MATHEMATICAL QUESTIONS

Question 1

Consider the triangle periodic signal shown in Fig. 1.

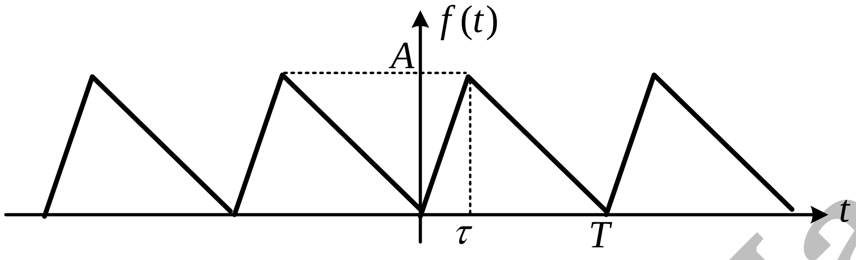


Figure 1: Triangle periodic signal.

(a) Express the periodic signal $f(t)$ in terms of elementary signals for $0 \leq t < T$.

For $0 \leq t \leq \tau$, we have

$$f(t) = \frac{A}{\tau}t$$

For $\tau < t \leq T$, we have,

$$f(t) = A - \frac{A}{T-\tau}(t-\tau)$$

So,

$$f(t) = \frac{A}{\tau}t(u(t) - u(t-\tau)) + (A - \frac{A}{T-\tau}(t-\tau))(u(t-\tau) - u(t-T))$$

$$f(t) = \frac{A}{\tau}r(t) - \frac{AT}{\tau(T-\tau)}r(t-\tau) + \frac{A}{T-\tau}r(t-T)$$

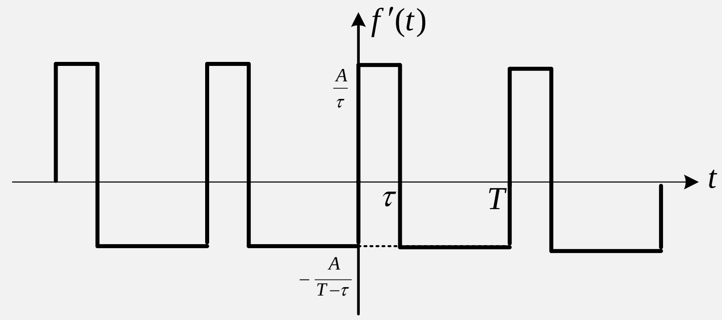
(b) Find the average and RMS values of $f(t)$.

$$f_{average} = \frac{1}{T}\int_0^T f(t)dt = \frac{1}{T}\int_0^\tau \frac{A}{\tau}tdt + \frac{1}{T}\int_\tau^T (A - \frac{A}{T-\tau}(t-\tau))dt = \frac{A}{2}$$

$$f_{rms} = \sqrt{\frac{1}{T}\int_0^T f^2(t)dt} = \sqrt{\frac{1}{T}\int_0^\tau (\frac{A}{\tau}t)^2dt + \frac{1}{T}\int_\tau^T (A - \frac{A}{T-\tau}(t-\tau))^2dt} = \frac{A}{\sqrt{3}}$$

(c) *Plot $f'(t)$, the derivative of $f(t)$.*

The derivative $f'(t) = \frac{A}{\tau}u(t) - \frac{AT}{\tau(T-\tau)}u(t-\tau) + \frac{A}{T-\tau}u(t-T)$ is plotted as



(d) *Let $g(t) = B + f(t)$, where B is a real number. Find the average and RMS values of $g(t)$.*

$$g_{av} = \frac{1}{T}\int_0^T (f(t)+B)dt = \frac{1}{T}\int_0^T Bdt + \frac{1}{T}\int_0^T f(t)dt = f_{average} + B = \frac{A}{2} + B$$

$$g_{rms} = \sqrt{\frac{1}{T}\int_0^T (f(t)+B)^2 dt} = \sqrt{\frac{1}{T}\int_0^T f^2(t)dt + \frac{1}{T}\int_0^T B^2 dt + \frac{1}{T}\int_0^T 2Bf(t)dt}$$

$$= \sqrt{f_{rms}^2 + B^2 + 2Bf_{av}} = \sqrt{\frac{A^2}{3} + B^2 + BA}$$

Question 2

Calculate the delivered power and voltage of the dependent source in Fig. 2.

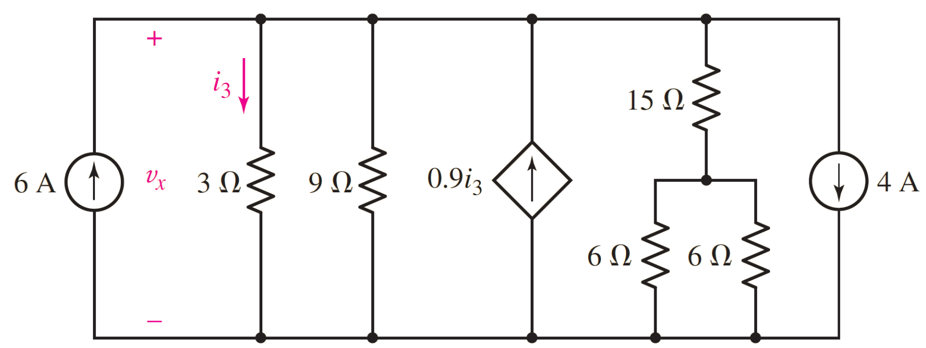


Figure 2: A circuit with dependent source.

Question 2 continued on next page. . .

$$i_3 = \frac{v_x}{3} \rightarrow v_x = 3i_3$$

$$KCL: -6 + \frac{v_x}{3} + \frac{v_x}{9} - 0.9i_3 + \frac{v_x}{15 + (6||6)} + 4 = 0$$

Hence,

$$-6 + i_3 + \frac{i_3}{3} - 0.9i_3 + \frac{i_3}{6} + 4 = 0 \rightarrow i_3 = \frac{10}{3} \rightarrow v_x = 10$$

So, the delivered power is

$$p = (0.9i_3)(v_x) = (0.9i_3)(3i_3) = (3)(10) = 30$$

Question 3

Consider the linear time-variant capacitor of Fig. 3 with the capacitance $C(t) = C_0 + C_1\cos(\omega_1 t)$.

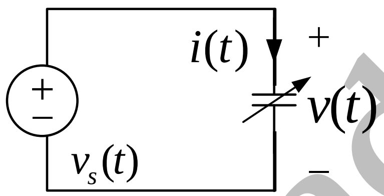


Figure 3: A linear time-variant capacitor with capacitance $C(t) = C_0 + C_1\cos(\omega_1 t)$.

(a) Find the capacitor current if $v_s(t) = A\cos(\omega_2 t)$. Further, calculate the energy stored in the capacitor during the interval $[0, t]$.

$$i(t) = \frac{d}{dt}(C(t)v(t)), \qquad v(t) = v_s(t) = A\cos(\omega_2 t)$$

$$\Rightarrow i(t) = \frac{d}{dt}[(c_0 + c_1\cos(\omega_1 t))(A\cos(\omega_2 t))] = \frac{d}{dt}[c_0 A\cos(\omega_2 t) + c_1 A\cos(\omega_1 t)\cos(\omega_2 t)]$$

$$= -c_0 A\omega_2 \sin(\omega_2 t) - c_1 A\omega_1 \sin(\omega_1 t)\cos(\omega_2 t) - c_1 A\omega_2 \sin(\omega_2 t)\cos(\omega_1 t)$$

$$w(0,t) = \int_0^t v(x)i(x)dx = \int_0^t -c_0 A^2\omega_2 \sin(\omega_2 x)\cos(\omega_2 x)dx$$

$$+ \int_0^t -c_1 A^2\omega_1 \sin(\omega_1 x)\cos^2(\omega_2 x)dx + \int_0^t -c_1 A^2\omega_2 \sin(\omega_2 x)\cos(\omega_1 x)\cos(\omega_2 x)dx =$$

$= \frac{c_0 A^2}{4}[\cos(2\omega_2 t) - 1] + c_1 A^2 \omega_1 [\frac{\cos((2\omega_2+\omega_1)t)-1}{4(2\omega_2+\omega_1)} + \frac{1-\cos((2\omega_2-\omega_1)t)}{4(2\omega_2-\omega_1)} + \frac{\cos(\omega_1 t)-1}{2\omega_1}] +$
$c_1 A^2 \omega_2 [\frac{\cos((2\omega_2+\omega_1)t)-1}{4(2\omega_2+\omega_1)} + \frac{\cos((2\omega_2-\omega_1)t)-1}{4(2\omega_2-\omega_1)}]$

(b) Find the capacitor current if $v_s(t) = A$. Further, calculate the energy stored in the capacitor during the interval $[0, t]$. Does the capacitor act like open circuit as $t \to \infty$?

$$i(t) = \frac{d}{dt}(C(t)v(t)), \qquad v(t) = v_s(t) = A$$

$$\Rightarrow i(t) = \frac{d}{dt}[A(c_0 + c_1 \cos(\omega_1 t))] = -c_1 A \omega_1 \sin(\omega_1 t)$$

$$w(0, t) = \int_0^t v(x) i(x) dx = \int_0^t -c_1 A^2 \omega_1 \sin(\omega_1 x) dx = c_1 A^2 (\cos(\omega_1 t) - 1)$$

Capacitor doesn't act like open circuit as $t \to \infty$ since the current doesn't converge to zero as $t \to \infty$.

(c) Find the capacitor current if $v_s(t) = A$ and $C_1 = 0$. Does the capacitor act like open circuit as $t \to \infty$?

$$i(t) = \frac{d}{dt}(C(t)v(t)), \qquad v(t) = v_s(t) = A$$

$$\Rightarrow i(t) = \frac{d}{dt}[Ac_0] = 0$$

Capacitor acts like open circuit as $t \to \infty$ since the current is always zeros and therefore, converges to zero as $t \to \infty$.

Question 4

In the circuit shown in Fig. 4, $v_s(t) = A\cos(\omega t)u(t)$ and $i_s(t) = B(1 - e^{-\alpha t})u(t)$. Calculate $v_L(t)$ and $i_C(t)$. Is $v_L(t)$ continuous? How about $i_C(t)$?

Question 4 continued on next page...

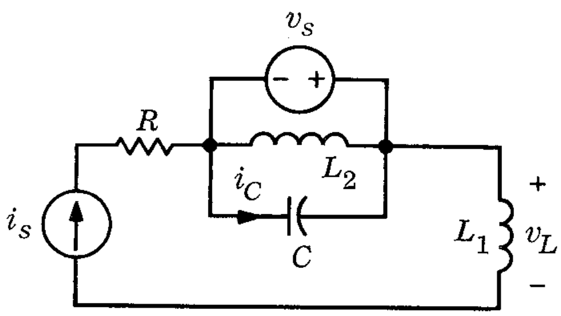


Figure 4: A circuit with LTI elements.

$$i_C = C\frac{dv_C}{dt}, \quad v_C = -v_s = -A\cos(\omega t)u(t)$$

$$\rightarrow i_C(t) = AC[\omega \sin(\omega t)u(t) - \delta(t)]$$

$$v_L = L_1\frac{di_{L_1}}{dt}, \quad i_{L_1} = i_s = B(1 - e^{-\alpha t})u(t)$$

$$\rightarrow v_L(t) = L_1B\alpha e^{-\alpha t}u(t) + L_1B(1 - e^{-\alpha t})\delta(t) = L_1B\alpha e^{-\alpha t}u(t)$$

$v_L(t)$ is not continuous due to function $u(t)$ which creates a discontinuity at $t = 0$
$i_C(t)$ is not continuous due to function $\delta(t)$ which creates a discontinuity at $t = 0$.

SOFTWARE QUESTIONS

Question 5

Write a simple MATLAB program that calculates the average and RMS values of a given periodic signal $f(t)$. The function $f(t)$ is represented by a function handle in its fundamental period $t \in [0, T]$.

Here is a sample MATLAB implementation of the desired function.

```
function [avg, rms] = avg_rms_cal(f, T)

% average
avg = integral(f,0,T)/T;
```

Question 5 continued on next page...

```

% rms
fsqr = @(t) f(t).^2;
rms = sqrt(integral(fsqr,0,T)/T);

end
```

You may use the following mfile to call the developed function and see its results.

```
clear all
clc

% sample average and rms calculation
[avg, rms] = avg_rms_cal(@sin, 2*pi);
avg
rms

% sample average and rms calculation
f = @(t) t;
[avg, rms] = avg_rms_cal(f, 1);
avg
rms
```

The mfile prints the average values of -1.6289×10^{-17} and 0.5000 and the RMS values of 0.7071 and 0.5774 for the periodic functions $f(t) = \sin(t), 0 \leq t \leq 2\pi$ and $f(t) = t, 0 \leq t \leq 1$, respectively.

BONUS QUESTIONS

Question 6

Find the RMS and average values of the real periodic signal $f(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k \cos(\frac{2\pi k}{T}t) + b_k \sin(\frac{2\pi k}{T}t)\right]$ **expanded in its Fourier series form.**

Before answering the main problem, let us express a lemma.
Lemma: For integer numbers n and m,

$$\int_0^T \cos(\frac{2\pi m}{T}t)\cos(\frac{2\pi n}{T}t)dt = \begin{cases} T/2 & m = n \neq 0 \\ T/2 & m = -n \neq 0 \\ T & m = n = 0 \\ 0 & |m| \neq |n| \end{cases} \tag{1}$$

$$\int_0^T \sin(\frac{2\pi m}{T}t)\sin(\frac{2\pi n}{T}t)dt = \begin{cases} T/2 & m = n \neq 0 \\ -T/2 & m = -n \neq 0 \\ 0 & m = n = 0 \\ 0 & |m| \neq |n| \end{cases} \tag{2}$$

$$\int_0^T \sin(\frac{2\pi m}{T}t)\cos(\frac{2\pi n}{T}t)dt = 0 \tag{3}$$

Proving the lemma is straightforward and it is suggested to do it yourself.

Average:
We know that integral of a sinusoidal signal on a period is zero (check it!). So,

$$f_{av} = \frac{1}{T}\int_0^T f(t)dt = \frac{1}{T}\int_0^T a_0 dt + \frac{1}{T}\int_0^T \underbrace{\sum_{k=1}^{\infty}\left[a_k\cos(\frac{2\pi k}{T}t) + b_k\sin(\frac{2\pi k}{T}t)\right]}_{y(t)} dt$$

$$= a_0 + y_{av} = a_0 + \frac{1}{T}\sum_{k=1}^{\infty}\underbrace{\int_0^T \left[a_k\cos(\frac{2\pi k}{T}t) + b_k\sin(\frac{2\pi k}{T}t)\right]dt}_{0} = a_0$$

Note that $y_{av} = 0$.
RMS:
The signal $f(t)$ consists of a DC term a_0 and an AC term $y(t)$ with zero average $y_{av} = 0$ as

$$f(t) = a_0 + \underbrace{\sum_{k=1}^{\infty}\left[a_k\cos(\frac{2\pi k}{T}t) + b_k\sin(\frac{2\pi k}{T}t)\right]}_{y(t)}$$

So,

$$f_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T f^2(t)dt} = \sqrt{a_0^2 + y_{\text{rms}}^2 + 2a_0 y_{av}} = \sqrt{a_0^2 + y_{\text{rms}}^2}$$

, where

$$y_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T y^2(t)dt}$$

Note that all the terms of $y(t)$ are periodic with the period T. First, we should compute $y^2(t)$:

$$y^2(t) = (\sum_{i=1}^{\infty}\left[a_i\cos(\frac{2\pi i}{T}t) + b_i\sin(\frac{2\pi i}{T}t)\right])(\sum_{j=1}^{\infty}\left[a_j\cos(\frac{2\pi j}{T}t) + b_j\sin(\frac{2\pi j}{T}t)\right])$$

$$= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i a_j\cos(\frac{2\pi i}{T}t)\cos(\frac{2\pi j}{T}t) + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} b_i b_j\sin(\frac{2\pi i}{T}t)\sin(\frac{2\pi j}{T}t)$$

$$+ \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i b_j\cos(\frac{2\pi i}{T}t)\sin(\frac{2\pi j}{T}t) + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} b_i a_j\sin(\frac{2\pi i}{T}t)\cos(\frac{2\pi j}{T}t)$$

Next, we should compute $\int_0^T y^2(t)dt$ which means putting an integral before each of above terms (don't be afraid of these long terms, it is just an expansion).

$$\int_0^T y^2(t)dt = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i a_j\int_0^T \cos(\frac{2\pi i}{T}t)\cos(\frac{2\pi j}{T}t)dt + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} b_i b_j\int_0^T \sin(\frac{2\pi i}{T}t)\sin(\frac{2\pi j}{T}t)dt$$

$$+ \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i b_j\int_0^T \cos(\frac{2\pi i}{T}t)\sin(\frac{2\pi j}{T}t)dt + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} b_i a_j\int_0^T \sin(\frac{2\pi i}{T}t)\cos(\frac{2\pi j}{T}t)dt$$

Question 6 continued on next page...

Now, we use the lemma. Last two terms are zero because they are made of the product of sines and cosines which are orthogonal in a period as given in (3). From first two terms, only the same-frequency sines and cosines remain as can be interpreted from (1) and (2). So,

$$\begin{aligned}\int_0^T y^2(t)dt &= \sum_{i=1}^{\infty} a_i^2 \frac{T}{2} + \sum_{i=1}^{\infty} b_i^2 \frac{T}{2} \\ &= \frac{T}{2}\sum_{i=1}^{\infty}(a_i^2 + b_i^2)\end{aligned}$$

So far, we have proven the Parseval's theorem which is a famous theorem and you will learn its details in the engineering mathematics course. Rest of the solution is quite simple.

$$\begin{aligned}\frac{1}{T}\int_0^T y^2(t)dt &= \frac{1}{2}\sum_{i=1}^{\infty}(a_i^2 + b_i^2) \\ \Rightarrow y_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T y^2(t)dt} &= \frac{\sqrt{\sum_{i=1}^{\infty}(a_i^2 + b_i^2)}}{\sqrt{2}}\end{aligned}$$

Hence

$$f_{\text{rms}} = \sqrt{a_0^2 + \frac{1}{2}\sum_{i=1}^{\infty}(a_i^2 + b_i^2)}$$

Question 7

Return your answers by filling the LaTeXtemplate of the assignment.

EXTRA QUESTIONS

Question 8

Feel free to solve the following questions from the book *"Basic Circuit Theory"* by C. Desoer and E. Kuh.

1. **Chapter 2, question 2.**
2. **Chapter 2, question 7.**
3. **Chapter 2, question 8.**
4. **Chapter 2, question 10.**
5. **Chapter 2, question 15.**

Question 8 continued on next page...

6. **Chapter 2, question 16.**
7. **Chapter 2, question 17.**
8. **Chapter 2, question 18.**