MATHEMATICAL QUESTIONS

Question 1

Calculate the power content of the SSB signal $u(t) = A_c m(t)\cos(2\pi f_c t) \mp A_c \hat{m}(t)\sin(2\pi f_c t)$.

$$P_u = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |u(t)|^2 dt$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} [A_c m(t)\cos(2\pi f_c t) \mp A_c \hat{m}(t)\sin(2\pi f_c t)]^2 dt$$

$$= \lim_{T\to\infty}\frac{A_c^2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} [m^2(t)\cos^2(2\pi f_c t) + \hat{m}^2(t)\sin^2(2\pi f_c t) \mp 2m(t)\hat{m}(t)\cos(2\pi f_c t)\sin(2\pi f_c t)]dt$$

$$= \lim_{T\to\infty}\frac{A_c^2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} [m^2(t)\frac{1+\cos(4\pi f_c t)}{2} + \hat{m}^2(t)\frac{1-\cos(4\pi f_c t)}{2} \mp m(t)\hat{m}(t)\sin(4\pi f_c t)]dt$$

$$= \lim_{T\to\infty}\frac{A_c^2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1}{2}m^2(t)dt + \lim_{T\to\infty}\frac{A_c^2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1}{2}\hat{m}^2(t)dt$$

$$+ \lim_{T\to\infty}\frac{A_c^2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} [\frac{1}{2}m^2(t)\cos(4\pi f_c t) - \frac{1}{2}\hat{m}^2(t)\cos(4\pi f_c t) \mp m(t)\hat{m}(t)\sin(4\pi f_c t)]dt \tag{1}$$

Clearly,

$$\lim_{T\to\infty}\frac{A_c^2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1}{2}m^2(t)dt = \frac{A_c^2}{2}P_m \tag{2}$$

Further,

$$\lim_{T\to\infty}\frac{A_c^2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1}{2}\hat{m}^2(t)dt = \frac{A_c^2}{2}P_m \tag{3}$$

, where we have used the fact that the power of a signal and its Hilbert transform is equal. Now, we show that the remaining three terms in (1) are equal to zero. We have

$$\lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} m^2(t)\cos(4\pi f_c t)dt = \int_{-\infty}^{\infty} m(t).[m(t)\cos(4\pi f_c t)]^* dt \tag{4}$$

Let $M(f) = \mathcal{F}\{m(t)\}$. According to Parseval's theorem,

$$\int_{-\infty}^{\infty} m(t).[m(t)\cos(4\pi f_c t)]^* dt = \int_{-\infty}^{\infty} M(f) \times \frac{1}{2}[M(f-2f_c) + M(f+2f_c)]^* df \tag{5}$$

The message signal $m(t)$ has a bandwidth of $W \ll f_c$. So, there is not an overlap between $M(f)$ and its shifted versions $M(f \pm 2f_c)$ and therefore, $M(f) \times M^*(f \pm 2f_c) = 0$ and (5) equals zero. As a result,

$$\lim_{T\to\infty}\frac{A_c^2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1}{2}m^2(t)\cos(4\pi f_c t)dt = 0 \tag{6}$$

Question 1 continued on next page...

Using a similar approach, we can show that

$$\lim_{T\to\infty}\frac{A_c^2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\frac{1}{2}\hat{m}^2(t)\cos(4\pi f_c t)dt = 0 \tag{7}$$

$$\lim_{T\to\infty}\frac{A_c^2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\frac{1}{2}m(t)\hat{m}(t)\sin(4\pi f_c t)dt = 0 \tag{8}$$

Noting (2), (3), (6), (7), and (8), we conclude that

$$P_u = \frac{A_c^2}{2}P_m + \frac{A_c^2}{2}P_m + 0 + 0 + 0 = A_c^2 P_m \tag{9}$$

Question 2

Find expressions for the in-phase and quadrature components, $x_c(t)$ and $x_s(t)$, as well as the envelope and phase, $V(t)$ and $\Theta(t)$ for DSB and SSB signals.

1. A DSB modulated signal is expressed as $u(t) = Am(t)\cos(2\pi f_0 t)$. Hence,

$$\begin{aligned} x_c(t) &= Am(t) \\ x_s(t) &= 0 \\ V(t) &= A|m(t)| \\ \Theta(t) &= 0 \end{aligned}$$

2. An SSB signal is written as $u_{\mathrm{SSB}}(t) = Am(t)\cos(2\pi f_0 t) \mp A\hat{m}(t)\sin(2\pi f_0 t)$. For the USSB signal (minus sign)

$$\begin{aligned} x_c(t) &= Am(t) \\ x_s(t) &= A\widehat{m}(t) \\ V(t) &= \sqrt{A^2\left(m^2(t)+\widehat{m}^2(t)\right)} = A\sqrt{m^2(t)+\widehat{m}^2(t)} \\ \Theta(t) &= \arctan\left(\frac{\widehat{m}(t)}{m(t)}\right) \end{aligned}$$

. For the LSSB signal (plus sign)

$$\begin{aligned} x_c(t) &= Am(t) \\ x_s(t) &= -A\widehat{m}(t) \\ V(t) &= \sqrt{A^2\left(m^2(t)+\widehat{m}^2(t)\right)} = A\sqrt{m^2(t)+\widehat{m}^2(t)} \\ \Theta(t) &= -\arctan\left(\frac{\widehat{m}(t)}{m(t)}\right) \end{aligned}$$

Question 3

The message signal $m(t)$ is applied to the system shown in Fig. 1 to generate the signal $y(t)$.

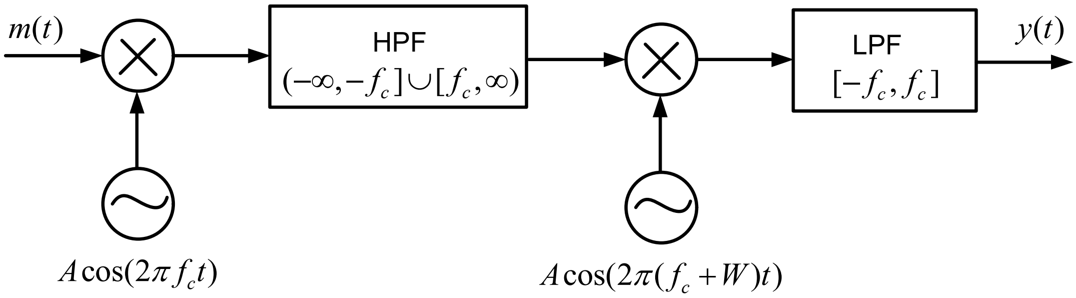

Figure 1: A sample scrambler.

(a) *Find the spectrum of* $y(t)$*, i.e.* $Y(f)$*.*

The spectrum of the modulated signal $Am(t)\cos(2\pi f_c t)$ is

$$V(f) = \frac{A}{2}[M(f - f_c) + M(f + f_c)]$$

. The spectrum of the signal at the output of the highpass filter is

$$U(f) = \frac{A}{2}[M(f + f_c)u(-f - f_c) + M(f - f_c)u(f - f_c)]$$

. Multiplying the output of the HPF with $A\cos(2\pi(f_c + W)t)$ results in the signal $z(t)$ with the spectrum

$$Z(f) =$$
$$\frac{A}{2}[M(f + f_c)u(-f - f_c) + M(f - f_c)u(f - f_c)] * \frac{A}{2}[\delta(f - (f_c + W)) + \delta(f + f_c + W)] =$$
$$\frac{A^2}{4}[M(f + f_c - f_c - W)u(-f + f_c + W - f_c) + M(f + f_c - f_c + W)u(f + f_c + W - f_c)$$
$$+ M(f - 2f_c - W)u(f - 2f_c - W) + M(f + 2f_c + W)u(-f - 2f_c - W)] =$$
$$\frac{A^2}{4}[M(f - W)u(-f + W) + M(f + W)u(f + W) + M(f - 2f_c - W)u(f - 2f_c - W)$$
$$+ M(f + 2f_c + W)u(-f - 2f_c - W)]$$

The lowpass filter will cut the double frequency components, leaving the spectrum:

$$Y(f) = \frac{A^2}{4}[M(f - W)u(-f + W) + M(f + W)u(f + W)]$$

(b) *Show that if* $y(t)$ *is transmitted, the receiver can pass it through a replica of the system shown in Fig. 1 to obtain* $m(t)$ *back.*

As it is observed from the spectrum $Y(f)$, the system shifts the negative frequency components to the positive frequency axis and the positive frequency components to the negative frequency axis. If we transmit the signal $y(t)$ through the system, we will get a scaled version of the original spectrum $M(f)$.

(c) How can the system be used as a simple scrambler to enhance communication privacy?

Clearly, the system reorders the frequency components of the message spectrum such that a positive component is located in the negative axis and vice versa. Now, if an intruder wants to eavesdrop the message, the reordering mechanism needs to be known. Otherwise, the intruder cannot extract the message.

Question 4

In Fig. 2, the transmitted power is 125 mW and the minimum acceptable SNR at the receiver is 30 dB. Assuming the noise power spectral density $N_0 = 4 \times 10^{-21}$ W/Hz and the bandwidth $B = 400$ kHz, answer the following questions.

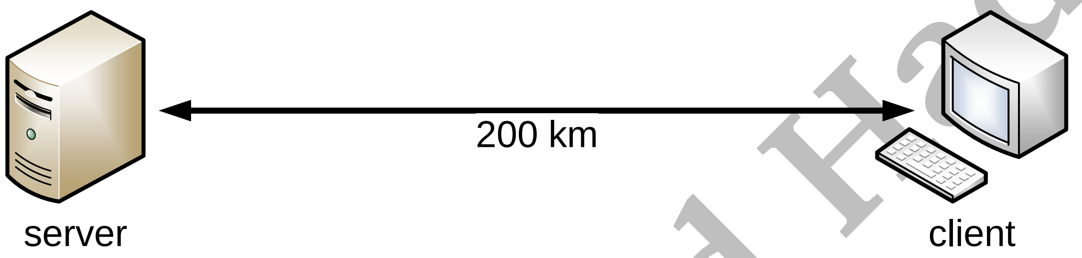

Figure 2: Server and Client.

(a) Suppose the server is connected to the client with a copper cable, and the transmission loss of the line is 2 dB/km. Is it possible to have a successful connection without using any amplifier?

$$L = 200 \times 2 = 400 \text{ dB} = 10\log_{10}\left(\frac{P_{in}}{P_{out}}\right) = 10\log_{10}\left(\frac{125 \text{ mW}}{P_{out}}\right) \Rightarrow P_{out} = 125 \times 10^{-40} \text{ mW}$$

$$P_{noise} = N_0 \times BW = 4 \times 10^{-21} \times 400 \times 10^3 = 16 \times 10^{-13} \text{ mW}$$

Hence,

$$\text{SNR} = 10\log_{10}\left(\frac{P_{out}}{P_{noise}}\right) = 10\log_{10}\left(\frac{125 \times 10^{-40}}{16 \times 10^{-13}}\right) = -261.07 \text{ dB} < 30 \text{ dB}$$

Thus, it is not possible to have a successful connection without using any amplifiers.

(b) Assume that we have m identical amplifiers, each with a gain of 10 dB. If your answer to the last part is no, find the minimum of m to have a successful connection.

$$P_{out} = P_{in}\frac{G_t}{L} \Rightarrow P_{out,dBm} = P_{in,dBm} + G_{t,dB} - L_{dB}$$

$$\text{SNR} \geq 30 \text{ dB} \Rightarrow 10\log_{10}\left(\frac{P_{out}}{P_{noise}}\right) \geq 30 \Rightarrow P_{out} \geq 16\times10^{-10} \text{ mW} \Rightarrow P_{out,dBm} \geq -87.96 \text{ dBm}$$

$$L_{dB} = 200 \times 2 = 400 \text{ dB}, \quad P_{in,dBm} = 10\log_{10}(125) = 20.97 \text{ dBm}$$

Question 4 continued on next page...

Now,

$P_{out,dBm} = P_{in,dBm} + G_{t,dB} - L_{dB} = 20.97 - 400 + G_{t,dB} \geq -87.96$ dBm $\Rightarrow G_{t,dB} \geq 291.07$

So, at least $m = \lceil 291.07/10 \rceil = 30$ amplifiers are required.

(c) Repeat the previous parts assuming the server and the client are connected with an optical fiber, and the transmission loss of the line is 0.3 dB/km.

$$P_{out} = P_{in}\frac{G_t}{L} \Rightarrow P_{out,dBm} = P_{in,dBm} + G_{t,dB} - L_{dB}$$

SNR ≥ 30 dB $\Rightarrow 10\log_{10}\big(\frac{P_{out}}{P_{noise}}\big) \geq 30 \Rightarrow P_{out} \geq 16\times 10^{-10}$ mW $\Rightarrow P_{out,dBm} \geq -87.96$ dBm

$$L_{dB} = 200 \times 0.3 = 60 \text{ dB}, \quad P_{in,dBm} = 10\log_{10}(125) = 20.97 \text{ dBm}$$

Now,

$P_{out,dBm} = P_{in,dBm} + G_{t,dB} - L_{dB} = 20.97 - 60 + G_{t,dB} \geq -87.96$ dBm $\Rightarrow G_{t,dB} \geq -48.93$

So, no amplifier is required.

Question 5

Fig. 3 shows the schematic of a ring modulator.

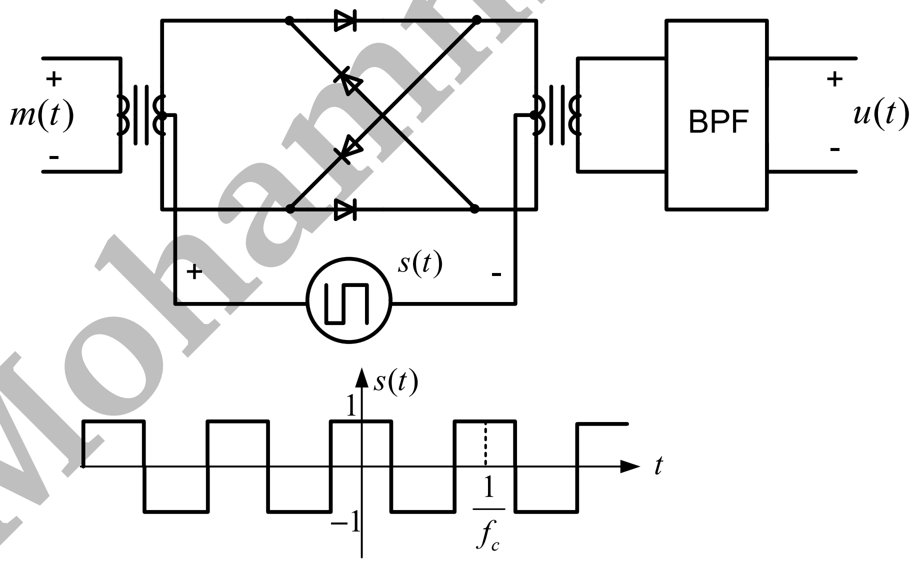

Figure 3: Ring modulator.

Question 5 continued on next page. . .

(a) *Show that if $|m(t)| \ll 1$, then $u(t)$ is a DSB-modulated signal.*

Clearly, if $|m(t)| \ll 1$, the diodes are turned on or off according to the periodic signal $s(t)$. So, the input signal to the BPF is

$$v(t) = \begin{cases} m(t) & s(t) \geqslant 0 \\ -m(t) & s(t) < 0 \end{cases} = m(t)s(t)$$

Applying Fourier series expansion,

$$v(t) = m(t)\frac{4}{\pi}\sum_{1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos(2\pi f_c(2n-1)t)$$

Passing $v(t)$ through a BPF with the center frequency $f = f_c$ and the bandwidth $2W$,

$$u(t) = \frac{4}{\pi}m(t)\cos(2\pi f_c t)$$

(b) *What are the main roles of the three-winding transformers in the ring modulator?*

The transfor is a versatile element that helps to remove the DC of the message, amplify or attenuate the message, and provide float reference points to connect the signal $s(t)$.

Question 6

An SSB-modulated signal having the bandwidth W and power P_m passes through a distortion-less channel with the attenuation A and delay D, as shown in Fig. 4.

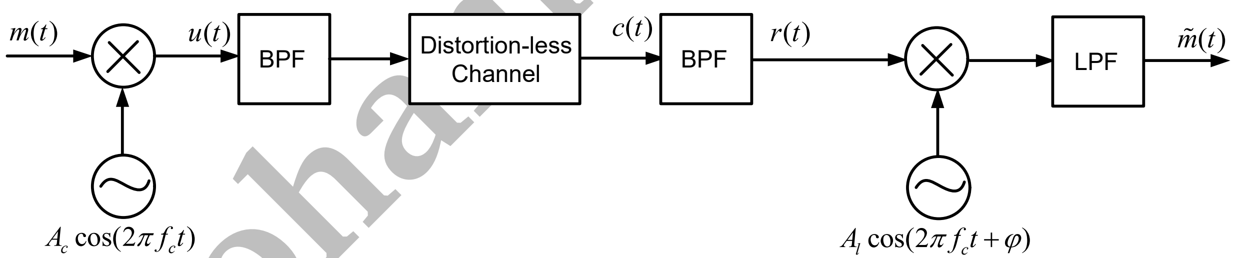

Figure 4: Block diagram of an SSB communication system.

(a) *Find the power of the detected signal $\tilde{m}(t)$.*

Without loss of generality, we assume that the signal is USSB.

$$u(t) = m(t) \times A_c\cos(2\pi f_c t)$$

$$c(t) = A \times m(t-D) \times A_c\cos(2\pi f_c(t-D)) - A \times \hat{m}(t-D) \times A_c\sin(2\pi f_c(t-D))$$

Question 6 continued on next page...

$$\tilde{m}(t) = \text{LPF}\big\{(A \times m(t-D) \times A_c \cos(2\pi f_c(t-D) - A \times \hat{m}(t-D) \times A_c \sin(2\pi f_c(t-D)))$$
$$\times A_l \cos(2\pi f_c t + \phi)\big\}$$
$$= \frac{AA_cA_l}{2}(m(t-D)\cos(\phi + 2\pi f_c D) + \hat{m}(t-D)\sin(\phi + 2\pi f_c D))$$

$$P_{\tilde{m}} = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}(\frac{AA_cA_l}{2}(m(t-D)\cos(\phi + 2\pi f_c D) + \hat{m}(t-D)\sin(\phi + 2\pi f_c D)))^2 dt$$
$$= \frac{A^2A_c^2A_l^2}{4}(P_m\cos^2(\phi + 2\pi f_c D) + P_m\sin^2(\phi + 2\pi f_c D)) = \frac{A^2A_c^2A_l^2}{4}P_m$$

(b) Find the power of the signal part in the detected signal $\tilde{m}(t)$.

$$\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\frac{AA_cA_l}{2}(m(t-D)\cos(\phi + 2\pi f_c D))^2 dt$$
$$= \frac{A^2A_c^2A_l^2}{4}\cos^2(\phi + 2\pi f_c D)P_m$$

(c) Calculate the value of the phase shift ϕ resulting in the maximum power of the signal part in the detected signal $\tilde{m}(t)$.

For the maximum power, $\cos^2(\phi + 2\pi f_c D) = 1$. So,

$$\phi = n\pi - 2\pi f_c D$$

SOFTWARE QUESTIONS

Question 7

Develop a MATLAB/Python code that accepts the message signal $m(t)$ and generates time- and frequency-domain plots of the corresponding DSB-, LSSB-, USSB-, and AM-modulated signals. The input arguments of the function are message signal $m(t)$, carrier frequency f_c, carrier amplitude A_c, AM modulation index a, and modulation type. Investigate the outputs of the function for a sample message signal.

Here is a sample implementation.

```
clear all; close all; clc;

ts = 1/100;
t = -10 : ts : 10;
m = (sinc(25*t)).^2;

modulation(t, m, ts, 50, 1, 1, 'AM');
modulation(t, m, ts, 50, 1, 1, 'DSB');
modulation(t, m, ts, 50, 1, 1, 'USSB');
modulation(t, m, ts, 50, 1, 1, 'LSSB');

function [] = modulation(t, m, ts, fc, Ac, a, modulationType)

    f = (-length(m)/2:length(m)/2-1)/length(m)/ts;

    if modulationType == 'AM'
        xc = Ac .* (1 + a*m) .* cos(2*pi*fc*t);
        xf = abs(ts.*fftshift(fft(xc)));
        figure;
        subplot(2, 1, 1);
        plot(t, xc);
        ylabel('xc');
        xlabel('t');
        title('AM');
        subplot(2, 1, 2);
        plot(f, xf);
        ylabel('Xf');
        xlabel('f');


    elseif modulationType == 'DSB'
        xc = Ac .* m .* cos(2*pi*fc*t);
        xf = abs(ts.*fftshift(fft(xc)));
        figure;
        subplot(2, 1, 1);
        plot(t, xc);
        ylabel('xc');
        xlabel('t');
        title('DSB');
        subplot(2, 1, 2);
        plot(f, xf);
        ylabel('Xf');
        xlabel('f');

    elseif modulationType == 'LSSB'
        hilbert_m = hilbert(m);
        xc = Ac .* m .* cos(2*pi*fc*t) + Ac .* hilbert_m .* sin(2*pi*fc*t);
        xf = abs(ts.*fftshift(fft(xc)));
        figure;
        subplot(2, 1, 1);
        plot(t, abs(xc));
        ylabel('xc');
        xlabel('t');
        title('LSSB');
        subplot(2, 1, 2);
        plot(f, xf);
        ylabel('Xf');
        xlabel('f');

    elseif modulationType == 'USSB'
        hilbert_m = hilbert(m);
        xc = Ac .* m .* cos(2*pi*fc*t) - Ac .* hilbert_m .* sin(2*pi*fc*t);
        xf = abs(ts.*fftshift(fft(xc)));
        figure;
        subplot(2, 1, 1);
        plot(t, abs(xc));
        ylabel('xc');
```

```
        xlabel('t');
        title('USSB');
        subplot(2, 1, 2);
        plot(f, xf);
        ylabel('Xf');
        xlabel('f');
    end
end
```

Figs. 5-8 illustrate the time- and frequency-domain plots of the DSB-, AM, USSB-, and LSSB-modulated signals for a sinc-squared sample message.

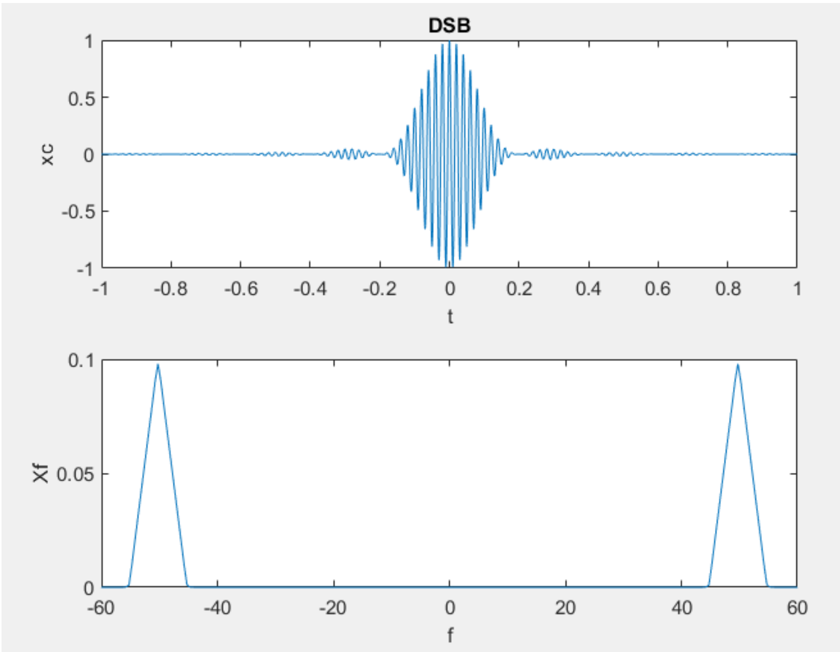

Figure 5: Time- and frequency-domain plots for a DSB-modulated signal.

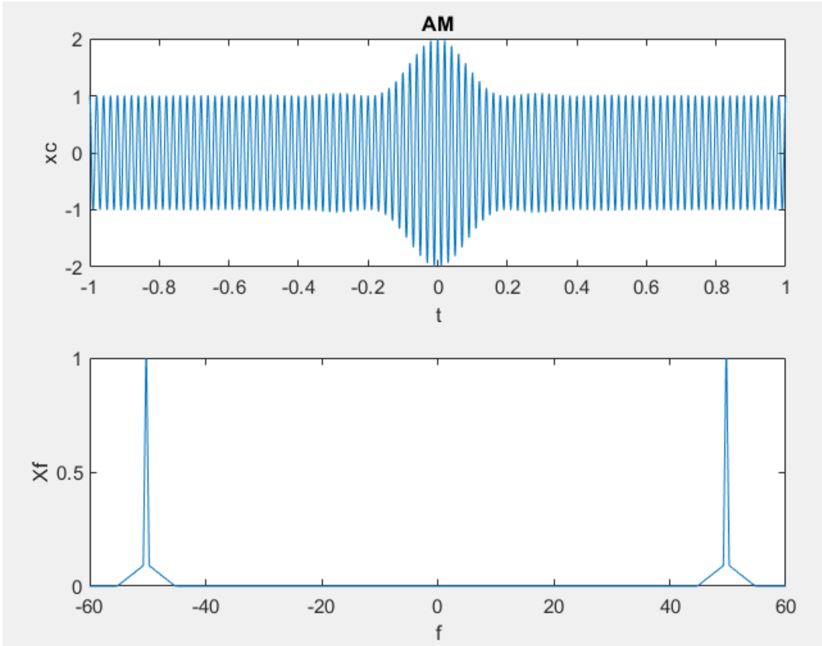

Figure 6: Time- and frequency-domain plots for an AM-modulated signal..

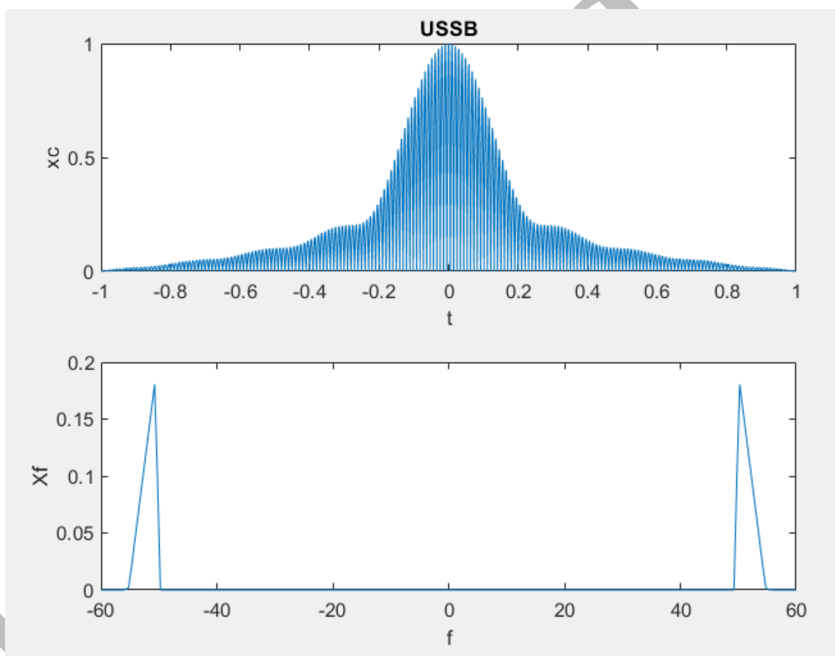

Figure 7: Time- and frequency-domain plots for a USSB-modulated signal..

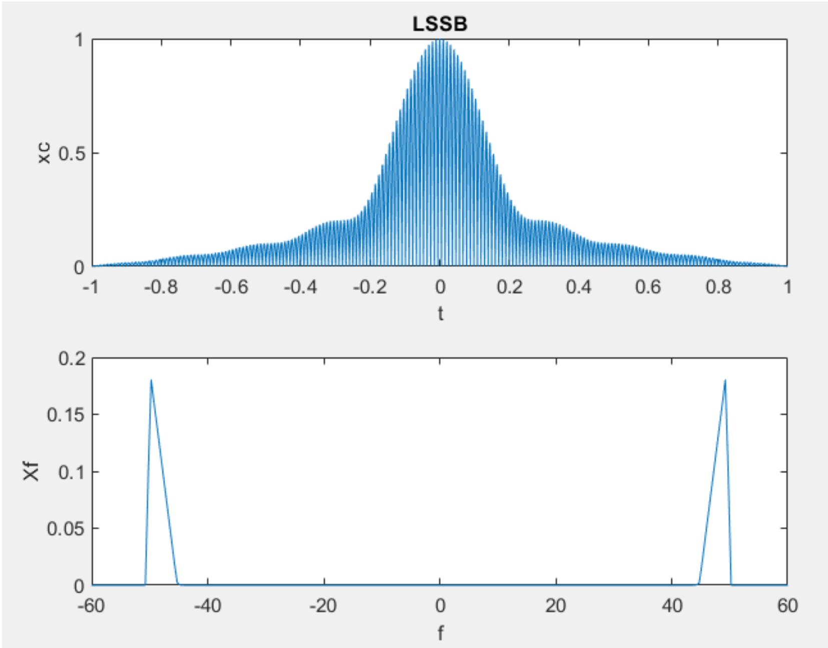

Figure 8: Time- and frequency-domain plots for an LSSB-modulated signal..

BONUS QUESTIONS

Question 8

Prove that the time duration σ_t and spectral bandwidth σ_f satisfy the inequality

$$\sigma_t \sigma_f \geqslant \frac{1}{4\pi}$$

, where the time duration

$$\sigma_t^2 = \frac{\int_{-\infty}^{\infty} (t-\bar{t})^2 |x(t)|^2 dt}{\int_{-\infty}^{\infty} |x(t)|^2 dt}, \quad \bar{t} = \frac{\int_{-\infty}^{\infty} t|x(t)|^2 dt}{\int_{-\infty}^{\infty} |x(t)|^2 dt}$$

and the spectral bandwidth

$$\sigma_f^2 = \frac{\int_{-\infty}^{\infty} (f-\bar{f})^2 |X(f)|^2 df}{\int_{-\infty}^{\infty} |X(f)|^2 df}, \quad \bar{f} = \frac{\int_{-\infty}^{\infty} f|X(f)|^2 df}{\int_{-\infty}^{\infty} |X(f)|^2 df}$$

such that $X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$. For which signal does the equality hold? Can you provide an intuition for this inequality?

Question 8 continued on next page...

Clearly,

$$(C+C^*) = 2\Re(C) \leqslant 2|C| => (C+C^*)^2 \leqslant 4|C|^2$$

For $C = \int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt$, we have

$$\Big(\int_{-\infty}^{\infty} [x_1(t)x_2^*(t) + x_1^*(t)x_2(t)]dt\Big)^2 \leqslant 4\Big|\int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt\Big|^2$$

According to Schwartz's inequality,

$$\Big|\int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt\Big|^2 \leqslant \int_{-\infty}^{\infty} |x_1(t)|^2 dt \int_{-\infty}^{\infty} |x_2(t)|^2 dt$$

So,

$$\Big(\int_{-\infty}^{\infty} [x_1(t)x_2^*(t) + x_1^*(t)x_2(t)]dt\Big)^2 \leqslant 4\int_{-\infty}^{\infty} |x_1(t)|^2 dt \int_{-\infty}^{\infty} |x_2(t)|^2 dt$$

Now, let $x_1(t) = tx(t)$ and $x_2(t) = \frac{dx(t)}{dt}$ and assume that $\bar{t}_x = 0$ and $\bar{f}_x = 0$ for $x(t)$ and $X(f)$,

$$\Big(\int_{-\infty}^{\infty} [tx^*(t)\frac{dx(t)}{dt} + tx(t)\frac{dx^*(t)}{dt}]dt\Big)^2 \leqslant 4\int_{-\infty}^{\infty} t^2|x(t)|^2 dt \int_{-\infty}^{\infty} |\frac{dx(t)}{dt}|^2 dt$$

Obviously,

$$\Big(\int_{-\infty}^{\infty} [tx^*(t)\frac{dx(t)}{dt} + tx(t)\frac{x^*(t)}{dt}]dt\Big)^2 = \Big(\int_{-\infty}^{\infty} t\frac{d}{dt}[x(t)x^*(t)]dt\Big)^2$$

and using integration by parts,

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = tx(t)x^*(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} t\frac{d}{dt}[x(t)x^*(t)]dt = -\int_{-\infty}^{\infty} t\frac{d}{dt}[x(t)x^*(t)]dt$$

since the time duration σ_t is limited. Therefore,

$$\Big(\int_{-\infty}^{\infty} |x(t)|^2 dt\Big)^2 \leqslant 4\int_{-\infty}^{\infty} t^2|x(t)|^2 dt \int_{-\infty}^{\infty} |\frac{dx(t)}{dt}|^2 dt$$

Finally, the Parseval's theorem yields

$$\int_{-\infty}^{\infty} |x(t)|^2 dt \int_{-\infty}^{\infty} |X(f)|^2 df \leq 4\int_{-\infty}^{\infty} t^2|x(t)|^2 dt \int_{-\infty}^{\infty} 4\pi^2 f^2|X(f)|^2 df$$

$$\frac{\int_{-\infty}^{\infty} t^2|x(t)|^2 dt}{\int_{-\infty}^{\infty} |x(t)|^2 dt} \frac{\int_{-\infty}^{\infty} f^2|X(f)|^2 df}{\int_{-\infty}^{\infty} |X(f)|^2 df} \geq \frac{1}{16\pi^2}$$

$$\sigma_t\sigma_f \geq \frac{1}{4\pi}$$

Now, assume that $\bar{t}_x \neq 0$ or $\bar{f}_x \neq 0$ for some $x(t)$ and $X(f)$. Let $y(t) = e^{-j2\pi\bar{f}_x(t+\bar{t}_x)}x(t+\bar{t}_x)$. Clearly, $Y(f) = X(f+\bar{f}_x)e^{j2\pi f\bar{t}}$. We have

$$\bar{t}_y = \frac{\int_{-\infty}^{\infty} \tau|y(\tau)|^2 d\tau}{\int_{-\infty}^{\infty} |y(\tau)|^2 d\tau} = \frac{\int_{-\infty}^{\infty} \tau|x(\tau+\bar{t}_x)|^2 d\tau}{\int_{-\infty}^{\infty} |x(\tau+\bar{t}_x)|^2 d\tau} = \frac{\int_{-\infty}^{\infty} (t-\bar{t}_x)|x(t)|^2 dt}{\int_{-\infty}^{\infty} |x(t)|^2 dt} = \bar{t}_x - \bar{t}_x = 0$$

Question 8 continued on next page...

$$\sigma_{t,y}^2 = \frac{\int_{-\infty}^{\infty} \tau^2 |y(\tau)|^2 d\tau}{\int_{-\infty}^{\infty} |y(\tau)|^2 d\tau} = \frac{\int_{-\infty}^{\infty} \tau^2 |x(\tau+\bar{t}_x)|^2 d\tau}{\int_{-\infty}^{\infty} |x(\tau+\bar{t}_x)|^2 d\tau} = \frac{\int_{-\infty}^{\infty} (t-\bar{t}_x)^2 |x(t)|^2 dt}{\int_{-\infty}^{\infty} |x(t)|^2 dt} = \sigma_{t,x}^2$$

$$\bar{f}_y = \frac{\int_{-\infty}^{\infty} u |Y(u)|^2 du}{\int_{-\infty}^{\infty} |Y(u)|^2 du} = \frac{\int_{-\infty}^{\infty} u |X(u+\bar{f}_x)|^2 du}{\int_{-\infty}^{\infty} |X(u+\bar{f}_x)|^2 du} = \frac{\int_{-\infty}^{\infty} (f-\bar{f}_x) |X(f)|^2 df}{\int_{-\infty}^{\infty} |X(f)|^2 df} = \bar{f}_x - \bar{f}_x = 0$$

$$\sigma_{f,y}^2 = \frac{\int_{-\infty}^{\infty} u^2 |Y(u)|^2 du}{\int_{-\infty}^{\infty} |Y(u)|^2 du} = \frac{\int_{-\infty}^{\infty} u^2 |X(u+\bar{f}_x)|^2 du}{\int_{-\infty}^{\infty} |X(u+\bar{f}_x)|^2 du} = \frac{\int_{-\infty}^{\infty} (f-\bar{f}_x)^2 |X(f)|^2 df}{\int_{-\infty}^{\infty} |X(f)|^2 df} = \sigma_{f,x}^2$$

Since, $\bar{t}_y = 0$ and $\bar{f}_y = 0$,

$$\sigma_{t,x}\sigma_{f,x} = \sigma_{t,y}\sigma_{f,y} \geq \frac{1}{4\pi}$$

One can easily check that the equality happens for the Gaussian signal $x(t) = Ae^{-\frac{(t-\bar{t})^2}{2\sigma_t^2}}$. The inequality justifies that the time duration and frequency bandwidth of a signal cannot be squeezed simultaneously below a certain value.

Question 9

Return your answers by filling the LaTeXtemplate of the assignment.