MATHEMATICAL QUESTIONS

Question 1

Use the integral definitions of the singular functions to prove the following identities.

(a) $tu_1(t) = -u_0(t)$.

(b) $f(t)\delta'(t) = -f'(0)\delta(t) + f(0)\delta'(t)$.

(c) $\delta^{(n)}(-t) = (-1)^n\delta^{(n)}(t)$.

Question 2

Prove that

(a) $\delta(t) = \frac{1}{\pi}\int_0^{+\infty}\cos(\alpha t)d\alpha$.

(b) $\sum_{n=-\infty}^{\infty} x(t+2nl) = \frac{1}{2l}\sum_{n=-\infty}^{\infty} e^{\frac{j\pi nt}{l}} X(\frac{n}{2l})$, *where* $X(f) = \mathcal{F}\{x(t)\}$.

Question 3

Assume that $X(f) = \mathcal{F}\{x(t)\}$ and $Y(f) = \mathcal{F}\{y(t)\}$. Show that

(a) $\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df$.

(b) $\int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau = \int_{-\infty}^{\infty} X(f)Y(f)e^{j2\pi ft}df$.

Question 4 continued on next page...

Question 4

The analytic signal $x_a(t)$ of the real signal $x(t)$ is a signal with the spectrum $2X(f)u(f)$, where $X(f)$ is the Fourier transform of $x(t)$.

(a) Show that the real and imaginary parts of $x_a(t)$ relates to $x(t)$ and its Hilbert transform $\hat{x}(t)$.

(b) Find the analytic signal of $x(t) = A\cos(2\pi f_0 t + \theta)$.

(c) How does the analytic signal generalize the concept of phasors?

(d) Surf the help page of MATLAB and describe how its Hilbert command works?

Question 5

Let $\{\phi_i(t)\}_{i=1}^N$ be an orthogonal set of N signals, i.e.,

$$\int_{-\infty}^{\infty} \phi_i(t)\phi_j^*(t)dt = 0, \quad 1 \le i, j \le N, \quad i \ne j$$

and

$$\int_{-\infty}^{\infty} |\phi_i(t)|^2 = 1, \quad 1 \le i \le N$$

. Let $\hat{x}(t) = \sum_{i=1}^N \alpha_i \phi_i(t)$ be the linear approximation of an arbitrary signal $x(t)$ in terms of $\{\phi_i(t)\}_{i=1}^N$, where α_i's are chosen such that

$$\epsilon^2 = \int_{-\infty}^{\infty} |x(t) - \hat{x}(t)|^2 dt$$

is minimized.

(a) Show that the minimizing α_i's satisfy

$$\alpha_i = \int_{-\infty}^{\infty} x(t)\phi_i^*(t)dt$$

Question 5 continued on next page...

(b) Show that

$$\epsilon_{min}^2 = \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{N} |\alpha_i|^2$$

(c) How does this general linear approximation relate to the Fourier series expansion?

Question 6

The generalized Fourier transform of the singular function $y(t)$ is defined as the function $Y(f)$ satisfying the integral equation

$$\int_{-\infty}^{\infty} Y(\alpha)x(\alpha)d\alpha = \int_{-\infty}^{\infty} y(\beta)X(\beta)d\beta$$

, where $x(t)$ is any test function such that the existence of its Fourier transform $X(f)$ is guaranteed under Dirichlet sufficient conditions.
Hint: It can be shown that the properties of the normal Fourier transform remain valid for the generalized Fourier transform.

(a) Discuss the reasons behind the definition.

(b) Use the definition to find the Fourier transform of $u(t)$.

(c) Use the definition to find the Fourier transform of sgn(t).

SOFTWARE QUESTIONS

Question 7

Write a MATLAB/Python code to calculate and plot the magnitude response, phase response, and impulse response of Butterworth filters with the frequency responses $H_n(f) = \frac{1}{B_n(j2\pi f)}$, where $B_n(s)$ is given in Tab. 1. Note that you should create three plots for the magnitude,

Question 7 continued on next page…

Order n	**Butterworth Polynomial** $B_n(s)$
1	$s+1$
2	$s^2+\sqrt{2}s+1$
3	$(s+1)(s^2+s+1)$
4	$(s^2+\sqrt{2-\sqrt{2}}s+1)(s^2+\sqrt{2+\sqrt{2}}s+1)$
5	$(s+1)(s^2+\frac{2}{1+\sqrt{5}}s+1)(s^2+\frac{1+\sqrt{5}}{2}s+1)$

Table 1: Butterworth polynomials of order $n=1,2,\cdots,5$.

phase, and impulse responses. In each plot, you should have 5 curves, each corresponding to a different value of the filter order $n=1,2,\cdots,5$. Describe what happens when the filter order increases.

BONUS QUESTIONS

Question 8

Verify that the amplitude response $|H_n(f)|=\frac{1}{\sqrt{1+(2\pi f)^{2n}}}$ for each row of Tab. 1.

Question 9

Return your answers by filling the LaTeXtemplate of the assignment.