MATHEMATICAL QUESTIONS

Question 1

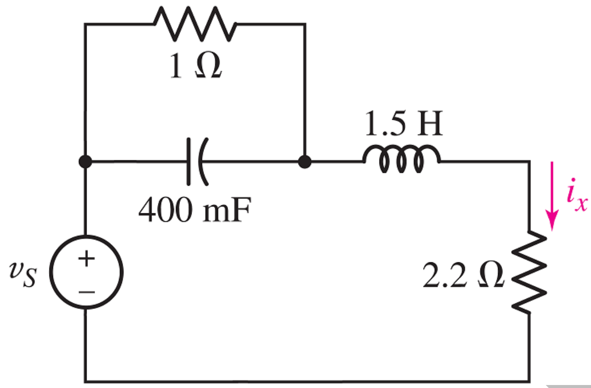


Figure 1: An LTI circuit.

Consider the LTI circuit shown in Fig. 1.

(a) Find the differential equation relating $i_x(t)$ to $v_s(t)$.

Pay attention to Fig. 2.

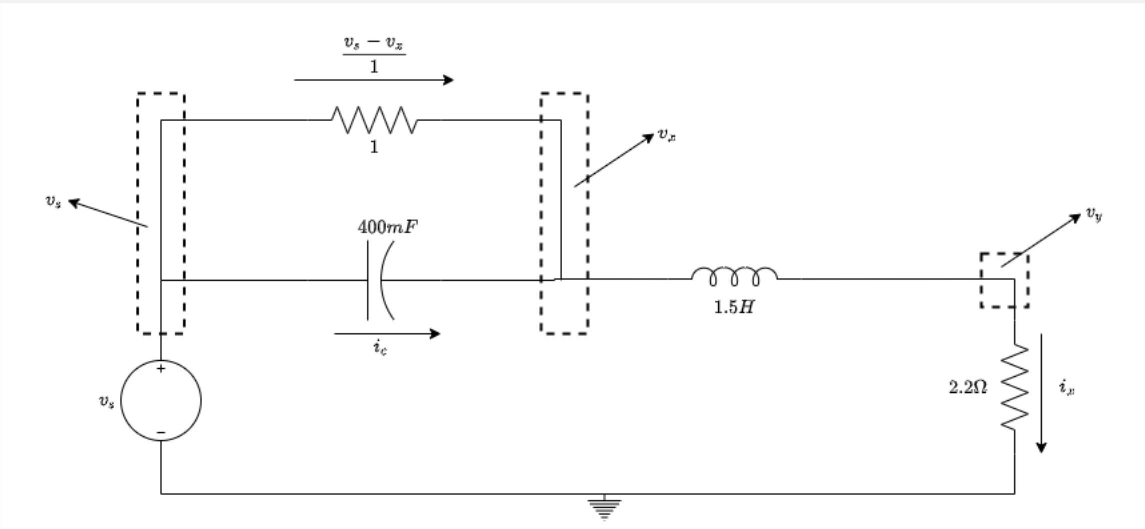


Figure 2: The labeled and annotated version of the circuit.

Question 1 continued on next page...

Writing KCL at node v_x,

$$\frac{v_s - v_x}{1} + i_c = i_x$$

Further, we know that

$$i_c = c\frac{dv_c}{dt} = 400 \times 10^{-3} \times \frac{d}{dt}(v_s - v_x)$$

Substituting the second equation into the first one,

$$v_s - v_x + 0.4\frac{dv_s}{dt} - 0.4\frac{dv_x}{dt} = i_x \tag{1}$$

We also know that

$$v_x - v_y = L\frac{di_L}{dt} = 1.5 \times \frac{di_x}{dt}, \quad v_y = 2.2i_x$$

which yields

$$v_x = 1.5\frac{di_x}{dt} + 2.2i_x \tag{2}$$

Substituting (2) in (1), we obtain

$$0.6\frac{d^2i_x}{dt^2} + 2.38\frac{di_x}{dt} + 3.2i_x = 0.4\frac{dv_s}{dt} + v_s$$

(b) Obtain the impulse response of $i_x(t)$.

The left hand side of the obtained differential equation has higher order than its right hand side. So, there is no impulse or its derivatives in the impulse response and it has a form like

$$h(t) = (Ae^{s_1t} + Be^{s_2t})u(t)$$

, where A and B are coefficients to be determined and s_1 and s_2 are the roots of the characteristic equation

$$0.6s^2 + 2.38s + 3.2 = 0 \Rightarrow s_1, s_2 = -1.9833 \pm j1.1831$$

We should substitute $h(t)$ into the obtain differential equation and find the coefficients. First, we calculate the derivatives of $h(t)$.

$$h'(t) = (As_1e^{s_1t} + Bs_2e^{s_2t})u(t) + (Ae^{s_1t} + Be^{s_2t})\delta(t)$$
$$h''(t) = (As_1^2e^{s_1t} + Bs_2^2e^{s_2t})u(t) + 2(As_1e^{s_1t} + Bs_2e^{s_2t})\delta(t) + (Ae^{s_1t} + Be^{s_2t})\delta'(t)$$

Remember that for any continuous function $f(t)$

$$f(t)\delta(t) = f(0)\delta(t)$$

$$f(t)\delta'(t) = f(0)\delta'(t)$$

Hence,

$$h'(t) = (As_1e^{s_1t} + Bs_2e^{s_2t})u(t) + (A + B)\delta(t)$$
$$h''(t) = (As_1^2e^{s_1t} + Bs_2^2e^{s_2t})u(t) + 2(As_1 + Bs_2)\delta(t) + (A + B)\delta'(t)$$

Question 1 continued on next page. . .

Substituting $h(t)$ and its derivatives in the differential equation,

$$\Big[\underbrace{(0.6s_1^2+2.38s_1+3.2)}_{0}Ae^{s_1t}+\underbrace{(0.6s_2^2+2.38s_2+3.2)}_{0}Be^{s_2t}\Big]u(t)$$
$$+\Big[(1.2s_1+2.38)A+(1.2s_2+2.38)B\Big]\delta(t)+(0.6A+0.6B)\delta'(t)=0.4\delta'(t)+\delta(t)$$

So, we have the following system of equations

$$\begin{cases}(1.2s_1+2.38)A+(1.2s_2+2.38)B=1\\ 0.6A+0.6B=0.4\end{cases}$$

with the solution

$$A=0.3333-j0.3522=0.4849\underline{/-46.5793^\circ},\quad B=0.3333+j0.3522=A^*$$

Finally,

$$h(t)=(Ae^{s_1t}+Be^{s_2t})u(t)=0.9698e^{-1.9833t}\cos(1.1831t-46.5793^\circ)u(t)$$

(c) Calculate the steady state response of $i_x(t)$ if $v_s(t)=A\cos(\omega t+\theta)$.

In phasor domain, the differential equation becomes

$$0.6(j\omega)^2I_x+2.38(j\omega)I_x+3.2I_x=0.4(j\omega)V_s+V_s$$

and we know that $V_s=A\underline{/\theta}$ if $A\geq 0$. Hence,

$$(3.2-0.6\omega^2+j2.38\omega)I_x=(1+j0.4\omega)A\underline{/\theta}\Rightarrow I_x=\frac{(1+j0.4\omega)A\underline{/\theta}}{3.2-0.6\omega^2+j2.38\omega}$$

and

$$i_{xss}(t)=A\frac{\sqrt{1+0.4^2\omega^2}}{\sqrt{(3.2-0.6\omega^2)^2+2.38^2\omega^2}}\cos\left(\omega t+\underline{/I_x}\right)$$

, where

$$\underline{/I_x}=\theta+\tan^{-1}\left(0.4\omega\right)-\begin{cases}-\pi+\tan^{-1}\left(\frac{2.38\omega}{3.2-0.6\omega^2}\right), & \omega\leq-\frac{4\sqrt{3}}{3}\\ \tan^{-1}\left(\frac{2.38\omega}{3.2-0.6\omega^2}\right), & -\frac{4\sqrt{3}}{3}<\omega\leq 0\\ \tan^{-1}\left(\frac{2.38\omega}{3.2-0.6\omega^2}\right), & 0<\omega\leq\frac{4\sqrt{3}}{3}\\ \pi+\tan^{-1}\left(\frac{2.38\omega}{3.2-0.6\omega^2}\right), & \omega\geq\frac{4\sqrt{3}}{3}\end{cases}$$

Question 2

Determine the Thevenin equivalent seen by $-j10\,\Omega$ impedance of Fig. 3 and use this to compute V_1.

Question 2 continued on next page...

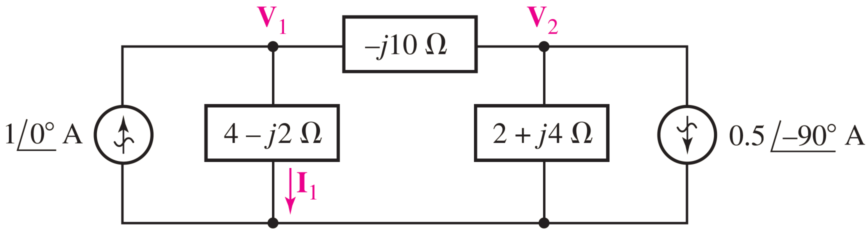


Figure 3: A circuit for which Thevenin equivalent seen by $-j10\,\Omega$ impedance is desired.

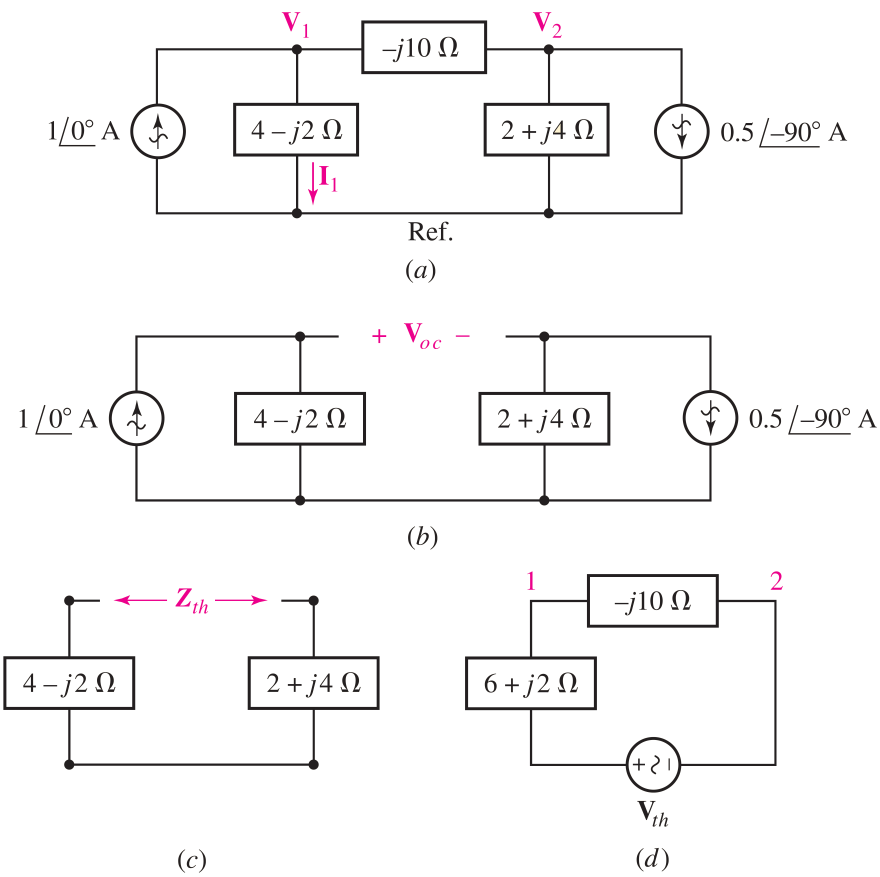


Figure 4: (a) The Thevenin equivalent seen by the $-j10\,\Omega$ impedance is desired. (b) V_{oc} is defined. (c) Z_{th} is defined. (d) The circuit is redrawn using the Thevenin equivalent.

The open-circuit voltage, defined in Fig. 4(b), is

$$V_{oc} = (1\angle 0^\circ)(4-j2) - (-0.5\angle -90^\circ)(2+j4) = 4 - j2 + 2 - j1 = 6 - j3\ \text{V}$$

. The impedance of the inactive circuit of Fig. 4(c) as viewed from the load terminals is simply the sum of the two remaining impedances. Hence,

$$Z_{th} = 6 + j2\,\Omega$$

Question 2 continued on next page...

. When we reconnect the circuit as in Fig. 4(d), the current directed from node 1 toward node 2 through the $-j10\ \Omega$ load is

$$I_{12} = \frac{6-j3}{6+j2-j10} = 0.6 + j0.3 \text{ A}$$

. We now know the current flowing through the $-j10\ \Omega$ impedance of Fig. 4(a). Note that we are unable to compute V_1 using the circuit of Fig. 4(d) as the reference node no longer exists. Returning to the original circuit, then, and subtracting the $0.6 + j0.3$ A current from the left source current, the downward current through the $4 - j2\ \Omega$ branch is found

$$I_1 = 1 - 0.6 - j0.3 = 0.4 - j03 \text{ A}$$

and, thus,

$$V_1 = (4-j2)(0.4-j03) = 1 - j2 \text{ V}$$

.

Question 3

A compensating capacitor parallel to the voltage source can be added to the circuit of Fig. 5 to make its overall power factor closer to 1. Calculate the capacitance value C of the compensating capacitor such that the overall power factor is 0.95. Assume that $Z_1 = 1.5\Omega$, $Z_2 = j\Omega$, and $Z_3 = (1-j2)\Omega$.

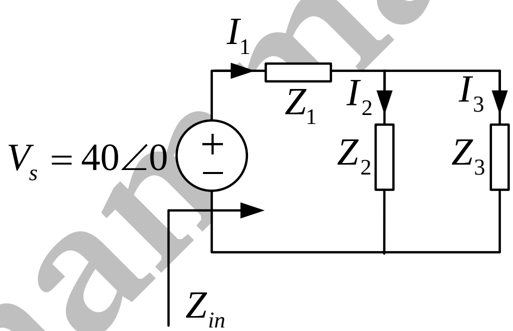


Figure 5: A circuit in phasor domain.

$Z_{inold} = [(j)||(1-j2)] + (1.5) = 2 + 1.5j$

$I_1 = \dfrac{40 < 0}{2+1.5j} = 16 < (-36.9)$

$P_{Z_{inold}} = \frac{1}{2} Z_{inold} |I_1|^2 = 256 + 192j \Rightarrow P_{avold} = 256, Q_{old} = 192, PF_{old} = 0.8$

$Z_{innew} = (2+1.5j)||(\dfrac{1}{jC\omega})$

$P_{Z_{innew}} = (256 + jQ_{new}) \Rightarrow P_{avnew} = 256, Q_{new} = ?, PF_{new} = 0.95$

$Q_{new} = P_{av} \tan(\cos^{-1}(0.95)) = 84.14$

$Q_{new} = Q_{old} + Q_c \Rightarrow Q_c = 84.14 - 192 = -107.86$

$P_c = 0 + jQ_c = \frac{1}{2} Y_c^* V_s^2 = -\frac{1}{2} j |V_s|^2 C\omega \Rightarrow -107.86 = -(40)^2 C\omega \Rightarrow C = \frac{0.067}{\omega}$

Question 4

Consider the one port shown in Fig. 6.

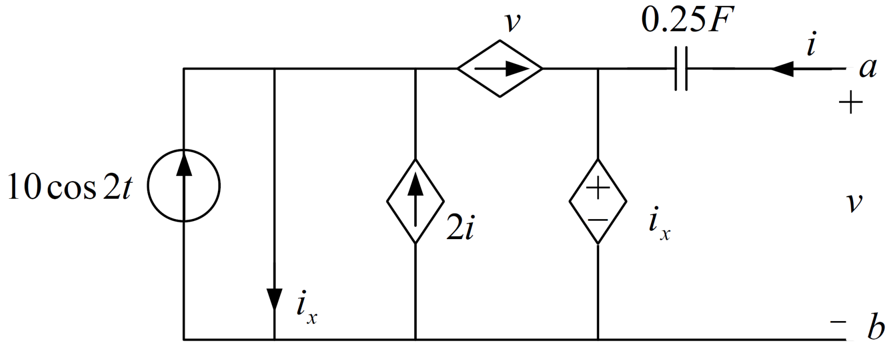


Figure 6: A circuit in sinusoidal steady state for which the Thevenin and Norton equivalent circuits are required.

(a) Find the equivalent Thevenin and Norton circuit seen from the port ab.

In phasor domain,

$$10cos(2t) \equiv 10, \quad Z_c = \frac{1}{j\omega C} = -2j$$

KCL gives

$$I_x = 10 + 2I - V$$

while KVL yields

$$V = I_x + IZ_c = I_x - 2jI$$

So,

$$V = 10 + 2I - V - 2jI$$
$$V = (1 - j)I + 5$$

Finally,

$$V_{oc} = 5, \quad Z_{th} = 1 - j \Rightarrow I_{sc} = \frac{V_{oc}}{Z_{th}} = \frac{5}{1-j} = 2.5 + 2.5j$$

(b) Calculate the impedance of the load Z_L absorbing the maximum power from the port ab.

The maximum amount of power will be dissipated by a load whose impedance is equal to the conjugate if the Thevenin impedance. So,

$$Z_L = Z_{th}^* = 1 + j$$

Question 5 continued on next page...

Question 5

Let $H_1(j\omega) = \frac{I}{I_{s1}}\Big|_{\substack{V_{s2}=0\\ Vs3=0}} = \frac{2+j\omega}{1+j\omega}$**,** $H_2(j\omega) = \frac{I}{V_{s2}}\Big|_{\substack{I_{s1}=0\\ Vs3=0}} = \frac{3-2j\omega}{1+j\omega}$**, and** $H_3(j\omega) = \frac{I}{V_{s3}}\Big|_{\substack{I_{s1}=0\\ Vs2=0}} = \frac{4+j\omega}{1+j\omega}$ **in the circuit shown in Fig. 7.**

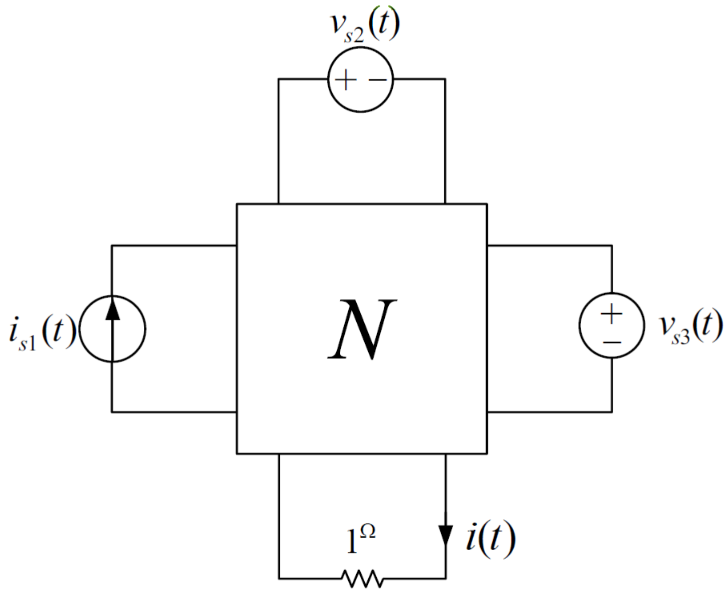


Figure 7: A multi-input circuit in sinusoidal steady state.

(a) Calculate the average power consumed by the $1\,\Omega$ *resistor if* $i_{s1}(t) = \cos(t)$, $v_{s2}(t) = 2\cos(t)$, *and* $v_{s3}(t) = 3\cos(t)$.

The sources have identical frequency so, we can use superposition in phasor domain.
$I_1 = H_1(j1).I_{s1} = \frac{2+j1}{1+j1}.1\underline{/0^\circ} = 1.5 - 0.5j$
$I_2 = H_2(j1).V_{s2} = \frac{3-2j}{1+j1}.2\underline{/0^\circ} = 1 - 5j$
$I_3 = H_3(j1).V_{s3} = \frac{4+j1}{1+j1}.3\underline{/0^\circ} = 7.5 - 4.5j$
$I = I_1 + I_2 + I_3 = 10 - 10j = 10\sqrt{2}\underline{/-45^\circ}$
$\Rightarrow P_{av} = \frac{1}{2}R|I|^2 = \frac{1}{2}1(10\sqrt{2})^2 = 100W$

(b) Calculate the average power consumed by the $1\,\Omega$ *resistor if* $i_{s1}(t) = \cos(2t)$, $v_{s2}(t) = 2\cos(3t)$, *and* $v_{s3}(t) = 3\cos(2t)$.

The sources have different frequencies so, we cannot use superposition in phasor power.
$I_1 = H_1(j2).I_{s1} = \frac{2+2j}{1+2j}.1\underline{/0^\circ} = 1.2 - 0.4j$
$I_2 = H_2(j3).V_{s2} = \frac{3-6j}{1+3j}.2\underline{/0^\circ} = -3 - 3j$
$I_3 = H_3(j2).V_{s3} = \frac{4+2j}{1+2j}.3\underline{/0^\circ} = 4.8 - 3.6j$
$\omega_1 = \omega_3 \neq \omega_2$
$\Rightarrow P_{av} = P_{av1,3} + P_{av2} = \frac{1}{2}R|I_1 + I_3|^2 + \frac{1}{2}R|I_2|^2 = \frac{1}{2}1(7.2)^2 + \frac{1}{2}1(4.2)^2 = 34.74W$

Question 6

Consider the circuit shown in Fig. 10.

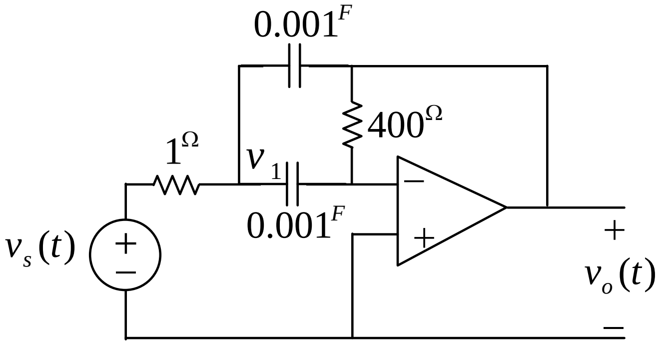


Figure 8: An op-amp circuit in sinusoidal steady state.

(a) Calculate the network function $H(j\omega) = \frac{V_o(j\omega)}{V_s(j\omega)}$.

KCL:

$$\frac{0 - V_o(j\omega)}{400} + \frac{0 - V_1(j\omega)}{\frac{1}{j\omega 0.001}} = 0$$

KCL:

$$\frac{V_1(j\omega) - V_s(j\omega)}{1} + \frac{V_1(j\omega) - V_o(j\omega)}{\frac{1}{j\omega 0.001}} + \frac{V_1(j\omega) - 0}{\frac{1}{j\omega 0.001}} = 0$$

By eliminating $V_1(j\omega)$ from the above equations, we get

$$H(j\omega) = \frac{V_o(j\omega)}{V_s(j\omega)} = \frac{-200}{1 + j10(\frac{\omega}{50} - \frac{50}{\omega})}$$

(b) Calculate and plot the amplitude and phase of the network function.

$$|H(j\omega)| = \frac{200}{\sqrt{1 + 100(\frac{\omega}{50} - \frac{50}{\omega})^2}}$$

Question 6 continued on next page...

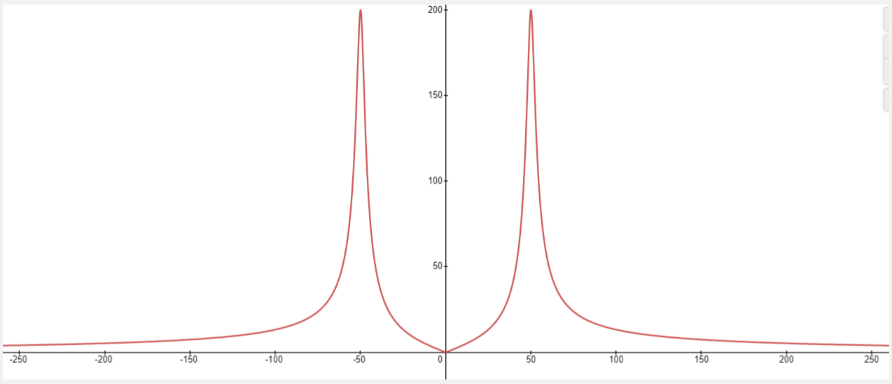


Figure 9: Amplitude of the network function.

$$\underline{/(j\omega)} = \pi - (\tan^{-1}(10\frac{\omega}{50} - 10\frac{50}{\omega}))$$

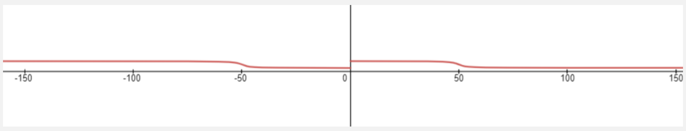


Figure 10: Phase of the network function.

(c) Investigate the filtering response of the network function and calculate its describing parameters such as central frequency, bandwidth, and quality factor.

The network function describes a second-order bandpass filter with the network function $H(j\omega) = \frac{R}{1+jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}$. Equating the two equations,

$$Q = 10, \ \omega = \omega_r = \omega_0 = 50 \text{ rad/s}$$

$$Q = \frac{\omega_0}{\Delta\omega} \Rightarrow \Delta\omega = 5 \text{ rad/s}, \quad |H(j\omega)|_{Max} = 200$$

$$\omega_1 \approx 47.5 \text{ rad/s}, \quad \omega_2 \approx 52.5 \text{ rad/s}$$

SOFTWARE QUESTIONS

Question 7

Use AC sweep simulation of PSpice to investigate how the values of the elements affect the filtering response of a series RLC circuit. Particularly, analyze the impact of the circuit element values on the bandwidth and central frequency of the filtering response.

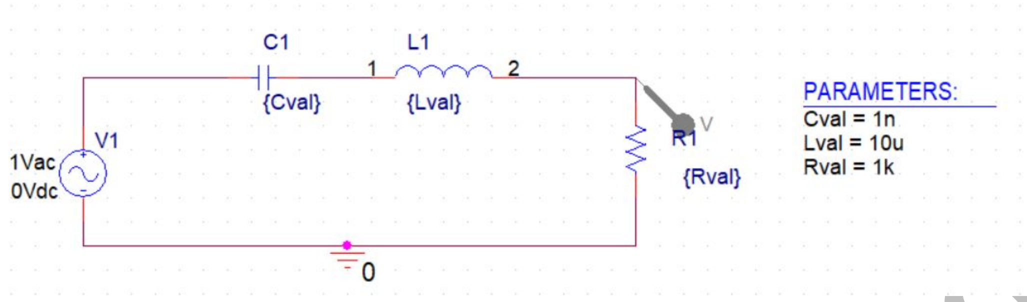


Figure 11: Schematic of the series RLC circuit.

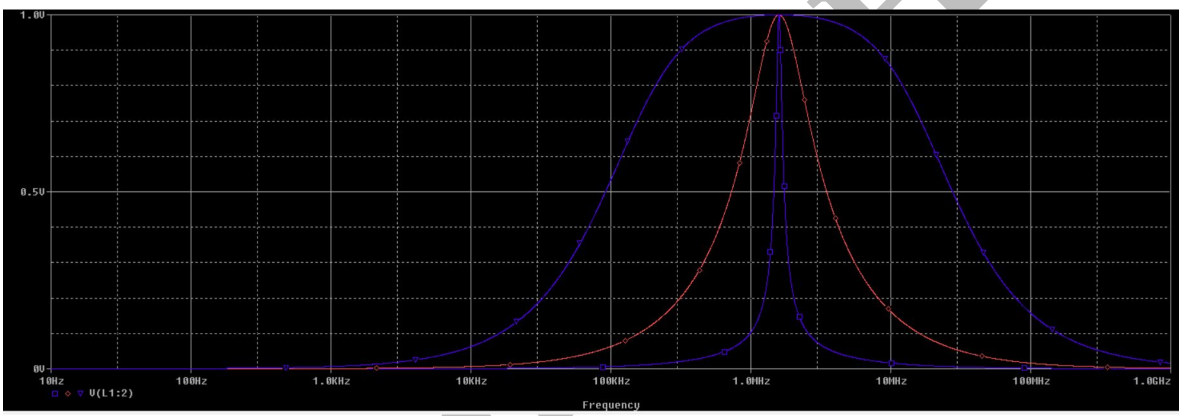


Figure 12: Amplitude response of the series RLC circuit for $R = 10, 100, 1000\ \Omega$.

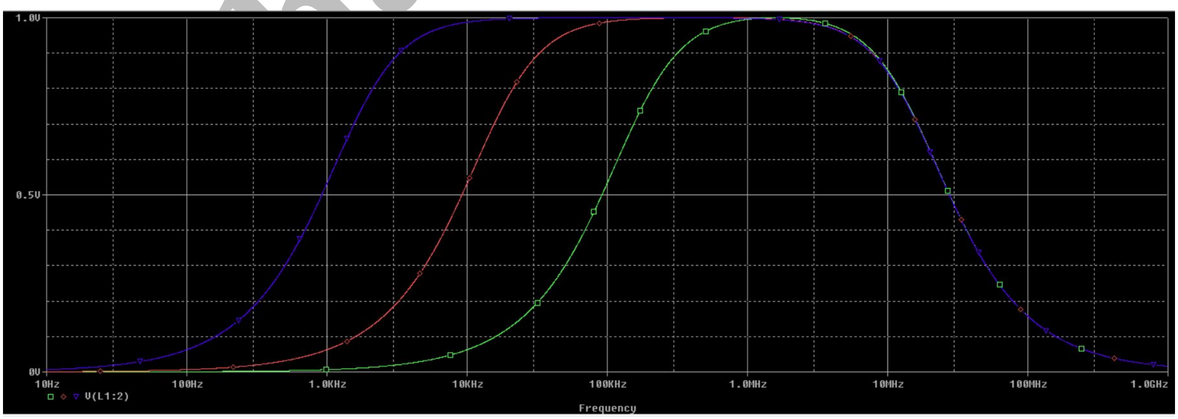


Figure 13: Amplitude response of the series RLC circuit for $C = 1, 10, 100$ nF.

Question 7 continued on next page...

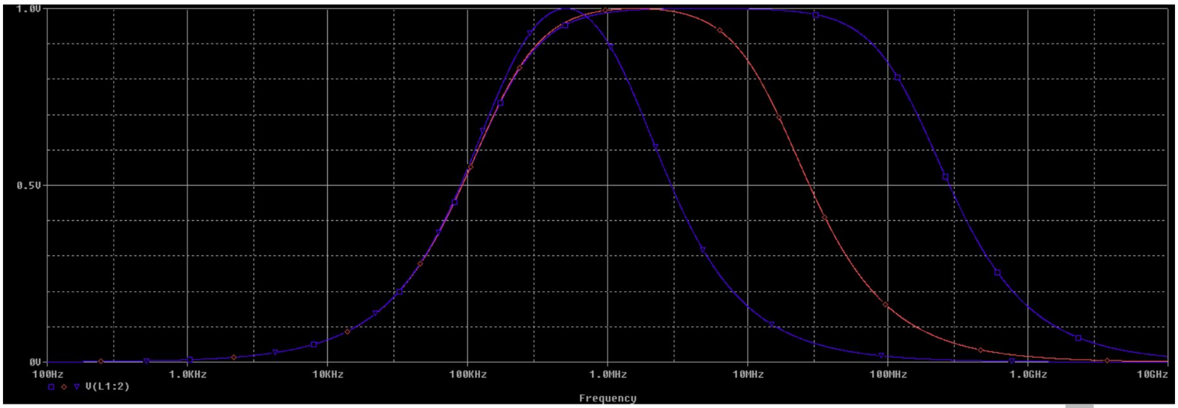


Figure 14: Amplitude response of the series RLC circuit for $L = 1, 10, 100$ μH.

The schematic of the series RLC circuit is shown in Fig. 11 while Figs. 12-14 investigate the impact of circuit elements of the filtering response. Clearly, increasing the resistance widens the filtering bandwidth while changing the inductance or capacitance shift the filtering central frequency.

BONUS QUESTIONS

Question 8

Consider the circuit shown in Fig. 15.

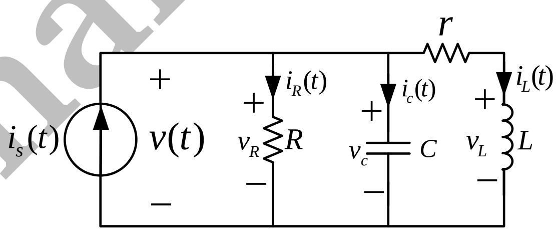


Figure 15: A parallel RLC circuit with real inductor.

(a) Calculate the network function $H(j\omega) = \frac{V(j\omega)}{I_s(j\omega)}$.

The network function can be obtained By current division:

$$V(j\omega) = RI_R(j\omega) = RI_S(j\omega)\frac{\frac{1}{R}}{\frac{1}{R} + j\omega C + \frac{1}{r+j\omega L}}$$

Question 8 continued on next page...

$$V(j\omega) = \frac{R(j\omega L + r)}{RCL(j\omega)^2 + (RCr + L)j\omega + (R + r)} I_S(j\omega)$$

So,

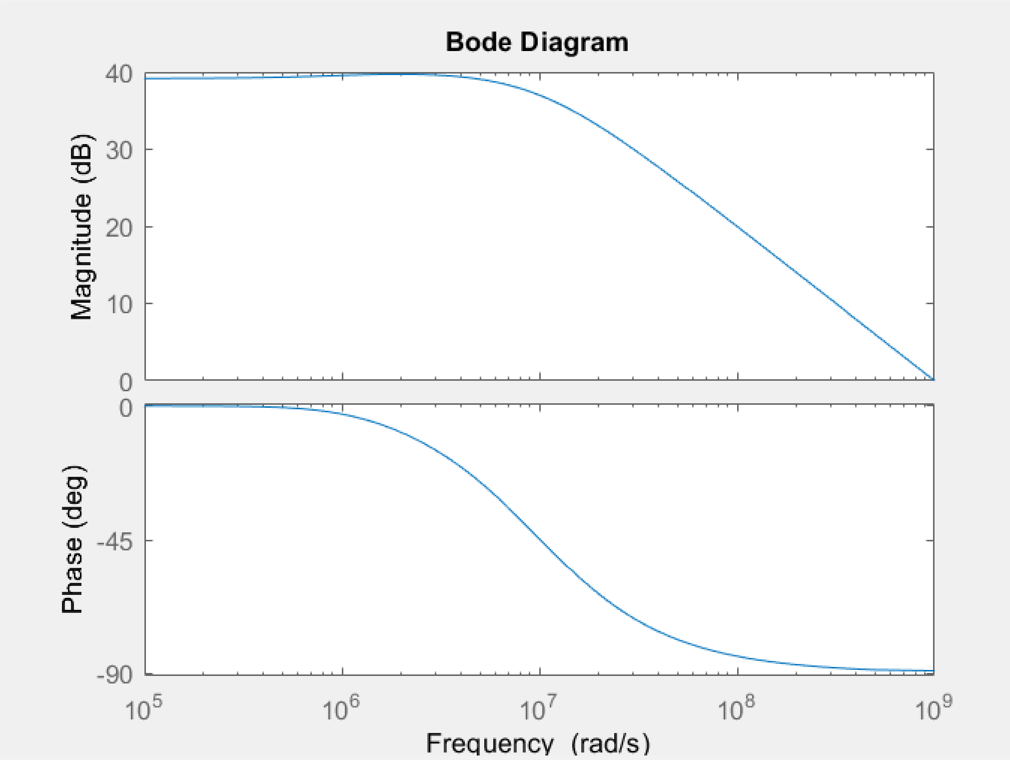
$$H(j\omega) = \frac{V(j\omega)}{I_S(j\omega)} = \frac{R(j\omega L + r)}{RCL(j\omega)^2 + (RCr + L)j\omega + (R + r)}$$

(b) *Calculate and plot the typical amplitude and phase curves of the network function.*

$$|H(j\omega)| = \frac{R\sqrt{r^2 + \omega^2 L^2}}{\sqrt{(R + r - RCL\omega^2)^2 + (RCr + L)^2\omega^2}}$$

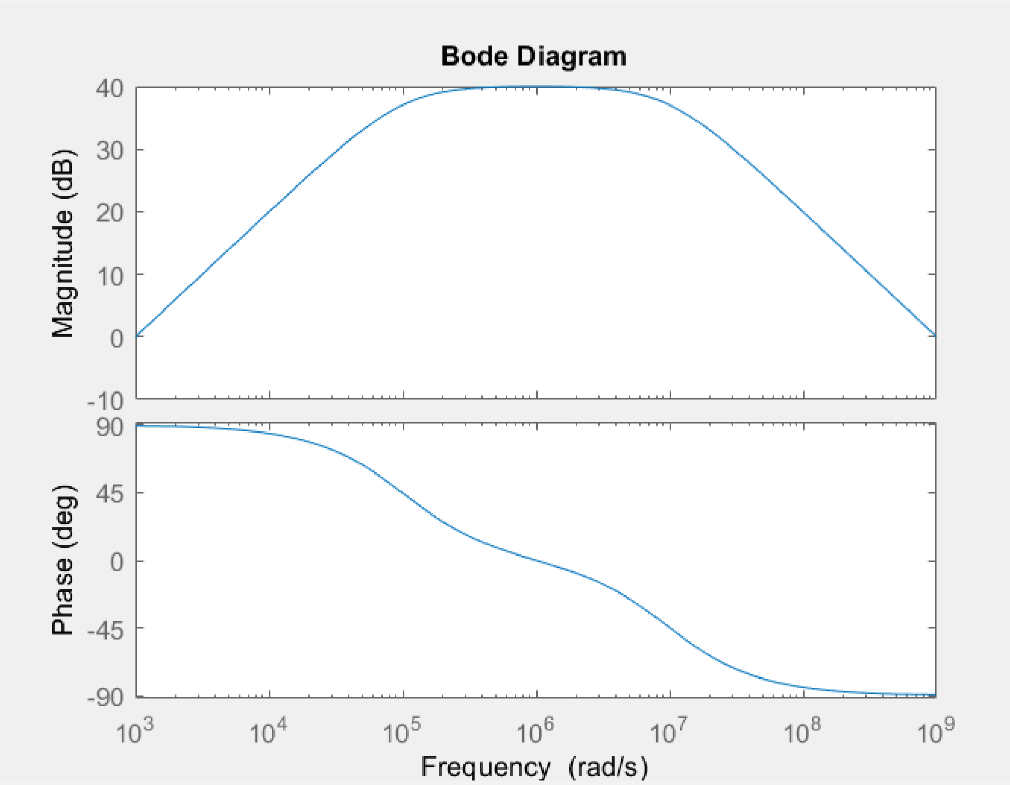
$$\underline{/H(j\omega)} = \tan^{-1}(\frac{\omega L}{r}) - \tan^{-1}(\frac{(RCr + L)\omega}{R + r - RCL\omega^2})$$

The amplitude and phase curves of the network function for $R = 100\ \Omega$, $C = 1$ nF, $L = 1$ mH, and $r = 1000\ \Omega$ is as follows.



(c) *Compare the filtering response with an RLC circuit with $r = 0$.*

The amplitude and phase curves of the network function for $R = 100\ \Omega$, $C = 1$ nF, $L = 1$ mH, and $r = 0\ \Omega$ is as follows.



As can be seen from the result of the previous part, the amplitude of $H(j\omega)$ is 40 dB for low frequencies and it converge to zero as we decrease the frequency, so the circuit acts as a lowpass filter.

If $r = 0$,the amplitude of $H(j\omega)$ is almost zero in very low and high frequencies but it's amplitude is 40 dB for 10^4 to 10^8 rad/s, so the circuit will be a bandpass filter.

In fact, the value of r severely affects the filtering response. For small values of r, the circuit resembles an RLC circuit with bandpass filtering response.

Question 9

Return your answers by filling the LaTeXtemplate of the assignment.

EXTRA QUESTIONS

Question 10

Feel free to solve the following questions from the book *"Engineering Circuit Analysis"* by W. Hayt, J. Kemmerly, and S. Durbin.

1. **Chapter 10, question 10.**
2. **Chapter 10, question 15.**

Question 10 continued on next page...

3. **Chapter 10, question 35.**
4. **Chapter 10, question 40.**
5. **Chapter 10, question 44.**
6. **Chapter 10, question 47.**
7. **Chapter 10, question 48.**
8. **Chapter 10, question 60.**
9. **Chapter 10, question 61.**
10. **Chapter 10, question 66.**
11. **Chapter 10, question 68.**
12. **Chapter 10, question 74.**
13. **Chapter 10, question 76.**