f) * \left[\sum_{n=-\infty}^{\infty} \delta\left(f-\frac{n}{2T_s}\right) - \sum_{n=-\infty}^{\infty} \delta\left(f-\frac{n}{2T_s}\right) e^{-j2\pi f T_s}\right]$$

$$= \frac{1}{2T_s}\left[\sum_{n=-\infty}^{\infty} X\left(f-\frac{n}{2T_s}\right) - \sum_{n=-\infty}^{\infty} X\left(f-\frac{n}{2T_s}\right)(-1)^n\right]$$

$$= \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X\left(f-\frac{2n+1}{2T_s}\right) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X\left(f-\frac{n}{T_s}-\frac{1}{2T_s}\right)$$

To have the spectrum copies separated, we should we,

$$\frac{1}{2T_s} + W \le \frac{3}{2T_s} - W \Rightarrow 2W \le \frac{1}{T_s}$$

, where W is the bandwidth of the signal $X(t)$.

(b) Can $x(t)$ be reconstructed from $y(t)$ by using a linear time-invariant system? Why?

The answer is NO. Here a copy of the spectrum of the original signal is not located at $f = 0$. So, we cannot separate the spectrum of the original signal using an LTI system which can shift the spectrum of its input.

(c) Can $x(t)$ be reconstructed from $y(t)$ by using a linear time-varying system? How?

Yes. First we use a bandpass filter to isolate $X_1(f) = X(f-\frac{1}{2T_s}) + X(f+\frac{1}{2T_s})$. Then, mix the filtered signal with $\cos(2\pi\frac{1}{2T_2}t)$ that gives

$$\frac{1}{2}[X_1(f+\frac{1}{2T_s}) + X_1(f-\frac{1}{2T_s})] = \frac{1}{2}[X(f) + X(f+\frac{2}{2T_s}) + X(f) + X(f-\frac{2}{2T_s})]$$

Applying a lowpass filter with bandwidth $W' \in [W, \frac{1}{T_s} - W]$, the original spectrum of $X(f)$ appears.

Question 6 continued on next page...

Question 6

A stationary source is distributed according to the triangle probability density function $f_X(x) = 0.5\Lambda(0.5x)$. This source is quantized using the four-level uniform quantizer

$$Q(x) = \begin{cases} 1.5, & 1 < x \leq 2 \\ 0.5, & 0 < x \leq 1 \\ -0.5, & -1 < x \leq 0 \\ -1.5, & -2 \leq x \leq -1 \end{cases}$$

Determine the probability density function of the random variable representing the quantizer error $X - Q(X)$.

Clearly, for $|e| > 0.5$, $f_E(e) = 0$. For $|e| \leq 0.5$,

$$\begin{aligned} F_E(e) = P(E \leq e) &= P(1 < X < 1.5 + e) + P(0 < X < 0.5 + e) \\ &\quad + P(-1 < X < -0.5 + e) + P(-2 < X < -1.5 + e) \\ &= F_X(1.5 + e) - F_X(1) + F_X(0.5 + e) - F_X(0) \\ &\quad + F_X(-0.5 + e) - F_X(-1) + F_X(-1.5 + e) - F_X(-2) \end{aligned}$$

$$f_E(e) = \frac{dF_E(e)}{de} = f_X(1.5 + e) + f_X(0.5 + e) + f_X(-0.5 + e) + f_X(-1.5 + e)$$

So,

$$f_E(e) = \frac{1}{2}[\Lambda(\frac{e + \frac{3}{2}}{2}) + \Lambda(\frac{e + \frac{1}{2}}{2}) + \Lambda(\frac{e - \frac{1}{2}}{2}) + \Lambda(\frac{e - \frac{3}{2}}{2})] = 1$$

Finally,

$$f_E(e) = \begin{cases} 0, & |e| > 0.5 \\ 1, & |e| \leq 0.5 \end{cases}$$

SOFTWARE QUESTIONS

Question 7

A quantizer with 2^ν quantized levels working over the input range $[-1, 1]$ is fed with a zero-mean Gaussian random variable having the variance σ^2. Develop a MATLAB/Python code to calculate the signal to quntization noise ratio when the quntization intervals are uniformly distributed and when the quantization intervals are nonuniformly distributed according to A-law companding method with the parameter A. Discuss the results for different values of ν, σ^2, and A. Feel free to plot any suitable curve to better describe the observations.

Question 7 continued on next page...

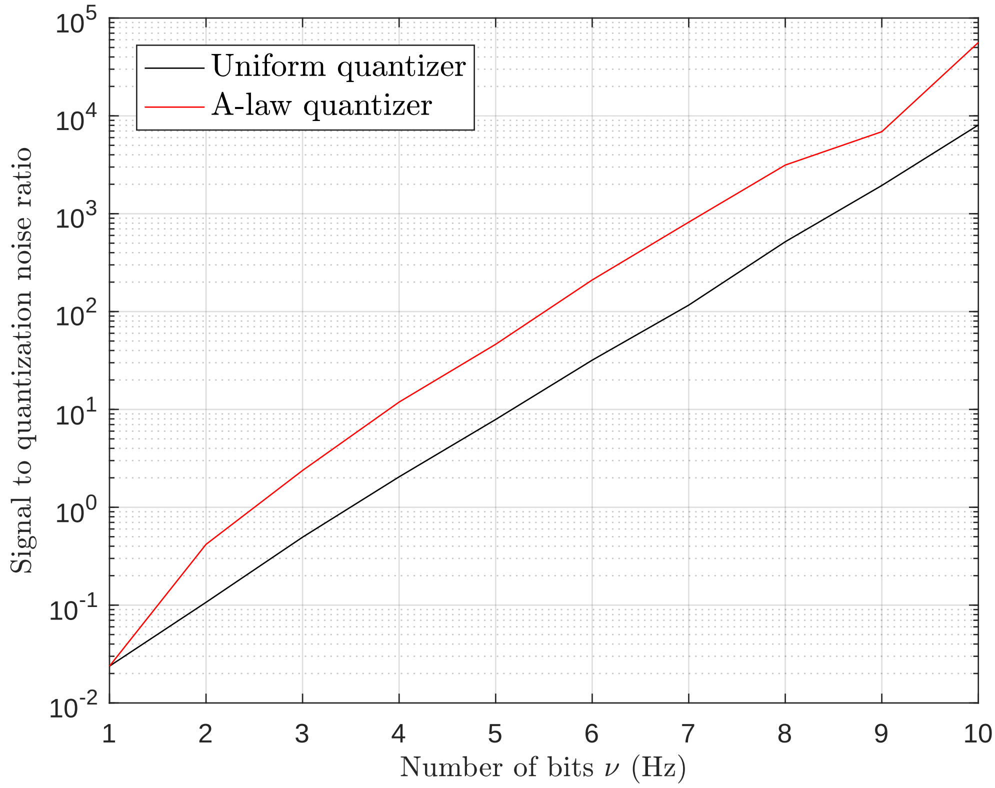

Figure 1: SQNR versus number of bits ν for $\sigma^2 = 0.01$ and $A = 9$.

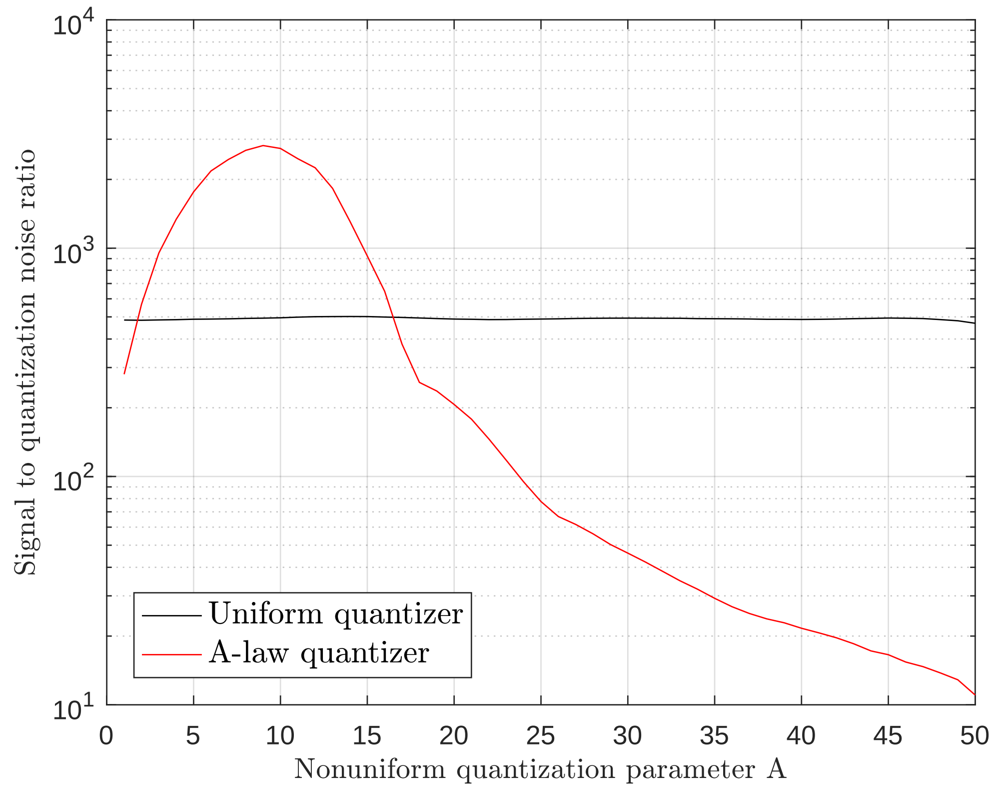

Figure 2: SQNR versus A-law parameter A for $\sigma^2 = 0.01$ and $\nu = 8$.

Question 7 continued on next page. . .

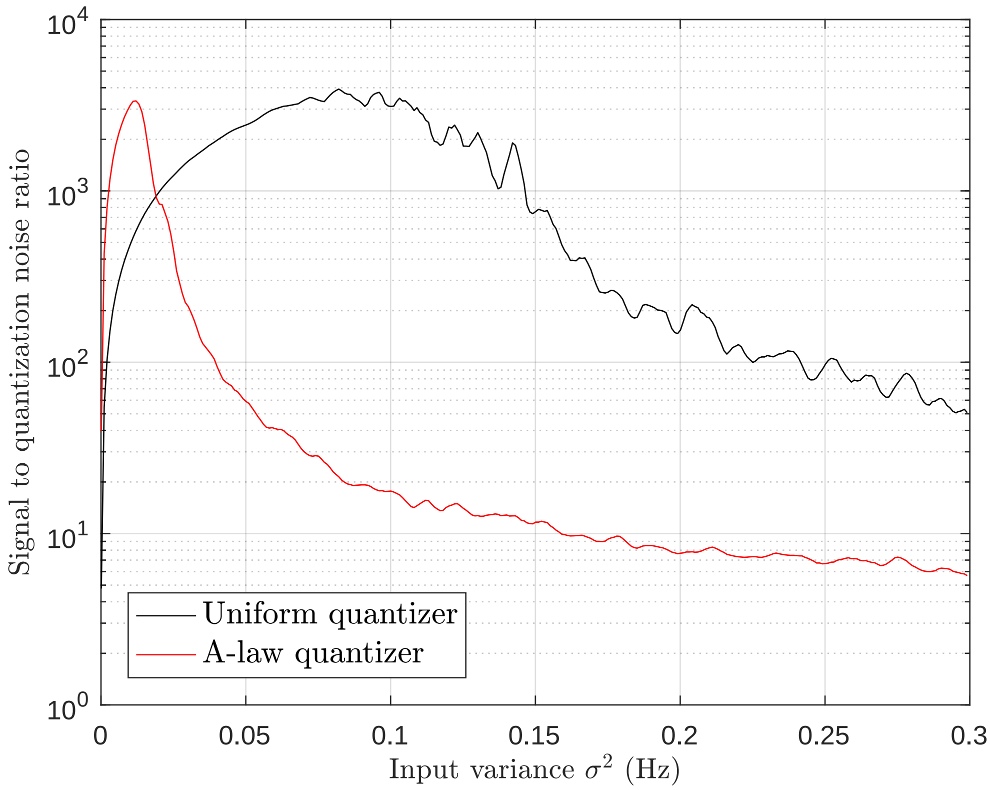

Figure 3: SQNR versus input variance σ^2 for $A = 9$ and $\nu = 8$.

The uniform quantizer can be implemented as.

```
function [value, level, bit] = uniform_quantizer(x, nu)
% quantization intervals
a=linspace(-1,1,2^nu+1);
a(1)=-Inf;
a(end)=[];
% quantization level
level=sum(x>=a)-1;
% quantization value
a(1)=-1;
value = a(level+1);
% quantization bit
bit=de2bi(bin2gray(level, 'pam', 2^nu), nu, 'left-msb');
end
```

A possible implementation of the A-law quantizer is

```
function [value, level, bit] = alaw_quantizer(x, nu, A)
% quantization intervals
a=linspace(-1,1,2^nu+1);
a(1)=-Inf;
a(end)=[];
b=zeros(size(a));
for m=1:1:(2^nu)
    if(abs(a(m)) < 1/abs(1+log(a(m))))
        b(m)= (abs(a(m))*(1+log(A))) / A.*sign(a(m));
    else
        b(m)= exp(-1+(abs(a(m))*(1+log(A)))) / A.*sign(a(m));
    end
end
% quantization level
level=sum(x>=b)-1;
% quantization value
```

Question 7 continued on next page...

```
b(1)=-1;
value = b(level+1);
% quantization bit
bit=de2bi(bin2gray(level, 'pam', 2^nu), nu, 'left-msb');
end
```

The following mfile calls the quantizers and computes their signal to quantization noise ratio.

```
%%
clear
clc
close all
% settings
sigma = sqrt(0.01);
A = 9;
nu = 8;
N = 100;

% average noise calculation
uniform_err = 0;
Alaw_err = 0;
for n=1:N
    x = sigma*randn;
    uniform_err = ((n-1)*uniform_err + (x-uniform_quantizer(x,nu))^2)/n;
    Alaw_err = ((n-1)*Alaw_err + (x-alaw_quantizer(x,nu,A))^2)/n;
end

% snr calculation
uniform_snr = sigma^2/uniform_err;
Alaw_snr = sigma^2/Alaw_err;

% snr gain
Alaw_snr/uniform_snr
```

As you can see in Fig. 1, increasing the number of bits ν improves SQNR. The SQNR of the A-law quantizer can be 7 times the SQNR of the uniform quantizer. According to Fig. 2, the SQNR of the A-law quantizer is maximized for a suitable value of A. Finally, if the input variance is high, meaning that the input of the quantizer is not concentrated around zero, the A-law quantizer may work worse than the uniform quantizer, as can be seen in Fig. 3.

BONUS QUESTIONS

Question 8

The DCT of an $N \times N$ picture with luminance function $x(m,n), 0 \le m,n \le N-1$ can be obtained as

$$X(0,0) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} x(k,l)$$

$$X(u,v) = \frac{2}{N}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} x(k,l)\cos\left[\frac{(2k+1)u\pi}{2N}\right]\cos\left[\frac{(2l+1)v\pi}{2N}\right], \quad u,v \ne 0$$

Question 8 continued on next page...

The $X(0,0)$ coefficient is usually called the DC component, and the other coefficients are called the AC components. Find the DCT of a constant picture having $x(m,n) = C, 0 \leq m,n \leq N-1$.

For the DC component,

$$X(0,0) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} C = NC$$

And for the AC component,

$$X(u,v) = \frac{2}{N}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} x(k,l)\cos\big[\frac{(2k+1)u\pi}{2N}\big]\cos\big[\frac{(2l+1)v\pi}{2N}\big], \quad u,v \neq 0$$

$$= \frac{2C}{N}\sum_{k=0}^{N-1}\cos\big[\frac{(2k+1)u\pi}{2N}\big]\sum_{l=0}^{N-1}\cos\big[\frac{(2l+1)v\pi}{2N}\big]$$

But $\sum_{l=0}^{N-1}\cos\big[\frac{(2l+1)v\pi}{2N}\big] = 0$ since for integer v

$$\sum_{l=0}^{N-1}\cos\big[\frac{(2l+1)v\pi}{2N}\big] = \sum_{l=0}^{N-1}\Re\{e^{j\frac{(2l+1)v\pi}{2N}}\}$$

$$= \Re\{e^{j\frac{v\pi}{2N}}\sum_{l=0}^{N-1}e^{j\frac{lv\pi}{N}}\} = \Re\{e^{j\frac{v\pi}{2N}}\frac{1-(e^{j\frac{v\pi}{N}})^N}{1-e^{j\frac{v\pi}{N}}}\} = \Re\{e^{j\frac{v\pi}{2N}}\frac{1-e^{jv\pi}}{1-e^{j\frac{v\pi}{N}}}\}$$

$$= \Re\{e^{j\frac{v\pi}{2N}}\frac{e^{\frac{jv\pi}{2}}}{e^{\frac{jv\pi}{2N}}}\frac{e^{\frac{-jv\pi}{2}}-e^{\frac{jv\pi}{2}}}{e^{-\frac{jv\pi}{2N}}-e^{\frac{jv\pi}{2N}}}\} = \Re\{e^{\frac{jv\pi}{2}}\frac{\sin(\frac{v\pi}{2})}{\sin(\frac{v\pi}{2N})}\} = \cos(\frac{v\pi}{2})\frac{\sin(\frac{v\pi}{2})}{\sin(\frac{v\pi}{2N})} = \frac{\sin(2v\pi)}{2\sin(\frac{v\pi}{2N})} = 0$$

So, the AC component is zero.

Question 9

Return your answers by filling the LaTeXtemplate of the assignment.