MATHEMATICAL QUESTIONS

Question 1

Use the integral definitions of the singular functions to prove the following identities.

(a) $tu_1(t) = -u_0(t)$.

$$\int_{-\infty}^{+\infty} \delta'(t)(tx(t))dt = -(tx(t))'|_{t=0} = -x(0) = -\int_{-\infty}^{+\infty} \delta(t)x(t)dt = \int_{-\infty}^{+\infty} -\delta(t)x(t)dt$$

$$\Rightarrow t\delta'(t) = -\delta(t) \Rightarrow tu_1(t) = -u_0(t)$$

(b) $f(t)\delta'(t) = -f'(0)\delta(t) + f(0)\delta'(t)$.

$$\int_{-\infty}^{+\infty} \delta'(t)(f(t)x(t))dt = -(f(t)x(t))'|_{t=0} = -f'(0)x(0) - x'(0)f(0)$$

$$= -f'(0)\int_{-\infty}^{+\infty} \delta(t)x(t)dt + f(0)\int_{-\infty}^{+\infty} \delta'(t)x(t)dt = \int_{-\infty}^{+\infty} (-f'(0)\delta(t) + f(0)\delta'(t))x(t)dt$$

$$\Rightarrow \delta'(t)f(t) = -f'(0)\delta(t) + f(0)\delta'(t)$$

(c) $\delta^{(n)}(-t) = (-1)^n\delta^{(n)}(t)$.

$$\int_{-\infty}^{+\infty} \delta^{(n)}(-t)x(t)dt = \int_{-\infty}^{+\infty} \delta^{(n)}(t)x(-t)dt = (-1)^n x^{(n)}(-t)|_{t=0} = (-1)^{2n}x^{(n)}(0)$$

$$= (-1)^n[(-1)^n x^{(n)}(0)] = (-1)^n \int_{-\infty}^{+\infty} \delta^{(n)}(t)x(t)dt = \int_{-\infty}^{+\infty} [(-1)^n\delta^{(n)}(t)]x(t)dt$$

$$\Rightarrow \delta^{(n)}(-t) = (-1)^n\delta^{(n)}(t)$$

Question 2

Prove that

Question 2 continued on next page...

(a) $\delta(t) = \frac{1}{\pi}\int_0^{+\infty}\cos(\alpha t)d\alpha$.

We know that the Fourier transform of $\delta(t)$ is 1. So if we write the inverse transform,

$$\delta(t) = \int_{-\infty}^{+\infty} 1e^{j2\pi ft}df = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 1e^{j\omega t}d\omega = \frac{1}{2\pi}(\int_0^{+\infty} e^{j\omega t}d\omega + \int_{-\infty}^{0} e^{j\omega t}d\omega)$$

$$= \frac{1}{2\pi}(\int_0^{+\infty} e^{j\omega t}d\omega + \int_0^{+\infty} e^{-j\omega t}d\omega) = \frac{1}{2\pi}\int_0^{+\infty}(e^{j\omega t}+e^{-j\omega t})d\omega = \frac{1}{2\pi}\int_0^{+\infty}(2\cos(\omega t))d\omega$$

$$= \frac{1}{\pi}\int_0^{+\infty}\cos(\omega t)d\omega$$

(b) $\sum_{n=-\infty}^{\infty} x(t+2nl) = \frac{1}{2l}\sum_{n=-\infty}^{\infty} e^{\frac{j\pi nt}{l}}X(\frac{n}{2l})$*, where* $X(f) = \mathcal{F}\{x(t)\}$.

It is obvious that $g(t) = \sum_{n=-\infty}^{\infty} x(t+2nl)$ is a periodic function with period $2l$. Let's write the Fourier series of this function. First, we need to calculate Fourier coefficients.

$$c_k = \frac{1}{2l}\int_0^{2l}(\sum_{-\infty}^{+\infty} x(t+2nl)e^{-jk\frac{2\pi t}{2l}})dt = \frac{1}{2l}\sum_{n=-\infty}^{+\infty}(\int_0^{2l} x(t+2nl)e^{-jk\frac{\pi t}{l}}dt)$$

$$= \frac{1}{2l}\sum_{n=-\infty}^{+\infty}\int_{2nl}^{2l(n+1)} x(t)e^{-jk\frac{\pi t}{l}}dt = \frac{1}{2l}\int_{-\infty}^{+\infty} x(t)e^{-jk\frac{\pi t}{l}}dt = \frac{1}{2l}X(\frac{k}{2l})$$

Now, we write the Fourier series :

$$g(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\frac{\pi t}{l}} = \sum_{k=-\infty}^{+\infty}\frac{1}{2l}X(\frac{k}{2l})e^{jk\frac{\pi t}{l}} = \frac{1}{2l}\sum_{n=-\infty}^{+\infty} X(\frac{n}{2l})e^{jn\frac{\pi t}{l}} = \sum_{n=-\infty}^{+\infty} x(t+2nl)$$

Question 3

Assume that $X(f) = \mathcal{F}\{x(t)\}$ and $Y(f) = \mathcal{F}\{y(t)\}$. Show that

(a) $\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df$.

This is Parseval's theorem and is proven as

$$x(t) = \int_{-\infty}^{+\infty} X(f_1)e^{j2\pi f_1 t}df_1, y^*(t) = \int_{-\infty}^{+\infty} Y^*(f_2)e^{-j2\pi f_2 t}df_2$$

$$\Rightarrow \int_{-\infty}^{+\infty} x(t)y^*(t)dt = \int_{-\infty}^{+\infty}(\int_{-\infty}^{+\infty} X(f_1)e^{j2\pi f_1 t}df_1)(\int_{-\infty}^{+\infty} Y^*(f_2)e^{-j2\pi f_2 t}df_2)dt$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} X(f_1)Y^*(f_2)(\int_{-\infty}^{+\infty} e^{j2\pi t(f_1-f_2)}dt)\, df_2\, df_1$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} X(f_1)Y^*(f_2)\delta(f_1-f_2)\, df_2\, df_1 = \int_{-\infty}^{+\infty} X(f_1)(\int_{-\infty}^{+\infty} Y^*(f_2)\delta(f_1-f_2)\, df_2)\, df_1$$

$$= \int_{-\infty}^{+\infty} X(f_1)Y^*(f_1)\,df_1 = \int_{-\infty}^{+\infty} X(f)Y^*(f)\,df$$

(b) $\int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau = \int_{-\infty}^{\infty} X(f)Y(f)e^{j2\pi ft}df$.

Defining $g(\tau) = y^*(t-\tau)$,

$$\begin{aligned}
\mathcal{F}\{y(\tau)\} &= Y(f) \\
\mathcal{F}\{y(-\tau)\} &= Y(-f) \\
\mathcal{F}\{y(-\tau+t)\} = \mathcal{F}\{y(-(\tau-t))\} &= Y(-f)e^{-j2\pi ft} \\
G(f) = \mathcal{F}\{g(\tau)\} = \mathcal{F}\{y^*(-\tau+t)\} &= Y^*(f)e^{-j2\pi ft}
\end{aligned}$$

Now, according to the proven Parseval's theorem in the previous part,

$$\int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau = \int_{-\infty}^{\infty} x(\tau)g^*(\tau)d\tau = \int_{-\infty}^{\infty} X(f)G^*(f)df = \int_{-\infty}^{\infty} X(f)Y(f)e^{j2\pi ft}df$$

Question 4

The analytic signal $x_a(t)$ of the real signal $x(t)$ is a signal with the spectrum $2X(f)u(f)$, where $X(f)$ is the Fourier transform of $x(t)$.

(a) Show that the real and imaginary parts of $x_a(t)$ relate to $x(t)$ and its Hilbert transform $\hat{x}(t)$.

$$\begin{aligned}
x_a(t) &\longleftrightarrow 2X(f)u(f) \\
x_a(t) &\longleftrightarrow X(f)(1+\text{sgn}(f)) \\
x_a(t) &\longleftrightarrow X(f)(1-jj\text{sgn}(f)) \\
x_a(t) &\longleftrightarrow X(f) + j\big[-j\text{sgn}(f)X(f)\big]
\end{aligned}$$

So,

$$x_a(t) = x(t) + j\hat{x}(t)$$

(b) Find the analytic signal of $x(t) = A\cos(2\pi f_0 t + \theta)$.

We know that $\hat{x}(t) = A\sin(2\pi f_0 t + \theta)$. So,

$$x_a(t) = x(t) + j\hat{x}(t) = A\cos(2\pi f_0 t + \theta) + jA\sin(2\pi f_0 t + \theta) = Ae^{j(2\pi f_0 t + \theta)} = Ae^{j\theta}e^{j2\pi f_0 t}$$

(c) How does the analytic signal generalize the concept of phasors?

Clearly, for $x(t) = A\cos(2\pi f_0 t + \theta)$, $x_a(t)e^{-j2\pi f_0 t}$ equals the equivalent phasor of $x(t)$, i.e., $x_l = Ae^{j\theta}$. This can be simply generalized to the real signal $x(t) = A(t)\cos(2\pi f_0 t + \theta(t))$ with a time-varying amplitude and phase. In fact, the time-varying phasor of $x(t)$ is defined as $x_l(t) = x_a(t)e^{-j2\pi f_0 t}$.

(d) Surf the help page of MATLAB and describe how its Hilbert command works?

The Hilbert command returns the analytic part of its input argument. The analytic signal for a sequence x_r has a one-sided Fourier transform. That is, the transform vanishes for negative frequencies. To approximate the analytic signal, the Hilbert calculates the Fast Fourier Transform (FFT) of the input sequence, replaces those FFT coefficients that correspond to negative frequencies with zeros, and calculates the inverse FFT of the result. The Hilbert uses a four-step algorithm:

1. Calculate the FFT of the input sequence x_r, storing the result in a vector x.
2. Create a vector h whose elements $h(i)$ have the values:

$$h(i) = \begin{cases} 1 \text{ for } i = 1, (n/2) + 1 \\ 2 \text{ for } i = 2, 3, ..., (n/2) \\ 0 \text{ for } i = (n/2) + 2, ..., n \end{cases}$$

3. Calculate the element-wise product of x and h.
4. Calculate the inverse FFT of the sequence obtained in step 3 and returns the first n elements of the result.

The technique assumes that the input signal, x_r, is a finite block of data. This assumption allows the function to remove the spectral redundancy in x_r exactly. Methods based on FIR filtering can only approximate the analytic signal, but they have the advantage that they operate continuously on the data.

Question 5

Let $\{\phi_i(t)\}_{i=1}^N$ be an orthogonal set of N signals, i.e.,

$$\int_{-\infty}^{\infty} \phi_i(t)\phi_j^*(t)dt = 0, \quad 1 \le i, j \le N, \quad i \ne j$$

and

$$\int_{-\infty}^{\infty} |\phi_i(t)|^2 = 1, \quad 1 \le i \le N$$

. Let $\hat{x}(t) = \sum_{i=1}^N \alpha_i \phi_i(t)$ be the linear approximation of an arbitrary signal $x(t)$ in terms of $\{\phi_i(t)\}_{i=1}^N$, where α_i's are chosen such that

$$\epsilon^2 = \int_{-\infty}^{\infty} |x(t) - \hat{x}(t)|^2 dt$$

Question 5 continued on next page...

is minimized.

(a) Show that the minimizing α_i's satisfy

$$\alpha_i = \int_{-\infty}^{\infty} x(t)\phi_i^*(t)dt$$

$$\epsilon^2 = \int_{-\infty}^{\infty} |x(t) - \sum_{i=1}^{N} \alpha_i\phi_i(t)|^2 dt = \int_{-\infty}^{\infty} (x(t) - \sum_{i=1}^{N} \alpha_i\phi_i(t))(x^*(t) - \sum_{j=1}^{N} \alpha_j^*\phi_j^*(t)dt)$$

$$= \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{N} \alpha_i \int_{-\infty}^{\infty} \phi_i(t)x^*(t)dt - \sum_{j=1}^{N} \alpha_j^* \int_{-\infty}^{\infty} x(t)\phi_j^*(t)dt$$

$$+ \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i\alpha_j^* \int_{-\infty}^{\infty} \phi_i(t)\phi_j^*(t)dt$$

$$= \int_{-\infty}^{\infty} |x(t)|^2 dt + \sum_{i=1}^{N} |\alpha_i|^2 - \sum_{i=1}^{N} \alpha_i \int_{-\infty}^{\infty} \phi_i(t)x^*(t)dt - \sum_{j=1}^{N} \alpha_j^* \int_{-\infty}^{\infty} x(t)\phi_j^*(t)dt$$

Completing the square in terms of α_i, we obtain

$$\epsilon^2 = \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{N} \Big| \int_{-\infty}^{\infty} \phi_i^*(t)x(t)dt \Big|^2 + \sum_{i=1}^{N} \Big|\alpha_i - \int_{-\infty}^{\infty} \phi_i^*(t)x(t)dt\Big|^2 \tag{1}$$

The first two terms are independent of α_i and the last term is always positive. Therefore the minimum is achieved for

$$\alpha_i = \int_{-\infty}^{\infty} x(t)\phi_i^*(t)dt$$

(b) Show that

$$\epsilon_{\mathit{min}}^2 = \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{N} |\alpha_i|^2$$

With this choice of α_i, the last term of (1) vanishes and we get

$$\epsilon_{min}^2 = \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{N} \Big| \int_{-\infty}^{\infty} \phi_i^*(t)x(t)dt \Big|^2 = \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{N} |\alpha_i|^2$$

(c) How does this general linear approximation relate to the Fourier series expansion?

Taking $\phi_i(t) = e^{j2\pi it/T_0}$, $\hat{x}(t)$ roughly takes the form of the Fourier series expansion while the minimizing α_i's are very similar to the coefficients of the Fourier series expansion.

Question 6

The generalized Fourier transform of the singular function $y(t)$ is defined as the function $Y(f)$ satisfying the integral equation

$$\int_{-\infty}^{\infty} Y(\alpha)x(\alpha)d\alpha = \int_{-\infty}^{\infty} y(\beta)X(\beta)d\beta$$

, where $x(t)$ is any test function such that the existence of its Fourier transform $X(f)$ is guaranteed under Dirichlet sufficient conditions.
Hint: It can be shown that the properties of the normal Fourier transform remain valid for the generalized Fourier transform.

(a) Discuss the reasons behind the definition.

Assume that $X(f)$ and $Y(f)$, the Fourier transform of $x(t)$ and $y(t)$, exist. We have

$$\begin{aligned}
&\int_{-\infty}^{\infty} Y(\alpha)x(\alpha)d\alpha \\
&= \int_{\alpha=-\infty}^{\infty} Y(\alpha) \int_{\beta=-\infty}^{\infty} X(\beta)e^{j2\pi\beta\alpha}d\beta d\alpha \\
&= \int_{\alpha=-\infty}^{\infty} \int_{\beta=-\infty}^{\infty} Y(\alpha)X(\beta)e^{j2\pi\beta\alpha}d\beta d\alpha \\
&= \int_{\beta=-\infty}^{\infty} X(\beta) \int_{\alpha=-\infty}^{\infty} Y(\alpha)e^{j2\pi\beta\alpha}d\alpha d\beta \\
&= \int_{-\infty}^{\infty} X(\beta)y(\beta)d\beta \\
&= \int_{-\infty}^{\infty} y(\beta)X(\beta)d\beta
\end{aligned}$$

, which is another form of the Parseval's theorem.
Now, let $y(t)$ be a singular function, which does not satisfy Dirichlet sufficient conditions. Further, assume that $x(t)$ is an arbitrary signal, whose Fourier transform exists under Dirichlet sufficient conditions. Obviously, if this integral equation holds for all pairs of $x(t) \leftrightarrow X(f)$, $Y(f)$ can be considered as the generalized Fourier transform of $y(t)$.

(b) Use the definition to find the Fourier transform of $u(t)$.

At first, we use the definition of the generalized Fourier transform to show that $\mathcal{F}\{\delta(t)\} = 1$.

$$\int_{-\infty}^{\infty} Y(\alpha)x(\alpha)d\alpha = \int_{-\infty}^{\infty} y(\beta)X(\beta)d\beta = \int_{-\infty}^{\infty} \delta(\beta)X(\beta)d\beta = X(0) = \int_{-\infty}^{\infty} x(\alpha)d\alpha$$

Question 6 continued on next page...

So, $Y(f) = \mathcal{F}\{\delta(t)\} = 1$ by the definition of the equality of singular functions. Using the duality property, we conclude that $\mathcal{F}\{1\} = \delta(-f) = \delta(f)$. Now, we use the properties of the Fourier transform to show that $\mathcal{F}\{u(t)\} = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$. We have

$$u(t) + u(-t) = 1 \Rightarrow U(f) + U(-f) = \mathcal{F}\{1\} = \delta(f)$$

. Let $U(f) = B(f) + k\delta(f)$. We have

$$\delta(f) = U(f) + U(-f) = B(f) + B(-f) + k\delta(f) + k\delta(-f) = B(f) + B(-f) + 2k\delta(f)$$

Therefore,

$$k = \frac{1}{2}, \quad B(f) = -B(-f)$$

. To find $B(f)$,

$$1 = \mathcal{F}\{\delta(t)\} = \mathcal{F}\{u'(t)\} = j2\pi f\mathcal{F}\{u(t)\} = j2\pi f(B(f) + \frac{1}{2}\delta(f)) = j2\pi fB(f)$$

So, $B(f) = \frac{1}{j2\pi f}$ and

$$U(f) = B(f) + k\delta(f) = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$

.

(c) *Find the Fourier transform of sgn(t).*

$$\text{sgn}(t) = 2u(t) - 1 \Rightarrow \mathcal{F}\{\text{sgn}(t)\} = \mathcal{F}\{2u(t) - 1\} = 2\mathcal{F}\{u(t)\} - \mathcal{F}\{1\}$$
$$\Rightarrow 2(\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)) - \delta(f) = \frac{1}{j\pi f}$$

SOFTWARE QUESTIONS

Question 7

Write a MATLAB/Python code to calculate and plot the magnitude response, phase response, and impulse response of Butterworth filters with the frequency responses $H_n(f) = \frac{1}{B_n(j2\pi f)}$, where $B_n(s)$ is given in Tab. 1. Note that you should create three plots for the magnitude, phase, and impulse responses. In each plot, you should have 5 curves, each corresponding to a different value of the filter order $n = 1, 2, \cdots, 5$. Describe what happens when the filter order increases.

Here is a sample implementation.

```
clc; clear; close all;
```

Order n	**Butterworth Polynomial** $B_n(s)$
1	$s+1$
2	$s^2+\sqrt{2}s+1$
3	$(s+1)(s^2+s+1)$
4	$(s^2+\sqrt{2-\sqrt{2}}s+1)(s^2+\sqrt{2+\sqrt{2}}s+1)$
5	$(s+1)(s^2+\frac{2}{1+\sqrt{5}}s+1)(s^2+\frac{1+\sqrt{5}}{2}s+1)$

Table 1: Butterworth polynomials of order $n = 1, 2, \cdots, 5$.

```
% defining variables

j = sqrt(-1);
syms f;
s = j*2*pi*f;

% defining an array with Butterworth Polynomials

B(1) = s+1;
B(2) = s^2 + sqrt(2)*s + 1;
B(3) = (s + 1)*(s^2 + s + 1);
B(4) = (s^2 + sqrt(2-sqrt(2))*s + 1)*(s^2 + sqrt(2+sqrt(2))*s + 1);
B(5) = (s + 1)*(s^2 + 2/(1+sqrt(5))*s + 1)*(s^2 + ...
    +(1+sqrt(5))/2*s + 1);

% Plotting magnitude responses

figure;
for i=1:5
    fplot(abs(1/B(i)),'linewidth',2);
    hold on;
end
grid minor;
ylabel('$$|H_n(f)|$$','interpreter','LaTeX','FontSize',25);
xlabel('$$f$$','interpreter','LaTeX','FontSize',25);
ylim([0 1.1]);
xlim([-1 1]);
legend('n=1','n=2','n=3','n=4','n=5','interpreter','LaTeX');


% Plotting phase responses

figure;
for i=1:5
    fplot(phase(1/B(i)),'linewidth',2);
    hold on;
end
grid minor;
ylabel('$$ \angle H_n(f)$$','interpreter','latex','FontSize',25);
xlabel('$$f$$','interpreter','LaTeX','FontSize',25);
ylim([-10 10]);
xlim([-2 2]);
legend('n=1','n=2','n=3','n=4','n=5','interpreter','LaTeX');


% Plotting impulse responses

f = figure;
for i=1:5
    fplot(ifourier(1/B(i)),'linewidth',2);
    hold on;
end
grid minor;
```

Question 7 continued on next page...

```
ylabel('$$h_n(t)$$','interpreter','LaTeX','FontSize',25);
xlabel('$$t$$','interpreter','LaTeX','FontSize',25);
ylim([-0.05 0.2]);
xlim([0 50]);
legend('n=1','n=2','n=3','n=4','n=5','interpreter','LaTeX');
```

The magnitude and phase responses are shown in Fig. 1 and 2 while the impulse response is plotted in Fig. 3. Clearly, increasing the filter order n, the filtering response approximates the ideal filter response better.

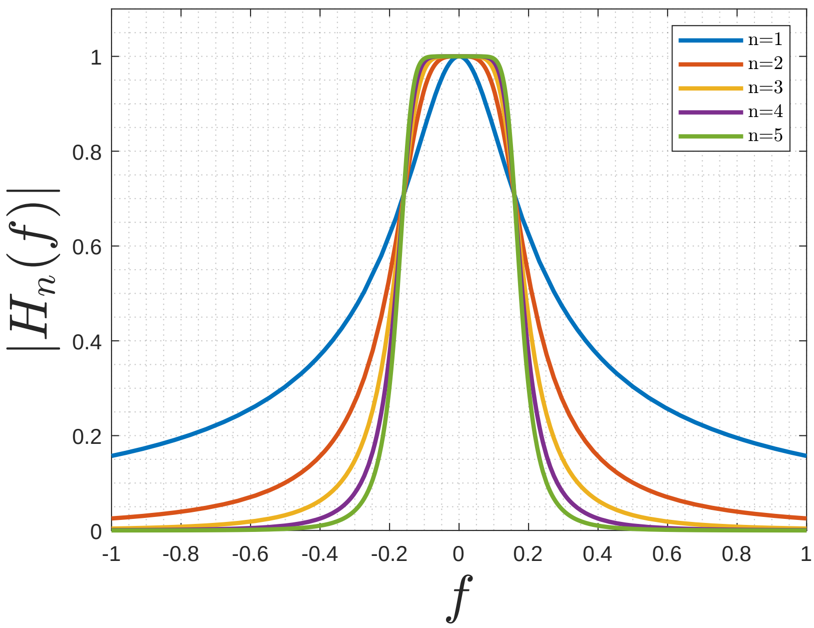

Figure 1: Magnitude response versus frequency for various values of filter order n.

Question 7 continued on next page...

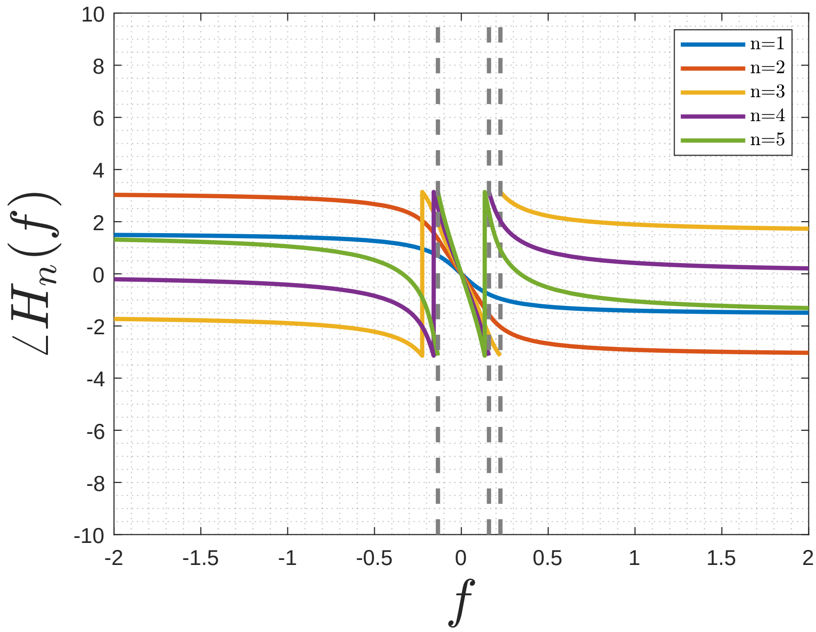

Figure 2: Phase response versus frequency for various values of filter order n.

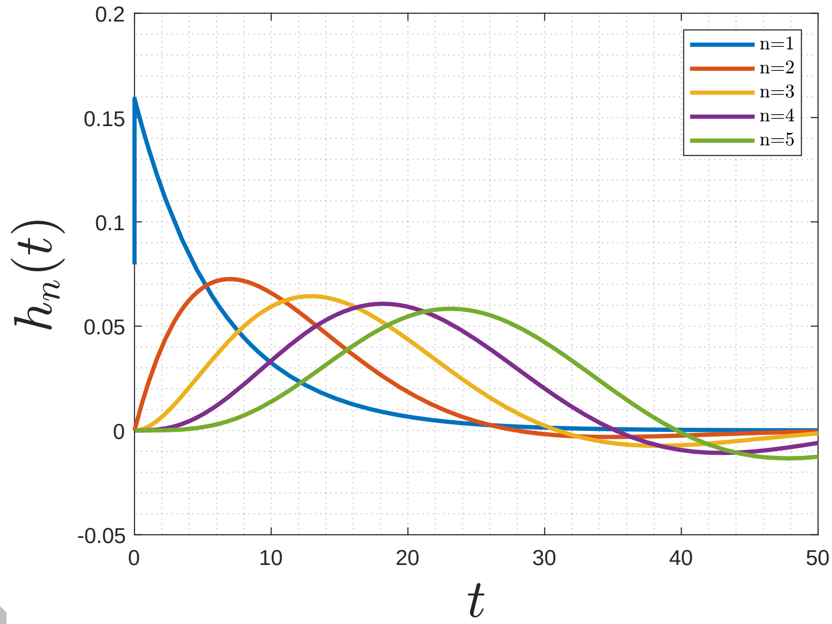

Figure 3: Impulse response versus time for various values of filter order n.

BONUS QUESTIONS

Question 8

Verify that the amplitude response $|H_n(f)| = \frac{1}{\sqrt{1+(2\pi f)^{2n}}}$ for each row of Tab. 1.

$$H_n(f) = \frac{1}{B_n(j2\pi f)} \Rightarrow |H_n(f)| = \frac{1}{|B_n(j2\pi f)|}$$

For $n = 1$,

$$\frac{1}{|B_n(j2\pi f)|} = \frac{1}{|j2\pi f + 1|} = \frac{1}{\sqrt{1^2 + (2\pi f)^2}} = \frac{1}{\sqrt{1 + (2\pi f)^2}}$$

For $n = 2$,

$$\frac{1}{|B_n(j2\pi f)|} = \frac{1}{|(j2\pi f)^2 + \sqrt{2}(j2\pi f) + 1|} = \frac{1}{|1 - (2\pi f)^2 + j2\sqrt{2}\pi f|}$$

$$= \frac{1}{\sqrt{(1 - (2\pi f)^2)^2 + (2\sqrt{2}\pi f)^2}} = \frac{1}{\sqrt{1 + (2\pi f)^4 - 8(\pi f)^2 + 8(\pi f)^2}} = \frac{1}{\sqrt{1 + (2\pi f)^4}}$$

For $n = 3$,

$$B_n(s) = s^3 + 2s^2 + 2s + 1 \Rightarrow B_n(j2\pi f) = (j2\pi f)^3 + 2(j2\pi f)^2 + 2(j2\pi f) + 1$$

$$= -8j(\pi f)^3 - 8(\pi f)^2 + 4j(\pi f) + 1 = (1 - 8(\pi f)^2) + j(-8(\pi f)^3 + 4(\pi f))$$

$$\Rightarrow |B_n(j2\pi f)| = \sqrt{1 + 64(\pi f)^4 - 16(\pi f)^2 + 64(\pi f)^6 + 16(\pi f)^2 - 64(\pi f)^4}$$

$$= \sqrt{1 + 64(\pi f)^6} = \sqrt{1 + (2\pi f)^6} \Rightarrow |H_n(f)| = \frac{1}{|B_n(j2\pi f)|} = \frac{1}{\sqrt{1 + (2\pi f)^6}}$$

For $n = 4$,

$$B_n(s) = s^4 + (\sqrt{2+\sqrt{2}} + \sqrt{2-\sqrt{2}})s^3 + (2+\sqrt{2})s^2 + (\sqrt{2+\sqrt{2}} + \sqrt{2-\sqrt{2}})s + 1$$

$$\Rightarrow (j2\pi f)^4 + (\sqrt{2+\sqrt{2}} + \sqrt{2-\sqrt{2}})(j2\pi f)^3 + (2+\sqrt{2})(j2\pi f)^2 + (\sqrt{2+\sqrt{2}} + \sqrt{2-\sqrt{2}})(j2\pi f)$$

$$+1 = (2\pi f)^4 - j(\sqrt{2+\sqrt{2}} + \sqrt{2-\sqrt{2}})(2\pi f)^3 - (2+\sqrt{2})(2\pi f)^2 + j(\sqrt{2+\sqrt{2}} + \sqrt{2-\sqrt{2}})(2\pi f)$$

$$+1 = ((2\pi f)^4 - (2+\sqrt{2})(2\pi f)^2 + 1) - j(\sqrt{2+\sqrt{2}} + \sqrt{2-\sqrt{2}})(2\pi f)((2\pi f)^2 - 1)$$

$$\Rightarrow |B_n(j2\pi f)| = \sqrt{[(2\pi f)^8 - 2(2+\sqrt{2})(2\pi f)^6 + 4(2+\sqrt{2})(2\pi f)^4 - 2(2+\sqrt{2})(2\pi f)^2 + 1] \ldots}$$

$$\overline{\ldots + [2(2+\sqrt{2})((2\pi f)^6 - 2(2\pi f)^4 + (2\pi f)^2)]} = \sqrt{1 + (2\pi f)^8}$$

$$\Rightarrow |H_n(f)| = \frac{1}{|B_n(j2\pi f)|} = \frac{1}{\sqrt{1 + (2\pi f)^8}}$$

For $n = 5$,

$$B_n(s) = s^5 + (1+\sqrt{5})s^4 + (3+\sqrt{5})s^3 + (3+\sqrt{5})s^2 + (1+\sqrt{5})s + 1$$

$$\Rightarrow B_n(j2\pi f) = (j2\pi f)^5 + (1+\sqrt{5})(j2\pi f)^4 + (3+\sqrt{5})(j2\pi f)^3 + (3+\sqrt{5})(j2\pi f)^2 + (1+\sqrt{5})(j2\pi f)$$

$$+1 = j(2\pi f)^5 + (1+\sqrt{5})(2\pi f)^4 - j(3+\sqrt{5})(2\pi f)^3 - (3+\sqrt{5})(2\pi f)^2 + j(1+\sqrt{5})(2\pi f) + 1$$

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$$= [(1+\sqrt{5})(2\pi f)^4 - (3+\sqrt{5})(2\pi f)^2 + 1] + j[(2\pi f)^5 - (3+\sqrt{5})(2\pi f)^3 + (1+\sqrt{5})(2\pi f)]$$
$$\Rightarrow |B_n(j2\pi f)| = ([(1+\sqrt{5})(2\pi f)^4 - (3+\sqrt{5})(2\pi f)^2 + 1]^2$$
$$+[(2\pi f)^5 - (3+\sqrt{5})(2\pi f)^3 + (1+\sqrt{5})(2\pi f)]^2)^{\frac{1}{2}} = (2(3+\sqrt{5})(2\pi f)^8 - 8(2+\sqrt{5})(2\pi f)^6$$
$$+8(2+\sqrt{5})(2\pi f)^4 - 2(3+\sqrt{5})(2\pi f)^2 + 1 + (2\pi f)^{10} - 2(3+\sqrt{5})(2\pi f)^8 + 8(2+\sqrt{5})(2\pi f)^6$$
$$-8(2+\sqrt{5})(2\pi f)^4 + 2(3+\sqrt{5})(2\pi f)^2)^{\frac{1}{2}} = (1+(2\pi f)^{10})^{\frac{1}{2}}$$
$$\Rightarrow |H_n(f)| = \frac{1}{|B_n(j2\pi f)|} = \frac{1}{\sqrt{1+(2\pi f)^{10}}}$$

Question 9

Return your answers by filling the LaTeXtemplate of the assignment.